Optimal investment, consumption and life insurance in a Lévy market

by

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Declaration

I, the undersigned declare that the dissertation, which I hereby submit for the degree *Magister Scientiae in Mathematics of Finance* at the University of Pretoria, is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other university.

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(Calisto Justino Guambe)

Date: 23rd October 2015

To

Justino Guambe & Luisa Lissave

Abstract

Optimal investment, consumption and life insurance in a Lévy market

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The purpose of this dissertation is to solve an optimal investment, consumption and life insurance problem described by jump-diffusion processes in two settings.

First, we consider a problem with random parameters of a wage earner who wants to save to his beneficiary for his death. Using one risk-free asset and one risky asset price given by a geometric jump-diffusion process, we obtain the optimal strategy via the dynamic programming approach, combining the Hamilton-Jacobi-Bellman equation with a backward stochastic differential equation with jumps.

Secondly, we discuss the optimal investment, consumption and life insurance problem with capital constraints. The problem consists of one risk-free asset and two risky asset prices defined in an independent Brownian motion and Poisson process. We derive the optimal strategy of the unconstrained problem via martingale approach, from which, the problem with capital constraint is solved applying the option based portfolio insurance method.

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I am also grateful to my parents for their unmeasurable support in my life.

Contents

Notations

Throughout this dissertation, we will assume the following notations:

- $\mathcal{B}(E)$ the Borel σ-field of any set $E \subset \mathbb{R}$ and $\mathcal{P}$ the predictable σ-field on $\Omega \times [0,T]$, where $\mathbb{R}$ denote the set of real numbers.

- $\mathcal{C}$ the space of continuous functions.

- $\mathbb{L}^2(\mathbb{R})$- the space of random variables $\xi : \Omega \mapsto \mathbb{R}$, such that $\mathbb{E}[\,|\xi|^2] < \infty$.

- $L^2_\nu(\mathbb{R})$- the space of measurable functions $v : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\int_{\mathbb{R}} |v(z)|^2 \nu(dz) < \infty \,,$$

  where ν is a σ-finite measure.

- $\mathbb{S}^2(\mathbb{R})$- the space of adapted càdlàg processes $Y : \Omega \times [0,T] \mapsto \mathbb{R}$ such that

$$\mathbb{E}[\sup |Y(t)|^2] < \infty \,.$$

- $\mathbb{H}^2(\mathbb{R})$- denote the space of predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\Big[\int_0^T |Z^2(t)|dt\Big] < \infty.$$

- $\mathbb{H}^2_N(\mathbb{R})$- the space of predictable processes $\Upsilon : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, such that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\Upsilon(t,z)|^2 \nu(dz) dt\right] < \infty \,.$$

- $x \wedge y := \min\{x, y\}$.

- A^T denote the transpose of the matrix A.

- U^- is the negative part of U defined by $U^- := \max\{-U, 0\}$ and U^+ is the positive part of U given by $U^+ := \max\{U, 0\}$.

- $\langle \cdot, \cdot \rangle$ is the inner product defined as follows:

$$\langle a, b \rangle := \sum_{k=1}^{n} a_k b_k, \quad a, b \in \mathbb{R}^n .$$

- $\mathbf{1}_A$ is a characteristic function defined by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } \ x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 1

Introduction

1.1 Background information

The problem of an investor who wants to invest in order to maximize his expected utility has received much attention in mathematical finance due to a variety of reasons. For instance, the solution to the problem is a major concern of an investor (individual or institutional) who need to allocate the wealth in each security over a certain/uncertain time horizon. Besides, it is a stochastic optimal control problem, which can be solved via different approaches, such as, the dynamic programming approach, the martingale method and the maximum principle.

Since the mean-variance analysis by Markowitz in the early 1960's, this problem has received numerous studies. The first results in continuous time optimal investment-consumption problem were obtained by Merton [25, 26] via dynamic programming approach. An alternative martingale method was developed later by Karatzas *et al.* [18], Karatzas *et al.* [19], Karatzas and Shreve [20], among others.

Concerning investments, one natural question may arise: *what will happen to investor's dependent if a premature death occur*? This is an interesting question during the investor's planning because his death may affect the well being of his dependent. Thus, it suggests an inclusion of a new variable in the investment-consumption problem, *a life insurance*[1]. Life insurance is an important tool to solve the question of uncertain lifetime. Richard [32], extended the Merton's optimal investment-consumption problem to include a life insurance purchase. Pliska and Ye [30], studied the optimal investment, consumption and life insurance problem for a wage earner with a random lifetime. Similar works include Huang and Milevsky [16], Kwak *et al.* [23], Duarte *et al.* [10], Shen and Wei [35], Kronborg and Steffensen [22].

In all these works, the problem has been solved assuming a market in which the asset prices are described by a continuous time process. However, as was pointed out by Merton, the analysis of the price evolution reveals some sudden and rare breaks (jumps) caused by external information flow. These behaviours constitute a very real concern of most investors and they can be modeled by a Poisson process, which has jumps occurring at rare and unpredictable time. In this dissertation, we solve an optimal investment, consumption and life insurance problem described by jump-diffusion[2] processes. For further reading in jump-diffusion models, see e.g., Jeanblanc-Picque and Pontier [17], Benth *et al.* [2], Runggaldier [34], Daglish [7], Oksendal and Sulem [29], among others.

Essentially, the optimization problem consists of three elements, namely decision variables, the objective function and the constraints. The problem with lack of constraints is called *unconstrained problem*, while the others are referred to as *constrained optimization problems*. This dissertation focus in both unconstrained and constrained problems.

1.2 Objectives

The main objective of this dissertation is to solve an optimal investment, consumption and life insurance problem in a jump-diffusion framework. As mentioned in the abstract, we consider two settings namely a model with

[1]See Definition 3.4.1.
[2]See Section 2.2

random parameters and a model with capital constraints.
We first solve the optimal investment, consumption and life insurance problem of a wage earner with random coefficient parameters, which include jumps, where his/her preference follows a power utility function. The parameters under consideration are the interest rate, the appreciation rate, the force of mortality, the dispersion rates, premium insurance ratio and discount rate. These parameters are not necessarily bounded. We consider a financial market described by one risk-free asset and one geometric jump-diffusion risky asset, and an insurance market, where the life insurance is given by infinitesimally small terms. The aim of the wage earner is to choose an optimal strategy that maximizes the expected discounted utility derived from consumption, legacy and terminal wealth over an uncertain time horizon. Motivated by [35], where a similar problem was solved over a diffusion framework and the theory of backward stochastic differential equations (BSDEs) with jumps studied by Delong [9], we obtain the optimal solution based on a dynamic programming approach, using a combination of the Hamilton-Jacobi-Bellman (HJB) equation and a BSDE with jumps. We do so, since the value function of our model cannot be determined from the partial differential equation as usual due to the parameters randomness. We conclude this problem providing the closed form solution to the BSDE related to the problem in special examples of geometric jump-diffusion mortality rate and the appreciation rate with jumps.

Then, we solve the optimal investment, consumption and life insurance problem when the investor is restricted to fulfil the American capital constraints. The capital constraints were first introduced in the optimization problem by Tepla [37] and then studied by El Karoui *et al.* [11]. They can be considered for many reasons, such as, the restrictions of the savings to become negative, that is, a non-borrowing constraints or the existence of a minimum return in savings, i.e., the interest rate guarantee. As in [17], we consider a financial market described by one risk-free asset and two risky assets. The risky assets are constructed on a space of pair processes, consisting of independent Brownian motion and Poisson processes. A single risky asset consists of two source of randomness, which implies incompleteness of the market and infinitely many martingale measures. Therefore, defining two risky assets, the number of risky assets is equal to the number of driving processes, thus the market is complete and the martingale measure is unique ([34]). In addition, we suppose existence of insurance market, where the sum insured[3] is to be paid out upon death before the time horizon. Using the martingale approach developed by Karatzas *et al.* [18], Karatzas *et al.* [19], Karatzas and Shreve [20], we solve the unrestricted control problem. This is because the solution to the restricted capital guarantee problem is based on terms derived from the martingale method. The optimal solution to the restricted problem is derived from the unrestricted optimal solution, applying the option based portfolio insurance (OBPI) method developed by El Karoui *et al.* [11]. These results are an extension of the results in [22].

1.3 Structure of the dissertation

The rest of the dissertation is structured as follows:
We start in Chapter 2, by presenting a review of relevant concepts used in this dissertation. We focus on random measures, compensated random measures and Lévy processes. We also give the Itô's formula for Lévy SDEs, the Girsanov's theorem, the HJB equation as well as the introduction of BSDEs with jumps and the utility functions.

Chapter 3 is devoted to the derivation of the wealth process. We start by deriving the wealth in the presence of investment-consumption in the market, then we consider the case were the investor is having external sources. Finally, we consider the case where, in addition to investment and consumption, the investor is paying a life insurance.

In Chapter 4, we solve the optimization problem with random parameters. We obtain the optimal solution using the combination of HJB equation and

[3]See page 32

BSDE with jumps. We conclude this chapter by giving two special examples. The results of this chapter have been published in *Insurance: Mathematics and Economics* Journal.

In Chapter 5, we solve the constrained optimization problem. First, we obtain the optimal solution for the unconstrained control problem, then the constrained optimal solution is derived from the unconstrained optimal solution using the option based portfolio insurance method.

Finally, in Chapter 6, we conclude.

Chapter 2

Review on stochastic calculus

In this chapter, we review important results in stochastic calculus which we use in this dissertation. We begin with stochastic and Lévy processes in Section 2.1. Section 2.2 deals with Itô calculus for Lévy stochastic integrals and stochastic differential equations. In Section 2.3, we give the concept of a martingale and the Girsanov's theorem for Itô-Lévy processes. Backward stochastic differential equations with jumps are considered in Section 2.4. Section 2.5 deals with stochastic control for optimization problems. Finally, in Section 2.6, we introduce the utility functions, which are very important in the resolution of our optimization problems in Chapters 4 and 5.

2.1 Stochastic processes

In this section, we present the key concepts of this dissertation. We mainly focus on the concepts of a probability space, conditional expectation, random measures, compensated random measures and Lévy processes. The definitions in this section are taken from ([1], Chapter 1 and [9], Chapter 2), unless otherwise stated.

Definition 2.1.1.

Let Ω be a non-empty set and $\mathcal{F}$ a collection of subsets of Ω. We call $\mathcal{F}$ a *σ-algebra* if the following hold:

(1) $\Omega \in \mathcal{F}$,

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(3) if $(A_n,\, n \in \mathbb{N})$ is a sequence of subsets in $\mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a *measurable space*.

Definition 2.1.2.
A *measure* on $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty]$ satisfying

(1) $\mu(\emptyset) = 0$;

(2)
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$
for every sequence $(A_n,\, n \in \mathbb{N})$ of mutually disjoint sets in $\mathcal{F}$.

The triple $(\Omega, \mathcal{F}, \mu)$ is called a *measure space*. If $\mu(\Omega) < \infty$, μ is said to be finite. More generally, a measure μ is *σ-finite* if we can find a sequence $(A_n,\, n \in \mathbb{N})$ in $\mathcal{F}$ such that $\Omega = \bigcup_{n=1}^{\infty} A_n$ and each $\mu(A_n) < \infty$.

If $\mu(\Omega) = 1$, the triple $(\Omega, \mathcal{F}, \mu)$ is called a *probability space*. In a probability space, μ is usually denoted by $\mathbb{P}$.

Definition 2.1.3.
Let $\mathcal{F}$ be a σ-algebra of subsets of a given set Ω. A family $(\mathcal{F}_t,\, t \geq 0)$ of sub σ-algebras of $\mathcal{F}$ is called a *filtration* if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \text{whenever} \quad s \leq t.$$

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with such a family $(\mathcal{F}_t,\, t \geq 0)$ is said to be *filtered*.

Throughout this dissertation, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$. We assume that the filtration satisfies the usual conditions ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero and $\mathcal{F}_t$ is *right continuous*, i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$).

Definition 2.1.4.
A *stochastic process* $X = X(\omega, t)$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is a collection of random variables defined on $\Omega \times [0,T]$.

Two stochastic processes $X = (X(t))_{t\in[0,T]}$ and $Y = (Y(t))_{t\in[0,T]}$ are *independent* if, for all $m, n \in \mathbb{N}$, all $0 < t_1 < t_2 < \cdots < t_n = T$ and all $0 < s_1 < s_2 < \cdots < s_m = T$, the σ-algebras $\sigma(X(t_1), X(t_2), \ldots, X(t_n))$ and $\sigma(X(s_1), X(s_2), \ldots, X(s_m))$ are independent. Similarly, a stochastic process $X = (X(t))_{t\in[0,T]}$ and a σ-algebra $\mathcal{F}$ are *independent* if $\mathcal{F}$ and $\sigma(X(t_1), X(t_2), \ldots, X(t_n))$ are independent for all $n \in \mathbb{N}$, $0 < t_1 < t_2 < \cdots < t_n = T$.

Definition 2.1.5.
Let $X = (X(t), t \in [0,T])$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is *adapted to the filtration* (or *$\mathcal{F}_t$-adapted*) if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$.

Definition 2.1.6.
Let $X = (X(t), t \in [0,T])$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is *progressively measurable* with respect to a filtration $\mathcal{F}_t$ if the function $X(t, \omega) : [0,T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}([0,T]) \times \mathcal{F}_t)$-measurable for each $t \in [0,T]$.

Definition 2.1.7.
An $\mathcal{F}$-adapted process $W := (W(t)\,,\, 0 \le t \le T)$ is called a *Brownian motion* if

(i) $W(0) = 0$ a.s.;

(ii) for $0 \le s < t \le T$, $W(t) - W(s)$ is independent of $\mathcal{F}_s$;

(iii) for $0 \le s < t \le T$, $W(t) - W(s)$ is a Gaussian random variable with mean zero and variance $t - s$, i.e., $W(t) - W(s) \sim \mathcal{N}(0\,,\, t-s)$;

(iv) for any $\omega \in \Omega\,$, $W(t)$ is a continuous function.

We then introduce the concept of conditional expectation.

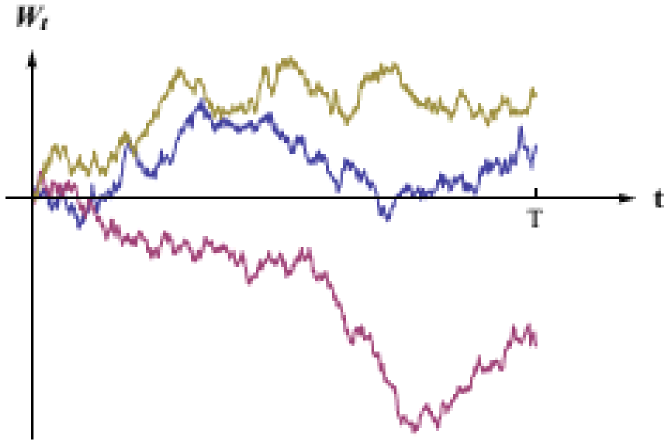

Figure 2.1: Paths of simulated Brownian motion.

Definition 2.1.8. ([4], Definition 2.4)
Let X be an $\mathcal{F}_T$-measurable integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The *conditional expectation* of X given $\mathcal{F}_t$ is defined to be a random variable $\mathbb{E}[X \mid \mathcal{F}_t]$ such that:

1. $\mathbb{E}[X \mid \mathcal{F}_t]$ is almost surely $\mathcal{F}_t$-measurable;
2. for any $A \in \mathcal{F}_t$,
$$\int_A \mathbb{E}[X \mid \mathcal{F}_t] dP = \int_A X d\mathbb{P}.$$

The following proposition gives the general properties of the conditional expectation.

Proposition 2.1.1. ([4], Proposition 2.4) Let $\mathcal{F}_t$ be a filtration on Ω and X, Y integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The conditional expectation has the following properties:

(1) $\mathbb{E}[aX + bY \mid \mathcal{F}_t] = a\mathbb{E}[X \mid \mathcal{F}_t] + b\mathbb{E}[Y \mid \mathcal{F}_t]$ *(linearity)*, $a\,, b \in \mathbb{R}$;

(2) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t]] = \mathbb{E}[X]$;

$\mathbb{E}[XY \mid \mathcal{F}_t] = X\mathbb{E}[Y \mid \mathcal{F}_t]$ *a.s. if* X *is* $\mathcal{F}_t$*-measurable and* XY *integrable;*

(3) $\mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[X]$ *if* X *is independent of* $\mathcal{F}_t$*;*

(4) $\mathbb{E}[X \mid \mathcal{F}_0] = \mathbb{E}[X]$ *and* $\mathbb{E}[X \mid \mathcal{F}_T] = X$ *almost surely;*

(5) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[X \mid \mathcal{F}_s]$ *a.s. for* $s < t$ *(tower property);*

(6) *If* $X \geq 0$*, then* $\mathbb{E}[X \mid \mathcal{F}_t] \geq 0$ *(positivity).*

Proof. See [4], Proposition 2.4. □

Definition 2.1.9.

A function N defined on $\Omega \times [0, T] \times \mathbb{R}$ is called a *random measure* if

(i) for any $\omega \in \Omega$, $N(\omega, \cdot)$ is a σ-finite measure on $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$;

(ii) for any $A \in \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$, $N(\cdot, A)$ is a random variable on $(\Omega, \mathcal{F}, P)$.

Remark.

$N(\omega, [0, T], A)$ may be equal to infinity.

Example 2.1.1.

Let $(T_n)_{n\geq 1}$ denote the sequence of jump times of a Poisson process. The function

$$N(\omega, [s, t]) = \sharp\{n \geq 1,\, T_n \in [s, t]\}\,,\; 0 \leq s < t \leq T,$$

which counts the number of jumps of the Poisson process in the interval $[s, t]$ defines a random measure.

If we fix ω, then the sequence of jump times $(T_n)_{n\geq 1}$ is given on the time axis and N as a function of $[s, t]$ is finite measure which counts the number of $(T_n)_{n\geq 1}$ are in the interval $[s, t]$.

If we fix $[s, t]$, then N is a Poisson distributed random variable which counts the number of random jump times $(T_n)_{n\geq 1}$ in the interval $[s, t]$.

Definition 2.1.10.

A random measure N is called $\mathcal{F}$-*predictable* if for any $\mathcal{F}$-predictable[1] process X such that $\int_0^T \int_{\mathbb{R}} |X(t, z)| N(dt, dz)$ exists, the process $\left(\int_0^t \int_{\mathbb{R}} X(s, z) N(ds, dz)\,,\, 0 \leq t \leq T\right)$ is $\mathcal{F}$-predictable.

[1] A predictable process is a real-valued stochastic process whose values are known, in a sense just in advance of time. Predictable processes are also called *previsible*.

Definition 2.1.11.
For a random measure N, we define

$$E_N(A) = \mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}} \mathbf{1}_A(\omega,t,z)N(\omega,dt,dz)\Big],\ A \in \mathcal{F}\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}).$$

If there exists an $\mathcal{F}$-*predictable* random measure ν such that

(i) E_ν is a σ-finite measure on $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$;

(ii) the measures E_N and E_ν are identical on $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$.

Then we say that the random measure N has a compensator ν.

Given the compensator ν of a random measure N, we define the compensated random measure by

$$\widetilde{N}(\omega,dt,dz) := N(\omega,dt,dz) - \nu(\omega,dt,dz)\,. \tag{2.1}$$

Remark.
The compensator is uniquely determined ([15], pp 295–297) and the random measures are usually related to jumps of discontinuous processes.

Definition 2.1.12.
A *Lévy process* is an $\mathcal{F}_t$-adapted process $\eta := (\eta(t)\,, 0 \le t \le T)$ such that

(i) $\eta(0) = 0$ a.s.;

(ii) for $0 \le s < t \le T$, $\eta(t) - \eta(s)$ is independent of $\mathcal{F}_s$;

(iii) for $0 \le s < t \le T$, $\eta(t) - \eta(s)$ has the same distribution as $\eta(t-s)$;

(iv) the process η is continuous in probability, i.e., for any $t \in [0,T]$ and $\epsilon > 0$,

$$\lim_{s\to t} P(|\eta(t) - \eta(s)| > \epsilon) = 0.$$

Next, we consider two special examples of Lévy processes.

Example 2.1.2. (Brownian motion)
A Brownian motion in $\mathbb{R}^N$ is a Lévy process $W = (W(t))_{t\in[0,T]}$ for which

(i) $W(t) \sim \mathcal{N}(0, tI)$ for each $t \in [0, T]$,

(ii) W has continuous sample paths.

Example 2.1.3. (The Poisson process)
The Poisson process of intensity $\lambda > 0$ is a Lévy process N taking values $\mathbb{N} \cup \{0\}$, so that

$$\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

for each $0, 1, 2, \ldots$.

The *compensated Poisson process* is given by $\widetilde{N} = (\widetilde{N}(t))_{t\in[0,T]}$ where each $\widetilde{N}(t) := N(t) - \lambda t$. Note that $\mathbb{E}[\widetilde{N}(t)] = 0$ and $\mathbb{E}[(\widetilde{N}(t))^2] = \lambda t$. This will be useful in Chapter 4.

We conclude this section giving the concept of Lévy measure.

Definition 2.1.13.
Let ν be a Borel measure[2] defined on $\mathbb{R}^N \setminus \{0\} = \{x \in \mathbb{R}^N,\, x \neq 0\}$. We say that ν is a *Lévy measure* if

$$\int_{\mathbb{R}^N \setminus \{0\}} \big(|y|^2 \wedge 1\big)\, \nu(dy) < \infty\,.$$

2.2 Itô calculus and Lévy stochastic differential equations

The purpose of this dissertation is to obtain the optimal strategy of an investor whose wealth is given by stochastic differential equation (SDE) and the Itô's formula plays a very important role in solving such equations. In this section, we give the Itô's formula for one-dimensional as well as for multidimensional equations. Furthermore, we give the theorem about existence and uniqueness of solutions of the Lévy SDE. For detailed information see e.g. ([1], Chapters 4 and 6 or [29], Sections 1.2-1.3).

[2]A measure defined o a Borel σ-algebra of a set Ω. See ([1], page 2 or [8], Chapter 2) for more details.

Definition 2.2.1. (Itô-Lévy processes)
Let $W(t)$, $0 \leq t \leq T$ be a Brownian motion and $N(dt,dz)$ a random measure with the compensated random measure $\widetilde{N}(dt,dz)$. *Itô-Lévy process* (or stochastic integral) is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$\begin{aligned} X(t) &= X(0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\widetilde{N}(ds,dz), \end{aligned} \tag{2.2}$$

where $\alpha : [0,T] \times \Omega \to \mathbb{R}$, $\beta : [0,T] \times \Omega \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy the following conditions:

$$\int_0^t |\alpha(s,\omega)|ds < \infty; \ \int_0^t \beta^2(s,\omega)ds < \infty; \ \int_0^t \int_{\mathbb{R}} \gamma^2(s,z,\omega)\nu(dz)ds < \infty.$$

The Equation (2.2) can be written in a differential form as

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\widetilde{N}(ds,dz), \tag{2.3}$$

or equivalently

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + \gamma(t,z,\omega)d\widetilde{N}(s). \tag{2.4}$$

The Equation (2.3) ((2.4)) is called *stochastic differential equation* (SDE).

Remark.
In some literatures, we can also find the form

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + dJ(t),$$

where $J(t) := \sum_{k=1}^{\widetilde{N}(t)} \gamma(T_k, \zeta_k)$. Here $\{(T_K, \zeta_k),\ k \in \{1,2,\ldots,\widetilde{N}(t)\}\}$ is the sequence of pairs of jump times and corresponding marks generated by the Poisson random measure.

The following theorem gives the Itô's formula for a one dimensional space.

Theorem 2.2.1. (The 1-dimensional Itô's formula). *Suppose that $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\widetilde{N}(ds,dz),$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ *and*

$$\widetilde{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt, & \text{if } |z| < a; \\ N(dt, dz), & \text{if } |z| \geq a, \end{cases}$$

for some $a \in [0, \infty]$. *Let* $f \in \mathcal{C}^2([0, T] \times \mathbb{R})$. *Then* $Y(t) = f(t, X(t))$ *is also an Itô-Lévy process and*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))\Big[\alpha(t,\omega)dt + \beta(t,\omega)dW(t)\Big] \\ &+ \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt + \int_{|z|<a}\Big[f(t, X(t^-) + \gamma(t,z,\omega)) \\ &- f(t, X(t^-)) - \frac{\partial f}{\partial x}(t, X(t))\gamma(t,z,\omega)\Big]\nu(dz)dt \\ &+ \int_{|z|<a}\Big[f(t, X(t^-) + \gamma(t,z,\omega)) - f(t, X(t^-))\Big]\widetilde{N}(dt, dz)\,. \end{aligned}$$

Theorem 2.2.2. (Itô-Lévy isometry) *Let* $X(t) \in \mathbb{R}$, $X(0) = 0$ *be a SDE* (2.3), *for* $\alpha = 0$. *Then*

$$\mathbb{E}[X^2(t)] = \mathbb{E}\Big[\int_0^t \beta^2(s)ds + \int_0^t \int_{\mathbb{R}} \gamma^2(s,z)\nu(dz)ds\Big]$$

provided that the right hand side is finite.

Proof. From Theorem 1.2.1. applied to $f(t, x) = x^2$. □

We then formulate the multidimensional version of the Itô's formula.

Theorem 2.2.3. *Let* $X_i(t) \in \mathbb{R}\,, i = 1, ..., N$ *be an Itô-Lévy process of the form*

$$dX_i(t) = \alpha_i(t,\omega)dt + \sum_{j=1}^{M}\beta_{ij}(t,\omega)dW_j(t) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j,\omega)\widetilde{N}_j(dt, dz_j), \tag{2.5}$$

where $\alpha_i : [0, T] \times \Omega \to \mathbb{R}$, $\beta_i : [0, T] \times \Omega \to \mathbb{R}^M$ *and* $\gamma_i : [0, T] \times \mathbb{R}^\ell \times \Omega \to \mathbb{R}^\ell$ *are adapted processes such that the integrals exist. Here* $W_j(t)\,, j = 1, ..., M$ *is* 1*-dimensional Brownian motion and*

$$\widetilde{N}_j(dt, dz_j) = N_j(dt, dz_j) - \mathbf{1}_{|z_j|<a_j}\nu_j(dz_j)dt,$$

where N_j are independent Poisson random measures with Lévy measures ν_j coming from ℓ independent (1-dimensional) Lévy processes $\eta_1, ..., \eta_\ell$ and $\mathbf{1}_{|z_j|<a_j}$ is a characteristic function, for some $a_j \in [0, \infty]$. Let $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^N)$. Then $Y(t) = f(t, X_1(t), ..., X_N(t))$ is also an Itô-Lévy process and

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\alpha_i dt + \beta_i dW(t)) + \frac{1}{2}\sum_{i,j=1}^{N}(\beta\beta^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}dt \\ &+ \sum_{k=1}^{\ell}\int_{|z_k|<a_k}\Big[f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-)) \\ &\qquad - \sum_{i=1}^{N}\gamma_i^{(k)}(t, z_k)\frac{\partial f}{\partial x_i}(X(t^-))\Big]\nu_k(dz_k)dt \\ &+ \sum_{k=1}^{\ell}\int_{|z_k|<a_k}\Big[f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-))\Big]\widetilde{N}_k(dt, dz_k), \end{aligned}$$

where $X(t) = (X_1(t), ..., X_N(t))$, $\beta \in \mathbb{R}^{N\times M}$, $W(t) = (W_1(t), ..., W_M(t))$ and $\gamma^{(k)} \in \mathbb{R}^\ell$ is the column number k of the $N \times \ell$ matrix γ.

Proof. See [1], Theorem 4.4.7. □

The following theorem states the existence and uniqueness of the solution of the SDE driven by Lévy processes.

Theorem 2.2.4. (Existence and uniqueness of solutions of Lévy SDEs). *Consider the following Lévy SDE in $\mathbb{R}^N$: $X(0) = x_0 \in \mathbb{R}^N$ and*

$$dX(t) = \alpha(t, X(t))dt + \beta(t, X(t))dW(t) + \int_{\mathbb{R}}\gamma(t, X(t), z)\widetilde{N}(ds, dz),$$

where $\alpha : [0,T] \times \Omega \to \mathbb{R}^N$, $\beta : [0,T] \times \Omega \to \mathbb{R}^{N\times M}$ and $\gamma : [0,T] \times \mathbb{R}^\ell \times \Omega \to \mathbb{R}^{N\times\ell}$ satisfy the following conditions:

(At most linear growth) *there exists a constant $C_1 < \infty$, such that*

$$|\alpha(t,x)|^2 + \|\beta(t,x)\|^2 + \int_{\mathbb{R}}\sum_{k=1}^{\ell}|\gamma_k(t,x,z_k)|^2\nu_k(dz_k) \le C_1(1 + |x|^2)$$

for all $x \in \mathbb{R}^N$;

(Lipschitz continuity) *there exists a constant $C_2 < \infty$, such that*

$$\begin{aligned} &|\alpha(t,x)-\alpha(t,y)|^2 + \|\beta(t,x)-\beta(t,y)\|^2 \\ &+ \int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k(t,x,z_k)-\gamma_k(t,y,z_k)|^2 \nu_k(dz_k) \;\leq\; C_2|x-y|^2, \end{aligned}$$

for all $x, y \in \mathbb{R}^N$.

Then there exists a unique càdlàg[3] *adapted solution $X(t)$ such that*

$$\mathbb{E}[|x(t)|^2] < \infty, \quad \forall t \in [0,T].$$

Proof. See [1], Theorem 6.2.3. □

The solution of a Lévy SDE in the time-homogeneous case, i.e., $\alpha(t,X) = \alpha(X)$, $\beta(t,X) = \beta(X)$ and $\gamma(t,x,z) = \gamma(x,z)$ is called *jump-diffusion process* or *Lévy-diffusion process.*

Next we introduce the concept of a generator operator A of X, where X is a solution of a Lévy SDE (2.5).

Definition 2.2.2.
Let $X(t) \in \mathbb{R}^N$ be a jump-diffusion process. Then the *generator* A of X is defined on functions $f : \mathbb{R}^N \to \mathbb{R}$ by

$$Af(x) = \lim_{t\to 0^+} \frac{1}{t}\{\mathbb{E}^x[f(X(t))] - f(x)\} \text{ (if the limit exists)},$$

where $\mathbb{E}^x[f(X(t))] = \mathbb{E}[f(X^{(x)}(t))]$, $X^{(x)}(0) = x$.

The following theorem gives the solution of $Af(x)$.

Theorem 2.2.5. *Consider $f \in \mathcal{C}_0^2(\mathbb{R}^N)$. Then $Af(x)$ exists and is given by*

$$\begin{aligned} Af(f) &= \sum_{i=1}^{n} \alpha_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} (\beta\beta^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &+ \int_{\mathbb{R}} \sum_{k=1}^{\ell} \left[f(x+\gamma^{(k)}(x,z_k)) - f(x) - \nabla f(x)\cdot\gamma^{(k)}(x,z_k)\right] \nu_k(dz_k). \end{aligned} \tag{2.6}$$

[3]right continuous with left limit.

Proof. Let $X \in \mathbb{R}^N$ be given by

$$\begin{aligned} dX_i(t) &= \alpha_i(x)dt + \sum_{j=1}^{M} \beta_{ij}(x)dW_j(t) + \sum_{j=1}^{\ell} \int_{\mathbb{R}} \gamma_{ij}(x, z_j)\widetilde{N}_j(dt, dz_j), \\ X_i(0) &= x_i, \end{aligned}$$

for $i = 1, \dots, N$. Define $Y = f(X)$. By Itô's formula (Theorem 2.2.3.), we have

$$\begin{aligned} dY(t) &= Af(x)dt + \sum_{i=1}^{N} \beta_i(x)\frac{\partial f}{\partial x_i}(x)dW(t)) \\ &\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} \Big[f(x + \gamma^{(k)}(x, z_k)) - f(x)\Big]\widetilde{N}_k(dt, dz_k), \end{aligned}$$

where $Af(x)$ is given by (2.6). Integrating the above equation we obtain

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t Af(X(s))ds + \int_0^t \sum_{i=1}^{N} \beta_i(X(s))\frac{\partial f}{\partial x_i}(x)dW(s)) \\ &\quad + \sum_{k=1}^{\ell} \int_0^t \int_{\mathbb{R}} \Big[f(X(s) + \gamma^{(k)}(X(s), z_k)) - f(X(s))\Big]\widetilde{N}_k(ds, dz_k). \end{aligned}$$

Taking expectation on both sides we obtain

$$\mathbb{E}[f(X(t))] - f(X(0)) = \mathbb{E}\left[\int_0^t Af(X(s))ds\right]. \tag{2.7}$$

From Lebesgue convergence theorem, it follows that

$$\frac{d}{dt}\mathbb{E}[f(X(t))] \mid_{t=0} = \lim_{t\to 0} \frac{\mathbb{E}[f(X(t))] - f(X(0))}{t}. \tag{2.8}$$

Then combining (2.7) and (2.8), leads to

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[f(X(t))] \mid_{t=0} &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[\int_0^t Af(X(s))ds\right] \\ &= Af(x). \end{aligned}$$

□

2.3 Martingales and Girsanov's theorem

In Financial Mathematics, martingales are crucial for option pricing models, for instance, in Chapter 5, we obtain the optimal strategy for a model restricted to satisfy the American put guarantee. It is through martingales that we solve the backward stochastic differential equation in Chapter 4. In this section, we introduce the concept of martingales and give the so-called Girsanov's theorem for Itô-Lévy processes. This section is adapted from ([1], Chapter 2; [9], Section 2.5 and [29], Section 1.4).

Definition 2.3.1.
Given a filtered measure space $(\Omega, \mathcal{F})$, we say that a random time $T : \Omega \to [0, \infty]$ is a *stopping time* of the filtration $(\mathcal{F}_t)$ if the event $(T \leq t) \in \mathcal{F}_t$ for each $t \geq 0$.

Definition 2.3.2.
Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An adapted process $X = (X(t))_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a *martingale* if

(i) $\mathbb{E}[|X(t)|] < \infty$, for all $t \in [0, T]$;

(ii) $\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s)$ a.s., for all $s \leq t$, $s, t \in [0, T]$.

If, for all $0 \leq s \leq t < \infty$, $\mathbb{E}[X(t) \mid \mathcal{F}_s] \geq X(s)$ a.s., then X is a *submartingale* and a *supermartingale* if $-X$ is a submartingale.
We define a *local martingale* as an adapted process $X = (X(t), t \in [0, T])$ for which there exists a sequence of stopping times $\tau_1 \leq \cdots \leq \tau_n \to T$ (a.s.) such that each of the process $(X(t \wedge \tau_n), t \in [0, T])$ is a martingale.

We introduce below the concept of uniform integrability

Definition 2.3.3. Let $X = \{X_i, \, i \in \mathcal{I}\}$ be a family of random variables, for some index $\mathcal{I}$. We say that X is *uniformly integrable* if

$$\lim_{n\to\infty} \sup_{i\in\mathcal{I}} \mathbb{E}\Big[|X_i| \mathbf{1}_{|X_i|>n}\Big] = 0$$

or equivalently
if X_i is bounded in L^1 and $\forall\epsilon > 0$, $\exists\delta > 0$: $\forall A \in \mathcal{F}$, $\mathbb{P}(A) < \delta \Rightarrow \sup_{i\in\mathcal{I}} \mathbb{E}\Big[|X_i|\mathbf{1}_A\Big] < \epsilon$.

Definition 2.3.4.
It is said that a measure ν is *absolutely continuous* with respect to μ (denoted by $\nu \ll \mu$), if $\mu(A) = 0$ implies that $\nu(A) = 0$, for any $A \in \mathcal{F}_t$.

Theorem 2.3.1. (Radon-Nikodym theorem). *Let μ and ν be σ-finite measures on space $(\Omega, \mathcal{F})$. If $\nu \ll \mu$, then there is a function $f \in \mathcal{F}$ such that for all $A \in \mathcal{F}$,*

$$\int_A f d\mu = \nu(A).$$

The function f is usually denoted by $\frac{d\nu}{d\mu}$ and is called Radon-Nikodym derivative.

Proof. See [8], pp. 139-141, Theorem 5. □

Definition 2.3.5.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered probability space and $\mathbb{Q}$ an other probability measure on $\mathcal{F}_T$. We say that $\mathbb{Q}$ *is equivalent to* $(\mathbb{P} \mid \mathcal{F}_T)$ if $(\mathbb{P} \mid \mathcal{F}_T) \ll \mathbb{Q}$ and $\mathbb{Q} \ll (\mathbb{P} \mid \mathcal{F}_T)$, i.e., $\mathbb{P}$ and $\mathbb{Q}$ have the same zero sets in $\mathcal{F}_T$.

Remark.
By the Radon-Nikodym theorem, $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T)$ and $\frac{d\mathbb{P}}{d\mathbb{Q}} = Z^{-1}(T)$ on $\mathcal{F}_T$, for some $\mathcal{F}_T$-measurable variable $Z(T) > 0$ almost surely.

Theorem 2.3.2. (Girsanov's Theorem for Itô-Lévy processes). *Let W and N be $(\mathbb{P}, \mathcal{F})$-Brownian motion and $(\mathbb{P}, \mathcal{F})$-random measure with compensator $\nu(dz)$. Moreover, consider $X(t)$ be a 1-dimensional Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\widetilde{N}(dt,dz),\ \ 0 \le t \le T\,.$$

Assume there exist predictable processes $\theta(t) = \theta(t,\omega) \in \mathbb{R}$ and $\psi(t,z) = \psi(t,z,\omega) \in \mathbb{R}$ such that

$$\beta(t)\theta(t) + \int_{\mathbb{R}} \gamma(t,z)\psi(t,z)\nu(dz) = \alpha(t),$$

for a.s. $(t,\omega) \in [0,T] \times \Omega$ *and such that the process*

$$\begin{aligned} Z(t) &:= \exp\Big[-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \ln(1-\psi(s,z))\widetilde{N}(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1-\psi(s,z)) + \psi(s,z)\}\nu(dz)ds\Big]\,,\ 0 \le t \le T \end{aligned}$$

is well defined and satisfies $\mathbb{E}[Z(T)] = 1$*. Furthermore, define the probability measure* $\mathbb{Q}$ *on* $\mathcal{F}_T$ *by* $d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega)$*. Then* $X(t)$ *is a local martingale with respect to* $\mathbb{Q}$ *and*

$$\begin{aligned} W^Q(t) &= W(t) + \int_0^t \theta(s)ds\,,\ 0 \le t \le T, \\ \widetilde{N}^Q(t,A) &= N(t,A) - \int_0^t \int_{\mathbb{R}} (1+\psi(s,z))\nu(dz)ds\,,\ 0 \le t \le T,\ A \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

are $(\mathbb{Q},\mathcal{F})$*-Brownian motion and* $(\mathbb{Q},\mathcal{F})$*-compensated random measure respectively.*

Proof. See [29], Theorem 1.31 and [9], Theorem 2.5.1. □

2.4 Backward stochastic differential equations with jumps

In this section we introduce the concept of backward stochastic differential equation (BSDE). In this type of Lévy SDEs, instead of an initial condition $Y(0) = y_0$ a.s., we impose a final condition $Y(T) = \xi$ a.s. For more details see ([9], Chapter 3).

Given the data (ξ, f), where $\xi : \Omega \to \mathbb{R}$ is an $\mathcal{F}_T$-measurable random variable and f is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable function. We consider the following backward stochastic differential equation (BSDE)

$$\begin{aligned} dY(t) &= -f(t,Y(t),Z(t),\Upsilon(t,z))dt + Z(t)dW(t) \\ &\quad + \int_{\mathbb{R}} \Upsilon(t,z)\widetilde{N}(dt,dz)\,; \\ Y(T) &= \xi\,, \end{aligned} \tag{2.9}$$

where the processes Z and Υ are called *control processes*. They control an adapted process Y so that Y satisfies the terminal condition.

Definition 2.4.1.
A triple $(Y, Z, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ is said to be a solution to a BSDE (2.9) if

$$\begin{aligned} Y(t) &= \xi + \int_t^T f(s, Y(s-), Z(s-), \Upsilon(s-, \cdot))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}} \Upsilon(s, z)\widetilde{N}(ds, dz)\,,\; 0 \le t \le T\,. \end{aligned}$$

Definition 2.4.2.
A pair (ξ, f) is said to be a standard data for BSDE (2.9), if the following conditions hold:

(C1) the terminal value $\xi \in \mathbb{L}^2(\mathbb{R})$;

(C2) the generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \mapsto \mathbb{R}$ is predictable, i.e., $f \in \mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(L^2_\nu(\mathbb{R}))$ and Lipschitz continuous in the sense that,

$$\begin{aligned} |f(\omega, t, y, z, v) - f(\omega, t, y', z', v')|^2 &\le K(|y - y'|^2 + |z - z'|^2 \\ &\quad + \int_{\mathbb{R}} |v(z) - v'(z)|^2 \nu(dz))\,, \end{aligned}$$

a.s., $(\omega, t) \in \Omega \times [0, T]$ a.e. for all (y, z, v), $(y', z', v') \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R})$;

(C3)

$$\mathbb{E}[\int_0^T |f(t, 0, 0)|^2 dt] < \infty\,.$$

We state below the theorem of existence and uniqueness of the solution to the BSDE (2.9).

Theorem 2.4.1. *Let* (ξ, f) *be a standard data. Then the BSDE* (2.9) *has a unique solution* $(Y, Z, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$.

Proof. See [9], Theorem 3.1.1. □

2.5 Stochastic control

In Chapter 4, we solve the optimal investment-consumption-insurance problem using the combination of BSDE with jumps introduced in the previous section and Hamilton-Jacobi-Bellman (HJB) equation we consideer in this section. For more details concerning to the HJB equation see e.g. ([29], Chapter 3 or [38], Chapter 3).

Consider the domain $\mathcal{S} \subset \mathbb{R}^N$ (the *solvency region*) and $X(t)$ and $\mathbb{R}^N$-valued stochastic process of the form

$$\begin{aligned} dX(t) &= \alpha(X(t),u(t))dt + \beta(X(t),u(t))dW(t) \\ &\quad + \int_{\mathbb{R}} \gamma(X(t^-),u(t^-),z)\widetilde{N}(dt,dz)\,; \\ X(0) &= x \in \mathbb{R}^N, \end{aligned} \tag{2.10}$$

where $\alpha : \mathbb{R}^N \times U \to \mathbb{R}^N$, $\beta : \mathbb{R}^N \times U \to \mathbb{R}^{N\times M}$ and $\gamma : \mathbb{R}^N \times U \times \mathbb{R}^N \to \mathbb{R}^{N\times\ell}$ are given functions. Here $U \subset \mathbb{R}^k$ is a given set.

A measurable map $u(\cdot) : [0,T] \to U$ is called a *control*. u is assumed to be *càdlàg* and *adapted*. A solution of (2.10) is called a *controlled jump diffusion*.

Given a functional measuring the performance of the controls

$$J^{(u)}(x) = \mathbb{E}^x\left[\int_0^{\tau_s} f(X(t),u(t))dt + g(X(\tau_s))\right], \tag{2.11}$$

where $\tau_s < \infty$ denote a stopping time and $f : \mathcal{S} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given continuous functions. The first and second terms on the right hand side of (2.11) are called *running cost* and *terminal cost* respectively.

Definition 2.5.1.
We say that the control process u is *admissible* and write $u \in \mathcal{A}$ if:

(1) the SDE (2.10) has a unique strong solution $X(t)$, for all $x \in \mathcal{S}$;

(2)
$$\mathbb{E}^x\left[\int_0^{\tau_s} f^-(X(t),u(t))dt + g^-(X(\tau_s))\right] < \infty\,.$$

The stochastic control problem can be stated as follows:

Problem (P1).
Find the value function $V(x)$ and an optimal control $u^* \in \mathcal{A}$ defined by

$$V(x) = \sup_{u \in \mathcal{A}} J^{(u)}(x) = J^{(u^*)}(x) \, .$$

Under mild conditions ([28], Theorem 11.2.3), it suffices to consider Markov[4] controls, i.e. $u(t) = u(X(t-))$. Note that if $u = u(x)$ is a Markov control, then $X(t) = X^u(t)$ is a jump diffusion process with a generator

$$\begin{aligned} A\phi(x) = A^u\phi(x) &= \sum_{i=1}^{N} \alpha_i(x, u(x)) \frac{\partial \phi}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{N} (\beta\beta^T)_{ij}(x, u(x)) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \\ &+ \int_{\mathbb{R}} \sum_{k=1}^{\ell} \Big[\phi(x + \gamma^{(k)}(x, u(x), z_k)) - \phi(x) \\ &\qquad - \nabla\phi(x) \cdot \gamma^{(k)}(x, u(x), z_k) \Big] \nu_k(dz_k) \, . \end{aligned}$$

We then formulate a verification theorem for the optimal control problem **(P1)** analogous to the classical Hamilton-Jacobi-Bellman (HJB) for Itô diffusion processes ([28], Chapter 11).

Theorem 2.5.1. (HJB for optimal control of jump diffusion process). *Let $\mathcal{S}$ be a solvency region and $\overline{S}$ the closure of $\mathcal{S}$. Denote by $\partial\mathcal{S}$, the boundary of $\mathcal{S}$ and by U, a control set.*

(a) *Suppose $\phi \in \mathcal{C}^2 \cap \mathcal{C}(\overline{S})$ satisfies the following conditions:*

(*i*) $A^u\phi(x) + f(x, u) \leq 0$, *for all $x \in \mathcal{S}$ and $u \in U$;*

(*ii*) $X(\tau_s) \in \partial\mathcal{S}$, *a.s. on $\tau_s \in [0, T]$ and*

$$\lim_{t \to \tau_s^-} \phi(X(t)) = g(X(\tau_s)) \ a.s.,$$

for all $u \in \mathcal{A}$ and g a function defined in (2.11)*;*

[4]A Markov process is a stochastic model that has the Markov property, i.e., $\mathbb{P}(X_t \in A \mid \mathcal{F}_s) = \mathbb{P}(X_t \in A \mid X_s)$, where $s < t$.

(iii)

$$\mathbb{E}^x\left[|\phi(X(\tau))| + \int_0^{\tau_s} |A\phi(X(t))|dt\right] < \infty\,,$$

for all admissible $u \in \mathcal{A}$ *and all* $\tau \in \mathcal{T}$*, where* $\mathcal{T}$ *is a set of stopping times;*

(iv) $\{\phi^-(X(\tau))\}_{\tau \leq \tau_s}$ *is uniformly integrable for all* $u \in \mathcal{A}$ *and* $X \in \mathcal{S}$*.*

Then $\phi(x) \geq V(x)$*, for all* $x \in \mathcal{S}$*;*

(b) *Moreover, suppose that for each* $x \in \mathcal{S}$ *there exists* $u = \widehat{u} \in U$ *such that*

(v) $A^{\widehat{u}(x)}\phi(x) + f(x, \widehat{u}(x)) = 0$ *and*

(vi) $\{\phi^-(X^{\widehat{u}}(\tau))\}_{\tau \leq \tau_s}$ *is uniformly integrable.*

Define $u^* := \widehat{u}(X(t^-)) \in \mathcal{A}$*. Then* u^* *is an optimal control and*

$$\phi(x) = V(x) = J^{(u^*)}(x)\,, \qquad \forall x \in \mathcal{S}.$$

Proof. See [29], Theorem 3.1. □

We then summarize the key points of the stochastic control approach.

The stochastic control or dynamic programming approach follows the following steps:

1. Introduce the problem;
2. define the value function;
3. derive the principle of optimality;
4. derive the (HJB) equation and then follow the Steps 1–3 to obtain an optimal pair.

2.6 Utility functions

In this section, we develop the properties of the utility function to be considered. For more details see e.g. [19] or ([20], Chapter 3).

Definition 2.6.1.
A *utility function* is a concave, non-decreasing, upper semi-continuous function $U : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(i) the half-line $dom(U) := \{x \in (0, \infty) : U(x) > -\infty\}$ is a nonempty subset of $[0, \infty)$;

(ii) the derivative U' is continuous, positive and strictly decreasing on the interior of $dom(U)$ and

$$U'(0) := \lim_{x \to 0} U'(x) = \infty\,, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0\,. \tag{2.12}$$

Definition 2.6.2.
Define a function $\lambda : \mathbb{R} \to \mathbb{R}$ by

$$\lambda(x) := -x\frac{U''(x)}{U'(x)}\,.$$

A utility function U is said to be of *Constant Relative Risk Aversion (CRRA)* type if λ is a constant.

Example 2.6.1.
A CRRA utility function to be used in this dissertation is of the form

$$U^{(\delta)}(x) := \begin{cases} x^{\delta}/\delta, & \text{if } x > 0, \\ \lim_{\epsilon \to 0} \epsilon^{\delta}/\delta, & \text{if } x = 0, \\ -\infty, & \text{if } x < \infty, \end{cases} \tag{2.13}$$

for $\delta \in (-\infty, 1) \setminus \{0\}$.

Definition 2.6.3.
Let U be a utility function. We define a strictly decreasing, continuous

inverse function $I : (0, \infty) \to (0, \infty)$ by $I(y) := (U'(y))^{-1}$. By analogous with (2.12), I satisfies

$$I(0) := \lim_{y \to 0} I(y) = \infty\,, \qquad I(\infty) := \lim_{y \to \infty} I(y) = 0\,. \tag{2.14}$$

Define a function

$$\widetilde{U}(y) := \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y)\,, \quad 0 < y < \infty\,, \tag{2.15}$$

which is the convex dual of $-U(-x)$, with U extended to be $-\infty$ on the negative real axis. The function $\widetilde{U}$ is strictly decreasing, strictly convex and satisfies

$$\begin{aligned} \widetilde{U}'(y) &= -I(y)\,, \qquad 0 < y < \infty \\ U(x) &= \min_{y>0}[\widetilde{U}(y) + xy] = \widetilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty\,. \end{aligned} \tag{2.16}$$

Then from (2.15) and (2.16), we have the following useful inequalities:

$$U(I(y)) \ \geq \ U(x) + y[I(y) - x]\,, \qquad \forall x > 0, y > 0\,, \tag{2.17}$$

$$\widetilde{U}(U'(x)) \ \leq \ \widetilde{U}(y) - x[U'(x) - y]\,, \qquad \forall x > 0, y > 0\,. \tag{2.18}$$

Chapter summary

As mentioned at the beginning of the chapter, we reviewed some important results that are used in this dissertation. The random measures and compensated random measures were considered. We also provided the Itô's formula for one-dimensional and multidimensional cases as well. The theorem of existence and uniqueness solution of a Lévy SDE was established. We introduced the concepts of martingale, BSDE with jumps as well as the HJB equation for jump diffusion processes. We ended the chapter considering the utility functions and their properties. In the next chapter, we shall derive the wealth process in three different contexts.

Chapter 3

Portfolio dynamics and life insurance

The aim of this chapter is to derive the wealth process of an investor in the presence of the triple (investment, consumption and life insurance). We start by defining the general financial market under consideration in this dissertation. Then we derive the wealth process when we just have portfolios and consumption in the market. Other than the sources received from the investments, an investor may have some external sources. This case is considered in Section 3.3. In Section 3.4, we obtain a wealth process when an investor, in addition to the consumption, pays a life insurance.

3.1 Financial market

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is given an M-dimensional Brownian motion $W(t) = (W_1(t), \dots, W_M(t))$ and an ℓ-dimensional Poisson random measure $N(t, A) = (N_1(t, A), \dots, N_\ell(t, A))$ with a Lévy measure $\nu(A) = (\nu_1(A), \dots, \nu_\ell(A))$, such that W and N are independent. Here, $W(0) = 0$ and $N(0, \cdot) = 0$ almost surely. This section is adapted from ([20], section 1.1).

We introduce a risk-free share with a price $S_0(t)$, $0 \le t \le T$ strictly positive, $\mathcal{F}_t$-adapted and continuous defined by

$$dS_0(t) = r(t)S_0(t)dt\,, \;\; S_0(0) = 1, \quad \forall t \in [0,T]\,, \tag{3.1}$$

or equivalently

$$S_0(t) = \exp\left(\int_0^t r(s)ds\right); \quad \forall t \in [0,T]\,,$$

where $r(t)$ is called the risk-free interest rate at time $t \in [0,T]$. The risk-free rate process $r(\cdot)$ is a random and time-dependent, $\mathcal{F}_t$-measurable.

Next, we introduce N stocks with price per share $S_1(t); \ldots; S_N(t)$ which are continuous, strictly positive and satisfy the following Lévy stochastic differential equation

$$\begin{aligned} dS_n(t) &= S_n(t)\Big[\alpha_n(t)dt + \sum_{m=1}^{M} \beta_{nm}(t)dW_m(t) \\ &\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} \gamma_{nk}(t,z_k)\widetilde{N}_k(dt,dz_k)\Big]; \quad \forall t \in [0,T]\,, \\ S_n(0) &= s_n > 0\,, \end{aligned} \tag{3.2}$$

where $\alpha_n : [0,T]\times\Omega \to \mathbb{R}$, $\beta_n : [0,T]\times\Omega \to \mathbb{R}^M$ and $\gamma_n : [0,T]\times\mathbb{R}^\ell\times\Omega \to \mathbb{R}^\ell$ are adapted processes, for $n = 1,\ldots,N$. By Itô's formula (Theorem 1.2.3), the solution of (3.2) is given by

$$\begin{aligned} S_n(t) &= s_n \exp\Big\{\int_0^t \Big[\alpha_n(s) - \frac{1}{2}\sum_{m=1}^{M}\beta_{nm}^2(s)\Big]ds + \sum_{m=1}^{M}\int_0^t \beta_{nm}(s)dW_m(s) \\ &\quad + \sum_{k=1}^{\ell}\int_0^t\int_{|z_k|<a_k} \{\ln(1+\gamma_{nk}(s,z_k)) - \gamma_{nk}(s,z_k)\}\nu_k(z_k)ds \\ &\quad + \sum_{k=1}^{\ell}\int_0^t\int_{\mathbb{R}} \ln(1+\gamma_{nk}(s,z_k))\widetilde{N}_k(ds,dz_k)\Big\}\,. \end{aligned}$$

Definition 3.1.1.
A *financial market*, hereafter denoted by $\mathcal{M}$, consists of

(i) a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

(ii) a positive constant T called the terminal time;

(iii) an M-dimensional Brownian motion $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ and an ℓ-dimensional Poisson random measure $\{N(t,\cdot), \mathcal{F}_t; 0 \leq t \leq T\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{0\leq t\leq T}$ is a filtration, with W independent of N;

(iv) a progressively measurable risk-free rate process $r(\cdot)$ satisfying

$$\int_0^T |r(t)|dt < \infty\,, \quad \text{a.s.}\,;$$

(v) a progressively measurable N-dimensional mean-rate of return process $\alpha(t)$ satisfying

$$\int_0^T \|\alpha(t)\|dt < \infty\,, \quad \text{a.s.}\,;$$

(vi) a progressively measurable, $N \times M$-matrix-valued volatility process $\beta(t)$ satisfying

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\int_0^T \beta_{nm}^2(t)dt < \infty\,, \quad \text{a.s.}\,;$$

(vii) a progressively measurable, $N \times \ell$-matrix-valued jump-coefficients process $\gamma(t,\cdot)$ satisfying

$$\sum_{n=1}^{N}\sum_{k=1}^{\ell}\int_0^T \gamma_{nk}^2(t,z_k)\nu_k(dz_k)dt < \infty\,, \quad \text{a.s.}\,;$$

$(viii)$ a vector of positive constant initial stock prices $S(0) = (s_1,\ldots,s_N)^T$.

Remark.
When $\gamma_{nk} = 0$, for any $n = 1,\ldots,N$ and $k = 1,\ldots,\ell$, we have a diffusion financial market considered in [20].

3.2 Portfolio and Gain processes

We consider a financial market $\mathcal{M}$ consisting of one risk-free asset (money market) given by (3.1) and N risky shares given by (3.2). The main objective of this section is that of deriving the dynamics of the value of a so-called *self-financing* portfolio in continuous time. This section is adapted from ([3], Chapter 6 and [20], Section 1.2), were a diffusion framework has been done.

As in [3, 20], We start by studying a model in discrete time, then let the length of the time step tend to zero, thus obtaining the continuous time.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the interval $[0, T]$.

Assumption 3.1.

$h_n(t_m)$ = the number of shares of stock n held during the period $[t_m, t_{m+1})$, for $n = 1, \ldots, N$ and $m = 0, \ldots, M-1$;

$h_0(t_m)$ = the number of shares held in the risk-free asset;

$c(t_m)$ = the amount spent on consumption during the period $[t_m, t_{m+1})$;

We also assume that for $n = 0, 1, \ldots, N$, the random variable $h_n(t_m)$ is $\mathcal{F}_{t_m}$-measurable, i.e., anticipation of the future is not permitted.

Let us define the value of the portfolios V by the stochastic difference equation

$$\begin{aligned} V(0) &= 0\,; \\ V(t_{m+1}) - V(t_m) &= \sum_{n=0}^{N} h_n(t_m)\,[S_n(t_{m+1}) - S_n(t_m)]\,; \quad m = 0, \ldots, M-1\,. \end{aligned}$$

Then $V(t_m)$ is the amount of the portfolios during the period $[0, t_m]$. On the other hand, the value of the portfolios at today's price is given by

$$V(t_m) = \sum_{n=0}^{N} h_n(t_m) S_n(t_m)\,; \quad m = 0, \ldots, M\,,$$

if and only if there is no exogenous infusion or withdrawal of funds on the interval $[0, T]$. In this case, the trading is called *self-financing*.

Suppose that $h(\cdot) = (h_0(\cdot), \ldots, h_N(\cdot))^T$ is an $\mathcal{F}_t$-adapted process defined on the interval $[0, T]$, not just on the partition points $t_0, \ldots, t_M$. The associated value process is now defined by the initial condition $V(0) = 0$ and the stochastic differential equation

$$dV(t) = \sum_{n=0}^{N} h_n(t) dS_n(t); \quad \forall t \in [0,T]\,. \tag{3.3}$$

If we consider that the cost for the consumption rate $c(t_m)$ given by $c(t_m)(t_{m+1} - t_m)$, the value process in continuous time becomes

$$dV(t) = \sum_{n=0}^{N} h_n(t) dS_n(t) - c(t)dt; \quad \forall t \in [0,T]\,. \tag{3.4}$$

We then give a mathematical definition of the central concepts.

Definition 3.2.1.
Let $S_0(t)$ be a risk-free price process given by (3.1) and $(S_n(t),\, t \in [0,T])$ be the risky price process given by (3.2), $n = 1, \ldots, N$.

(1) A *portfolio strategy* $(h_0(\cdot), h(\cdot))$ for the financial market $\mathcal{M}$ consists of an $\mathcal{F}_t$-progressively measurable real valued process $h_0(\cdot)$ and an $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^N$-valued process $h(\cdot) = (h_1(\cdot), \ldots, h_N(\cdot))^T$;

(2) the portfolio $h(\cdot)$ is said to be *Markovian* if it is of the form $h(t, S(t))$, for some function $h : [0,T] \times \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$;

(3) the *value process* V corresponding to the portfolio h is given by

$$V(t) = \sum_{n=0}^{N} h_n(t) S_n(t)\,;$$

(4) a *consumption process* is an $\mathcal{F}_t$-adapted one dimensional process $\{c(t);\, t \in [0,T]\}$;

(5) a portfolio-consumption pair (h, c) is called *self-financing* if the value process V satisfies the condition

$$dV(t) = \sum_{n=0}^{N} h_n(t) dS_n(t) - c(t)dt; \quad \forall t \in [0,T]\,.$$

For computational purposes it is often convenient to describe a portfolio in relative terms, i.e., we specify the relative proportion of the total portfolio value which is invested in the stock.

Define

$$\pi_n(t) := \frac{h_n(t)S_n(t)}{V(t)}; \quad n = 1, \ldots, N$$

and $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_N(\cdot))^T$, where

$$\pi_0(t) = 1 - \sum_{n=1}^{N} \pi_n(t).$$

From (3.1) and (3.2), the value process (3.4) becomes

$$\begin{aligned} dV(t) &= \left[V(t)\left(r(t) + \sum_{n=1}^{N} \pi_n(t)(\alpha_n(t) - r(t))\right) - c(t)\right] dt \\ &+ V(t)\sum_{n=1}^{N}\sum_{m=1}^{M} \pi_n(t)\beta_{nm}(t)dW_m(t) \\ &+ V(t)\sum_{n=1}^{N}\sum_{k=1}^{\ell} \pi_n(t)\int_{\mathbb{R}} \gamma_{nk}(t,z_k)\widetilde{N}_k(dt,dz_k);\ 0 \le t \le T, \end{aligned} \tag{3.5}$$

or equivalently

$$\begin{aligned} V(t) &= \int_0^t \left[V(t)\left(r(t) + \sum_{n=1}^{N} \pi_n(t)(\alpha_n(t) - r(t))\right) - c(t)\right] dt \\ &+ \sum_{n=1}^{N}\sum_{m=1}^{M} \int_0^t V(t)\pi_n(t)\beta_{nm}(t)dW_m(t) \\ &+ \sum_{n=1}^{N}\sum_{k=1}^{\ell} \int_0^t V(t)\pi_n(t)\int_{\mathbb{R}} \gamma_{nk}(t,z_k)\widetilde{N}_k(dt,dz_k);\ 0 \le t \le T, \end{aligned} \tag{3.6}$$

where

$$\int_0^T |\pi^T(t)(\alpha(t) - r(t)\mathbf{1})|dt < \infty; \quad \int_0^T \|\pi(t)\beta(t)\|^2 dt < \infty \tag{3.7}$$

and

$$\int_0^T \int_{\mathbb{R}} \|\pi(t)\gamma(t,z)\|^2 \nu(dz)dt < \infty \tag{3.8}$$

hold almost surely, where $\mathbf{1}$ represents an N-dimensional vector of units $\mathbf{1} = (1, \ldots, 1)$ and $\alpha \in \mathbb{R}^N$, $\beta \in \mathbb{R}^{N \times M}$ and $\gamma \in \mathbb{R}^{N \times \ell}$.

Remark.
The definition of the value process in (3.6) does not take into account any cost for trading. A market in which there are no transaction costs is called *frictionless*.
The Conditions (3.7)-(3.8) are imposed in order to ensure the existence of the integrals in (3.6).
If $\pi_0(\cdot) < 0$, means that the investor is borrowing money from the money market. The position $\pi_n(\cdot)$; $n = 1, \ldots, N$ in stock n may be negative, which corresponds to the short-selling of the stocks.

In this dissertation, we consider a frictionless market and only the case where borrowing money from the market and short-selling are not permitted, i.e., $\pi_n(t) \geq 0$; $\forall t \in [0, T]$; $n = 0, \ldots, N$.

3.3 Income and wealth processes

An investor may have sources of income and expenses other than those from investments in the assets discussed in the previous section. Here, we include this possibility in the model. This section is mainly adapted from ([20], Section 1.3).

Definition 3.3.1.
Let $\mathcal{M}$ be a financial market. A *cumulative income process* $(\Gamma(t); 0 \leq t \leq T)$ is a cumulative wealth received by an investor on the time interval $[0, T]$. In particular, the investor is given initial wealth $\Gamma(0)$. If Γ has the structure

$$d\Gamma(t) = i(t)dt, \quad 0 \leq t \leq T \tag{3.9}$$

or equivalently

$$\Gamma(t) = \int_0^t i(s)ds, \quad 0 \leq t \leq T\,, \tag{3.10}$$

for some progressively measurable process $i(\cdot)$, representing and income rate, then we say that the investor receives an income continuously.

Definition 3.3.2.
Let $\mathcal{M}$ be a financial market, $\Gamma(\cdot)$ in (3.9), a cumulative income process and $(\pi_0(\cdot), \pi(\cdot))$ a portfolio process. The wealth process associated with $(\Gamma(\cdot), \pi_0(\cdot), \pi(\cdot))$ is

$$dX(t) := dV(t) + d\Gamma(t)\,,$$

where $V(\cdot)$ is the value process defined by (3.3).

The portfolio-consumption $(\pi_0(\cdot), \pi(\cdot), c(\cdot))$ is said to be Γ-*financed* if

$$dX(t) = dV(t) + d\Gamma(t)\,,$$

where dV is given by (3.5) and $d\Gamma$ by (3.9).

3.4 Life insurance process

In this dissertation, we solve the optimal investment, consumption and life insurance problem. This section is devoted to introduce the concept of life insurance contract and the hazard function. For more details see e.g. [30], ([33], Chapter 7) and ([24], Chapter 3).

Definition 3.4.1.
A *general life insurance contract* is a vector $((\xi(t), \delta(t))_{t\in[0,T]})$ of t-portfolios, where for any $t \in [0, T]$, the portfolio $\xi(t)$ is interpreted as a payment of the insurer to the insurant (*benefit*) and $\delta(t)$ as a payment of the insurant to the insurer (*premium*), respectively taking place at time t.

As in the previous section, in the presence of life insurance contract in the portfolio, if the premium is continuously given by $\delta(t)$, from (3.6) and (3.10), the portfolio-consumption and life insurance $(\pi_0(\cdot), \pi(\cdot), c(\cdot), \delta(\cdot))$ is said to be Γ-*financed* if

$$
\begin{aligned}
V(t) &= \int_0^t \left[V(t) \left(r(t) + \sum_{n=1}^{N} \pi_n(t)(\alpha_n(t) - r(t)) \right) + i(t) - c(t) - \delta(t) \right] dt \\
&+ \sum_{n=1}^{N} \sum_{m=1}^{M} \int_0^t V(t)\pi_n(t)\beta_{nm}(t) dW_m(t) \\
&+ \sum_{n=1}^{N} \sum_{k=1}^{\ell} \int_0^t V(t)\pi_n(t) \int_{\mathbb{R}} \gamma_{nk}(t, z_k)\widetilde{N}_k(dt, dz_k);\ 0 \le t \le T\,.
\end{aligned}
$$

3.4.1 Survival function and force of mortality

Let τ be the random *lifetime* or *age-at-death* of an individual. Set $F(t) = P(\tau < t)$, the distribution function of τ. We assume that an individual is alive at time $t = 0$, that is, once has been born, his/her lifetime is not equal to zero ($F(0) = 0$). We define the *survival function* $\bar{F}(t)$, by

$$\bar{F}(t) = P(\tau > t \mid \mathcal{F}_t) = 1 - F(t)\,,$$

where $\mathcal{F}_t$ is the filtration at time t. Clearly, $\bar{F}(0) = 1$ because $F(0) = 0$.

Hereafter, we assume that the distribution function $F(t)$ is continuous, thus the distribution has density $f(t) = F'(t)$. For an infinitesimal $\epsilon > 0$, we have that

$$P(t < \tau \le t + \epsilon) = f(t) \cdot \epsilon\,. \tag{3.11}$$

Consider $P(t < \tau \le t + \epsilon \mid \tau \ge t)$, the probability that the individual under consideration will die within the interval $[t, t + \epsilon]$, given that he/she has survived t years, i.e., $\tau \ge t$. From (3.11), the *force of mortality* or *a* *hazard function* of τ is defined by

$$\begin{aligned}\mu(t) &:= \lim_{\epsilon\to 0}\frac{P(t<\tau\le t+\epsilon\mid\tau\ge t)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{P(t<\tau\le t+\epsilon)}{\epsilon P(\tau\ge t)}\\ &= \frac{1}{\bar F(t)}\lim_{\epsilon\to 0}\frac{F(t+\epsilon)-F(t)}{\epsilon}\\ &= \frac{f(t)}{\bar F(t)}\\ &= -\frac{d}{dt}(\ln(\bar F(t)))\,, \end{aligned} \tag{3.12}$$

provided that $\bar F(t)\neq 0$, $\forall t$. If $\bar F(t)=0$, the force of mortality $\mu(t)=\infty$ by definition. The larger $\mu(t)$ is equivalent to the larger the probability that an individual of age t will die soon, i.e., within a small time interval $[t,t+\epsilon]$.

From (3.12), the survival function of an individual is given by

$$\bar F(t)=\exp\left(-\int_0^t\mu(s)ds\right) \tag{3.13}$$

and consequently, the conditional probability density of death of the individual under consideration at time t, by

$$f(t)=F'(t)=\mu(t)\exp\left(-\int_0^t\mu(s)ds\right). \tag{3.14}$$

Remark.

The filtration $\mathcal{F}_t$ is defined in such a way that it includes the information from the market as well as the information of lifetime of an individual.

Under a life insurance contract, the benefit insured consists of a single payment, called *the sum insured*. The time and amount of the payment may be random variables. Let $\phi(t)$ be the sum insured to be paid out upon death time $t<T$ and X, the policyholder's wealth. Choosing ϕ, the *policyholder*[1] agrees to hand over the amount of money $X-\phi$ to the pension company

[1] A policyholder is an individual who pays an amount of money to the insurance company

upon death, i.e., the pension company keeps the wealth X for themselves and pays out ϕ as a life insurance. If the contract is frictionless, the risk premium rate to pay for the life insurance ϕ at time t is $\mu(t)(\phi(t) - X(t))dt$ ([22]). Then the wealth process becomes

$$\begin{aligned} dV(t) &= \left[V(t)\left(r(t) + \sum_{n=1}^{N} \pi_n(t)(\alpha_n(t) - r(t))\right) - c(t) - \mu(t)(\phi(t) - X(t))\right] dt \\ &\quad + V(t) \sum_{n=1}^{N} \sum_{m=1}^{M} \pi_n(t) \beta_{nm}(t) dW_m(t) \\ &\quad + V(t) \sum_{n=1}^{N} \sum_{k=1}^{\ell} \pi_n(t) \int_{\mathbb{R}} \gamma_{nk}(t, z_k) \widetilde{N}_k(dt, dz_k);\ 0 \le t \le T\,. \end{aligned} \tag{3.15}$$

Chapter summary

In this chapter, we have derived the wealth process of an investor facing three different scenarios: first we obtained a wealth process for investment-consumption portfolios, then we included an income and a life insurance as well. In the first two, was an extension of [20], where the similar cases were considered in a geometric diffusion model. The inclusion of the life insurance is similar to that in [30].

In the next two chapters, we shall solve the optimal investment, consumption and life insurance problem when the the investor's wealth is given by (3.15), for $N = M = \ell = 1$ and $N = 2$, $M = \ell = 1$ respectively.

Chapter 4

Optimal investment, consumption and life insurance problem with random parameters

In this chapter, we extend the results in [35] to a geometric Itô-Lévy jump process. Our modelling framework is very general as it allows random parameters which are unbounded and involves some jumps. It also covers parameters which are both Markovian and non-Markovian functionals. Unlike in [35] who considered a diffusion framework, ours solves the problem using a novel approach, which combines the Hamilton-Jacobi-Bellman (HJB) introduced in Section 2.5 and a backward stochastic differential equation (BSDE) with jumps in Section 2.4. In Section 4.1, we provide the modeling dynamics and problem formulation. In Section 4.2, we provide a verification theorem for the combined HJB equation and the BSDE related to our problem. In Section 4.3, we give the main result of this chapter and finally, in Section 4.4, we give two special examples to illustrate the main result. This chapter is based on results from [13].

4.1 The Model formulation

We consider a frictionless financial market $\mathcal{M}$ consisting of a risk-free asset $S_0 := (S_0(t)_{t\in[0,T]})$ and a risky asset $S := (S(t))_{t\in[0,T]}$ defined as follows:

$$
\begin{aligned}
dS_0(t) &= r(t)S_0(t)dt, \quad S_0(0) = 1\,, && (4.1)\\
dS(t) &= S(t)\left[\alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz)\right], && (4.2)\\
S(0) &= s > 0\,,
\end{aligned}
$$

where $r(t), \alpha(t), \beta(t)$ are $\mathbb{R}^+$-valued and $\gamma(t,\cdot) > -1$ are both $\mathcal{F}_t$-adapted and predictable processes. $\widetilde{N}$ is the compensated Poisson random measure defined by (2.1).

We assume that the wage earner is alive at time $t = 0$, whose lifetime is a non-negative random variable τ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From subsection 3.4.1, the conditional survival probability of the wage earner is given by (3.13) and the conditional survival probability density of the death of the wage earner by (3.14).

We suppose existence of an insurance contract, where the term life insurance is continuously traded. We assume that the wage earner is paying premiums at the rate $p(t)$, at time t for the life insurance contract and the insurance company will pay $p/\eta(t)$ to his beneficiary for his death, where the $\mathcal{F}_t$-adapted process $\eta(t) > 0$ is the premium insurance ratio. This parameter η is allowed to be stochastic due to stochastic mortality or safety loading. When the wage earner dies, the total legacy to his beneficiary is given by

$$
\ell(t) := X(t) + \frac{p(t)}{\eta(t)}\,,
$$

where $X(t)$ is the wealth process of the wage earner at time t and $p(t)/\eta(t)$ the insurance benefit paid by the insurance company to the beneficiary if death occurs at time t.

Let $c(t)$ be the consumption rate of the wage earner and $\pi(t)$ the fraction of the wage earner's wealth invested in the risky share. The wealth process

$X(t)$ is defined by the following stochastic differential equation (SDE):

$$\begin{aligned} dX(t) &= [X(t)(r(t)+\pi(t)\mu(t)) - c(t) - p(t)]\,dt + \pi(t)\beta(t)X(t)dW(t) \\ &+ \pi(t)X(t)\int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz), \quad t \in [0,\tau\wedge T]\,, \qquad (4.3) \\ X(0) &= x > 0\,, \end{aligned}$$

where $\mu(t) := \alpha(t) - r(t)$ is the appreciation rate.

The main problem of this chapter is to choose the optimal investment, consumption and life insurance problem, so that the wage earner maximizes the expected discounted utilities derived from the intertemporal consumption during $[0,\tau\wedge T]$, from the legacy if he dies before time $T<\infty$ and from the terminal wealth if he is alive until time T. We suppose that the discount process rate $\rho(t)$ is positive and $\mathcal{F}_t$-adapted process.

Given the utility function U for the intertemporal consumption, legacy and terminal wealth, the wage earner's problem is then to choose an investment, consumption and insurance strategy so as to optimize the following performance functional:

$$\begin{aligned} J(0,x_0;\pi,c,p) \;&:=\; \sup_{(\pi,c,p)\in\mathcal{A}} \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du}U(c(s))ds \qquad (4.4) \\ &+e^{-\int_0^\tau \rho(u)du}U(\ell(\tau))\mathbf{1}_{\{\tau\le T\}} + e^{-\int_0^T \rho(u)du}U(X(T))\mathbf{1}_{\{\tau>T\}}\Big]\,, \end{aligned}$$

where U is the utility function for the consumption, legacy and terminal wealth. We consider a utility function of the constant relative risk aversion (CRRA) type, given by (2.13).

From (3.13) and (3.14), the maximum utility (4.4) is equivalent to the following performance functional

$$\begin{aligned} J(0,x_0,\pi,c,p) &= \sup_{(\pi,c,p)\in\mathcal{A}} \mathbb{E}\Big[\int_0^T e^{-\int_0^s \rho(u)du}[\bar{F}(s)U(c(s)) + f(s)U(\ell(s))]ds \\ &+e^{-\int_0^T \rho(u)du}\bar{F}(T)U(X(T))\Big]\,. \end{aligned}$$

Hence,

$$
\begin{aligned}
J(0,x_0;\pi,c,p) &= \sup_{(\pi,c,p)\in\mathcal{A}} \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\lambda(u))du}[U(c(s))+\lambda(s)U(\ell(s))]ds \\
&\quad + e^{-\int_0^T(\rho(u)+\lambda(u))du}U(X(T))\Big]\,. \qquad (4.5)
\end{aligned}
$$

As we are studying a model with random parameters and jumps, the following assumptions are essential to guarantee the existence and uniqueness of a solution to the BSDE related to our problem in the next section. (See [35]):

(A1) the appreciation rate of the share is greater than the interest rate;

(A2) the interest rate, force of mortality, premium insurance ratio and discount rate are bounded away from zero, that is,

$$\exists \epsilon > 0 \,:\, |\Lambda(t)| \geq \epsilon\,, \quad \text{a.e.},$$

where $\Lambda := r, \lambda, \eta, \rho\,;$

(A3) the random parameters satisfy the exponential integrability conditions

$$\mathbb{E}\Big[\exp(\theta\int_0^T |\Lambda(t)|dt)\Big] < \infty\,,$$

for a sufficient large θ, where $\Lambda := r, \lambda, \eta, \rho$.

Definition 4.1.1.

The set of strategies $\mathcal{A} := \{(\pi,c,p) := (\pi(t),c(t),p(t))_{t\in[0,T]}\}$ is said to be admissible if the following conditions hold:

(1) a triple $(\pi,c,p) \in (0,1)\times\mathbb{R}^+\times\mathbb{R}$ is $\mathcal{F}$-adapted process such that

$$\int_0^T c(t)dt < \infty\,, \quad \int_0^T |p(t)|dt < \infty \ \text{ and } \int_0^T \pi^2(t)dt < \infty\,, \quad \mathbb{P}-\text{a.s.},$$

(2) the SDE (4.3) has a unique strong solution associated with (π,c,p) such that:

$$X(t) \geq 0\,, \quad \mathbb{P}-\text{a.s.},$$

(3) and finally:

$$\mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\lambda(u))du}[U^-(c(s))+\lambda(s)U^-(\ell(s))]ds + e^{-\int_0^T(\rho(u)+\lambda(u))du}U^-(X(T))\Big] \;<\; \infty\,.$$

To use the dynamic programming principle, we consider the dynamic version of the performance functional (4.5) given by

$$J(t,x,\pi,c,p) \;=\; \mathbb{E}_{t,x}\Big[\int_t^T e^{-\int_t^s(\rho(u)+\lambda(u))du}[U(c(s))+\lambda(s)U(\ell(s))]ds + e^{-\int_t^T(\rho(u)+\lambda(u))du}U(X(T))\Big]\,, \tag{4.6}$$

where $\mathbb{E}_{t,x}$ represents the conditional expectation $\mathbb{E}[\cdot\,|\,X(t)=x,\mathcal{F}_t]$. The wage earner wishes to maximize the dynamic performance functional (4.6) under the admissible set $\mathcal{A}$, subject to the wealth process (4.3). Therefore, the value function of the problem is given by

$$V(t,x)=\sup_{(\pi,c,p)\in\mathcal{A}} J(t,x,\pi,c,p)=J(t,x,\pi^*,c^*,p^*)\,,$$

where $(\pi^*,c^*,p^*)\in\mathcal{A}$ is the optimal investment, consumption and life insurance strategy to be determined in the next section. We point out that the value function $V(t,x)$ is an $\mathcal{F}_t$- measurable random variable, since all model parameters are random in our model, so the value function can not be determined from the partial differential equation as usual. To determine the value function, we will use a combination of the (HJB) equation and the (BSDE) equation. Although it is not possible to obtain an explicit solution to the optimal π^*, we will show the sufficient conditions to guarantee that the solution $\pi^*\in(0,1)$ exists. The optimal c^* and p^* are derived explicitly.

4.2 Combination of HJB equation with BSDE with jumps

In this section, we employ a combination of HJB equation introduced in Theorem 2.5.1. with BSDE with jumps introduced in section 2.4 to solve our optimal problem.

Let $\mathcal{S} := [0,T]\times\mathbb{R}\times\mathbb{R}$ be the solvency region. We define $\Psi : \mathcal{S} \mapsto \mathbb{R}$ such that $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,2}(\mathcal{S})$. We suppose that $\Psi_t, \Psi_x, \Psi_y, \Psi_{xx}, \Psi_{yy}$ are partial derivatives with respect to t, x, y respectively. We define the following partial differential generator:

$$
\begin{aligned}
& \mathcal{L}^{\pi,c,p}[\Psi(t,x,y)] \\
= \quad & -(\rho(t)+\lambda(t))\Psi(t,x,y) + \Psi_t(t,x,y) \\
& + \left[X(t)(r(t)+\pi(t)\mu(t)) - c(t) - p(t)\right]\Psi_x(t,x,y) \\
& -\Psi_y(t,x,y) f(t,Y(t),\Upsilon(t)) + \frac{1}{2}\pi^2(t)\beta^2(t)X^2(t)\Psi_{xx}(t,x,y) \\
& + \int_{\mathbb{R}}\Big[\Psi(t,x+\pi x\gamma(t,z),y) - \Psi(t,x,y) \\
& \quad -\pi x\gamma(t,z)\Psi_x(t,x,y)\Big]\nu(dz) + \int_{\mathbb{R}}\Big[\Psi(t,x,y+\Upsilon(t,z)) \\
& -\Psi(t,x,y) - \Upsilon(t,z)\Psi_x(t,x,y)\Big]\nu(dz)\,.
\end{aligned}
$$

The following theorem is a verification result for the combination of the HJB equation and the BSDE associated with our problem. We prove similarly as in ([35], Theorem 3.1.).

Theorem 4.2.1. (Verification theorem). *Suppose that* (ξ, f) *satisfy the conditions* $(\mathbf{C1}) - (\mathbf{C3})$*. Let* $\overline{\mathcal{S}}$ *be the closure of the solvency region* $\mathcal{S}$*. Moreover, let* Φ *denote the following function:*

$$\Phi(t,x,Y(t),\pi,c,p) := \mathcal{L}^{\pi,c,p}[\Psi(t,x,y)] + U(c) + \lambda(t)U(x+p/\eta(t))\,. \quad (4.7)$$

Suppose there is a function $\Psi \in \mathcal{C}^2$ *and an admissible control* $(\pi^*, c^*, p^*) \in \mathcal{A}$ *such that:*

(1) $\Phi(t,x,Y(t),\pi,c,p) \leq 0$ *for all* $(\pi,c,p) \in \mathcal{A}$;

(2) $\Phi(t,x,Y(t),\pi^*,c^*,p^*) = 0$;

(3) *for all* $(\pi,c,p) \in \mathcal{A}$,

$$\lim_{t\to T^-} \Psi(t,x,y) = U(x);$$

(4) *let* $\mathcal{K}$ *be the set of stopping times* $\kappa \leq T$. *The family* $\{\Psi(\kappa,X(\kappa),Y(\kappa))\}_{\kappa\in\mathcal{K}}$ *is uniformly integrable.*

Then

$$\begin{aligned}\Psi(t,x,y) &= \sup_{(\pi,c,p)\in\mathcal{A}} J(t,x;\pi,c,p)\\ &= J(t,x;\pi^*,c^*,p^*)\end{aligned}$$

and (π^*,c^*,p^*) *is an optimal control.*

Proof. To prove this theorem, we first define a sequence of localizing stopping times (see [1], Section 2.2 for more details) as follows

$$\begin{aligned}\mathcal{K}_{\mathcal{S}}^{(N)} &:= T\wedge\inf\{t\geq 0 \mid (t,X(t),Y(t))\notin\mathcal{S}\}\\ &\quad \wedge\inf\{t>0 \mid t\geq N \text{ or } |X(t)|\geq N \text{ or } \|Y(t)\|\geq N\}.\end{aligned}$$

Then, Dynkin formula ([29], Theorem 1.24.),

$$\begin{aligned}\Psi(t,x,Y(t)) &= \mathbb{E}_{t,x}\Big[e^{-\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}}[\rho(s)+\lambda(s)]ds}\Psi\left(\mathcal{K}_{\mathcal{S}}^{(N)},X(\mathcal{K}_{\mathcal{S}}^{(N)}),Y(\mathcal{K}_{\mathcal{S}}^{(N)})\right)\\ &\quad -\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}} e^{-\int_t^s[\rho(u)+\lambda(u)]du}\mathcal{L}^{\pi,c,p}[\Psi(s,X(s),Y(s))]ds\Big].\end{aligned} \tag{4.8}$$

Hence, for all $(\pi,c,p)\in\mathcal{A}$, using Condition 1 to (4.8) gives

$$\begin{aligned}&\Psi(t,x,Y(t))\\ \geq\ &\mathbb{E}_{t,x}\Big[\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}} e^{-\int_t^s[\rho(u)+\lambda(u)]du}[U(c(s))+\lambda(s)U(X(s)+p(s)/\eta(s))]ds\\ &+e^{-\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}}[\rho(s)+\lambda(s)]ds}\Psi\left(\mathcal{K}_{\mathcal{S}}^{(N)},X(\mathcal{K}_{\mathcal{S}}^{(N)}),Y(\mathcal{K}_{\mathcal{S}}^{(N)})\right)\Big].\end{aligned} \tag{4.9}$$

Using Conditions 3-4, Fatou's lemma ([8], Theorem 3. p 57) and the Dominated Convergence Theorem ([8], Theorem 10. p 63) yield

$$\begin{aligned}
&\Psi(t,x,Y(t)) \\
\geq\ & \liminf_{N\to\infty} \mathbb{E}_{t,x}\Big[\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}} e^{-\int_t^s[\rho(u)+\lambda(u)]du}[U(c(s))+\lambda(s)U(X(s)+p(s)/\eta(s))]ds \\
&\qquad + e^{-\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}}[\rho(s)+\lambda(s)]ds}\Psi\left(\mathcal{K}_{\mathcal{S}}^{(N)}, X(\mathcal{K}_{\mathcal{S}}^{(N)}), Y(\mathcal{K}_{\mathcal{S}}^{(N)})\right)\Big] \\
\geq\ & \mathbb{E}_{t,x}\Big[\liminf_{N\to\infty}\Big\{\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}} e^{-\int_t^s[\rho(u)+\lambda(u)]du}[U(c(s))+\lambda(s)U(X(s)+p(s)/\eta(s))]ds \\
&\qquad + e^{-\int_t^{\mathcal{K}_{\mathcal{S}}^{(N)}}[\rho(s)+\lambda(s)]ds}\Psi\left(\mathcal{K}_{\mathcal{S}}^{(N)}, X(\mathcal{K}_{\mathcal{S}}^{(N)}), Y(\mathcal{K}_{\mathcal{S}}^{(N)})\right)\Big\}\Big] \\
=\ & J(t,x,\pi,c,p)\,.
\end{aligned} \tag{4.10}$$

Similarly, we can use Conditions 2-4 to derive that

$$\Psi(t,x,Y(t)) = J(t,x,\pi^*,c^*,p^*)\,. \tag{4.11}$$

Consequently, combining (4.10) and (4.11) completes the proof. □

From the above theorem, the value function is the solution of the following system of the HJB equation and BSDE with jumps:

$$\begin{cases} \sup_{(\pi,c,p)\in\mathcal{A}} \Phi(t,x,Y(t),\pi,c,p) = 0, & \Psi(T,x,\xi)=U(x) \\ dY(t) = -f(t,Y(t),\Upsilon(t))dt + \int_{\mathbb{R}}\Upsilon(t,z)\tilde{N}(dt,dz), & Y(T)=\xi\,. \end{cases} \tag{4.12}$$

4.3 General solutions

In this section, we investigate the solutions to the investment, consumption and life insurance problem (4.12) for a wage earner with power utility (2.13).

Theorem 4.3.1. *Under the assumptions* (**A1**) – (**A3**), *the optimal investment, consumption and life insurance strategy of the problem is:*

$$c^* = xe^{-Y(t)}\,, \quad p^* = \eta(t)x\left\{\left[\frac{\lambda(t)}{\eta(t)}\right]^{\frac{1}{1-\delta}} e^{-Y(t)} - 1\right\},$$

and π^* *is the solution of the equation*

$$\mu(t) - (1-\delta)\pi\beta^2(t) - \int_{\mathbb{R}} \left[1 - (1+\pi\gamma(t,z))^{\delta-1}\right]\gamma(t,z)\nu(dz) = 0\,,$$

where $Y \in \mathbb{S}^2(\mathbb{R})$ *is given by*

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left\{\int_t^T \ln\left[1 + \int_t^s \left(1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right)du\right]K(s)ds\right\},$$

for $K(\cdot)$ *to be specified later.*

Proof. To obtain the optimal solution (π^*, c^*, p^*), from the terminal condition, we try the value function of the form:

$$V(t,x) = \Psi(t,x,Y(t)) = \frac{1}{\delta}x^{\delta}e^{(1-\delta)Y(t)}, \tag{4.13}$$

where Y is the solution of the BSDE (2.9), for $\xi = 0$, $Z = 0$ and f to be defined later. With this choice, it is clear that Φ in equation (4.7) fulfils the standard procedures to solve equation (4.12), i.e. $\Phi_{\pi\pi} < 0$, $\Phi_{cc} < 0$ and $\Phi_{pp} < 0$.

Applying the first order conditions of optimality to Φ with respect to (π, c, p) we obtain

$$-\Psi_x(t,x,Y(t)) + U'(c) = 0\,, \tag{4.14}$$

$$-\Psi_x(t,x,Y(t)) + \lambda(t)\frac{\partial U(x+p/\eta(t))}{\partial p} = 0 \tag{4.15}$$

and

$$\begin{aligned} \mu(t)x\Psi_x(t,x,Y(t)) - (1-\delta)\pi\beta^2(t)x^2\Psi_{xx}(t,x,Y(t)) \\ + \int_{\mathbb{R}}\left[\Psi_\pi(t,x+\pi x\gamma(t,z),Y(t)) - \gamma(t,z)x\Psi_x(t,x,Y(t))\right]\nu(dz) &= 0\,. \end{aligned} \tag{4.16}$$

Substituting (2.13) and (4.13) into (4.14)-(4.16) give the following optimal investment, consumption and life insurance strategy

$$c^*(t) = xe^{-Y(t)}\,, \tag{4.17}$$

$$p^*(t) = \eta(t)x\left\{\left[\frac{\lambda(t)}{\eta(t)}\right]^{\frac{1}{1-\delta}}e^{-Y(t)} - 1\right\} \tag{4.18}$$

and $\pi^*(t)$ is the solution of the following equation

$$
\begin{aligned}
x^{\delta}e^{(1-\delta)Y(t)}\Big\{\mu(t)-(1-\delta)\pi(t)\beta^2(t)\\
-\int_{\mathbb{R}}\left[1-(1+\pi(t)\gamma(t,z))^{\delta-1}\right]\gamma(t,z)\nu(dz)\Big\} &= 0\,. \qquad (4.19)
\end{aligned}
$$

Furthermore, from (4.18), the optimal legacy is given by

$$
\ell^*(t)=x\left[\frac{\lambda(t)}{\eta(t)}\right]^{\frac{1}{1-\delta}}e^{-Y(t)}\,.
$$

To ensure the existence of the unique solution $\pi\in(0\,,1)$ in (4.19) we follow the ideas in [2]. To this end, we define a function

$$
h(\pi)=\mu(t)-(1-\delta)\pi\beta^2(t)-\int_{\mathbb{R}}\left[1-(1+\pi\gamma(t,z))^{\delta-1}\right]\gamma(t,z)\nu(dz)\,.
$$

If $\pi=0$, $h(\pi)=\mu(t):=\alpha(t)-r(t)>0$ and

$$
h'(\pi)=-(1-\delta)\left[\beta^2(t)+\int_{\mathbb{R}}(1+\pi\gamma(t,z))^{\delta-2}\gamma^2(t,z)\nu(dz)\right]<0\,.
$$

Then, the solution exists and is unique if

$$
h(1)=\mu(t)-(1-\delta)\beta^2(t)-\int_{\mathbb{R}}\left[1-(1+\gamma(t,z))^{\delta-1}\right]\gamma(t,z)\nu(dz)<0\,,
$$

hence

$$
(1-\delta)\beta^2(t)+\int_{\mathbb{R}}\left[1-(1+\gamma(t,z))^{\delta-1}\right]\gamma(t,z)\nu(dz)>\alpha(t)-r(t)\,.
$$

To complete the proof, we need to obtain the function Y. Substituting (4.17), (4.18) and π^* into the HJB equation (4.12) gives the following expression

$$
\begin{aligned}
&\frac{1-\delta}{\delta}x^{\delta}e^{(1-\delta)Y}\Big\{-f-\frac{1}{1-\delta}(\rho+\lambda)+\frac{\delta}{1-\delta}(r+\eta)\\
&+\frac{\delta}{1-\delta}\Big\{\pi^*\mu-\frac{1-\delta}{2}(\pi^*)^2\beta^2+\frac{1}{\delta}\int_{\mathbb{R}}\left[(1+\pi^*\gamma(t,z))^{\delta}-1-\delta\pi^*\gamma(t,z)\right]\nu(dz)\Big\}\\
&+\left[1+\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right]e^{-Y}+\int_{\mathbb{R}}\Big\{\frac{1}{1-\delta}\left[e^{(1-\delta)\Upsilon}-1\right]-\Upsilon\Big\}\nu(dz)\Big\}=0\,.
\end{aligned}
$$

Then, taking the coefficient of $x^\delta e^{(1-\delta)Y}$ equal to zero, leads to

$$\begin{aligned} f(t,Y(t),\Upsilon(t)) &= K(t) + \left[1 + \frac{\lambda^{\frac{1}{1-\delta}}(t)}{\eta^{\frac{\delta}{1-\delta}}(t)}\right] e^{-Y(t)} \\ &\quad + \int_{\mathbb{R}} \left\{ \frac{1}{1-\delta}\left[e^{(1-\delta)\Upsilon(t,z)} - 1\right] - \Upsilon(t,z) \right\} \nu(dz)\,, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} K(t) &= -\frac{1}{1-\delta}(\rho(t)+\lambda(t)) + \frac{\delta}{1-\delta}(r(t)+\eta(t)) + \frac{\delta}{1-\delta}\Big\{\pi^*(t)\mu(t) \\ &\quad -\frac{1-\delta}{2}(\pi^*(t))^2\beta^2(t) + \frac{1}{\delta}\int_{\mathbb{R}}\Big[(1+\pi^*(t)\gamma(t,z))^\delta - 1 \\ &\qquad -\delta\pi^*(t)\gamma(t,z)\Big]\nu(dz)\Big\}\,. \end{aligned}$$

Under the Assumptions (**A1**) – (**A3**), the BSDE (2.9) with the generator (4.20), the control $Z = 0$ and the terminal value $\xi = 0$, satisfies the Conditions (**C1**) – (**C3**). Then, by Theorem 2.4.1, there exists a unique solution $(Y,\Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$. As in ([9], Chapter 11), to obtain this solution, we define the probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ as follows:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}(T)} = M\,, \tag{4.21}$$

where M is the Radon-Nikodym derivative given by the dynamics

$$\frac{dM(t)}{M(t)} := \int_{\mathbb{R}} \left[\frac{e^{(1-\delta)\Upsilon(s,z)} - 1}{(1-\delta)\Upsilon(s,z)} - 1\right] \widetilde{N}(ds,dz)\,.$$

Suppose that

$$\upsilon := \upsilon(t,z) = \frac{e^{(1-\delta)\Upsilon(s,z)} - 1}{(1-\delta)\Upsilon(s,z)} - 1\,. \tag{4.22}$$

By Theorem 2.3.2.,

$$\widetilde{N}^{\mathbb{Q}}(dt,dz) := N(dt,dz) - (1+\upsilon(t,z))\nu(dz)dt, \quad 0 \le t \le T\,,$$

is a $(\mathbb{Q},\mathcal{F})$- compensated random measure. Then, under the probability $\mathbb{Q}$, the BSDE (2.9) with generator (4.20) and $Z = 0$ becomes:

$$\begin{aligned} dY(t) &= -\left\{K(t) + \left[1 + \frac{\lambda^{\frac{1}{1-\delta}}(t)}{\eta^{\frac{\delta}{1-\delta}}(t)}\right] e^{-Y(t)}\right\} dt \\ &\quad + \int_{\mathbb{R}} \Upsilon(t,z)\widetilde{N}^{\mathbb{Q}}(dt,dz)\,; \\ Y(T) &= 0\,. \end{aligned} \tag{4.23}$$

By ([9], Theorem 11.2.1), the solution of (4.23) has the representation

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left\{\int_t^T \ln\left[1 + \int_t^s \left(1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right) du\right] K(s)ds\right\} \tag{4.24}$$

and Υ can be determined by the martingale representation theorem. □

As in [35], Y can be interpreted as an intuitive actuarial value process of a consumption rate from current to $\tau \wedge T$, $\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}$ insurance benefit paying at death time $\tau < T$ and a legacy at terminal T if the wage earner survives until time T.

4.4 Special Examples

In this section, we present two examples which are particular cases of Theorem 4.3.1. In each example, we consider a model with one random parameter and all others deterministic functions of t. We derive the explicit solution Y and consequently the Theorem 4.3.1. is verified.

In the first example, we consider a stochastic mortality with jumps. Jumps in a mortality process might occur for a variety of reasons: sudden changes in environmental conditions or a radical medical changes [5]. At the second example, we consider the case of stochastic appreciation rate with jumps. This case is motivated by the cointegrated model in [6], where the log-prices depend on the diffusion appreciation rate.

Example 4.4.1. We consider the force of mortality given by the following geometric jump-diffusion model:

$$d\lambda(s) = \lambda(s)[ads + bdW(s) + \int_{z>-1} z\widetilde{N}(ds, dz)], \quad \lambda(t) = \lambda > 0, \quad 0 \le t \le s \le T, \tag{4.25}$$

where $a, b \in \mathbb{R}$ are constants and $z > -1$.

Then, $Y(t)$ in (4.24) is represented as follows:

$$\begin{aligned} Y(t) &= \int_t^T \mathbb{E}^{\mathbb{Q}}\left\{\ln\left[1+\int_t^s \left(1+\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right)du\right]\right\} L(s)ds \\ &\quad -\frac{1}{1-\delta}\mathbb{E}^{\mathbb{Q}}\left\{\int_t^T \ln\left[1+\int_t^s \left(1+\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right)du\right]\lambda(s)ds\right\}, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} L(s) &:= -\frac{\rho(s)}{1-\delta}+\frac{\delta}{1-\delta}(r(s)+\eta(s))+\frac{\delta}{1-\delta}\Big\{\pi^*\mu(s)-\frac{1-\delta}{2}(\pi^*)^2\beta^2(s) \\ &\quad +\frac{1}{\delta}\int_{\mathbb{R}}\left[(1+\pi^*\gamma(s,z))^\delta-1-\delta\pi^*\gamma(s,z)\right]\nu(dz)\Big\} \end{aligned}$$

and $\mathbb{E}^{\mathbb{Q}}[\,\cdot\,]$ is the conditional expectation under $\mathbb{Q}$, given $\mathcal{F}_t$.

Under the probability $\mathbb{Q}$, the dynamics of the mortality process is given by

$$d\lambda(s)=\lambda(s)\left[\left(a+\int_{z>-1} zv(s,z)\nu(dz)\right)ds+bdW^{\mathbb{Q}}(s)+\int_{z>-1} z\widetilde{N}^{\mathbb{Q}}(ds,dz)\right].$$

Then, the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\lambda(s)]$, given $\mathcal{F}_t$ is given by

$$\mathbb{E}^{\mathbb{Q}}[\lambda(s)]=\lambda\exp\left\{\int_t^s\left[a+\int_{z>-1} zv(s,z)\nu(dz)\right]ds\right\}. \tag{4.27}$$

We define a new probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ as follows:

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}:=\frac{\lambda(s)}{\mathbb{E}^{\mathbb{Q}}[\lambda(s)]}=\exp\left\{\int_0^s\left[-\frac{1}{2}b^2+\int_{z>-1}(\ln(1+z)-z)\,\nu(dz)\right]du\right\}.$$

By change of measures, (4.26) becomes:

$$\begin{aligned} Y(t) &= \int_t^T \mathbb{E}^{\widetilde{\mathbb{Q}}}\left\{\ln\left[1+\int_t^s \left(1+\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right)du\right]\right\} L(s)ds \\ &\quad -\frac{1}{1-\delta}\int_t^T \mathbb{E}^{\widetilde{\mathbb{Q}}}\left\{\ln\left[1+\int_t^s \left(1+\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right)du\right]\right\}\cdot\mathbb{E}^{\mathbb{Q}}[\lambda(s)]ds\,. \end{aligned} \tag{4.28}$$

Applying the Itô's formula to $\lambda^{\frac{1}{1-\delta}}$ under $\widetilde{\mathbb{Q}}$, we obtain:

$$\begin{aligned} d\lambda^{\frac{1}{1-\delta}}(s) &= \lambda^{\frac{1}{1-\delta}}(s)\Big\{\frac{1}{1-\delta}\Big\{a + \frac{2-\delta}{2(1-\delta)}b^2 + \int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}} - 1\right] \\ &\quad + zv(s,z) - \delta z\Big\}\nu(dz)\Big\}ds + \frac{1}{1-\delta}bdW^{\widetilde{\mathbb{Q}}}(s) \\ &\quad + \int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}} - 1\right]\widetilde{N}^{\widetilde{\mathbb{Q}}}(ds,dz)\Big\}. \end{aligned}$$

Hence, taking expectation $\mathbb{E}[\,\cdot\,]$, under $\widetilde{\mathbb{Q}}$, gives:

$$\begin{aligned} \mathbb{E}^{\widetilde{\mathbb{Q}}}[\lambda^{\frac{1}{1-\delta}}(s)] &= \lambda^{\frac{1}{1-\delta}}\exp\Big\{\int_t^s \frac{1}{1-\delta}\Big\{a + \frac{2-\delta}{2(1-\delta)}b^2 \\ &\quad + \int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}} - 1\right] + zv(u,z) - \delta z\Big\}\nu(dz)\Big\}du\Big\}, \end{aligned}$$

since

$$\int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}} - 1\right] + zv(u,z) - \delta z\Big\}\nu(dz) < \infty\,.$$

To obtain the expectation $\mathbb{E}^{\widetilde{\mathbb{Q}}}[\,\cdot\,]$ in (4.28), we consider:

$$Z(s) = \int_t^s \left(1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\,.$$

By linearity of conditional expectation, we see that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[Z] = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_t^s \left(1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\right] = \int_t^s \left(1 + \frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\right]}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\,. \tag{4.29}$$

Then from [36], we know that, for a random variable X such that $\mathbb{E}[X] \gg 0$, the third derivative is small and the Taylor approximation series to $\mathbb{E}[\ln(1+X)]$ is very accurate, that is:

$$\mathbb{E}[\ln(1+X)] = \ln(1+\mathbb{E}[X]) - \frac{\mathbb{V}(X)}{2(1+\mathbb{E}[X])^2} + R_X\,,$$

where $R_X > 0$ is very small. Applying this techniques in (4.28), we have:

$$\begin{aligned} &\mathbb{E}^{\widetilde{\mathbb{Q}}}\left\{\ln\left[1+\int_t^s\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\right]\right\} \\ =\ &\ln\left[1+\int_t^s\left(1+\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\right]}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\right]-\Pi+R_X\,, \end{aligned} \tag{4.30}$$

where

$$\Pi=\frac{\mathbb{V}\left[\int_t^s\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\right]}{2(1+\int_t^s\left(1+\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\right]}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du)^2}\,.$$

We then need to obtain the variance

$$\mathbb{V}(Z):=\mathbb{E}^{\widetilde{\mathbb{Q}}}[Z^2]-\left(\mathbb{E}^{\widetilde{\mathbb{Q}}}[Z]\right)^2\,.$$

Provided that the integrand of Z is absolutely convergent, by Fubini's theorem we can change the order of integration and applying the linearity of conditional expectation, leads to

$$\begin{aligned} \mathbb{E}^{\widetilde{\mathbb{Q}}}[Z^2] &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_{v=t}^s\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(v)}{\eta^{\frac{\delta}{1-\delta}}(v)}\right)dv\cdot\int_{u=t}^s\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)du\right] \\ &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_{v=t}^s\int_{u=t}^s\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(v)}{\eta^{\frac{\delta}{1-\delta}}(v)}\right)\cdot\left(1+\frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)}\right)dudv\right] \\ &= \int_{v=t}^s\int_{u=t}^s\Bigg\{1+\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\right]}{\eta^{\frac{\delta}{1-\delta}}(u)}+\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(v)\right]}{\eta^{\frac{\delta}{1-\delta}}(v)} \\ &\quad+\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\cdot\lambda^{\frac{1}{1-\delta}}(v)\right]}{\eta^{\frac{\delta}{1-\delta}}(u)\cdot\eta^{\frac{\delta}{1-\delta}}(v)}\Bigg\}dudv\,. \end{aligned} \tag{4.31}$$

It remains to get the expectation $\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\cdot\lambda^{\frac{1}{1-\delta}}(v)\right]$ in the right hand side of (4.31). By Itô's formula, we know that

$$
\begin{aligned}
& d\left[\lambda^{\frac{1}{1-\delta}}(u)\cdot\lambda^{\frac{1}{1-\delta}}(v)\right] \\
= \; & \lambda^{\frac{1}{1-\delta}}(u)\cdot\lambda^{\frac{1}{1-\delta}}(v)\Big\{\frac{1}{1-\delta}\Big\{a+\frac{2-\delta}{2(1-\delta)}b^2+\int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}}-1\right] \\
& +zv(u,z)-\delta z\Big\}\nu(dz)\Big\}du+\frac{1}{1-\delta}bdW^{\widetilde{\mathbb{Q}}}(u) \\
& +\int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}}-1\right]\tilde{N}^{\widetilde{\mathbb{Q}}}(du,dz)+\frac{1}{1-\delta}\Big\{a+\frac{2-\delta}{2(1-\delta)}b^2 \\
& +\int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}}-1\right]+zv(v,z)-\delta z\Big\}\nu(dz)\Big\}dv \\
& +\frac{1}{1-\delta}bdW^{\widetilde{\mathbb{Q}}}(v)+\int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}}-1\right]\tilde{N}^{\widetilde{\mathbb{Q}}}(dv,dz) \\
& +\Big\{\frac{1}{(1-\delta)^2}b^2+\int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}}-1\right]^2\nu(dz)\Big\}du\wedge dv \\
& +\int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}}-1\right]^2\tilde{N}^{\widetilde{\mathbb{Q}}}(du\wedge dv,dz)\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\lambda^{\frac{1}{1-\delta}}(u)\cdot\lambda^{\frac{1}{1-\delta}}(v)\right] \\
= \; & \lambda^{\frac{2}{1-\delta}}\exp\Big\{\frac{1}{1-\delta}\int_t^u\Big\{a+\frac{2-\delta}{2(1-\delta)}b^2 \\
& +\int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}}-1\right]+zv(x,z)-\delta z\Big\}\nu(dz)\Big\}dx \\
& +\frac{1}{1-\delta}\int_t^v\Big\{a+\frac{2-\delta}{2(1-\delta)}b^2+\int_{z>-1}\Big\{(1-\delta)\left[(1+z)^{\frac{1}{1-\delta}}-1\right] \\
& +zv(x,z)-\delta z\Big\}\nu(dz)\Big\}dx \\
& +\int_t^{u\wedge v}\Big\{\frac{1}{(1-\delta)^2}b^2+\int_{z>-1}\left[(1+z)^{\frac{1}{1-\delta}}-1\right]^2\nu(dz)\Big\}dx\Big\}.
\end{aligned}
$$

Then, substituting (4.27) and (4.30) into (4.28) we obtain Y. Taking Y into Theorem 4.3.1., we obtain the optimal investment, consumption and life insurance strategy and the value function.

We conclude this example illustrating the effect of the jump in the mortality rate. We consider the following parameters $a = -0.035, b = 0.1, \lambda(0) =$

0.05 and $T = 40$ years. The equations to be simulated are: Given the geometric jump-diffusion model studied in this example,

$$d\lambda(s) = \lambda(s)[ads + bdW(s) + dN(t)], \quad \lambda(t) = \lambda > 0, \quad 0 \le t \le s \le T$$

and a geometric diffusion mortality given by

$$d\lambda(s) = \lambda(s)[ads + bdW(s)], \quad \lambda(t) = \lambda > 0, \quad 0 \le t \le s \le T.$$

Graphically, we see that the jump-diffusion mortality can capture the high rates of mortality (solid curve) in the figure given below. This can happen in the cases of radical change in environmental conditions such as war, earthquakes, sudden pandemics etc. Which can not be captured by a geometric brownian motion (dashed curve).

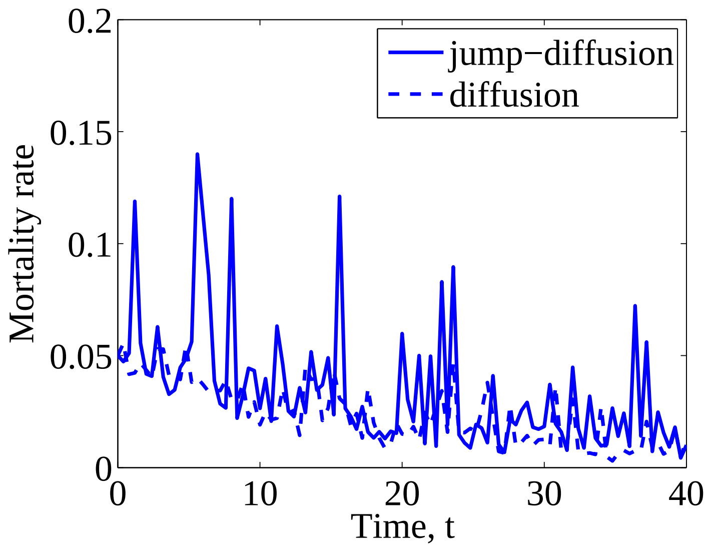

Figure 4.1: Mortalities with diffusion and jump-diffusion processes.

Example 4.4.2. We consider an appreciation rate with jump $\mu(s)$ governed by the following dynamics

$$d\mu(s) = (a(s) - b\mu(s))ds + \int_{\mathbb{R}} \psi(s,z)\tilde{N}(ds,dz), \quad \mu(t) = \mu^{+}; \quad 0 \le t \le s \le T,$$

where $a(t) \in \mathbb{R}$ is deterministic and uniformly bounded, $b \in \mathbb{R}$ is constant.

Under the probability $\mathbb{Q}$, the appreciation rate $\mu(s)$ follows the dynamics

$$d\mu(s) = \left[a(s) - b\mu(s) + \int_{\mathbb{R}} \psi(s,z)\upsilon(s,z)\nu(dz)\right] ds + \int_{\mathbb{R}} \psi(s,z)\tilde{N}^{\mathbb{Q}}(ds,dz)\,. \tag{4.32}$$

Then, taking the conditional expectation $\mathbb{E}[\,\cdot\,]$, under $\mathbb{Q}$, we obtain

$$\mathbb{E}^{\mathbb{Q}}[\mu(s)] = e^{-b(s-t)}\mu_0 + \int_t^s \left[a(u) + \int_{\mathbb{R}} \psi(u,z)\upsilon(u,z)\nu(dz)\right] e^{-b(u-s)}du\,, \tag{4.33}$$

since

$$\int_{\mathbb{R}} \psi(u,z)\upsilon(u,z)\nu(dz) < \infty\,.$$

The solution (4.24) becomes

$$Y(t) = \int_t^T \ln\left[1 + \int_t^s \left(1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}\right) du\right] \cdot \left(M(s) + \frac{\delta}{1-\delta}\pi^* \mathbb{E}^{\mathbb{Q}}[\mu(s)]\right) ds\,, \tag{4.34}$$

where

$$\begin{aligned} M(s) \quad = \quad & -\frac{\rho(s)+\lambda(s)}{1-\delta} + \frac{\delta}{1-\delta}(r(s)+\eta(s)) + \frac{\delta}{1-\delta}\Big\{-\frac{1-\delta}{2}(\pi^*)^2\beta^2(s) \\ & +\frac{1}{\delta}\int_{\mathbb{R}} \left[(1+\pi^*\gamma(s,z))^\delta - 1 - \delta\pi^*\gamma(s,z)\right]\nu(dz)\Big\}. \end{aligned}$$

Taking Y in (4.34) into Theorem 4.3.1., we obtain the optimal investment, consumption and life insurance strategy and the value function.

Chapter summary

In this chapter, we solved the optimal investment, consumption and life insurance problem with random parameters using a combination of HJB equation and BSDE with jumps method. We obtained the optimal strategy, where the BSDE is solved via martingale approach, representing its solution by the expected value under $\mathbb{Q}$ martingale measure. Then in section 4.4, we derived the explicit expected value in two cases. The remainder are similar to those in Examples 4.4.1 and 4.4.2.

Chapter 5

Optimal investment, consumption and life insurance problem with capital guarantee

In this chapter, based on the results in [22], we solve a geometric jump-diffusion optimization problem. We use the martingale approach applied in [19] to obtain the optimal solution to the unrestricted problem in section 5.2. In section 5.3, we obtain the solution to the restricted capital guarantee problem based on terms derived from the martingale method in the unrestricted problem.

5.1 Financial Model

We consider a real Brownian motion $W = \{W(t), \mathcal{F}_t^W; 0 \le t \le T\}$ associated to the complete filtered probability space $(\Omega^W, \mathcal{F}^W, (\mathcal{F}_t^W), \mathbb{P}^W)$ and a Poisson process $N = \{N(t), \mathcal{F}_t^N, 0 \le t \le T\}$ associated to the complete filtered probability space $(\Omega^N, \mathcal{F}^N, (\mathcal{F}_t^N), \mathbb{P}^N)$ with intensity $\lambda(t)$ and

$$\tilde{N}(t) := N(t) - \int_0^t \lambda(t)dt\,,$$

a $\mathbb{P}^N$-martingale compensated poisson process. We assume that the intensity is Lebesgue integrable on $[0, T]$.

We consider the product space:

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}) := (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, (\mathcal{F}_t^W \otimes \mathcal{F}_t^N), \mathbb{P}^W \otimes \mathbb{P}^N)\,,$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration introduced in Definition 2.1.3. On this space, W and N are independent processes.

We consider a frictionless financial market $\mathcal{M}$ consisting of a risk-free asset $S_0 := (S_0(t)_{t\in[0,T]})$ and a risky asset $S := (S_1(t),\ S_2(t))_{t\in[0,T]}$ defined by the following jump-diffusion model:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1\,, \tag{5.1}$$

$$dS_i(t) = S_i(t)\left[\alpha_i(t)dt + \beta_i(t)dW(t) + \gamma_i(t)dN(t)\right], \quad S(0) = s\,, \tag{5.2}$$

where $r(t), \alpha_i(t), \beta_i(t)$ and $\gamma_i(t)$, $i = 1, 2$ satisfy the following assumption:

Assumption 5.1.
The interest rate $r(t)$, the vector of mean rate of returns $\alpha(t) := (\alpha_1(t), \alpha_2(t))$, the dispersion coefficients $\beta(t) := (\beta_1(t), \beta_2(t))$ and $\gamma(t) := (\gamma_1(t), \gamma_2(t))$ are measurable $\mathcal{F}_t$-adapted uniformly bounded processes and $\gamma_i(t) > -1$ for $i = 1, 2$.

Let us consider a policyholder whose lifetime is a nonnegative random variable τ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As in Chapter 4, the conditional survival probability of the policyholder is given by (3.13) and the conditional survival probability density of the death of the policyholder by (3.14).

Let $c(t)$ be the consumption rate of the policyholder, $\pi := (\pi_1, \pi_2)$ the fraction of the policyholder's wealth invested in the risky assets S and $p(t)$ the sum insured paid for the life insurance.

Definition 5.1.1.
The consumption rate c is measurable, $\mathcal{F}_t$-adapted process, nonnegative and

$$\int_0^T c(t)dt < \infty, \quad \text{a.s.}$$

The allocation process $\pi := (\pi_1, \pi_2)$ is an $\mathcal{F}_t$-predictable process with

$$\sum_{i=1}^{2} \int_0^t \pi_i^2(t)dt < \infty, \quad \text{a.s.} \tag{5.3}$$

The insurance process p is measurable $\mathcal{F}_t$-adapted process, nonnegative and

$$\int_0^T p(t)dt < \infty, \quad \text{a.s.}$$

Suppose that the policyholder receives a labor income of rate $\ell(t) \geq 0$, $\forall t \in [0, \tau \wedge T]$. The wealth process $X(t)$ is defined by the following stochastic differential equation (SDE):

$$\begin{aligned} dX(t) &= [(r(t) + \mu(t))X(t) + \langle \pi(t), \phi(t) \rangle + \ell(t) - c(t) - \mu(t)p(t)]\, dt \\ &+ \langle \pi(t), \beta(t) \rangle dW(t) + \langle \pi(t), \gamma(t) \rangle dN(t), \quad t \in [0, \tau \wedge T], \\ X(0) &= x_0 > 0, \end{aligned} \tag{5.4}$$

where π satisfying (5.3) is the vector amount invested in the risky share $S := (S_1, S_2)$, $\phi := (\alpha_1 - r,\ \alpha_2 - r)$ is the vector of appreciation rate, $\beta(t) := (\beta_1(t), \beta_2(t))$, $\gamma(t) := (\gamma_1(t), \gamma_2(t))$. The expression $\mu(t)(p(t) - X(t))dt$ correspond to the risk premium rate introduced in subsection 3.4.1. Notice that choosing $p > X$ corresponds to buying a life insurance and $p < X$ corresponds to selling a life insurance, that is buying an annuity ([22]).

Assumption 5.1 and Definition 5.1.1. guarantee that the wealth process (5.4) is well defined and has a unique solution given by

$$\begin{aligned} X(t) &= x_0 e^{\int_0^t (r(s)+\mu(s))ds} + \int_0^t e^{\int_s^t (r(u)+\mu(u))du} \Big[\langle \pi(s), \phi(s) \rangle + \ell(s) - c(s) \\ &\quad - \mu(s)p(s) \Big] ds + \int_0^t \langle \pi(s), \beta(s) \rangle e^{\int_s^t (r(u)+\mu(u))du} dW(s) \\ &\quad + \int_0^t \langle \pi(s), \gamma(s) \rangle e^{\int_s^t (r(u)+\mu(u))du} dN(s). \end{aligned} \tag{5.5}$$

We define a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ in which S_i are local martingales. As in [34], the Radom-Nikodym derivative is given by:

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} := \Lambda(t) &= \exp\Big\{ \int_0^t [(1 - \psi(s))\lambda(s) - \frac{1}{2}\theta^2(s)]ds + \int_0^t \theta(s)dW(s) \\ &\quad + \int_0^t \ln(\psi(s))dN(s) \Big\}. \end{aligned} \tag{5.6}$$

Under $\mathbb{Q}$, we have that:

$$\begin{cases} dW^{\mathbb{Q}}(t) = dW(t) - \theta(t)dt, \\ d\widetilde{N}^{\mathbb{Q}}(t) = dN(t) - \psi(t)\lambda(t)dt\,. \end{cases}$$

where

$$\theta(t) = \frac{\gamma_1(t)(\alpha_2 - r) - \gamma_2(\alpha_1 - r)}{\beta_1\gamma_2 - \gamma_1\beta_2} \quad (5.7)$$

$$\psi(t)\lambda(t) = \frac{\beta_2(\alpha_1 - r) - \beta_1(\alpha_2 - r)}{\beta_1\gamma_2 - \gamma_1\beta_2}\,. \quad (5.8)$$

For existence of (5.7)-(5.8), we assume that $\beta_1\gamma_2 - \gamma_1\beta_2 \neq 0$.

Remark.

An asset price defined by jump-diffusion process consists of two sources of randomness, which implies infinitely many martingale measures. For instance, for $i = 1$ in (5.2), the Radom-Nikodym derivative is given by (5.6), where the market price of risk (MPR) θ is given by

$$\theta(t) = \frac{r(t) - \alpha_1(t) - \gamma_1(t)\psi(t)\lambda(t)}{\beta_1(t)}\,, \quad (5.9)$$

for an arbitrary $\psi(t) \geq 0$. Thus, if we consider another asset price S_2 in (5.2), for $i = 2$, as a price of a derivative asset with underlying S_1 and using the Itô's formula, we obtain the same MPR, i.e.,

$$\frac{r(t) - \alpha_1(t) - \gamma_1(t)\psi(t)\lambda(t)}{\beta_1(t)} = \frac{r(t) - \alpha_2(t) - \gamma_2(t)\psi(t)\lambda(t)}{\beta_2(t)}\,,$$

which gives (5.8). Substituting (5.8) into (5.9), gives (5.7) and consequently a unique martingale measure. See [34], for more details.

From (5.7) and (5.8), we have that:

$$[\langle \pi(s), \phi(s)\rangle + \langle \pi(s), \theta(s)\beta(s)\rangle + \langle \pi(s), \psi(s)\lambda(s)\gamma(s)\rangle] = 0\,,$$

then under $\mathbb{Q}$, the dynamics of the wealth process (5.4) is given by

$$\begin{aligned} dX(t) &= [(r(t) + \mu(t))X(t) + \ell(t) - c(t) - \mu(t)p(t)]\,dt \\ &\quad + \langle \pi(t), \beta(t)\rangle dW^{\mathbb{Q}}(t) + \langle \pi(t), \gamma(t)\rangle d\widetilde{N}^{\mathbb{Q}}(t)\,, \end{aligned}$$

which gives the following representation:

$$
\begin{aligned}
X(t) &= x_0 e^{\int_0^t (r(s)+\mu(s))ds} + \int_0^t e^{\int_s^t (r(u)+\mu(u))du}\Big[\ell(s) - c(s) - \mu(s)p(s)\Big]ds \\
&\quad + \int_0^t \langle \pi(s), \beta(s)\rangle e^{\int_s^t (r(u)+\mu(u))du} dW^{\mathbb{Q}}(s) \\
&\quad + \int_0^t \langle \pi(s), \gamma(s)\rangle e^{\int_s^t (r(u)+\mu(u))du} d\widetilde{N}^{\mathbb{Q}}(s)\,. \qquad (5.10)
\end{aligned}
$$

Definition 5.1.2.
Define the set of admissible strategies $\{\mathcal{A}\}$ as the consumption, investment and life insurance strategies for which the corresponding wealth process given by (5.10) is well defined and

$$X(t) + g(t) \geq 0, \quad \forall t \in [0,T]\,, \qquad (5.11)$$

where g is the time t actuarial value of future labor income defined by

$$g(t) := \int_t^T e^{-\int_t^s (r(u)+\mu(u))du}\ell(s)ds$$

and

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \langle \pi(s), \beta(s)\rangle e^{\int_s^t (r(u)+\mu(u))du} dW^{\mathbb{Q}}(s)\right] &= 0\,, \qquad (5.12)\\
\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \langle \pi(s), \gamma(s)\rangle e^{\int_s^t (r(u)+\mu(u))du} d\widetilde{N}^{\mathbb{Q}}(s)\right] &= 0\,. \qquad (5.13)
\end{aligned}
$$

From the conditions (5.12)-(5.13), we see that the last two terms in (5.10) are $\mathbb{Q}$ local martingales and from (5.11), a supermartingale (see Definition 2.3.2.). Then, the strategy (c, π_1, π_2, p) is admissible if and only if $X(T) \geq 0$ and $\forall t \in [0,T]$,

$$
\begin{aligned}
X(t) + g(t) &= \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^s (r(u)+\mu(u))du}[c(s) + \mu(s)p(s)]ds \\
&\quad + e^{-\int_t^T (r(u)+\mu(u))du} X(T) \mid \mathcal{F}_t\Big]\,. \qquad (5.14)
\end{aligned}
$$

At time zero this means that the strategies have to fulfil the following budget constraint:

$$\begin{aligned} X(0)+g(0) &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^t (r(u)+\mu(u))du}[c(t)+\mu(t)p(t)]dt \\ &\quad + e^{-\int_0^T (r(u)+\mu(u))du} X(T)\Big]. \end{aligned} \tag{5.15}$$

As in [22], the following remark is useful for the rest of the chapter.

Remark.
Define

$$\begin{aligned} Z(t) &:= \int_0^t e^{-\int_0^s (r(u)+\mu(u))du}[c(s)+\mu(s)p(s)-\ell(s)]ds \\ &\quad + X(t)e^{-\int_0^t (r(u)+\mu(u))du}, \quad t \in [0,T]. \end{aligned} \tag{5.16}$$

By (5.10) we have that the Conditions (5.12) and (5.13) are fulfilled if and only if Z is a martingale under $\mathbb{Q}$. The natural interpretation is that, under $\mathbb{Q}$, the discounted wealth plus discounted pension contributions should be martingales. It is useful to note that if Z is a martingale under $\mathbb{Q}$, the dynamics of X can be represented in the following form:

$$\begin{aligned} dX(t) &= [(r(t)+\mu(t))X(t)+\ell(t)-c(t)-\mu(t)p(t)]dt \\ &\quad + \phi(t)dW^{\mathbb{Q}}(t)+\varphi(t)d\widetilde{N}^{\mathbb{Q}}(t), \quad t \in [0,T], \end{aligned} \tag{5.17}$$

for some $\mathcal{F}_t^W$-adapted process ϕ and $\mathcal{F}_t^N$-adapted process φ, satisfying $\phi(t), \varphi(t) \in \mathcal{L}^2$, $\forall t \in [0,T]$, then under $\mathbb{Q}$, Z is a martingale.

The condition (5.11) allows the wealth to become negative, as long as it does not exceed in absolute value the actuarial value of future labor income. Doubling strategies are ruled out as this condition puts a lower boundary on the wealth process.

5.2 The Unrestricted control problem

We consider a power utility function $U : \mathbb{R} \to [-\infty, \infty)$, of constant relative risk aversion (CRRA) type given by (2.13).

The policyholder chooses his strategy $(c(t), \pi(t), p(t))$ in order to optimize the expected utility from consumption, legacy upon death and terminal pension. Similar to Chapter 4, his strategy fulfils the following:

$$\sup_{(\pi,c,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du} U(c(s))ds + e^{-\int_0^\tau \rho(u)du} U(p(\tau))\mathbf{1}_{\{\tau\leq T\}}$$
$$+e^{-\int_0^T \rho(u)du} U(X(T))\mathbf{1}_{\{\tau>T\}}\Big]. \tag{5.18}$$

Here, ρ is a deterministic function representing the policyholder's time preferences. $\mathcal{A}'$ is the subset of the admissible strategies (feasible strategies) given by:

$$\mathcal{A}' := \Big\{(c,\pi,p)\in\mathcal{A}\Big|\ \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du}\min(0,U(c(s)))ds$$
$$+e^{-\int_0^\tau \rho(u)du}\min(0,U(p(\tau)))\mathbf{1}_{\{\tau\leq T\}}$$
$$+e^{-\int_0^T \rho(u)du}\min(0,U(X(T)))\mathbf{1}_{\{\tau>T\}}\Big]> -\infty\Big\}. \tag{5.19}$$

The feasible strategy (5.19) means that it is allowed to draw an infinite utility from the strategy $(\pi, c, p) \in \mathcal{A}'$, but only if the expectation over the negative parts of the utility function is finite. It is clear that for a positive utility function, the sets $\mathcal{A}$ and $\mathcal{A}'$ are equal ([22]).

Using (3.13) and (3.14), we can rewrite the policyholder's optimization problem (5.18) as:

$$\sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s \rho(u)du}[\bar{F}(s)U(c(s)) + f(s)U(p(s))]ds$$
$$+e^{-\int_0^T \rho(u)du}\bar{F}(T)U(X(T))\Big].$$

Hence,

$$\sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s (\rho(u)+\mu(u))du}[U(c(s)) + \mu(s)U(p(s))]ds$$
$$+e^{-\int_0^T (\rho(u)+\mu(u))du} U(X(T))\Big]. \tag{5.20}$$

The following theorem, gives the optimal investment, consumption and life insurance strategy $(c^*(t), \pi_1^*(t), \pi_2^*(t), p^*(t))$, for any $t \in [0, T]$ of the unrestricted control problem (5.18).

Theorem 5.2.1. *Given the problem (5.18), the optimal investment, consumption and life insurance strategy* $(c^*(t), \pi_1^*(t), \pi_2^*(t), p^*(t))$*, for any* $t \in [0, T]$ *is given by the following expressions:*

$$\begin{aligned} c^*(t) &= c^*(0)\exp\Big\{\int_0^t \Big[r(s) - \tilde{r}(s) + \frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) \\ &\quad -\psi(s)\lambda(s)\Big(\psi^{-\frac{1}{1-\delta}}(s) - 1\Big)\Big]ds - \frac{1}{1-\delta}\int_0^t \theta(s)dW(s) \\ &\quad + \int_0^t \Big(\psi^{-\frac{1}{1-\delta}}(s) - 1\Big)\,dN(s)\Big\}, \end{aligned}$$

$$\begin{aligned} \pi_1^*(t) &= \frac{\Big(1 - \psi^{-\frac{1}{1-\delta}}(t)\Big)\beta_2(t) - \frac{1}{1-\delta}\theta(t)\gamma_2(t)}{\beta_1\gamma_2 - \beta_2\gamma_1}(X^*(t) + g(t)), \\ \pi_2^*(t) &= \frac{\frac{1}{1-\delta}\theta(t)\gamma_1(t) + \Big(\psi^{-\frac{1}{1-\delta}}(t) - 1\Big)\beta_1(t)}{\beta_1\gamma_2 - \beta_2\gamma_1}(X^*(t) + g(t)), \end{aligned}$$

and

$$\begin{aligned} p^*(t) &= p^*(0)\exp\Big\{\int_0^t \Big[r(s) - \tilde{r}(s) + \frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) \\ &\quad -\psi(s)\lambda(s)\Big(\psi^{-\frac{1}{1-\delta}}(s) - 1\Big)\Big]ds - \frac{1}{1-\delta}\int_0^t \theta(s)dW(s) \\ &\quad + \int_0^t \Big(\psi^{-\frac{1}{1-\delta}}(s) - 1\Big)\,dN(s)\Big\}, \end{aligned}$$

where

$$\begin{aligned} X^*(t) + g(t) &= (x_0 + g(0))\exp\Big\{\int_0^t \Big[r(s) + \mu(s) - \frac{1+\mu(s)}{f(s)} \\ &\quad + \frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) - \psi(s)\lambda(s)\Big(\psi^{-\frac{1}{1-\delta}}(s) - 1\Big)\Big]ds \\ &\quad - \frac{1}{1-\delta}\int_0^t \theta(s)dW(s) - \frac{1}{1-\delta}\int_0^t \ln\psi(s)dN(s)\Big\}, \end{aligned}$$

$$f(t) := \int_t^T e^{-\int_t^s(\tilde{r}(u)+\mu(u))du}(1+\mu(s))ds + e^{-\int_t^T(\tilde{r}(u)+\mu(u))du},$$

$$\begin{aligned} \tilde{r}(t) &:= -\frac{\delta}{1-\delta}r(t) + \frac{1}{1-\delta}\rho + \frac{\delta}{2(1-\delta)^2}\theta^2(t) \\ &\quad + \Big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1 + \frac{\delta}{1-\delta}(\psi(t) - 1)\Big)\lambda(t) \end{aligned}$$

and

$$g(t) := \int_t^T e^{-\int_t^s (r(u)+\mu(u))du} \ell(s) ds \,.$$

Proof. Consider the inverse of the derivative of the utility function U, $I : (0, \infty] \to [0, \infty)$ in (2.13), i.e., $I(x) = x^{-\frac{1}{1-\delta}}$. By the concavity of U, the inequality (2.17) is satisfied.

From (5.6), let us define the adjusted state price deflator Γ by

$$\begin{aligned}\Gamma(t) &:= \Lambda(t) e^{\int_0^t (\rho(s)-r(s))ds} \\ &= \exp\Big\{ \int_0^t [\rho(s) - r(s) - \frac{1}{2}\theta^2(s) + (1-\psi(s))\lambda(s)]ds + \int_0^t \theta(s) dW(s) \\ &\quad + \int_0^t \ln(\psi(s)) dN(s) \Big\}.\end{aligned}$$

Then the dynamics of Γ is given by:

$$d\Gamma(t) = \Gamma(t) \Big[(\rho(t) - r(t))dt + \theta(t) dW(t) + (\psi(t) - 1) d\widetilde{N}(t) \Big] \,. \tag{5.21}$$

We define ζ^* as a constant satisfying:

$$\begin{aligned}\mathcal{H}(\zeta^*) &:= \mathbb{E}^{\mathbb{Q}} \Big[\int_0^T e^{-\int_0^t (r(u)+\mu(u))du} \Big[I(\zeta^* \Gamma(t)) + \mu(t) I(\zeta^* \Gamma(t)) \Big] dt \\ &\quad + e^{-\int_0^T (r(u)+\mu(u))du} I(\zeta^* \Gamma(T)) \Big] = x_0 + g(0) \,. \end{aligned} \tag{5.22}$$

For any strategy $(c, \pi, p) \in \mathcal{A}'$ with corresponding wealth process $X(t)$ given, using (2.17), the budget constraint (5.15) and (5.22), we have:

$$\begin{aligned}
& \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U(c(s))+\mu(s)U(p(s))]ds \\
& \quad +e^{-\int_0^T(\rho(u)+\mu(u))du}U(X(T))\Big] \\
\leq\ & \mathbb{E}\Big[\int_0^T e^{-\int_0^t(\rho(u)+\mu(u))du}\Big[I(\zeta^*\Gamma(t))+\mu(t)I(\zeta^*\Gamma(t))\Big]dt \\
& \quad +e^{-\int_0^T(\rho(u)+\mu(u))du}I(\zeta^*\Gamma(T))\Big] \\
& -\mathbb{E}\Big[\int_0^T e^{-\int_0^t(\rho(u)+\mu(u))du}\zeta^*\Gamma(t)\Big[(I(\zeta^*\Gamma(t))-c(t))+\mu(t)\Big(I(\zeta^*\Gamma(t)) \\
& \quad -p(t)\Big)\Big]dt+\zeta^*\Gamma(T)e^{-\int_0^T(\rho(u)+\mu(u))du}\left(I(\zeta^*\Gamma(T))-X(T)\right)\Big] \\
=\ & \mathbb{E}\Big[\int_0^T e^{-\int_0^t(\rho(u)+\mu(u))du}\Big[I(\zeta^*\Gamma(t))+\mu(t)I(\zeta^*\Gamma(t))\Big]dt \\
& \quad +e^{-\int_0^T(\rho(u)+\mu(u))du}I(\zeta^*\Gamma(T))\Big]\,.
\end{aligned}$$

Then, since (c,π,p) was arbitrary, we obtain the candidate optimal strategy (c^*,π^*,p^*) given by:

$$c^*(t) = I(\zeta^*\Gamma(t))\,, \tag{5.23}$$

$$p^*(t) = I(\zeta^*\Gamma(t))\,, \tag{5.24}$$

$$X^*(T) = I(\zeta^*\Gamma(T))\,. \tag{5.25}$$

Since (c^*,π^*,p^*) by definition of ζ^* fulfils the budget constraints, it is well known that, in a complete market, there exists an allocation strategy π^* such that $X(T)=X^*(T)$ and (c^*,π^*,p^*) is admissible ([20]). We need to calculate the allocation strategy π^* and to obtain the explicit solutions to c^* and p^*.

From the definition of I, we get from (5.22) the following:

$$\begin{aligned}
\mathcal{H}(\zeta^*) &:= \mathbb{E}\Big[\int_t^T \Gamma(t)e^{-\int_0^t(\rho(u)+\mu(u))du}(\zeta^*\Gamma(t))^{-\frac{1}{1-\delta}}(1+\mu(t))dt \\
&\quad +\Gamma(T)e^{-\int_0^T(\rho(u)+\mu(u))du}(\zeta^*\Gamma(T))^{-\frac{1}{1-\delta}}\Big] \\
&= (\zeta^*)^{-\frac{1}{1-\delta}}f(0)\,,
\end{aligned} \tag{5.26}$$

where we have defined:

$$\begin{aligned} f(t) &= \mathbb{E}\Big[\int_t^T e^{-\int_t^s(\rho(u)+\mu(u))du}\left(\frac{\Gamma(s)}{\Gamma(t)}\right)^{-\frac{\delta}{1-\delta}}(1+\mu(t))dt \\ &\quad + e^{-\int_t^T(\rho(u)+\mu(u))du}\left(\frac{\Gamma(T)}{\Gamma(t)}\right)^{-\frac{\delta}{1-\delta}}\ \Big|\ \mathcal{F}_t\Big]. \end{aligned}$$

Note that from (5.21) and using Itô's formula (Theorem 2.2.1.), we get:

$$\begin{aligned} d\Gamma^{-\frac{\delta}{1-\delta}}(t) &= \Gamma^{-\frac{\delta}{1-\delta}}(t)\Big\{\Big[-\frac{\delta}{1-\delta}(\rho(t)-r(t)) + \frac{\delta}{2(1-\delta)^2}\theta^2(t) \\ &\quad + \big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1 + \frac{\delta}{1-\delta}(\psi(t)-1)\big)\lambda(t)\Big]dt - \frac{\delta}{1-\delta}\theta(t)dW(t) \\ &\quad + \left(\psi^{-\frac{\delta}{1-\delta}}(t)-1\right)d\widetilde{N}(t)\Big\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\Gamma^{-\frac{\delta}{1-\delta}}(t)] &= \exp\Big\{\int_t^s\Big[\frac{\delta}{1-\delta}(r(u)-\rho(u)) + \frac{\delta}{2(1-\delta)^2}\theta^2(u) \\ &\quad + \big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1 + \frac{\delta}{1-\delta}(\psi(t)-1)\big)\lambda(t)\Big]dt\Big\}. \end{aligned}$$

Then

$$f(t) = \int_t^T e^{-\int_t^s(\tilde{r}(u)+\mu(u))du}(1+\mu(s))ds + e^{-\int_t^T(\tilde{r}(u)+\mu(u))du}, \tag{5.27}$$

where

$$\begin{aligned} \tilde{r}(t) &= -\frac{\delta}{1-\delta}r(t) + \frac{1}{1-\delta}\rho + \frac{\delta}{2(1-\delta)^2}\theta^2(t) + \Big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1 \\ &\quad + \frac{\delta}{1-\delta}(\psi(t)-1)\Big)\lambda(t). \end{aligned} \tag{5.28}$$

Since $\mathcal{H}(\zeta^*) = x_0 + g(0)$, we get from (5.26) that

$$\zeta^* = (x_0 + g(0))^{\delta-1}f(0)^{1-\delta}.$$

Inserting this ζ^* into (5.23)-(5.25) and using the budget constraint (5.15) we obtain the following expressions:

$$c^*(t) = D^*(t) = \frac{X(t)+g(t)}{f(t)}, \tag{5.29}$$

$$X^*(T) = \frac{X(t)+g(t)}{f(t)}\left(\frac{\Gamma(T)}{\Gamma(t)}\right)^{-\frac{1}{1-\delta}}. \tag{5.30}$$

From (5.21), by Itô's formula, we know that

$$\begin{aligned}\left(\frac{\Gamma(T)}{\Gamma(t)}\right)^{-\frac{1}{1-\delta}} &= \exp\Big\{\frac{1}{1-\delta}\int_t^T \Big[r(s)+\frac{1}{2}\theta^2(s)-\rho(s)+[\psi(s)-1\\ &\quad -\ln\psi(s)]\lambda(s)\Big]ds-\frac{1}{1-\delta}\int_t^T\theta(s)dW(s)\\ &\quad -\frac{1}{1-\delta}\int_t^T\ln\psi(s)d\widetilde{N}(s)\Big\}.\end{aligned}$$

Then we have:

$$\begin{aligned}dX^*(t) &= \mathcal{O}dt-\frac{1}{1-\delta}\theta(t)(X^*(t)+g(t))dW(t)\\ &\quad +\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)(X^*(t)+g(t))dN(t)\,, \qquad (5.31)\end{aligned}$$

where $\mathcal{O} := \mathcal{O}(t, X^*(t), g(t))$. Comparing (5.31) with (5.4), we obtain the optimal allocation

$$\left\{\begin{array}{l}\langle\pi^*(t),\beta(t)\rangle = -\frac{1}{1-\delta}\theta(t)(X^*(t)+g(t))\\ \langle\pi^*(t),\gamma(t)\rangle = \left(\psi^{-\frac{1}{1-\delta}}-1\right)(X^*(t)+g(t))\end{array}\right\}$$

and

$$\pi_1^*(t) = \frac{\left(1-\psi^{-\frac{1}{1-\delta}}(t)\right)\beta_2(t)-\frac{1}{1-\delta}\theta(t)\gamma_2(t)}{\beta_1\gamma_2-\beta_2\gamma_1}(X^*(t)+g(t))\,, \quad (5.32)$$

$$\pi_2^*(t) = \frac{\frac{1}{1-\delta}\theta(t)\gamma_1(t)+\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)\beta_1(t)}{\beta_1\gamma_2-\beta_2\gamma_1}(X^*(t)+g(t))\,. \quad (5.33)$$

Inserting (5.29), (5.32) and (5.33) into (5.4) we obtain

$$\begin{aligned}\frac{d(X^*(t)+g(t))}{X^*(t)+g(t)} &= \Big[r(t)+\mu(t)-\frac{1+\mu(t)}{f(t)}+\frac{1}{1-\delta}\theta^2(t)\\ &\quad -\psi(t)\lambda(t)\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)\Big]dt-\frac{1}{1-\delta}\theta(t)dW(t)\\ &\quad +\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)dN(t)\,. \qquad (5.34)\end{aligned}$$

Hence, by Itô's formula, we get the following solution:

$$
\begin{aligned}
X^*(t)+g(t) &= (x_0+g(0))\exp\Big\{\int_0^t\Big[r(s)+\mu(s)-\frac{1+\mu(s)}{f(s)}+\frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) \\
&\quad -\psi(s)\lambda(s)\left(\psi^{-\frac{1}{1-\delta}}(s)-1\right)\Big]ds-\frac{1}{1-\delta}\int_0^t\theta(s)dW(s) \\
&\quad -\frac{1}{1-\delta}\int_0^t\ln\psi(s)dN(s)\Big\}. \qquad (5.35)
\end{aligned}
$$

Since f is bounded away from zero, $\forall t\in[0,T]$, we have that $X^*(t)$ is well defined and (5.11) is fulfilled. From (5.29), (5.27) and (5.34), by Itô's formula, we have that:

$$
\begin{aligned}
dc^*(t) &= \frac{d(X^*(t)+g(t))}{f(t)}-\frac{X^*(t)+g(t)}{f^2(t)}f'(t)dt \\
&= c^*(t)\Big\{\Big[r(t)-\tilde{r}(t)+\frac{1}{1-\delta}\theta^2(t)-\psi(t)\lambda(t)\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)\Big]dt \\
&\quad -\frac{1}{1-\delta}\theta(t)dW(t)+\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)dN(t)\Big\},
\end{aligned}
$$

which gives $\forall t\in[0,T]$, the following solution:

$$
\begin{aligned}
c^*(t) &= c^*(0)\exp\Big\{\int_0^t\Big[r(s)-\tilde{r}(s)+\frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) \\
&\quad -\psi(s)\lambda(s)\left(\psi^{-\frac{1}{1-\delta}}(s)-1\right)\Big]ds-\frac{1}{1-\delta}\int_0^t\theta(s)dW(s) \\
&\quad +\int_0^t\left(\psi^{-\frac{1}{1-\delta}}(s)-1\right)dN(s)\Big\} \qquad (5.36)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
p^*(t) &= p^*(0)\exp\Big\{\int_0^t\Big[r(s)-\tilde{r}(s)+\frac{\frac{1}{2}-\delta}{(1-\delta)^2}\theta^2(s) \\
&\quad -\psi(s)\lambda(s)\left(\psi^{-\frac{1}{1-\delta}}(s)-1\right)\Big]ds-\frac{1}{1-\delta}\int_0^t\theta(s)dW(s) \\
&\quad +\int_0^t\left(\psi^{-\frac{1}{1-\delta}}(s)-1\right)dN(s)\Big\}, \qquad (5.37)
\end{aligned}
$$

which complete the proof. □

5.3 The restricted control problem

In this section, we solve the optimal investment, consumption and life insurance problem for the constrained[1] control problem. We obtain an optimal strategy for the case of continuous constraints (American put options)[2] by using a so-called *option based portfolio insurance (OBPI) strategy*. The OBPI method consists in taking a certain part of capital and invest in the optimal portfolio of the unconstrained problem and the remaining part insures the position with American put. We prove the admissibility and the optimality of the strategy. For more details see e.g. [11, 22].

Consider the following problem

$$\sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U(c(s))+\mu(s)U(p(s))]ds \qquad (5.38)$$
$$+e^{-\int_0^T(\rho(u)+\mu(u))du}U(X(T))\Big]\,,$$

under the capital guarantee restriction

$$X(t)\geq k(t,Z(t)),\ \ \forall t\in[0,T]\,, \qquad (5.39)$$

and

$$Z(t):=\int_0^t h(s,X(s))ds\,,$$

where k and h are deterministic functions of time. The guarantees discussed above are covered by

$$k(t,z)=0 \qquad (5.40)$$

and

$$k(t,z)=x_0e^{\int_0^t(r^{(g)}(s)+\mu(s))ds}+ze^{\int_0^t(r^{(g)}(s)+\mu(s))ds}\,, \qquad (5.41)$$

with

$$h(s,x)=e^{-\int_0^s(r^{(g)}(u)+\mu(u))du}[\ell(s)-c(s,x)-\mu(s)p(s,x)]\,,$$

[1] The problem (5.38) with the restriction (5.39).

[2] An American option is an option contract in which not only the decision whether to exercise the option or not, but also the choice of the exercise time is at the discretion of the option's holder ([27]).

where $r^{(g)} \leq r$ is the minimum rate of return guarantee excess of the objective mortality μ. Then

$$k(t,z) = x_0 e^{\int_0^t (r^{(g)}(s)+\mu(s))ds} + \int_0^t e^{\int_s^t (r^{(g)}(u)+\mu(u))ds}[\ell(s) - c(s) - \mu(s)p(s)]ds\,. \tag{5.42}$$

We still denote by X^*, c^*, π^* and p^* the optimal wealth, optimal consumption, investment and life insurance for the unrestricted problem (5.18), respectively. The optimal wealth for the unrestricted problem $Y^*(t) := X^*(t) + g(t)$ has the dynamics

$$\begin{aligned} dY^*(t) &= Y^*(t)\Big\{\Big[r(t) + \mu(t) - \frac{1+\mu(t)}{f(t)} + \frac{1}{1-\delta}\theta^2(t) \\ &\quad -\psi(t)\lambda(t)\left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]dt - \frac{1}{1-\delta}\theta(t)dW(t) \\ &\quad + \left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)dN(t)\Big\} \\ &= Y^*(t)\Big\{\left[r(t) + \mu(t) - \frac{1+\mu(t)}{f(t)}\right]dt - \frac{1}{1-\delta}\theta(t)dW^{\mathbb{Q}}(t) \\ &\quad + \left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)d\widetilde{N}^{\mathbb{Q}}(t)\Big\}, \quad \forall t \in [0,T]\,; \\ Y^*(0) &= X^*(0) + g(0) = y_0\,, \end{aligned} \tag{5.43}$$

where $y_0 := x_0 + g(0)$. Let $P^a_{y,z}(t,T,k+g)$ denote the time-t value of an American put option with strike price $k(s, Z(s)) + g(s)$, $\forall s \in [t,T]$, where $Z(t) = z$ and maturity T written on a portfolio Y, where $Y(s)$, $s \in [t,T]$ is the solution to (5.43), with $Y(t) = y$. By definition ([11], Section 4), the price of such put option is given by

$$\begin{aligned} P^a_{y,z}(t,T,k+g) &:= \sup_{\tau_s \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^{\tau_s}(r(u)+\mu(u))du}[k(\tau_s, Z(\tau_s)) + g(\tau_s) \\ &\qquad -Y(\tau_s)]^+ \Big| Y(t) = y, Z(t) = z\Big]\,, \end{aligned} \tag{5.44}$$

where $\mathcal{T}_{t,T}$ is the set of stopping times $\tau_s \in [t,T]$.

Given an underlying unconstrained allocation (5.43) with an American put option (5.44), suppose that ϱ is a part of capital invested in the unconstrained problem and $1-\varrho$, the remaining part which insures the position with American put. The American put option-based portfolio insurance is given by

$$\widehat{X}^{(\varrho)}(t) := \varrho(t, Z(t))Y^*(t) + P^a_{\varrho Y^*,Z}(t, T, k+g) - g(t), \quad t \in [0, T], \tag{5.45}$$

where $\varrho \in (0, 1)$ is determined by the budget constraint

$$\varrho(t, Z(t))Y^*(0) + P^a_{\varrho Y^*,Z}(0, T, k+g) - g(0) = x_0. \tag{5.46}$$

By definition of an American put option, $P^a_{\varrho Y^*,Z}(t, T, k+g) \geq (k(t, z)+g(t)-\varrho Y^*(t))^+$, $\forall t \in [0, T]$. Hence

$$\begin{aligned}
\widehat{X}^{(\varrho)}(t) &:= \varrho(t, Z(t))Y^*(t) + P^a_{\varrho Y^*,Z}(t, T, k+g) - g(t) \\
&\geq \varrho(t, Z(t))Y^*(t) + (k(t, z) + g(t) - \varrho Y^*(t))^+ - g(t) \\
&\geq k(t, z), \ \forall t \in [0, T]\,,
\end{aligned} \tag{5.47}$$

i.e., $\widehat{X}^{(\varrho)}$ fulfils the American capital guarantee.

Consider the strategy $(\varrho c^*, \varrho\pi^*, \varrho p^*)$, where $\varrho(t)$ is defined by

$$\varrho(t) = \varrho_0 \vee \sup_{s \leq t} \left(\frac{b(s, Z(s))}{Y^*(s)} \right), \tag{5.48}$$

ϱ_0 is given by the budget constraint (5.46) and $b(t, Z(t))$ is the exercise boundary of the American put option given by

$$b(t, z) := \sup \left\{ y : P^a_{y,z}(t, T, k+g) = (k(t, z) + g(t) - y)^+ \right\}. \tag{5.49}$$

We recall some basic properties of American put options in a Black-Scholes market ([27], pp. 219-221)

$$\begin{aligned}
&P^a_{y,z}(t, T, k+g) = k(t, z) + g(t) - y, && \forall (t, y, z) \in \mathcal{W}^c \\
&\tfrac{\partial}{\partial y} P^a_{y,z}(t, T, k+g) = -1, && \forall (t, y, z) \in \mathcal{W}^c \\
&AP^a_{y,z}(t, T, k+g) = (r(t) + \mu(t))P^a_{y,z}(t, T, k+g), && \forall (t, y, z) \in \mathcal{W}\,,
\end{aligned}$$

where from (5.43), the generator operator A is given by (see Theorem 2.2.5)

$$\begin{aligned}
(A\phi)(y) &= \frac{\partial \phi}{\partial t} + \left(r(t) + \mu(t) - \frac{1+\mu(t)}{f(t)} \right) y \frac{\partial \phi}{\partial y} + \frac{1}{2(1-\delta)^2} \theta^2(t) y^2 \frac{\partial^2 \phi}{\partial y^2} \\
&+ \left[\phi(t, y\psi^{-\frac{1}{1-\delta}}, z) - \psi(t, y, z) - y \left(\psi^{-\frac{1}{1-\delta}} - 1 \right) \frac{\partial \phi}{\partial y} \right] \lambda(t)
\end{aligned}$$

and

$$\mathcal{W} := \{(t,y,z) : P^a_{y,z}(t,T,k+g) > (k(t,z)+g(t)-y)^+\}$$

defines the continuation region. $\mathcal{W}^c$ is the stopping region, that is the complementary of the continuation region $\mathcal{W}$. From the exercise boundary given in (5.49), we can write the continuation region by

$$\mathcal{W} = \{(t,y,z) : y > b(t,z)\}\,.$$

Define a function H by

$$H(t,y,z) := y + P^a_{y,z}(t,T,k+g) - g(t)\,,$$

then we have

$$\widehat{X}^{(\varrho)}(t) = H(t,\varrho(t,Z(t))Y^*(t),Z(t))\,.$$

From the properties of $P^a_{y,z}(t,T,k+g)$, we deduce that

$$\begin{aligned}
H(t,y,z) &= k(t,z), \quad \forall (t,y,z) \in \mathcal{W}^c,\\
\frac{\partial}{\partial y}H(t,y,z) &= 0, \quad \forall (t,y,z) \in \mathcal{W}^c \qquad (5.50)\\
AH(t,y,z) &= \frac{\partial}{\partial t}k(t,z) + h(t,z)\frac{\partial}{\partial z}k(t,z) \quad \forall (t,y,z) \in \mathcal{W}^c, \qquad (5.51)\\
AH(t,y,z) &= (r(t)+\mu(t))P^a_{y,z}(t,T,k+g) + \ell(t) - (r(t)+\mu(t))g(t)\\
&\quad + \left(r(t)+\mu(t) - \frac{1+\mu(t)}{f(t)}\right)y\\
&\quad + \left(P^a_{y\psi^{-\frac{1}{1-\delta}},z}(t,T,k+g) - P^a_{y,z}(t,T,k+g)\right)\lambda(t)\\
&= (r(t)+\mu(t))H(t,y,z) + \ell(t) - \frac{1+\mu(t)}{f(t)}y + \Big[H(t,y\psi^{-\frac{1}{1-\delta}},z)\\
&\quad -H(t,y,z) - y\left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]\lambda(t), \quad \forall (t,y,z) \in \mathcal{W}\,. \qquad (5.52)
\end{aligned}$$

Proposition 5.3.1. *The strategy $(\varrho c^*, \varrho\pi_1, \varrho\pi_2, \varrho p^*)$, where ϱ is defined by (5.48) and (5.46), is admissible.*

Proof. For ϱ constant and linearity of $Y^*(t), \forall t \in [0,T]$, we have that $\varrho Y^*(t)$ and $Y^*(t)$ have the same dynamics. Then, using Itô's formula, (5.51)-(5.52),

$(c^*(t), p^*(t))$ in Theorem 5.2.1 and the fact that ϱ increases only at the boundary, we obtain (here, $\frac{\partial}{\partial y}$ means differentiating with respect to the second variable)

$$
\begin{aligned}
& dH(t, \varrho(t, Z(t))Y^*(t), Z(t)) \\
= \; & [dH(t, \varrho Y^*(t), Z(t))] + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, Z(t))Y^*(t), Z(t))d\varrho(t, Z(t)) \\
= \; & AH(t, \varrho Y^*(t))dt - \frac{1}{1-\delta}\theta(t)\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), Z(t))dW^{\mathbb{Q}}(t) \\
& + \left[H(t, \varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t), Z(t)) - H(t, \varrho Y^*(t), Z(t))\right] d\widetilde{N}^{\mathbb{Q}}(t) \\
& + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, Z(t))Y^*(t), Z(t))d\varrho(t, Z(t)) \\
= \; & \Big\{(r(t) + \mu(t))H(t, \varrho Y^*(t), Z(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& + \Big[H(t, \varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t), Z(t)) - H(t, \varrho Y^*(t), Z(t)) \\
& - \varrho Y^*(t)\left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]\lambda(t)\Big\}\mathbf{1}_{(\varrho(t,Z(t))Y^*(t) > b(t,Z(t)))}dt \\
& \left[\frac{\partial}{\partial t}k(t, Z(t)) + h(t, Z(t))\frac{\partial}{\partial z}k(t, Z(t))\right]\mathbf{1}_{(\varrho(t,Z(t))Y^*(t) \leq b(t,Z(t)))}dt \\
& + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, Z(t))Y^*(t), Z(t))\mathbf{1}_{(\varrho(t,Z(t))Y^*(t) = b(t,Z(t)))}d\varrho(t, Z(t)) \\
& - \frac{1}{1-\delta}\theta(t)\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), Z(t))dW^{\mathbb{Q}}(t) \\
& + \left[H(t, \varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t), Z(t)) - H(t, \varrho Y^*(t), Z(t))\right] d\widetilde{N}^{\mathbb{Q}}(t)\,.
\end{aligned}
$$

From (5.50) we know that $\frac{\partial}{\partial y}H(t, \varrho(t, Z(t))Y^*(t), Z(t)) = 0$ on the set

$\{(t,\omega) : \varrho(t,Z(t))Y^*(t) = b(t,Z(t))\}$, then

$$\begin{aligned}
& dH(t,\varrho(t,Z(t))Y^*(t),Z(t)) \\
= & \Big\{(r(t)+\mu(t))H(t,\varrho Y^*(t),Z(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& +\Big[H(t,\varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t),Z(t)) - H(t,\varrho Y^*(t),Z(t)) \\
& -\varrho Y^*(t)\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)\Big]\lambda(t)\Big\}dt + \Big[\frac{\partial}{\partial t}k(t,Z(t)) \\
& +h(t,Z(t))\frac{\partial}{\partial z}k(t,Z(t)) - [(r(t)+\mu(t))k(t,Z(t)) + \ell(t) - \varrho(t,Z(t))c^*(t) \\
& -\varrho(t,Z(t))\mu(t)p^*(t)]\Big]\mathbf{1}_{(\varrho(t,Z(t))Y^*(t)\le b(t,Z(t)))}dt \\
& -\frac{1}{1-\delta}\theta(t)\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),Z(t))dW^{\mathbb{Q}}(t) \\
& +\left[H(t,\varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t),Z(t)) - H(t,\varrho Y^*(t),Z(t))\right]d\widetilde{N}^{\mathbb{Q}}(t).
\end{aligned}$$

Hence, since
$\{(t,\omega) : \varrho(t,Z(t))Y^*(t) \le b(t,Z(t))\} = \left\{(t,\omega) : \varrho(t,Z(t)) = \frac{b(t,Z(t))}{Y^*(t)}\right\}$ has a zero $dt \otimes d\mathbb{P}$-measure, we conclude that

$$\begin{aligned}
& dH(t,\varrho(t,Z(t))Y^*(t),Z(t)) \\
= & \Big\{(r(t)+\mu(t))H(t,\varrho Y^*(t),Z(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& +\Big[H(t,\varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t),Z(t)) - H(t,\varrho Y^*(t),Z(t)) \\
& -\varrho Y^*(t)\left(\psi^{-\frac{1}{1-\delta}}(t)-1\right)\Big]\lambda(t)\Big\}dt \\
& -\frac{1}{1-\delta}\theta(t)\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),Z(t))dW^{\mathbb{Q}}(t) \\
& +\left[H(t,\varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t),Z(t)) - H(t,\varrho Y^*(t),Z(t))\right]d\widetilde{N}^{\mathbb{Q}}(t),
\end{aligned}$$

i.e. by (5.17), the strategy $(\varrho c^*, \varrho\pi^*, \varrho p^*)$ is admissible. □

We then state the main result of this section, which we prove similarly as in [22].

Theorem 5.3.2. *Consider the strategy* $(\widehat{c}, \widehat{\pi}_1, \widehat{\pi}_2, \widehat{p})$, $\forall t \in [0,T]$ *given by*

$$\widehat{c} = \frac{\varrho(t, Z(t))Y^*(t)}{f(t)} = \varrho(t, Z(t))c^*(t), \tag{5.53}$$

$$\widehat{\pi}_i = \varrho(t, Z(t))\pi_i^*(t), \tag{5.54}$$

$$\widehat{p} = \frac{\varrho(t, Z(t))Y^*(t)}{f(t)} = \varrho(t, Z(t))p^*(t), \tag{5.55}$$

where the strategy (c^*, π_i^*, p^*), $i = 1, 2$ *is defined in Theorem 5.2.1. Combined with a position in an American put option written on the portfolio* $(\varrho(s)Y^*(s))$ *with strike price* $k(s, Z(s)) + g(s)$, $\forall s \in [t, T]$ *and maturity* T, *where* $\varrho(s), s \in [t, T]$ *is a function defined by (5.48) and (5.46).*

Then, the strategy is optimal for the American capital guarantee control problem given by (5.38)-(5.39).

Proof. Let (c, π_1, π_2, p) be any feasible strategy with corresponding wealth process $(X(t))_{t\in[0,T]}$ satisfying $X(0) = x_0$ and $X(t) \geq k(t, Z(t))$, $\forall t \in [0, T]$. Since u is concave by definition of a utility function (Definition 2.6.1), we get that

$$\begin{aligned}
& \int_0^T e^{-\int_0^t(\rho(s)+\mu(s))ds}[u(c(t)) + \mu(t)u(p(t))]dt + e^{-\int_0^T(\rho(s)+\mu(s))ds}u(X(T)) \\
& -\Big(\int_0^T e^{-\int_0^t(\rho(s)+\mu(s))ds}[u(\widehat{c}(t)) + \mu(t)u(\widehat{p}(t))]dt \\
& +e^{-\int_0^T(\rho(s)+\mu(s))ds}u(\widehat{X}^{(\varrho)}(T))\Big) \\
= & \int_0^T e^{-\int_0^t(\rho(s)+\mu(s))ds}[u(c(t)) - u(\widehat{c}(t)) + \mu(t)(u(p(t)) - u(\widehat{p}(t))]dt \\
& +e^{-\int_0^T(\rho(s)+\mu(s))ds}\Big(u(X(T)) - u(\widehat{X}^{(\varrho)}(T))\Big) \\
\leq & \int_0^T e^{-\int_0^t(\rho(s)+\mu(s))ds}[u'(\widehat{c}(t))(c(t) - \widehat{c}(t)) + \mu(t)u'(\widehat{p}(t))(p(t) - \widehat{p}(t))]dt \\
& +e^{-\int_0^T(\rho(s)+\mu(s))ds}u'(\widehat{X}^{(\varrho)}(T))\Big(X(T) - \widehat{X}^{(\varrho)}(T)\Big) \\
=: & (*)\,.
\end{aligned} \tag{5.56}$$

Since (c, π_1, π_2, p) was arbitrary chosen, we end the proof by showing that

$\mathbb{E}[(*)] \leq 0$. By the CRRA property $u'(xy) = u'(x)u'(y)$, we have

$$u'(\widehat{c}(t))(c(t) - \widehat{c}(t)) = u'(\lambda(t, Z(t)))u'(p^*(t))(c(t) - \widehat{c}(t)), \quad (5.57)$$
$$u'(\widehat{p}(t))(p(t) - \widehat{p}(t)) = u'(\lambda(t, Z(t)))u'(p^*(t))(p(t) - \widehat{p}(t)). \quad (5.58)$$

Observing that $Y^*(T) = X^*(T)$, the the terminal value becomes

$$\begin{aligned} \widehat{X}^{(\varrho)}(T) &= \varrho(T, Z(T))X^*(T) + [k(T, Z(T)) - \varrho(T, Z(T))X^*(T)]^+ \\ &= \max[\varrho(T, Z(T))X^*(T), k(T, Z(T))]. \end{aligned} \quad (5.59)$$

By use of (5.59) and using the fact that u' is a decreasing function (Definition 2.6.1.), we get that

$$\begin{aligned} & u'(\widehat{X}^{(\varrho)}(T)) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \\ = \; & \min[u'(\varrho(T, Z(T)))u'(X^*(T)), u'(k(T, Z(T)))] \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \\ = \; & u'(\varrho(T, Z(T)))u'(X^*(T)) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \\ & -[u'(\varrho(T, Z(T)))u'(X^*(T)) - u'(k(T, Z(T)))]^+(X(T) - k(T, Z(T))), \end{aligned}$$

where the last equality is established by using that $\widehat{X}^{(\varrho)}(T) = k(T, Z(T))$ on the set $\{(T, \omega) : u'(\varrho(T, Z(T))X^*(T)) \geq u'(k(T, Z(T)))\}$. Since by assumption $X(t) \geq k(t, Z(t))$, $\forall t \in [0, T]$, we conclude that

$$u'(\widehat{X}^{(\varrho)}(T)) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \leq u'(\varrho(T, Z(T)))u'(X^*(T)) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right). \quad (5.60)$$

Inserting (5.57), (5.58) and (5.60) and then (5.23)-(5.25) into (5.56), we get

$$\begin{aligned} \mathbb{E}[(*)] \; &\leq \; \mathbb{E}\Big[\int_0^T e^{-\int_0^t (\rho(s)+\mu(s))ds} \Big[u'(\varrho(t, Z(t)))u'(c^*(t))(c(t) - \widehat{c}(t)) \\ &\quad +\mu(t)u'(\varrho(t, Z(t)))u'(p^*(t))(p(t) - \widehat{p}(t))\Big] dt \\ &\quad +e^{-\int_0^T (\rho(s)+\mu(s))ds} u'(\varrho(T, Z(T)))u'(X^*(T)) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \Big] \\ &= \; \zeta^* \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t, Z(t)))(c(t) - \widehat{c}(t) + \mu(t)(p(t) \\ &\quad -\widehat{p}(t)))dt + e^{-\int_0^T (r(s)+\mu(s))ds} u'(\varrho(T, Z(T))) \left(X(T) - \widehat{X}^{(\varrho)}(T)\right) \Big]. \end{aligned}$$

Since $u'(\varrho(t,Z(t)))$ is a decreasing function[3], we can use the integration by parts formula to get

$$\begin{aligned}
&\mathbb{E}[(*)] \qquad (5.61)\\
= \;& \zeta^*\Bigg(\mathbb{E}^{\mathbb{Q}}\Bigg[\underbrace{\int_0^T e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t,Z(t)))(c(t)-\widehat{c}(t)+\mu(t)(p(t)-\widehat{p}(t)))dt}_{(\bigstar)}\\
&\underbrace{+\int_0^T u'(\varrho(t,Z(t)))d\left(e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t,Z(t)))\left(X(t)-\widehat{X}^{(\varrho)}(t)\right)\right)}_{(\bigstar\bigstar)}\Bigg]\\
&+\mathbb{E}^{\mathbb{Q}}\Bigg[\underbrace{\int_0^T e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t,Z(t)))\left(X(t)-\widehat{X}^{(\varrho)}(t)\right)du'(\varrho(t,Z(t)))}_{(\bigstar\bigstar\bigstar)}\Bigg]\Bigg).
\end{aligned}$$

The third term in (5.61) can be rewritten as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[(\bigstar\bigstar\bigstar)]\\
= \;& \mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t,Z(t)))\,(X(t)-k(t,Z(t)))\,du'(\varrho(t,Z(t)))\Bigg]\\
&+\mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T e^{-\int_0^t (r(s)+\mu(s))ds} u'(\varrho(t,Z(t)))\Big(k(t,Z(t))\\
&\qquad -\widehat{X}^{(\varrho)}(t)\Big)du'(\varrho(t,Z(t)))\Bigg].
\end{aligned}$$

The first term is non-positive since per definition $X(t) \geq k(t,Z(t))$, $\forall t \in [0,T]$ and $du'(\varrho(t,Z(t))) \leq 0$, $\forall t \in [0,T]$ (u' is decreasing and ϱ is increasing). The second term equals zero since $du'(\varrho(t,Z(t))) \neq 0$ only on the set $\{(t,\omega) : \widehat{X}^{(\varrho)}(t) = k(t,Z(t))\}$. We conclude that $\mathbb{E}^{\mathbb{Q}}[(\bigstar\bigstar\bigstar)] \leq 0$. The two first terms of (5.61) can be written as

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(\bigstar)+(\bigstar\bigstar)] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T u'(\varrho(t,Z(t)))dB_1(t)\right)\\
&\quad -\mathbb{E}^{\mathbb{Q}}\left[\int_0^T u'(\varrho(t,Z(t)))dB_2(t)\right),
\end{aligned}$$

[3]This ensures that the stochastic integral in (5.61) is well defined

where

$$B_1(t) := \int_0^t e^{-\int_0^t (r(s)+\mu(s))ds}(c(s)+\mu(s)p(s)-\ell(s))ds + e^{-\int_0^t (r(s)+\mu(s))ds}X(t)\,,$$
$$B_2(t) := \int_0^t e^{-\int_0^t (r(s)+\mu(s))ds}(\widehat{c}(s)+\mu(s)\widehat{p}(s)-\ell(s))ds + e^{-\int_0^t (r(s)+\mu(s))ds}\widehat{X}^{(\varrho)}(t)\,.$$

Since both strategies are admissible, we note that by (5.16), B_1 and B_2 are martingales under the equivalent measure $\mathbb{Q}$. Since $u'(\varrho(t,Z(t))) \leq u'(\varrho(0,z_0))$, $\forall t \in [0,T]$, we get that

$$\mathbb{E}^{\mathbb{Q}}[(\bigstar)+(\bigstar\bigstar)] = 0\,.$$

Finally, we conclude that

$$\mathbb{E}[(*)] = \mathbb{E}^{\mathbb{Q}}[(\bigstar)+(\bigstar\bigstar)] + \mathbb{E}^{\mathbb{Q}}[(\bigstar\bigstar\bigstar)] \leq 0\,.$$

□

Chapter summary

In this chapter, we have solved the optimization problem with American capital guarantee. We obtained the constrained optimal strategy from the unconstrained optimal solution using the so-called option based portfolio insurance approach. The unconstrained control problem was solved via martingale approach.

Chapter 6

Conclusion

6.1 Summary

In this dissertation, we solved an optimization problem under jump-diffusion framework in two settings, namely a problem with random parameters and a problem with American capital constraints.

Chapter 2 was devoted to the introduction of the relevant concepts used in this dissertation. We point out the following: random and compensated random measures, which constitute the key concepts to define a Lévy SDE, the Itô's formula, which is the important tool in solving the SDEs in our optimization problems, the Radon-Nikodym derivative used when solving an equation (SDE or BSDE) using a martingale approach, the conditions to obtain the solution of a BSDE, the verification theorem for optimization problem using a dynamic programming approach and finally the power utility functions and its properties.

In Chapter 3, we derived the wealth process in a market with investment, consumption, income process and life insurance respectively.

In Chapter 4, we obtained an optimal investment, consumption and life insurance in a problem with random parameters which include jumps. These random parameters need not to be bounded. By including jumps in parameters, we may cover all the possibilities that may occur in the real market. For instance, sudden changes in environmental conditions implies the inclusion of jumps in a mortality rate. An application a dynamic programming approach, combining the HJB equation with BSDE made it possible to characterize the optimal solution and the value function in terms of a unique solution of a BSDE with jumps. We have solved our problem using one risk-free and one risky asset in our modeling framework. We concluded this chapter by providing two special examples. By choosing the cases of random mortality and appreciation rates in our examples, we have covered all the other possibilities of parameters randomness. In fact, if the premium insurance ratio is random, the explicit solution of our BSDE can be derived similarly as in Example 4.4.1. On the other hand, if one of the parameters (interest rate, discount rate or the dispersion rates) is random, the explicit solution of the BSDE can be derived as in Example 4.4.2. The results in this chapter have been published in an accredited journal ([13]). Similar work can be done for a market with n risky assets.

Chapter 5 solved the optimal investment, consumption and life insurance problem with capital constraints, specifically the American capital guarantee. We have solved the unconstrained optimization problem applying the martingale approach. We did so, since the solution to the restricted capital guarantee problem is based on terms derived from the martingale method. Finally, we proved the admissibility of the solution in the restricted problem and its solution was obtained from the solutions of the unconstrained problem using the so-called OBPI approach. Our contribution in the existing literature is that of adding capital constraints in a model described by jump-diffusion processes. The results obtained in this chapter have been submitted for publication as well. They can also be extended to a market with $n + m$ risky assets in which the driving processes are an n-dimensional Brownian motion and m-dimensional Poisson processes.

6.2 Future research

In all our models, we have considered a power utility function. We would like to solve the similar problems in a general utility case. We also would like to take model risk into account and therefore study 'robust' optimal control for similar problems in both complete and incomplete markets. Furthermore, the stability of the optimization problem considered in Chapter 4 it is also an interesting question for a future research.

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