Distribution-free Phase II CUSUM Control Chart for Joint Monitoring of Location and Scale

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Abstract: Mukherjee and Chakraborti[1] proposed a single distribution-free (nonparametric) Shewhart-type chart based on the Lepage[2] statistic for simultaneously monitoring both the location and the scale parameters of a continuous distribution when both of these parameters are unknown. In the present work, we consider a single distribution-free CUSUM chart, based on the Lepage[2] statistic, referred to as the CUSUM-Lepage (denoted by CL) chart. The proposed chart is distribution-free (nonparametric) and therefore, the in control (denoted IC) properties of the chart remain invariant and known for all continuous distributions. Control limits are tabulated for implementation of the proposed chart in practice. The IC and out of control (denoted OOC) performance properties of the chart are investigated through simulation studies in terms of the average, the standard deviation, the median and some percentiles of the run length distribution. Detailed comparison with a competing Shewhart-type chart is presented. Several existing CUSUM charts are also considered in the performance comparison. The proposed CL chart is found to perform very well in the location-scale models. We also examine the effect of the choice of the reference value (k) of CUSUM chart on the performance of the CL chart. The proposed chart is illustrated with a real data set. Summary and conclusions are presented.

Keywords: Ansari-Bradley Statistic; Average Run Length; Phase I and II; CUSUM Lepage Chart; Nonparametric; Monte-Carlo simulation; Statistical process control; Shewhart Lepage Chart; Wilcoxon Rank-sum Statistic.

1. INTRODUCTION

Robustness of many of the available control charts for monitoring a process largely depends on the assumption of normality which is often difficult to justify in practice. In recent times, researchers have advocated using distribution-free (nonparametric) control charts, in particular when the process distribution is unknown, or known to be markedly different from the normal distribution. Nevertheless, most of the distribution-free control charts that are available in the literature are designed for monitoring either the process location or the scale parameter separately. Using separate charts for different process parameters can cause practical problems with regard to implementation and interpretation. Thus a single chart (as opposed to two separate charts) for joint monitoring of location and scale parameters has been recommended as it may be simpler and may have some performance advantages (see for example, McCraken et al.[3]). To understand the importance and impact of joint monitoring, readers may see the review by Cheng and Thaga[4] for coverage of literature until 2005 and the paper by McCracken and Chakraborti[5] for more recent advances.

The CUSUM charts were first introduced by Page[6]. Over the years, CUSUM charts have proven (see, for example, Hawkins[7], Woodall[8], Lucas[9], Chang and Gan[10]) to be useful in SPC and many other areas for monitoring processes over time. Various modifications and adaptations of CUSUM charts have also been proposed in the literature (see, for example, Reynolds et al.[11], Gan[12], Goel[13] and Mukherjeeet et al.[14]). The CUSUM chart is known to be superior to the Shewhart control chart in the sense that the CUSUM control charts tend to have smaller Average Run Lengths (ARL's) particularly for small changes in the parameters. While a Shewhart chart is better in detecting an immediate abrupt (transient) change, the cumulative sum (CUSUM) chart is more effective in detecting more sustained changes. The reader is referred to Hawkins and Olwell[15] and Gan[16] for a detailed discussion on CUSUM control charting literature.

While much work has been done and continues to be done in the parametric setting, it is now well recognized that the distribution-free (or nonparametric) charts are useful and expected to be superior when the model assumptions such as normality is difficult to validate. There has been a significant amount of work done in this area, particularly over the last decade. While Park and Reynolds[17] developed nonparametric control charts for monitoring the location parameter of a continuous process based on the linear placement statistic, McDonald[18] considered a CUSUM procedure for individual observations based on the sequential ranks. Bakir and Reynolds[19] and Amin et al.[20] proposed nonparametric CUSUM charts based on the signed-rank and the sign statistic, respectively. Run-length distribution of the CUSUM chart with estimated parameters was discussed in detail by Jones et al.[21]. Li et al.[22] proposed nonparametric CUSUM and EWMA charts based on the Wilcoxon Rank-sum statistic to detect shifts in the location parameter. Recently, Yang and Cheng[23] and Mukherjee et al.[14] developed nonparametric CUSUM charts to detect possible small shifts in the process mean. Ross et al.[24] discussed the problem of nonparametric monitoring of data streams for changes in location and scale and Ross and Adams[25] considered two nonparametric control charts for detecting arbitrary distributional changes in the process. For an overview of nonparametric control charts, see Chakraborti et al.[26-28]. The area of nonparametric (distribution-free) process monitoring and control continues to grow at a rapid pace. Among the more recent papers, see for example, Qiu and Li[29-30] and Chatterjee and Qiu[31]. There is also a good discussion on the subject in the recent book by Qiu[32]. Note that in this paper our focus is on the univariate case; there are several papers for multivariate distribution-free process monitoring, but we do not concern ourselves with such problems here.

However, none of these exisiting charts are specifically designed for monitoring the unknown location and the scale parameters in Phase II, using a reference sample from Phase I. For this important problem, Mukherjee and Chakraborti[1] considered a nonparametric Shewhart-Lepage (SL) chart based on the Lepage[2] statistic. For the same problem, Chowdhury et al.[33] considered a nonparametric Shewhart-Cucconi (SC) chart based on the Cucconi[34] statistic. Encouraged by these findings, in this paper, we take the work a step further in a new direction and consider a nonparametric CUSUM chart based on the Lepage[2] statistic. The proposed CL chart is expected to be more effective in detecting smaller, more sustained types of shifts.

The rest of the paper is organized as follows. Section 2 provides a brief background of nonparametric control charting procedures for joint monitoring of location and scale parameters on the basis of the Lepage[2] statistic, along with the statistical framework and preliminaries. The proposed CL control chart is introduced in Section 3. Section 4 is devoted to a brief discussion on the run length distribution, determination of the upper control limit (UCL) and examining the IC performance of the chart. The OOC performance of the CL chart, along with a detailed comparison with the SL chart, and some other nonparametric CUSUM charts for location shift and for general alternatives is presented in Section 5. The performance is based on various run length distribution characteristics obtained via Monte-Carlo simulation. In Section 6, we study the effect of the choice of a key parameter of a CUSUM chart, the so-called reference value, on the performance of the CL chart. The charting procedure is illustrated in Section 7 with a data set from Montgomery[35]. We conclude with a summary in Section 8.

2. STATISTICAL FRAMEWORK AND PRELIMINARIES

Simultaneous monitoring of location and scale parameters is useful in many applications, see, for example, McCracken et al.[3]. A distribution-free (nonparametric) control chart is useful when the assumption about the process distribution cannot be made or justified. While most of the distribution-free control charts have been devoted to monitoring the location parameter only, Mukherjee and Chakraborti[1] and Chowdhury et al.[33] proposed nonparametric control charts to monitor both the location and scale parameters. The Lepage[2] test is a combination of the Wilcoxon Rank-sum test (for location) and the Ansari-Bradley test (for scale), and has been used to test for the equality of location and scale parameters in the nonparametric literature. Mukherjee and Chakraborti[1] adapted the Lepage statistic in a Shewhart type control chart to monitor the location and the scale parameters. In this paper the Lepage statistic is used in a CUSUM chart and the resulting chart is called the CUSUM-Lepage (CL) chart.

Let $U_1, U_2, \ldots, U_m$ be a random sample (say the first sample) from an unknown univariate population with a continuous distribution function (cdf) $F(x)$. Further suppose that $V_1, V_2, \ldots, V_n$ be a second random sample mutually independent of the first, from a cdf $G(y) = F\left(\frac{x-\theta}{\delta}\right)$; $\theta \in \Re$; $\delta > 0$. The constants θ and δ represent the unknown location and scale parameters, respectively. Mukherjee and Chakraborti[1] consider the standardized WRS and the AB statistics, say, $S_1 = \frac{T_1-\mu_1}{\sigma_1}$ and $S_2 = \frac{T_2-\mu_2}{\sigma_2}$, repectively. Here, the Wilcoxon Rank-sum (WRS) statistic (T_1) is used to test the hypothesis $\theta = 0$ and is described as the sum of ranks of the second sample in the combined sample of size $N\ (= m + n)$. Further, the Ansari-Bradley (AB) test (T_2), a popular choice for testing the hypothesis $\delta = 1$ is the sum of the absolute deviation of ranks of the second sample in the combined sample from the average rank, that is $\frac{(N+1)}{2}$. Thereafter, define the Lepage test statistic as: $S_L^2 = S_1^2 + S_2^2$. Detailed expressions for the mean and the variance of T_1 and T_2 are given in Mukherjee and Chakraborti[1] and hence is omitted here.

It is easy to see that $E(S_1|IC) = E(S_2|IC) = 0$ and $E(S_1^2|IC) = E(S_2^2|IC) = 1$ and therefore, $E(S_L^2|IC) = 2 = \mu_L$, say. Note that, S_L^2 is nonnegative by definition. Also, it is easy to see that whenever θ deviates from 0, irrespective of the direction of the shift, the absolute value (ignoring the sign) of S_1 is expected to be larger on an average. Similarly, irrespective of the direction of the shift of δ from 1, S_2 is expected to be larger on an average. As a consequence, irrespective of the type and the direction of the shift, either in location or in scale or in both, S_L^2 is expected to take on larger values compared to μ_L when the process is OOC. Thus, in order to monitor small change(s) in the location and/or the scale parameter, we propose an upper one-sided CUSUM chart based on the Lepage[2] statistic.

3. CONSTRUCTION OF UPPER ONE-SIDED CUSUM-LEPAGE (CL) CHART

The upper one-sided CL chart may be constructed as follows:

Step1. Collect a reference sample $\boldsymbol{X_m} = (X_1, X_2, \dots, X_m)$ of size m from an IC process. Establishing a reference sample is itself a challenging problem and there are several Phase I control charts available in the literature for this purpose. In this paper, we assume that an appropriate reference sample is available a-priori.

Step 2. Collect $\boldsymbol{Y_{j,n}} = (Y_{j1}, Y_{j2}, \dots, Y_{jn})$, the *j*-th Phase II (test) sample of size n, $j = 1, 2, \dots$

Step 3. Identify the *U*'s with the *X*'s and the *V*'s with the *Y* 's respectively. Calculate the WRS statistic T_{1j} and the AB statistic T_{2j} using the reference sample and the *j*-th test sample and obtain the standardized WRS and AB statistics S_{1j} and S_{2j}, respectively, as described in Section 2. Finally calculate the Lepage statistic S_{Lj}^2 for the *j*-th test sample.

Step 4. Recall that $E(S_L^2|IC) = 2$ and therefore for the *j*-th test sample, $j = 1, 2, \dots$, the CL plotting statistic is given by: $C_j = max[0\ ,\ C_{j-1} + (S_{Lj}^2 - 2) - k]$; with the starting value $C_0 = 0$. Here, $k(\geq 0)$ is called a reference value.

Step 5. Plot C_j against an upper control limit (UCL) *H*. The lower control limit (LCL) is 0 by definition for the proposed upper one sided CUSUM chart.

Step 6. If C_j exceeds H, the process is declared OOC at the j-th test sample. If not, the process is thought to be IC and monitoring continues to the next test sample.

Step 7. Follow-up: Recently Chowdhury et al.[33] and McCracken et al.[3] introduced the idea of p-value based follow up when a process is declared OOC. We follow the same idea here and when the process is declared OOC, we compute the p-values for the Wilcoxon test for location and the Ansari-Bradley test for scale respectively, based on two samples; one with the m PhaseI observations, and the other with the n observations from the j-th test sample. Denote the p-values of the corresponding tests as p_1 and p_2, respectively. As in Chowdhury et al.[33] and McCracken et al.[3] we argue that if p_1 is very low but not p_2, a shift in only location is indicated. If p_1 is relatively high but p_2 is low; only a shift in scale is suspected. If both p-values are very low; a shift in both location and scale is declared.

4. RUN LENGTH DISTRIBUTION

Brook and Evans[36] and Woodall[8] among others proposed approximating the run length distribution of various CUSUM procedures when parameters are known by using the available results for a Markov process. The run length distribution of a Phase II CUSUM process may be approximated by a Markov Process given the reference sample $\boldsymbol{X_m}$, and the conditional average run length may be obtained using the properties of the Markov process. Since the conditional average run length is a random variable, one can find and use the unconditional average run length by integrating (averaging) over all possible conditional run lengths.

In the present context, however, the exact conditional distribution of the Lepage statistic is itself complicated with no clear explicit form. As a consequence, the exact unconditional run length distribution of the proposed CUSUM Lepage procedure appears intractable. However, interested readers may explore the possibility of deriving a suitable approximation for computing approximate conditional ARL under the IC set up via the Markov chain approach.

Given these complexities in the derivations of the conditional and the unconditional average run lengths, we employ Monte-Carlo simulations to evaluate the necessary quantities. Details of the simulations and the results are discussed in the subsequent sections and subsections.

4.1. Determination of *H*

Numerical computations in R.2.14.1 software, based on Monte-Carlo simulations, are used to determine H on the basis of 50,000 replicates. Because of the distribution-free nature of the CL chart, we generate m observations from a standard normal distribution for the PhaseI sample and n observations from the same distribution for each test sample. The results, which are displayed in Table 1, show a pretty stable and meaningful estimates of the IC average run length (ARL_0) and other percentiles of the IC run length distribution. We have chosen m = 30, 50, 100 and 150 for the reference sample size and n = 5 and 11 for the test sample size as in Mukherjee and Chakraborti[1]. The value of k is chosen to be 0, 3 or 6, respectively. Note that since $SD(S_i^2|IC) = 2,$ the values 3 and 6 represent 1.5 and 3 times the standard deviation of the plotting statistic. This range of values appear to be reasonable to cover different amount of shifts in either of the location or the scale parameters or the both. It was observed in the simulation studies that no substaintial changes in the ARL figures result from minor (less than 0.5 times the standard deviation of plotting statistic) changes in the k values; moreover, the needed H and ARL values for any k in the interval (0,6) can be obtained via interpolation using the three values of k used in this article. For any given triplet (m,n,k), a search is conducted to obtain the appropriate H value that ensures the ARL_0 is close to a nominal (target) value. The fourth to the sixth columns of Table 1 give the required H values for a target ARL_0 = 250, 370 and 500 respectively. Thus, for example, when 30 reference observations and test samples of size 5 are available with k=3 and an ARL_0 of 500 is desired, the upper control limit H for the CL chart is given by 4.617. To justify the nonparametric nature of the proposed chart, it can be easily verified that for a given combination of (m,n,k;ARL_0) the same H values as in Table 1 are valid for any other non-normal distribution. It can be seen later under Table 3-5 that for $\theta = 0$ and $\delta = 1$, run length percentiles as well as mean and standard deviations are almost same, except for minor sampling fluctuations for all three types of distributions, because of the distribution-free nature of the plotting statistic under IC set-up.

We see from Table 1 that that for any fixed combination of (m,n,k) values, the higher the nominal ARL_0 values, the higher the values of H. Further for fixed n and k, H increases with the increase in the reference sample size m but the H decreases with an increase in the test sample size n for fixed m and k in almost all the cases except for a very few sampling fluctuations. Finally, as m and n both increase, H values tend to stabilize as a function of the ratio $\lambda = \frac{n}{m+n}$.

4.2. IC performance of the Chart

Table 2 shows that the IC run length distribution is highly right skewed and consequently is worthwhile to study various summary measures viz. the mean, the SD and several percentiles including the first and the third quartiles. As mentioned earlier in 4.1, in the IC case, we simulate both the reference and the test samples from the standard normal distribution for m = 30, 50, 100, 150 and n = 5, 11 and k=0, 3, and 6. For a given triplet (m, n, k), we obtain H from Table 1, for a nominal (target) ARL_0 of 500 and simulate various characteristics of the IC run length distribution.

It is observed that the target ARL_0 value 500 is much higher than the medians for all (*m,n,k*) combinations. When *m* = 100 or 150, the median is more or less half of the target ARL_0 value of 500. As the reference sample size *m* increases from 30 to 150, for a fixed *n* and *k*, all the percentiles including the median increase except the 95th percentile and the SD decreases. The 95th percentile is seen to be more or less stable around 3.6 to 4.4 times the target ARL_0 = 500 except for the cases when *m*=30 and 50 with *k*=0. It is also observed that for fixed *m* and *k*, as test sample size increases, all percentiles except the 75th and the 95th percentile decrease and the SD increases. It is also worth mentioning that in the majority of the combinations of (*m,n,k*), the 75th percentiles of the run length distributions are closer to the nominal IC ARL_0 value of 500. This indicates that the IC run length distribution of the proposed chart, like that of many other control charts, is heavily skewed with a long right tail.

5. PERFORMANCE COMPARISONS

We consider two popular symmetric distributions and a well-known asymmetric distribution under the general location-scale family in order to facilitate the OOC performance comparisons. First (case I) we consider, the thin tailed symmetric normal distribution (N(θ, δ)) with pdf $f(u) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2\delta^2}(u-\theta)^2}$, $u \in (-\infty, \infty)$, second (case II), the heavy tailed symmetric Cauchy distribution (Cauchy(θ;δ)) with pdf $f(u) = \frac{\delta}{\pi(\delta^2+(u-\theta)^2)}$, $u \in (-\infty, \infty)$and finally (case III), we choose asymmetric lognormal distribution with pdf $f(u) = \frac{1}{\delta u\sqrt{2\pi}} e^{-\frac{1}{2\delta^2}(lnu-\theta)^2}$, $u \in (0, \infty)$. We examine the performance characteristics of the run length distribution when the IC sample in each case is taken from the corresponding standard distribution with θ = 0 and δ = 1. Thus in case I, the IC samples are taken from a N(0,1) distribution, with the OOC sample coming from a N(θ,δ) distribution. To examine the effects of shifts in the mean and the variance, 48 combinations of θ and δ values are considered. We choose θ = 0, 0.25, 0.5, 0.75, 1, 1.5, 2 and 3 along with δ = 0.5 (decreasing scale); 1 (scale invariant) and 1.25, 1.5, 1.75 and 2 (increasing scale) respectively. For symmetric distribution, for any given δ, OOC performance of the charts for shifts $+\tau$ and $-\tau$ for any $\tau > 0$ is same. Thus downward shifts in location are not separately considered for symmetric distributions. We not only study the performance of the CL charts for these three distributions, but also compare them with that of the SL chart of Mukherjee and Chakraborti[1] for the same combinations of shifts. For brevity, only the results for m=50, 100 and n = 5 are presented in Table 3. In case II, we examine the chart performance characteristics for the heavy tailed and symmetric Cauchy distribution for both the CL and SL charts, using the same combinations of the reference and test sample sizes and location and scale parameters (θ and δ), with the IC sample coming from a Cauchy(0,1) distribution. These results are shown in Table 4. Finally, results for the asymmetric (right skewed) lognormal distribution are presented in Table 5 where we choose $\theta = 0, \pm 0.25, \pm 0.5, \pm 0.75, \pm 1.0, \pm 1.5, \pm 2$ and ± 3 with six values of δ as before. Note that in Tables 3, 4 and 5 (and later in Tables 7, 8 and 9) the first row of each of the cells shows the ARL and (SDRL) values, whereas the second row shows the 5th, 25th, 50th, 75th and the 95th percentiles (in this order). All the shaded cells correspond to the best performing chart in that scenario.

In general, the simulation results reveal that the OOC run length distributions are also skewed to the right and this can be observed from Tables 3 and 4 for both charts. Moreover, except for minor sampling fluctuations, for fixed m, n, k and a given ARL_0, the OOC ARL values as well as the percentiles all decrease sharply with the increasing shift in the location and also with the increasing shift in the scale. This (expected) phenomenon is seen for both the CL and SL charts and this indicates that both distribution-free charts are reasonably effective in detecting shifts in the location and/or in the scale. However, the effectiveness of the chart (speed of detection) varies depending on the type of shift and the type of chart being considered. Both CL and SL charts detect a shift in the scale faster than that in the location.

For example, from Table 3, we see that for a 25% increase in the location parameter (θ) when the scale parameter (δ) is in IC, there is about a 41% reduction in the ARL for the SL chart and about 38-51% reduction in the CL chart with $m = 50$ and varying choices of k. Moreover, in the same situation, for m=100, we find nearly a 49% reduction in the ARL of the SL chart and about 50-63% reduction in that of the CL chart. However, for a 25% increase in the scale parameter when the location parameter is in IC, there is about 78.7% and 80% reduction in the ARL for the SL chart with both $m = 50$ and 100 respectively. On the other hand, we observe a reduction of 79-87% in the ARL of the CL chart for m=50 and nearly 81-89% reduction for m=100 and different choices of k. Finally, when both the location and scale parameters increase by 25%, the ARL for the SL chart decreases by nearly 85% and the same for the CL chart reduces by nearly 85-91% for $m = 50$. We also see that when the scale parameter decreases, the proposed chart can be effectively used with a suitable choice of k and this will be discussed further in Section 6.

The pattern is quite similar for the SDRL for both the charts. For example, we see that for a 25% increase in the location when the scale is IC, the SDRL decreases for an increase in the shift in both parameters but decreases at a faster rate for a shift in the scale parameter. For example, when $m = 100$, for a 25% increase in the location only, the SDRL decreases by 44.5% for the SL chart and around 38-51% for the CL chart, while for a 25% increase in the scale parameter only, the SDRL decreases by 83.2% for the SL chart and 82-93% for the CL chart.

From Table 3, it is interesting to note that the CL chart with a suitable value of *k*, performs better than the SL chart not only in detecting small shifts in the parameters, but for larger shifts as well.

Next, it is useful to examine the OOC chart performance for underlying distributions that have tails heavier than the normal since heavier-tailed symmetric distributions under the location-scale family, such as the Cauchy distribution arise in applications where extreme values can occur with higher probability. Keeping this in mind we repeat the simulation study with data from the Cauchy distribution. The performance characteristics of the run length distribution were evaluated when the IC sample is taken from a Cauchy(0,1) distribution, but the test samples are from a Cauchy(θ,δ) distribution. To study the impact of a shift in the location and scale, as in the normal case, we study the same 40 combinations of θ and δ values. From Table 4, it is seen that for the Cauchy distribution, the general patterns in the OOC ARL values remain the same as in the case of the normal distribution, but the magnitudes of the ARL values are much higher for a similar shift in the location or in the scale parameter, indicating a moderately slower detection of shifts under the heavier-tailed distribution. For example, when m = 100, k=0 and the location and scale both increase by 25%, the ARL is 161.7 compared to 39.3 in the normal case for the CL chart. Moreover, the percentiles as well as the SDRL values all increase under the Cauchy distribution.

It is interesting to note that the OOC performance of the asymmetric lognormal distribution behaves more or less similar (in fact in many cases better) to that of normal distribution. For example, when m = 100 and k=0, for i) 25% increase in location, assuming scale is in control, there is a 66% reduction in ARL for lognormal distrubution as compared to 63% for normal, ii) 25% increase in scale with no shift in location, lognormal distribution shows 88.9% reduction in ARL as compared to 88.6% for normal distribution, iii) 25% increase in both location and scale, ARL for the normal distribution is reduced by 91.8% and that of lognormal distribution is reduced by 92.2%.

In summary, Tables 3, 4 and 5 show that for any kind of shift in the process location and/or scale (variability), the proposed CL chart outperforms the SL chart. However, the variation in the performance of the CL chart can be further explained by the reference value, *k* which is discussed in the next section.

5.1. Comparison with other CUSUM Charts

The purpose of the paper is to study a distribution-free chart for detecting shifts in both location and scale parameters of a process in Phase-II when both these process parameters are unknown and are estimated from a Phase I reference sample. It should be clearly noted that there is no existing meaningful distribution-free CUSUM chart for this purpose. The key issue here is to ensure that the chart is distribution-free, that is, the IC performance of the chart remains the same for all univariate continuous distributions. The current work is different from other studies, say under a change point set up as discussed in Ross and Adams[25] among others. Nevertheless, we compare the proposed chart with some existing charts based on two well-known distribution-free statistics, namely, the Cramér–von Mises statistic (CvM) and the Kolmogorov-Smironov (KS) statistic, which may be used to detect a general shift in the process distribution as considered by Ross and Adams[25]. We further consider two popular CUSUM charts proposed to detect a location shift, one based on the Wilcoxon Rank-sum statistic proposed by Li et al[22] and also considered by Qiu and Zhang[37] and the other based on the exceedance statistic considered by Mukherjee et al.[14]. Additionally, we consider a conventional (parametric) CUSUM X-bar chart with modifications for standards unknown. We summarize the details of five charts considered for comparisons in Table 6.

For comparison purposes, we consider three distributions from the location-scale family, namely, the normal(θ, δ) as a representative of a symmetric thin tailed distribution, the Laplace(θ, δ) distribution as a heavy tailed distributions and lognormal (θ, δ) distribution as an asymmetric right skewed distribution. For sake of brevity, we consider $\theta = 0, 0.25, 0.5, 1$ and $\delta = 1, 1.5, 2$. As we mostly consider small shifts, we restrict ourselves to using only $k = 0$. Findings for the normal(θ, δ) distribution are summarized in Table 7, results for lognormal (θ, δ) are provided in Table 8 and that for Laplace(θ, δ) are shown in Table 9.

It is observed from Tables 7, 8 and 9 that the CL chart outperforms the competing charts, when, a shift in scale as well as in both the location and scale occur, irrespective of the magnitude of the changes in these process parameters and the process distribution. As far as a large shift in location is concerned, it is once again the CL chart which performs the best except possibly for some sampling fluctuation in the case of the Laplace distribution, where the CUSUM CvM chart performs the best. For $\theta = 0.25$, CUSUM KS charts perform best for both normal and lognormal distribution while for $\theta = 0.5$, CUSUM CvM chart performs the best. For the Laplace distribution, it is the CUSUM exceedance chart which outperforms the other charts for a small change in the location and is the CUSUM CvM chart which performs the best when there is a moderate shift in location.

Superiority of the Lepage chart (statistic) over the Rank-sum chart (statistic) may be explained heuristically from the fact that one of the components of the Lepage Statistic is the Rank-sum Statistic. Natually, any change that the Rank-sum statistic detects, is also highly likely to be detected by Lepage statistic. Moreover, the Lepage statistic was introduced in the 1970s much after the development of the KS or the CvM statistics and it became very popular for a class of location-scale family because of its better efficiency (power performance) than the KS or the CVM statistics. As a consequence, it is expected to be far superior than any other charts for location-scale models when there is a shift in scale with/without change in location.

6. EFFECT OF *k* ON THE PERFORMANCE OF THE CHART

It is well-known that the reference value, k, of a CUSUM chart chart has a significant impact on both the IC and OOC performance of the chart. From Lepage[2], it is easy to see that Lepage statistic approximately follows a chi-square distribution with 2 d.f. Therefore, the approximate variance of the Lepage statistic is 4 and the standard deviation (SD) is 2. Thus, for the CL chart we choose k to be 0, 1.5 and 3 times the SD, that is k is taken to be equal to 0, 3 and 6 respectively, to cover a wide range of shifts. For the sake of brevity, more choices of k are omitted. Looking at these results, it is possible to get a fair idea about the possible results for any k in the interval [0, 6]. In the IC set up, for fixed a m, n and ARL_0, we can see from Table 1 that the value of H decreases with an increase in k. Table 2 shows that the $SDRL_0$ is relatively higher when k=0, but stabilizes as k increases.

In the OOC set-up, it is evident from Table 3 that the CL chart is able to detect smaller shifts in location quite successfully with k=0, for all choices of m and n under the normal distribution when the variability is under control, which is quite expected for a CUSUM chart. But, interestingly, the CL chart outperforms the SL chart, even for moderate to large shifts in the location (that is, $\theta \geq 1$) with k = 3 and 6, except for $\theta = 3$, where both the charts perform similarly. The CL chart displays nearly the same behavior for a scale shift when the underlying distribution is normal and the process location is in control. We see that if $\theta = 0$, the CL chart with k=0 is the best for $1 < \delta \leq 2$. From Table 3, we further see that for small shifts in location along with small to large shifts in scale, the CL chart shows the best performance for $k = 0$ and for large shifts in location accompanied by small to large shifts in scale, the CL chart performs best for k=3. Note that the OOC performance of the CL chart with $k = 6$ is nearly similar to the OOC performance of the SL chart and both are usually the best choice to detect large shifts in the mean.

The OOC performance of the CL chart for the Cauchy distribution is presented in Table 4. The results show that the CL chart in general performs better than the SL chart in detecting small, moderate to large shifts in location or scale or both for only k=0, when m=50. There is only one unusual behavior we found for m=50 where k=6 gives the best result for the CL chart for the case $\theta = 0$ and $\delta = 1.25$. Also for m=100, the CL chart outperforms the SL chart for small to moderate shifts in location or/and scale for k=0 except in the case of large shift in location irrespective of shifts in scale where the CL chart performs better for k=3.

Table 5 on the OOC performance for the lognormal distribution reveals that for i) small to moderate shifts in location (both upward and downward) or in scale, the CL chart outperforms the SL chart for k=0, ii) large shift in location (both upward and downward) or in scale, the CL chart for k=3 and SL chart perform best, iii) small to moderate shifts in location (both upward and downward) and scale, the CL chart performs better than the SL chart for k=0 and iv) large shift in location (both positive or negative) and scale, the CL chart for $k=3$ and the SL chart demonstrate the best result, except for a few sampling fluctuations.

Thus, it is observed that with a few exceptions, the CL chart performs better in terms of both the ARL and the SDRL for k=0, for both the normal, the Cauchy and the lognormal distributions for shifts in location, or scale or both. It is also noted that the magnitude of reduction in the ARL or the SDRL is much higher in case of the normal and the lognormal distribution than for the Cauchy, for different types of shifts in process parameters. In general, it is recommended to consider smaller values of k, say k=0 for detecting small to medium shifts in location or/and scale in thin tailed and asymmetric distributions and smaller to larger shifts in case of heavy tailed distributions. If the target shift in location or/and scale is relatively large in a thin tailed and asymmetric distribution or very large in a heavy tailed distribution, one should use k=3 to get a quick OOC signal.

Finally, we can see from Table 7, that, when the target ARL_0 is 500, the reference sample size is 100; the test sample size=5 and k=0; the $SDRL_0$ values are about 760 for the CvM statistic, about 600 for the KS statistic and 850 for the Lepage Statistic. It is a fact that the $SDRL_0$ is marginally higher for the CL chart. This is a common problem in many Phase-II CUSUM charts when parameters are unknown. However, note that the $SDRL_0$ of the CL chart drops down to around 650 when m=300 and n=5 and below 600 when m=500 and n=5 even with k=0. Note that the choice of k=0 is worth considering since only this choice effectively detects small shifts in the location if the scale decreases. This can be easily seen from Tables 3, 4 and 5.

7. ILLUSTRATIVE EXAMPLE

Here, we illustrate the proposed nonparametric CL chart using the well-known piston ring data in Montgomery[35](Table 5.1 and 5.2, respectively). Piston rings for an automotive engine are produced by a forging process. The goal is to establish statistical control of the inside diameters of the rings manufactured by this process. Twenty five samples each of size 5, shown in Table 5.1 of Montgomery[35] are taken. A Phase I analysis in Montgomery[35] concluded that we may consider this data set, with 125 observations, as a set of reference data. Further, in Table 5.2 of Montgomery[35], fifteen Phase II samples (test samples) each of size 5 are given. That is, for the present purpose, n = 5. In order to apply the CL chart, using simulations, for a target ARL_0 of 500, for m = 125, n = 5 and for k = 0, 3 and 6, we find the H value to be 28.316879, 6.810828 and 3.474940, respectively. The lower control limit is 0 by default. The fifteen CL plotting statistics are given in Table 10 and shown in the following figures for the three choices of *k*.

Table 10: CL Plotting statistics for *m*=125 *n*=5 and *p*-values for Follow-Up

Sample no	*k* =0 *H*=28.32	*k*=3 *H*=6.81	*k* =6 *H*= 3.47	*p*-value for each sample	
				Wilcoxon Test	AB Test
1	1.68	0	0	0.2234	0.1418
2	0.00	0	0	0.8416	0.7803
3	2.19	0	0	0.0412	0.8748
4	0.74	0	0	0.5014	0.7434
5	2.52	0	0	0.3930	0.0829
6	1.92	0	0	0.2448	0.8652
7	1.10	0	0	0.3385	0.5937
8	2.06	0	0	0.3700	0.1457
9	4.11	0	0	0.0564	0.5356
10	6.92	0	0	0.0372	0.5283
11	5.22	0	0	0.7666	0.6363
12	16.77	8.55	5.55	0.0027	0.0390
13	30.64	19.42	13.42	0.0016	0.0180
14	50.41	36.19	27.19	0.0005	0.0024
15	53.09	35.87	23.87	0.0389	0.5524

The CL charts for *k*=3 and 6 show that the process stays in control for the first eleven test samples and goes OOC for the first time at sample number 12. The OOC signal persists in all the test samples from sample number 12 onwards till sample number 15. Note that, Mukherjee and Chakraborti[1] and Chowdhury et al.[33] also found the first signal at the 12th test sample. Following the signal from the chart at sample 12, it is of interest to diagnose if the signal is due to a shift in location, or scale or both. For this post signal follow-up diagnostic stage, we use step 7 of Section 4 and carry out a two-sided two-sample Wilcoxon Rank-sum test first for location between the 125 observations from the reference sample (Phase I) and the 5 (PhaseII) observations from the 12th test sample. From Table 10, we see that this test yields a *p*-value p_1 =

0.0027. Next, we conduct a two-sided two-sample Ansari-Bradley test for scale using the same data and find the p-value as p_2 = 0.0390. Thus, while p_1 is much smaller than 1%, p_2 lies between 1% and 5%. Hence, we conclude that there is strong evidence of a shift in location with some evidence of a shift in scale at test sample number 12.

Next, we consider test sample number 13 which has been signaled to be OOC. If we carry out the same tests between the reference samples and test sample 13, we see from Table 10, p_1 =0.001634 and p_2 =0.01798, again indicating strong evidence of a shift in location with some evidence of a shift in scale. Note that, if k=0 is used, the CL chart signals an OOC process from sample number 13 onwards without producing any signal at sample number 12 as in Mukherjee et al.[14]. This phenomenon of a delayed signal with k=0 may be explained with our findings in Table 3 noting that the Piston ring data as in Montgomery[35] is close to normal. Table 3 clearly shows that k=3 is the best choice for detecting moderate to large shift in both location and scale.

Further, sample number 15 shows OOC behavior irrespective of the choice of k and it is one such OOC signal which is not found in Mukherjee and Chakraborti[1] and Chowdhury et al.[33] in the context of joint monitoring with the same piston ring data. Note that the same signal is also not found in the Shewhart chart in Montgomery[35]. However, Mukherjee et al.[14] did observe the same OOC phenomenon at the 15th sample using the CUSUM X-bar chart and the nonparametric Exceedence CUSUM chart for detecting shift in the location parameter. In the context of joint monitoring, it is the CL chart which has been able to identify the fifteenth sample having come from an OOC process, although it is not known whether or not this is a genuine OOC signal or a false alarm. If we apply follow-up procedure for sample 15, we see from Table 10 while p_1 is marginally less than 5%, p_2 is significantly higher than 5%. Hence, we conclude that there is evidence of a shift in location and no evidence of a shift in scale at test sample number 15.

8. SUMMARY AND CONCLUSIONS

In this paper we consider a single phase II distribution-free CUSUM control chart based on the well-known Lepage[2] statistic for the joint monitoring of the location and scale parameters of a continuous distribution using a reference sample from a Phase I analysis. Our results show that the proposed CL chart has nice properties and is more effective than several existing distribution-free charts. The current work can lead to a series of possible future works. Construction of EWMA chart based on the Lepage Statistic will be challenging as there is no simple explicit form of conditional variance of Lepage statistic can be obtained. Further, implementation of the charts based on Lepage Statistic using the idea of guaranteed unconditional run length at Phase-II will be another interesting future research problem.

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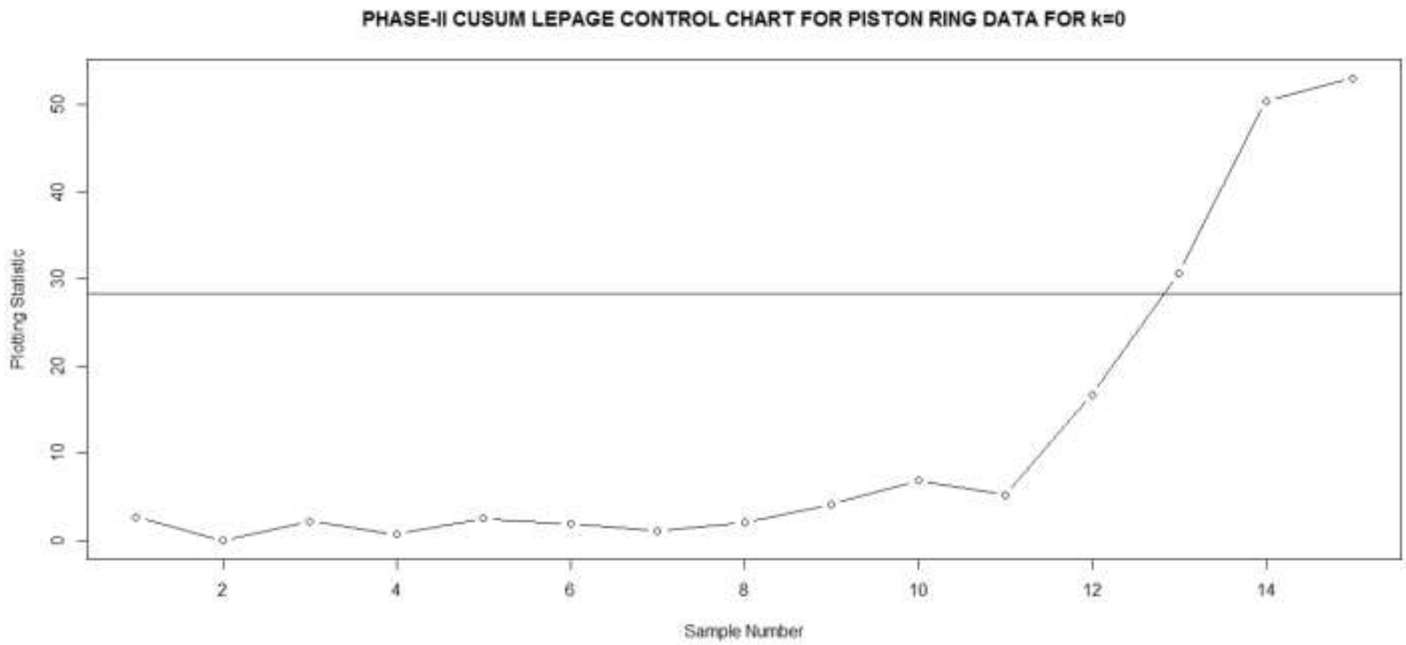
Table-1. Charting constant *H* for the CL chart, for various values of *m* and *n*, and for some standard (target) values of ARL_0

Chart Parameter	Reference Sample Size	Test Sample size	The Charting Constant (Upper Control Limit) : *H*		
k	*m*	*n*	Target $ARL_0 = 250$	Target $ARL_0 = 370$	Target $ARL_0 = 500$
0	30	5	13.548	15.529	17.183
		11	11.470	12.888	14.143
	50	5	16.705	18.919	21.188
		11	14.731	16.467	18.485
	100	5	20.227	23.730	26.551
		11	19.520	22.276	24.953
	150	5	22.320	26.062	29.432
		11	21.537	25.222	28.476
3	30	5	3.546	4.236	4.617
		11	3.607	4.089	4.457
	50	5	4.498	5.110	5.617
		11	4.326	4.908	5.366
	100	5	5.263	5.994	6.531
		11	5.197	5.882	6.435
	150	5	5.486	6.259	6.853
		11	5.507	6.277	6.859
6	30	5	0.543	1.150	1.568
		11	0.489	0.927	1.287
	50	5	1.277	1.961	2.379
		11	1.125	1.710	2.133
	100	5	2.016	2.689	3.279
		11	1.906	2.563	3.112
	150	5	2.214	2.989	3.596
		11	2.212	2.926	3.475

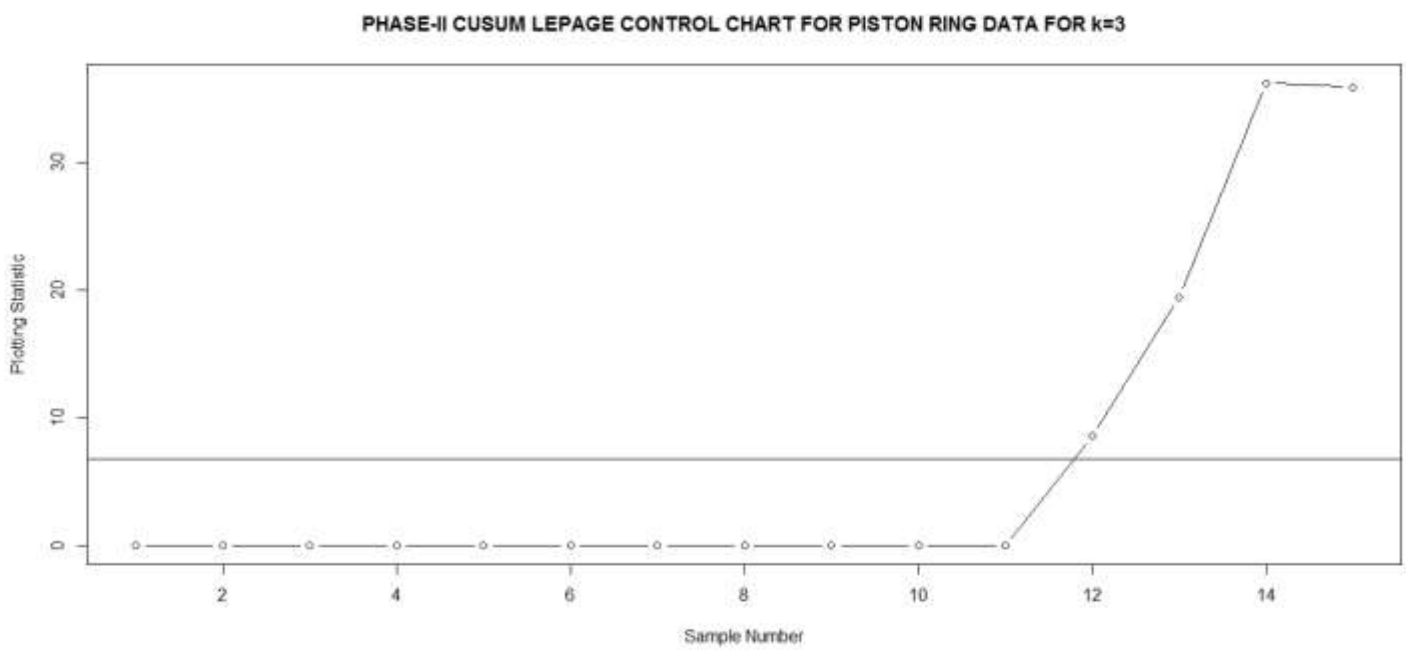
Table-2. IC performance characteristics of the CL chart for $ARL_0 = 500$

m	*n*	*H*	ARL_0	$SDRL_0$	5th Percentile	1st Quartile	Median	3rd Quartile	95th Percentile
Simulated values with *k*=0									
30	5	17.183	500.600	1049.699	13	40	108	365	2999
30	11	14.143	504.448	1066.059	8	27	87	372	3046
50	5	21.188	495.915	967.781	20	58	142	414	2523
50	11	18.485	503.024	997.102	13	42	122	419	2690
100	5	26.551	488.914	837.785	33	88	194	480	2052
100	11	24.953	508.989	889.857	26	75	179	494	2271
150	5	29.432	493.945	768.880	42	107	229	521	1888
150	11	28.476	504.454	810.384	35	95	215	528	2043
Simulated values with *k*=3									
30	5	4.6173	504.517	712.892	7	61	201	540	1952
30	11	4.4567	501.870	846.033	6	51	181	547	2200
50	5	5.617	500.868	787.318	10	72	217	566	1994
50	11	5.366	497.409	780.593	8	67	209	571	2019
100	5	6.531	506.972	695.658	15	100	267	620	1830
100	11	6.435	502.536	645.015	12	91	255	614	1876
150	5	6.853	501.232	612.544	17	108	281	636	1694
150	11	6.859	494.026	632.737	14	101	276	634	1733
Simulated values with *k*=6									
30	5	1.568	513.270	863.696	9	63	194	545	2204
30	11	1.287	497.279	817.785	8	58	192	553	2082
50	5	2.379	480.159	740.192	12	77	217	549	1873
50	11	2.133	501.533	759.076	11	74	224	590	1957
100	5	3.279	503.980	676.439	17	105	274	621	1801
100	11	3.111	506.052	678.493	16	97	269	634	1805
150	5	3.596	495.321	610.456	19	119	305	684	1754
150	11	3.475	503.862	641.108	17	107	285	644	1728

Figure-1.



PHASE-II CUSUM LEPAGE CONTROL CHART FOR PISTON RING DATA FOR k=0
Plotting Statistic
0
10
20
30
40
50
2
4
6
8
10
12
14
Sample Number



PHASE-II CUSUM LEPAGE CONTROL CHART FOR PISTON RING DATA FOR k=3
Plotting Statistic
0
10
20
30
2
4
6
8
10
12
14
Sample Number

PHASE-II CUSUM LEPAGE CONTROL CHART FOR PISTON RING DATA FOR k=6

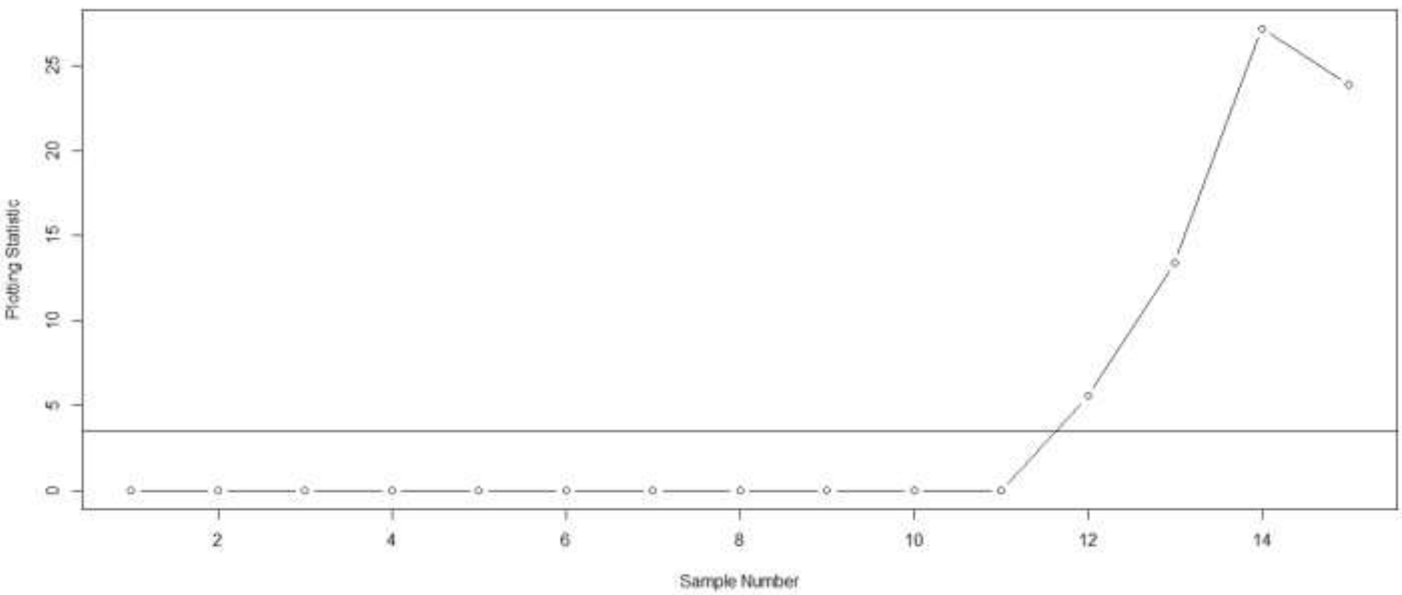


Table-3. Performance comparisons between the Shewhart type and CUSUM type Lepage charts for the Normal (θ,δ) distribution with ARL_0 = 500.

θ	*m*=50, *n*=5				*m*=100, *n*=5			
	ShewhartLepage Chart	Proposed CUSUM Lepage chart			ShewhartLepage Chart	Proposed CUSUM Lepage chart		
		k=0	*k*=3	*k*=6		*k*=0	*k*=3	*k*=6
δ =0.50								
0	1672.5 (1875.6) 21, 176, 714, 2980, 5000	93.5 (481.8) 7, 13, 19, 34, 155	1150.99 (1695.9) 7, 51, 258, 1408, 5000	1656.9 (1874.0) 19, 163, 697, 2956, 5000	2215.1 (1945.4) 57, 419, 1478, 4959	36.6 (154.0) 10, 15, 22, 33, 74	984.2 (1500.0) 68 269 1080 5000	2103.9 (1942.5) 48, 356, 1299, 4588, 5000
0.25	1808.8 (1896.0) 28, 231, 892, 3420, 5000	61.1 (335.0) 8, 14, 21, 33, 99	1404.8 (1793.8) 11, 95, 453, 2138, 5000	1798.3(1893.8) 27, 223, 876, 3395, 5000	2507.2(1960.7) 83,604, 1994,5000, 5000	31.3 (74.7) 12, 18, 24, 34, 63	1496.9 (1759.7) 19, 154, 632, 2323, 5000	2441.1 (1965.8) 73, 543, 1872, 5000, 5000
0.5	1361.9 (1745.0) 10, 103, 467, 1979, 5000	22.1 (75.4) 8, 13, 17, 23, 41	1165.1 (1659.4) 7, 65, 317, 1467, 5000	1337.7 (1730.6) 10, 96, 452, 1929, 5000	1670.6(1823.5) 23, 201, 782, 2845, 5000	22.4 (9.1) 11, 17, 21, 26, 38	1250.6 (1615.3) 14, 117, 475, 1694, 5000	1645.9 (1816.2) 22, 190, 761, 2790, 5000
0.75	415.9 (1029.8) 2, 10, 42, 218, 2757	11.2 (7.3) 4, 7, 10, 14, 22	346.2 (963.4) 2, 6, 23, 136, 2164	414.8 (1036.4) 2, 9, 41, 210, 2755	264.3 (712.4) 2, 13, 46, 167, 1254	12.2 (5.2) 5, 8, 11, 15, 22	168.5 (570.97) 2, 7, 20, 77, 730	257.6 (705.7) 2, 12, 42, 158, 1215
1.0	54.6 (312.1) 1, 2, 5, 17, 149	5.8 (3.1) 2, 4, 5, 7, 12	32.3 (253.8) 1, 2, 3, 8, 62	50.2 (303.6) 1, 2, 5, 15, 135	18.0 (76.6) 1, 2, 5, 14, 61	6.1 (2.6) 3, 4, 6, 7, 11	7.3 (33.1) 1, 2, 3, 6, 20	15.9 (65.9) 1, 2, 5, 12, 55
1.5	1.5 (3.8) 1,2, 5, 17, 149	2.4 (0.8) 2, 2, 2, 3, 4	1.3 (0.8) 1, 1, 1, 1, 2	1.5 (1.96) 1, 1, 1, 1, 3	1.3 (0.8) 1, 1, 1, 1, 3	2.6 (0.7) 2, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1,2	1.3 (0.7) 1, 1, 1, 1, 2
2	1.0 (0.1) 1, 1, 1, 1, 1	1.8 (0.4) 1, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0(0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.1) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
3	1.0 (0.0) 1, 1, 1, 1, 1	1.1 (0.3) 1, 1, 1, 1, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1	2.0 (0.0) 2, 2, 2, 2, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1
δ =1.00								
0	499.6 (918.9) 13, 78, 215, 534, 1886	504.0 (978.7) 20, 58, 144, 422, 2523	503.9 (797.4) 10, 73, 216, 565, 2014	487.4 (753.4) 12, 78, 225, 555, 1890	513.0 (738.9) 18, 106, 276, 635, 1792	481.1 (824.2) 33, 87, 192, 472, 1966	471.3 (640.9) 15, 95, 252, 584, 1646	503.98 (676.4) 17, 105, 274, 621, 1801
0.25	292.7 (641.3) 7, 40, 116, 308, 1124	248.9 (645.3) 12, 30, 64, 165, 1058	291.2 (550.9) 5, 35, 107, 300, 1170	301.0 (554.9) 6, 39, 114, 317, 1186	257. 6 (410.3) 9, 47, 127, 303, 917	177.8 (407.7) 18, 40, 75, 157, 598	242.0 (397.4) 5, 40, 113, 280, 890	252.2 (399.8) 7, 45, 120, 296, 920
0.5	94.7 (253.9) 2, 12, 34, 91, 351	44.1 (141.6) 6, 13, 22, 40, 125	87.9 (211.1) 2, 10, 30, 83, 346	91.3 (209.1) 2, 12, 34, 89, 345	66.5 (98.6) 3, 13, 35, 80, 237	32.9 (37.8) 8, 16, 24, 38, 82	56.6 (92.1) 2, 10, 28, 65, 203	65.1 (100.2) 2, 13, 33, 77, 232
0.75	26.9 (61.1) 1, 5, 12, 28, 96	13.3 (12.5) 3, 7, 11, 16, 32	22.1 (57.20 1, 3, 9, 22, 82	24.7 (45.7) 1, 4, 11, 26, 92	20.3 (27.3) 1, 5, 12, 25, 68	13.2 (7.7) 4, 8, 12, 16, 27	14.9 (20.6) 1, 3, 8, 18, 51	18.9 (25.5) 1, 4, 10, 24, 64
1.0	9.3 (18.6) 1, 2, 5,11, 31	7.1 (4.4) 2, 4, 6, 9, 15	6.9 (10.9) 1, 2, 4, 8, 22	8.7 (14.1) 1, 2, 5, 10, 29	7.7 (8.8) 1, 2, 5, 10, 24	7.4 (3.6) 2, 5, 7, 9, 14	5.3 (5.6) 1, 2, 3, 7, 16	6.99 (8.2) 1, 2, 4, 9, 22
1.5	2.3 (2.2) 1, 1, 2, 3, 6	3.4 (1.4) 2, 2, 3, 4, 6	1.99 (1.5) 1, 1, 2, 2, 5	2.2 (1.98) 1, 1, 2, 3, 6	2.1 (1.7) 1, 1, 2, 3, 5	3.6 (1.4) 2, 2, 3, 4, 6	1.9 (1.1) 1, 1, 2, 2, 4	2.0 (1.5) 1, 1, 2, 2, 5
2	1.3 (0.6) 1, 1, 1, 1, 2	2.3 (0.7) 1, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.6) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.5 (0. 7) 2, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2
3	1.0 (0.1) 1, 1, 1, 1, 1	1.6 (0.5) 1, 1, 2, 2, 2	1.0 (0.07) 1, 1, 1, 1, 1	1.0 (0.06) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.2) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
δ =1.25								
0	106.2 (197.8) 4, 21, 54 124, 369	64.7 (138.2) 9, 19, 34, 65, 200	95.1 (155.0) 2, 17, 47, 113, 340	103.7 (157.4) 3, 20, 53, 124, 366	102.9 (124.1) 5, 25, 62, 133, 337	54.8 (59.2) 12, 24, 39, 64, 144	86.1 (110.8) 2, 19, 50, 110, 291	99.8 (123.7) 4, 24, 60, 128, 330
0.25	73.6 (116.1) 3, 14, 37, 86, 261	46.4 (111.6) 7, 15, 26, 47, 131	70.3 (120.1) 2, 12, 33, 81, 256	74.5 (119.2) 2, 14, 37, 87, 270	70.6 (92.3) 3, 17, 41, 89, 232	39. 3 (36.7) 9, 19, 30, 47, 99	56.9 (75.7) 2, 12, 32, 71, 195	68.8 (89.5) 3, 16, 40, 87, 230

0.5	35.4 (55.5) 2, 7, 18, 42, 123	21.4 (30.3) 4, 10, 15, 25, 55	31.0 (54.5) 2, 6, 15, 35, 113	33.7 (51.4) 2, 7, 17, 40, 119	30.9 (38.8) 2, 8, 18, 40, 101	20.5 (14.0) 6, 12, 17, 25, 46	23.7 (32.8) 2, 5, 13, 29, 78	29.5 (37.3) 2, 7, 17, 38, 97
0.75	15.2 (22.5) 1, 4, 8, 18, 51	11.3 (8.5) 3, 6, 9, 14, 26	12.7 (19.5) 1, 3, 7, 15, 43	14.8 (21.5) 1, 3, 8, 18, 50	13.6 (15.6) 1, 4, 9, 18, 43	11.6 (6.5) 3, 7, 10, 15, 24	10.0 (11.8) 1, 3, 6, 13, 31	12.8 (14.99) 1, 3, 8, 17, 41
1.0	7.4 (9.3) 1, 2, 4, 9, 23	7.0 (4.2) 2, 4, 6, 9, 15	5.8 (7.1) 1, 2, 4, 7, 18	7.2 (9.2) 1, 2, 4, 9, 23	6.7 (7.0) 1, 2, 4, 9, 20	7.4 (3.7) 2, 5, 7, 10, 14	4.98 (4.9) 1, 2, 3, 6, 14	6.2 (6.7) 1, 2, 4, 8, 19
1.5	2.6 (2.4) 1, 1, 2, 3, 7	3.7 (1.7) 2, 2, 3, 5, 7	2.3 (1.7) 1, 1, 2, 3, 6	2.5 (2.2) 1, 1, 2, 3, 6	2.5 (2.1) 1, 1, 2, 3, 7	4.0 (1.7) 2, 3, 4, 5, 7	2.1 (1.4) 1, 1, 2, 3, 5	2.3 (1.9) 1, 1, 2, 3, 6
2	1.4 (0.9) 1, 1, 1, 2, 3	2.6 (0.9) 2, 2, 2, 3, 4	1.4 (0.7) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	2.8 (0.9) 2, 2, 3, 3, 5	1.4 (0.6) 1, 1, 1, 2, 3	1.4 (0.7) 1, 1, 1, 2, 3
3	1.0 (0.2) 1, 1, 1, 1, 1	1.7 (0.5) 1, 1, 2, 2, 2	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.15) 1, 1, 1, 1, 1	1.0 (0.2) 1, 1, 1, 1, 1	2.1 (0.3) 2, 2, 2, 2, 3	1.0 (0.2) 1, 1, 1 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
δ =1.50								
0	36.82 (46.98) 2, 9, 22, 47, 118	20.7 (17.7) 5, 10, 16, 25, 50	30.7 (42.99) 2, 6, 17, 38, 106	35.6 (46.3) 2, 8, 11, 45, 120	37.5 (42.2) 2, 10, 24, 50, 118	21.5 (13.3) 7, 13, 19, 27, 46	27.7 (33.2) 2, 7, 17, 36, 91	35.95 (41.7) 2, 9, 22, 47, 115
0.25	30.64 (38.81) 2, 7, 18, 39, 102	18.4 (15.8) 4, 9, 15, 23, 43	25.2 (35.4) 2, 5, 14, 31, 84	30.1 (39.3) 2, 7, 17, 38, 101	29.9 (34.2) 2, 8, 19, 39, 91	19.0 (11.4) 6,11, 17, 24, 40	22.2 (26.1) 2, 6, 14, 29, 71	27.97 (32.0) 2, 7, 18, 37, 89
0.5	19.0 (24.77) 1, 5, 11, 24, 64	13.4, (9.5) 3, 7, 11, 17, 30	15.4 (20.8) 1, 4, 9, 19, 52	18.6 (24.7) 1, 4, 11, 23, 62	17.8 (19.7) 1, 5, 12, 24, 55	13.9 (7.8) 4, 9, 12, 18, 28	13.2 (14.8) 1, 4, 8, 17, 42	16.99 (19.5) 1, 4, 11, 22, 54
0.75	10.78 (12.87) 1, 3, 7, 14, 34	9.2 (5.8) 2, 5, 8, 12, 20	8.5 (10.3) 1, 2, 5, 11, 27	10.4 (12.9) 1, 3, 6, 13, 33	10.2 (10.7) 1, 3, 7, 14, 30	9.9 (5.2) 3, 6, 9, 13, 19	7.5 (7.8) 1, 2, 5, 10, 23	9.6 (10.7) 1, 3, 6, 13, 30
1.0	6.5 (7.2) 1, 2, 4, 8, 19	6.7 (3.8) 2, 4, 6, 9, 14	5.0 (5.4) 1, 2, 3, 6, 15	5.98 (6.8) 1, 2, 4, 8, 18	6.1 (6.1) 1, 2, 4, 8, 18	7.2 (3.5) 2, 5, 7, 9, 14	4.6 (4.3) 1, 2, 3, 6, 13	5.6 (5.6) 1, 2, 4, 7, 17
1.5	2.8 (2.5) 1, 1, 2, 4, 8	4.0 (1.9) 2, 2, 4, 5, 8	2.4 (1.2) 1, 1, 2, 3, 6	2.7 (2.3) 1, 1, 2, 3, 7	2.7 (2.2) 1, 1, 2, 3, 7	4.3 (1.9) 2, 3, 4, 5, 8	2.3 (1.6) 1, 1, 2 3, 5	2.5 (2.0) 1, 1, 2, 3, 7
2	1.6 (1.1) 1, 1, 1, 2, 4	2.8 (1.1) 2, 2, 2, 3, 5	1.5 (0.86) 1, 1, 1, 2, 3	1.6 (0.99) 1, 1, 1, 2, 4	1.6 (1.0) 1, 1, 1, 2, 4	3.0 (1.1) 2, 2, 3, 4, 5	1.5 (0.8) 1, 1, 1, 2, 3	1.6 (0.9) 1, 1, 1, 2 3
3	1.1 (0.3) 1, 1, 1, 1, 2	1.8 (0.6) 1, 1, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	2.2 (0.4) 2, 2, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2
δ =1.75								
0	18.5 (20.7) 1, 5, 11, 24, 59	12.3 (7.7) 3, 7, 11, 16, 26	14.0 (17.2) 1, 4, 8, 18, 45	17.7 (21.1) 1, 4, 11, 23, 57	19.1 (20.3) 1, 5, 13, 26, 59	13.2 (6.8) 4, 9, 12, 17, 26	12.9 (14.1) 1, 4, 8, 17, 40	17.5 (19.0) 1, 5, 11, 24, 54
0.25	16.7 (20.4) 1, 4, 11, 22, 53	11.5 (7.1) 3, 7, 10, 14, 24	12.6 (14.9) 1, 3, 8, 16, 40	15.6 (18.3) 1, 4, 10, 20, 50	16.4 (17.2) 1, 5, 11, 22, 50	12.5 (6.4) 4, 8, 11, 16, 24,	11.5 (12.2) 1, 3, 8, 15, 35	15.5 (16.7) 1, 4, 10, 21, 48
0.5	12.1 (13.9) 1, 3, 8, 16, 37	9.6 (5.7) 3, 6, 9, 12, 20	9.3 (10.8) 1, 2, 6, 12, 29	11.6 (13.5) 1, 3, 7, 15, 37	12.1 (12.5) 1, 4, 8, 16, 37	10.5 (5.3) 3, 7, 10, 13, 20	8.6 (9.0) 1, 2, 6, 11, 26	11.2 (12.0) 1, 3, 7, 15, 35
0.75	8.3 (9.0) 1, 2, 5, 11, 25	7.8 (4.4) 2, 5, 7, 10, 16	6.4 (6.99) 1, 2, 4, 8, 20	7.9 (8.9) 1, 2, 5, 10, 25	8.4 (8.3) 1, 3, 6, 11, 24	8. 5 (4.2) 3, 6, 8, 11, 16	5.97 (5.8) 1, 2, 4, 8, 17	7.5 (7.6) 1, 2, 5, 10, 22
1.0	5.7 (5.8) 1, 2, 4, 7, 17	6.2 (3.3) 2, 4, 6, 8, 12	4.4 (4.3) 1, 2, 3, 6, 12	5.3 (5.6) 1, 2, 3, 7, 16	5.5 (5.2) 1, 2, 4, 7, 16	6.7 (3.2) 2, 4, 6, 8, 13	4.2 (3.7) 1, 2, 3, 5, 11	5.1 (4.9) 1, 2, 3, 7, 15
1.5	2.9 (2.6) 1, 1, 2, 4, 8	4.1 (1.9) 2, 2, 4, 5, 8	2.5 (1.9) 1, 1, 2, 3, 6	2.7 (2.4) 1, 1, 2, 3, 7	2.8 (2.4) 1, 1, 2, 4, 7	4.4 (1.9) 2, 3, 4, 6, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.6 (2.1) 1, 1, 2, 3 7
2	1.8 (1.2) 1, 1, 1, 2, 4	3.0 (1.3) 2, 2, 3, 4, 5	1.7 (1.0) 1, 1, 1, 2, 4	1.7 (1.1) 1, 1, 1, 2, 4	1.8 (1.2) 1, 1, 1, 2, 4	3.2 (1. 3) 2, 2, 3, 4, 6	1.6 (0.9) 1, 1, 1, 2, 3	1.7 (1.1) 1, 1, 1, 2 4
3	1.1 (0.4) 1, 1, 1, 1, 2	2.0 (0.7) 1, 2, 2, 2, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	2.3 (0.6) 2, 2, 2, 3, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2
δ =2.00								
0	11.3 (12.3)	8.9 (4.9)	8.4 (9.1)	10.8 (12.0)	11.5 (11.9)	9.8 (4.6)	7.8 (7.7)	10.6 (11.1)

	1, 3, 7, 15, 34	3, 6, 8, 11, 18,	1, 2, 5, 11, 26	1, 3, 7, 14, 34	1, 3, 8, 15, 35	3, 7, 9, 12, 18	1, 2, 5, 10, 23	1, 3, 7, 14, 32
0.25	10.3 (11.0) 1, 3, 7, 14, 32	8.6 (4.6) 2, 5, 8, 11, 17	7.6 (8.1) 1, 2, 5, 10, 23	9.9 (10.9) 1, 3, 6, 13, 31	10.8 (10.9) 1, 3, 7, 15, 32	9.5 (4.5) 3, 6, 9, 12, 18	7.3 (7.0) 1, 2, 5, 10, 21	9.8 (10.2) 1, 3, 6, 13, 20
0.5	8.5 (9.0) 1, 3, 6, 11, 25	7.7 (4.2) 2, 5, 7, 10, 16	6.4 (6.7) 1, 2, 4, 8, 19	8.2 (8.9) 1, 2, 5, 11, 25	8.6 (8.5) 1, 3, 6, 11, 25	8.5 (4.0) 3, 6, 8, 11, 16	6.1 (5.7) 1, 2, 4, 8, 17	7.98 (8.1) 1, 2, 5, 11, 24
0.75	6.5 (6.4) 1, 2, 4, 9, 19	6.7 (3.6) 2, 4, 6, 9, 13	5.1 (4.9) 1, 2, 3, 7, 14	6.3 (6.6) 1, 2, 4, 8, 19	6.6 (6.5) 1, 2, 5, 9, 19	7.3 (3.4) 2, 5, 7, 9, 14	4.8 (4.3) 1, 2, 3, 6, 13	6.0 (5.9) 1, 2, 4, 8, 18
1.0	4.9 (4.8) 1, 2, 3, 7, 14	5.7 (2.9) 2, 4, 5, 7, 11	3.9 (3.6) 1, 2, 3, 5, 11	4.7 (4.7) 1, 2, 3, 6, 14	4.8 (4.5) 1, 2, 3, 6, 14	6.2 (2.9) 2, 4, 6, 8, 11	3.7 (3.2) 1, 2, 3, 5, 10	4.5 (4.2) 1, 2, 3, 6, 13
1.5	2.9 (2.5) 1, 1, 2, 4, 8	4.1 (2.0) 2, 2, 4, 5, 8	2.5 (1.9) 1, 1, 2, 3, 6	2.7 (2.4) 1, 1, 2, 3, 7	2.9 (2.5) 1, 1, 2, 4, 8	4.5 (2.0) 2, 3, 4, 6, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.7 (2.1) 1, 1, 2, 3, 7
2	1.9 (1.4) 1, 1, 1, 2, 5	3.1 (1.3) 2, 2, 3, 4, 6	1.7 (1.1) 1, 1, 1, 2 4	1.7 (1.1) 1, 1, 1, 2, 4, 16	1.9 (1.3) 1, 1, 1, 2, 4	3.4 (1.4) 2, 2, 3, 4, 6	1.7 (1.0) 1, 1, 1, 2 ,4	1.8 (1.2) 1, 1, 1, 2, 4
3	1.2 (0.5) 1, 1, 1, 1, 2	2.1 (0.8) 1, 2, 2, 2, 4	1.2 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.4 (0.7) 2, 2, 2, 3, 4	1.2 (0.4) 1, 1, 1, 1, 2	1.2 (0.4) 1, 1, 1, 1, 2

Table-4. Performance comparisons between the Shewhart and CUSUM type Lepage charts for the Cauchy (θ,δ) distribution with ARL_0 = 500.

θ	*m*=50, *n*=5				*m*=100, *n*=5			
	ShewhartLepage Chart	Proposed CUSUM Lepage chart			ShewhartLepage Chart	Proposed CUSUM Lepage chart		
		k=0	*k*=3	*k*=6		*k*=0	*k*=3	*k*=6
δ =0.50								
0	1206.8 (1476.5) 25, 174, 557, 1608, 5000	151.7 (510.7) 12, 23, 40, 83, 499	952.5 (1325.1) 15, 107, 372, 1159, 4956	1228.1 (1494.9) 24, 175, 566, 1670, 5000	1716.8 (1663.6) 58, 366, 1060, 2669, 5000	86.5 (237.3) 16, 28, 44, 76, 230	1036.7 (1317.1) 24, 154, 483, 1337, 4704	1632.4 (1627.2) 54, 337, 987, 2478, 5000
0.25	1367.9 (1583.0) 28, 200, 660, 1940, 5000	156.5 (506.2) 13, 26, 45, 94, 504	1183.6 (1509.5) 19, 139, 496, 1557, 5000	1368.3 (1586.7) 27, 197, 658, 1957, 5000	1911.2 (1733.2) 70, 441, 1271, 3176, 5000	93.5 (213.4) 18, 33, 52, 90, 255	1386.7 (1530.1) 36, 240, 746, 1998, 5000	1860.9 (1714.8) 65, 422, 1213, 3028, 5000
0.5	1509.1 (1700.1) 23, 193, 725, 2319, 5000	107.7 (340.2) 12, 26, 44, 81, 304	1390.6 (1670.3) 17, 150, 600, 2021, 5000	1506.0 (1712.5) 23, 189, 705, 2302, 5000	1951.5(1806.4) 55, 390, 1244, 3451, 5000	80.5 (112.7) 20, 37, 56, 89, 206	1683.0 (1713.2) 40, 293, 968, 2688, 5000	1939.0 (1798.3) 56, 393, 1235, 3379, 5000
0.75	1389.440 (1739.8) 10, 105, 509, 2092, 5000	57.96 (161.9) 9, 19, 33, 57, 154	1342.2 (1744.9) 7, 74, 433, 1995, 5000	1389.8 (1738.8) 10, 103, 508, 2083, 5000	1482.8 (1707.4) 22, 181, 673, 2253, 5000	53.5 (56.5) 13, 26, 41, 64, 128	1363.0 (1681.7) 14, 126, 543, 1989, 5000	1492.4 (1721.4) 20, 172, 665, 2304, 5000
1.0	1067.4 (1631.1) 4, 38, 220, 1270, 5000	31.0 (49.87) 5, 11, 20, 37, 87	1036.4 (1659.9) 3, 21, 153, 1161, 5000	1073.9 (1636.5) 4, 36, 223, 1274, 5000	933.4 (1443.4) 8, 61, 257, 1024, 5000	30.3 (24.7) 8, 14, 23, 38, 76	798.3 (1393.9) 4, 28, 144, 758, 5000	913.2 (1425.7) 7, 56, 250, 1002, 5000
1.5	500.0 (1190.0) 1, 6, 32, 245, 4290	11.9 (12.2) 3, 5, 8, 14, 34	415.5 (1154.2) 1, 3, 10, 89, 3992	504.4 (1203.7) 1, 5, 29, 237, 4466	292.8 (796.5) 1, 8, 35, 164, 1513	11.2 (8.1) 4, 6, 9, 13, 26	166.6 (658.7) 2, 4, 9, 36, 716	276.9 (780.1) 1, 7, 28, 142, 1431
2	224.1 (805.7) 1, 2, 6, 43, 1169	6.4 (5.5) 2, 3 5, 7, 16	143.6 (696.8) 1, 2, 3, 8, 404	215.9 (798.2) 1, 2, 5, 35, 1106	83.9 (387.2) 1, 2, 6, 27, 302	6.2 (3.3) 3, 4, 5, 7, 12	25.6 (239.2) 1, 2 3, 6, 31	73.2 (378.4) 1, 2, 5, 18, 243
3	49.2 (361.0) 1, 1, 1, 1, 3, 84	3.4 (1.9) 2, 2, 3 4, 7	18.4 (248.6) 1, 1, 2, 2, 7	41.2 (332.3) 1, 1, 1, 3, 54	9.3 (101.1) 1, 1, 1, 3, 17	3.5 (1.2) 2, 3, 3 4, 6	2.0 (17.1) 1, 1 1 2 4	6.3 (85.1) 1, 1 1, 2, 8
δ =1.00								
0	468.9 (725.6)	513.8 (993.4)	487.7 (769.5)	491.2 (759.0)	502.1 (663.9)	493.9 (859.8)	478.8 (651.1)	504.2 (676.7)

	13, 76, 212, 536, 1823	20, 58, 144, 427, 2605	10, 72, 211, 555, 1974	12, 79, 225, 565, 1890	18, 106, 277, 626, 1780	34, 88, 192, 474, 2038	14, 93, 255, 591, 1705	17, 104, 272, 625, 1784
0.25	468.3 (745.4) 11, 69, 203, 529,1869	426.5 (900.7) 17, 46, 113, 331, 2088	476.4 (767.6) 9, 66, 197, 531, 1921	472.4 (751.8) 10, 67, 198, 534, 1899	464.0 (642.2) 16, 90, 241, 570, 1681	389.8 (741.8) 27, 67, 145, 350, 1601	441.5 (630.1) 11, 80, 220, 538, 1611	451.5 (644.1) 14, 86, 229, 547, 1650
0.5	396.0 (690.8) 7,47, 149, 423, 1649	272.9 (690.5) 11, 29, 64, 183, 1198	408.8 (730.2) 5, 41, 145, 431, 1755	429.1 (756.2) 7, 48, 154, 453, 1835	379.4 (603.1) 10, 60, 172, 438,1445	207.1 (489.6) 17, 38, 73, 167, 765	346.8 (564.9) 6, 49, 148, 402, 1351	367.3 (584.3) 9, 58, 164, 420, 1405
0.75	343.6 (691.8) 4,29,98, 327, 1547	138.4 (456.9) 7, 17, 34, 82, 493	346.98 (726.4) 2, 21, 84, 312, 1611	340.1 (683.3) 4, 27, 97, 324, 1489	274.4 (507.0) 6, 35, 106, 287, 1093	86.2 (246.8) 10, 22, 38, 72, 258	229.9 (453.2) 3, 24, 78, 233, 956	265.0 (493.9) 5, 33, 101, 281, 1030
1.0	270.1 (641.5) 3,16, 58,211, 1257	60.7 (235.6) 5, 11, 20, 41, 180	258.6 (657.0) 2, 10, 42, 142, 1235	273.1 (643.5) 2, 15, 56, 214, 1295	173.3 (377.5) 3, 19, 57, 164, 719	35.8 (80.5) 7, 14, 22, 37, 96	140.9 (351.2) 2, 11, 37, 118, 598	173.9 (396.3) 2, 17, 53, 161, 721
1.5	149.2 (500.4) 1,5, 18, 73, 670	15.2 (27.6) 3, 6, 9, 16, 42	121.7 (473.6) 1, 3, 10, 41, 526	139.9 (470.4) 1, 5, 16, 67, 646	66.98 (214.7) 1, 6, 17, 50, 257	12.99 (13.3) 4, 7, 10, 15, 30	39.6 (162.3) 1, 3, 8, 24, 146	62.6 (198.4) 1, 5, 14, 46, 249
12	76.2 (347.2) 1,2, 7, 25, 283	7.8 (8.2) 2, 4, 6, 9, 19	52.5 (324.7) 1, 2, 4, 10, 122	72.5 (359.4) 1, 2, 6, 21, 243	26.5 (116.2) 1,2, 6, 17, 93	7.3 (4.3) 3, 5, 6, 9, 15	10.7 (62.7) 1, 2, 3, 7, 26	23.3 (127.4) 1, 2, 5, 13, 77
3	20.9 (179.4) 1, 1, 2, 5, 42	4.1 (2.5) 2, 3, 4, 5, 8	8.96 (121.2) 1, 1, 2, 3, 9	16.8 (146.4) 1, 1, 2, 4, 32	5.0 (22.3) 1, 1, 2, 4, 13	4.1 (1.7) 2, 3, 4, 5, 7	2.4 (7.1) 1, 1, 2, 2, 5	3.9 (18.9) 1, 1, 2, 3, 9
δ =1.25								
0	248.9 (440.7) 7, 40, 108, 271, 948	255.5 (657.1) 13, 32, 67, 173, 1080	250.5 (449.2) 5, 35, 102, 270, 940	245.7 (426.2) 6, 38, 108, 270, 940	238.9 (344.3) 9, 51, 128, 293, 810	192.7 (421.2) 19, 43, 82, 174, 664	216.8 (315.6) 6, 42, 114, 263, 773	239.2 (337.2) 8, 49, 129, 294, 836
0.25	232.9 (417.97) 6, 36, 101, 258, 879	231.3 (639.1) 11, 28, 60, 153, 989	240.2 (427.7) 4, 32, 94, 248, 911	235.6 (424.2) 5, 35, 99, 257, 894	221.8 (318.5) 8, 45, 117, 272, 775	161.7 (370.4) 16, 36, 68, 142, 566	195.6 (296.4) 4, 36, 99, 236, 704	216.6 (318.3) 7, 43, 116, 265, 741
0.5	205.2 (392.9) 5, 27, 80, 216, 819	164.4 (513.5) 8, 20, 41, 99, 595	214.5 (439.7) 2, 22, 73, 215, 877	211.9 (412.2) 4, 26, 79, 219, 846	185.4 (290.1) 6, 34, 91, 217, 670	107.1 (289.97) 12, 26, 46, 90, 330	160.2 (264.2) 3, 25, 73, 184, 600	179.5 (282.4) 5, 32, 87, 213, 643
0.75	178.9 (397.5) 3, 18, 57, 167, 738	95.6 (336.6) 6, 14, 26, 58, 320	175.3 (422.6) 2, 14, 46, 151, 745	179.6 (393.2) 2, 17, 56, 170, 739	135.8 (234.9) 4, 22, 60, 152, 516	53.2 (119.3) 9, 18, 29, 52, 151	114.4 (217.7) 2, 25, 44, 119, 454	135.9 (254.6) 3, 20, 57, 148, 525
1.0	148.1 (383.4) 2,11,37, 117, 620	53.1 (233.5) 5, 10, 18, 34, 142	136.9 (391.6) 2, 8, 27, 98, 593	144.96 (371.4) 2, 11, 36, 118, 613	97.4 (196.1) 3, 14, 38, 99, 385	30.2 (77.7) 6, 12, 19, 31, 76	71.9 (169.1) 2, 8, 24, 68, 288	93.0 (189.4) 2, 12, 35, 94, 368
1.5	81.98 (271.8) 1, 5, 15, 50, 353	16.4 (58.2) 3, 6, 9, 15, 41	72.6 (310.40 1, 3, 9, 32, 277	83.7 (312.8) 1, 4, 13, 47, 332	42.9 (122.1) 1, 5, 14, 37, 159	12.5 (10.5) 4, 7, 10, 15, 28	25.7 (86.5) 1, 3, 8, 20, 93	40.4 (123.2) 1, 4, 12, 34, 151
2	46.6 (232.2) 1, 2, 6, 20, 150	8.1 (9.9) 2, 4, 6, 9, 19	31.2 (195.8) 1, 2, 4, 10, 86	45.8 (233.1) 1, 2, 6, 18, 162	18.7 (67.2) 1, 3, 6, 15, 65	7.6 (4.2) 3, 5, 7, 9, 15	8.8 (34.3) 1, 2, 4, 8, 26	16.3 (55.8) 1, 2, 5, 13, 58
3	14.1 (106.2) 1, 1, 2, 5, 33	4.3 (2.6) 2, 3, 4, 5, 8	6.95 (74.8) 1, 1, 2, 3, 10	14.1 (130.1) 1, 1, 2, 4, 26	4.3 (12.4) 1, 1, 2, 4, 13	4.3 (1.8) 2, 3, 4, 5, 8	2.5 (4.4) 1, 1, 2, 3, 6	3.9 (36.8) 1, 1, 2, 3, 10
δ =1.50								
0	138.2 (243.0) 4, 24, 65, 155, 493	117.0 (393.6) 9, 20, 36, 75, 372	134.9 (273.6) 2, 19, 55, 144, 502	138.9 (251.2) 4, 23, 63, 153, 514	131.3 (175.6) 6, 29, 75, 166, 444	72.9 (149.2) 13, 25, 42, 73, 212	108.9 (156.7) 3, 22, 58, 135, 375	127.2 (175.1) 4, 27, 71, 159, 438
0.25	133.5 (248.7) 4, 22, 59, 145, 497	103.7 (341.7) 8, 18, 33, 71, 341	128.8 (250.6) 2, 17, 51, 135, 493	133.1 (245.9) 3, 21, 58, 146, 495	122.6 (167.4) 5, 27, 68, 154, 417	66.7 (156.3) 11, 23, 38, 66, 185	103.3 (153.6) 2, 19, 52, 125, 367	120.2 (172.5) 4, 25, 66, 147, 417
0.5	122.7 (245.4) 3,18, 50,127, 463	79.5 (275.4) 7, 15, 27, 54, 251	115.7 (253.4) 2, 14, 42, 116, 447	120.6 (237.3) 3, 17, 49, 127, 464	105.9 (156.9) 4, 21, 56, 128, 372	48.6 (91.7) 10, 19, 30, 51, 133	86.6 (142.9) 2, 15, 41, 102, 315	101.9 (155.2) 3, 20, 53, 122, 357
0.75	103.8 (224.6) 2, 13, 37, 104, 406	58.0 (240.9) 5, 12, 20, 39, 158	96.8 (237.6) 2, 9, 29, 87, 396	103.2 (230.6) 2, 12, 37, 103, 403	84.4 (134.4) 3, 16,41, 99, 304	33.9 (58.5) 7, 14, 22, 37, 87	64.3 (110.3) 2, 10, 29, 71, 246	78.9 (131.4) 2, 13, 37, 91, 291
1.0	84.1 (218.2) 2, 9, 26, 75, 332	36.1 (161.5) 4, 9, 15, 27, 93	74.6 (209.7) 2, 6, 19, 59, 307	84.8 (209.5) 2, 9, 26, 78, 341	61.3 (116.2) 2, 10, 28, 66, 225	22.5 (48.8) 6, 11, 16, 25, 54	44.8 (93.6) 2, 6, 18, 46, 174	59.96 (116.5) 2, 9, 25, 64, 221
1.5	51.3 (155.9) 1, 4, 12, 37, 206	14.8 (56.1) 3, 6, 9, 14, 35	43.4 (171.1) 1, 3, 8, 24, 165	52.9 (179.8) 1, 4, 11, 37, 214	30.0 (65.2) 1, 5, 12, 30, 112	11.9 (9.2) 4, 7, 10, 14, 26	18.4 (51.6) 1, 3, 7, 17, 66	28.2 (77.0) 1, 4, 11, 28, 104
2	31.3 (142.8) 1, 2, 6, 18, 112	8.1 (9.6) 2, 4, 6, 9, 19	20.2 (124.5) 1, 2, 4, 10, 61	30.8 (144.5) 1, 2, 5, 16, 107	14.5 (38.1) 1, 3, 6, 14, 52	7.7 (4.4) 3, 5, 7, 9, 15	7.6 (20.1) 1, 2, 4, 8, 23	12.9 (36.2) 1, 2, 5, 12, 45
3	12.1 (98.8) 1, 1, 2, 5, 29	4.4 (2.7) 2, 3, 4, 5, 9	6.5 (86.1) 1, 1, 2, 3, 11	10.3 (79.99) 1, 1, 2, 5, 24	4.4 (10.3) 1, 1, 2, 5, 13	4.5 (1.9) 2, 3, 4, 5, 8	2.6 (3.7) 1, 1, 2, 3, 6	3.7 (8.8) 1, 1, 2, 4, 10

δ =1.75								
0	85.6 (151.3) 3, 15, 41, 98, 304	52.7 (182.7) 6, 14, 23, 42, 140	76.97 (151.8) 2, 12, 33, 84, 290	86.2 (155.2) 2, 15, 40, 97, 313	81.1 (104.2) 4,19, 47, 104, 272	32.2 (58.4) 9, 17, 27, 42, 94	63.3 (96.8) 2, 13, 34, 79, 220	79.8 (109.4) 3, 18, 45, 99, 271
0.25	83.6 (149.9) 3, 15, 39, 94, 306	50.97 (190.6) 6, 13, 22, 40, 139	75.7 (154.6) 2, 11, 31, 80, 287	82.8 (148.9) 2, 14, 38, 92, 299	78.6 (105.9) 4, 18, 45, 98, 262	34.4 (43.7) 9, 16, 25, 39, 84	60.3 (89.7) 2, 12, 31, 73, 212	75.9 (109.1) 2, 16, 42, 94, 258
0.5	79.6 (155.9) 3, 13, 34, 84, 299	43.5 (181.5) 6, 12, 19, 35, 110	69.0 (148.5) 2, 9, 26, 70, 268	76.1 (143.5) 2, 12, 33, 83, 281	68.9 (95.4) 3, 15, 38, 85, 233	29.4 (36.8) 7, 15, 22, 34, 71	52.2 (82.1) 2, 10, 26, 61, 190	64.4 (88.9) 2, 13, 35, 80, 223
0.75	68.4 (140.5) 2, 10, 27, 71, 257	34.5 (150.2) 5, 10, 16, 28, 84	58.8 (143.3) 2, 7, 20, 56, 230	67.9 (142.7) 2, 9, 27, 69, 261	56.2 (81.9) 2, 12, 30, 68, 197	23.3 (27.8) 6, 12, 18, 27, 55	40.8 (71.7) 2, 7, 19, 46, 149	54.8 (85.7) 2, 10, 27, 64, 197
1.0	57.6 (143.1) 2, 7,20, 55, 216	24.7 (101.7) 4, 8, 13, 22, 61	46.6 (117.9) 2, 5, 14, 41, 188	57.6 (136.3) 2, 7, 20, 55, 224	43.8 (70.4) 2, 9, 22, 51, 154	18.1 (21.9) 5, 10, 15, 22, 40	28.8 (52.7) 2, 5, 13, 31, 104	41.4 (71.5) 2, 7, 20, 47, 153
1.5	38.2 (112.4) 1, 4, 11, 31, 149	13.5 (52.4) 3, 6, 9, 14, 30	29.0 (102.5) 1, 3, 7, 20, 108	35.1 (101.2) 1, 4, 10, 28, 138	23.5 (41.7) 1, 5, 11, 26, 83	11.1 (7.3) 4, 7, 10, 13, 23	14.2 (30.4) 1, 3, 6, 15, 49	22.1 (43.6) 1, 4, 10, 23, 81
2	22.7 (90.4) 1, 2, 6, 16, 81	8.1 (11.3) 2, 4, 6, 9, 18	15.5 (75.5) 1, 2, 4, 9, 49	21.9 (89.1) 1, 2, 5, 15, 80	12.9 (26.1) 1, 3, 6, 13, 45	7.6 (4.0) 3, 5, 7, 9, 15	7.1 (15.7) 1, 2, 4, 7, 21	11.4 (24.3) 1, 2, 5, 12, 39
3	8.9 (50.4) 1, 1, 2, 6, 24	4.6 (3.3) 2, 3, 4, 6, 9	5.5 (53.7) 1, 1, 2, 4, 11	8.8 (71.97) 1, 1, 2, 5, 21	4.3 (7.9) 1, 1, 2, 5, 13	4.6 (1.99) 2, 3, 4, 6, 8	2.8 (3.3) 1, 1, 2, 3, 7	3.7 (6.5) 1, 1, 2, 4, 11
δ =2.00								
0	57.2 (94.8) 2,11, 29, 66, 204	30.7 (115.4) 5, 11, 17, 29, 75	49.3 (92.8) 2, 8, 22, 53, 183	57.8 (100.1) 2, 10, 28, 67, 206	56.6 (70.8) 3, 14, 33, 72, 188	24.5 (25.6) 7, 13, 20, 29, 56	38.96 (53.7) 2, 8, 21, 48, 134	52.8 (68.0) 2, 12, 31, 68, 177
0.25	56.6 (97.7) 2, 11, 28, 65, 198	28.2 (91.7) 5, 10, 17, 28, 70	47.95 (92.0) 2, 7, 20, 50, 178	56.7 (105.1) 2, 10, 27, 64, 199	52.98 (68.1) 3, 12, 31, 68, 173	23.4 (24.2) 7, 13, 19, 28, 52	37.7 (52.7) 2, 8, 21, 46, 131	50.4 (72.3) 2, 11, 29, 63, 172
0.5	53.6 (98.1) 2, 10, 25, 60, 193	25.4 (65.6) 5, 10, 15, 25, 65	44.5 (98.7) 2, 6, 18, 45, 167	52.4 (96.4) 2, 9, 24, 58, 196	48.5 (63.3) 2, 11, 27, 61, 165	21.1 (18.4) 6, 12, 17, 25, 47	33.2 (49.7) 2, 7, 17, 40, 116	45.6 (61.9) 2, 10, 25, 57, 156
0.75	47.6 (92.3) 2, 8, 21, 51, 174	21.3 (62.95) 4, 9, 13, 22, 53	38.2 (84.7) 2, 5, 15, 38, 140	45.6 (82.2) 2, 7, 20, 49, 169	40.9 (57.2) 2, 9, 22, 50, 141	18.3 (16.2) 6, 10, 15, 22, 41	27.6 (46.3) 2, 6, 14, 32, 97	38.2 (56.4) 2, 8, 21, 46, 132
1.0	40.3 (86.7) 2, 6, 16, 41, 149	17.99 (62.98) 4, 7, 11, 18, 43	32.4 (80.7) 2, 4, 11, 30, 128	40.99 (94.4) 1, 6, 16, 40, 152	33.3 (51.2) 2, 7, 17, 40, 115	15.0 (10.4) 5, 9, 13, 18, 33	20.98 (34.8) 2, 4, 10, 24, 74	31.1 (50.8) 2, 6, 16, 37, 110
1.5	28.6 (73.9) 1, 4, 10, 26, 109	11.2 (22.9) 3, 6, 8, 12, 26	20.4 (56.6) 1, 2, 6, 16, 79	27.4 (75.4) 1, 3, 9, 24, 104	19.89 (33.2) 1, 4, 10, 23, 69	9.2 (6.2) 4, 6, 9, 13, 21	11.6 (19.5) 1, 3, 6, 13, 39	17.9 (33.8) 1, 3, 9, 20, 62
2	18.2 (70.0) 1, 2, 6, 14, 66	7.7 (9.6) 2, 4, 6, 9, 17	12.1 (49.5) 1, 2, 4, 9, 38	16.8 (61.1) 1, 2, 5, 13, 61	11.5 (19.8) 1, 3, 6, 13, 39	7.5 (3.9) 3, 5, 7, 9, 15	6.4 (9.9) 1, 2, 4, 7, 20	10.3 (22.7) 1, 2, 5, 11, 35
3	7.6 (28.1) 1,1, 3, 6, 24	4.7 (2.8) 2, 3, 4, 6, 9	5.3 (43.8) 1, 1, 2, 4, 11	6.9 (34.0) 1, 1, 2, 5, 20	4.6 (7.7) 1, 1, 3, 5, 14	4.8 (2.1) 2, 3, 4, 6, 9	2.8 (2.9) 1, 1, 2, 3, 7	3.8 (5.9) 1, 1, 2, 4, 11

Table-5. Performance comparisons between the Shewhart and CUSUM type Lepage charts for the Lognormal (θ,δ) distribution with ARL_0 = 500.

θ	*m*=50, *n*=5				*m*=100, *n*=5			
	ShewhartLepage Chart	Proposed CUSUM Lepage chart			ShewhartLepage Chart	Proposed CUSUM Lepage chart		
		k=0	*k*=3	*k*=6		*k*=0	*k*=3	*k*=6
δ =0.50								
-3.0	1.0 (0.0) 1, 1, 1, 1, 1	1.4 (0.5) 1, 1, 1, 2, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1 1, 1	2.0 (0.0) 2, 2, 2, 2, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1

-2.0	1.0 (0.1) 1, 1, 1, 1, 1	1.96 (0.3) 1, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.1) 2, 2, 3, 3, 4	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 4
-1.5	1.8 (4.5) 1, 1, 1, 2, 4	2.6 (0.8) 2, 2, 2, 3, 4	1.4 (1.0) 1, 1, 1, 1, 2	1.7 (4.8) 1, 1, 1, 2, 4	1.4 (1.0) 1, 1, 1, 1, 3	2.7 (0.7) 2, 2, 3, 3, 4	1.3 (0.5) 1, 1, 1, 1, 2	1.3 (0.8) 1, 1, 1, 1 3
-1.0	82.2 (414.4) 1, 2, 7, 25, 249	6.0 (2.8) 3, 4, 5, 7, 11	50.3 (328.8) 1, 2, 4, 10, 114	97.99 (464.7) 1, 2, 7, 27, 324	26.3 (123.0) 1, 3, 7, 18, 88	6.3 (2.5) 3, 5, 6, 8, 11	9.2 (57.7) 1, 2, 4, 7, 24	22.4 (111.3) 1, 2, 6, 15, 74
-0.75	554.96 (1179.7) 2, 14, 66, 363, 4285	10.4 (4.7) 4, 7, 10, 13, 18	413.2 (1041.0) 2, 8, 34, 206, 2770	632.1 (1290.1) 2, 16, 78, 443, 5000	366.1 (879.7) 3, 18, 65, 242, 1951	12.0 (4.6) 6, 9, 12, 15, 20	208.7 (646.2) 2, 8, 25, 102, 975	356.96 (869.9) 3, 16, 59, 233, 1884
-0.50	1479.2 (1789.2) 14, 131, 573, 2300, 5000	17.7 (38.6) 8, 11, 15, 19, 33	1139.9 (1632.7) 8, 70, 322, 1398, 5000	1585.2 (1840.3) 15, 148, 649, 2652, 5000	1903.1 (1885.6) 32, 283, 1059, 3636, 5000	20.5 (7.9) 11, 16, 19, 24, 34	1295.7 (1631.9) 17, 136, 516, 1773, 5000	1859.5 (1875.2) 29, 260, 1011, 3470, 5000
-0.25	1767.4 (1881.9) 26, 213, 856, 3245, 5000	49.5 (299.2) 8, 12, 18, 27, 76	1243.3 (1715.9) 9, 72, 347, 1651, 5000	1766.0 (1899.4) 25, 203, 815, 3344, 5000	2501.4 (1964.2) 81, 587, 1999, 5000, 5000	28.1 (66.8) 11, 16, 22, 31, 55	1371.8 (1714.9) 17, 129, 515, 1984, 5000	2409.7 (1966.1) 68, 517, 1821, 5000, 5000
0	1675.7 (1871.5) 22, 178, 725, 2985, 5000	95.2 (495.2) 7, 12, 19, 34, 160	1131.6 (1676.1) 7, 52, 257, 1371, 5000	1669.1 (1877.4) 20, 167, 710, 2966, 5000	2210.4 (1946.8) 57, 418, 1469, 4957, 5000	36.5 (156.4) 10, 15, 22, 33, 73	974.3 (1493.1) 10, 67, 259, 1070, 5000	2087.0 (1933.1) 48, 357, 1264, 4466, 5000
0.25	1817.7 (1899.2) 29, 230, 897, 3457, 5000	61.4 (339.4) 9, 14, 21, 33, 98	1392.5 (1779.8) 11, 95, 455, 2094, 5000	1777.4 (1888.4) 26, 213, 854, 3319, 5000	2517.3 (1962.6) 82, 598, 2038, 5000, 5000	31.3 (65.9) 12, 18, 24, 34, 63	1491.3 (1762.6) 20, 155, 620, 2319, 5000	2446.7 (1968.3) 73, 535, 1887, 5000, 5000
0.5	1351.9 (1739.98) 10, 100, 461, 1946, 5000	22.6 (85.8) 8, 13, 17, 23, 42	1164.5 (1653.5) 7, 65, 320, 1470, 5000	1343.7 (1738.5) 9, 98, 456, 1917, 5000	1662.5 (1816.4) 24, 201, 792, 2813, 5000	22.5 (9.5) 12, 17, 21, 26, 38	1236.5 (1605.1) 14, 120, 468, 1648, 5000	1656.2 (1815.98) 24, 195, 781, 2818, 5000
0.75	419.5 (1038.6) 2, 10, 42, 217, 2798	11.2 (6.3) 4, 7, 10, 14, 22	340.6 (956.3) 2, 6, 22, 136, 2174	413.0 (1035.3) 2, 9, 39, 214, 2759	270.2 (727.6) 2, 13, 47, 168, 1265	12.1 (5.1) 5, 8, 11, 15, 22	168.6 (572.9) 2, 7, 20, 78, 708	258.7 (714.3) 2, 12, 41, 153, 1210
1.0	50.0 (291.2) 1, 2, 5, 17, 141	5.8 (3.1) 2, 4, 5, 7, 12	35.3 (278.1) 1, 2, 3, 8, 64	48.5 (286.4) 1, 2, 5, 15, 138	18.3 (86.3) 1, 2, 5, 14, 61	6.1 (2.6) 3, 4, 6, 7, 11	7.4 (49.6) 1, 2, 3, 6, 20	16.8 (91.5) 1, 2, 5, 12, 54
1.5	1.5 (3.5) 1, 1, 1, 1, 3	2.4 (0.7) 2, 2, 2, 3, 4	1.3 (0.8) 1, 1, 1, 1, 2	1.4 (1.8) 1, 1, 1, 1, 3	1.3 (0.8) 1, 1, 1, 1, 3	2.6 (0.7) 2, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1, 2	1.3 (0.7) 1, 1, 1, 1, 2
2.0	1.0 (0.1) 1, 1, 1, 1, 1	1.8 (0.4) 1, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.1) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
3.0	1.0 (0.0) 1, 1, 1, 1, 1	1.1 (0.3) 1, 1, 1, 1, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1	2.0 (0.0) 2, 2, 2, 2, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.0) 1, 1, 1, 1, 1
δ =1.00								
-3.0	1.0 (0.1) 1, 1, 1, 1, 1	1.7 (0.5) 1, 1, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.1) 2, 2, 2, 2, 2	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
-2.0	1.3 (0.7) 1, 1, 1, 1, 3	2.3 (0.6) 2, 2, 2, 3, 3	1.3 (0.5) 1, 1, 1, 1, 2	1.3 (0.6) 1, 1, 1, 1, 3	1.2 (0.6) 1, 1, 1, 1, 2	2.4 (0.6) 2, 2, 2, 3, 4	1.0 (0.1) 1, 1, 1, 1, 1	1.2 (0.5) 1, 1, 1, 1, 2
-1.5	2.5 (2.4) 1, 1, 2, 3, 7	3.4 (1.3) 2, 2, 3, 4, 6	2.1 (1.6) 1, 1, 2, 3, 5	2.5 (2.4) 1, 1, 2, 3, 7	2.3 (1.9) 1, 1, 2, 3, 6	3.6 (1.2) 2, 3, 3, 4, 6	1.2 (0.5) 1, 1, 1, 1, 2	2.2 (1.7) 1, 1, 2, 3, 5
-1.0	9.98 (15.7) 1, 2, 5, 12, 34	7.3 (4.2) 3, 5, 6, 9, 15	7.9 (13.3) 1, 2, 4, 9, 25	10.6 (18.2) 1, 2, 5, 12, 36	8.3 (9.98) 1, 2, 5, 11, 26	7.5 (3.5) 3, 5, 7, 9, 14	1.9 (1.2) 1, 1, 2, 2, 4	7.8 (9.4) 1, 2, 5, 10, 24
-0.75	30.0 (68.5) 1, 5, 13, 31, 107	13.8 (13.2) 7.3 (4.2)	26.1 (66.8) 1, 4, 11, 26, 91	32.7 (73.5) 1, 5, 14, 33, 119	22.6 (31.1) 1, 5, 13, 28, 76	13.5 (7.9) 5, 8, 12, 17, 28	5.96 (6.5) 1, 2, 4, 7, 17	21.9 (31.3) 1, 5, 12, 27, 73
-0.50	105.9 (237.3) 3, 14, 39, 103, 404	47.6 (167.8) 7, 14, 23, 41, 126	104.3 (253.5) 3, 12, 35, 96, 408	119.2 (271.4) 3, 15, 43, 114, 462	78.8 (126.6) 3, 15, 39, 93, 281	33.6 (36.2) 9, 16, 25, 39, 82	16.8 (23.7) 1, 4, 9, 20, 55	76.7 (125.6) 3, 15, 39, 89, 270
-0.25	327.4 (577.1) 8, 45, 132, 353, 1279	251.4 (643.0) 14, 33, 68, 175, 1016	341.8 (617.6) 7, 43, 130, 359, 1384	354.0 (624.1) 8, 49, 140, 376, 1413	292.7 (447.8) 9, 53, 145, 345, 1056	182.5 (405.5) 19, 42, 77, 160, 628	65.2 (108.5) 3, 12, 32, 74, 235	287.5 (449.1) 9, 52, 140, 335, 1044
0	470.6(728.0) 12, 74, 212, 536, 1828	496.8 (969.4) 21, 59, 142, 409, 2553	493.7 (771.1) 12, 76, 217, 562, 1971	502.3 (777.2) 13, 80, 225, 569, 1982	513.5 (684.3) 18, 107, 278, 634, 1823	494.9 (847.9) 34, 89, 194, 480, 2082	492.3 (669.7) 17, 100, 262, 610, 1736	506.6 (670.8) 19, 105, 276, 632, 1778

0.25	259.2 (487.6) 6, 35, 99, 270, 1023	231.5 (610.3) 12, 30, 62, 159, 927	271.9 (535.4) 6, 33, 98, 273, 1089	266.1 (507.2) 6, 34, 101, 273, 1052	242.8(382.8) 8, 44, 117, 284, 878	169.4 (377.5) 18, 40, 74, 152, 576	224.7 (361.8) 7, 39, 106, 260, 828	235.0 (368.9) 8, 43, 114, 275, 848
0.5	78.0 (178.4) 2, 11, 30, 78, 288	42.9 (139.0) 6, 13, 21, 38, 117	74.1 (177.4) 2, 10, 26, 70, 279	79.2 (178.5) 2, 11, 30, 78, 298	63.6(100.8) 3, 13, 32, 75, 221	32.3 (39.3) 8, 16, 24, 38, 81	54.2 (90.2) 2, 11, 27, 62, 193	62.3 (99.4) 3, 12, 32, 73, 218
0.75	22.6 (43.0) 1, 4, 10, 24, 80	12.9 (13.2) 4, 7, 10, 15, 31	19.3 (46.1) 1, 4, 8, 20, 68	22.2 (49.7) 1, 4, 10, 24, 77	19.0(25.6) 1, 5, 11, 23, 63	12.98 (7.6) 5, 8, 11, 16, 27	14.6 (19.9) 1, 4, 8, 18, 48	18.4 (24.97) 1, 4, 11, 23, 61
1.0	8.2 (12.4) 1, 2, 4, 10, 27	6.8 (4.0) 2, 4, 6, 8, 14	6.5 (9.7) 1, 2, 4, 7, 20	7.9 (11.6) 1, 2, 4, 9, 26	7.3(8.4) 1, 2, 5, 9, 23	7.2 (3.3) 3, 5, 7, 9, 13	5.4 (5.6) 1, 2, 4, 7, 16	6.9 (7.99) 1, 2, 4, 9, 21
1.5	2.2 (2.0) 1, 1, 1, 3, 6	3.2 (1.3) 2, 2, 3, 4, 6	1.9 (1.4) 1, 1, 2, 2, 4	2.1 (1.8) 1, 1, 1, 3, 5	2.1(1.7) 1, 1, 2, 3, 5	3.5 (1.2) 2, 3, 3, 4, 6	1.8 (1.1) 1, 1, 2, 2, 4	2.0 (1.5) 1, 1, 2, 2, 5
2.0	1.2(0.6) 1, 1, 1, 1, 2	2.1 (0.6) 1, 2, 2, 2, 3	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.6) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.4 (0.6) 2, 2, 2, 3, 3	1.2 (0.4) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2
3.0	1.0 (0.1) 1, 1, 1, 1, 1	1.4 (0.5) 1, 1, 1, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.04) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.05) 1, 1, 1, 1, 1
δ =1.25								
-3.0	1.0 (0.2) 1, 1, 1, 1, 1	1.8 (0.5) 1, 2, 2, 2, 2	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.2) 1, 1, 1, 1, 1	2.1 (0.3) 2, 2, 2, 2, 3	1.0 (0.0) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
-2.0	1.5 (0.9) 1, 1, 1, 2, 3	2.5 (0.8) 2, 2, 2, 3, 4	1.4 (0.7) 1, 1, 1, 2, 3	1.5 (0.9) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	2.98 (0.97) 2, 2, 3, 4, 5	1.0 (0.2) 1, 1, 1, 1, 1	1.4 (0.8) 1, 1, 1, 2, 3
-1.5	2.8 (2.6) 1, 1, 2, 3, 8	3.8 (1.6) 2, 3, 4, 5, 7	2.4 (1.9) 1, 1, 2, 3, 6	2.7 (2.5) 1, 1, 2, 3, 7	2.6 (2.2) 1, 1, 2, 3, 7	4.3 (1.7) 2, 3, 4, 5, 7	2.2 (1.5) 1, 1, 2, 3, 5	2.5 (1.97) 1, 1, 2, 3, 6
-1.0	8.2 (10.8) 1, 2, 5, 10, 26	7.3 (4.1) 3, 5, 6, 9, 15	6.7 (8.4) 1, 2, 4, 8, 20	8.4 (10.9) 1, 2, 5, 10, 27	7.3 (7.7) 1, 2, 5, 9, 22	7.3 (3.3) 3, 5, 7, 9, 13	5.5 (5.3) 1, 2, 4, 7, 15	6.9 (7.2) 1, 2, 5, 9, 21
-0.75	17.3 (26.6) 1, 4, 9, 21, 58	11.9 (8.6) 4, 7, 10, 14, 27	14.6 (23.1) 1, 4, 8, 17, 49	17.9 (28.7) 1, 4, 10, 21, 61	14.8 (17.1) 1, 4, 9, 19, 46	10.2 (4.99) 4, 7, 9, 13, 20	11.4 (13.1) 1, 3, 7, 14, 35	14.3 (16.8) 1, 4, 9, 18, 45
-0.50	39.4 (63.9) 2, 8, 20, 46, 137	22.7 (34.9) 5, 10, 16, 26, 58	36.5 (66.5) 2, 7, 17, 40, 129	42.3 (67.4) 2, 8, 21, 49, 150	34.1 (43.4) 2, 8, 20, 44, 112	14.4 (7.8) 5, 9, 13, 18, 29	27.6 (35.3) 2, 7, 16, 34, 91	32.8 (41.2) 2, 8, 19, 42, 107
-0.25	82.1 (130.6) 3, 16, 42, 97, 289	47.3 (99.5) 8, 16, 28, 49, 135	77.1 (126.0) 3, 14, 37, 88, 277	86.2 (135.6) 3, 17, 44, 101, 305	76.2 (97.6) 3, 18, 45, 97, 252	19.4 (11.3) 7, 12, 17, 24, 41	64.2 (84.2) 3, 15, 37, 81, 216	74.5 (95.2) 3, 17, 44, 95, 250
0	103.0 (153.7) 4, 21, 55, 124, 360	64.4 (136.4) 10, 20, 34, 64, 196	99.4 (160.7) 4, 19, 50, 116, 352	107.4 (160.7) 4, 21, 55, 129, 380	104.5(129.1) 5, 25, 63, 135, 345	54.99 (60.9) 13, 25, 40, 65, 143	90.1 (114.9) 4, 21, 53, 115, 297	102.4 (124.4) 5, 25, 62, 132, 338
0.25	67.7 (107.7) 3, 13, 34, 80, 239	43.5 (88.8) 8, 15, 26, 46, 122	63.8 (110.1) 3, 12, 31, 73, 227	68.8 (109.6) 3, 13, 34, 79, 248	66.8(85.6) 3, 16, 40, 85, 220	38.6 (34.9) 10, 19, 29, 47, 95	56.7 (73.7) 3, 13, 33, 72, 188	64.8 (83.5) 3, 15, 38, 82, 215
0.5	31.4 (50.4) 2, 7, 16, 37, 111	20.5 (28.8) 5, 9, 15, 24, 52	27.97 (47.1) 2, 6, 14, 32, 97	30.99 (49.1) 2, 6, 16, 37, 108	28.9(35.4) 2, 7, 18, 37, 94	20.2 (13.7) 6, 11, 17, 25, 45	23.6 (29.5) 2, 6, 14, 30, 77	28.3 (35.1) 2, 7, 17, 36, 93
0.75	13.6(19.3) 1, 3, 8, 17, 45	10.8 (8.0) 3, 6, 9, 13, 25	11.6 (16.9) 1, 3, 7, 14, 38	13.4 (18.98) 1, 3, 8, 16, 44	12.9(15.0) 1, 4, 8, 17, 40	11.4 (6.2) 4, 7, 10, 14, 23	10.1 (11.3) 1, 3, 7, 13, 31	12.5 (14.5) 1, 3, 8, 16, 39
1.0	6.8 (8.3) 1, 2, 4, 8, 21	6.7 (3.9) 2, 4, 6, 8, 14	5.6 (6.3) 1, 2, 4, 7, 17	6.5 (7.9) 1, 2, 4, 8, 20	6.5 (6.7) 1, 2, 4, 8, 19	7.2 (3.4) 3, 5, 7, 9, 14	5.1 (4.9) 1, 2, 4, 7, 14	6.2 (6.4) 1, 2, 4, 8, 18
1.5	2.5(2.2) 1, 1, 2, 3, 7	3.5 (1.5) 2, 2, 3, 4, 6	2.2 (1.6) 1, 1, 2, 3, 5	2.4 (2.1) 1, 1, 2, 3, 6	2.4(2.0) 1, 1, 2, 3, 6	3.8 (1.5) 2, 3, 4, 5, 7	2.1 (1.4) 1, 1, 2, 3, 5	2.3 (1.8) 1, 1, 2, 3, 6
2.0	1.4(0.8) 1, 1, 1, 2, 3	2.4 (0.8) 1, 2, 2, 3, 4	1.4 (0.7) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	2.6 (0.8) 2, 2, 2, 3, 4	1.4 (0.6) 1, 1, 1, 2, 3	1.4 (0.7) 1, 1, 1, 2, 3
3.0	1.0 (0.5) 1, 1, 1, 1, 1	1.5 (0.5) 1, 1, 2, 2, 2	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	1.0(0.1) 1, 1, 1, 1, 1	2.0 (0.1) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
δ =1.50								
-3.0	1.1 (0.3) 1, 1, 1, 1, 2	1.9 (0.5) 1, 2, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	2.2 (0.4) 2, 2, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2
-2.0	1.7 (1.1)	2.8 (1.0)	1.6 (0.9)	1.7 (1.1)	1.6 (1.1)	3.2 (1.1)	1.6 (0.8)	1.6 (0.98)

	1, 1, 1, 2, 4	2, 2, 3, 3, 5	1, 1, 1, 2, 3	1, 1, 1, 2, 4	1, 1, 1, 2, 4	2, 2, 3, 4, 5	1, 1, 1, 2, 3	1, 1, 1, 2, 4
-1.5	2.9 (2.6) 1, 1, 2, 4, 8	4.0 (1.7) 2, 3, 4, 5, 7	2.6 (1.99) 1, 1, 2, 3, 6	2.9 (2.6) 1, 1, 2, 4, 8	2.8 (2.4) 1, 1, 2, 4, 8	4.4 (1.8) 2, 3, 4, 5, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.7 (2.2) 1, 1, 2, 3, 7
-1.0	6.9 (7.9) 1, 2, 4, 9, 21	6.9 (3.6) 3, 4, 6, 9, 14	5.8 (6.2) 1, 2, 4, 7, 17	6.99 (7.9) 1, 2, 4, 9, 22	6.4 (6.4) 1, 2, 4, 8, 19	6.8 (3.0) 3, 5, 6, 8, 12	5.0 (4.6) 1, 2, 4, 7, 14	6.1 (6.1) 1, 2, 4, 8, 18
-0.75	11.9 (14.7) 1, 3, 7, 15, 38	9.7 (6.0) 3, 6, 8, 12, 20	9.8 (11.7) 1, 3, 6, 12, 31	12.0 (14.97) 1, 3, 7, 15, 39	10.9 (11.6) 1, 3, 7, 14, 33	8.6 (3.97) 4, 6, 8, 11, 16	8.4 (8.5) 1, 3, 6, 11, 24	10.5 (11.2) 1, 3, 7, 14, 32
-0.50	20.9 (27.1) 1, 5, 12, 26, 69	14.0 (9.8) 4, 8, 12, 17, 31	17.5 (23.1) 1, 5, 10, 21, 57	21.4 (27.6) 1, 5, 13, 27, 70	19.4 (21.5) 1, 5, 13, 26, 60	10.8 (5.1) 4, 7, 10, 13, 20	14.9 (16.6) 1, 4, 10, 19, 46	18.7 (20.9) 1, 5, 12, 25, 59
-0.25	32.4 (41.4) 2, 8, 19, 41, 107	18.9 (16.5) 5, 10, 15, 23, 44	27.5 (36.97) 2, 7, 16, 34, 93	33.2 (43.1) 2, 8, 19, 41, 111	31.7 (36.2) 2, 8, 20, 42, 101	12.7 (6.1) 5, 8, 12, 16, 24	24.2 (27.3) 2, 7, 15, 32, 76	30.8 (35.1) 2, 8, 20, 41, 97
0	36.9 (46.7) 2, 9, 22, 47, 122	21.2 (18.9) 6, 11, 16, 26, 51	31.6 (41.7) 2, 8, 18, 39, 106	37.1 (48.8) 2, 9, 22, 46, 124	38.2(42.9) 2, 10, 24, 51, 120	21.8 (13.0) 8, 13, 19, 27, 46	29.1 (33.1) 2, 8, 19, 38, 92	36.7 (41.9) 2, 10, 23, 48, 116
0.25	28.6 (36.7) 2, 7, 17, 36, 94	18.1 (14.7) 5, 9, 14, 22, 43	24.4 (32.97) 2, 6, 14, 30, 81	28.5 (36.9) 2, 7, 16, 36, 95	29.2 (33.2) 2, 8, 19, 38, 92	18.9 (11.0) 7, 11, 16, 24, 40	22.6 (25.6) 2, 6, 14, 30, 71	28.1 (31.8) 2, 8, 18, 37, 89
0.5	17.2(21.5) 1, 4, 10, 22, 56	12.9 (8.97) 4, 7, 11, 16, 29	14.7 (18.9) 1, 4, 9, 18, 48	16.9 (21.8) 1, 4, 10, 21, 55	17.2 (19.1) 1, 5, 11, 23, 54	13.8 (7.5) 5, 9, 12, 17, 28	13.5 (14.8) 1, 4, 9, 17, 41	16.6 (18.4) 1, 5, 11, 22, 52
0.75	9.8 (11.9) 1, 3, 6, 12, 31	8.96 (5.4) 3, 5, 8, 11, 19	8.2 (9.5) 1, 3, 5, 10, 25	9.6 (11.6) 1, 3, 6, 12, 30	9.7 (10.3) 1, 3, 6, 13, 29	9.7 (4.9) 4, 6, 9, 12, 19	7.8 (7.8) 1, 3, 5, 10, 23	9.3 (9.8) 1, 3, 6, 12, 28
1.0	5.9(6.4) 1, 2, 4, 7, 18	6.4 (3.4) 2, 4, 6, 8, 13	5.0 (5.0) 1, 2, 3, 6, 14	5.7 (6.2) 1, 2, 4, 7, 17	5.9(5.8) 1, 2, 4, 8, 17	7.0 (3.2) 3, 5, 6, 9, 13	4.7 (4.2) 1, 2, 3, 6, 13	5.6 (5.5) 1, 2, 4, 7, 16
1.5	2.6 (2.3) 1, 1, 2, 3, 7	3.7 (1.6) 2, 3, 3, 5, 7	2.4 (1.7) 1, 1, 2, 3, 6	2.6 (2.2) 1, 1, 2, 3, 7	2.6 (2.1) 1, 1, 2, 3, 7	4.1 (1.6) 2, 3, 4, 5, 7	2.3 (1.6) 1, 1, 2, 3, 5	2.5 (2.0) 1, 1, 2, 3, 6
2.0	1.6(1.0) 1, 1, 1, 2, 4	2.6 (0.97) 1, 2, 2, 3, 4	1.5 (0.8) 1, 1, 1, 2, 3	1.6 (0.97) 1, 1, 1, 2, 3	1.6(1.0) 1, 1, 1, 2, 4	2.9 (0.9) 2, 2, 3, 3, 5	1.5 (0.8) 1, 1, 1, 2, 3	1.6 (0.9) 1, 1, 1, 1, 2
3.0	1.1(0. 3) 1, 1, 1, 1, 2	1.7 (0.6) 1, 1, 2, 2, 2	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.2) 1, 1, 1, 1, 2	1.1(0.3) 1, 1, 1, 1, 2	2.1 (0.3) 2, 2, 2, 2, 3	1.1 (0.2) 1, 1, 1, 1, 2	1.1 (0.2) 1, 1, 1, 1, 2
δ =1.75								
-3.0	1.1 (0.4) 1, 1, 1, 1, 2	2.0 (0.6) 1, 2, 2, 2, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	2.2 (0.4) 2, 2, 2, 2, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2
-2.0	1.9 (1.3) 1, 1, 1, 2, 4	2.99 (1.1) 2, 2, 3, 4, 5	1.8 (1.1) 1, 1, 1, 2, 4	1.8 (1.3) 1, 1, 1, 2, 4	1.8 (1.2) 1, 1, 1, 2, 4	3.2 (1.1) 2, 2, 3, 4, 5	1.7 (0.97) 1, 1, 1, 2, 4	1.8 (1.2) 1, 1, 1, 2, 4
-1.5	3.0 (2.7) 1, 1, 2, 4, 8	4.2 (1.8) 2, 3, 4, 5, 8	2.7 (2.1) 1, 1, 2, 3, 7	3.0 (2.6) 1, 1, 2, 4 8	2.9 (2.5) 1, 1, 2, 4, 8	4.4 (1.8) 2, 3, 4, 5, 8	2.5 (1.8) 1, 1, 2, 3, 6	2.8 (2.3) 1, 1, 2, 4, 7
-1.0	6.0 (6.3) 1, 2, 4, 8, 18	6.4 (3.2) 3, 4, 6, 8, 12	5.0 (4.9) 1, 2, 4, 6, 14	6.1 (6.4) 1, 2, 4, 8, 18	5.7 (5.5) 1, 2, 4, 8, 17	6.8 (3.0) 3, 5, 6, 8, 12	4.5 (3.9) 1, 2, 3, 6, 12	5.5 (5.2) 1, 2, 4, 7, 16
-0.75	8.9 (9.8) 1, 3, 6, 12, 27	8.1 (4.3) 3, 5, 7, 10, 16	7.3 (7.8) 1, 2, 5, 9, 22	9.0 (10.0) 1, 3, 6, 12, 28	8.6 (8.8) 1, 3, 6, 11, 26	8.6 (3.97) 4, 6, 8, 11, 16	6.5 (6.0) 1, 2, 5, 9, 18	8.1 (8.1) 1, 3, 6, 11, 24
-0.50	12.99 (14.9) 1, 4, 8, 17, 41	10.1 (5.7) 4, 6, 9, 13, 21	10.6 (12.0) 1, 3, 7, 13, 33	12.97 (14.9) 1, 4, 8, 17, 41	12.7 (13.4) 1, 4, 8, 17, 38	10.8 (5.1) 4, 7, 10, 13, 20	9.5 (9.5) 1, 3, 7, 12, 28	12.1 (12.7) 1, 4, 8, 16, 37
-0.25	17.2 (19.9) 1, 5, 11, 22, 54	11.8 (6.9) 4, 7, 10, 15, 25	13.6 (15.8) 1, 4, 9, 17, 43	17.3 (20.4) 1, 5, 11, 22, 55	17.2 (18.3) 1, 5, 11, 23, 53	12.7 (6.1) 5, 8, 12, 16, 24	12.4 (12.8) 1, 4, 8, 16, 37	16.5 (17.5) 1, 5, 11, 22, 51
0	18.4 (21.0) 1, 5, 12, 24, 58	12.4 (7.6) 4, 7, 11, 15, 26	14.6 (16.8) 1, 4, 9, 19, 46	18.3 (21.8) 1, 5, 11, 24, 58	19.1 (20.1) 1, 5, 13, 26, 59	13.4 (6.6) 5, 9, 12, 17, 26	13.7 (14.3) 1, 4, 9, 18, 41	18.2 (19.5) 1, 5, 12, 24, 56
0.25	15.6 (18.2) 1, 4, 10, 20, 49	11.4 (6.7) 4, 7, 10, 14, 24	12.6 (14.8) 1, 4, 8, 16, 39	15.5 (18.1) 1, 4, 10, 20, 49	16.4 (17.6) 1, 5, 11, 22, 50	12.4 (6.0) 5, 8, 11, 15, 24	12.0 (12.2) 1, 4, 8, 16, 36	15.5 (16.3) 1, 5, 10, 21, 47
0.5	11.2 (12.8) 1, 3, 7, 15, 35	9.5 (5.4) 3, 6, 8, 12, 20	9.3 (10.3) 1, 3, 6, 12, 28	11.1 (12.5) 1, 3, 7, 14, 35	11.6(12.1) 1, 3, 8, 15, 35	10.4 (4.98) 4, 7, 10, 13, 20	8.8 (8.6) 1, 3, 6, 12, 25	11.0 (11.3) 1, 3, 7, 15, 33
0.75	7.6 (8.1) 1, 2, 5, 10, 23	7.5 (4.1)	6.4 (6.5)	7.5 (8.2)	7.7 (7.8) 1, 2, 5, 10, 23	8.3 (3.9) 3, 6, 8, 10, 15	6.1 (5.6)	7.5 (7.5)

		3, 5, 7, 9, 15	1, 2, 4, 8, 19	1, 2, 5, 10, 23			1, 2, 4, 8, 17	1, 2, 5, 10, 22
1.0	5.2 (5.3) 1, 2, 3, 7, 15	5.9 (3.0) 2, 4, 5, 7, 12	4.4 (4.1) 1, 2, 3, 6, 12	5.1 (5.1) 1, 2, 3, 7, 15	5.3(5.0) 1, 2, 4, 7, 15	6.6 (2.9) 3, 4, 6, 8, 12	4.3 (3.7) 1, 2, 3, 6, 11	5.0 (4.7) 1, 2, 4, 7, 14
1.5	2.7 (2.3) 1, 1, 2, 3, 7	3.8 (1.7) 2, 3, 4, 5, 7	2.4 (1.8) 1, 1, 2, 3, 6	2.7 (2.2) 1, 1, 2, 3, 7	2.8(2.3) 1, 1, 2, 4, 7	4.3 (1.7) 2, 3, 4, 5, 17	2.4 (1.7) 1, 1, 2, 3, 6	2.7 (2.1) 1, 1, 2, 3, 7
2.0	1.7 (1.2) 1, 1, 1, 2, 4	2.8 (1.1) 1, 2, 3, 3, 5	1.7 (0.96) 1, 1, 1, 2, 3	1.7 (1.1) 1, 1, 1, 2, 4	1.7 (1.2) 1, 1, 1, 2, 4	3.1 (1.1) 2, 2, 3, 4, 5	1.6 (0.9) 1, 1, 1, 2, 3	1.7 (1.1) 1, 1, 1, 2, 4
3.0	1.1 (0.4) 1, 1, 1, 1, 2	1.8 (0.6) 1, 1, 2, 2, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	2.2 (0.4) 2, 2, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2
δ =2.00								
-3.0	1.2 (0.5) 1, 1, 1, 1, 2	2.1 (0.7) 1, 2, 2, 2, 3	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.3 (0.6) 2, 2, 2, 3, 3	1.0 (0.0) 1, 1, 1, 1, 1	1.2 (0.5) 1, 1, 1, 1, 2
-2.0	1.98 (1.4) 1, 1, 1, 2, 5	3.1 (1.2) 2, 2, 3, 4, 5	1.9 (1.2) 1, 1, 2, 2, 4	1.98 (1.4) 1, 1, 1, 2, 5	1.9 (1.4) 1, 1, 1, 2, 5	3.4 (1.2) 2, 2, 3, 4, 6	1.8 (1.1) 1, 1, 1, 2, 4	1.9 (1.3) 1, 1, 1, 2, 4
-1.5	3.0 (2.6) 1, 1, 2, 4, 8	4.2 (1.8) 2, 3, 4, 5, 7	2.7 (2.0) 1, 1, 2, 3, 7	3.0 (2.6) 1, 1, 2, 4, 8	3.0 (2.5) 1, 1, 2, 4, 8	4.5 (1.8) 2, 3, 4, 5, 8	2.6 (1.8) 1, 1, 2, 3, 6	2.9 (2.3) 1, 1, 2, 4, 7
-1.0	5.2 (5.1) 1, 2, 4, 7, 15	5.9 (2.7) 2, 4, 5, 7, 11	4.4 (3.9) 1, 2, 3, 6, 12	5.2 (5.2) 1, 2, 4, 7, 15	5.1 (4.8) 1, 2, 4, 7, 15	6.3 (2.7) 3, 4, 6, 8, 11	4.1 (3.3) 1, 2, 3, 5, 11	4.9 (4.5) 1, 2, 3, 7, 14
-0.75	7.0 (7.3) 1, 2, 5, 9, 21	6.9 (3.3) 3, 5, 6, 9, 13	5.7 (5.4) 1, 2, 4, 7, 16	6.9 (7.2) 1, 2, 5, 9, 21	6.8 (6.6) 1, 2, 5, 9, 20	7.5 (3.2) 3, 5, 7, 9, 13	5.2 (4.6) 1, 2, 4, 7, 14	6.5 (6.2) 1, 2, 5, 9, 19
-0.50	9.1 (9.7) 1, 3, 6, 12, 28	7.9 (3.99) 3, 5, 7, 10, 15	7.2 (7.2) 1, 3, 5, 9, 21	8.99 (9.6) 1, 3, 6, 12, 27	9.0 (9.1) 1, 3, 6, 12, 27	8.6 (3.8) 4, 6, 8, 11, 16	6.6 (6.1) 1, 3, 5, 9, 18	8.6 (8.5) 1, 3, 6, 11, 25
-0.25	10.8 (11.7) 1, 3, 7, 14, 33	8.8 (4.5) 4, 6, 8, 11, 18	8.4 (8.8) 1, 3, 6, 11, 25	10.7 (11.5) 1, 3, 7, 14, 33	11.1 (11.2) 1, 3, 8, 15, 33	9.6 (4.2) 4, 7, 9, 12, 17	7.9 (7.4) 1, 3, 6, 10, 22	10.4 (10.5) 1, 3, 7, 14, 31
0	11.3(12.2) 1, 3, 7, 15, 34	9.0 (4.6) 4, 6, 8, 11, 18	8.7 (8.99) 1, 3, 6, 11, 26	11.1 (11.99) 1, 3, 7, 15, 34	11.8 (12.1) 1, 4, 8, 16, 35	9.9 (4.4) 4, 7, 9, 12, 18	8.3 (7.8) 1, 3, 6, 11, 23	11.0 (11.1) 1, 3, 8, 15, 33
0.25	10.2(10.9) 1, 3, 7, 13, 31	8.6 (4.4) 3, 5, 8, 11, 17	7.99 (8.1) 1, 3, 5, 10, 23	9.9 (10.6) 1, 3, 6, 13, 30	10.7(10.8) 1, 3, 7, 14, 32	9.4 (4.2) 4, 6, 9, 12, 17	7.6 (7.1) 1, 3, 5, 10, 22	10.0 (10.0) 1, 3, 7, 13, 30
0.5	8.1 (8.6) 1, 3, 5, 11, 24	7.6 (3.8) 3, 5, 7, 9, 15	6.6 (6.5) 1, 2, 5, 8, 19	7.98 (8.4) 1, 3, 5, 10, 24	8.5 (8.5) 1, 3, 6, 11, 25	8.4 (3.7) 4, 6, 8, 10, 15	6.4 (5.8) 1, 2, 5, 8, 18	8.05 (7.9) 1, 3, 6, 11, 24
0.75	6.1 (6.2) 1, 2, 4, 8, 18	6.5 (3.2) 3, 4, 6, 8, 12	5.1 (4.7) 1, 2, 4, 7, 14	6.1 (6.1) 1, 2, 4, 8, 18	6.4(6.1) 1, 2, 4, 8, 19	7.2 (3.2) 3, 5, 7, 9, 13	4.9 (4.2) 1, 2, 4, 6, 13	6.1 (5.8) 1, 2, 4, 8, 18
1.0	4.6 (4.5) 1, 2, 3, 6, 13	5.4 (2.6) 2, 4, 5, 7, 10	3.9 (3.4) 1, 2, 3, 5, 11	4.5 (4.3) 1, 2, 3, 6, 13	4.8(4.5) 1, 2, 3, 6, 13	6.1 (2.6) 3, 4, 6, 8, 11	3.8 (3.1) 1, 2, 3, 5, 10	4.6 (4.1) 1, 2, 3, 6, 13
1.5	2.7 (2.3) 1, 1, 2, 3, 7	3.9 (1.7) 2, 3, 4, 5, 7	2.5 (1.8) 1, 1, 2, 3, 6	2.7 (2.2) 1, 1, 2, 3, 7	2.8 (2.3) 1, 1, 2, 4, 7	4.3 (1.7) 2, 3, 4, 5, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.7 (2.2) 1, 1, 2, 3, 7
2.0	1.8 (1.3) 1, 1, 1, 2, 4	2.9 (1.2) 1, 2, 3, 4, 5	1.8 (1.1) 1, 1, 1, 2, 4	1.8 (1.3) 1, 1, 1, 2, 4	1.9(1.3) 1, 1, 1, 2, 4	3.3 (1.2) 2, 2, 3, 4, 5	1.7 (1.0) 1, 1, 1, 2, 4	1.8 (1.2) 1, 1, 1, 2, 4
3.0	1.2(0.5) 1, 1, 1, 1, 2	1.9 (0.7) 1, 1, 2, 2, 3	1.2 (0.4) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	1.2(0.5) 1, 1, 1, 1, 2	2.3 (0.5) 2, 2, 2, 3, 3	1.2 (0.4) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2

Table 6: Details of the CUSUM Statistics Considered for a Comparative Study

Types of CUSUM Chart	Pivot Statistic	Statistic to be Plotted against specified UCL
Conventional X-bar (with corrections for unknown parameters)	$C_i^{C+} = \sqrt{\frac{mn(m+n-2)}{(m+n)}} \frac{(\overline{y_\iota} - \bar{x}))}{\sqrt{(m-1)S_x^2 + (n-1)S_{yi}^2}}$; $C_i^{C-} = -C_i^{C+}$; $i = 1,2,\ldots$ Where $\bar{x}$ and $\overline{y_\iota}$ are the means of reference sample and i-th test sample; S_x^2 and S_{yi}^2 are the variances of reference sample and i-th test sample respectively.	Let $\mathbb{C}_o^{C+} = \mathbb{C}_o^{C-} = 0$; Compute, $\mathbb{C}_i^{C+} = \max\{0, \mathbb{C}_{i-1}^{C+} + C_i^{C+} - k\}$; $\mathbb{C}_i^{C-} = \max\{0, \mathbb{C}_{i-1}^{C-} + C_i^{C-} - k\}$ and plot $\mathbb{C}_i^{C} = \max\{\mathbb{C}_i^{C+}, \mathbb{C}_i^{C-}\}, i = 1,2,\ldots$
Wilcoxon Rank-sum	$C_i^{W+} = S_{1i}$ and $C_i^{W-} = -C_i^{W+}$; $i = 1,2,\ldots$ Where S_{1i} is the standardized sum of ranks of i-th test sample in the combined $(m+n)$ samples taken together with reference samples, as defined as in Section 2.	Let $\mathbb{C}_o^{W+} = \mathbb{C}_o^{W-} = 0$; Compute, $\mathbb{C}_i^{W+} = \max\{0, \mathbb{C}_{i-1}^{W+} + C_i^{W+} - k\}$; $\mathbb{C}_i^{W-} = \max\{0, \mathbb{C}_{i-1}^{W-} + C_i^{W-} - k\}$ and plot $\mathbb{C}_i^{W} = \max\{\mathbb{C}_i^{W+}, \mathbb{C}_i^{W-}\}, i = 1,2,\ldots$

Exceedance	$C_i^{E+} = \left(E_i - \frac{n}{2}\right);\ C_i^{E-} = -C_i^{E+};\ i = 1,2,\ldots$ Where E_i is the number of i th test sample exceeding Median of reference Sample	Let $\mathbb{C}_o^{E+} = \mathbb{C}_o^{E-} = 0$; Compute, $\mathbb{C}_i^{E+} = \max\{0, \mathbb{C}_{i-1}^{E+} + C_i^{E+} - k\}; \mathbb{C}_i^{W-} = \max\{0, \mathbb{C}_{i-1}^{W-} + C_i^{W-} - k\}$ and plot $\mathbb{C}_i^{W} = \max\{\mathbb{C}_i^{W+}, \mathbb{C}_i^{W-}\}, i = 1,2,\ldots$
Kolmogorov-Smironov	$C_i^{KS+} = \frac{D_{m,n,i} - \mu_D}{\sigma_D};\ C_i^{KS-} = \frac{D'_{m,n,i} - \mu_D}{\sigma_D};\ \ i = 1,2,\ldots$ where $D_{m,n,i} = \max_x(F_m(x) - G_{n,i}(x))$; $D'_{m,n,i} = \max_x(G_{n,i}(x) - F_m(x))$; evaluated over all observed points. Further, $\mu_D = E\left(D_{m,n,i} \middle| F = G\right) = E(D'_{m,n,i}\|F = G)$ and $\sigma_D = SD\left(D_{m,n,i} \middle| F = G\right) = SD(D'_{m,n,i}\|F = G)$; which can be evaluated via bootstrapping.	Let $\mathbb{C}_o^{KS+} = \mathbb{C}_o^{KS-} = 0$; Compute, $\mathbb{C}_i^{KS+} = \max\left\{0, \mathbb{C}_{i-1}^{KS+} + C_i^{KS+} - k\right\}; \mathbb{C}_i^{KS-} = \max\left\{0, \mathbb{C}_{i-1}^{KS-} + C_i^{KS-} - k\right\}$ and plot $\mathbb{C}_i^{KS} = \max\left\{\mathbb{C}_i^{KS+}, \mathbb{C}_i^{KS-}\right\}, i = 1,2,\ldots$
Cramér–von Mises	$C_i^{CvM} = \frac{T_{Ui} - \mu_D}{\sigma_D};\ i = 1,2,\ldots$ where $T_{Ui} = \frac{T_{Ui}}{mn(m+n)} - \frac{4mn-1}{6(m+n)}$ and W_{Ui} is defined as$W_{Ui} = m\sum_{k=1}^{m}(r_{Uk} - i)^2 + n\sum_{j=1}^{n}(s_{Uj} - j)^2$; $r_{Uk}{}'s$ being ordered rank	Let $\mathbb{C}_o^{CvM} = 0$; Compute and plot $\mathbb{C}_i^{CvM} = \max\left\{0, \mathbb{C}_{i-1}^{CvM} + C_i^{CvM} - k\right\}$;

Table-7. Comparisons of various CUSUM Procedures under normal (θ, δ) distribution for $m=100$, $n=5$, $k=0$ with ARL_0 = 500.

Shift θ	Shift δ	CUSUM X-BAR	CUSUM RANK-SUM	CUSUM EXCEDANCE	CUSUM CVM	CUSUM KS	CUSUM LEPAGE
0.00	1.0	501.47 (522.54) 110, 191, 315, 592, 1579	496.02 (512.03) 110, 189, 312, 594, 1505	501.42 (506.19) 112, 191, 316, 600, 1565	496.71(766.89) 41, 106, 225, 527, 1955	500.76 (595.15) 100, 179, 299, 568, 1586	498.98 (851.2) 35, 89, 198, 485, 2101
0.25	1.0	129.08 (149.71) 54, 73, 96, 133, 281	135.22 (179.53) 55, 74, 96, 135, 300	186.10(248.78) 63, 89, 121, 182, 484	157.50 (322.46) 20, 41, 73, 150, 527	127.00 (162.74) 50, 69, 92, 131, 287	170.8 (384.4) 18, 40, 73, 151, 578
0.50	1.0	51.37 (14.26) 34, 42, 49, 58, 77	52.53(14.49) 35, 43, 50, 59, 79	67.75(23.57) 43, 53, 63, 76, 107	29.37 (27.49) 9, 16, 23, 35, 69	48.43 (14.87) 31, 38, 45, 55, 75	32.3 (38.8) 8, 16, 24, 38, 80
1.00	1.0	25.06 (3.87) 19, 22, 25, 27, 32	26.59(3.40) 22, 24, 26, 29, 33	35.67 (4.61) 29, 33, 35, 38, 44	7.40 (2.94) 4, 5, 7, 9, 13	23.38(3.64) 18, 21, 23, 26, 30	7.2 (3.4) 3, 5, 7, 9, 14
0.00	1.5	375.15 (303.44) 99, 172, 280, 470, 977	503.86(427.17) 129, 222, 360, 637, 1383	648.43(590.56) 161, 273, 442, 808, 1823	43.22(27.02) 14, 26, 37, 53, 92	190.77 (97.29) 83, 124, 168, 232, 370	21.7 (13.1) 8, 13, 19, 27, 46
0.25	1.5	121.28 (98.28) 51, 72, 95, 134, 269	160.68(152.96) 66, 92, 122, 172, 362	260.12(302.64) 89, 128, 175, 269, 691	35.02 (20.88) 12, 21, 30, 43, 73	101.06 (51.02) 50, 69, 88, 117, 195	18.9 (10.96) 7, 11, 17, 23, 39
0.50	1.5	52.50 (16.48) 33, 41, 49, 60, 82	66.55 (22.75) 43, 53, 62, 75, 103	100.79(40.38) 61, 77, 92, 115, 168	21.77 (11.96) 8, 14, 19, 27, 44	53.47 (15.92) 34, 43, 50, 61, 82	13.9 (7.5) 5, 9, 12, 17, 28
1.00	1.5	25.75 (4.61) 19, 23, 25, 28, 34	33.11(5.19) 26, 29, 32, 36, 43	49.48(8.11) 39, 44, 48, 54, 64	9.27(3.97) 4, 6, 9, 11, 17	27.82(4.78) 21, 24, 27, 31, 36	7.0 (3.2) 3, 5, 6, 9, 13
0.00	2.0	291.60 (206.40) 85, 148, 233, 371, 698	515.76(388.78) 142, 250, 395, 660, 1288	755.71 (635.91) 194, 339, 541, 953, 2009	21.79 (9.88) 9, 15, 20, 27, 40	101.33(29.28) 60, 81, 98, 118, 154	9.8 (4.3) 4, 7, 9, 12, 18
0.25	2.0	120.03 (92.40) 47, 70, 95, 137, 273	187.33(157.64) 77, 109, 145, 207, 418	329.15 (324.12) 116, 167, 230, 353, 888	19.97 (9.00) 8, 14, 18, 25, 37	76.02(24.41) 45, 59, 72, 89, 122	9.4 (4.1) 4, 6, 9, 12, 17
0.50	2.0	53.77 (18.87) 31, 41, 50, 63, 88	81.68(27.05) 50, 64, 76, 93, 132	131.06 (56.56) 77, 98, 120, 149, 219	16.59(7.41) 7, 11, 15, 21, 30	50.44(13.06) 33, 41, 48, 57, 75	8.4 (3.7) 4, 6, 8, 10, 15
1.00	2.0	26.52 (5.66) 18, 23, 26, 30, 37	40.62(7.33) 30, 35, 40, 45, 54	64.19(12.05) 48, 56, 63, 71, 86	9.90 (4.20) 4, 7, 9, 12, 18	30.06(5.45) 22, 26, 29, 33, 40	6.1 (2.6) 3, 4, 6, 8, 11

Table-8. Comparisons of various CUSUM Procedures under Lognormal (θ, δ) distribution for $m=100$, $n=5$, $k=0$ with ARL_0 = 500.

Shift θ	Shift δ	CUSUM X-BAR	CUSUM RANK-SUM	CUSUM EXCEDANCE	CUSUM CVM	CUSUM KS	CUSUM LEPAGE
0.00	1.0	513.42 (548.03) 108, 197, 321, 604, 1612	499.26 (520.14) 108, 188, 314, 597, 1574	500.45 (528.60) 112, 191, 316, 587, 1553	484.12 (750.08) 43, 105, 224, 517, 1835	504.53 (597.16) 100, 177, 299, 581, 1604	494.9 (847.9) 34, 89, 194, 480, 2082
0.25	1.0	221.27(402.39) 46, 73, 111, 193, 743	133.63 (165.45) 55, 75, 97, 136, 298	179.67 (225.91) 64, 89, 119, 180, 476	160.04 (328.36) 20, 40, 72, 146, 558	129.36 (190.23) 49, 69, 92, 129, 290	170.67 (378.71) 18, 40, 74, 154, 576
0.50	1.0	61.32 (71.50) 26, 37, 49, 68, 127	52.58 (14.47) 35, 43, 50, 59, 79	68.51(33.08) 43, 53, 63, 76, 110	28.96(27.56) 9, 15, 23, 34, 66	48.76(15.72) 31, 39, 46, 55 76	32.18 (35.43) 8, 16, 24, 38, 80
1.00	1.0	22.11 (9.06) 13, 17, 20, 25, 36	26.62(3.40) 22, 24, 26, 29, 33	35.67(4.58) 29, 33, 35, 38, 44	7.36(2.86) 4, 5, 7, 9, 13	23.41(3.62) 18, 21, 23, 25, 30	7.22 (3.38) 3, 5, 7, 9, 14
0.00	1.5	66.48 (73.25) 30, 42, 56, 75, 130	501.93 (421.41) 130, 221, 357, 641 1344	646.66(587.62) 159, 271, 448, 800, 1827	43.52(27.29) 14, 25, 37, 54, 93	190.41 (95.14) 83, 125, 170, 231, 365	21.70(12.96) 7, 13, 19, 27, 46
0.25	1.5	38.48 (16.47) 21, 28, 35, 45, 67	156.27 (143.58) 67 92 120 170 344	258.05 (281.23) 92, 129, 176, 267, 705	34.91 (20.82) 12, 21, 30, 43, 74	100.51(50.78) 51 68 88 116 193	18.88 (11.11) 7, 11, 16, 23, 39
0.50	1.5	26.40 (9.17) 15, 20, 25, 30, 43	66.41(20.23) 43, 53, 62, 75, 103	99.66(38.63) 60, 77, 91, 113, 163	21.89 (11.84) 8, 14, 19, 27, 45	53.23(15.46) 34, 42, 50, 61, 82	13.87 (7.55) 5, 9, 12, 17, 28

1.00	1.5	15.62 (4.28) 10, 13, 15, 18, 23	33.11(5.12) 26, 29, 33, 36, 42	49.56 (8.19) 39, 44, 49, 54, 64	9.23(4.01) 4, 6, 9, 11, 17	27.77(4.88) 21, 24, 27, 31, 36	6.97 (3.24) 3, 5, 6, 9, 13
0.00	2.0	28.59 (9.23) 17, 22, 27, 33, 45	509.40 (383.66) 144, 245, 388, 654, 1277	757.51(636.90) 197, 339, 541, 948, 2077	21.72 (9.82) 9, 15, 20, 27, 40	101.18(28.52) 61, 81, 98, 118, 153	9.86 (4.34) 4, 7, 9, 12, 18
0.25	2.0	22.33(6.48) 14, 18, 21, 26, 34	188.17(155.41) 76, 109, 146, 208, 427	328.48(334.39) 116, 166, 231, 351, 867	19.97 (9.06) 8, 13, 19, 25, 37	75.27 (23.89) 44, 58, 71, 88, 120	9.42(4.15) 4, 6, 9, 12, 17
0.50	2.0	18.15 (4.71) 12, 15, 17, 21, 27	81.97 (27.81) 50, 64, 77, 94, 131	130.64(51.83) 77, 99, 120, 149, 218	16.55 (7.39) 7, 11, 15, 20, 30	50.34(12.91) 33, 41, 48, 57, 74	8.43(3.71) 4, 6, 8, 10, 15
1.00	2.0	13.10 (2.84) 9, 11, 13, 15, 18	40.51 (7.29) 30, 35, 40, 45, 54	64.30 (11.97) 48, 56, 63, 71, 86	9.90 (4.28) 4, 7, 9, 12, 18	30.02 (5.42) 22, 26, 29, 33, 40	6.10(2.61) 3, 4, 6, 8, 11

Table-9. Comparisons of various CUSUM Procedures under Laplace (θ, δ) distribution for $m=100$, $n=5$, $k=0$ with ARL_0 = 500.

Shift		CUSUM X-BAR	CUSUM RANK-SUM	CUSUM EXCEDANCE	CUSUM CVM	CUSUM KS	CUSUM LEPAGE
θ	δ						
0.00	1.0	498.70 (527.50) 112, 190, 314, 590, 1524	502.47 (531.23) 110, 190, 310, 590, 1553	502.15 (512.35) 112, 191, 320, 600, 1537	489.33 (758.72) 42, 106, 226, 520, 1870	505.02 (600.01) 101, 179, 302, 572, 1629	502.2 (862.8) 34, 89, 195, 483, 2118
0.25	1.0	221.07 (325.80) 65, 94, 132, 211, 657	167.04 (234.46) 60, 83, 111, 164, 412	145.74 (166.16) 64, 84, 107, 147, 327	187.77(413.45) 21, 44, 79, 165, 643	151.13(231.14) 51, 72, 99, 148, 386	266.7 (572.7) 21, 48, 97, 230, 1008
0.50	1.0	77.01 (53.04) 42, 55, 67, 87, 136	61.49 (23.59) 39, 48, 57, 69, 98	64.17 (17.14) 45, 53, 61,71, 93	36.42 (50.01) 10, 17, 26, 41, 89	53.17 (22.50) 32, 41, 49, 60, 87	60.2 (138.9) 10, 20, 33, 59, 171
1.00	1.0	35.59(7.72) 25, 30, 35, 40, 49	31.18(4.92) 24, 28, 31, 34, 40	38.37(4.36) 32, 35, 38, 41, 46	8.99(3.87) 4, 6, 8, 11, 16	25.35(4.53) 19, 22, 25, 28, 34	10.1 (5.99) 4, 6, 9, 12, 21
0.00	1.5	378.31(314.44) 100, 172, 277, 473, 1007	502.39 (446.48) 126, 217, 350, 631, 1410	644.40 (579.70) 153, 265, 441, 811, 1850	60.14(46.15) 17, 32, 48, 74, 140	247.94 (179.65) 90, 142, 203, 296, 543	34.1 (28.5) 10, 18, 27, 41, 80
0.25	1.5	193.56(204.53) 61, 91, 129, 206, 559	197.92 (223.03) 68, 99, 135, 204, 526	205.27 (204.95) 87, 118, 153, 215, 472	47.50 (35.85) 14, 26, 39, 58, 110	129.54(101.24) 54, 76, 102, 146, 294	28.8 (23.4) 9, 16, 23, 35, 67
0.50	1.5	77.52(41.52) 41, 55, 68, 89, 141	75.90 (33.13) 46, 58, 69, 86, 125	88.70 (24.74) 59, 72, 84, 99, 135	27.38(17.66) 9, 16, 23, 34, 60	60.65 (22.78) 36, 46, 56, 69, 100	19.6 (13.8) 6, 11, 16, 24, 43
1.00	1.5	36.36 (9.03) 24, 30, 35, 41, 53	37.81 (6.82) 28, 33, 37, 42, 50	50.03(7.11) 40, 45, 49, 54, 63	10.99(4.96) 5, 7, 10, 13, 20	30.22(6.03) 22, 26, 29, 34, 41	8.8 (4.5) 4, 6, 8, 11, 17
0.00	2.0	299.82 (216.99) 87, 151, 236, 381, 727	505.71(402.97) 135, 234, 375, 640, 1323	738.46(609.45) 191, 332, 539, 935, 1982	30.41 (16.40) 11, 19, 27, 38, 61	136.53 (53.22) 70, 99, 127, 162, 235	14.5 (7.5) 6, 9, 13, 18, 28
0.25	2.0	173.01(149.57) 57, 87, 126, 198, 459	221.39 (220.80) 79, 114, 157, 236, 595	263.76 (261.24) 108, 151, 199, 282, 605	27.44 (14.30) 10, 17, 25, 34, 55	98.05(43.30) 50, 68, 88, 117, 180	13.5 (6.9) 5, 9, 12, 17, 26
0.50	2.0	79.31(47.15) 39, 53, 69, 91, 151	90.66 (41.65) 52, 67, 83, 104, 155	114.19(36.21) 73, 91, 107, 129, 176	20.89(10.40) 8, 13, 19, 26, 40	59.81(20.18) 36, 46, 56, 69, 96	11.5 (5.7) 5, 7, 10, 14, 22
1.00	2.0	37.48(10.42) 24, 30, 36, 43, 57	44.73(9.10) 32, 38, 44, 50, 62	62.14(10.00) 36, 48, 55, 61, 68, 80	11.59 (5.09) 5, 8, 11, 14, 21	32.84 (6.90) 23, 28, 32, 37, 45	7.4 (3.3) 3, 5, 7, 9, 13

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