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INVERSE BOUNDARY PROBLEM FOR ESTIMATING THE BOUNDARY CONDITIONS OF A RADIANT ENCLOSURE USING THE MEASURMENT OF TEMPERATURE ON A SOLID OBJECT WITHIN THE ENCLOSURE

Dehghani M., Sarvari S.M.H.*, and Ajam H.
*Author for correspondence
Department of Mechanical Engineering,
University of Sistan & Baluchestan,
Zahedan, 98135-161,
Iran,
E-mail: sarvari@hamoon.usb.ac.ir

ABSTRACT

In this paper, we present an inverse analysis to estimate the thermal boundary conditions over a two-dimensional radiant enclosure from the knowledge of the measured temperatures for some points on a solid object within the enclosure. The conduction heat transfer in the solid object and the radiative heat transfer between the surface elements of the enclosure are formulated by the finite volume method and the net radiative method, respectively. The resultant set of nonlinear equations, including the energy equation for the solid object, the energy conservation along the boundary surface of the solid object, and the radiative exchange between surface elements of the radiant enclosure, are solved by the Newton's method. The inverse method for estimation of boundary conditions over the radiant enclosure surface is solved by the conjugate gradient method through minimizing an objective function which is expressed as the sum of square residuals between measured and estimated temperatures for some sampling points on the solid object. The performance of the present technique of inverse analysis is evaluated by several numerical experiments, and the effects of some parameters, such as the number and the positions of sensors, and the measurement errors over the accuracy of the inverse solution are investigated. The results show that the temperature profile over the wall of the radiant enclosure can be recovered accurately, even for sharp gradient profiles and noisy input data.

INTRODUCTION

Inverse heat transfer problems are concerned with the determination of the thermal properties, the initial condition, the boundary conditions, and the strength of heat source from the knowledge of the temperature or heat flux measurements taken at the interior or the boundary points of the domain. They have been widely used in many design and manufacturing applications, especially when direct measurements of the surface condition are not possible. Many studies of the inverse problems with conduction and radiation have been reported [1-5]. Inverse problems have also received much attention in recent years for the cases with multimode heat transfer [6,7]. A comprehensive study of inverse heat transfer problems has been reported in [8].

NOMENCLATURE

d	[-]	Direction of descent
E	[-]	Error
f	[-]	Objective function
F_{j-k}	[-]	Geometric configuration factor
M	[-]	Number of elements over the bottom wall of the radiative enclosure
N	[-]	Number of sampling points on the solid object
q	[W/m^2]	Heat flux
R	[-]	Number of radiative surfaces
S	[-]	Sensitivity coefficient
T	[K]	Temperature
V	[-]	Number of finite volumes over the solid object
Y	[K]	Measured temperature of sampling points on the solid object
Z	[-]	Number of surface elements over the boundary surface of the solid object

Special characters

β	[-]	Search step size
ε	[-]	Emissivity
φ	[K]	Unknown elemental temperature over the surface of the radiant enclosure
γ	[-]	Conjugation coefficient
κ	[W/mK]	thermal conductivity
σ	[W/m^2 K^4]	Stefan-Boltzman constant
Θ	[-]	Normal distributed random variable
μ	[-]	Standard deviation of the measurement errors

Subscripts

J,k	[-]	Radiative element in the enclosure
m	[-]	Element on the bottom wall of the radiant enclosure
n	[-]	Sampling point on the solid object
z	[-]	Element number on the boundary surface of the solid object

Superscripts

v	[-]	Iteration number

In the present work, we deal with the inverse problem of estimating the boundary conditions over the boundary surface of a radiant enclosure by measuring the temperatures of some points on a solid object within the enclosure. The applications may be seen in manufacturing, thermal treatment and food industries where we are interested to know the strength of radiant heaters located on the wall surface of the radiant oven by the measurement of temperatures over some points of product surface.

Heat is transferred in the solid object by conduction, and the dominant mode of heat transfer in the enclosure is radiation. The solid object is subdivided into control volumes and the boundary surface of the solid object and the radiant enclosure are subdivided into surface elements.

For the direct problem, the conduction heat transfer in the solid object is formulated by the finite volume method, and the radiation heat transfer between surface elements are formulated by the net radiation method. The complete set of nonlinear equations is then solved by Newton's method. For the inverse problem, the temperature distribution over some parts of boundary surface of the radiant enclosure is regarded as unknown, and the temperatures for some sampling points over the solid object are considered to be available by the measurement. The conjugate gradient method is used for minimization of the objective function which is defined as the sum of square deviations between the measured and estimated values of temperatures on the solid object.

Finally, the performance and the accuracy of the present method for recovering the boundary temperature distribution over the radiant enclosure from the knowledge of measured temperatures of solid object is examined by considering some examples with different temperature distributions over the radiant enclosure. The effects of the location of sensors and noisy input data on the accuracy of the inverse solution are investigated by several numerical experiments.

DESCRIPTION OF PROBLEM

Consider a two-dimensional square enclosure A, and the square solid object B within it, as depicted in Figure 1.

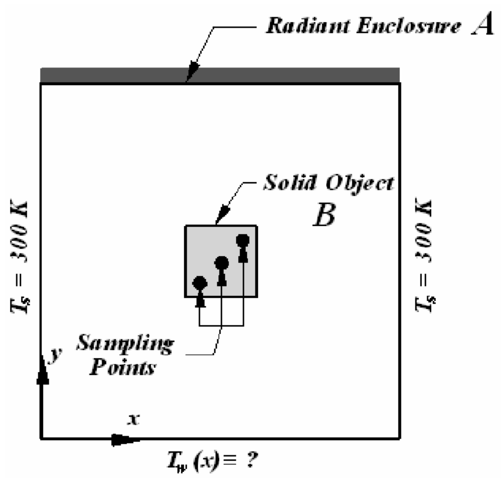


Figure 1 Two-dimensional square enclosure A, and the square solid object B within it

All the internal walls of the enclosure A, and the boundary surfaces of the solid object B are diffuse-gray. The enclosure filled with a transparent medium. All the thermal properties are assumed to be constant. Heat is transferred by radiation throughout the enclosure A, and is transferred in the solid object B with conduction. The side walls of the enclosure are at constant temperature of $T_s = 300K$ and the top wall of the enclosure is kept insulated. No boundary condition is specified over the bottom wall of the enclosure. The aim of the inverse problem is to find the boundary conditions over the bottom wall of the enclosure from the knowledge of measured temperatures for some sampling points on the solid object B.

DIRECT PROBLEM

The steady state conduction heat transfer in the square solid object B is governed by the Fourier's law of conduction as follows:

$$\nabla \cdot (\kappa \nabla T) = 0 \tag{1}$$

Equation (1) is solved by the finite volume method. In this method, the domain of interest is subdivided into V finite volumes. Writing energy balance for all finite volumes leads to a set of V linear algebraic equations, which can be solved by conventional solvers such as LU decomposition approach, provided that the boundary conditions over all boundary surfaces are known. However, here no boundary conditions are known a priori. Hence, the number of unknowns is $V + Z$, where Z is the number of surface elements over the boundary surface of the solid object B.

The radiative exchange for surface elements can be described by the following equation [9]

$$\sum_{j=1}^{R} \left(\frac{\delta_{kj}}{\varepsilon_j} - \frac{1-\varepsilon_j}{\varepsilon_j} F_{k-j} \right) q_j = \sum_{j=1}^{R} \left(\delta_{kj} - F_{k-j} \right) \sigma T_j^4 \quad , \ 1 \le k \le R \tag{2}$$

where R is the total number of radiative surface elements in the enclosure and F_{k-j} is the geometric configuration factor which can be calculated by the Hottel's crossed-string method [10]. Here, δ_{kj} is the Kronicker delta defined by

$$\delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \ne j \end{cases} \tag{3}$$

The boundary conditions over the boundary surfaces of the enclosure may be written as follows

$$q(x,1) = 0.0 \tag{4a}$$

$$T(0,y) = T(1,y) = T_s \tag{4b}$$

$$T(x,0) = \sum_{m=1}^{M} \varphi_m \, \delta_d (x - x_m) \tag{4c}$$

where δ_d is the Dirac delta function, x_m is the mean location of the m-th element on the wall surface and φ_m's are the known coefficients.

If the elemental temperatures over the boundary surfaces of the solid object B are specified, then the set of R equations (2) can be solved to calculate the unknown elemental temperatures or heat fluxes. However, since no boundary condition is known over the boundary surfaces of the solid object B, the number of unknowns is increased to $R+Z$.

As discussed above, we conclude that the conduction heat transfer equation through the solid object medium and the radiative transfer equation in the enclosure cannot be solved independently. Considering the interface condition over all the boundary surface elements of the solid object made by the following equation

$$\kappa\frac{\partial T_z}{\partial n_z}+q_z=0\ ,\ z=1,\ldots,Z \tag{5}$$

where n_z is the normal direction to the boundary surface element z, we now have a set of total $V+R+Z$ nonlinear equations with the same number of unknowns. Because of nonlinearity due to the fourth power of temperatures in the radiative transfer equation, the conventional solvers of linear set of algebraic equations cannot be used for solving the set of equations. Hence, the set of equations must be solved through an iterative approach, such as Newton's method. The Newton's method for solving the set of nonlinear equations is described in detail in [11], and will not be repeated.

INVERSE PROBLEM

For the inverse problem, the temperature distribution over the bottom wall of the radiant enclosure is regarded as unknown, and the measured values of temperatures for some sampling points are available for the analysis. The objective function is expressed by the sum of square residuals between the estimated and measured values of temperatures for sampling points over the solid object as follows:

$$f(\mathbf{\Phi})=\left[\mathbf{Y}-\mathbf{T}(\mathbf{\Phi})\right]^T\left[\mathbf{Y}-\mathbf{T}(\mathbf{\Phi})\right] \tag{6}$$

where $\mathbf{\Phi}=\{\varphi_1,\ldots,\varphi_M\}$ is the vector of unknown parameters. $\mathbf{Y}=\{Y_1,\ldots,Y_N\}$ and $\mathbf{T}(\mathbf{\Phi})=\{T_1(\mathbf{\Phi}),\ldots,T_N(\mathbf{\Phi})\}$ are the vectors of measured and estimated temperatures of sampling points on the solid object, respectively. M and N are the number of elements over the bottom wall of the enclosure and the number of sampling points on the solid object, respectively.

The CGM is an iterative procedure in which at each iteration a suitable step size, β, is taken along a direction of descent, d, in order to minimize the objective function, so that

$$\mathbf{\Phi}^{\nu+1}=\mathbf{\Phi}^{\nu}-\beta^{\nu}\mathbf{d}^{\nu} \tag{7}$$

where the subscript ν is the iteration number. The direction of descent can be determined as a conjugation of the gradient direction, ∇f, and the direction of descent from the previous iteration as follows:

$$\mathbf{d}^{\nu}=\nabla f(\mathbf{\Phi}^{\nu})+\gamma^{\nu}\mathbf{d}^{\nu-1} \tag{8}$$

where γ is the conjugation coefficient given by

$$\gamma^{\nu}=\frac{\left[\nabla f(\mathbf{\Phi}^{\nu})\right]^T\left[\nabla f(\mathbf{\Phi}^{\nu})-\nabla f(\mathbf{\Phi}^{\nu-1})\right]}{\left[\nabla f(\mathbf{\Phi}^{\nu-1})\right]^T\left[\nabla f(\mathbf{\Phi}^{\nu-1})\right]}\quad \text{with}\quad \gamma^0=0 \tag{9}$$

The gradient direction is determined as:

$$\nabla f(\mathbf{\Phi}^{\nu})=-2\left[\mathbf{S}^{\nu}\right]^T\left[\mathbf{Y}-\mathbf{T}(\mathbf{\Phi}^{\nu})\right] \tag{10}$$

where **S** is the sensitivity (or Jacobian) matrix. The elements of the sensitivity matrix are:

$$S^{\nu}_{nm}(\mathbf{\Phi}^{\nu})=\frac{\partial T^{\nu}_n(\mathbf{\Phi}^{\nu})}{\partial\varphi^{\nu}_m} \tag{11}$$

The estimated temperatures can be linearized with a Taylor series expansion and then the minimization with respect to step size, β^{ν}, is performed to yield the following expression for the step size:

$$\beta^{\nu}=\frac{\left[\mathbf{S}^{\nu}(\mathbf{\Phi}^{\nu})\mathbf{d}^{\nu}\right]^T\left[\mathbf{T}(\mathbf{\Phi}^{\nu})-\mathbf{Y}\right]}{\left[\mathbf{S}^{\nu}(\mathbf{\Phi}^{\nu})\mathbf{d}^{\nu}\right]^T\left[\mathbf{S}^{\nu}(\mathbf{\Phi}^{\nu})\mathbf{d}^{\nu}\right]} \tag{12}$$

SENSITIVITY PROBLEM

To minimize the objective function given by equation (6), we need to calculate the sensitivity coefficients, S^{ν}_{mn}, given by equation (11). Differentiation of governing equation of heat transfer in the solid object with respect to φ^{ν}_m leads to

$$\nabla\cdot(\kappa\nabla T'_m)=0 \tag{13}$$

where $\nabla T'_m=\partial T/\partial\varphi^{\nu}_m$. Similarly, differentiation of radiative transfer equation with respect to φ^{ν}_m, we obtain:

$$\sum_{j=1}^{R}\left(\frac{\delta_{kj}}{\varepsilon_j}-\frac{1-\varepsilon_j}{\varepsilon_j}F_{k-j}\right)q'_{m,j}=4\sum_{j=1}^{R}\left(\delta_{kj}-F_{k-j}\right)\sigma T_j^3T'_{m,j}\ ,\ 1\le k\le R \tag{14}$$

Here, $T'_{m,j}=\partial T_j/\partial\varphi^{\nu}_m$ and $q'_{m,k}=\partial q_j/\partial\varphi^{\nu}_m$. Differentiation of the governing equation for the interface boundary condition with respect to φ^{ν}_m leads to:

$$\kappa\frac{\partial T'_{m,z}}{\partial n_z}+q'_{m,z}=0\ ,\ z=1,\ldots,Z \tag{15}$$

Differentiation of boundary conditions over the boundary surface elements of the enclosure with respect to φ^{ν}_m leads to

$$q'_m(x,1)=\frac{\partial q(x,1)}{\partial\varphi^{\nu}_m}=0 \tag{16a}$$

$$T'_m(0,y)=T'_m(1,y)=\frac{\partial T_s}{\partial\varphi^{\nu}_m}=0 \tag{16b}$$

$$T'_m(x,0)=\delta_d(x-x_m) \tag{16c}$$

The set of equations (13)-(15) can be solved in a similar manner with the direct problem, however, another iterative procedure is needed because of the presence of $T^3_{m,j}$ in equation

(14) whose values must be imposed from the solution of the direct problem. Solving the set of equations (13)-(15), the values of $T'_n = \partial T_n / \partial \varphi_m^v$, $n = 1,\ldots,N$ are in fact the components of m-th column in the sensitivity matrix.

STOPPING CRITERION

The stopping criterion for no measurement error is

$$f(\mathbf{\Phi}) < \xi \tag{17-a}$$

where ξ is a small specified positive number. However, for measurement errors in temperatures, the discrepancy principle is used [12]

$$f(\mathbf{\Phi}) < M\mu^2 \tag{17-b}$$

where μ is the standard deviation of the measurement errors.

COMPUTATIONAL ALGORITHM

The computational procedure for the inverse problem is summarized as follows:

Step 1- Set $v = 0$ and assume a set of elemental temperatures over the bottom wall of the radiant enclosure, $\mathbf{\Phi}^v$, stored at the nodes of the domain.

Step 2- Solve the direct problem given by equations (1)-(5) and compute the temperatures for sampling points on the solid object.

Step 3- Calculate the objective function $f(\mathbf{\Phi}^v)$ given by equation (6). Terminate the iteration procedure if the stopping criterion is satisfied, otherwise go to step 4.

Step 4- Solve the sensitivity problem given by equations (13)-(15) and compute the sensitivity matrix, $\mathbf{S}^v$.

Step 5- Compute the gradient direction $\nabla f(\mathbf{\Phi}^v)$ from equation (10), then compute the conjugate coefficient γ^v from equation (9).

Step 6- Compute the direction of descent $\mathbf{d}^v$ from equation (8).

Step 7- Compute the search step size β^v from equation (12).

Step 8- Knowing β^v and $\mathbf{d}^v$, compute the new set of elemental temperatures over the bottom wall of the radiant enclosure, $\mathbf{\Phi}^v$.

Step 9- Replace v by $v+1$ and go back to step 2.

RESULTS AND DISCUSSION

In order to check the performance and the accuracy of the present method to recover the boundary temperatures over the radiant enclosure from the knowledge of the measured temperatures for some sampling points on the solid object, we now try to recover three boundary temperature distributions along the bottom wall of the radiant enclosure. These profiles are: (i) step distribution, (ii) cosine distribution, and (iii) triangular distribution. Figure 2 shows these three profiles.

The measured temperatures, Y_n $(n = 1,\ldots,N)$, are generated by:

$$Y_{n,measured} = Y_{n,exact} + \mu\,\Theta \tag{18}$$

where Θ is a normal distributed random variable.

Figure 2 Three boundary temperature distributions along the bottom wall of the radiant enclosure

For normally distributed errors, there is a 99% probability of a value of Θ lying in the range $-2.576 \le \Theta \le 2.576$. The errors are estimated based on both the relative error, E_{rel} , and the root mean square error, E_{rms} , which are defined as follows:

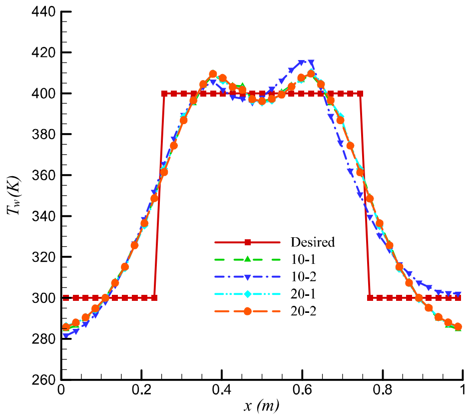
$$E_{rel,m} = \left| \left(\varphi_{m,est} - \varphi_{m,ex}\right) / \varphi_{m,ex} \times 100 \right| \tag{19}$$

$$E_{rms} = \left\{ \frac{1}{M} \sum_{m=1}^{M} \left[\left(\varphi_{m,est} - \varphi_{m,ex}\right) / \varphi_{m,ex} \right]^2 \times 100 \right\}^{1/2} \tag{20}$$

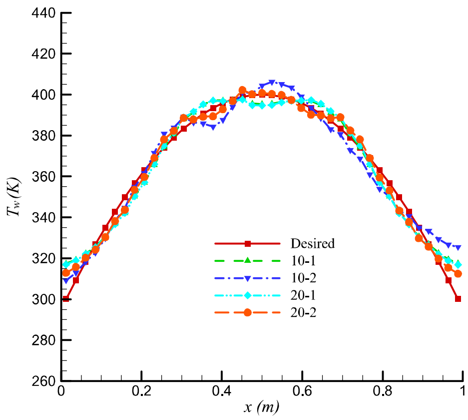
The relative error measures the deviation between estimated and exact values of temperatures for each surface element, whereas the root mean square error measures the deviation between estimated and exact values of temperatures over all the boundary surface elements.

Figure 3 Four cases of sensor positions: (a) 10-1, (b) 10-2, (c) 20-1, (d) 20-2

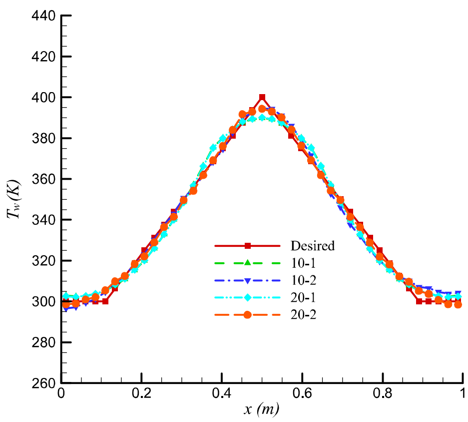
The inverse estimation is built for two cases with 10 and 20 sensors. In order to examine the effects of the sensor positions on inverse estimation, we consider two sets of sensor positions for each case (figures 3a-d). The results are compared for the case of no measurement error, $\mu = 0$.



(a)



(b)



(c)

Figure 4 The comparison of the results using four cases of sensor positions for (a) step, (b) cosine, and (c) triangular profiles.

Figures (4a-c) show the comparison of recovered and desired temperature profiles for cases with different number and arrangement of sensors. As shown, the cosine and triangular profiles are well recovered even using 10 sensors, however, the inverse estimation is less accurate for the case of step profile. Table 1 shows the values of maximum and root mean square errors for the cases with different sensors locations. Although the maximum errors for some cases are large, however, the values of the root mean square errors are acceptable for engineering applications.

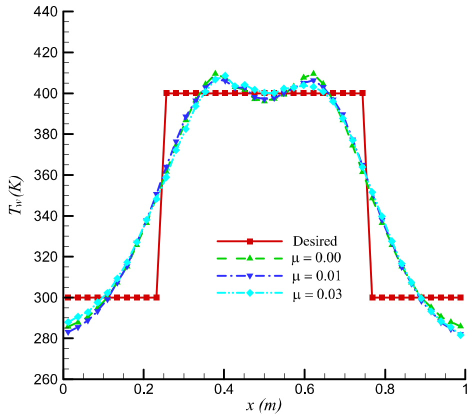
Table 1- The maximum relative errors and the root mean square errors for estimation of boundary temperatures over the bottom wall of the radiant furnace

Profile	Sensor Arrangement	E_{rms}	$E_{rel,\max}$
Step	10-1	5.980	16.423
	10-2	6.057	17.257
	20-1	5.840	16.220
	20-2	5.942	16.208
Cosine	10-1	1.836	5.849
	10-2	2.167	8.492
	20-1	1.822	5.666
	20-2	1.350	4.267
Triangular	10-1	1.179	2.540
	10-2	1.056	2.372
	20-1	1.161	2.474
	20-2	0.724	1.797

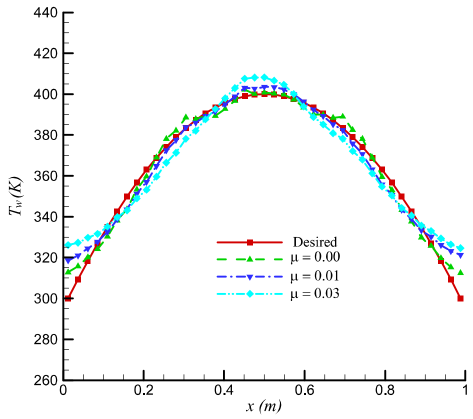
The effects of the measurement errors are investigated by comparing the results obtained with different values of standard deviations of measured data. The measured data are generated by equation (18) for three standard deviations of $\mu = 0, 0.01, 0.03$. The measurements are built using 20 sensors which are arranged as indicated in figure (3d). Table 2 shows the maximum relative errors and the root mean square errors for different values of standard deviations. As indicated the results are still acceptable, even with noisy data. The reconstructed temperature profiles for noisy measurement data are compared with the exact temperature profiles in figures (5a-c).

Table 2 The maximum relative errors and the root mean square errors for estimation of boundary temperatures over the bottom wall of the radiant furnace for different values of measurement error.

Profile	Standard deviation of measurement errors	E_{rms}	$E_{rel,\max}$
Step	$\mu = 0.00$	5.942	16.208
	$\mu = 0.01$	6.103	17.025
	$\mu = 0.03$	6.172	17.186
Cosine	$\mu = 0.00$	1.351	4.267
	$\mu = 0.01$	2.015	7.084
	$\mu = 0.03$	2.811	8.725
Triangular	$\mu = 0.00$	0.724	1.797
	$\mu = 0.01$	0.915	2.001
	$\mu = 0.03$	0.920	1.910



(a)



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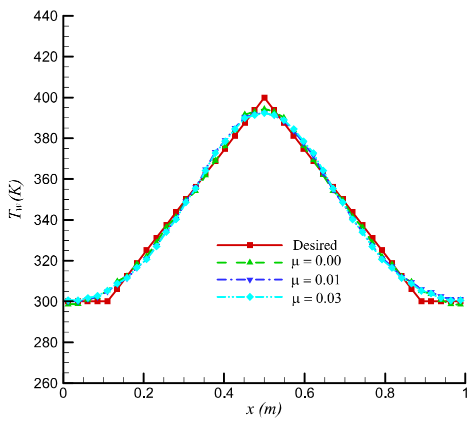

(c)

Figure 5 The comparison of the results using different standard deviations for (a) step, (b) cosine, and (c) triangular profiles.

CONCLUSION

An inverse analysis was built for estimation of boundary conditions over the wall surface of a radiant enclosure from the knowledge of measured temperatures on a solid object within the enclosure. The set of nonlinear equations obtained by differencing the conduction heat transfer equation and the set of net radiation equations were solved by the Newton's method. For the inverse problem, the temperature distribution over the wall surface of the enclosure was regarded as unknown, and the temperatures measured by sensors for some sampling points were considered to be available. The conjugate gradient method was used for minimization of an objective function, which is defined as the sum of square deviations between measured and estimated values of temperatures for sampling points on the solid object. The sensitivity coefficients were estimated by solving a set of boundary value problems which were obtained by differentiation of the governing equations of the direct problem with respect to elemental temperatures over the boundary surfaces of the radiant enclosure. The inverse problem was solved for different locations of the solid object and different positions of sensors. Comparing the results estimated using different arrangements of 10 and 20 sensors show that the results obtained by 10 sensors were less accurate. The effects of standard deviation of measured data on the accuracy of the inverse problem were examined by comparing the results for different values of standard deviations. The results implied that the recovered boundary conditions over the radiant enclosure were acceptable, even for noisy data.

References

[1] Volle F., Maillet D., Gradeck M., Kouachi A., Lebouché M., Practical application of inverse heat conduction for wall condition estimation on a rotating cylinder, *International Journal of Heat and Mass Transfer*, Vol. 52, 2009, pp. 210–221.

[2] Kuang Chen C.O., Su C.R., Inverse estimation for temperatures of outer surface and geometry of inner surface of furnace with two layer walls, *Energy Conversion and Management*, Vol. 49, 2008, pp. 301–310.

[3] Huang C.H., Wang S.P., A three-dimensional inverse heat conduction problem in estimating surface heat flux by conjugate gradient method, *International Journal of Heat and Mass Transfer*, Vol. 42, 1999, pp. 3387-3403.

[4] Park H.M., Yoon T.Y., Solution of the inverse radiation problem using a conjugate gradient method, *International Journal of Heat and Mass Transfer*, Vol. 43, 2000, pp. 1767–1776.

[5] Sarvari S.M.H., Mansouri S.H., Howell J.R., Inverse boundary design radiation problem in absorbing–emitting media with irregular geometry, *Numerical Heat Transfer A*, Vol. 43, 2003, pp. 565–584.

[6] Sarvari S.M.H., Howell J.R., Mansouri S.H., Inverse boundary design conduction–radiation problem in irregular two-dimensional domains, *Numerical Heat Transfer B*, Vol. 44, 2003, pp. 209–224.

[7] Sarvari S.M.H., Inverse determination of heat source distribution in conductive–radiative media with irregular geometry, *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 93, 2005, pp. 383–395.

[8] Özisik M.N., Orlande R.B., Inverse Heat Transfer, Fundamentals and Applications, Taylor and Francis, New York, 2000.

[9] Siegel R., Howell J.R., Thermal Radiation Heat Transfer, Taylor & Francis, New York, 2002.

[10] Hottel H. C., Radiant Heat Transmission, William H. McAdams, Heat Transmission, McGraw-Hill, New York, 1954.

[11] Burden, R.L., Faires, J.D., Numerical analysis, Boston, PWS Publishing company, 1993.

[12] Alifanov O.M., Inverse Heat Transfer Problems, Springer-Verlag, New York, 1994.