FIXED FUZZY POINTS OF FUZZY MAPPINGS IN HAUSDORFF FUZZY METRIC SPACES WITH APPLICATION

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Abstract. Recently, Phiangsungnoen et al. [J. Inequal. Appl. 2014:201 (2014)] studied fuzzy mappings in the framework of Hausdorff fuzzy metric spaces. Following this direction of research, we establish the existence of fixed fuzzy points of fuzzy mappings. An example is given to support the result proved herein; we also present a coincidence and common fuzzy point result. Finally, as an application of our results, we investigate the existence of solution for some recurrence relations associated to the analysis of quicksort algorithms.

1. **Introduction and Preliminaries**

Uncertainty is an essential part of the real world problems, then Zadeh [36] initiated the study of fuzzy set theory to deal with the phenomenon of uncertainty in control systems. Consequently, fuzzy set theory has been successfully applied in many branches of scientific and social sciences. In particular, combining the notion of fuzziness with the distance structure on a nonempty set, Kramosil and Michalek [18] introduced fuzzy metric spaces as a generalization and extension of the classical metric spaces. The conditions which they formulated were modified later by George and Veeramani [9, 10] to obtain Hausdorff topology induced by such fuzzy metric. Successively, Rodríguez-López and Romaguera [28] introduced Hausdorff fuzzy metric on the family of nonempty compact subsets of a given fuzzy metric space. Then, fixed point theory in fuzzy metric spaces has been studied by a number of authors (see, for example, [5, 7, 11, 20, 22, 23, 27, 29, 31, 34]). On the other hand, it is well-known that a Baire metric provides mathematical models in denotational semantics of programming languages (see [2, 3, 4, 16, 21]). Following this line of research, Romaguera et al. [29] obtained an application of fixed point results to the domain of words endowed with a fuzzy metric space induced by a Baire metric. Also, to deal with the fuzziness in the functional equations, Heilpern [15] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem of Nadler for multivalued mappings [25]. A number of authors extended the results of Heilpern and discussed the existence of fixed points for fuzzy generalized contractive conditions (see [6, 33, 35]). Very recently, Phiangsungnoen et al. [26] initiated the

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study of fuzzy fixed point theory for fuzzy mappings in Hausdorff fuzzy metric spaces.

In this paper, we obtain fixed fuzzy points of fuzzy mappings in Hausdorff fuzzy metric spaces under generalized contractive conditions. Our results extend, unify and generalize the comparable results in [15, 17, 25, 26]. As an immediate application of our results, we obtain the existence of solution for some recurrence relations associated to the analysis of Quicksort algorithms.

Consistent with [32], [10], and [28], the following definitions and results will be needed in the sequel.

Let X be a set of points whose generic element is denoted by x and $I = [0, 1]$. A fuzzy set A in X is characterized by a mapping $A : X \to I$ (called grade membership function). Thus every crisp set A can be considered as a fuzzy set with Ψ_A (characteristic function of a set A) as its grade membership function.

Definition 1.1. [32] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous $t - norm$ if the following conditions hold:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all a in $[0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.2. [9] Let X be a nonempty set and $*$ a continuous $t-$norm. A fuzzy set M on $X^2 \times (0, +\infty)$ is called a fuzzy metric on X if for all $x, y, z \in X$ and $s, t > 0$, the following conditions hold:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(v) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

The $3-$tuple $(X, M, *)$ is called a fuzzy metric space. For $t > 0$, $\varepsilon \in (0, 1)$, the set

$$B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$$

is called an open ball with center at x and radius equal to ε. A set A in X is called an open set if for each x in A, one may find an open ball $B_M(x, \varepsilon, t)$ such that $B_M(x, \varepsilon, t) \subseteq A$. The collection τ_M of all open subsets of X is a topology on X called the topology induced by a fuzzy metric M. It is known from [12] that every fuzzy metric space $(X, M, *)$ is metrizable, that is, there exists a metric d on X such that a topology induced by d agrees with τ_M. If (X, d) is a metric space and $M_d : X \times X \times (0, +\infty) \to (0, 1]$ is defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$, then $(X, M_d, *)$ is a fuzzy metric space (called the standard fuzzy metric induced by the metric d (see [9]), where $a * b = \min\{a, b\}$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be

(a) convergent to a point $x \in X$, with respect to τ_M, if and only if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$;

(b) Cauchy sequence if for each $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.

A fuzzy metric space $(X, M, *)$ is said to be complete (see [10]) if every Cauchy sequence in X is convergent. A subset $A \subseteq X$ is said to be

(i) closed if for every sequence $\{x_n\}$ in A with $x_n \to x$, we have that $x \in A$;

(ii) compact if every sequence $\{x_n\}$ in A has a convergent subsequence to a point in A.

A fuzzy metric M is said to be continuous on $X^2 \times (0, +\infty)$ if $\lim_{n\to+\infty} M(x_n, y_n, t_n) = M(x, y, t)$, whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, +\infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, +\infty)$, that is,

$$\lim_{n\to+\infty} M(x_n, x, t) = \lim_{n\to+\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to+\infty} M(x, y, t_n) = M(x, y, t).$$

It is known that M is continuous on $X^2 \times (0, +\infty)$ and $M(x, y, \cdot)$ is nondecreasing on $(0, +\infty)$. For more details, we refer to [9, 10, 11, 12, 13].

Rodríguez-López and Romaguera [28] introduced Hausdorff fuzzy metric on a family of nonempty compact subsets of a given fuzzy metric space $(X, M, *)$. Let $K(X)$ be the family of all nonempty compact subsets of X. For $A, B \in K(X)$, $x \in X$ and $t > 0$ define:

$$M(x, A, t) = \sup_{a\in A} M(x, a, t)$$

and

$$H_M(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(b, A, t)\}$$

for all $t \geq 0$. The 3−tuple $(K(X), H_M, *)$ is a fuzzy metric space and the fuzzy metric H_M is called the Hausdorff fuzzy metric induced by the fuzzy metric M.

Lemma 1.3. [14] *Let X be a nonempty set and $g : X \to X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Let A be a fuzzy set in X. If $\alpha \in (0, 1]$, then the α−level set A_α of A is defined as:

$$A_\alpha = \{x : A(x) \geq \alpha\}.$$

For $\alpha = 0$, we have $A_0 = \overline{\{x \in X : A(x) > 0\}}$, where $\overline{B}$ denotes the closure of the set B. A fuzzy set A is said to be more accurate than fuzzy set B, denoted by $A \subset B$ if and only if $A(x) \leq B(x)$ for each x in X. It is obvious that if $0 < \alpha \leq \beta \leq 1$, then $A_\beta \subseteq A_\alpha$. Corresponding to each $\alpha \in [0, 1]$ and $x \in X$, the fuzzy point $(x)_\alpha$ of X is a fuzzy set $(x)_\alpha : X \to [0, 1]$ given by

$$(x)_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, we have the following indicator function

$$(x)_1(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$W_\alpha(X) = \{A \in I^X : A_\alpha \text{ is nonempty and compact}\}.$$

For $A,B \in W_\alpha(X)$ and $\alpha \in [0,1]$, define

$$\begin{cases} M_\alpha(x,B,t) = \sup_{y \in B_\alpha} M(x,y,t), \\ H_{M_\alpha}(A,B,t) = H_M(A_\alpha, B_\alpha, t). \end{cases}$$

Note that M_α is a nonincreasing function of α and H_{M_α} is the Hausdorff fuzzy metric induced by the fuzzy metric M on $W_\alpha(X)$.

Let Y be an arbitrary subset of a fuzzy metric space $(X,M,*)$. A mapping $F : Y \to W_\alpha(X)$ is called a fuzzy mapping over the set Y, that is $Fy \in W_\alpha(X)$ for each y in Y. As a fuzzy set Fy in X is characterized by a membership function $Fy : X \to [0,1]$, so $Fy(x)$ is the membership of x in Fy. Thus a fuzzy mapping F over Y is a fuzzy subset of $Y \times X$ having membership function $Fy(x)$.

In a more general sense than given in [15], a mapping $F : X \to I^X$ is a fuzzy mapping over X [33].

Definition 1.4. [6] A fuzzy point x_α in X is called a fixed fuzzy point of a fuzzy mapping F if $(x)_\alpha \subset Fx$, that is, $Fx(x) \geq \alpha$ or $x \in (Fx)_\alpha$, that is the fixed degree of x in Fx is at least α. If $(x)_1 \subset Fx$, then x is a fixed point of fuzzy mapping F.

Very recently, Ali and Abbas [1] gave the following definitions.

Definition 1.5. [1] Let $F : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$. A fuzzy point $(x)_\alpha$ in X is called

(a) coincidence fuzzy point of hybrid pair (g,F) if $(gx)_\alpha \subset Fx$, that is $Fx(gx) \geq \alpha$ or $gx \in (Fx)_\alpha$ (the fixed degree of gx in Fx is at least α);
(b) common fixed fuzzy point of hybrid pair (g,F) if $(x)_\alpha = (gx)_\alpha \subset Fx$, that is $x = gx \in (Fx)_\alpha$ (the fixed degree of x and gx in Fx is the same and is at least α).

The sets of all fixed fuzzy points, coincidence fuzzy points and common fixed fuzzy points of the hybrid pair (g,F) are denoted by $F_\alpha(F)$, $C_\alpha(F,g)$ and $F_\alpha(F,g)$, respectively.

Definition 1.6. [1] Let $F : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$. Then

(c) the hybrid pair (g,F) is called $w-$fuzzy compatible if $g(Fx)_\alpha \subseteq (Fgx)_\alpha$ whenever $x \in C_\alpha(F,g)$;
(d) a mapping g is called $F-$fuzzy weakly commuting at some point $x \in X$ if $g^2x \in (Fgx)_\alpha$.

The following lemma is crucial to proving the main result in this paper.

Lemma 1.7. *Let $(X,M,*)$ be a fuzzy metric space, $x,y \in X$ and $A,B \in W_\alpha(X)$. Then the following conditions hold:*

(i) *for each $x \in X$, $B \in W_\alpha(X)$ and $t > 0$, there is $(b_0)_\alpha \subset B$ such that*

$$M_\alpha(x,B,t) = M(x,b_0,t);$$

(ii) $M_\alpha(x, A, t) = 1$ *implies that* $(x)_\alpha \subset A$;
(iii) *for* $(y_x)_\alpha \subset B$ *with* $M(x, y_x, t) = M_\alpha(x, B, t)$, *we have* $M_\alpha(x, C, t+s) \geq M_\alpha(x, B, t) * M_\alpha(y_x, C, s)$.

Proof. We proceed point by point.
(i) As the function $y \mapsto M(x, y, t)$ is continuous (see [28]) and B_α is compact, so there exists $(b_0)_\alpha \subset B$ such that

$$\sup_{b \in B_\alpha} M(x, b, t) = M(x, b_0, t).$$

(ii) As $M_\alpha(x, A, t) = 1$, so by (i), there exists $(y)_\alpha \subset A$ such that $M_\alpha(x, A, t) = M(x, y, t) = 1$. Thus, it follows that $x = y$ and hence $(x)_\alpha = (y)_\alpha \subset A$.

(iii) From (i), we have a point $(y_x)_\alpha \subset B$ such that $M_\alpha(x, B, t) = M(x, y_x, t)$. Now for each $(z)_\alpha \subset C$, we obtain

$$M_\alpha(x, C, t+s) \geq M(x, z, t+s) \geq M(x, y_x, t) * M(y_x, z, s).$$

By the continuity of the $t-$norm we have

$$M_\alpha(x, C, t+s) \geq M_\alpha(x, B, t) * M_\alpha(y_x, C, s).$$

□

Let S be the class of functions $\psi : (0, 1] \to (0, 1]$ satisfying $\limsup_{n \to +\infty} \psi(x_n) < 1$ whenever $\{x_n\} \subseteq (0, 1]$ is nondecreasing or $\lim_{n \to +\infty} x_n = 1$.

Example 1.8. Define the mappings $\psi_1, \psi_2 : (0, 1] \to (0, 1]$ by

$$\psi_1(x) = 1 - x \text{ and } \psi_2(x) = 1 - e^{x-1}.$$

Then $\psi_i \in S$ for $i = 1, 2$.

A sequence $\{t_n\}$ of positive real numbers is said to be $s-$increasing (see [13]) if there exists $n_0 \in \mathbb{N}$ such that

$$t_{m+1} \geq t_m + 1$$

for all $m \geq n_0$. Also, in a fuzzy metric space, infinite product is denoted by

$$M(x, y, t_1) * M(x, y, t_2) * \cdots * M(x, y, t_n) * \cdots = \prod_{i=1}^{+\infty} M(x, y, t_i).$$

Then an $s-$increasing sequence $\{t_n\}$ is said to satisfy the condition (T-1), with respect to $x, y \in X$, if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\prod_{n \geq n_0}^{+\infty} M(x, y, t_n) \geq 1 - \varepsilon$. Throughout this paper, we assume that an $s-$increasing sequence $\{t_n\}$ satisfies the condition (T-1).

2. Fixed Fuzzy Point Theorems

In this section we prove a fixed fuzzy point theorem for fuzzy mappings in fuzzy metric spaces.

Theorem 2.1. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* $F : X \to W_\alpha(X)$ *a fuzzy mapping. Suppose that, for all* $x, y \in X$ *and* $t > 0$, *the following condition holds:*

$$H_{M_\alpha}(Fx, Fy, \psi(M(x, y, t))t) \geq M(x, y, t), \tag{1}$$

where $\psi \in S$. *Then* F *has a fixed fuzzy point.*

Proof. Let x_0 be a given point in X and $x_1 \in (Fx_0)_\alpha$. Since $(Fx_1)_\alpha$ is compact, in view of Lemma 1.7, we can choose $x_2 \in (Fx_1)_\alpha$ such that

$$\begin{aligned} M(x_2, x_1, t) &\geq M_\alpha(x_2, x_1, \psi(M(x_1, x_0, t))t) \\ &= \sup_{y \in (Fx_1)_\alpha} M(y, x_1, \psi(M(x_1, x_0, t))t) \\ &\geq H_{M_\alpha}(Fx_1, Fx_0, \psi(M(x_1, x_0, t))t) \\ &\geq M(x_1, x_0, t). \end{aligned}$$

Thus, we have

$$M(x_2, x_1, t) \geq M(x_1, x_0, t).$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in (Fx_n)_\alpha$ for all $n \geq 0$, and

$$M(x_{n+2}, x_{n+1}, t) \geq M(x_{n+1}, x_n, t).$$

Thus the sequence $\{M(x_{n+2}, x_{n+1}, t)\}$ is nondecreasing. If follows from an assumption on ψ that

$$\limsup_{n \to +\infty} \psi(M(x_{n+2}, x_{n+1}, t)) < 1$$

and hence there exist $k < 1$ and $n_1 \in \mathbb{N}$ such that

$$\psi(M(x_{n+1}, x_n, t)) < k, \text{ for all } n > n_1. \tag{2}$$

As $M(x, y, \cdot)$ is nondecreasing, then from (2) we have

$$\begin{aligned} M(x_{n+2}, x_{n+1}, kt) &\geq M(x_{n+2}, x_{n+1}, \psi(M(x_{n+1}, x_n, t))t) \\ &\geq H_{M_\alpha}(Fx_{n+1}, Fx_n, \psi(M(x_{n+1}, x_n, t))t) \\ &\geq M(x_{n+1}, x_n, t) \end{aligned}$$

for all $n \in \mathbb{N}$ with $n > n_1$. Hence we have

$$M(x_{n+2}, x_{n+1}, t) \geq M(x_{n+1}, x_n, \frac{t}{k}), \text{ for all } n \in \mathbb{N} \text{ with } n > n_1.$$

Continuing this way, for all $n > n_1$, we can obtain

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, \frac{t}{k}) \\ &\geq M(x_{n-2}, x_{n-1}, \frac{t}{k^2}) \\ &\geq ... \\ &\geq M(x_{n_1}, x_{n_1+1}, \frac{t}{k^{n-n_1}}). \end{aligned}$$

Now, we note that the sequence $t_n = \dfrac{t}{n(n+1)k^{n-n_1}}$ is an $s-$increasing sequence, since $\lim_{n \to +\infty} (t_{n+1} - t_n) = +\infty$, for all $t > 0$. Next fix $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ with $n_0 > n_1$ such that $\prod_{j \geq n}^{+\infty} M(x_{n_1}, x_{n_1+1}, t_j) \geq 1 - \varepsilon$ for all $n \geq n_0$. Then, for all

$m > n \geq n_0$ and $h_j = \dfrac{1}{j(j+1)}$ for $j \in \{n, n+1, \ldots, m-1\}$, since $\sum_{j=n}^{m-1} h_j < 1$, we write

$$
\begin{aligned}
& M(x_n, x_m, t) \\
& \geq M(x_n, x_m, \sum_{j=n}^{m-1} h_j t) \\
& \geq M(x_n, x_{n+1}, h_n t) * M(x_{n+1}, x_{n+2}, h_{n+1} t) * \cdots * M(x_{m-1}, x_m, h_{m-1} t) \\
& \geq M(x_{n_1}, x_{n_1+1}, \frac{h_n}{k^{n-n_1}} t) * M(x_{n_1}, x_{n_1+1}, \frac{h_{n+1}}{k^{n-n_1+1}} t) \\
& \quad * \cdots * M(x_{n_1}, x_{n_1+1}, \frac{h_{m-1}}{k^{m-n_1-1}} t) \\
& \geq \prod_{j=n}^{+\infty} M(x_{n_1}, x_{n_1+1}, \frac{h_j}{k^{j-n_1}} t) \geq 1 - \varepsilon.
\end{aligned}
$$

Hence $M(x_n, x_m, t) \geq 1 - \varepsilon$ for all $m > n \geq n_0$ and the sequence $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, we have

$$\lim_{n \to +\infty} M(x_n, \bar{x}, t) = 1 \text{ for each } t > 0, \tag{3}$$

for some $\bar{x}$ in X. By given assumption, we have

$$\limsup_{n \to +\infty} \psi(M(x_n, \bar{x}, t)) < 1.$$

Thus there exists λ with $k < \lambda < 1$ such that

$$\limsup_{n \to +\infty} \psi(M(x_n, \bar{x}, t)) < \lambda. \tag{4}$$

Now, by (1) and (4), we have

$$
\begin{aligned}
H_{M_\alpha}(Fx_n, F\bar{x}, \lambda t) &\geq H_{M_\alpha}(Fx_n, F\bar{x}, \psi(M(x_n, \bar{x}, t))t) \\
&\geq M(x_n, \bar{x}, t).
\end{aligned}
$$

On taking limit as $n \to +\infty$, we obtain $\lim_{n \to +\infty} H_{M_\alpha}(Fx_n, F\bar{x}, t) = 1$. Since $x_{n+1} \in (Fx_n)_\alpha$, it follows that $\lim_{n \to +\infty} \sup_{v \in (F\bar{x})_\alpha} M(x_{n+1}, v, t) = 1$, and hence there exists a sequence $\{y_n\}$ with $y_n \in (F\bar{x})_\alpha$ such that

$$\lim_{n \to +\infty} M(x_n, y_n, t) = 1 \tag{5}$$

for each $t > 0$. Now, for each $n \in \mathbb{N}$, we have

$$M(y_n, \bar{x}, t) \geq M(y_n, x_n, \frac{t}{2}) * M(x_n, \bar{x}, \frac{t}{2}). \tag{6}$$

On taking limit as $n \to +\infty$ in (6) and by (3) and (5), we have $\lim_{n \to +\infty} M(y_n, \bar{x}, t) = 1$, that is, $\lim_{n \to +\infty} y_n = \bar{x}$. Since $(F\bar{x})_\alpha$ is compact, $\bar{x} \in (F\bar{x})_\alpha$, that is $(\bar{x})_\alpha \subset F\bar{x}$. Hence $\bar{x}$ is a fixed fuzzy point of F. □

As an immediate consequence of Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let $(X, M, *)$ be a complete fuzzy metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that, for all $x, y \in X$ and $t > 0$, the following condition holds:*

$$H_{M_\alpha}(Fx, Fy, kt) \geq M(x, y, t)$$

where $k \in (0, 1)$. Then F has a fixed fuzzy point.

Remark 2.3. Let $\beta : (0, 1] \to [0, +\infty)$ be the function defined by $\beta(x) = \frac{t}{x} - t$, and $\gamma : [0, +\infty) \to [0, 1)$ any function satisfying $\limsup_{r \to t^+} \gamma(r) < 1$ for all $t \in [0, +\infty)$. If $\psi(x) = (\gamma \circ \beta)(x)$, then $\psi \in S$. For example, if $\gamma(x) = \dfrac{x}{x+t}$ then $\psi(x) = 1 - x$. Thus the following corollary of Theorem 2.1 is a generalization of [17, Theorem 2.4] for fuzzy mappings in fuzzy metric spaces, while [17, Theorem 2.4] is a generalization of the fixed point results in [13], and [24] in the case of compact sets.

Corollary 2.4. *Let $(X, M, *)$ be a complete fuzzy metric space, $F : X \to W_\alpha(X)$ a fuzzy mapping and $\gamma : [0, +\infty) \to [0, 1)$ a function such that $\limsup_{r \to t^+} \gamma(r) < 1$ for all $t \in [0, +\infty)$. Suppose that, for all $x, y \in X$ and $t > 0$, the following condition holds:*

$$H_{M_\alpha}(Fx, Fy, \gamma(\frac{t}{M(x, y, t)} - t)t) \geq M(x, y, t)$$

Then F has a fixed fuzzy point.

One can prove the next theorem by following the same lines of proof of Theorem 2.1 and hence, to avoid repetitions, we omit the details.

Theorem 2.5. *Let $(X, M, *)$ be a complete fuzzy metric space, $F : X \to I^X$ a fuzzy mapping and $\alpha : X \to (0, 1]$ a function such that $(Fx)_{\alpha(x)}$ is a nonempty compact subset of X for all $x \in X$. Suppose that, for all $x, y \in X$ and $t > 0$, the following condition holds:*

$$H_M((Fx)_{\alpha(x)}, (Fy)_{\alpha(y)}, \psi(M(x, y, t))t) \geq M(x, y, t),$$

where $\psi \in S$. Then F has a fixed fuzzy point.

Recently Phiangsungnoen [26] proved the following theorem.

Theorem 2.6. [26] *Let $(X, M, *)$ be a complete fuzzy metric space and $\alpha : X \to (0, 1]$ a mapping such that $(Fx)_{\alpha(x)}$ is a nonempty compact subset of X for all $x \in X$, where $F : X \to I^X$ is a fuzzy mapping such that*

$$H_M((Fx)_{\alpha(x)}, (Fy)_{\alpha(y)}, kt) \geq M(x, y, t)$$

holds for all $t > 0$, where $k \in (0, 1)$. If there exist $x_0 \in X$ and $x_1 \in (Fx_0)_{\alpha(x_0)}$ such that

$$\lim_{n \to +\infty} \prod_{i=n}^{+\infty} M(x_0, x_1, th^i) = 1 \tag{7}$$

for all $t > 0$ and $h > 1$, then F has a fixed fuzzy point.

Remark 2.7. Take $t_n = th^n$ in Theorem 2.6. As $h > 1$ then $\lim_{n\to+\infty}(t_{n+1} - t_n) = \lim_{n\to+\infty} t(h-1)h^n = +\infty$ and $\{t_n\}$ is an $s-$increasing sequence. Thus (7) implies that, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\prod_{n=1}^{+\infty} M(x_0, x_1, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$. Consequently condition (T-1) holds true with respect to x_0, x_1.

In the light of above remark, Theorem 2.6 is a special case of Theorem 2.5.

Remark 2.8. Let (X, d) be a metric space, $a * b = ab$ and $(X, M, *)$ be a fuzzy metric space, where $M(x, y, t) = \dfrac{t}{t + d(x, y)}$. Note that $(X, M, *)$ is complete if and only if (X, d) is complete. Since $M_\alpha(x, B, t) = \sup_{y\in B_\alpha} M(x, y, t)$ and

$$H_{M_\alpha}(A, B, t) = \min\{\inf_{x\in A_\alpha} M(x, B_\alpha, t), \inf_{y\in B_\alpha} M(y, A_\alpha, t)\},$$

then we have

$$\begin{aligned}
&H_{M_\alpha}(Fx, Fy, \psi(M(x, y, t))t) \\
&= \min\left\{\inf_{x\in A_\alpha} \frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + d(x, B_\alpha)}, \inf_{y\in B_\alpha} \frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + d(A_\alpha, y)}\right\} \\
&= \min\left\{\frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + \sup\limits_{x\in A_\alpha} d(x, B_\alpha)}, \frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + \sup\limits_{y\in B_\alpha} d(A_\alpha, y)}\right\} \\
&= \frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + D_\alpha(A, B)},
\end{aligned}$$

where

$$D_\alpha(A, B) = \max\{\sup_{x\in A_\alpha} d(x, B_\alpha), \sup_{x\in B_\alpha} d(x, A_\alpha)\}.$$

Hence $H_{M_\alpha}(Fx, Fy, \psi(M(x, y, t))t) \geq M(x, y, t)$ implies that

$$\frac{\psi(M(x, y, t))t}{\psi(M(x, y, t))t + D_\alpha(Fx, Fy)} \geq \frac{t}{t + d(x, y)}.$$

Consequently,

$$D_\alpha(Fx, Fy) \leq \psi(M(x, y, t))d(x, y).$$

If $\psi(x) = k$, where $k \in (0, 1)$, then we have

$$D_\alpha(Fx, Fy) \leq kd(x, y),$$

which is the contractive condition given in [6]. Since $D(Fx, Fy) = \sup_\alpha D_\alpha(Fx, Fy)$, therefore we obtain $D(Fx, Fy) \leq kd(x, y)$, which is the contractive condition in [15]. Thus Theorem 2.1 generalizes the results in [6, 15].

Example 2.9. Let $X = \{0, 1, 2\}$ be endowed with the fuzzy metric M defined by

$$M(x, y, t) = \frac{t^2}{t^2 + d(x, y)},$$

where

$$d(x,x)=0 \text{ for any } x\in X,\ d(0,1)=10,\ d(0,2)=12,\ d(1,2)=16,$$
$$d(x,y)=d(y,x) \text{ for all } x,y\in X.$$

Note that $(X,M,*)$, where $a*b=ab$, is a complete fuzzy metric space. Also, for any $s-$increasing sequence $\{t_n\}$, $\prod_{n=1}^{+\infty} M(x,y,t_n)$ is convergent (see [13, Examples 4.10 and 4.11]). Let $\alpha\in(0,\frac{1}{3})$. Define a fuzzy mapping $F: X\to W_\alpha(X)$ as follows:

$$(F0)(x) = \begin{cases}\alpha & \text{if } x=0,\\ \frac{\alpha}{2} & \text{if } x=1,\\ 0 & \text{if } x=2,\end{cases} \quad (F1)(x)=\begin{cases}\alpha & \text{if } x=0\\ 0 & \text{if } x=1,\\ \frac{\alpha}{3} & \text{if } x=2,\end{cases}$$

$$(F2)(x) = \begin{cases}\frac{\alpha}{3} & \text{if } x=0,\\ \alpha & \text{if } x=1,\\ \frac{\alpha}{4} & \text{if } x=2.\end{cases}$$

Note that $(F0)_\alpha=(F1)_\alpha=\{0\}$, $(F2)_\alpha=\{1\}$. Thus for any $x,y\in\{0,1\}$, we have $H_M((Fx)_\alpha,(Fy)_\alpha,kt)=1$. Also, we consider the following cases:

(i) If $x=0$ and $y=2$, we obtain

$$\begin{aligned} H_{M_\alpha}((F0)_\alpha,(F2)_\alpha,kt) &= \min\{\inf_{x\in(F0)_\alpha} M(x,(F2)_\alpha,kt), \inf_{y\in(F2)_\alpha} M(y,(F0)_\alpha,kt)\}\\ &= \frac{k^2t^2}{k^2t^2+10}\\ &\geq \frac{t^2}{t^2+12}=M(0,2,t), \end{aligned}$$

whenever $k\geq\sqrt{\frac{5}{6}}$.

(ii) If $x=1$ and $y=2$, we have

$$\begin{aligned} H_{M_\alpha}((F1)_\alpha,(F2)_\alpha,kt) &= \min\{\inf_{x\in(F1)_\alpha} M(x,(F2)_\alpha,kt), \inf_{y\in(F2)_\alpha} M(y,(F1)_\alpha,kt)\}\\ &= \frac{k^2t^2}{k^2t^2+10}\\ &\geq \frac{t^2}{t^2+16}=M(1,2,t), \end{aligned}$$

whenever $k\geq\sqrt{\frac{5}{8}}$.

Consequently whenever $k\geq\sqrt{\frac{5}{6}}$, for each $x,y\in X$ and $t>0$, we have

$$H_{M_\alpha}(Fx,Fy,kt)\geq M(x,y,t).$$

Thus, all the conditions of Theorem 2.2 are satisfied; here $x = 0$ is a fixed fuzzy point of F. Indeed, for $x = 0$, we have $(x)_\alpha \subset Fx$ as $(F0)(0) \geq \alpha$, therefore $0 \in (F0)_\alpha$.

3. Consequence and Application

3.1. **Coincidence and Common Fixed Fuzzy Point Theorem.** Let $g : X \to X$ be a mapping and $F : X \to W_\alpha(X)$ be a fuzzy mapping and denote $(F(X))_\alpha = \bigcup_{x\in X}(Fx)_\alpha$. We call (g, F) a hybrid fuzzy pair. Now we prove the following theorem for the existence of coincidence fuzzy points and common fixed fuzzy points of hybrid fuzzy pair as an application of Theorem 2.1.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space and (g, F) a hybrid fuzzy pair such that $(F(X))_\alpha \subseteq g(X)$. Suppose that, for all $x, y \in X$ and $t > 0$, the following condition holds:*

$$H_{M_\alpha}(Fx, Fy, \psi(M(x, y, t))t) \geq M(gx, gy, t)$$

where $\psi \in S$. Then $C_\alpha(g, F) \neq \emptyset$ provided that for each $\varepsilon > 0$ and an $s-$increasing sequence $\{t_n\}$, there exists $n_0 \in \mathbb{N}$ such that $\prod\limits_{n\geq n_0}^{+\infty} M(gx, gy, t_n) \geq 1 - \varepsilon$. Moreover, the hybrid fuzzy pair (g, F) has a common fixed fuzzy point if one of the following conditions holds:

- (i) *F and g are $w-$fuzzy compatible, $\lim\limits_{n\to+\infty} g^n x = y$ for some $x \in C_\alpha(g, F)$, $y \in X$ and g is continuous at y.*
- (ii) *g is $F-$fuzzy weakly commuting for some $x \in C_\alpha(g, F)$ and gx is a fixed point of g, that is $g^2x = gx$.*
- (iii) *g is continuous at x for some $x \in C_\alpha(g, F)$ and for some $y \in X$, $\lim\limits_{n\to+\infty} g^n y = x$.*

Proof. By Lemma 1.3, there exists $E \subseteq X$ such that $g : E \to X$ is one-to-one and $g(E) = g(X)$. Define a mapping $A : g(E) \to W_\alpha(X)$ by

$$A(gx) = Fx \text{ for all } gx \in g(E).$$

As g is one-to-one on E, then A is well defined. Since

$$\begin{aligned} H_{M_\alpha}(A(gx), A(gy), \psi(M(x, y, t))t) &= H_{M_\alpha}(Fx, Fy, \psi(M(x, y, t))t) \\ &\geq M(gx, gy, t), \end{aligned}$$

the mapping A satisfies (1) and hence all the conditions of Theorem 2.1. By Theorem 2.1, A has a fixed fuzzy point $u \in g(E)$ such that $(u)_\alpha \subset A(u)$. Thus $u \in (A(u))_\alpha$. Since $(F(X))_\alpha \subseteq g(X)$, then there exists $v \in X$ such that $(gv)_\alpha = (u)_\alpha$. Hence $(gv)_\alpha \subset A(gv) = Fv$, that is $gv \in (A(gv))_\alpha = (Fv)_\alpha$ and $C_\alpha(F, g)$ is nonempty.

Now, if (i) holds, then for some $x \in C_\alpha(F, g)$ and $y \in X$, we have $\lim\limits_{n\to+\infty} g^n x = y$. As g is continuous at y, so y is a fixed point of g. Since F and g are $w-$fuzzy compatible, $g^n x \in C_\alpha(F, g)$ for all $n \geq 1$, that is, for all $n \geq 1$, we have $(g^n x)_\alpha \subset F(g^{n-1}x)$ and

$$\begin{aligned} M_\alpha(gy,Fy,t) &\geq M(gy,g^nx,\frac{t}{2})*M_\alpha(g^nx,Fy,\frac{t}{2}) \\ &\geq M(gy,g^nx,\frac{t}{2})*H_{M_\alpha}(F(g^{n-1}x),Fy,\frac{t}{2}) \\ &\geq M(gy,g^nx,\frac{t}{2})*H_{M_\alpha}(F(g^{n-1}x),Fy,\psi(M(g^{n-1}x,y,\frac{t}{2}))\frac{t}{2}) \\ &\geq M(gy,g^nx,\frac{t}{2})*M(g^nx,gy,\frac{t}{2}). \end{aligned}$$

On taking limit as $n\to+\infty$, we have

$$M_\alpha(y,Fy,t)\geq M(y,y,\frac{t}{2})*M(y,y,\frac{t}{2})=1.$$

This implies that $y\in(Fy)_\alpha$ and $(y)_\alpha=(gy)_\alpha\subset(Fy)$. Hence y is a common fixed fuzzy point of F and g.
If (ii) holds, then $gx=g^2x\in(F(gx))_\alpha$. Hence, gx is a common fixed fuzzy point of F and g.
If (iii) holds then by the continuity of g at x, we get $x=gx\in(Fx)_\alpha$. Hence, x is a common fixed fuzzy point of F and g. □

3.2. **Application to Domain of Words.** In this subsection, we give an application of our main result in theoretical computer science. Let $\sum$ be a nonempty alphabet and $\sum^\infty$ the set of all finite and infinite sequences (words) over $\sum$. By convention, we denote the empty sequence (word) by $\emptyset$ and assume $\emptyset\in\sum^\infty$. Let the prefix order $\sqsubseteq$ on $\sum^\infty$ be defined as follows:

$$x\sqsubseteq y \text{ if and only if } x \text{ is a prefix of } y.$$

For each sequence (word) $x\neq\emptyset$ in $\sum^\infty$, let $l(x)\in[1,+\infty]$ be the length of x and assume that $l(\emptyset)=0$. Also, if $x\in\sum^\infty$ has length $n<+\infty$, we write $x:=x_1x_2\cdots x_n$, otherwise (i.e., in the case of infinite sequence) we write $x:=x_1x_2\cdots$. Now, if $x,y\in\sum^\infty$, then $x\sqcap y$ denotes the common prefix of x and y. Note that $x=y$ if and only if $x\sqsubseteq y$ and $y\sqsubseteq x$ and $l(x)=l(y)$. Define a mapping $d_\sqsubseteq:\sum^\infty\times\sum^\infty\to[0,+\infty)$ by

$$d_\sqsubseteq(x,y)=\begin{cases}0 & \text{iff } x=y,\\ 2^{-l(x)} & \text{if } x\sqsubseteq y,\\ 2^{-l(y)} & \text{if } y\sqsubseteq x,\\ 2^{-l(x\sqcap y)} & \text{otherwise.}\end{cases}$$

In view of the fact that if $x\sqsubseteq y$ then $x\sqcap y=x$ and if $y\sqsubseteq x$ then $x\sqcap y=y$, therefore for all $x,y\in\sum^\infty$ we can write

$$d_\sqsubseteq(x,y)=\begin{cases}0 & \text{iff } x=y,\\ 2^{-l(x\sqcap y)} & \text{otherwise.}\end{cases}$$

It is well known that $d_\sqsubseteq$ is a Baire metric and a complete metric on $\sum^\infty$ (see [29]). Define a fuzzy metric on $\sum^\infty$ by

$$M_{d_\sqsubseteq}(x,y,t)=\frac{t^2}{t^2+d_\sqsubseteq(x,y)}.$$

Then $(\sum^{\infty}, M, *)$ is a complete fuzzy metric space, where $a * b = ab$. For any $s-$increasing sequence $\{t_n\}$, $\prod_{n=1}^{+\infty} M(x, y, t_n)$ is convergent. Next, we consider the average case time complexity analysis of the sorting algorithm called Quicksort, according to [8, 19, 29, 30]. More precisely, it yields the following recurrence relation:

$$\begin{cases} T(1) = 0, \\ T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1), \ n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

For $\sum = [0, +\infty)$, say the set of nonnegative real numbers, in correspondence to T we consider the functional $\phi : \sum^{\infty} \to \sum^{\infty}$ that associates $\phi(x) := (\phi(x))_1 (\phi(x))_2 \cdots$ to $x := x_1 x_2 \cdots$, and is defined as follows:

$$\begin{cases} (\phi(x))_1 = 0, \\ (\phi(x))_n = \dfrac{2(n-1)}{n} + \dfrac{n+1}{n} x_{n-1}. \end{cases}$$

This implies that $l((\phi(x))) = l(x) + 1$ for all $x \in \sum^{\infty}$ and in particular $l((\phi(x))) = +\infty$ whenever $l(x) = +\infty$.

Next, we show that the functional ϕ has a fixed point by using our Theorem 2.1. Then, we consider the fuzzy mapping $F : \sum^{\infty} \to W_{\alpha}(\sum^{\infty})$ defined by $Fx = (\phi(x))_{\alpha}$ for all $x \in \sum^{\infty}$.
Let $\psi(r) = \frac{1}{\sqrt{2}}$ for all $r \in (0, 1]$ so that $\psi \in S$. Thus we distinguish the following two cases:

Case 1: If $x = y$, then we have

$$H_{M_{d_{\sqsubseteq}}}((\phi(x))_{\alpha}, (\phi(x))_{\alpha}, \frac{t}{\sqrt{2}}) = 1 = M(x, x, t).$$

Case 2: If $x \neq y$, then for all $t > 0$, we have

$$\begin{aligned} H_{M_{d_{\sqsubseteq}}}((\phi(x))_{\alpha}, (\phi(y))_{\alpha}, \frac{t}{\sqrt{2}}) &= M_{d_{\sqsubseteq}}((\phi(x))_{\alpha}, (\phi(y))_{\alpha}, \frac{t}{\sqrt{2}}) \\ &= \left\{ \frac{t^2}{t^2 + 2 d_{\sqsubseteq}((\phi(x))_{\alpha}, (\phi(y))_{\alpha})} \right\} \\ &= \left\{ \frac{t^2}{t^2 + 2 \cdot 2^{-(l((\phi(x))_{\alpha} \sqcap (\phi(y))_{\alpha}))}} \right\} \\ &\geq \left\{ \frac{t^2}{t^2 + 2 \cdot 2^{-(l(\phi(x \sqcap y)))}} \right\} \\ &= \frac{t^2}{t^2 + 2 \cdot 2^{-(l(x \sqcap y) + 1)}} \\ &= \frac{t^2}{t^2 + 2^{-l(x \sqcap y)}} = M(x, y, t). \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and F has a fixed fuzzy point $z = z_1 z_2 \cdots$, that is $z \in (Fz)_{\alpha}$. By definition of F, z is a fixed point of ϕ and so a solution to the recurrence relation for T. Hence we obtain

$$\begin{cases} z_1 = 0, \\ z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}, \ n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

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