An investigation into how the undergraduate mathematics topic of sibling curves of functions can be developed and used for student enrichment

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Submitted in partial fulfilment of the requirements for the degree

Philosophiae Doctor

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Declaration

I, the undersigned, declare that the thesis which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own, independent work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature: ....

Name: Harry Zandberg Wiggins

Date: ....

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Abstract

The aim of this research study is to investigate how undergraduate mathematics can be enhanced through research and how this enriched content can be exploited for the enrichment of academically stronger students. The study offers a unique blend of mathematical and educational research.

In the first part of the study a problem stemming from teaching the undergraduate topic of complex numbers, namely on how to represent the zeroes of functions, particularly polynomials, is researched. The notion of *sibling curves* ([51], [52]) offers an elegant and natural way to represent the zeroes of a polynomial, which is explored and expanded in this thesis. A library of sibling curves for well-known functions is developed and presented. Significant research results are that every polynomial of degree n has n sibling curves. This result gives a more geometric interpretation of the roots of a polynomial than the Fundamental Theorem of Algebra. I then focus on quadratic polynomials with complex coefficients and prove that, although the siblings are not always parabolas, the two sibling curves are always congruent and that they lie on a hyperbolic paraboloid determined by the coefficients of the polynomial. Some of these results are reported on in [115].

The second part of the study centres on utilising the researched knowledge on sibling curves for student enrichment. A group of first year students were guided through a number of designed activities using an inquiry-based learning approach to explore polynomials, complex numbers and ultimately sibling curves. Implementation of the programme as well as experiences are reported on, following a research approach of *evaluation research*. Student as well as facilitator experiences, discoveries and learning curves were captured in order to analyse perspectives. Results show that there is a need to stimulate and challenge academically strong undergraduate students. The study further shows that all the participants of this enrichment programme benefited from this experience. The students were engaged with the work and had the opportunity to delve deeper into the mathematical topic while sharpening their problem solving skills. I, as facilitator, had the opportunity to interact closely with academically strong students and experience their needs first hand, which added a new dimension to my teaching. This research

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also demonstrates how enrichment programmes can be a vehicle to expose enriched content to academically strong students.

The dual value of the study is that it adds not only to the knowledge base of complex number theory, but also to the body of reported experiences on student enrichment in undergraduate mathematics teaching.

It is envisaged that research findings reported on in this study will lead to an increased focus on student enrichment at tertiary level. This study exposes this element of teaching academically strong students and offers possible avenues of challenging these students.

Keywords: Undergraduate mathematics, complex numbers, zeroes of functions, polynomials, sibling curves, Fundamental Theorem of Algebra, academically strong students, student enrichment, inquiry-based learning, evaluation research, effects of an enrichment programme at university level.

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Outcomes of the study

One research paper [115] has resulted from the research with another [116] submitted, both in ISI journals. A third paper, based on the enrichment phase of the study, is in preparation, aimed to be submitted to a mathematical educational research journal. I was awarded the prize for best presentation at the departmental 2014 postgraduate seminar series for a presentation based on the reserch in this study. The research has been reported on at various conferences and international forums.

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The work presented in this thesis is a product of many people to whom I owe much gratitude.

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If I have been able to see further than other men, it is because I have stood on the shoulders of giants. (Sir Isaac Newton)

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Glossary

Calculus 1A - Calculus 1A course refers to the WTW114 course presented in the first semester of the first year for students in the mathematics stream at the University of Pretoria.

Calculus 1B - Calculus 1B course refers to the WTW128 course presented in the second semester of the first year for students in the mathematics stream at the University of Pretoria.

Calculus 2A - Calculus 2A course refers to the WTW218 course presented in the first semester of the second year for students in the mathematics stream at the University of Pretoria.

Linear Algebra 1 - Linear Algebra 1 course refers to the WTW126 course presented in the second semester of the first year for students in the mathematics stream at the University of Pretoria.

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Chapter 1

Introduction

1.1 Background

> *The good teacher makes the poor student good and the good student superior.*
> (Marva Collins)

Large classes are rapidly becoming the norm at the University of Pretoria where this study is set. Increased student numbers coupled with limited resources lead to large classes. This is especially true for many undergraduate courses. In these undergraduate courses teachers are faced with a large group of students with a broad spectrum of abilities and levels of interest in the subjects concerned. An environment is created where lecturers have the complex task of creating a learning environment suited to every student in the classroom and to also cater for the needs of every student, in particular for academically stronger students.

> *Education should aim to elevate each student to the level of his or her maximum ability – and do so in an atmosphere of genuine respect, integrity, trust, and compassion.* [107]

> *...high ability students need an appropriately challenging and supportive education environment where the instruction is within their zone of proximal development - neither too easy, nor too hard.* [73]

Academically stronger students face a higher level of boredom in schools than their peers as reported in educational literature ([31], [41], [62], [98]). More recent studies indicate that a lack

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of challenge in the classroom is largely responsible the this boredom of academically stronger students ([32], [90]). This lack of challenge at school level is often due to *Dumbed Down Curriculum* ([27], [35], [90]) or lack of meaningful differentiation ([2], [3]).

At university level the content is strong, but there is still a lack of meaningful differentiation. Often academically stronger students are in danger of not reaching their potential as content is commonly taught with the average student in mind. It is possible to spend valuable resources and effort on average to below average students but leaving the academically stronger students unchallenged and perhaps neglected.

Student enrichment is a way of nurturing academically stronger students to keep them enthusiastic and to help them discover their potential [100]. This study will investigate the experiences that student enrichment had on a select group of undergraduate students in mathematics.

1.2 Problem statement

My interest in student enrichment grew from my role as a lecturer and co-course coordinator teaching first year mathematics courses at the University of Pretoria. More and more students are being drawn to higher education, resulting in an increase in the numbers in first year undergraduate classes. The increase in the size of first year mathematics classes puts a strain on the teaching team; the teaching load increases; the tutoring load increases; the marking load increases and it becomes increasingly difficult to address the needs of specific subgroups.

Lecturers have a defined syllabus that they strictly stick to and thus rarely get the opportunity to explore an interesting topic or an idea that links up with a concept that is being taught in class. Furthermore time constraints to teach a certain topic within a specified number of lesson periods also put pressure on the lecturer for teaching sound problem solving skills instead of algorithmic type of solution methods.

> *When students acquire knowledge with understanding, they can apply that knowledge to learn new topics and solve new and unfamiliar problems.* [17]

Teaching time is limited and valuable. Lecturers need to cover much foundational material that will impact on later years. So the opportunity to enrich material is limited.

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Another challenge that lecturers of large classes face is that students come from diverse socio-economic backgrounds. Some students attended top private schools, others attended public schools, often situated in underpriviliged, remote rural areas. Some students had excellent high school mathematics teachers, other students were less fortunate. The mathematical skills and maturity and interest of the students vary greatly. It is not an easy task for the lecturer to pitch a lesson at the correct level. If the lecturer aims to teach for the average student, top students are not stimulated and may lose interest. If the lecturer spends time on enhancing a topic, s/he could lose a portion of the class as some students might not be mature enough to handle this.

Lecturers of undergraduate classes typically face a class of students who are studying for many different degrees. If students are planning to major in mathematics you would need to teach them not only the foundational knowledge, but also sound problem solving skills to prepare them for later years. If students are studying economics, physics, chemistry or biology you would like to introduce relevant examples where the mathematical concepts connect with their chosen fields to make sure that these concepts not just a mathematical curiosity but a language in which they can explain physical phenomena in other subjects.

Thus teaching in the current context is a complex task and we need to find creative ways to keep talented students stimulated. Mathematical enrichment is thought to offer a solution to this problem.

1.3 Aim and purpose of the study

The dual aim of this research study is to investigate how undergraduate content can be enhanced through research and how this enriched content can be exploited for student enrichment for academically stronger students.

To achieve this dual aim, I begin by taking a problem from an undergraduate mathematics topic and expand on the theory. Using this knowledge, I implement a student enrichment programme that involves first year students using evaluation research. The experiences of the participants are recorded using several instruments to evaluate the impact this enrichment case study had on them.

The driver of this study is the enhancement of mathematical content and the resulting enrichment possiblites are then illustrated through a single case study.

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The expectation is that the research findings reported on in this study will lead to more focus on student enrichment. It will expose this element of teaching mathematics students of high ability and offer possible avenues for challenging these students.

The purpose of the study is to add to the knowledge base of the topic of complex numbers and to the body of reported experiences on student enrichment in undergraduate mathematics.

1.4 Research questions

The research questions for this study are as follows:

1. **How can the undergraduate mathematical content be enhanced through research?**
2. **How can the enhanced undergraduate mathematical content be used for enrichment by means of inquiry-based learning?**
3. **What are the educational experiences of this particular student enrichment programme?**

To answer these research questions I report on two tasks undertaken. The first task was to expand on the undergraduate mathematics topic of complex numbers. One of the problems encountered when teaching complex numbers arises from an inability to visualize the complex roots of a polynomial in a convincing manner. *Sibling curves* ([50], [51]) offer an elegant and natural way to represent the zeroes of a polynomial. By expanding on this work, new and interesting results were obtained, which can be visually demonstrated using technology [115]. The work reported on in this paper forms a significant part of this thesis.

The second task was to design and implement an enrichment programme, based on the mathematics developed in the first part of the study, for a select group of students and then to investigate their perspective on the enrichment. A group of first year students was firstly exposed to the concept of *sibling curves* and was then guided to explore polynomials, complex numbers and ultimately sibling curves. Their experiences, discoveries and learning curves were captured in this study in order to analyse their perspectives on the student enrichment project.

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1.5 Context of this study

This section applies to the second part of the project, in which I conducted an enrichment case study involving first year students at the University of Pretoria, South Africa.

1.5.1 University of Pretoria

The study was undertaken at the University of Pretoria between May 2014 and December 2014. The University of Pretoria is a major research-oriented South African university that attracts students from diverse socio-economic backgrounds and from a wide range of secondary schools in South Africa and abroad. It is one of the largest universities in South Africa, hosting approximately 60 000 students.

1.5.2 Department of Mathematics and Applied Mathematics

The Department of Mathematics and Applied Mathematics at the University of Pretoria is one of the largest mathematics departments in South Africa. It was started in 1913 with only 38 students, but today has enrolment numbers of around 20 000 students in various mathematics modules. The Department of Mathematics and Applied Mathematics at the University of Pretoria offers three-year degree programmes in mathematics or applied mathematics, as well as service mathematics courses for other degrees such as engineering.

1.5.3 First year mainstream mathematics

The mainstream first year mathematics major course has a minimum entry level of a 7 (translating to a final mark of at least 80%) in the National Senior Certificate (NSC) and an Admission Point Score (APS) of 32 (out of a possible 42). The NSC is the main school-leaving certificate in South Africa. The calculation of the APS is based on a student's achievement in six recognised subjects by using a seven-point rating scale.

In the first semester students enrol for WTW114 (Calculus 1A). In the second semester they enrol for WTW128 (Calculus 1B) and WTW126 (Linear Algebra 1). Together these three courses are intended for students wishing to become professional mathematicians or secondary school mathematics teachers or for students who need to complete these courses as a co-requisite for their degree in actuarial science, physics, computer science, statistics, chemistry, etcetera.

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Due to a large number of students (see Table 1.1), the teaching in the first year is done typically in large groups (150+). The courses present weekly tutorials, in which students are given problems on the previous week's work. The tutorial groups are of a smaller size where students are given the opportunity to solve problems by themselves with the help of the teaching assistants available. Enrolment figures as well as pass rates for these courses for the period 2012 – 2014 are given in Table 1.1.

	Enrolment 2012	Pass rate 2012	Enrolment 2013	Pass rate 2013	Enrolment 2014	Pass rate 2014
Calculus 1A	833	63.4%	1029	68.2%	946	64.0%
Calculus 1B	870	57.4%	1106	71.8%	849	65.2%
Linear Algebra 1	1027	57.6%	1058	56.6%	967	77.6%

Table 1.1: Enrolment and pass rates for first year mathematics courses at the University of Pretoria

1.6 Structure of the thesis

This thesis is designed to explore the stated research questions mentioned in Section 1.4. The background of the study is sketched in Section 1.1, followed by the problem statement in Section 1.2. The aim of this study is outlined in Section 1.3. In Section 1.1 I indicated that my research focus was to expand on the theory of sibling curves and then to proceed to use this knowledge to implement a student enrichment project. This thesis consists of two parts: Part A (Chapters 2 to 5) documents the mathematical research on which I base the student enrichment project in part B (Chapters 6 to 9). In Part A I took a problem stemming from the teaching of the topic of complex numbers in mathematics, leading to expanding the mathematics involved. The knowledge is then ploughed back for enrichment of students as a project for first year mathematics students, which is documented in part B.

Part A consists of four chapters. In Chapter 2 we briefly look at various number systems and discuss the history of complex numbers as well as ways to visually represent the roots of polynomials, where the idea of sibling curves originated from. In Chapter 3 I create a library of examples of sibling curves for a variety of functions. Chapter 4 investigates the sibling curves of polynomials of any degree. This chapter includes methods to parametrize sibling curves and an enlightening result for any complex polynomial. In Chapter 5 we restrict ourselves to quadratic polynomials which I investigate in depth by looking at real quadratics and complex quadratics

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and I also extend the idea of sibling curves to the more general θ-*siblings*.

Part B consists of four chapters. In Chapter 6 I look at the existing literature on student enrichment to position my study. Chapter 7 discusses the methodology of the student enrichment study undertaken. Chapter 8 documents the results of the study. Chapter 9 concludes this study by discussing conclusions, value of the study, recommendations and further research that can be done on student enrichment.

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Chapter 2

Representing the zeroes of polynomials

2.1 Introduction

When solving for the zeroes of polynomials you quickly realise that most often one or more of the zeroes are not real numbers, but are complex numbers. The topic of complex numbers forms an essential part of any first year mathematics syllabus at university. One of the problems encountered when teaching this topic arises from an inability to visualize the complex zeroes of a polynomial. In this chapter we look at the following question:

How can we represent the zeroes of polynomials?

Before we tackle this important question, we look at various number systems in Section 2.2. We start by looking at the natural numbers. Then we move to the integers, rationals, real numbers, complex numbers and even hypercomplex numbers. In Section 2.3 we take snapshots of the development of complex numbers. In Section 2.4 we finally tackle the question on how to represent the zeroes of polynomials by looking at various attempts to do this.

2.2 Number systems

$$\mathbb{N}$$

As humankind evolved it was no longer enough to have a sense of more or less, even or odd, etcetera. It was important to count. Thus the need for natural numbers $1, 2, 3, \ldots$ arose. The Sumerians around 3000 BC were the first to develop a counting system to keep track of their stock of goods, cattle, horses, donkeys and other possessions. The Sumerian system was passed

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along to the Akkadians around 2500 BC and then to the Babylonians in 2000 BC.

Today we are still using the Hindu-Arabic numeral system which was invented between the 1st and 4th centuries AD by Indian mathematics, $1, 2, 3, \ldots,$ the natural numbers. The set of natural numbers is denoted by $\mathbb{N}$.

$$\mathbb{N} \subseteq \mathbb{Z}$$

The Babylonians [60] first conceived a mark to indicate that a number was absent from a column, like the number 0 in 1203 signifies that there are no tens in that number. This was the basis for the positional number system used today. In this system the few symbols $0, 1, 2, \ldots, 9$ are sufficient for writing down any positive integer. In contrast the Maya Long Count Calendar required the use of a zero as a place-holder within its vigesimal (base 20 number system) positional number system. The Mayans used a shell glyph as a zero symbol. See Figure 2.1. Although this was a good start, it took a few centuries before the Indian mathematicians began to understand zero. Brahmagupta around 650 AD wrote down standard rules for dealing with zero.

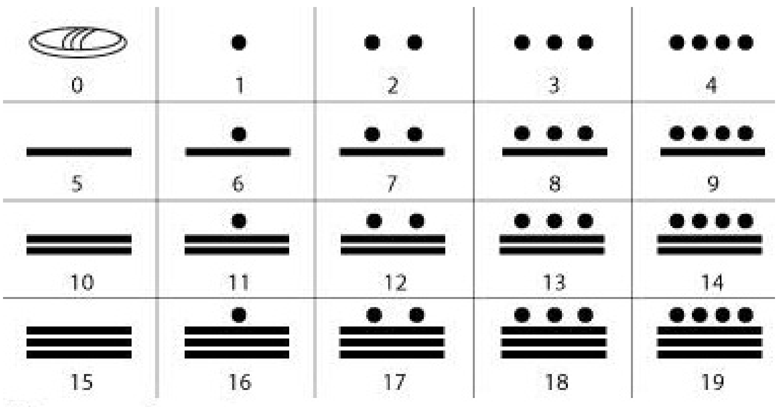

Figure 2.1 : Maya number system

The acceptance of zero and negative numbers was also motivated by the emergence of economy to deal with the concepts of profit, loss and breaking even. For a while negative numbers were considered "false". Diophantus, a third century Greek mathematician, referred to an equation that was equivalent to $4z + 20 = 0$ in *Arithmetica* as an *absurd* equation, because it had a negative solution.

Historically negative numbers first appeared in *Nine Chapters on the Mathematical Art*, which dates from the period of the Han Dynasty approximately 202 BC - AD 220. They used red rods

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for positive numbers and black rods for negative numbers. The natural numbers, supplemented by zero and the negative numbers form the set of integers which we denote by $\mathbb{Z}$. In 1759 English Mathematician Baron Francis Masères ([61], p. 119) wrote the following about negative integers :

> *They serve only, as far as I am able to judge, to puzzle the whole doctrine of equations and to render obscure and mysterious things that are in their own nature exceedingly plain and simple...It were to be wished therefore that negative roots had never been admitted into algebra or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many learned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed.*

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

The English word *fraction* comes from the Latin word *fractus* which means *broken*. The earliest fractions were reciprocals of integers, e.g. $\frac{1}{2}$ or $\frac{1}{3}$. The Egyptians called them *Egyptian fractions*. However, the method of placing one number below the other and working with fractions in this way is found in Indian mathematician Aryabbhatta's work in AD 499. Another strong motivation for the development of fractions comes from dealing with equations like $4z = 7$ and $10z = 17$. Today we represent all *rational numbers*, that is those fractions whose numerator and denominator are both integers, by the set $\mathbb{Q}$.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

It is rather impressive that ancient civilisations were aware that there are numbers other than rational numbers. A Babylonian clay tablet with writing in cuneiform labelled YBC 7289 dating from 1800-1600 BC gives an approximation of $\sqrt{2}$ to four sexagesimal (base 60) figures as $1, 24, 51, 10$. This number can be converted to $1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \frac{30547}{21600} = 1.41421\overline{296}$. This approximation is accurate to about six decimal digits.

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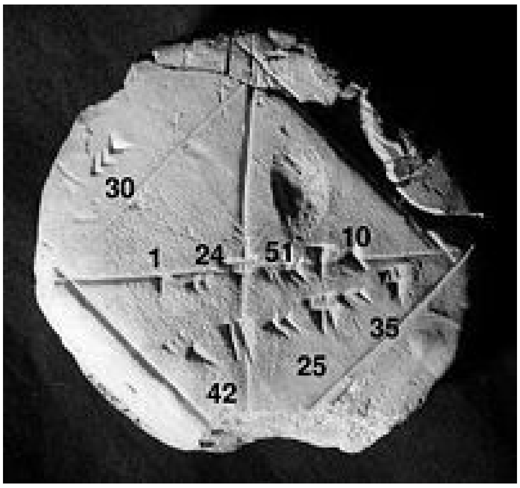

Figure 2.2 : Babylonian clay table YBC 7289 with annotations

Pythagoras of Samos was a Greek philosopher and mathematician. He was also the founder of the religious movement called Pythagoreanism around 5 BC. Little is known with certainty about the discovery that the square root of two is *irrational*, but the name of Hippasus of Metapontum is often mentioned in this regard. The Pythagoreans treated the fact that $\sqrt{2}$ was an irrational number as a secret. However it is rumoured that Hippasus was murdered for divulging this secret.

It turns out there are uncountable many irrational numbers. In 1744 Euler proved that e is an irrational number and in 1764 Johann Heinrich Lambert proved that π is an irrational number. Combining the set of rational and irrational numbers we get the set of *real numbers* denoted by $\mathbb{R}$. Axiomatically the reals can uniquely be defined and are sometimes called the only *complete ordered field*.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

When solving equations we realise that $z^2 - 1 = 0$ has two real roots, but $z^2 + 1 = 0$ has no real roots. So if we want to create solutions for $z^2 + 1 = 0$, we need a larger number system. This new number system is called the set of *complex numbers* and is denoted by $\mathbb{C}$.

Surprisingly the discovery of complex numbers arose from the desire to solve cubic equations and not from the need of solving quadratic equations. A short history of complex numbers appears in the next section. Another interesting property of complex numbers is that any polynomial with complex coefficients always has complex roots. This was not the case for real polynomials. For example $z^2 + 1 = 0$ is a polynomial with real coefficients, but it has no real roots. However we lose the property of order when enlarging from real numbers to complex

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numbers.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O} \subseteq \mathbb{S}$$

Sir William Rowan Hamilton generalised complex numbers to *quaternions* in 1843. This larger set is denoted by $\mathbb{H}$. In this new number system multiplication is no longer commutative and the equation $z^2 = 1$ now has infinitely many solutions. Other examples of hypercomplex numbers include the *octonions* denoted by $\mathbb{O}$. The octonians are an expansion of the set of quaternions and in this number system multiplication is no longer associative. Applying the Cayley-Dickson construction to the octonions, you can form an even larger number system called the *sedenions* usually denoted by $\mathbb{S}$. These are not the only examples of hypercomplex numbers. In 1876 William Kingdon Clifford introduced more examples of hypercomplex numbers called *Clifford algebras*.

2.3 History

Here I used ([8], [25], [45], [49], [72], [74], [77], [81], [113]) to compile a short history of complex numbers in terms of snapshots that capture the development of complex numbers.

- An ancient Egyptian papyrus from around 1200 BC contains the first known solution to a quadratic equation. This document also contains some medical knowledge such as the first known documented pregnancy test. In this papyrus [45] they posed the following problem, which suggests that they had some knowledge of quadratic equations.

  > *The area of a square of* 100 *is equal to that of the two smaller squares, the side of one square is* $\frac{3}{4}$ *of the other. What are the sides of the two unknown squares?* $(x^2 + (\frac{3}{4}x)^2 = 100)$

  The Babylonians of Mesopotamia (presently Iraq) prepared the groundwork for Al-Khwarizmi. Al-Khwarizmi (about 790-850) was an Islamic mathematician who wrote about Hindu-Arabic numerals and in his algebra treatise *Hisab al-jabr w´al-muqabala* gave the solution for quadratic equations of various types. He restricted himself to equations with positive solutions. His proofs were geometric in nature.

- Omar Khayyam (1050-1123) was a Persian astronomer, famous poet and leading mathematician of his time. In 1079 he solved cubic equations geometrically by intersecting parabolas and circles. He incorrectly speculated the non-existence of a general formula to solve cubic equations.

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- Gerard of Cremona (1114-1187) was a famous Italian translator of scientific works from Arabic into Latin. He translated 87 books from the Arabic language including the mathematical work of Al-Khwarizmi into Latin placing Italian mathematics on a strong footing.
- Around 1225 Emporer Frederick II held court in Sicily. Johannes of Palermo, a member of Emperor Frederick's court posed several problems. One of the problems was to solve $z^3+2z^2+10z=20$. Leonardo da Pisa, also known as Fibonacci, showed that the solution of the equation was not a whole number, nor a rational number, nor any of the Euclidean irrational magnitudes. Fibonacci did however find a rational approximation to this cubic equation.
- Great strides towards the development of complex numbers were made by Italians during the Renaissance period. The general cubic equation $z^3+az^2+bz+c=0$ was changed to a simpler form $z^3+pz+q=0$ by introducing a new variable $z'=z+\frac{1}{3}a$. This substitution appeared for the first time in two anonymous manuscripts near the end of the 14th century.

If we only desire terms with positive coefficients, then there are three cases to consider. They are collectively known as depressed cubics :

$$\begin{aligned} z^3+pz &= q \\ z^3 &= pz+q \\ z^3+q &= pz \end{aligned}$$

The first to solve $z^3+pz=q$ was Scipione del Ferro, a professor at the University of Bologna. On his deathbed in 1526 del Ferro confided his formula to his student Antonio Mario Fiore. During this period it was custom to keep mathematical discoveries secret [49] in order to obtain an advantage over fellow mathematicians by proposing problems beyond their capabilities.

Fiore decided to challenge Nicolo Fontana (1506-1557) to a mathematical contest in 1535. The rules were that each gives the other 30 problems to solve within 40 to 50 days. The winner would be the one who had solved the most problems. When Fontana was a boy, he was badly cut by a French soldier resulting in a speech impediment, and he was sometimes called Tartaglia, the stammerer. He was too poor to attend school, so he taught himself mathematics and became a teacher of this subject, first in Verona and then later in Venice.

The night before the contest Tartaglia rediscovered the formula to solve all depressed cubics and won the contest by solving all of Fiore's problems within two hours. One of

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the problems posed by Fiore was the following:

> *A man sells a sapphire for* 500 *ducats, making a profit of the cube root of his capital. How much is the profit?* ($x^3 + x = 500$)

Girolamo Cardano (1501-1576) was impressed by the victory and visited Tartaglia. He persuaded Tartaglia to confide the formula to him which he did in the form of a capitolo poem. (If the lines of the poem are represented by letters, and the same letter means that those lines rhyme, the pattern of the poem is aba,bcb,cdc, etcetera.)

Cardano had to sign an oath of secrecy to keep it secret until Tartaglia would publish it himself. Cardano was able to reconstruct the proof from knowing the formula. Cardano also learned that del Ferro had the formula. He verified this information by interviewing relatives who gave him access to del Ferro's work. Cardano then published his solution in *Ars Magna*, 1545. It should be noted that Cardano gave credit to del Ferro by making him the first author and Tartaglia who obtained the formula later in an independent manner.

For the solution of $z^3 = pz + q$ we replace $z = u + v$ in the equation. We obtain $z^3 - pz = u^3 + v^3 + 3uv(u + v) - p(u + v) = q$. By setting $3uv = p$, we obtain $u^3 + v^3 = q$ and $u^3v^3 = (\frac{p}{3})^3$. That is the sum and the product of u^3 and v^3 is known. These two equations can now be used to form a quadratic equation that can easily be solved. It turns out that $z = u + v = \sqrt[3]{\frac{q}{2} + w} + \sqrt[3]{\frac{q}{2} - w}$ where $w = \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}$.

The so-called *casus irreducibili* is when the expression under the square root sign of w is negative. Cardano avoids dealing with this case in his *Ars Magna*. Cardano was the first to introduce complex numbers as $a + \sqrt{-b}$ into algebra. He had misgivings about this, but in Chapter 37 he asks to divide 10 into two parts whose product is 40. His answers were $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Adding them you get 10 and "putting aside the tortures involved", by multiplying you get

$$\begin{aligned}(5 + \sqrt{-15})(5 - \sqrt{-15}) &= 25 - 5\sqrt{-15} + 5\sqrt{-15} - (\sqrt{-15})^2 \\ &= 25 - (-15) \\ &= 40\end{aligned}$$

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However Cardano's formula did not work for the equation $z^3 = 15z + 4$. Applying his formula, you get $w = \sqrt{2^2 - 5^3} = \sqrt{-121}$ and $z = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. Rafael Bombelli resolved this *casus irreducibili* in his *l'Algebra* (1572-1579), a set of three books. Bombelli considered complex numbers as a linear combination with positive coefficients with four basis elements : "piu" being +1, "meno" being -1, "piu di meno" being $+\sqrt{-1}$ and "meno de meno" being $-\sqrt{-1}$.

Bombelli observed that the cubic has $z = 4$ as a solution and explains this by showing $\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$ and $\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}$, since $(2 + \sqrt{-1})^3 = 2 + 11\sqrt{-1}$ and $(2 - \sqrt{-1})^3 = 2 - 11\sqrt{-1}$. After doing this Bombelli commented:

> *At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof.*

Thus $z = 4$ is a solution of $z^3 - 15z - 4 = 0$. Factoring yields $z^3 - 15z - 4 = (z - 4)(z^4 + 4z + 1)$. The remaining two real roots can now be found using the quadratic formula which has roots $-2 \pm \sqrt{3}$. So starting with a problem using real numbers we needed the use of complex arithmetic to find all the real solutions.

- René Descartes (1596-1650) was a French philosopher whose work, *La Géométrie*, demonstrated his application of algebra to geometry from which we obtained the Cartesian plane. Descartes associated imaginary numbers with the geometric impossibility by using a geometric construction to solve the equation $z^2 = az - b^2$ where a and b^2 are both positive. According to [8] Descartes coined the term *imaginary*. Descartes also had a rule of signs to determine the maximum number of positive real and negative real roots a polynomial can possess.

- John Wallis (1616-1703) was an English mathematician who published his notes on *Algebra* in 1685. In this document he rejected the absurd idea that negative numbers were nothing. Wallis gave a physical explanation based on a line with a zero mark. Positive numbers are numbers at a distance from the zero point to the right, whilst negative numbers are numbers at a distance to the left of zero. So he is generally credited for being the originator of the number line. He also made some progress in giving a geometric interpretation to $\sqrt{-1}$.

- Abraham de Moivre (1667-1754) was a French mathematician who left France to seek refuge in London. There de Moivre befriended Isaac Newton. In 1698 he mentioned that Newton knew an equivalent expression to what is known today as de Moivre's theorem:

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$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

where n is an integer. This formula gives an important connection between complex numbers and trigonometric functions. Supposedly Newton used this formula to calculate cube roots that appeared in Cardano's formulas. This formula can also be used to show that $\sqrt{i}$ can be written in the form $a + bi$ where a and b are real numbers, and thus proved that Leibniz incorrectly thought in 1702 that $z^4 + 1$ was a counter-example to the Fundamental Theorem of Algebra.

- Leonard Euler (1707-1783) was a Swiss mathematician who introduced the notation $i = \sqrt{-1}$ [25] and visualized complex numbers as points with rectangular coordinates. Euler also used the equation $x+iy = r(\cos\theta+i\sin\theta)$ to connect the rectangular coordinates with polar coordinates and visualized the roots of $z^n = 1$ as the vertices of a regular polygon. He also defined the complex exponential and proved the identity $e^{i\theta} = \cos\theta + i\sin\theta$. Using this formula it follows that $i^i = (e^{\frac{i\pi}{2}})^i = e^{\frac{-\pi}{2}} = \frac{1}{\sqrt{e^\pi}}$, showing that an imaginary power of an imaginary number can be a real number.

- Caspar Wessel (1745-1818) was a Norwegian mathematician who obtained and published a suitable presentation of complex numbers. Wessel presented in 1797 to the Royal Danish Academy of Sciences his paper *On the Analytic Representation of Direction: An Attempt.* The quality of the paper was judged so high that it was the first paper to be accepted for publication by a non-member of the academy. Wessel's approach used what we call *vectors* today. Wessel used the geometric addition of vectors (parallelogram law) and defined multiplication of vectors in terms of what we call adding the polar angles and multiplying magnitudes.

- Jean-Robert Argand (1768-1822) was a Parisian bookkeeper. He produced a pamphlet in 1806 titled *Essai sur une maniére les quantités imaginaires dans les constructions géométriques* (Essay on the geometrical interpretation of imaginary quantities). One of the copies ended up in the hands of mathematician Adrien-Marie Legendre who in turn mentioned it in a letter to Francois Francais, a professor of mathematics. His brother Jaques, a professor of military art and a mathematician, inherited his papers when he died. Jacques found Legendre's letter describing Argand's mathematical results, but Legendre failed to credit Argand. Jaques then published an article in 1813 in the *Annales de Mathémathiques* giving the basics of complex numbers. In the very last paragraph Jaques acknowledged Legendre's letter and encouraged the unknown author to come for-

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ward. Argand heard about this and his response appeared in the next edition of the journal. Accompanying Argand's reply was a short note by Francais in which he declared Argand to be the first to develop the geometry of the imaginaries.

Peter Rothe wrote in his book *Arithmetica Philosophica* in 1608 that a polynomial equation of degree n (with real coefficients) may have n solutions. Albert Girard in his book *L'invention nouvelle en l'Algèbre* in 1629 claims that a polynomial equation of degree n has n solutions, but Girard does not state that they had to be real numbers. Many tried their hand at this, including Euler (1749), de Foncenex (1759), Lagrange (1772) and Laplace (1795). Near the end of the 18th century, 1788, James Wood published his attempt, but it had an algebraic gap.

This result is known as the *Fundamental Theorem of Algebra*. Argand is also known for producing an elegant proof of the Fundamental Theorem of Algebra in his 1814 work *Réflexions sur la nouvelle théorie d'analyse* (Reflections on the new theory of analysis). Argand was the first to complete a rigorous proof of this theorem. He was also the first to generalize the theorem to include polynomials with complex coefficients. Argand's proof was merely an existence proof. However in 1940 German mathematician Hellmuth Kneser modified Argand's proof into a constructive proof.

- Niels Hendrik Abel (1802-1829) was a Norwegian mathematician who proved that there is no general formula for the roots of a polynomial of degree 5 or greater in terms of its coefficients. Abel proved that polynomials of degree 5 or more cannot be solved by radicals, that is using arithmetical operations and surds. This result is known as the *Abel-Ruffini theorem*. This was a surprising result for polynomials, as the formula for quadratic equations was known long ago and in the 16th century Cardano and Targalia found formulas for cubic and quartic equations.

- Sir William Rowan Hamilton (1805-1865) was an Irish mathematician who used ordered pairs of real numbers (a, b) to define complex numbers as a couple in his 1831 memoir. Hamilton then defines addition and multiplication of couples as $(a, b)+(c, d) = (a+c, b+d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This is in fact an algebraic definition of complex numbers. However he is better known for inventing the quaternions.

- Johann Carl Friedrich Gauss (1777-1855) was a famous German mathematician. His doctoral dissertation was on the Fundamental Theorem of Algebra. He ended up providing four proofs of the Fundamental Theorem of Algebra. His first attempt published in 1799 was mainly geometric, but it had a topological gap. Alexander Ostrowski filled this gap

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in 1920. Gauss is also credited with introducing the term *complex number* and in a letter to Bessel, Gauss points out a result that was later known later as Cauchy's theorem. This result was not published and was later rediscovered by Cauchy and Weierstrass.

- Augustin-Louis Cauchy (1789-1857) was a French mathematician who developed complex function theory in an 1814 memoir submitted to the French Académie des Sciens. Cauchy's idea of *analytic functions* was not explicitly mentioned in this memoir, but the concept was basically there. Cauchy also published a paper in 1825 on *contour integrals*, but Cauchy was not the first. Poisson published a 1820 paper that contained a path that was not on the real line.

Cauchy also showed in 1847 that the complex numbers are isomorphic to a quotient ring of real polynomials factored by the ideal generated by the polynomial $z^2 + 1$. In this construction we "force" the equation $z^2 + 1 = 0$ to have two solutions. Cauchy also said the following about complex numbers:

> *We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give it.*

2.4 Representation of zeroes

In this section we look at some attempts to visualize the roots of polynomials. We consider various attempts from ([5], [54], [75], [65], [66], [67], [78], [91], [103], [110], [111]). Our final idea [30] leads to the concept of sibling curves which is the main focus of this thesis. Other attempts can be found ([4], [11], [22], [23], [29], [48], [89], [96], [108]).

2.4.1 Rotating the parabola

In [75] Narayan focuses on quadratic real polynomials with no real roots. If the quadratic $Az^2 + Bz + C = 0$ has no real roots when $B^2 - 4AC < 0$. Note the roots of $Az^2 + Bz + C = 0$ and the monic polynomial $z^2 + bz + c = 0$ are the same when $b = \frac{B}{A}$ and $c = \frac{C}{A}$. So they focus on this 'Standard Complex Quadratic Function' $z^2 + bz + c = 0$ where $b^2 - 4c < 0$.

Narayan created a three dimensional axis system, where the X-axis is the real axis and the Y-axis is the imaginary axis. Narayan then sketches the parabola in the XZ-plane. Drawing the cylinder $x^2 + y^2 = c$, Narayan visualises the roots of the four equations $\pm(Az^2 \pm Bz + C) = 0$ by the intersection of this cylinder with the plane $x = \frac{-b}{2}$.

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For example, consider the polynomial $3z^2 - 6z + 12 = 0$. The standard form is $z^2 - 2z + 4 = 0$. Now we draw the cylinder $x^2 + y^2 = 4$. The plane $x = -\frac{-2}{2} = 1$ cuts cylinder at $y = \pm\sqrt{3}$. Thus the roots are $1 \pm \sqrt{3}i$ as shown in the diagram below at points R1 and R2.

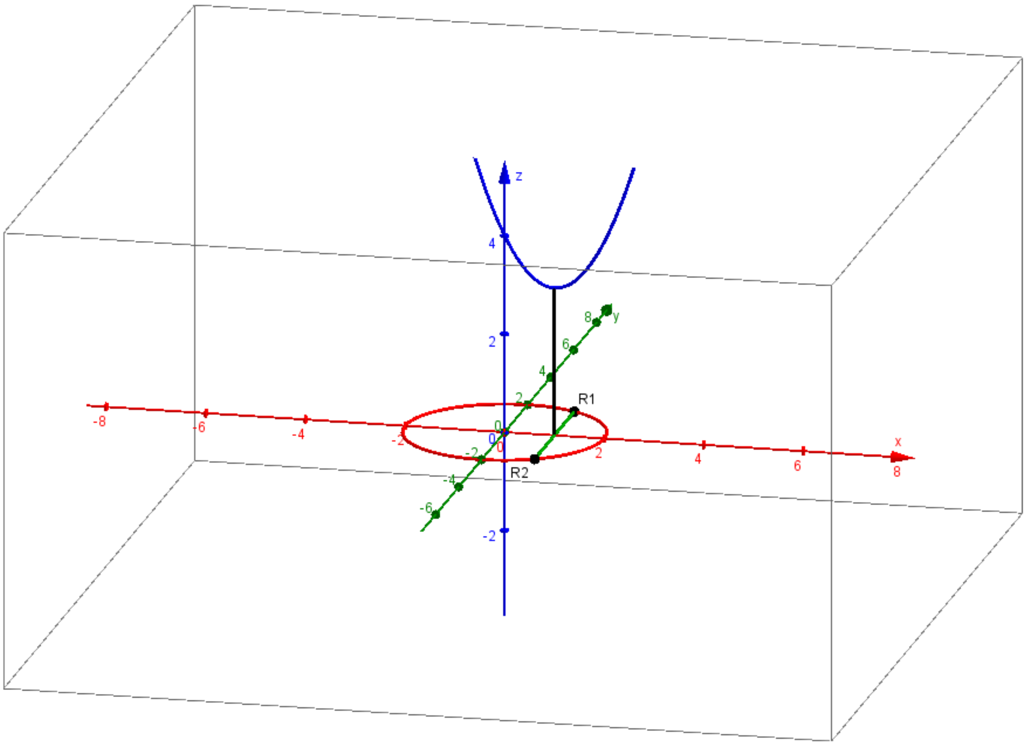

Figure 2.4.1 : Roots of $z^2 - 2z + 4 = 0$.

This method uses the idea that the real part of the roots are the critical numbers, that is those numbers which make the derivative zero. If we consider the cubic $z^3 - 1 = 0$ we see that 0 is the only critical number. However the complex roots are $\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$. Note the real parts of the complex roots are not equal to the critical numbers.

Although this method allows us to visualize the zeroes in three dimensions, the down side is that it only works for certain quadratic equations. If the roots are real, you need to use the normal two dimensional way of visualising the roots. If the roots are complex, you can use this three dimensional way of visualising the roots. The other downside to this method is that it is not easily extended to polynomials of degree three or higher.

2.4.2 Superimposing the Argand plane onto the Cartesian plane

A very popular idea is to superimpose the Argand plane onto the Cartesian plane. That is to read the complex roots from a graph by relating the point (x, y) to the complex number $x + iy$, transforming the point (x, y) into the complex number $x + iy$.

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In [78] Norton considers $f(z) = (z-a)^2 + b^2$ where b is a non-zero real number. This quadratic has two complex roots $a - bi$ and $a + bi$. So if we sketch $f(z)$ in the Cartesian plane, it does not intersect the x-axis. For the method the author reflects the curve $f(z)$ about its turning point. See the Figure 2.4.2.1. The new curve cuts the x-axis twice. Proceed to take the midpoint of these two points as the centre of a circle passing through these two points. Now we pretend we are in the Argand plane. The topmost and bottommost point will be the roots of this quadratic.

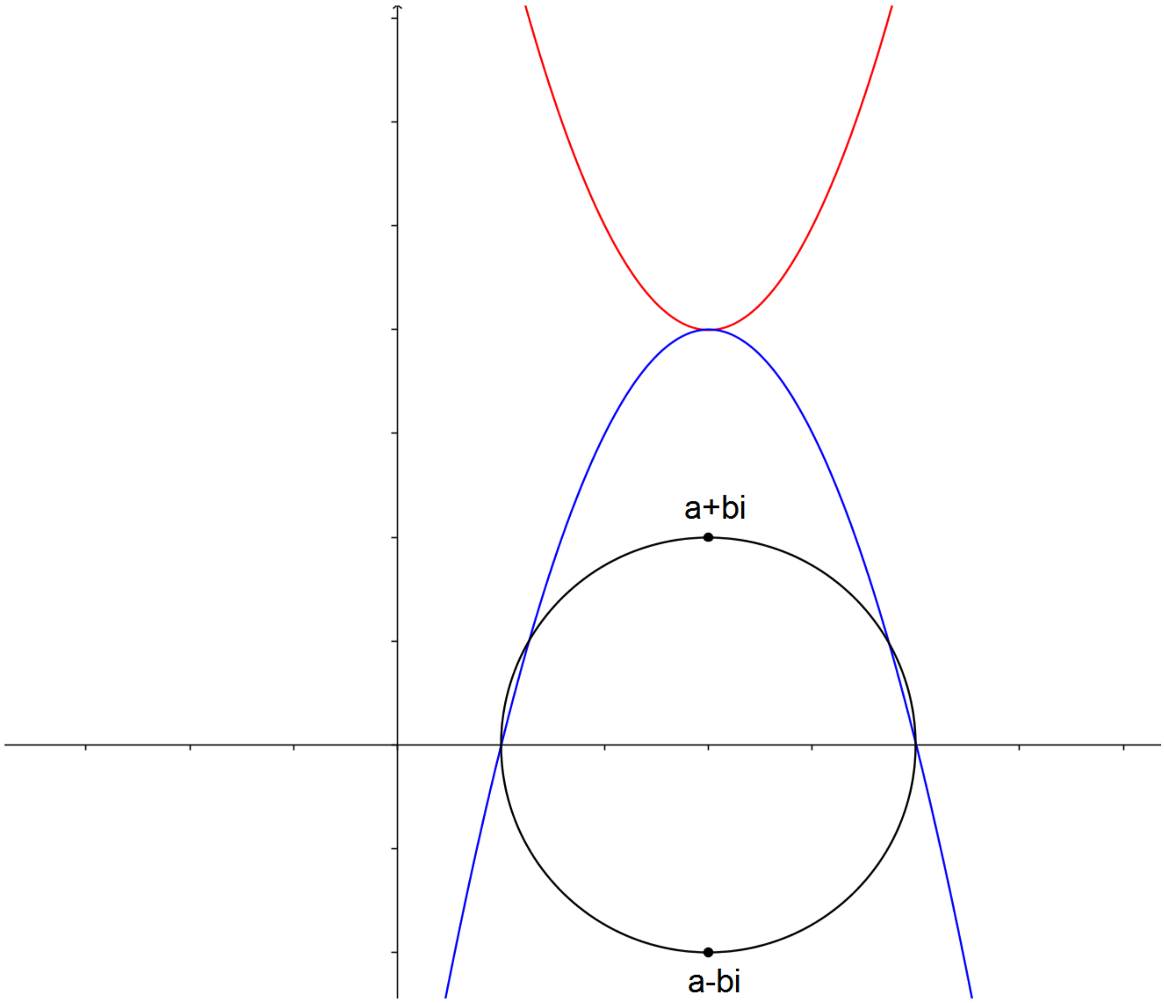

Figure 2.4.2.1 : Roots of a real quadratic polynomial

It should be noted that for a quadratic equation $f(z) = (z-a)^2 + b^2$, we see the turning point has coordinates (a, b^2). Now label a, the abscissa as the real part of the roots. The ordinate b^2, is the square of the imaginary part, giving the two complex roots $a - bi$ and $a + bi$. So from the turning point, the complex roots can be deduced.

In papers by Gleason, Irwin and Crawley ([46], [55], [22]) they do similar constructions to find the roots for cubic polynomials. By sketching a cubic function in the Cartesian plane, we will find all the real roots. Suppose a cubic function has one real root $z = p$ and two complex

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$z = a \pm bi$. To find the complex roots we draw a tangent line to the curve through the point $P(p; 0)$. This line cuts the curve at the point T. Here T has abscissa a and the line has slope b^2. This construction is illustrated in Figure 2.4.2.2.

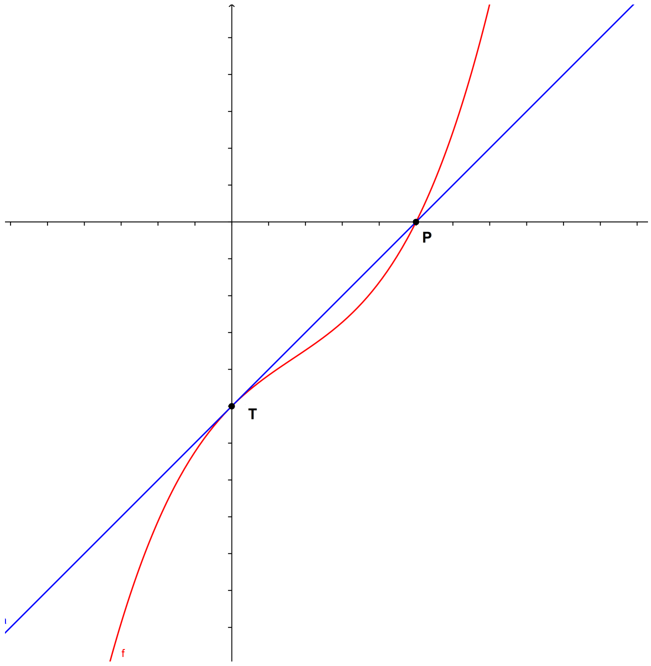

Figure 2.4.2.2 : Roots of a real cubic polynomial

This approach was extended by Yanosik [119] to the quartic case. Suppose $p(z)$ is a polynomial of degree 4 with two real roots r_1 and r_2. There is a unique parabola of the form $y = m(z - r_1)(z - r_2)$ that is tangent to the curve $y = p(z)$ at point T. Let a be the abscissa of T and let b^2 be the slope of the tangent line at T. Then the two remaining complex roots are $a \pm bi$. This is depicted Figure 2.4.2.3.

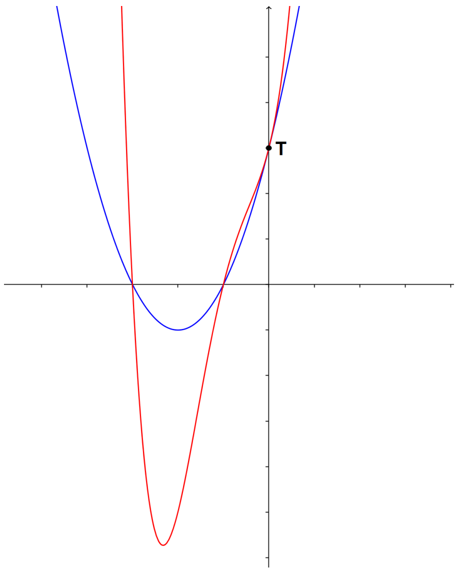

Figure 2.4.2.3 : Roots of a real quartic polynomial

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In [43] Gehman proves a more general result : Suppose $p(z)$ is a polynomial of degree $n \geq 2$ with complex roots $a \pm bi$. Suppose $p(z) = (z^2 - 2az + a^2 + b^2)f(z)$ where $f(z)$ is a polynomial of degree $n-2$ with real coefficients. Then there is a unique curve $y = mf(z)$ which intersects the curve $y = f(z)$ on the x-axis in the $n-2$ real or complex points whose abscissa satisfies the equation $f(z) = 0$ and this curve is tangent to the curve $y = p(z)$ at a point T having the property that the derivatives of the two functions at the point are equal from the first to the kth where k is the order of the highest ordered non-vanishing derivative at the point T. The abscissa of T is a and the slope of the tangent line is b^2.

2.4.3 Moving between the Cartesian and Argand planes

In this section we look at a new idea from Ward [114] published in 1937. Ward moved between the Cartesian and Argand planes to find the roots of a cubic equation. Earlier on we saw that any monic cubic can be reduced to $f(z) = z^3 + pz + t$ by a linear transformation.

Suppose $u+iv$ is a root of $z^3+pz+t = 0$. Substituting into this cubic gives $v(-v^2+3u^2+p) = 0$. So if the root is complex then v is not zero and we must have

$$-v^2 + 3u^2 + p = 0$$

This equation is a hyperbola and is the locus of the complex roots of the cubic in the Argand plane. Ward's approach is to sketch $y = -x^3 - px$ in the Cartesian plane. The intersection of this graph with $y = t$ gives the real root of the cubic. Suppose point A is the intersection and the coordinates are (r, t). This is indicated in Figure 2.4.3.1.

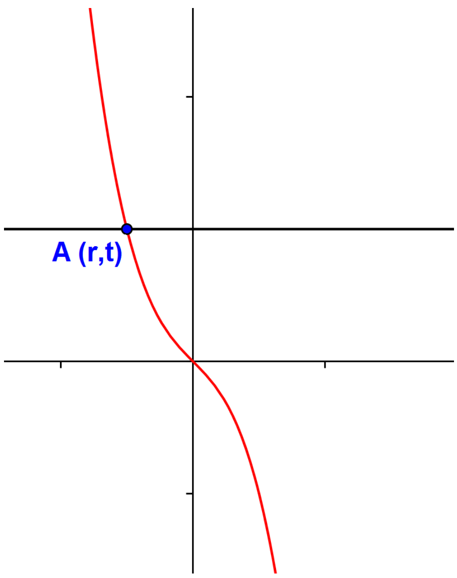

Figure 2.4.3.1 : Graph of $y = -x^3 - px$ in the Cartesian plane

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Viéta's formula states that the sum of the n roots of the polynomial $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 + a_0$ is $\frac{-a_{n-1}}{a_n}$. Using the Viéta formula, we see that the sum of the three roots of $z^3 + pz + t = 0$ is 0. If the polynomial had real coefficients, then the roots will form a conjugate pair and hence the other two complex roots will have real part $\frac{-r}{2}$. Now we move onto the Argand plane. We draw the line $x = r$ in the Argand plane, then we draw the line $x = \frac{r}{2}$ and then we reflect in the imaginary axis to draw the line $x = \frac{-r}{2}$.

We then proceed to sketch the hyperbola $-v^2 + 3u^2 + p = 0$ in the Argand plane. The imaginary roots, R1 and R2 can now be formed by the intersection with this line and the hyperbola. The intersection gives the complex roots of this cubic as seen in Figure 2.4.3.2.

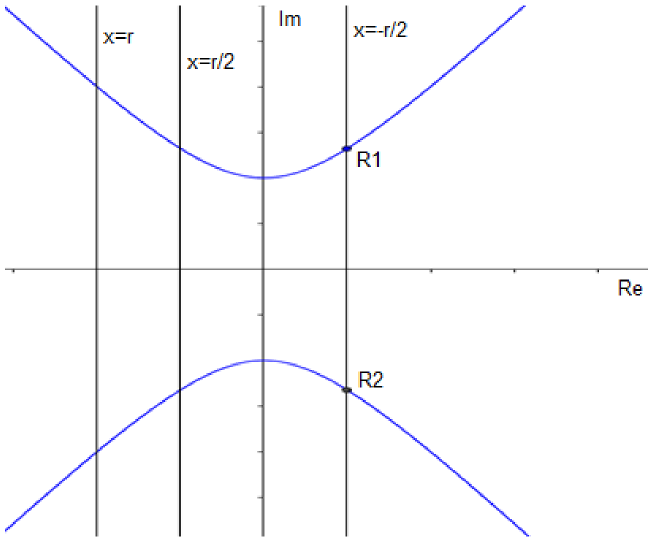

Figure 2.4.3.2 : Complex roots of $z^3 + pz + t$

A similar procedure can be followed for quartic polynomials. Considering the reduced quartic equation $z^4 + pz^2 + qz + t = 0$. We can again suppose $u + vi$ is a complex root and by substituting we obtain

$$4uv^3 - 4u^3 - pu - q = 0$$

This is the locus of the complex roots in the Argand plane. This graph has three fixed asymptotes, $u = 0$, $u = v$ and $u = -v$ which are independent of p and q. Therefore this graph has three distinct branches and will contain the complex roots of the quartic equation. This also shows that the higher the degree of the polynomial, the more computational effort is needed to represent the complex roots.

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2.4.4 Modulus surface

This method ([65], [66], [67]) uses the modulus of a complex number $z = x + iy$, that is $|z| = |x + iy| = \sqrt{x^2 + y^2}$. In this approach the horizontal plane is the Argand plane and the vertical axis is a real axis. So again this approach gives us a three dimensional way of representing the roots of the polynomial.

For each complex number z we sketch the points $(z, |f(z)|)$ in three dimensions, thus making use of this three dimensional set of axes to have a representation of a function. The graph of the function $|f(z)|$ is called a modulus surface. The roots of the polynomials are represented when the modulus surface meets the Argand plane.

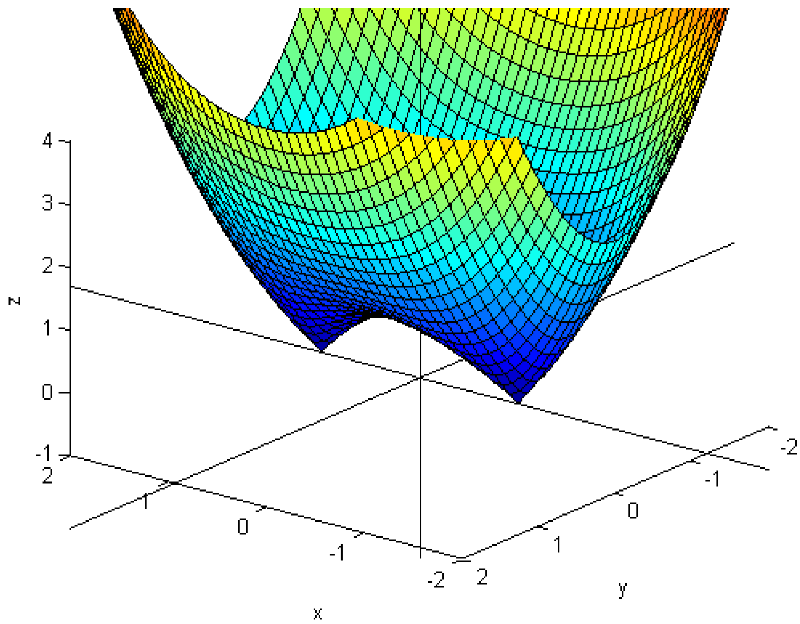

Figure 2.4.4 : Modulus surface of $f(z) = z^2 + 1$

The visual advantage of using a modulus surface representation is clear. It gives a three-dimensional picture where the roots can be found when the modulus surface has minimum points or when the surface touches the Argand plane. Surfaces are not easily drawn by hand but with the aid of computer graphics they are easy to draw. Technology also allows us to visualize how the zeroes of a polynomial change when coefficients are changed. In fact Long acknowledges [67] that some of the main theorems he discovered were making use of computer experimentation.

Computer graphics inspired the work of Velleman [110] who uses colour coding to give a visual representation of a function mapping complex numbers onto complex values.

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Although both representations are visual and illustrative, they do not give a clear picture of functions that map complex numbers onto complex numbers. It should be noted that using the modulus of complex numbers, the function becomes distorted and is no longer a representation of the original function. The roots are presented but not directly related to the function.

2.4.5 Fehr's idea

In [51] Harding and Engelbrecht explore an idea that appeared in an American secondary school textbook [30]. The author, Howard Fehr, was a past president of the National Council for Teachers in Mathematics in the USA and a professor at Columbia University. His idea was quite simple. Fehr considered functions f that map complex numbers onto complex numbers. Fehr then restricted the domain to those complex numbers that map the function onto real values. On this new domain the function has a range consisting of real values and can be represented in three dimensions by taking the domain as the horizontal plane and the range as the vertical axis.

So the horizontal plane acts as the Argand plane, whereas the vertical axis is the new range. This allows one to give a three dimensional picture of the four dimensional function that can be derived from the function that maps complex numbers onto complex numbers. Furthermore this three dimensional cut of a four dimensional space produces all the roots of the function f, because it contains all the values for which $f(z) = 0$.

Let us reconsider Fehr's example [30]. Fehr noted that if you draw $y = x^2 + 3x + 4$ in the Cartesian plane, you obtain a graphical representation of this function. Since this graph does not intersect the x-axis, we see that there are no real roots. See Figure 2.4.5.1 taken from [30]. Thus the roots are complex and this method does not visualize the complex roots.

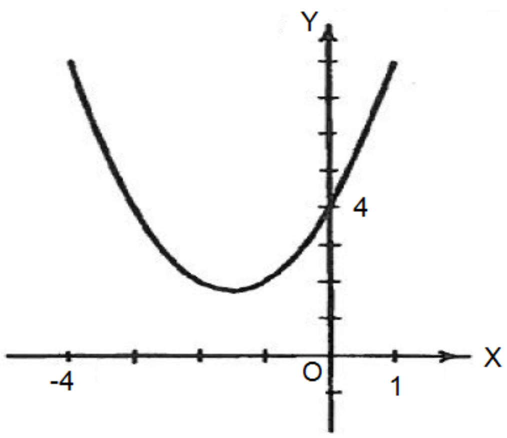

Figure 2.4.5.1 : $y = x^2 + 3x + 4$ in the Cartesian plane

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Then Fehr [30] proceeded to ask for which complex values of z would z^2+3z+4 be a real number. By letting $z = a+bi$ you get $z^2+3z+4 = (a+bi)^2+3(a+bi)+4 = (a^2-b^2+3a+4)+ib(2a+3)$. Therefore $b = 0$ or $a = -\frac{3}{2}$. Sketching these points forms two curves in three dimensions as seen in Figure 2.4.5.2 taken from [30]. These new curves cut the horizontal plane in the complex roots of this function labelled R_1 and R_2.

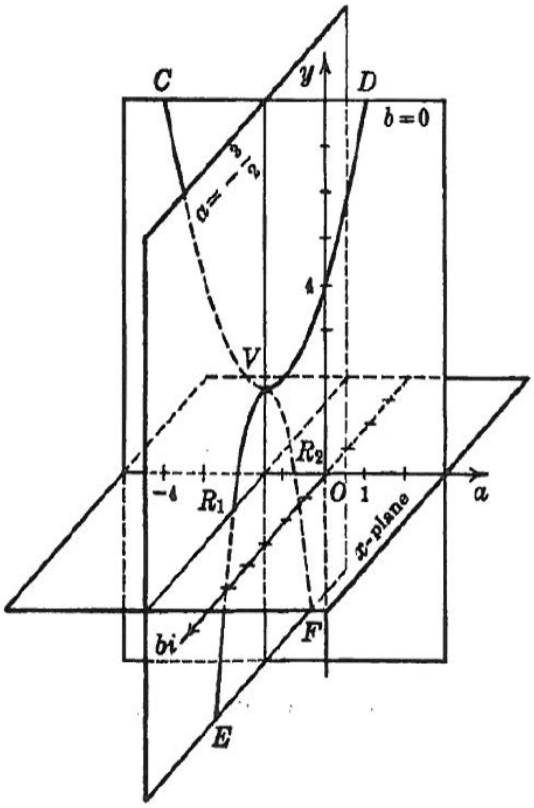

Figure 2.4.5.2 : Curves in three dimensions

In the next chapter more functions are considered to investigate Fehr's idea. Each time, when we find the set of complex numbers for which $f(z)$ is real, we obtain a selection of curves which authors Harding and Engelbrecht ([50], [51], [52]) coined **sibling curves**. These three-dimensional curves give us a very elegant way to visualize the zeroes of polynomials as the zeroes are special points on these curves.

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Chapter 3

Library of sibling curves

3.1 Introduction

In this chapter the following question is dealt with:

What do the sibling curves of complex valued functions look like?

The question is answered by producing a library of examples by considering several different kinds of functions. In each case we find the sibling curves and sketch them.

In Section 3.2 polynomials of various degrees are considered. In Section 3.3 a few rational functions are considered. In Section 3.4 we consider exponential and trigonometric functions. In Section 3.5 we consider multi-valued functions. The functions investigated are listed below. Examples 3.2.3, 3.2.5, 3.2.6, 3.2.8, 3.2.10, 3.4.1, 3.4.2, 3.4.3, 3.4.5, 3.4.6 and 3.5.2 were also considered by Harding and Engelbrecht in [51].

3.2 **Polynomials**

3.2.1 $f(z) = 2z + 3$

3.2.2 $f(z) = z^2$

3.2.3 $f(z) = z^2 + 2z + 2$

3.2.4 $f(z) = z - z^2$

3.2.5 $f(z) = z^3 - 1$

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3.2.6 $f(z) = z^3 - 4z + 1$

3.2.7 $f(z) = z^4 + 1$

3.2.8 $f(z) = z^4 + 2z^2 + z + 2$

3.2.9 $f(z) = z^5$

3.2.10 $f(z) = z^6 - 1$

3.3 Rational functions

3.3.1 $f(z) = \frac{1}{z}$

3.3.2 $f(z) = \frac{1}{z^2+1}$

3.3.3 $f(z) = \frac{z}{z^2+1}$

3.4 Exponential and trigonometric functions

3.4.1 $f(z) = e^z$

3.4.2 $f(z) = \sin z$

3.4.3 $f(z) = \cos z$

3.4.4 $f(z) = \tan z$

3.4.5 $f(z) = \sinh z$

3.4.6 $f(z) = \cosh z$

3.4.7 $f(z) = \tanh z$

3.5 Multi-valued functions

3.5.1 $f(z) = \log z$ or $e^{f(z)} = z$

3.5.2 $z^2 + f(z)^2 = k^2$ where $k > 0$

3.5.3 $z^3 + f(z)^3 = k$

3.5.4 $f(z) = \sqrt{z}$

3.5.5 $f(z) = z^{1.5}$

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3.2 Polynomials

Example 3.2.1. *The sibling curves for* $f(z) = 2z + 3$.

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x + iy) = 2(x + iy) + 3 = 2x + 3 + 2iy$. Thus if we want $f(x + iy)$ to be a real number then $y = 0$.

Parametrization: Hence the sibling curves are $(t, 2t + 3)$ where $t \in \mathbb{R}$.

Remark: Note there is one continuous curve. It is a straight line that cuts the plane $f(z) = k$ for any real number k exactly once.

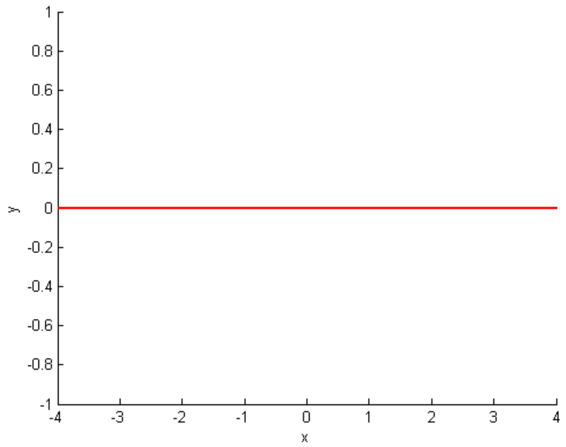

Figure 3.2.1.1 : Projections of sibling curves of $2z + 3$

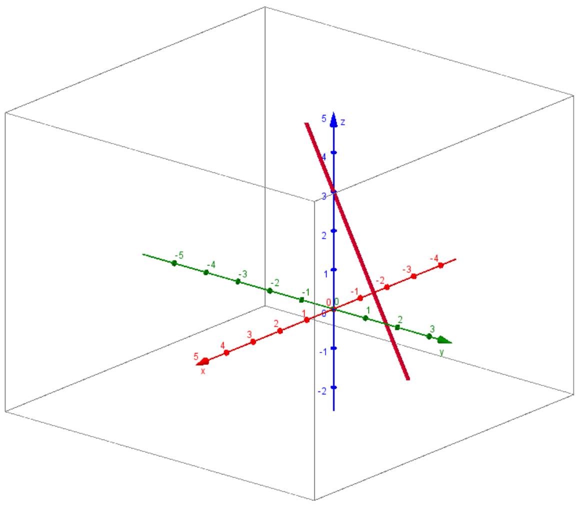

Figure 3.2.1.2 : Sibling curves of $2z + 3$

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Example 3.2.2. *The sibling curves for* $f(z) = z^2$.

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x + iy) = x^2 - y^2 + 2ixy$. Thus if $f(x+iy)$ is a real number we obtain $2xy = 0$. This gives $x = 0$ or $y = 0$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^2)$ and $g_2(t) = (it, -t^2)$ where $t \in \mathbb{R}$.

Remark: In this example there are two curves. Each curve is a parabola and they intersect at the origin. Also note the parabolas are in planes which are perpendicular.

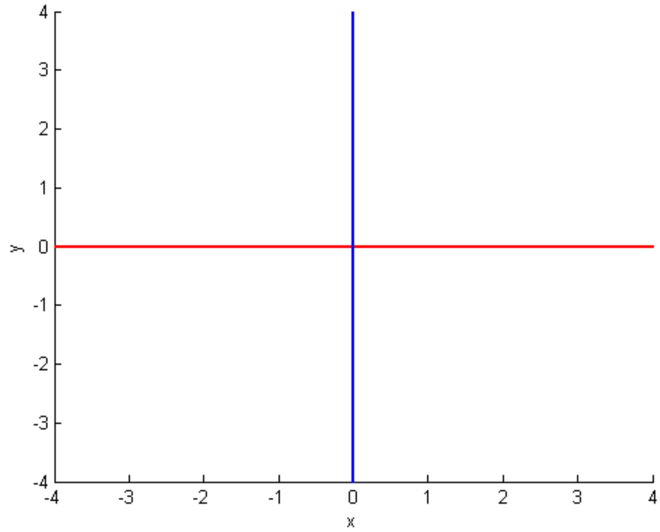

Figure 3.2.2.1 : Projections of sibling curves of z^2.

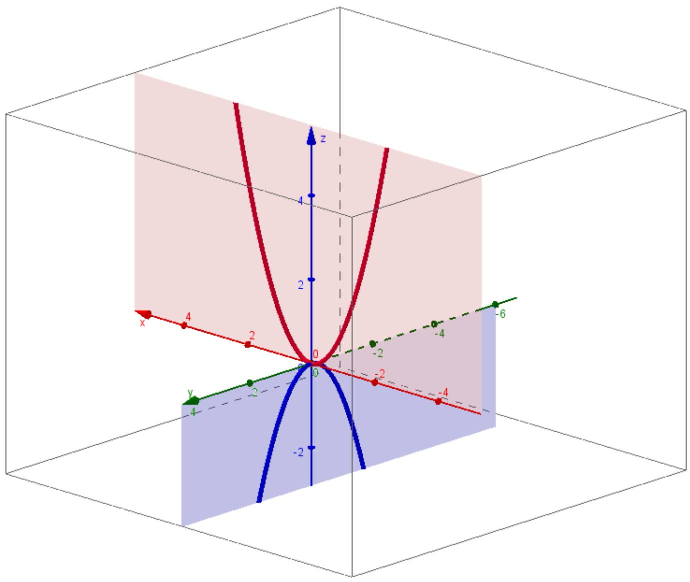

Figure 3.2.2.2 : Sibling curves of z^2

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Example 3.2.3. *The sibling curves for* $f(z) = z^2 + 2z + 2$

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x+iy) = (x^2 - y^2 + 2x + 2) + i(2xy + 2y)$. Thus if $f(x+iy)$ is a real number we obtain $2y(x+1) = 0$. This gives $x = -1$ or $y = 0$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^2 + 2t + 2)$ and $g_2(t) = (-1 + it, 1 - t^2)$ where $t \in \mathbb{R}$.

Remark: In this example there are two curves. Each curve is again a parabola and they intersect at the point $(0, 1)$. Also note the parabolas are again in planes which are perpendicular.

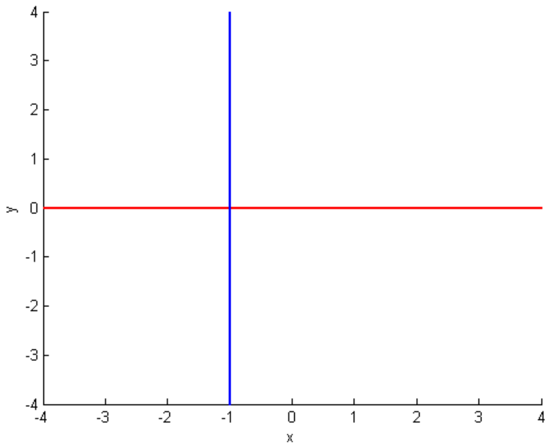

Figure 3.2.3.1: Projection of sibling curves of $z^2 + 2z + 2$

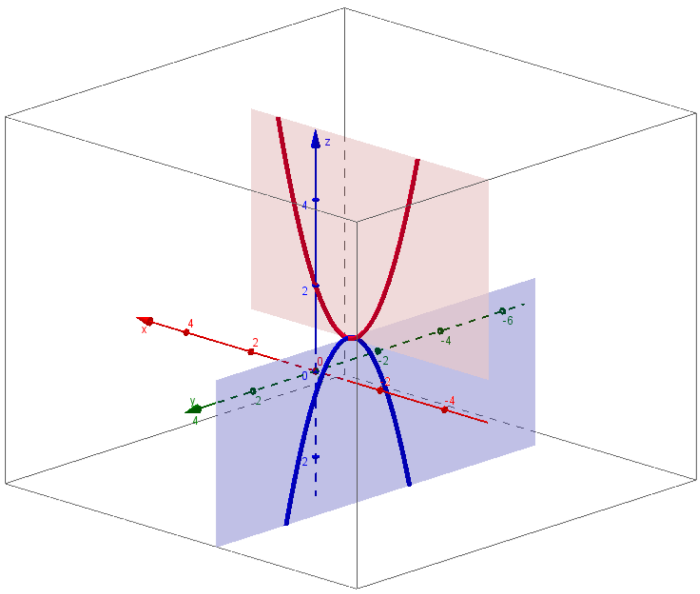

Figure 3.2.3.2: Sibling curves of $z^2 + 2z + 2$

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Example 3.2.4. *The sibling curves for* $f(z) = z - z^2$.

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x + iy) = (x - x^2 + y^2) + i(y - 2xy)$. Thus if $f(x + iy)$ is a real number we obtain $y(1 - 2x) = 0$. This gives $y = 0$ or $x = \frac{1}{2}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t - t^2)$ and $g_2(t) = (\frac{1}{2} + it, t^2 + \frac{1}{4})$ where $t \in \mathbb{R}$.

Remark: In this example there are also two curves. Each curve is again a parabola and they intersect at the point $(\frac{1}{2}, \frac{1}{4})$. Also note the parabolas are again in planes which are perpendicular.

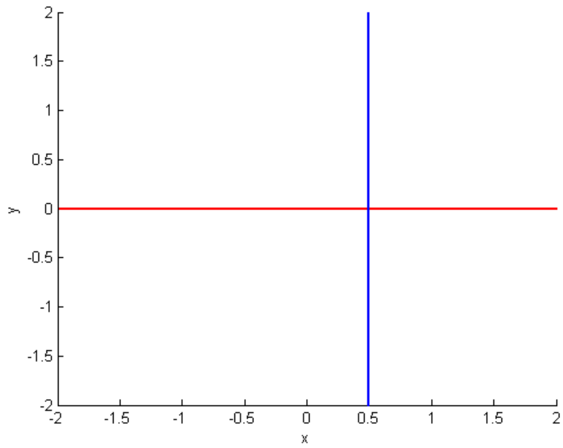

Figure 3.2.4.1: Projection of sibling curves of $z - z^2$

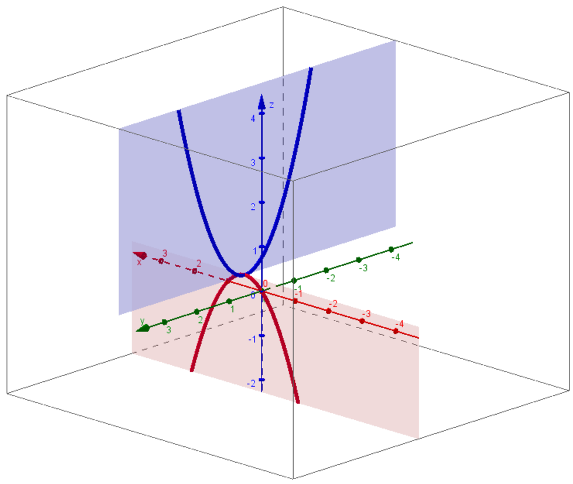

Figure 3.2.4.2: Sibling curves of $z - z^2$

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Example 3.2.5. *The sibling curves for* $f(z) = z^3 - 1$.

Calculation: Now $f(z) \in \mathbb{R}$ gives $z^3 \in \mathbb{R}$. Suppose $z = re^{i\theta}$ for some real numbers r and $0 \leq \theta < \pi$. Thus $z^3 = r^3 e^{3i\theta}$. So if z^3 is a real number, then so is $e^{3i\theta}$. This gives $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^3 - 1)$ and $g_2(t) = (te^{i\frac{\pi}{3}}, -t^3 - 1)$ and $g_3(t) = (te^{i\frac{2\pi}{3}}, t^3 - 1)$ where $t \in \mathbb{R}$.

Remark: Here there are three curves. Each one is a continuous curve and the projection is three straight lines that divide the horizontal plane equally. Also note that all three sibling curves pass through the point $(0, -1)$.

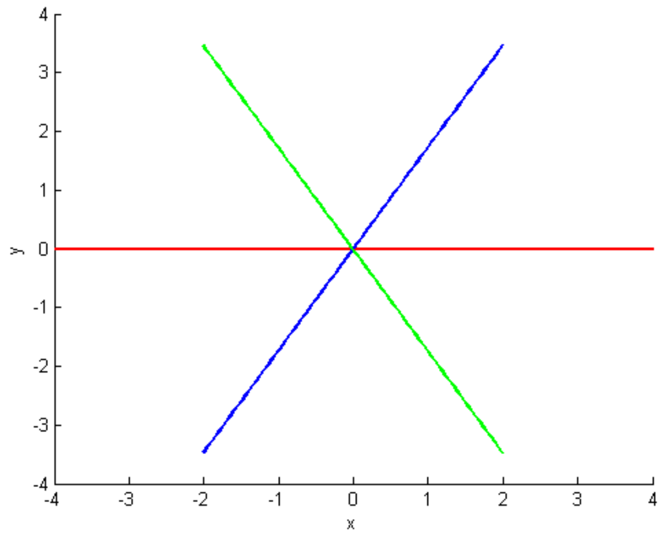

Figure 3.2.5.1: Projection of sibling curves of $z^3 - 1$

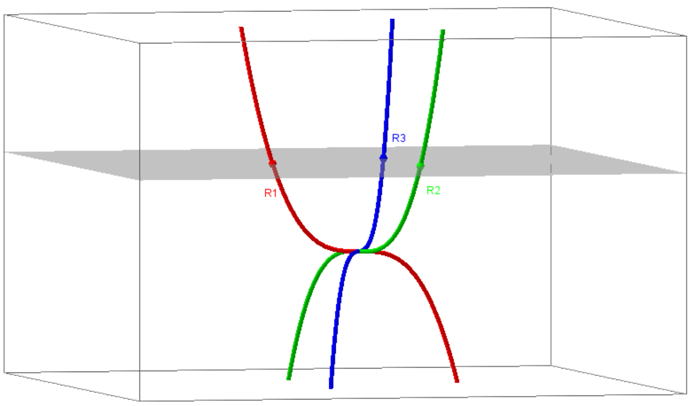

Figure 3.2.5.2: Sibling curves of $z^3 - 1$

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Example 3.2.6. *The sibling curves for $f(z) = z^3 - 4z + 1$.*

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x+iy) = (x^3 - 3xy^2 - 4x + 1) + i(3x^2y - 3y^3 - 4y)$. So if $f(x+iy)$ is a real number, we obtain $y(3x^2 - 3y^2 - 4) = 0$. So $y = 0$ or $x^2 - y^2 = \frac{4}{3}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^3 - 4t + 1)$ and $g_2(t) = (\frac{2}{\sqrt{3}} \sec t + i\frac{2}{\sqrt{3}} \tan t, \frac{8}{3\sqrt{3}} \sec^3 t - \frac{8}{\sqrt{3}} \tan t \sec t - \frac{8}{\sqrt{3}} \sec t + 1)$ where $\frac{-\pi}{2} < t < \frac{\pi}{2}$ and $g_3(t) = (\frac{2}{\sqrt{3}} \sec t + i\frac{2}{\sqrt{3}} \tan t, \frac{8}{3\sqrt{3}} \sec^3 t - \frac{8}{\sqrt{3}} \tan t \sec t - \frac{8}{\sqrt{3}} \sec t + 1)$ where $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Remark: Note again there are three continuous curves, but the picture is very different this time. One is a normal cubic function and the other two are curves lying on the hyperbolic surface $x^2 - y^2 = \frac{4}{3}$. Furthermore the sibling curves intersect in two points $(\frac{-2}{\sqrt{3}}, \frac{3\sqrt{3}+16}{3\sqrt{3}})$ and $(\frac{2}{\sqrt{3}}, \frac{3\sqrt{3}-16}{3\sqrt{3}})$.

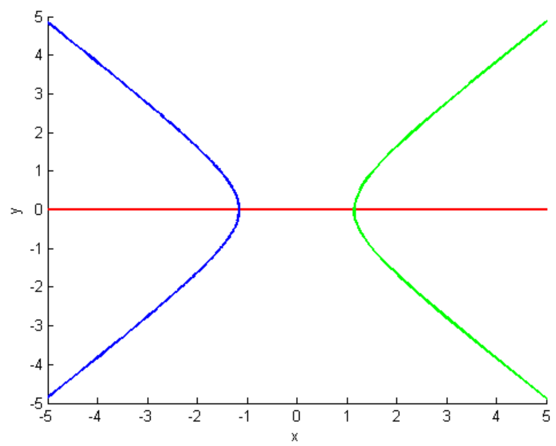

Figure 3.2.6.1: Projection of sibling curves of $z^3 - 4z + 1$

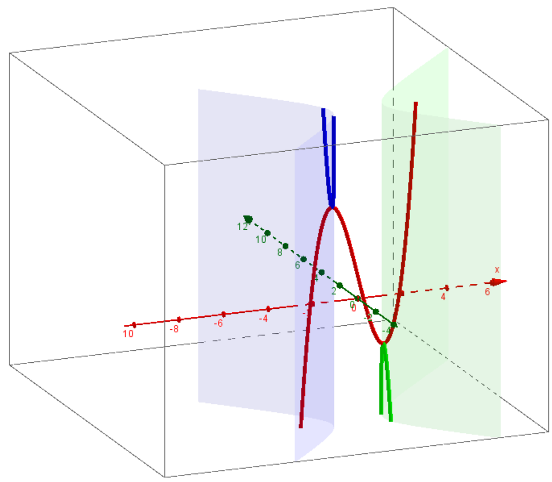

Figure 3.2.6.2: Sibling curves of $z^3 - 4z + 1$

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Example 3.2.7. *The sibling curves for* $f(z) = z^4 + 1$.

Calculation: Now $f(z) \in \mathbb{R}$ gives $z^4 \in \mathbb{R}$. Suppose $z = re^{i\theta}$ for some real numbers r and $0 \leq \theta < \pi$. Thus $z^4 = r^4 e^{4i\theta}$. So if z^4 is a real number, then so is $e^{4i\theta}$, giving $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^4 + 1)$ and $g_2(t) = (it, -t^4 + 1)$, $g_3(t) = (-t, t^4 + 1)$ and $g_4(t) = (-it, -t^4 + 1)$ where $t \in \mathbb{R}$.

Remark: Note, there are four continuous curves. They all pass through the point $(0, 1)$. Also the projection is four straight lines that divide the horizontal plane equally.

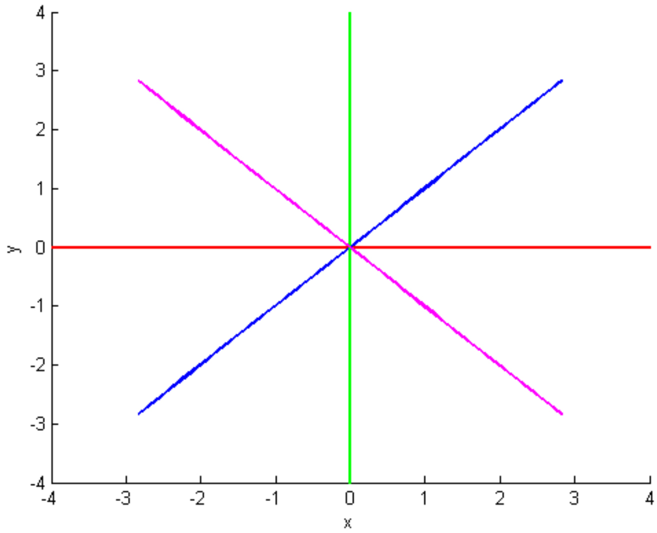

Figure 3.2.7.1: Projection of sibling curves of $z^4 + 1$

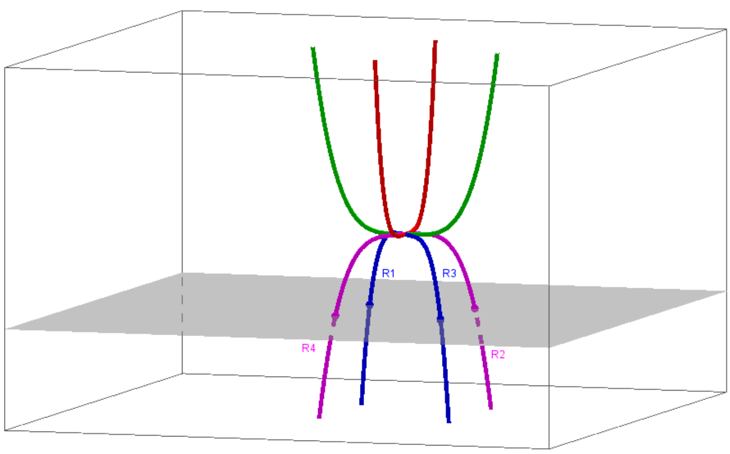

Figure 3.2.7.2: Sibling curves of $z^4 + 1$

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Example 3.2.8. *The sibling curves for* $f(z) = z^4 + 2z^2 + z + 2$.

Calculation: Let $z = x + iy$ where x and y are real numbers. Now $f(x+iy) = (x^4 - 6x^2y^2 + y^4 + 2x^2 - 2y^2 + x + 2) + i(4x^3y - 4xy^3 + 4xy + y)$. So if $f(x+iy) \in \mathbb{R}$ then $y(4x^3 - 4xy^2 + 4x + 1) = 0$. That is $y = 0$ or $4xy^2 = 4x^3 + 4x + 1$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^4 + 2t^2 + t + 2)$ where t is a real number. Suppose a is the only real solution to $4x^3 + 4x + 1 = 0$, that is $a \approx -0.237$. So if $t \leq a$, then $g_2(t) = (t \pm i\sqrt{\frac{4t^3+4t+1}{4t}}, t^4 - 6t^2\frac{4t^3+4t+1}{4t} + (\frac{4t^3+4t+1}{4t})^2 + t^2 - 2\frac{4t^3+4t+1}{4t} + t + 2)$. If $t > 0$ we get $g_3(t) = (t + i\sqrt{\frac{4t^3+4t+1}{4t}}, t^4 - 6t^2\frac{4t^3+4t+1}{4t} + (\frac{4t^3+4t+1}{4t})^2 + t^2 - 2\frac{4t^3+4t+1}{4t} + t + 2)$. If $t > 0$ we get $g_4(t) = (t - i\sqrt{\frac{4t^3+4t+1}{4t}}, t^4 - 6t^2\frac{4t^3+4t+1}{4t} + (\frac{4t^3+4t+1}{4t})^2 + t^2 - 2\frac{4t^3+4t+1}{4t} + t + 2)$.

Remark: Again there are four continuous curves, but the picture is very different. Only two sibling curves intersect. The coordinates of the point are approximately $(-0.237, 1.878)$. The other two sibling curves never intersect any of the other sibling curves.

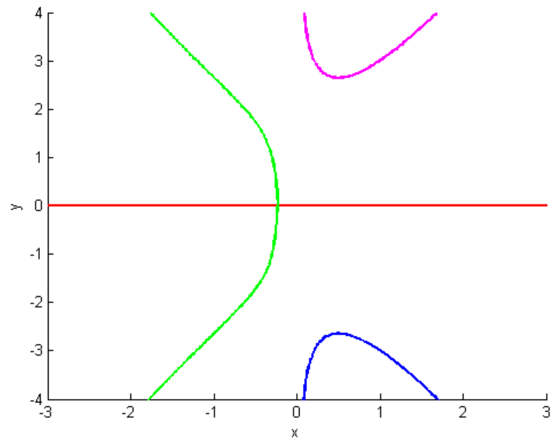

Figure 3.2.8.1: Projection of sibling curves of $z^4 + 2z^2 + z + 2$

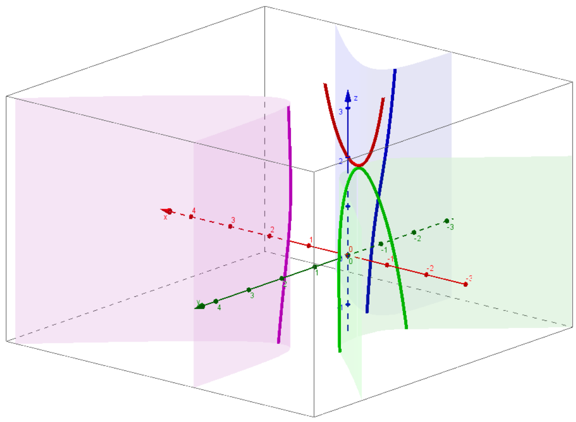

Figure 3.2.8.2: Sibling curves of $z^4 + 2z^2 + z + 2$

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Example 3.2.9. *The sibling curves for $f(z) = z^5$.*

Calculation: Now $f(z) \in \mathbb{R}$ gives $z^5 \in \mathbb{R}$. Suppose $z = re^{i\theta}$ for some real numbers r and $0 \leq \theta < \pi$. Thus $z^5 = r^5 e^{5i\theta}$. So if z^5 is a real number then $e^{5i\theta}$ is also a real number. This means $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^5 - 1)$ and $g_2(t) = (te^{i\frac{\pi}{5}}, -t^5 - 1)$, $g_3(t) = (te^{i\frac{2\pi}{5}}, t^5 - 1)$, $g_4(t) = (te^{i\frac{3\pi}{5}}, -t^5 - 1)$ and and $g_5(t) = (te^{i\frac{4\pi}{5}}, t^5 - 1)$ where $t \in \mathbb{R}$.

Remark: Note, there are five continuous curves intersecting. They all pass through the origin. Also the projection is five straight lines that divide the horizontal plane equally.

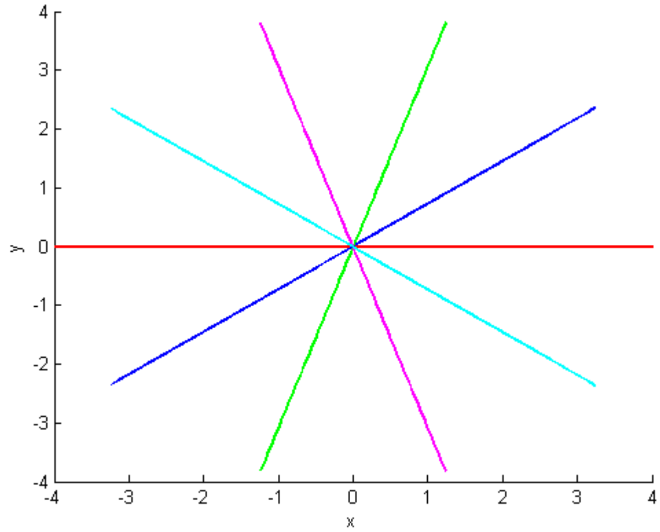

Figure 3.2.9.1: Projection of sibling curves of z^5

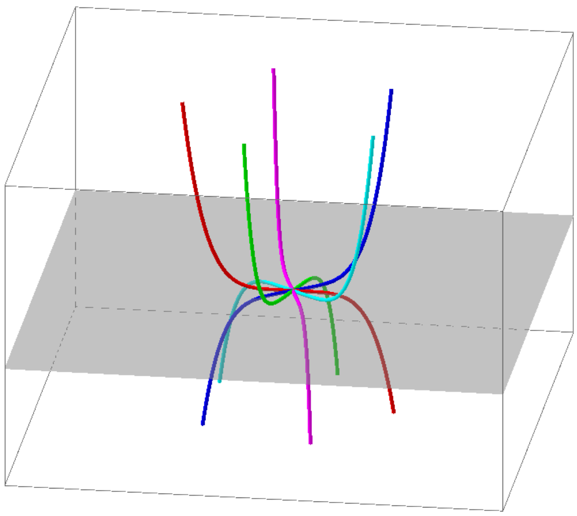

Figure 3.2.9.2: Sibling curves of z^5

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Example 3.2.10. *The sibling curves for* $f(z) = z^6 - 1$.

Calculation: Now $f(z) \in \mathbb{R}$ gives $z^6 \in \mathbb{R}$. Suppose $z = re^{i\theta}$ for some real numbers r and $0 \leq \theta < \pi$. Thus $z^6 = r^6 e^{6i\theta}$. So if z^6 is a real number then so is $e^{6i\theta}$. This gives $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, t^6 - 1)$ and $g_2(t) = (te^{i\frac{\pi}{6}}, -t^6 - 1)$, $g_3(t) = (te^{i\frac{\pi}{3}}, t^6 - 1)$, $g_4(t) = (te^{i\frac{\pi}{2}}, -t^6 - 1)$,$g_5(t) = (te^{i\frac{2\pi}{3}}, t^6 - 1)$ and $g_6(t) = (te^{i\frac{5\pi}{6}}, -t^6 - 1)$ where $t \in \mathbb{R}$.

Remark: Note, there are six continuous curves. They all pass through the point $(0, -1)$. The projection is six straight lines that divide the horizontal plane equally.

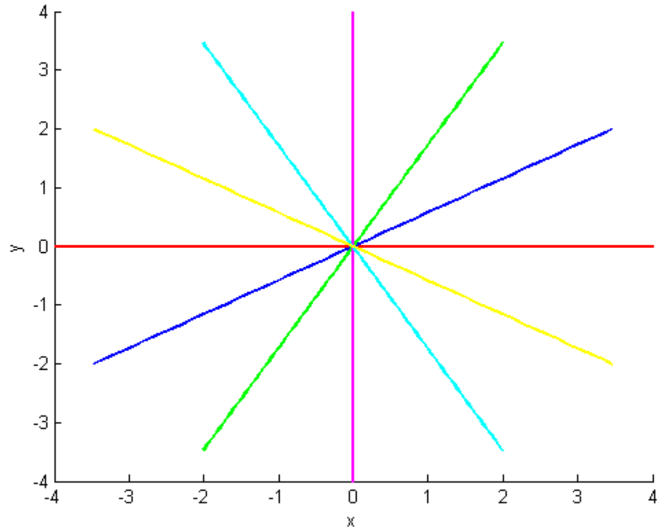

Figure 3.2.10.1: Projection of sibling curves of $z^6 - 1$

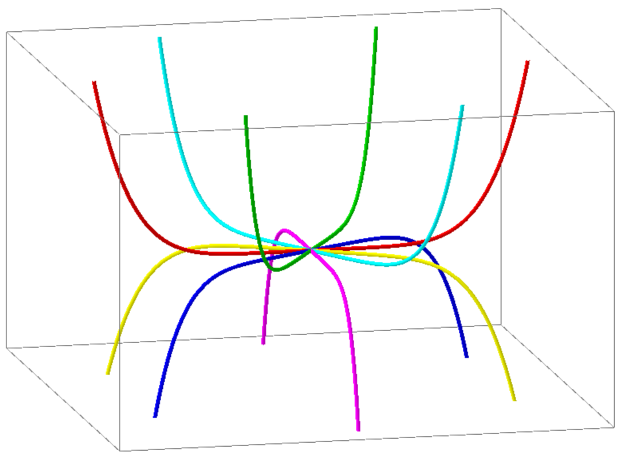

Figure 3.2.10.2: Sibling curves of $z^6 - 1$

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3.3 Rational functions

Example 3.3.1. *The sibling curves for* $f(z) = \frac{1}{z}$.

Calculation: We take the domain $\mathbb{C}\backslash\{0\}$. Let $z = x + iy$ where x and y are real numbers. Now $f(z) = f(x+iy) = \frac{x-iy}{x^2+y^2}$. Thus if $f(x+iy)$ is a real number obtain $y = 0$ and also $x^2 + y^2 \neq 0$. This gives projections $\{(x, 0) : x > 0\}$ and $\{(x, 0) : x < 0\}$.

Parametrization: Hence the sibling curves are $g_1(t) = (t, \frac{1}{t})$ and $g_2(t) = (-t, \frac{-1}{t})$ where $t > 0$.

Remark: Note, like the example of $f(z) = 2z + 3$, we see the only values which makes $f(z)$ a real number, are real numbers. We end up with the hyperbola in the $\mathbb{C} \times \mathbb{R}$ space. This function is also interesting as it does not have zeroes. This can be observed by noticing the sibling curves do not cut the horizontal plane.

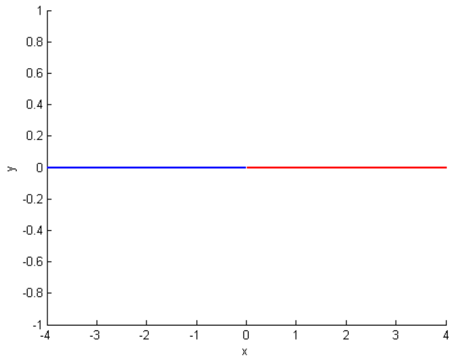

Figure 3.3.1.1: Projection of sibling curves of $\frac{1}{z}$

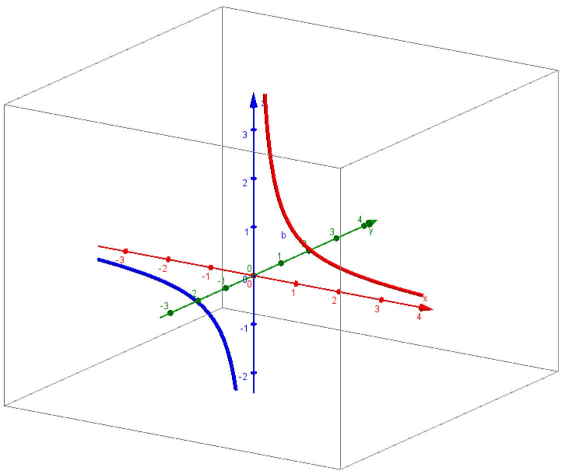

Figure 3.3.1.2: Sibling curves of $\frac{1}{z}$

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Example 3.3.2. *The sibling curves for* $f(z) = \frac{1}{z^2+1}$.

Calculation: Consider the domain $\mathbb{C}\backslash\{-i, i\}$. Now $f(z) = \frac{(\bar{z})^2+1}{(z^2+1)((\bar{z})^2+1)}$. Thus $f(z)$ is real iff $(\bar{z})^2 + 1$ is real. That is when $(\bar{z})^2$ is real. This happens when $z \in \mathbb{R}$ or $z = it$ where $t \in \mathbb{R} \setminus \{-1, 1\}$. This produces four continuous curves.

Parametrization: So $g_1(t) = (t, \frac{1}{t^2+1})$. Also $g_2(t) = (it, \frac{1}{1-t^2})$ where $t < -1$, $g_2(t) = (it, \frac{1}{1-t^2})$ where $-1 < t < 1$ and $g_3(t) = (it, \frac{1}{1-t^2})$ where $t > 1$.

Remark: We end up with four continuous curves. Two of the sibling curves intersect at the point $(0, 1)$. Note that each sibling curve is planar, meaning there is a plane containing that sibling curve. This function is also interesting as it does not have zeroes. This can be observed by noticing the sibling curves do not cut the horizontal plane.

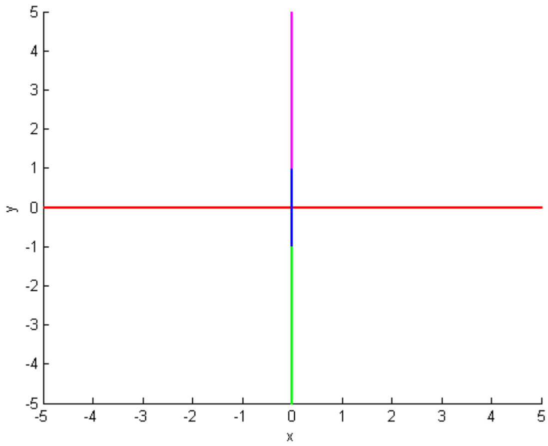

Figure 3.3.2.1: Projection of sibling curves of $\frac{1}{z^2+1}$

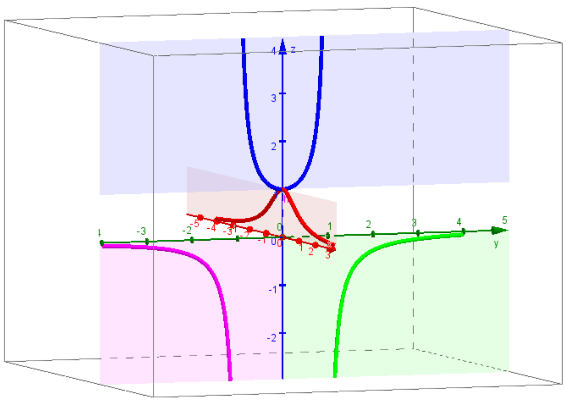

Figure 3.3.2.2: Sibling curves of $\frac{1}{z^2+1}$

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Example 3.3.3. *The sibling curves for* $f(z) = \frac{z}{z^2+1}$.

Calculation: Consider the domain $\mathbb{C}\backslash\{-i, i\}$. Let $z = x+iy$ where x and y are real numbers. Now $f(z) = f(x+iy) = \frac{x+iy}{x^2-y^2+1+2ixy} = \frac{x(x^2+y^2+1)+iy(1-x^2-y^2)}{(x^2-y^2+1)^2+4x^2y^2}$. Thus if $f(x+iy)$ is a real then $y(1-x^2-y^2) = 0$. That is $y = 0$ or $x^2+y^2 = 1$.

Parameterization: This produces three sibling curves $g_1(t) = (t, \frac{t}{t^2+1})$, $g_2(t) = (\cos(t), \sin(t), \frac{2cos(t)}{(\cos^2 t - \sin^2 t + 1)^2 + 4\sin^2 t \cos^2 t})$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $g_3(t) = (\cos(t), \sin(t), \frac{2cos(t)}{(\cos^2 t - \sin^2 t + 1)^2 + 4\sin^2 t \cos^2 t})$ where $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Remark: Thus there are three curves. One is the expected rational function and the other two are curves laying on the cylinder $x^2+y^2 = 1$. Two sibling curves intersect at the point $(1, \frac{1}{2})$ and two sibling curves intersect at the point $(-1, -\frac{1}{2})$.

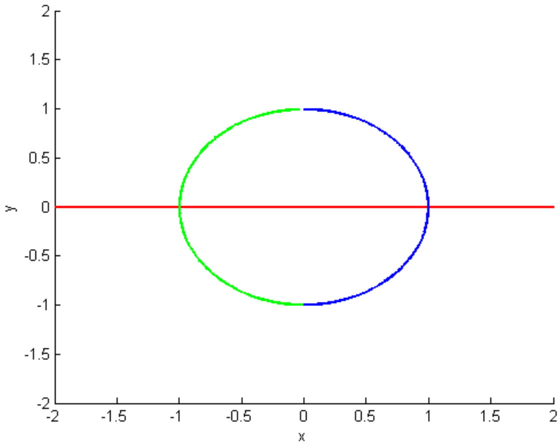

Figure 3.3.3.1: Projection of sibling curves of $\frac{z}{z^2+1}$

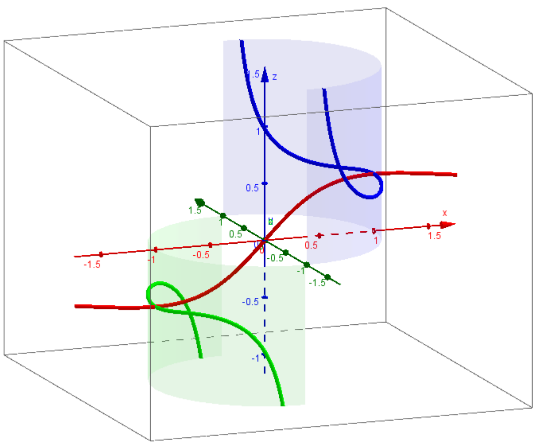

Figure 3.3.3.2: Sibling curves of $\frac{z}{z^2+1}$

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3.4 Exponential and trigonometric functions

Example 3.4.1. *The sibling curves for* $f(z) = e^z$.

Calculation: Suppose $z = x + iy$ where x and y are real numbers.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x.e^{iy} \\ &= \cos y.e^x + i\sin y.e^x \end{aligned}$$

Thus if $e^z \in \mathbb{R}$, then $\sin y.e^x = 0$. Since $e^x > 0$, we must have $\sin y = 0$. This gives $y = n\pi$ where $n \in \mathbb{Z}$.

Parametrization: Thus the sibling curves are $g(t) = (t + in\pi, (-1)^n.e^t)$, where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: Unlike the previous examples, we end up with infinitely many sibling curves. All of them have the shape of e^z curve.

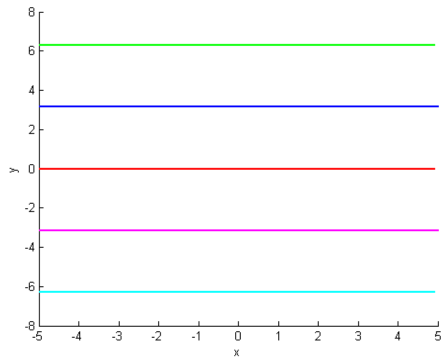

Figure 3.4.1.1: Projection of sibling curves of e^z

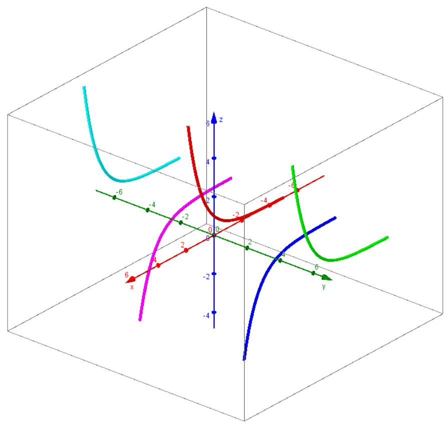

Figure 3.4.1.2: Sibling curves of e^z

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Example 3.4.2. *The sibling curves for* $f(z) = \sin z$.

Calculation: Suppose $z = x + iy$ where x and y are real numbers.

$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \frac{e^{-y+ix} - e^{y-ix}}{2i} \\ &= \frac{(e^{-y} - e^{y})\cos x + i(e^{y} + e^{-y})\sin x}{2i}\end{aligned}$$

Thus if $\sin z \in \mathbb{R}$, then $(e^{-y} - e^{y})\cos x = 0$. Thus $y = 0$ or $\cos x = 0$. The latter gives $x = \frac{(2n+1)\pi}{2}$ where $n \in \mathbb{Z}$.

Parametrization: Thus the sibling curves are $g(t) = (t, \sin t)$ or $(\frac{(2n+1)\pi}{2} + it, (-1)^n \cosh t)$, where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: We end up with infinitely many sibling curves. One is the normal sine curve as expected and then infinitely many curves in the shape of the cosh curve. The points where sibling curves meet are $(\frac{(2n+1)\pi}{2}, (-1)^n)$ where n is a positive integer.

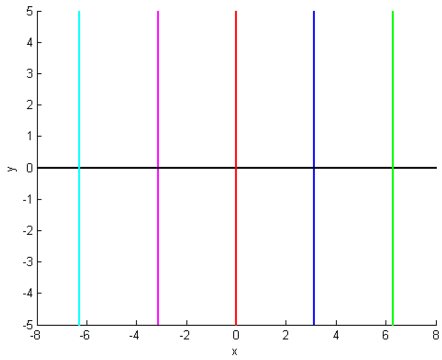

Figure 3.4.2.1: Projection of sibling curves of $\sin z$

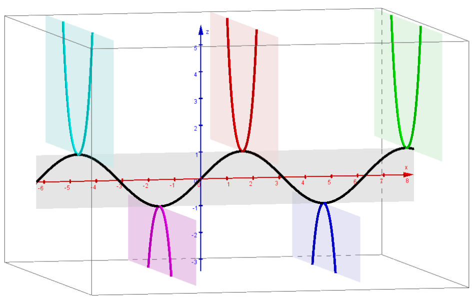

Figure 3.4.2.2: Sibling curves of $\sin z$

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Example 3.4.3. *The sibling curves for* $f(z) = \cos z$*.*

Calculation: Suppose $z = x + iy$ where x and y are real numbers.

$$\begin{aligned}\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \frac{e^{-y+ix} + e^{y-ix}}{2} \\ &= \frac{(e^{-y} + e^{y})\cos x + i(-e^{y} + e^{-y})\sin x}{2}\end{aligned}$$

Thus if $\cos z \in \mathbb{R}$, then $(-e^{y} + e^{-y})\sin x = 0$. This gives $y = 0$ or $\sin x = 0$. The latter gives $x = n\pi$ where $n \in \mathbb{Z}$.

Parametrization: Thus the sibling curves are $g(t) = (t, \cos t)$ where $t \in \mathbb{R}$ and $g(t) = (n\pi + it, (-1)^n \cosh t)$, where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: We end up with infinitely many sibling curves. One has the shape of the cos curve. The rest have the shape of the cosh curve. The sibling curves meet at the points $(n\pi, (-1)^n)$ where n is an integer.

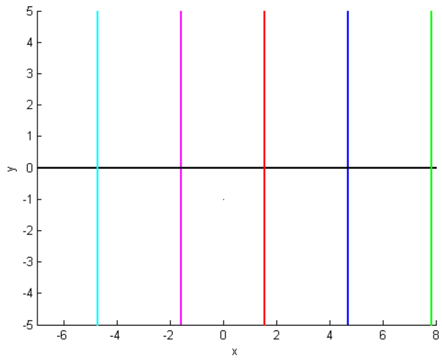

Figure 3.4.3.1: Projection of sibling curves of $\cos z$

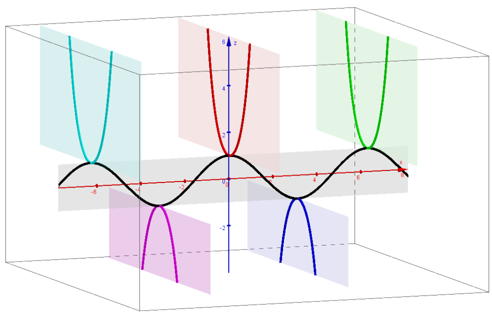

Figure 3.4.3.2: Sibling curves of $\cos z$

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Example 3.4.4. *The sibling curves for* $f(z) = \tan z$.

Calculation: Now $\tan z = \frac{\sin z}{\cos z}$. Therefore $\tan z$ has domain $\mathbb{C} \setminus \{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$. Suppose $z = x + iy$ where x and y are real numbers, then

$$\begin{aligned}
\tan(z) &= \frac{\sin z}{\cos z} \\
&= \frac{e^{2ix} - e^{2y}}{ie^{2ix} + ie^{2y}} \\
&= \frac{e^{2ix} - e^{2y}}{ie^{2ix} + ie^{2y}} \times \frac{e^{-2ix} + e^{2y}}{e^{-2ix} + e^{2y}} \\
&= \frac{1 - e^{4y} + e^{2y}(e^{2ix} - e^{-2ix})}{i(e^{2ix} + e^{2y})(e^{-2ix} + e^{2y})} \\
&= \frac{1 - e^{4y} + 2ie^{2y}\sin(2x)}{i(e^{2ix} + e^{2y})\overline{e^{2ix} + e^{2y}}}
\end{aligned}$$

Notice the denominator is imaginary. So if $\tan z$ is real, then the numerator must also be imaginary. This happens when $1 - e^{4y} = 0$ is real. That is $y = 0$.

Parameterization: Hence the sibling curves are precisely the tan curve, that is $g(t) = (t, \tan t)$ where $t \in \mathbb{R} \setminus \{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$.

Remark: Note, like the example of $f(z) = 2z + 3$, we see the only values which make $f(z)$ a real number, are real numbers. So we end up with the tan curve again in the $\mathbb{C} \times \mathbb{R}$ space.

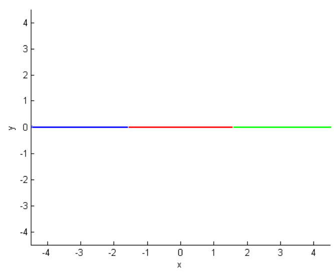

Figure 3.4.4.1: Projection of sibling curves of $\tan z$

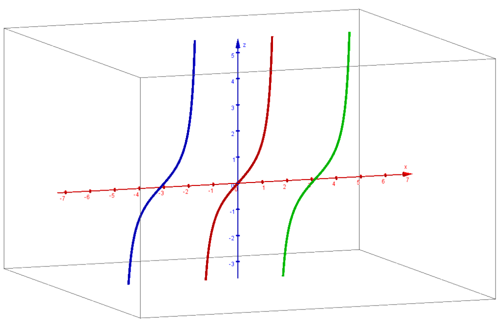

Figure 3.4.4.2: Sibling curves of $\tan z$

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Example 3.4.5. *The sibling curves for* $f(z) = \sinh z$.

Calculation: Suppose $z = x + iy$, then

$$\begin{aligned}
\sinh z &= \frac{e^z - e^{-z}}{2} \\
&= \frac{e^{x+iy} - e^{-x-iy}}{2} \\
&= \frac{e^x(\cos(y) + i\sin(y)) - e^{-x}(\cos(y) - i\sin(y))}{2} \\
&= \frac{\cos y(e^x - e^{-x}) + i\sin y(e^x + e^{-x})}{2}
\end{aligned}$$

Thus $\sinh z \in \mathbb{R}$ if $\sin y(e^x + e^{-x}) = 0$. Since $e^x + e^{-x} > 0$, we obtain $\sin y = 0$. That is $y = n\pi$ where $n \in \mathbb{Z}$.

Parameterization: Hence the sibling curves are the curves given by $g(t) = (t, n\pi, \sinh(t + in\pi))$ where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: Since $\sin y = 0$, we have $\cos y = -1$ or $\cos y = 1$. This explains the two kinds of sibling curves as seen in Figure 3.4.5.2.

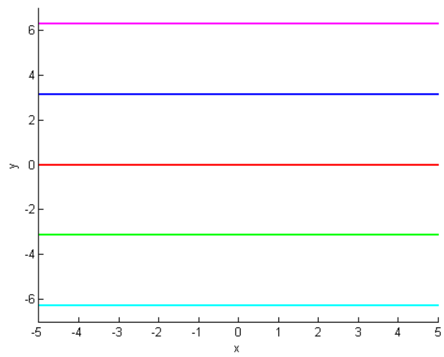

Figure 3.4.5.1 : Projection of sibling curves of $\sinh z$

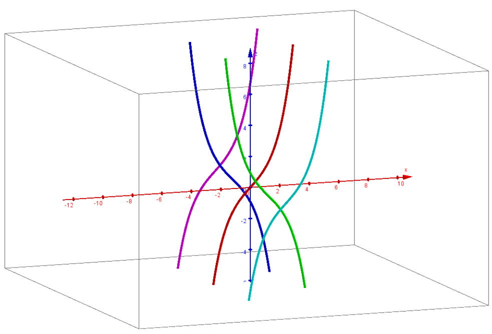

Figure 3.4.5.2 : Sibling curves of $\sinh z$

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Example 3.4.6. *The sibling curves for* $f(z) = \cosh z$.

Calculation: Suppose $z = x + iy$, then

$$\begin{aligned}
\sinh z &= \frac{e^z + e^{-z}}{2} \\
&= \frac{e^{x+iy} + e^{-x-iy}}{2} \\
&= \frac{e^x(\cos(y) + i\sin(y)) + e^{-x}(\cos(y) - i\sin(y))}{2} \\
&= \frac{\cos y(e^x + e^{-x}) + i\sin y(e^x - e^{-x})}{2}
\end{aligned}$$

Thus $\cosh z \in \mathbb{R}$ if $\sin y(e^x - e^{-x}) = 0$. Thus $x = 0$ or $\sin y = 0$. The latter gives $y = n\pi$ where $n \in \mathbb{Z}$.

Parameterization: Hence the sibling curves are the curves given by $g(t) = (0, t, \cos(t)$ where $t \in \mathbb{R}$ or $g(t) = (t, n\pi, \cosh(t + in\pi))$ where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: So the sibling curves are the normal cos curve and other curves touching the cos curve at y-values $n\pi$ where $n \in \mathbb{Z}$. The points where sibling curves meet are $(n\pi, (-1)^n)$.

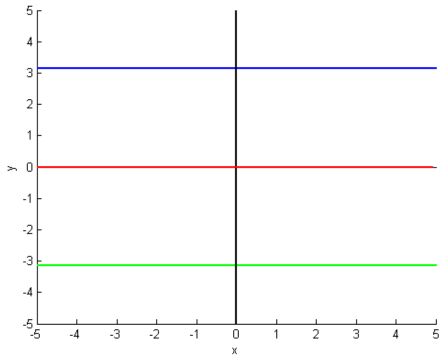

Figure 3.4.6.1: Projection of sibling curves of $\cosh(z)$

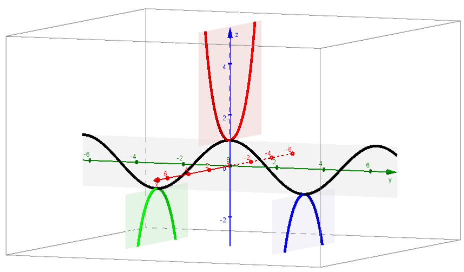

Figure 3.4.6.2: Sibling curves of $\cosh(z)$

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Example 3.4.7. *The sibling curves for* $f(z) = \tanh z$*.*

Calculation: Suppose $z = x + iy$ where x and y are real.

$$\begin{aligned} \tan z &= \frac{e^{2z} - 1}{e^{2z} + 1} \\ &= \frac{e^{2x}e^{2iy} - 1}{e^{2x}e^{2iy} + 1} \times \frac{e^{2x}e^{-2iy} + 1}{e^{2x}e^{-2iy} + 1} \\ &= \frac{e^{4x} - 1 + ie^{2x}(2\sin y)}{(e^{2x}e^{2iy} + 1)\overline{(e^{2x}e^{2iy} + 1)}} \end{aligned}$$

Thus if $f(z)$ is real, we must have $\sin y = 0$, giving $y = n\pi$ where $n \in \mathbb{Z}$.

Parameterization: So the sibling curves are given by $g(t) = (t + in\pi, \tanh(t + in\pi))$ where $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Remark: In this example we ended up with infinitely many sibling curves. They all have the shape of the tanh curve and there are no intersections.

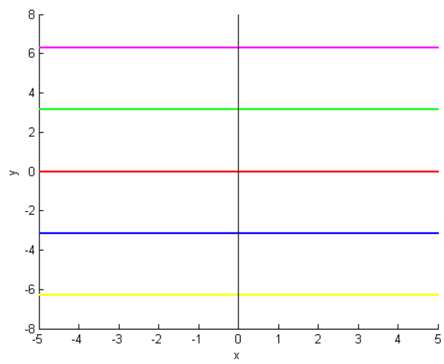

Figure 3.4.7.1: Projection of sibling curves of $\tanh z$

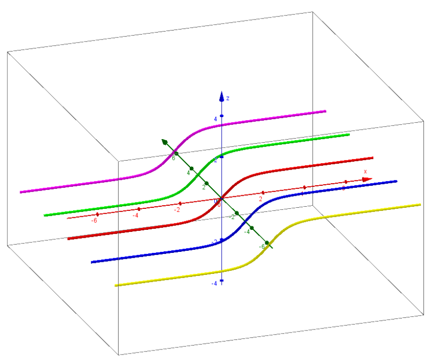

Figure 3.4.7.2: Sibling curves of $\tanh z$

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3.5 Multi-valued functions

We can also find the sibling curves to multi-valued functions or those surfaces given by an equation. Thus given $g(z, f(z)) = 0$, we are looking for all the points on the surface where $f(z) \in \mathbb{R}$. The first example is $e^{f(z)} = z$.

Example 3.5.1. *The sibling curves for* $f(z) = \log(z)$.

Calculation: It should be noted that $\log(z)$ is a multi-valued function. The domain is $\mathbb{C}\backslash\{0\}$ and $z \mapsto \log|z| + i(arg\, z + 2\pi k)$ where k is an integer. If the imaginary part is zero, we obtain all the points (a, b) where $e^b = a$.

Parameterization: Thus if $t \in \mathbb{R}$ then $b = t$ and $a = e^t$ giving the sibling curve (e^t, t).

Remark: Again, like the example of $f(z) = 2z + 3$, we see the only values that make $f(z)$ a real number, are real numbers. We thus end up with the log curve in the $\mathbb{C} \times \mathbb{R}$ space.

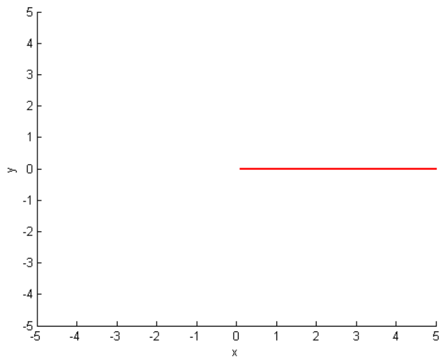

Figure 3.5.1.1: Projection of sibling curves of $\log(z)$

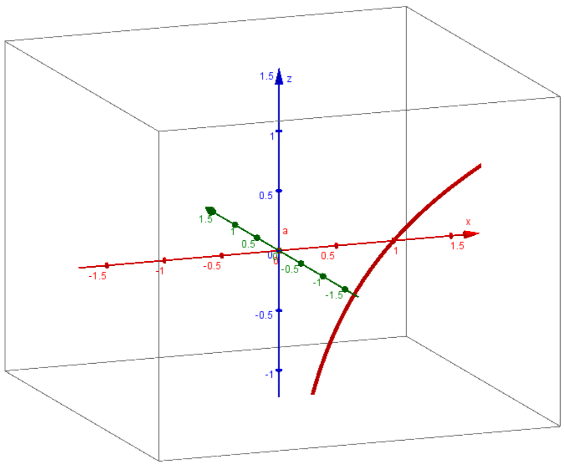

Figure 3.5.1.2: Sibling curves of $\log(z)$

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Example 3.5.2. *The sibling curves for $z^2 + (f(z))^2 = k^2$ where k is a positive real number.*

Calculation: If $-k \leq f(z) \leq k$, then $(f(z))^2 = k^2 - z^2 \geq 0$. Parameterizing you get the circle $(k\cos(t), k\sin(t))$ for some real t. If $|f(z)| > k$, then $z^2 = k^2 - (f(z))^2 < 0$. Hence $iz \in \mathbb{R}$.

Parameterization: Thus $z = ti$ for some real number t. Hence $(ti, \pm\sqrt{k^2+t^2})$ and $(-ti, \pm\sqrt{k^2+t^2})$ are two more sibling curves to this equation. Note $(k\cos(t), k\sin(t))$ is the third sibling curve.

Remark: We end up with three sibling curves. Notice the projection on the horizontal plane is a straight line and the projections overlap. One sibling curve is a circle and the other two are curves above and below the circle. The sibling curves intersect at the points $(-k, 0)$ and $(k, 0)$.

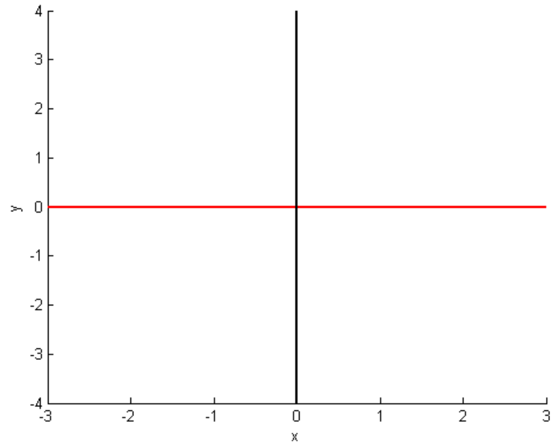

Figure 3.5.2.1: Projection of sibling curves of a circle (k=1)

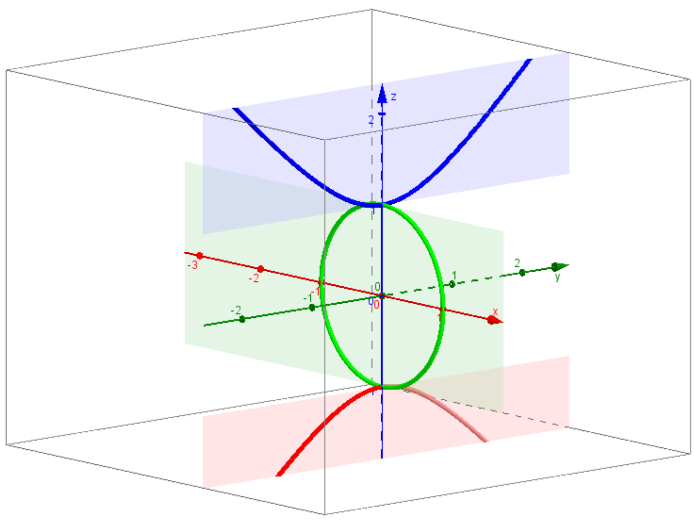

Figure 3.5.2.2: Sibling curves of a circle (k=1)

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Example 3.5.3. *The sibling curves for* $z^3 + (f(z))^3 = k$ *where* k *is a real number.*

Calculation: Suppose $f(z) = t$. Thus $z^3 = k - t^3$. Now for each real number r, $z^3 = r$ has three solutions $\sqrt[3]{r}$, $\sqrt[3]{r}e^{i\frac{2\pi}{3}}$ and $\sqrt[3]{r}e^{i\frac{4\pi}{3}}$.

Parameterization: Thus the parametrization of the sibling curves is given by the formulae below where t is a real number.

$$\begin{aligned} g_1(t) &= (\sqrt[3]{k - t^3}, t) \\ g_2(t) &= (\sqrt[3]{k - t^3}e^{i\frac{2\pi}{3}}, t) \\ g_3(t) &= (\sqrt[3]{k - t^3}e^{i\frac{4\pi}{3}}, t) \end{aligned}$$

Remark: We end up with three sibling curves. Notice the projection on the horizontal plane is three straight lines.

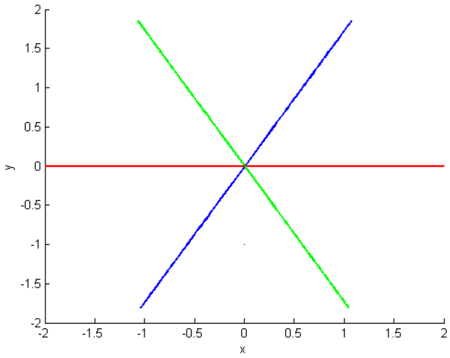

Figure 3.5.3.1: Projection of sibling curves of $z^3 + f(z)^3 = 1$

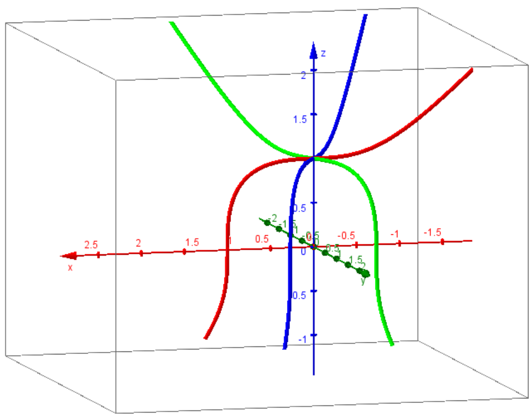

Figure 3.5.3.2: Sibling curves of $z^3 + f(z)^3 = 1$

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Example 3.5.4. *The sibling curves for* $f(z) = \sqrt{z}$.

Calculation: This multi-valued function $f(z) = \sqrt{z}$ can be rewritten as $z = (f(z))^2$. So if $f(z) = t$ where t is real then $z = t^2$.

Parameterization: Thus the parameterization of the sibling curves is given by (t^2, t) where t is a real number.

Remark: This multi-valued function $f(z) = \sqrt{z}$ or $f(z) = z^{0.5}$ only has one sibling curve. This sibling curve is a parabola. A similar calculation shows that $f(z) = z^{1/3}$ also only has one sibling curve.

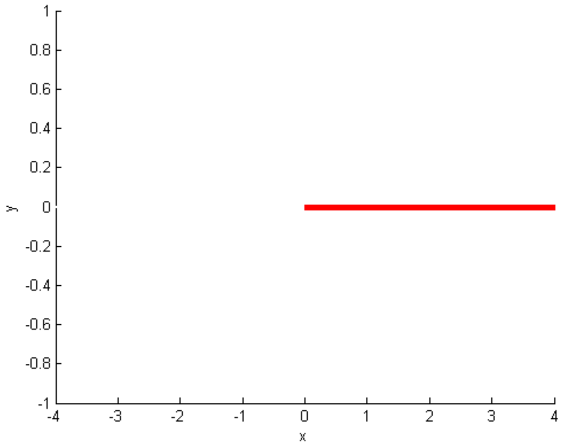

Figure 3.5.4.1: Projection of sibling curves of $f(z) = \sqrt{z}$

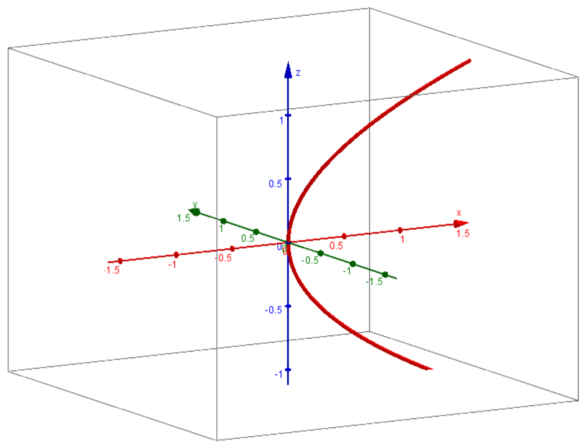

Figure 3.5.4.2: Sibling curves of $f(z) = \sqrt{z}$

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Example 3.5.5. *The sibling curves for* $f(z) = z^{1.5}$.

Calculation: This multi-valued function $f(z) = z^{1.5}$ can be rewritten as $z^3 = (f(z))^2$. Suppose $z = re^{i\theta}$ where $r \geq 0$ and $0 \leq \theta < 2\pi$. Thus $z^3 = r^3 e^{i\theta}$. If $f(z)$ is real, then z^3 is real and $z^3 \geq 0$. Thus $\theta = 0$ or $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Parameterization: Thus the parametrization of the sibling curves is given by $g_1(t) = (t, \sqrt{t^3})$, $g_2(t) = (t, -\sqrt{t^3})$, $g_3(t) = (te^{\frac{2\pi}{3}i}, \sqrt{t^3})$, $g_4(t) = (te^{\frac{2\pi}{3}i}, -\sqrt{t^3})$, $g_5(t) = (te^{\frac{4\pi}{3}i}, \sqrt{t^3})$ and $g_6(t) = (te^{\frac{4\pi}{3}i}, -\sqrt{t^3})$ where t is a non-negative real number.

Remark: This multi-valued function $f(z) = z^{1.5}$ therefore has six sibling curves. They all pass through the point $(0, 0)$.

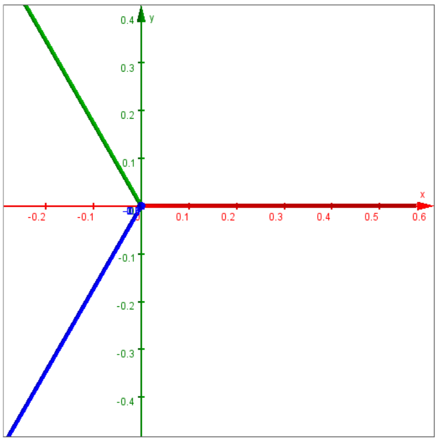

Figure 3.5.4.1: Projection of sibling curves of $f(z) = z^{1.5}$

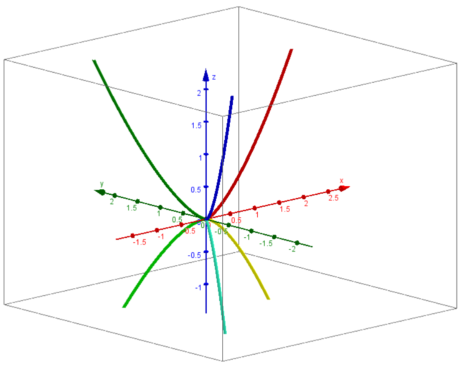

Figure 3.5.4.2: Sibling curves of $f(z) = z^{1.5}$

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Chapter 4

Sibling curves of polynomials

4.1 Introduction

A real polynomial of degree n has n complex roots. This is a result that follows from the Fundamental Theorem of Algebra. For example if $f(z) = z^2 + 1$, we get two solutions i and $-i$. The roots in this case are imaginary and the question of how to visually represent these zeroes is answered by means of the concept of *sibling curves*. This term was coined in papers [51] and [52].

However note that the Cartesian plane can only represent the real roots. Thus the idea of sibling curves is a more powerful visual tool to represent the roots of a polynomial. It also does not matter whether the roots are real or complex.

In Example 3.2.1, we saw that a real polynomial of degree 1 has 1 sibling curve. In Example 3.2.2 – 3.2.4 we saw that a real polynomial of degree 2 has 2 sibling curves. In Example 3.2.5 and Example 3.2.6 we saw that a real polynomial of degree 3 has 3 sibling curves. In Example 3.2.7 and Example 3.2.8 we saw that a real polynomial of degree 4 has 4 sibling curves. In Example 3.2.9 we saw that a real polynomial of degree 5 has 5 sibling curves. In Example 3.2.10 we saw that a real polynomial of degree 6 has 6 sibling curves. These observations led to the following questions:

1. **Does a real polynomial of degree n have n sibling curves?**
2. **What happens if we allow the coefficients to be complex?**

We proceed to answer these two questions in this chapter. In Section 4.2 we start off by looking at sub-parametrization, an important idea for this chapter. In Section 4.3 we prove a few lemmas before we resolve these questions in Section 4.4.

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4.2 An example of a sub-parametrization around a point

Now to show that the number of sibling curves depends only on the degree of the polynomial, we need to be able to form parametrizations of sibling curves. We start off by considering the simplest polynomials of degree n.

Example 4.2.1. We will show that the function $f : \mathbb{C} \to \mathbb{C}$ where $f(z) = z^n$ for some positive integer n has n sibling curves. To find the sibling curves of $f(z) = z^n$, we want to find all the values of z such that $f(z) \in \mathbb{R}$ and then form sibling curves using these points. To determine these values, we use De Moivre's Theorem. Assume $z = re^{i\theta}$ for some real numbers r and θ. If $f(z) = z^n = r^n e^{ni\theta} \in \mathbb{R}$, then $n\theta = \pi j$ for some integer j. Hence $\theta = \frac{\pi j}{n}$.

The projection of these points on the Argand plane gives us n straight lines parameterized by $g(t) = te^{\frac{\pi i j}{n}}$ where $t \in \mathbb{R}$ and $j = 0, 1, \ldots, n-1$. The curves defined on these n lines are the n sibling curves. They are parametrized by $g_j : \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ where

$$g_j(t) = (te^{\frac{\pi i j}{n}}, t^n(-1)^j), \quad j = 0, 1, 2, \ldots, n-1; t \in \mathbb{R}.$$

Notice that these n sibling curves contain all points z such that $f(z) \in \mathbb{R}$.

The parametrization in this example was easy. Furthermore the parametrization works for the entire sibling curve. It some cases you can only find a partial parametrization of a sibling curve using a power series. We demonstrate this idea to form a partial parametrization of a sibling curve. We take a specific polynomial at a specific point in the next example.

Example 4.2.2. Consider $f(z) = z - z^2$. We saw earlier on that this quadratic has two sibling curves. We focus on the point $z = 0$ and find a sub-parametrization of the sibling curve containing the point $z = 0$.

To find a parametrization for the portion of the sibling curve around $z = 0$, we consider a parametrization of the form $g(t) = (z, f(z))$ with $f(z) \in \mathbb{R}$. We use real number $f(z) = t$ as the parameter, so $g(t) = (h(t), t)$ with $f(h(t)) = t \in \mathbb{R}$. We try $h(t)$ as a power series about 0, that is

$$h(t) = 0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \ldots$$

for t for which it converges. Then

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$(h(t))^2 = a_1^2t^2 + (2a_1a_2)t^3 + (2a_1a_3 + a_2^2)t^4 + (2a_1a_4 + 2a_2a_3)t^5 + (2a_1a_5 + 2a_2a_4 + a_3^2)t^6 + \ldots.$

We desire $f(h(t)) = h(t) - (h(t))^2 = t$. This gives us

$$\begin{aligned} a_1 &= 1 \\ a_2 - a_1^2 &= 0 \\ a_3 - (2a_1a_2) &= 0 \\ a_4 - (2a_1a_3 + a_2^2) &= 0 \\ a_5 - (2a_1a_4 + 2a_2a_3) &= 0 \\ a_6 - (2a_1a_5 + 2a_2a_4 + a_3^3) &= 0 \\ &\ldots \end{aligned}$$

Solving, we obtain $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 5, a_5 = 14, a_6 = 42$, etcetera. Therefore a portion of one sibling can be parametrized by

$$g(t) = (t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + \ldots, t)$$

if the radius of convergence is greater than 0.

Coincidentally these coefficients are the well-known Catalan numbers [13] named after Eugéne Catalan (1814 – 1894). They appear in various counting problems, for example counting the number of triangulations of a convex polygon, the number of ways to tile a stair step shape of height n with n rectangles, the number of non-isomorphic trees with $n+1$ vertices, number of expressions containing n pairs of parentheses which are correctly matched, etcetera. Catalan numbers are formally defined as $C_1 = 1$ and then recursively by

$$C_{n+1} = C_1C_n + C_2C_{n-1} + \ldots + C_{n-1}C_2 + C_nC_1, n \in \mathbb{N}.$$

Applying Stirling approximations to this series, it can be shown that it has a radius of convergence $\frac{1}{4}$. Hence this is indeed a sub-parametrization for the sibling curve containing 0. The red curve is the sibling curve, the blue curve is an approximation of the parametrization using a polynomial of degree 3.

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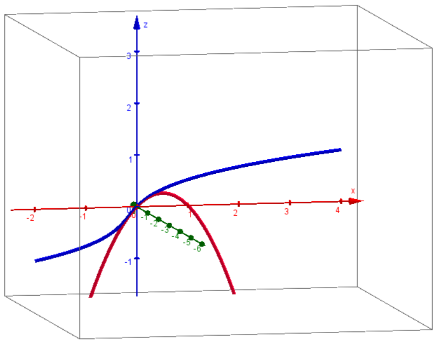

Figure 4.2.1: Partial sibling curves of $f(z) = z - z^2$ around 0

4.3 A few lemmas

The proof that a polynomial of degree n has n sibling curves needs the existence of sub-parametrizations at any point on a sibling curve. Lemma 4.3.1 and Lemma 4.3.2 show that these parametrizations have non-zero radii of convergence. This is needed in Lemma 4.3.3 to produce the sub-parametrizations when a point has a certain multiplicity.

Lemma 4.3.1. *For all positive integers k, n we have*

$$\binom{kn}{n} \leq k^{kn}.$$

Proof. The result is certainly true for $k = 1$. Fix some integer $k \geq 2$. We proceed by induction on n. Note if $n = 1$ then $\binom{k}{1} = k \leq k^k$.

Note $\frac{kn+j}{kn-n+j} \leq k$ if $k \geq 2$ and j any positive integer. This follows from the fact $2kn \leq k^2 n$ and $j \leq kj$ which gives $kn + j \leq k^2 n - kn + kj = k(kn - n + j)$. Using this, we get

$$\begin{aligned} \binom{k(n+1)}{n+1} &= \binom{kn}{n} \frac{(kn+1) \cdot (kn+2) \cdot \ldots \cdot (kn+k)}{(kn-n+1) \cdot (kn-n+2) \cdot \ldots \cdot (kn-n+k)} \\ &\leq k^{kn} \cdot k \cdot k \cdot \ldots \cdot k \\ &= k^{k(n+1)}. \end{aligned}$$

This completes our induction. ☐

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Lemma 4.3.2. *Assume $a_1, a_2, \ldots, a_n \in \mathbb{C}$ are the first n terms of a sequence. For complex $\alpha_1, \alpha_2, \ldots, \alpha_n$, define the rest of the sequence recursively by*

$$\begin{aligned} a_{m+1} &= \alpha_1 \sum_{j_p \geq 1} a_{j_1} a_{m+1-j_1} + \alpha_2 \sum_{j_p \geq 1} a_{j_1} a_{j_2} a_{m+1-j_1-j_2} + \ldots \\ &+ \alpha_n \sum_{j_p \geq 1} a_{j_1} a_{j_2} \ldots a_{j_{n-1}} a_{m+1-j_1-i_2 \ldots -j_{n-1}} \end{aligned}$$

where $m \geq n$. Then $\sqrt[m]{|a_m|}$ is a bounded sequence.

Proof. We start by eliminating the coefficients α_i, by defining a new sequence b_i that does not need these coefficients. We then proceed to define another sequence d_i whose recurrence equation is even simpler. With the help of Catalan numbers we then show $\sqrt[m]{d_m}$ is bounded. This then proves $\sqrt[m]{b_m}$ and $\sqrt[m]{|a_m|}$ are both bounded sequences.

We start off by finding a positive real number t such that $|\alpha_j t| \leq t^{j+1}$ for $j = 1, 2, \ldots, n$. Hence by the triangle inequality if $m \geq n$,

$$\begin{aligned} |ta_{m+1}| &\leq \sum_{j_p \geq 1} |\alpha_1 t a_{j_1} a_{m+1-j_1}| + \sum_{j_p \geq 1} |\alpha_2 t a_{j_1} a_{j_2} a_{m+1-j_1-j_2}| + \ldots \\ &+ \sum_{j_p \geq 1} |\alpha_m t a_{j_1} a_{j_2} \ldots a_{j_{n-1}} a_{m+1-j_1-j_2 \ldots -j_{n-1}}| \\ &\leq \sum_{j_p \geq 1} |(ta_{j_1})(ta_{m+1-j_1})| + \sum_{j_p \geq 1} |(ta_{j_1})(ta_{j_2})(ta_{m+1-j_1-j_2})| + \ldots \\ &+ \sum_{j_p \geq 1} |(ta_{j_1})(ta_{j_2}) \ldots (ta_{j_{n-1}})(ta_{m+1-j_1-j_2 \ldots -j_{n-1}})| \end{aligned}$$

Define $b_j = |ta_j|$. If $m \geq n$, then

$$\begin{aligned} b_{m+1} &\leq \sum_{j_p \geq 1} b_{j_1} b_{m+1-j_1} + \sum_{j_p \geq 1} b_{j_1} b_{j_2} b_{m+1-j_1-j_2} + \ldots \\ &+ \sum_{j_p \geq 1} b_{j_1} b_{j_2} \ldots b_{j_{n-1}} b_{m+1-j_1-j_2 \ldots -j_{n-1}}. \end{aligned}$$

Define $d_1 = \max\{1, b_1\}$ and $d_j = b_j$ if $j = 2, 3, \ldots, n$. If $m \geq n$, define d_{m+1} as follows

$$\begin{aligned} d_{m+1} &= \sum_{j_p \geq 1} d_{j_1} d_{m+1-j_1} + \sum_{j_p \geq 1} d_{j_1} d_{j_2} d_{m+1-j_1-j_2} + \ldots \\ &+ \sum_{j_p \geq 1} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} d_{m+1-j_1-j_2 \ldots -j_{n-1}}. \end{aligned}$$

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An easy induction argument shows that $b_j \leq d_j$ for all positive integers j. Now choose a positive real number k such that $n^2 \leq k$ and $d_j \leq k^{j-1} d_1^j C_j$ for $j = 1, 2, \ldots, n$ where C_j are the Catalan numbers defined in Example 4.2.2. Also for convenience define $d_0 = 1$. Assume for induction that $d_j \leq k^{j-1} d_1^j C_j$ for all $j \leq m$ where $m \geq n$. Consider d_{m+1}.

$$d_{m+1} = \sum_{0 \leq j_p \leq m, j_1 + \ldots + j_n = m+1} d_{j_1} d_{j_2} \ldots d_{j_n}$$

For each term one of the d_{j_p} terms has to have $j_p > 0$, that is $j_p = 1, 2, \ldots, m$. It can occur at n places, that is $p = 1, 2, \ldots, n$. Thus

$$d_{m+1} \leq n[d_1 \sum_{0 \leq j_p \leq m, j_1 + \ldots + j_{n-1} = m} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} + \ldots + d_m \sum_{0 \leq j_p \leq 1, j_1 + \ldots + j_{n-1} = 1} d_{j_1} d_{j_2} \ldots d_{j_{n-1}}]$$

Consider each d_k term. It is possible that one $j_p = m + 1 - k$. This can occur $(n-1)$ times, giving a total of $(n-1) d_{m+1-k}$. However if none of the $j_p = m + 1 - k$, then all the j_p is less than $m + 1 - k$ and

$$\sum_{0 \leq j_p < m+1-k, j_1 + \ldots + j_{n-1} = m+1-k} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} \leq d_{m+1-k}$$

which shows

$$d_{m+1} \leq n[d_1(nd_m) + d_2(nd_{m-1}) + \ldots + d_m(nd_1)]$$

Each term now contains n^2 and $n^2 \leq k$. Using the induction hypothesis, we obtain

$$d_{m+1} \leq k[(d_1 C_1)(k^{m-1} d_1^m C_m) + (k^1 d_1^2 C_2)(k^{m-2} d_1^{m-1} C_{m-1}) + \ldots + (k^{m-1} d_1^m C_m)(d_1 C_1)]$$

Simplifying and using the Catalan recursion, we obtain

$$\begin{aligned} d_{m+1} &\leq k^m d_1^{m+1}(C_1 C_m + C_2 C_{m-1} + \ldots + C_m C_1) \\ d_{m+1} &\leq k^m d_1^{m+1} C_{m+1} \end{aligned}$$

This completes the induction and therefore $\sqrt[m]{d_m} \leq k^{1-1/m} d_1 \sqrt[m]{C_m}$. Using $C_m = \frac{\binom{2m}{m}}{m+1} \leq \frac{4^m}{m+1}$

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by Lemma 4.3.1, we see that $\sqrt[m]{d_m}$ is a bounded sequence. Since $0 \leq b_m \leq d_m$, it follows that $\sqrt[m]{b_m}$ is bounded and consequently $\sqrt[m]{|a_m|}$ is also a bounded sequence. □

Now we are ready to produce the sub-parametrizations when a point has a certain multiplicity. This is a very important result and will be used in the next section.

Lemma 4.3.3. *If f is a polynomial with a root of multiplicity m at the origin, then there are m distinct sub-parametrizations of sibling curves around $z = 0$.*

Proof. Since f is a polynomial with a root of multiplicity m at the origin $f(z) = c_n z^n + c_{n-1}z^{n-1} + \ldots c_m z^m$ for some complex coefficients $c_m, \ldots, c_n$, $c_m \neq 0$ and $m \leq n$.

Suppose $h(t) = a_1 t + a_2 t^2 + a_3 t^3 + \ldots$ for some coefficients a_j. The aim here is to find $h(t)$ so that $f(h(t)) = t^m$ and then to manipulate it into forming m sub-parametrizations.

Looking at the t^m term of $f(h(t))$ we want $c_m a_1^m = 1$. Select any of the m values for a_1 to satisfy this equation. To proceed, consider the t^k term where $k \geq m+1$. We want

$$\begin{aligned} 0 &= c_n \sum_{j_p \geq 1} a_{j_1} a_{j_2} \ldots a_{j_{n-1}} a_{k-j_1-j_2-\ldots-j_{n-1}} + \ldots \\ &+ c_m \sum_{j_p \geq 1} a_{j_1} a_{j_2} \ldots a_{j_{m-1}} a_{k-j_1-j_2-\ldots-j_{m-1}} \end{aligned}$$

Using this equation, we can solve a_k uniquely by induction. Continuing in this manner, we form a series that satisfies the equation $f(h(t)) = t^m$ and by Lemma 4.3.2, we know that $\limsup |a_n|^{1/n}$ exists, thus the power series has a non-zero radius of convergence.

Using this power series, we can now form m sub-parametrizations of sibling curves around 0. In the formula below $j = 0, 1, 2, \ldots, m-1$ and

$$g_j(t) = \begin{cases} (h(\sqrt[m]{t}e^{\frac{\pi i j}{m}}), t(-1)^j) & \text{if } t \geq 0 \\ (h(\sqrt[m]{-t}e^{\frac{\pi i j}{m}+\pi i}), t(-1)^{j+m+1}) & \text{if } t < 0. \end{cases}$$

It should be noted that each g_j is continuous and differentiable on the interval of convergence. Note if m is even, then the sub-parametrization produces curves with either non-negative or non-positive z values. And if m is odd then the sub-parametrization produces curves with both negative and positive z values. □

Before we prove the general result, let us demonstrate this lemma by an example.

Example 4.3.4. Consider $f(z) = z^2 - z^3$. To find two sub-parametrizations around 0, we take

$$h(t) = a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

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Then

$$\begin{aligned} (h(t))^2 &= a_1^2t^2 + 2a_1a_2t^3 + (a_2^2 + 2a_1a_3)t^4 + \dots \\ (h(t))^3 &= a_1^3t^3 + (3a_1^2a_2)t^4 + \dots \end{aligned}$$

Thus if $f(h(t)) = t^2$, then

$$\begin{aligned} a_1^2 &= 1 \\ 2a_1a_2 - a_1^3 &= 0 \\ a_2^2 + 2a_1a_3 - 3a_1^2a_2 &= 0 \\ &\dots \end{aligned}$$

Selecting $a_1 = 1$, we can solve the other coefficients uniquely, as $a_2 = \frac{1}{2}, a_3 = \frac{5}{8}, \dots$. Thus $h(t) = t + \frac{1}{2}t^2 + \frac{5}{8}t^3 + \dots$. By Lemma 4.3.2 we know this series has a radius of convergence bigger than 0. Using $h(t)$ we now form two sub-parametrizations around 0.

$$g_0(t) = \begin{cases} (\sqrt{t} + \frac{1}{2}t + \frac{5}{8}t\sqrt{t} + \dots, t) & \text{if } t \geq 0 \\ (-\sqrt{-t} - \frac{1}{2}t + \frac{5}{8}t\sqrt{-t} + \dots, -t) & \text{if } t < 0. \end{cases}$$

and

$$g_1(t) = \begin{cases} (i\sqrt{t} - \frac{1}{2}t - \frac{5}{8}it\sqrt{t} + \dots, -t) & \text{if } t \geq 0 \\ (-i\sqrt{-t} + \frac{1}{2}t - \frac{5}{8}it\sqrt{-t} + \dots, t) & \text{if } t < 0. \end{cases}$$

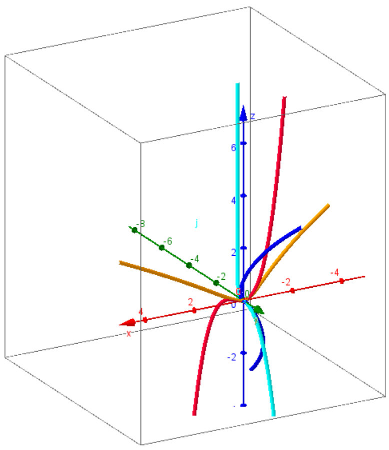

Figure 4.3.4: Partial sibling curves of $f(z) = z^2 - z^3$ around 0

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In this diagram the dark blue and red curves are the sibling curves going through the origin. The light blue curve and orange curve are approximations of the parametrization of the sibling curves through the origin, given by g_0 and g_1 to three terms.

4.4 The general case

With the help of Lemma 4.3.3, we are ready to prove the main result of this chapter.

Theorem 4.4.1. *If $f(z)$ is a complex polynomial of degree n, then f has n sibling curves.*

Proof. We will show for each real value of w that there are always n portions of sibling curves containing the solutions of $f(z) = w$. If these subcurves are glued together, we get the desired n sibling curves.

For some fixed $w \in \mathbb{R}$, we know $f(z) = w$ has n solutions by the fundamental theorem of algebra. Some may have multiplicity higher than 1. Take solution z_1 with multiplicity m. Then $f(z) - w = q(z - z_1)$ where $q(z) = c_m z^m + \ldots + c_n z^n$. Hence, if $q(z) = 0$ then $f(z + z_1) = w$. So we only need to show that we get m sibling subcurves containing the solutions $q(z) = 0$. Note there are at most $n - 1$ real values w such that $g(z) = f(z) - w$ has a root with multiplicity higher than one.

Noting $q(z) = f(z_1) - w = 0$, the solution now lies in using Lemma 4.3.3. There we proved that it is possible to define m sibling subcurves of q around 0. By [19] they contain all the solutions in that neighbourhood. Furthermore each of them has the same non-zero radius of convergence. Thus each value of w produces n sub-parametrizations.

Now for any real value w, we have n sub-parametrizations. Suppose R is the smallest radius of convergence for these sub-parametrizations. That is each sub-parametrization is valid on the interval $(w - R, w + R)$. Now consider the real values $w - \frac{R}{2}$ and $w + \frac{R}{2}$. They each have n sub-parametrizations. By glueing two sub-parametrizations that overlap together, we form n piece-wise functions on the interval $[w - \frac{R}{2}, w + \frac{R}{2}]$. Continuing in this manner we produce the n sibling curves. □

It should be noted that the proof above is not only true for real polynomials like $z^4 + 2z^2 + z + 2$ or $z^6 - 1$, but that it holds true for any polynomial, including those with complex coefficients, like $z^8 + iz + 3$ or $z^7 - (i + 2)z^3 + z^2 + i$.

This result gives us a richer and more visual understanding of the roots of a polynomial. The Fundamental Theorem of Algebra merely tells us that a polynomial of degree n has n roots.

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Now with the aid of this theorem we see that each polynomial of degree n has n special curves associated with it and they contain the zeroes of the polynomial. Moreover, we obtain a better visual understanding of the four dimensional graph of a complex function of complex variables by looking at the three dimensional cut of the graph on which the function values are real.

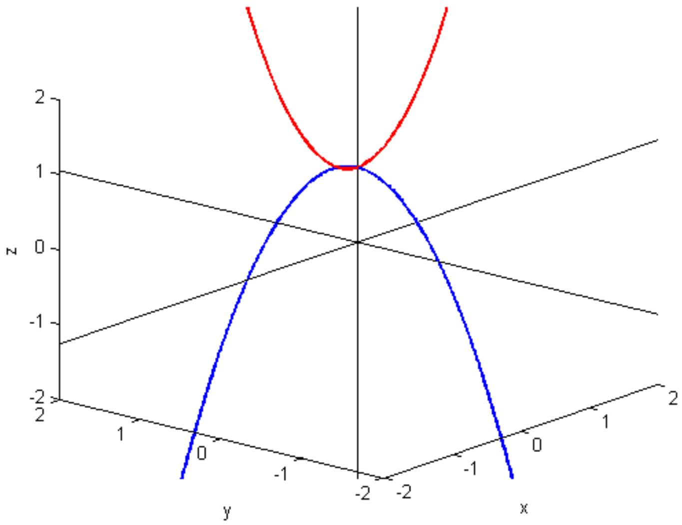

Figure 4.4.1: Sibling curves of $z^2 + 1$.

In the graph above the roots of z^2+1 are $-i$ and i are shown. They both lie on the blue sibling curve. However, if we relax the smoothness of sibling curves, we can rebranch the sibling curves to form the next figure.

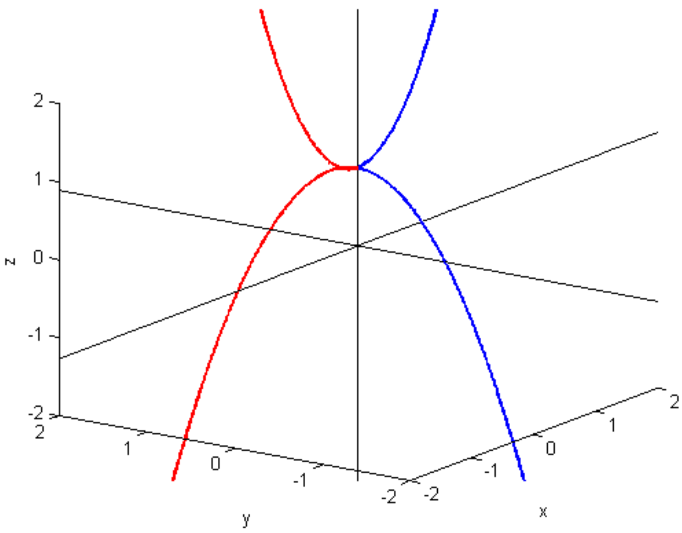

Figure 4.4.2: Rebranching of the sibling curves of $z^2 + 1$

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Now we have two curves, both containing a root of $z^2 + 1 = 0$. In general it can be shown by rebranching the sibling curves, we can end up with n sibling curves, each containing one zero of the polynomial.

Theorem 4.4.2. *If $f(z)$ is a complex polynomial of degree n then the sibling curves can be rebranched into n continuous curves each containing a root.*

Proof. We start with $f(z) = 0$. Knowing $f(z) = 0$ has n solutions and by the previous result we create n sub-parametrizations that contain these n roots. We continue the glueing process by considering $f(z) = w$ where $w > 0$ and $w < 0$.

The only values for w where the glueing process needs consideration is when the sibling curves meet. This happens when $f'(v) = 0$ and $f(v) = w$ for some complex number v, since $f'(v) \neq 0$ implies that $w = f(v)$ can be viewed as a coordinate change in a neighborhood of v. This happens only a finite number of times. If this happens around v, we have m parameterizations that is above v, label them $T_1, T_2, \ldots, T_m$ and there are m parameterizations below v, label them $B_1, B_2, \ldots, B_m$. by glueing B_i with T_i, we keep continuity but we might lose differentiability. Continuing in this manner, we ultimately create n curves that each contain a root.

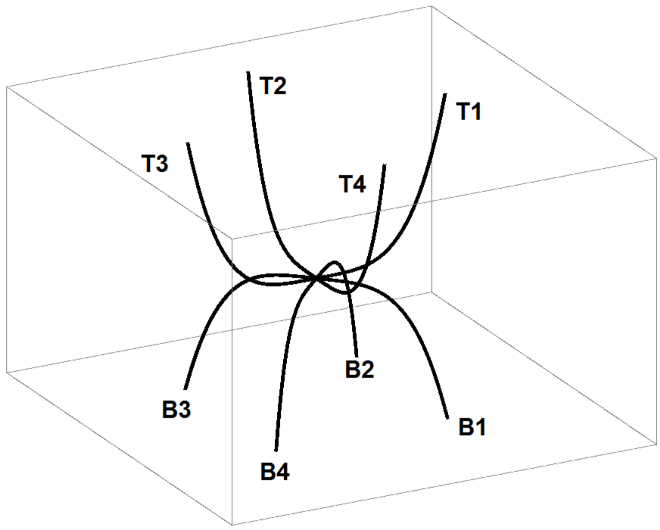

Figure 4.4.3: Rebranching of sibling curves if four meet

In this example four sibling curves meet. See Figure 4.4.3. Typically to preserve differentiability B1 and B3, B2 and B4, T1 and T3, T2 and T4 gets glued together. This can also be confirmed by the angles of the projections on the horizontal plane. However if we glue B1 and T1, B2 and T2, B3 and T3, B4 and T4 we end up creating four curves that each contains a root, they might not be differentiable, but they are still continuous. □

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Chapter 5

Sibling curves of quadratic polynomials

5.1 Introduction

Given a quadratic polynomial, we have a formula to determine its roots. However, if we calculate the sibling curves of a quadratic polynomial, we have a more effective way of visualizing the roots of a quadratic polynomial.

In this chapter we focus solely on quadratic equations and answer the following two questions:

1. **What do the sibling curves of quadratic polynomials look like?**
2. **What geometric properties do the sibling curves of quadratic polynomials have?**

Section 5.2 commences by only considering real quadratics. Section 5.3 moves on to considering complex quadratics. In Section 5.4 we look at congruence which is observed in the previous two sections. In Section 5.5 we look at a surface on which all the quadratic sibling curves lie. Finally in Section 5.6 we extend the idea of sibling curves to θ-sibling curves.

5.2 Real quadratics

To get an understanding of the sibling curves of quadratic polynomials, we start our investigation by looking at quadratic polynomials where the coefficients are real. The next result shows that in this case we get two sibling curves that always meet in one point.

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Theorem 5.2.1. Let $f(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$. The sibling curves are $(t, at^2 + bt + c)$ and $(\frac{-b}{2a} + it, \frac{4ac-b^2}{4a} - at^2)$ in $\mathbb{C} \times \mathbb{R}$.

Proof. Let $z = x + iy$. Then $f(z) = az^2 + bz + c = (ax^2 - ay^2 + bx + c) + i(2axy + by)$. So if $f(z) \in \mathbb{R}$ then $2axy + by = 0$, ie. $y = 0$ or $x = \frac{-b}{2a}$. Hence the projection of each sibling curve into the domain $\mathbb{C}$ is a straight line. Using this we get two intersecting sibling curves. One is $(t, at^2 + bt + c)$, the original parabola. The other is $(\frac{-b}{2a} + it, \frac{4ac-b^2}{4a} - at^2)$. □

It should be noted that these two sibling curves meet at the point $(\frac{-b}{2a}, \frac{4ac-b^2}{4a})$, the turning point of the original parabola. From Figure 5.2.1 it appears that the other sibling curve is also a parabola. The next result confirms this suspicion and proves something slightly stronger. For this theorem we need to define the idea of congruency.

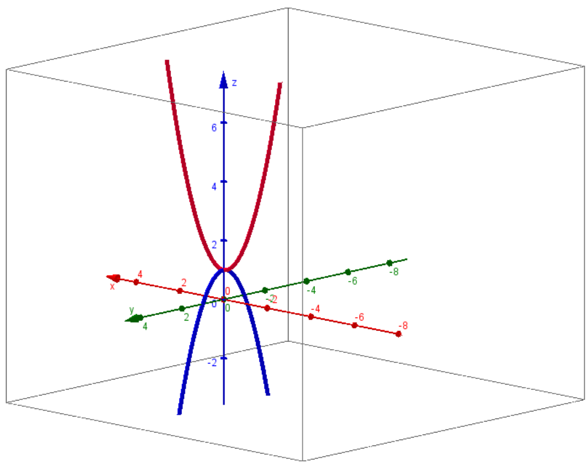

Figure 5.2.1: Sibling curves of $f(z) = z^2 + 1$

Definition 5.2.2. Suppose $A, B \subseteq \mathbb{C} \times \mathbb{R}$. We say sets A and B are congruent if there is a bijection $\phi : A \to B$ and furthermore whenever $a_1, a_2 \in A$ then $d(a_1, a_2) = d(\phi(a_1), \phi(a_2))$ where d is the usual distance metric in the space $\mathbb{C} \times \mathbb{R}$.

So two sets are congruent if there is a distance preserving bijection, also known as an isometry. Now we are ready to reconsider the sibling curves of quadratic polynomials with real coefficients.

Theorem 5.2.3. If $f(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$, then the two sibling curves are congruent parabolas.

Proof. From the previous result, it follows that the sibling curves are $(t, at^2 + bt + c)$ and $(\frac{-b}{2a} + it, \frac{4ac-b^2}{4a} - at^2)$. Consider the map $\phi : (t, at^2 + bt + c) \mapsto (\frac{-b}{2a} + i(t + \frac{b}{2a}), -at^2 - bt - c)$. Note this map is a bijection between the two sibling curves.

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To show this map preserves distance, consider two points $P(t, at^2+bt+c)$ and $Q(s, as^2+bs+c)$ for some real numbers t and s. Note $d(P,Q) = \sqrt{(t-s)^2+(at^2+bt-as^2-bs)^2}$, $\phi(P) = (\frac{-b}{2a}+i(t+\frac{b}{2a}), -at^2-bt-c)$ and $\phi(Q) = (\frac{-b}{2a}+i(s+\frac{b}{2a}), -as^2-bs-c)$. Thus $d(\phi(P),\phi(Q)) = \sqrt{(t-s)^2+(at^2+bt-as^2-bs)^2}$ showing $d(P,Q) = d(\phi(P),\phi(Q))$ which proves that this map preserves distance.

Therefore the two sibling curves are congruent. Hence they are both parabolas. □

Observing Figure 5.2.1 again, it is clear that the two sibling curves are congruent and can be obtained from each other via two rotations in $\mathbb{C}\times\mathbb{R}$.

5.3 Complex quadratics

We now look at quadratic polynomials where the coefficients are complex numbers. We first face the question when do the sibling curves of a quadratic polynomial intersect?

Theorem 5.3.1. Consider $f(z) = az^2+bz+c$ where $a,b,c \in \mathbb{C}$ and $a \neq 0$. The two sibling curves intersect if and only if $\frac{4ac-b^2}{4a}$ is a real number.

Proof. Two sibling curves intersect if and only if there is a z so that $f'(z) = 0$ and $f(z) \in \mathbb{R}$. If $f'(z) = 0$ then $z = \frac{-b}{2a}$. Now $f(z)$ is a real number if $\frac{b^2}{4a} - \frac{2b^2}{4a} + c = \frac{4ac-b^2}{4a}$ is a real number. □

Note $\frac{4ac-b^2}{4a} = \frac{4|a|^2c-b^2\overline{a}}{4|a|^2}$. So alternatively if $4|a|^2c-b^2\overline{a}$ is a real number, the sibling curves meet.

Suppose $a = a_1+a_2i, b = b_1+b_2i$ and $c = c_1+c_2i$ where a_1,a_2,b_1,b_2,c_1,c_2 are all real numbers. It turns out that for the two sibling curves to intersect that $4(a_1^2+a_2^2)c_2-2b_1b_2a_1+b_1^2a_2-b_2^2a_2 = 0$ or

$$c_2 = \frac{2b_1b_2a_1+b_2^2a_2-b_1^2a_2}{4(a_1^2+a_2^2)}$$

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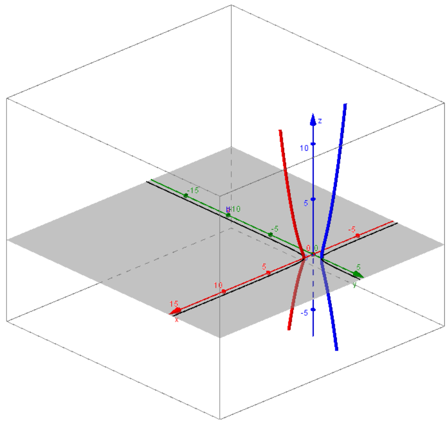

Figure 5.3.1: Sibling curves of $f(z) = (-0.2 - 0.1i)z^2 + (1.4 + 0.3i)z + (3.4 + 1.8i)$

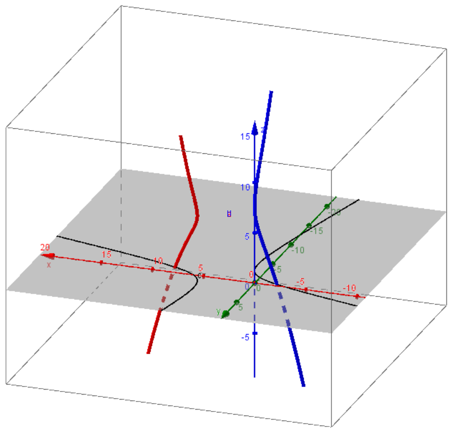

Figure 5.3.2: Sibling curves of $f(z) = (z - 1)(z - i)$

Sketching the sibling curves of $f(z) = (-0.2 - 0.1i)z^2 + (1.4 + 0.3i) + (3.4 + 1.8i)$ and $f(z) = (z - 1)(z - i)$ we notice the projection on the domain is a hyperbola (Figure 5.3.1 and Figure 5.3.2) and not two intersecting straight lines as was the case with real quadratic polynomials. This observation leads us to ask what do the projections of sibling curves look like in the general quadratic case?

Theorem 5.3.2. Consider $f(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{C}$ and $a \neq 0$. The projections of the sibling curves are always hyperbolic on the domain. Furthermore they only meet when $4|a|^2c - b^2\overline{a}$ is a real number.

Proof. Assume $a = a_1 + a_2i, b = b_1 + b_2i, c = c_1 + c_2i, z = x + iy$ where $a_1, a_2, b_1, b_2, c_1, c_2, x, y \in \mathbb{R}$. Thus $z^2 = (x^2 - y^2) + i2xy$. So if $f(z) \in \mathbb{R}$ then

$$a_2x^2 + a_1 2xy - a_2y^2 + b_1y + b_2x + c_2 = 0.$$

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Now finding the discriminant, we get $(2a_1)^2 - 4(a_2)(-a_2) = 4(a_1^2 + a_2^2) > 0$. So the projections of the sibling curves are always hyperbolic.

If $a_2 \neq 0$ then

$$x = \frac{-2a_1y - b_2 \pm \sqrt{(4a_1^2 + 4a_2^2)y^2 + (4a_1b_2 - 4a_2b_1)y + b_2^2 - 4a_2c_2}}{2a_2}$$

Thus if the quadratic inside the square root sign has two equal roots, the sibling curves meet. This happens if $c_2 = \frac{2b_1b_2a_1 + b_2^2a_2 - b_1^2a_2}{4(a_1^2 + a_2^2)}$, otherwise they do not meet.

If $a_2 = 0$, we get $2a_1xy + b_1y + b_2x + c_2 = 0$ or

$$(2a_1x + b_1)(2a_1y + b_2) = b_1b_2 - 2a_1^2c_2$$

Thus if $c_2 = \frac{b_1b_2}{2a_1^2}$, the sibling curves meet in one point.

□

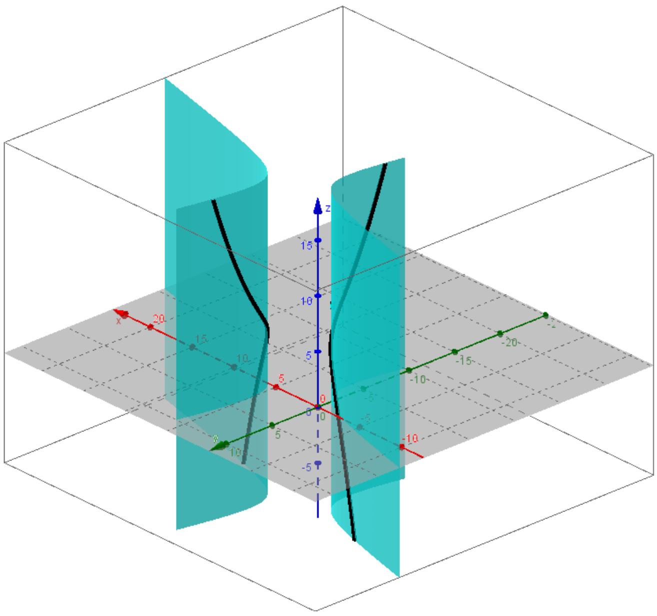

Figure 5.3.3: Sibling curves of $f(z) = (-0.2 - 0.1i)z^2 + (1.4 + 0.3i)z + (3.4 + 1.8i)$ shown on surface created by the projections

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5.4 Congruence

Earlier we noticed that if the coefficients are real, the sibling curves are congruent. We now prove that this is the case for any quadratic polynomial giving us a stronger result than Theorem 5.2.3.

Theorem 5.4.1. *The two sibling curves of any quadratic polynomial are congruent.*

Proof. Consider $f(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{C}$. Let $g(z) = f(z - \frac{b}{2a}) = Az^2 + C$ where $A = a$ and $C = \frac{b^2}{4a^2} - \frac{b^2}{2a} + c$. So we only need to show that the sibling curves of $g(z)$ are congruent.

Since $A \neq 0$, there is a complex number α so that $\alpha^2 = \frac{1}{A}$. Let $h(z) = g(\alpha z) = A\alpha^2 z^2 + C = z^2 + C$. So we only need to show the sibling curves of $h(z)$ are congruent to show the sibling curves of $g(z)$ are congruent.

If $C \in \mathbb{R}$ then the sibling curves are congruent according to Theorem 5.2.3. Suppose now $C = c_1 + c_2 i$ where $c_2 \neq 0$. In this case the sibling curves are $(t - \frac{c_2}{2t}i, t^2 - \frac{c_2^2}{4t^2})$ where $t > 0$ and $(t - \frac{c_2}{2t}i, t^2 - \frac{c_2^2}{4t^2})$ where $t < 0$. They are again congruent.

So using rotation, scaling and translation it follows that all quadratic sibling curves are congruent. □

Sibling curves of higher degree polynomials are not always congruent as the next example illustrates.

Example 5.4.2. Consider $f(z) = z^3 - z$. A simple calculation shows the sibling curves are $(t, t^3 - t)$, $(\sqrt{\frac{t^2+1}{3}} + it, \frac{-8t^2-2}{3}\sqrt{\frac{t^2+1}{3}})$ and $(-\sqrt{\frac{t^2+1}{3}} + it, \frac{8t^2+2}{3}\sqrt{\frac{t^2+1}{3}})$ where t is a real number.

Now consider the following curves in $\mathbb{C} \times \mathbb{R}$ parametrized by t, $a(t) = (t, t^3 - t)$ and $b(t) = (\sqrt{\frac{t^2+1}{3}} + it, \frac{-8t^2-2}{3}\sqrt{\frac{t^2+1}{3}})$.

It is clear, $a(t)$ is planar, it lies in the $y = 0$ plane. Now $b(\pm\sqrt{2}) = (1 \pm \sqrt{2}i, \frac{10}{3})$ and $b(\pm\sqrt{11}) = (1 \pm \sqrt{11}i, -60)$. So the plane that goes through these four points is $170x + 3z = 160$. It is not hard to see this plane does not contain $b(0) = (\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. So the curve $b(t)$ is not planar, so no isometry exists between sibling curves a and b.

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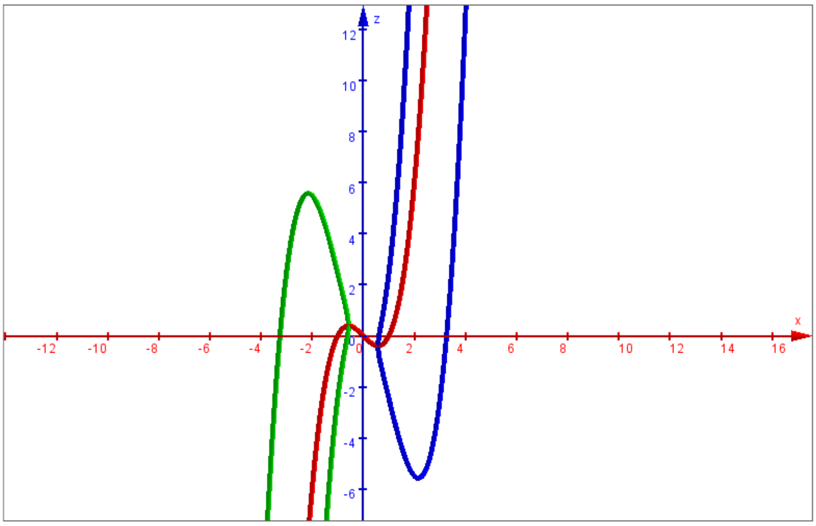

Figure 5.4.1: “Front view” of sibling curves of $f(z) = z^3 - z$

From Figure 5.4.1 it looks as if the three sibling curves intersect in two points. This is in fact true. They intersect when $f'(z) = 0$, and the coordinates are $(\frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}} - 1)$ and $(\frac{-1}{\sqrt{3}}, \frac{-1}{3\sqrt{3}} - 1)$. The next figure again illustrates that the sibling curves are not all congruent.

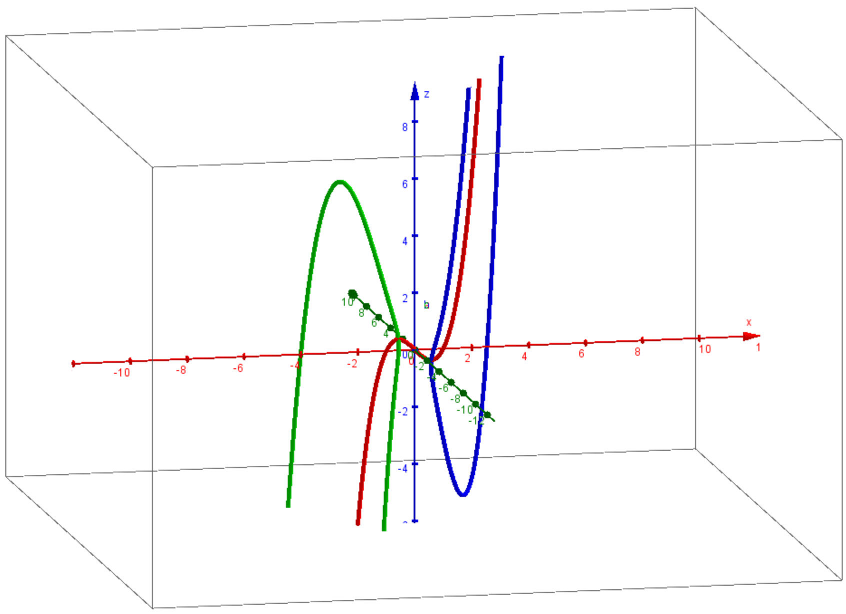

Figure 5.4.2: Sibling curves of $f(z) = z^3 - z$

However, looking at the curvature, we can find a way to distinguish the situation when the siblings curves of a quadratic polynomial meet or not meet.

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Theorem 5.4.3. If k is the curvature of the sibling curves of a quadratic polynomial, then the two sibling curves meet iff $1/k^2$ is a polynomial of degree 6.

Proof. Earlier we saw, we only need to consider $f(z) = z^2 + C$. If C is a real number, then the sibling curves meet. As they are congruent, we only consider one parametrization.

$$\begin{aligned} r(t) &= (t, 0, t^2 + C) \\ r'(t) &= (1, 0, 2t) \\ r''(t) &= (0, 0, 2) \end{aligned}$$

Thus $r'(t) \times r''(t) = (0, -2, 0)$. So

$$\begin{aligned} k(t) &= \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \\ &= \frac{2}{\sqrt{(1+4t^2)^3}} \\ \frac{1}{(k(t))^2} &= \frac{(1+4t^2)^3}{4} \end{aligned}$$

Thus $1/k^2$ is a polynomial of degree 6 if the sibling curves meet.

If the sibling curves do not meet, then $C = a - 2bi$ for some real numbers a and b and $b \neq 0$. Thus $f(x + iy) = x^2 - y^2 + a + i2(xy - b)$. For the sibling curves we must have $xy = b$. Thus one parametrization of a sibling curve is given below where $t > 0$.

$$\begin{aligned} r(t) &= (t, \frac{b}{t}, t^2 - \frac{b^2}{t^2} + a) \\ r'(t) &= (1, \frac{-b}{t^2}, 2t + \frac{2b^2}{t^3}) \\ r''(t) &= (0, \frac{2b}{t^3}, 2 - \frac{6b^3}{t^4}) \end{aligned}$$

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To find the curvature, we need

$$\begin{aligned} r'(t) \times r''(t) &= \left(\frac{-2b}{t^2} + \frac{6b^3}{t^6}, -2 + \frac{6b^2}{t^4}, \frac{2b}{t^3}\right) \\ |r'(t) \times r''(t)| &= \frac{1}{t^6}\sqrt{4t^{12} - 20b^2t^8 + 4b^2t^6 + 12b^4t^4 + 36b^6} \\ |r'(t)| &= \frac{1}{t^3}\sqrt{4t^8 + t^6 + 8b^2t^4 + b^2t^2 + 4b^4} \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{(k(t))^2} &= \frac{|r'(t)|^6}{|r'(t) \times r''(t)|^2} \\ &= \frac{(4t^8 + t^6 + 8b^2t^4 + b^2t^2 + 4b^4)^3}{4t^{18} - 20b^2t^{14} + 4b^2t^{12} + 12b^4t^{10} + 36b^6t^6} \end{aligned}$$

Notice that the numerator has a constant term of $64b^{12}$ which is not zero, since $b \neq 0$. Also the denominator can be factorised as t^6 times a polynomial of degree 12, thus showing that $\frac{1}{(k(t))^2}$ cannot be a polynomial of degree 6 if the sibling curves do not meet.

$\square$

This theorem is interesting in its own right, as it gives a geometric property of the sibling curves of quadratic polynomials in terms of curvature.

5.5 Hyperbolic paraboloid

In summary we noticed in the case of quadratic polynomials that one of two things can happen. One scenario is that the sibling curves meet. In this case we get two parabolas which are planar, implying that each sibling curve lies in its own plane. The second scenario is that the sibling curves never meet. Here we end up with two curves which are not parabolas. In either situation the sibling curves are congruent to each other.

From the previous section it follows that we really only need to consider the quadratic polynomial $f(z) = z^2 + C$ for some complex number C. If $z = x + iy$, then $f(z) = (x + iy)^2 + C = x^2 - y^2 + Re(C) + i(2xy + Im(C))$. Hence the sibling curves are on the surface $z = x^2 - y^2 + Re(C)$ or on the hyperbolic paraboloid (taco-shaped surface) $z = x^2 - y^2$ after an appropriate rotation, scaling and translation. The two typical possibilities are depicted in Figure 5.5.1 and Figure 5.5.2.

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In Figure 5.5.1 we see the typical scenario when the two sibling curves meet. We know in this case we get two congruent parabolas that meet in a point. Notice that the two parabolas meet in the saddle point of the hyperbolic paraboloid.

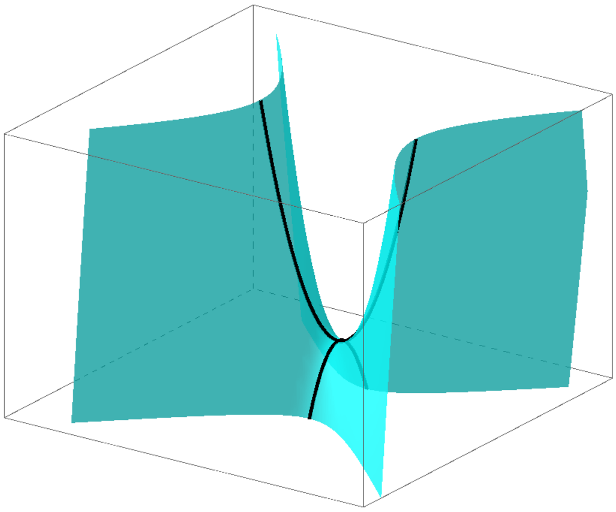

Figure 5.5.1: Sibling curves of $f(z) = z^2 - 1$ on a hyperbolic paraboloid

In Figure 5.5.2 we see the typical scenario when the two sibling curves do not meet. We know in this case we do not get two parabolas. However the sibling curves are still congruent and they lie opposite each other on the surface of the hyperbolic paraboloid.

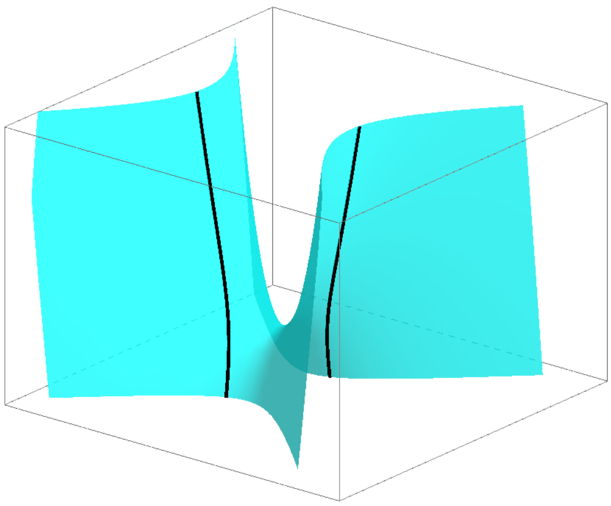

Figure 5.5.2: Sibling curves of $f(z) = z^2 - 1 + 1.8i$ on a hyperbolic paraboloid

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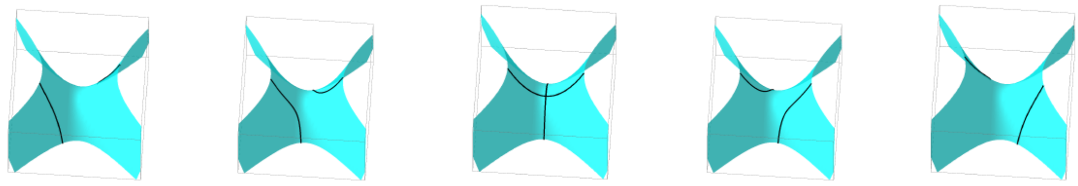
Below we sketched the sibling curves of $f(z) = z^2 + 2z + (1 + ki)$ for various values of k. By varying k from negative to positive, we see the evolution of sibling curves on the taco-like surface or the hyperbolic paraboloid. Notice when $k = 0$ the two sibling curves intersect in a point. They are both parabolas and it is the only value of k for which this happens. The reader can also visit https://cardanogroup.wordpress.com/ to see an animation of these sibling curves on this hyperbolic paraboloid.



(a) $k = -2$ (b) $k = -1$ (c) $k = 0$ (d) $k = 1$ (e) $k = 2$

Figure 5.5.3: Snapshots of various sibling curves on hyperbolic paraboloid

5.6 θ-sibling curves

So far we have only considered the values for which $f(z)$ is real. In so doing we formed the sibling curves of the complex valued function $f(z)$ which allows us to visualize the zeroes of the function $f(z)$. Now we pose, the question for which values will $f(z)$ be purely imaginary, that is the real part of $f(z)$ is zero. Or even more general, when is $f(z) = e^{i\theta}k$ for some real values θ and k.

Definition 5.6.1. We say $(g(t), p(g(t)))$ is a θ-sibling of p if $p(g(t)) = e^{i\theta}k$ for some real value k.

Note that 0-sibling curves correspond to the normal sibling curves that we have been studying the whole time since $e^{i0}k = 1 \times k = k$. Also note that all the θ-siblings contain the zeroes of the function. That is the intersection of the θ-siblings is the zeroes of the function. Each θ-sibling gives a different three dimensional cut of the complex valued function $f(z)$, which is four dimensional.

Theorem 5.6.2. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$, then the θ-siblings of p are the siblings of $q(z) = \frac{a_n}{e^{i\theta}}z^n + \frac{a_{n-1}}{e^{i\theta}}z^{n-1} + \ldots + \frac{a_0}{e^{i\theta}}$.

Proof. If $a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0 = e^{i\theta}k$ then $\frac{a_n}{e^{i\theta}}z^n + \frac{a_{n-1}}{e^{i\theta}}z^{n-1} + \ldots + \frac{a_0}{e^{i\theta}} = k$. So the θ-siblings are the normal siblings of the polynomial $q(z) = a_n e^{-i\theta}z^n + a_{n-1}e^{-i\theta}z^{n-1} + \ldots + a_0 e^{-i\theta}$. □

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Theorem 5.6.3. Consider $f(z) = az^2 + bz + c$ where $a, b, c \in \mathbb{C}$ and $a \neq 0$. The θ-siblings either always intersect or the θ-sibling curves only intersect for two values of θ.

Proof. Suppose $az^2 + bz + c = e^{i\theta}k$ for some real value k. Then $e^{-i\theta}az^2 + e^{-i\theta}bz + e^{-i\theta}c$ is a real number. Now $T = e^{-i\theta}(4|a|^2c - b^2\overline{a})$. Thus if $4|a|^2c - b^2\overline{a} = 0$, then T is always zero, giving two sibling curves that always intersect. One such example is $f(z) = z^2$. This occurs when $c = \frac{b^2}{4a}$.

However if $4|a|^2c - b^2\overline{a} \neq 0$, then for only two values of θ will T be a real number, producing two sibling curves that intersect. For the other values of θ the sibling curves do not intersect. One such example is $f(z) = z^2 + 1$. □

Thus the θ-siblings have the same geometric properties as the normal sibling curves. Each θ-sibling gives us a three dimensional cut of the four dimensional graph given by a complex valued function. Therefore we do get a full representation of what this four dimensional graph looks like.

5.7 Further research

Chapters 2 to 4, provide some interesting examples and results regarding sibling curves. This theory enriches the theory of complex numbers and adds a more visual component to this already rich theory. Sibling curves allow us to represent the roots of polynomials in a more natural way, which can be shown to students using technology.

In Chapter 3 we proved that a complex polynomial of degree n has n sibling curves. These special curves contain all the n roots of the polynomial and are therefore a visual way of illustrating the roots of a polynomial. Thus sibling curves give us a geometric way to represent a polynomial of degree n as n curves in three-space. This can lead to further geometric questions to be asked about sibling curves.

For example the Abel-Ruffini theorem (also known as Abel's impossibility theorem) states that there is no general solution using radicals to polynomial of degree five or higher. One possible investigation is to see what geometric properties sibling curves of polynomials of degree five or higher have, that can help to answer the following question:

Is it possible to use sibling curves theory to prove the Abel-Ruffini theorem?

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Studying the examples in Chapter 2, you find an examples of sibling curves that are planar and not planar. For example the sibling curves of $f(z) = z^2$ are planar, but some of the sibling curves of $f(z) = z^4+2z^2+z+2$ are not. So one possible research question to investigate is:

Under what conditions is a sibling curve planar?

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Chapter 6

Literature review

6.1 Introduction

This chapter synthesises the literature to set the background for the research on student enrichment.

6.2 Student enrichment

There have been several attempts to define enrichment within an educational setting. Correll [21] defines enrichment as any experience that substitutes, supplements or extends instruction beyond that normally offered by schools. Another definition by Stanley [100] defines enrichment as any educational procedure beyond the usual ones for the subject or grade or age of the student that does not accelerate or retard the student's placement in the subject or grade. Eyre and Marjoram [28] define enrichment as any sort of activity or learning that is outside the core of learning which most students undertake. Clendening and Davies [18] define enrichment as:

> *Any learning experience that replaces, supplements, or extends instruction beyond the restrictive boundaries of course content, textbook and classroom and that includes depth of understanding, breadth of understanding and relevance to the student and to the world in which he or she lives.* ([33], p. 2)

This definition is clearly adapted from Correll's definition. This emphasis on depth, breadth and relevance tries to illuminate the essence of student enrichment which is supported by other scholars ([118], [69]).

> *Enrichment as a way of giving better educational opportunities to the mentally advanced child...* ([118], p. 98)

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In their definition Clendening and Davies [18] try to remove the "aura of vagueness and confusion" that seems to surround the term according to Barbe ([6], p. 521) so that it does not become a label that educators hide behind according to Gold [47].

To further clarify what the type of activities are that can be considered as student enrichment, we provide some examples. Examples 1 and 4 are taken from Stanley [100].

1. **Busy-work.** Many teachers use this method to occupy their bright and fast working students. Teachers would give these students extra work, that is the same work, but in greater quantity than what is typically given to the average student. Instead of giving them a few selected problems as homework, the teacher would give the entire list to the precocious students to keep them busy. Although these students are benefiting by sharpening their arithmetic skills and speed of solving mundane problems, they get oppressed creatively by not being challenged.

2. **Olympiads.** Teachers have their students participating in olympiads, such as mathematics olympiads, computer olympiads, physics olympiads, chemistry olympiads, biology olympiads, linguistics olympiads, astronomy olympiads, geography olympiads, philosophy olympiads [80], to list but a few. Olympiads generate excitement among students and they get the opportunity to test their problem solving skills by solving problems not typically seen in the classroom. These problems are different from the typical curriculum problems and some students who are not exposed to a little training beforehand can become demotivated as these problems can be considered as tough.

3. **Courses and programmes over the holiday.** One such example is the Johnson County Community College [58] summer programme for high school students. Their slogan is "Show your children they are college material". They expose high school students to a wide variety of topics and students get to choose from an exciting list of classes in mathematics, physics, technology, entrepreneurship, music, literature, archaeology, etcetera. These students also get the opportunity to interact with like-minded students and get to appreciate their subject more than what is being achieved in class. These programmes also have the scope of interdisciplinary education which is typically not achieved in the school curriculum.

4. **Cultural activities.** A teacher can organise trips and visits. A field trip or museum visit makes students excited to get out of the classroom. It could make the subject livelier. Teachers could organise a talk from an expert or a university lecturer in a relevant area. Teachers could give their students a project to stimulate further interest and which they can present to the class or display at a parent-teachers evening. Many project ideas can be found at

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http://metagifted.org/products/enrichmentProjectList/. This type of enrichment allows teachers to tap into skills like music, art, drama, dance, foreign languages, which are not typically tested in early years, and give students the opportunity to be creative.

In summary given the daily expectations and constraints on students and teachers, student enrichment is an opportunity to create a learning experience that will add value to the student's education.

6.3 Enrichment, acceleration and gifted education

The idea of student enrichment was influenced by thinking about *gifted education.*

> *Gifted education is the development of appropriate programming options to help students who exhibit gifted and talented behaviours best apply their strengths and interests to meet their learning potential.* ([15], p. 29)

Fox [36] juxtaposed student enrichment and gifted education to showcase their complementary nature, but more often the differences are highlighted by Clendening and Davies [18], Terman and Oden [104] and Witty [117]. Whilst gifted education only caters for the brightest students, student enrichment can cater for the whole class or those that the have potential to excel in a certain subject.

Student enrichment should not be confused with acceleration. Whereas enrichment refers to the presentation of curriculum content with more depth, breadth, complexity, or abstractness than in the general curriculum, acceleration refers to the practice of presenting curriculum content earlier or at a faster pace. Some ways of acceleration include acceleration in one or more subject areas, grade skipping, advanced placement programs, college courses offered in high school, early graduation from high school or early entrance into college [12].

Martinson [69], promoting enrichment, explains that enrichment is a means of providing pupils with greater freedom and latitude of inquiry and hence greater fulfilment and intellectual satisfaction than generally given in the basic school curriculum. Stanley [100] is more in favour of acceleration. He argues that by exposing pupils to more advanced subject matter or higher-order treatment of regular concepts, enrichment only serves to postpone boredom. Therefore Stanley explains that appropriate enrichment must accompany acceleration at school level. Others like Fox [36] see acceleration and enrichment as two techniques to be used flexibly to

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cater for the pupil's educational needs.

6.4 Enrichment for whom?

Academically strong students learn differently from their peer groups and typically are not being challenged and their learning needs are not being met [70]. Mathematical and science curricula that are taught in the traditional manner are often inappropriate, because it is highly repetitive and provides little depth ([56], [57], [90]).

Caine and Caine [16] showed that research indicates that learning takes place when students' abilities and interests are stimulated by appropriate levels of challenge. According to research by Schultz et al. [97].

> *When tasks are not sufficiently challenging, the brain does not release enough of the chemicals needed for learning: dopamine, noradrenalin, serotonin and other neurochemicals.* ([101], p. 9)

McAllister and Plourde [70] feel that challenge is a very important component of effective curriculum and instruction. If the content or tasks are too simple, these academically strong students will not become engaged in the work and consequently they will not be learning. So there is a definite need for enrichment for academically strong students.

The question of 'Enrichment for whom?' also depends on time and educational resources of teachers, parents and schools. Feng [33] acknowledges that all learners can benefit from enrichment and that it is not only for gifted students. Feng provides an example of how enrichment can take place in the 'classroom setting' by using 'extended planning' [105], whereby teachers introduce content not in the syllabus to the whole class in order to update the syllabus or integrate with other subjects or to include topics that are of current interest.

In the USA curriculum enrichment initially received attention ninety years ago out of concern that the educational needs of some students were not met [6]. In 1983 *A Nation at Risk: The Imperative for Educational Reform* a report of American President Ronald Reagan's National Commission on Excellence in Education was released [95]. Its publication is considered a landmark event in American educational history and it reports the ever-growing assertion that

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American schools were failing. It ignited a spark of local, state, and federal reform efforts.

One such effort is the Schoolwide Enrichment Model (SEM) created by J.S. Renzulli ([93], [94]). The SEM model was developed to encourage and create opportunities for creative productivity in young people by taking into account each student's academic strengths, interests, learning styles and preferred modes of expression. The software generates a profile of each student and using this data, a highly sophisticated search engine matches internet resources to specific student strengths. The SEM is based on a learning theory called the Enrichment Triad Model, which was developed in 1977 [92]. The Enrichment Triad Model was designed to encourage advanced level learning and creative productivity [92] by

- exposing students to various areas of interest, and fields of study in which they want to become interested in exploring or developing,
- providing students with the skills and resources needed to gain advanced-level content and thinking skills, and
- creating opportunities for students to apply their skills to problems and interests that they want to follow.

The Renzulli System is an example of an enrichment tool, but it only caters for high school students. Limited enrichment resources are available for university and college students. My interest in student enrichment is at university level, specifically in undergraduate courses, where I feel there is a need to stimulate academically strong students in these courses, and I would like to draw attention to enrichment at university level in this study.

6.5 Student enrichment and inquiry-based learning

Inquiry-based learning or enquiry-based learning or problem-based learning is primarily a pedagogical method, developed during the discovery learning movement of the 1960s as an answer to the traditional methods of instruction. The philosophy of inquiry-based learning finds its roots in constructivist learning theories, such as the work of Bruner [14], Dewey [24], Vygotsky [112], and Freire [40] among others.

> *Tell me and I forget, show me and I remember, involve me and I understand.* (John Gay, English poet)

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This adage captures the philosophy of inquiry-based learning. Inquiry-based teaching is an approach to instruction that begins with exploring curriculum content rather than teaching the content. It provides a framework for students to experiment, ask their own questions and be active participants in the learning process. This is especially applicable to science subjects such as mathematics and physics, where it is easy to create scenarios for students to investigate and create their own questions for exploration.

When designing learning activities, it can be useful to consider the information processing approach to learning by Anderson [1]. Anderson highlights three principles that play a big role in acquiring new information: Activation of prior knowledge, encoding specificity and elaboration of knowledge. Education should help students in activating relevant prior knowledge, creating context that resembles the future contexts as closely as possible and to stimulate students to elaborate on their knowledge.

According to Barrows and Kelson [7] some of the goals of inquiry-based learning are that it is designed to help students to

- construct an extensive and flexible knowledge base;
- develop effective problem solving skills;
- develop self-directed, lifelong learning skills;
- become effective collaborators and;
- become intrinsically motivated to learn.

The power of an inquiry-based approach to teaching and learning lies in its potential to increase intellectual engagement and nurture deep understanding. Inquiry honours the multifaceted, interconnected nature of knowledge construction and creates an opportunity for both teacher and students to collaboratively build, test and reflect on their learning [102].

These goals are similar to the goals of enrichment activities listed by Freeman ([38], [39]). Freeman states that enrichment activities 'often lack clear goals' and he identified three important goals for implementing enrichment:

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- Increased analytical and problem solving skills.
- Development of profound, durable and worthwhile interests.
- Stimulation of originality, initiative and self-direction.

These ideas are echoed by Gibson and Efinger ([44], p. 53) who state that a good enrichment program should provide "all students the opportunity to develop higher order thinking skills, pursue more rigorous content, and engage in first-hand investigation". These goals and benefits are very much aligned and show that inquiry-based learning is a fruitful approach to take to create enrichment programmes, especially in the science field.

6.6 Student enrichment in mathematics

In this section we focus on student enrichment in mathematics specifically. We provide examples of student enrichment in mathematics and the possible benefits of implementing student enrichment in mathematics.

6.6.1 Stimulating interest in mathematics

One of the reasons for encouraging enrichment is to keep students engaged and interested in mathematics. Enrichment can also highlight the links between mathematics and other areas of study such as history, art or science that can put mathematics in context for students.

An online example of mathematics enrichment is PLUS [85]. PLUS is an internet magazine for students and teachers. It focuses on the engagement of pupils in mathematics and the promotion and popularisation of mathematics as central roles to enrichment. PLUS claims that disinterested or underachieving students are also in need of enrichment as are the bright students. It contains resources to introduce readers to the beauty and practical applications of mathematics. It contains articles, podcasts and other useful resources.

Some countries have reported that there is a decline in student's positive attitude towards Mathematics ([42], [99]). This apathy and disinterest are resulting in fewer students deciding to major in mathematics ([68], [76], [64], [63]). Jones and Simons [59] pointed out that in the UK these concerns have led to enrichment activities for these students to stimulate their interest in mathematics and who then could possibly study mathematics after completing school.

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Feng [34] further explains that there are other benefits from mathematics enrichment. Feng looked at four different case studies: a Mathematics Summer, a series of Mathematics Master classes, an after-school outreach and enrichment programme and the UK Mathematics Trust's Maths Challenge participation. Feng reports that students benefited in multiple ways from these enrichment projects. Some reported that their mathematical knowledge broadened and they understood mathematics better. Some students report personal and social development whereas others were given exposure to higher education and were inspired to study mathematics further after leaving school.

Jones and Simons [59] note that this newly gained appreciation of mathematics will raise the profile of mathematics and will demonstrate to students that mathematics is worthwhile to pursue beyond school. This was also noted in a Dutch enrichment programme where this positive effect was carried over to university, where the participants choose to study more challenging fields on average [10].

6.6.2 Problem solving

Feng [33] explains that if teachers use enrichment to present their students with a stimulating experience of Mathematics it becomes a means of fostering mathematical thinking and problem solving. So by using enrichment in mathematics classes we meet the educational needs of the interested students. This can also be a way to correct some of the shortfalls in the curriculum and to encourage investigatory activities ([34], p. 4).

Another provider of Mathematics enrichment is NRICH [79], a mathematics project that was started in 1996 by the University of Cambridge and that has problem solving as one of its main focuses. Originally it was started to support capable, bright young students in Mathematics from communities that have little mathematical support. It has now developed into an online community where students and teachers across the planet can obtain resources for their classrooms. NRICH offers students challenging and engaging activities to develop their mathematical thinking and problem solving skills. These NRICH activities will enrich their experience of mathematics and they will get an opportunity to see mathematics in more meaningful contexts than those typically delivered at school. Below is a quote from their website.

> *The NRICH Project aims to enrich the mathematical experiences of all learners. To support this aim, members of the NRICH team work in a wide range of capacities, including providing professional development for teachers wishing to embed rich mathematical tasks into everyday classroom practice.* [79]

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According to Piggott ([82], [83]), mathematics enrichment serves to support problem solving approaches, to improve students' attitude towards mathematics, to increase an appreciation for mathematics as a discipline and to develop the conceptual structures that support mathematical understanding and thinking. Enrichment is therefore a flexible approach to teaching mathematics that encourages experimentation and communication.

Piggott ([82], [83]) discusses the position of enrichment versus mathematical thinking versus problem solving. These concepts are intractably interwoven. Problem solving is the process of solving problems or learning how to solve problems. Mathematical thinking is more involved as it could include generalizing, decomposing, modelling, visualising or looking at other problems that may have similar features. Therefore mathematical thinking is a deeper way of exploring mathematics. Mathematics enrichment according to Piggott ([82], [83]) has to capitalise on mathematical thinking and problem solving to benefit the students and enrich the subject.

George Pólya was a prolific Hungarian mathematician who made important contributions to combinatorics, number theory, analysis and statistics [9]. He also had a deep interest in mathematics education. In his book *How to Solve it* [86], Pólya focussed on the culture of problem solving and how it can be nurtured among students. This book has been translated into many languages. Pólya uses the following steps when faced with a mathematical problem:

(1) First, understand the problem; ask yourself what is the unknown, what has to be proved?

(2) Make a plan.

(3) Carry out the plan.

(4) Look back on your work. Ask yourself: How could it be better?

Pólya advises "If you can't solve a problem, then there is an easier problem you can solve. Find it." ([86], p. 114) Or "If you cannot solve the proposed problem, try to first solve some related problem." The process of problem solving as such is then a process of enrichment in the sense that it requires the student to investigate beyond the set problem.

Next, I provide two examples from literature that demonstrates ways to create problems or scenarios that will require problem solving skills. These problems require the student to apply the steps outlined by Pólya. One is an example of a non-routine problem and the other is an example of an open-ended problem.

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Non-routine problem. Piggott [84] discusses one of the NRICH problems that was posted on their website for students around the age of twelve. She looked at a problem that can be considered challenging for the typical twelve year old student. It was a non-standard problem that encourages their mathematical thinking and mathematical knowledge as it combines different concepts into an unfamiliar situation.

The problem was to find two primes of which the sum is a perfect square. First it was noted that 2+2=4, 2+7=9 and 5+11=16. Students were then tasked to try out all the squares of the numbers between 4 and 20 themselves. Then students were asked if they found any square numbers which cannot be made by adding two prime numbers.

In this problem students had to combine isolated ideas of prime numbers, square numbers and arithmetic to solve an unusual problem. This non-standard problem immediately requires students to not only have the required mathematical knowledge but also to be able to "think around the problem". Once they have solved the problem they have to be able to communicate their results to the teacher or fellow students.

Piggott suggests that non-standard problems are important to challenge students' preconceptions by supplying them with unfamiliar situations in which standard procedures may not necessarily work. Instead they are now forced to think about what they know mathematically wise and apply this to this new problem. This process allows them to develop their ability to make mathematical sense of the problem and to increase their knowledge and understanding of the related mathematical ideas, techniques and strategies needed to solve the problem.

Open-ended problem. Yee [120] wrote that teachers can facilitate mathematical thinking and understanding of mathematical concepts by using open-ended problems. Teachers typically use routine mathematical questions to test students' algorithmic skills and arithmetic skills. This traditional approach creates the illusion that mathematics is a one-step, two-step or many-step procedure to solve problems, which it is not.

Open-ended problems are sometimes called "ill-structured problems" as they are not clearly formulated with all the information. The problem solver has to fill in the missing data or make assumptions to solve the problem. There is no fixed method to solve these problems and these problems often reflect the real word situation better. Examples include determining the number of television sets owned in South Africa or determining the volume of water usage of a city in a year.

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An example of a closed question that typically appears as a word problem for primary school students is:

> *A polar bear weighs about 20 times as heavy as Ali. If Ali weighs 25kg, what is the mass of the polar bear?* ([120], p. 22)

Most students would see the key phrase "20 times as heavy" as an instruction to multiply This problem clearly has one answer and the calculation is one multiplication problem. Yee then transforms this well-structured problem to the following short open-ended situation:

> *A polar bear weighs 500 kg. How many children do you need to have the same mass?*

Suddenly the calculation process is not so clear. There is no key phrase that gives the solution away and to solve this problem requires that students fill in the missing data. Students are now forced to devise a plan to solve and to make a contribution to this problem by making assumptions on the weight of children. This leads to many possible correct answers that typically occur with open-ended problems.

Turning standard problems into short open-ended problems creates an enriched mathematical experience for students. It showcases the importance of mathematics that allows teachers to create more realistic situations for which mathematics is needed. Using open-ended problems also allows teachers to tap into the students' thinking as opposed to the teacher's own thinking through closed questions that have predetermined methods and answers.

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6.7 Positioning this study: What, who, how.

What? In this enrichment case study, I will adopt the definition of enrichment from Clendening and Davies [18] that defines enrichment as:

> *Any learning experience that replaces, supplements, or extends instruction beyond the restrictive boundaries of course content, textbook and classroom and that includes depth of understanding, breadth of understanding and relevance to the student and to the world in which he or she lives.*

This definition sufficiently shows that an enrichment programme should create positive experiences outside the classroom with the goal of deeper learning and having value for the participants in the study.

Who? The enrichment study is aimed at mathematics students at university level. A group of first year academically strong students was invited to participate in this enrichment study to capture their experiences by exposing them to enhanced undergraduate content.

How? The goal of the enrichment programme is to open the world of complex numbers using technology and sibling curves by means of inquiry-based learning activities. Using principles of inquiry-based learning, the students will get the opportunity to broaden their knowledge and sharpen their problem solving skills.

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Chapter 7

Research design and methodology

7.1 Introduction

The research design is presented to showcase the implementation of an enrichment programme and to capture student experiences regarding the programme. When referring to "the study" I refer to the study on student enrichment from Chapter 7 onwards.

Section 7.2 exposes the research design of this study. Section 7.3 discusses the first phase of the study, which involves designing and implementing mathematical activities, my personal journal of all activities and creating a Facebook page for communication. Section 7.4 discusses the second phase of the study, which involves conducting a survey, presenting a tutorial test and interviewing a select group of students that participated in this study. Lastly, in Section 7.5 issues regarding ethics are discussed.

7.2 Research design

This study draws on students' perspectives of student enrichment for undergraduate mathematics students. The research methodology followed is that of evaluation research ([87], [109]) with elements of action research ([71], [88]) embedded within. The student enrichment project went through a planning phase leading to implementation and was followed by an evaluation phase, the two phases typical of evaluation research. With the implementation, action research principles were followed when I reflected, planned and issued an assignment, then observed student experiences before embarking on the next assignment, once again following the steps of a cycle of action research. The study was designed to explore and examine student experiences regarding student enrichment following the two phases of research methodology.

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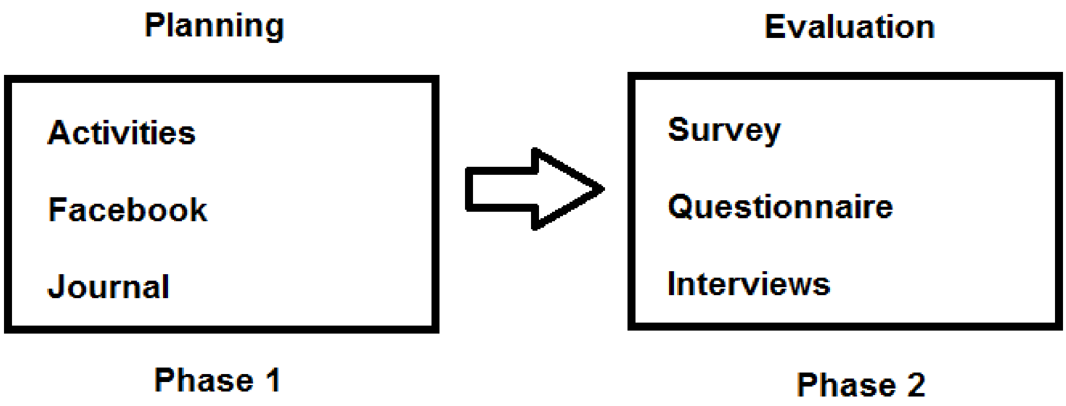

Figure 7.1: Two phases of the study

The purpose of the first phase was to create an opportunity for student enrichment. This involved designing mathematical activities that were assigned for completion by a selected group of students. They also had an opportunity to interact with each other and me on *Facebook*. To keep track of my thoughts and my interaction with participating students I kept a personal journal during Phase 1. This is explained in more detail in Section 7.3.

The purpose of the second phase was to capture students' experiences regarding the enrichment activities. A survey was conducted after the termination of the first phase to capture quantitative data. A comparison was also drawn between performance of participants in the study and the rest of the group in a tutorial test on complex numbers. See Section 7.4.1-7.4.2 for further discussion.

To conclude phase two, participants' perspectives of their experiences of the enrichment programme were captured. This was done by obtaining qualitative data through short interviews with a select group of students who participated in this study. See Section 7.4.3-7.4.5 for further discussion.

7.3 Phase 1: Planning phase

The purpose of Phase 1 was to design activities that would form the backbone of the enrichment programme. In Phase 1, I created an environment where students were exposed to enrichment. They had the opportunity to do mathematics outside the classroom and curriculum, aimed at benefiting mathematically from it.

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7.3.1 Selection and recruitment

At the University of Pretoria the mainstream first year Calculus 1A course (WTW114) has an enrolment figure of approximately 1000. This course is presented in the first semester of the year and is a prerequisite for the two follow-up modules Calculus 1B course (WTW128) and Linear Algebra 1 course (WTW126). The latter two courses are presented in the second semester. The topic of complex numbers is presented in the Linear Algebra 1 course, eight weeks into the second semester. From the Calculus 1A course a group of 30 students were selected to take part in the enrichment programme. These students were selected eight weeks into the semester on the grounds of performance in the first of two semester tests.

The criteria for participating in the project were that a student should have achieved 70% or more in both the semester tests and that some diversity in the study fields of the students should be evident. When two students showed the same performance, their study fields were considered for compiling a diverse participant group. All of these students were planning to enrol for the follow up Calculus 1B course and Linear Algebra 1 course.

Participation in this project was voluntary. Students were informed that this would be an enrichment programme involving the topic of complex numbers and that they would benefit mathematically from it. They were also told that the project would be time consuming and that it would involve extra work outside their normal university course load.

Students were invited via e-mail and were requested to meet me in my office for further information and participation. Out of the 30 that were invited 23 students chose to participate in this project. A few students never replied to the e-mail and others replied saying that they were too busy with university work and other activities already.

7.3.2 Mathematical activities

Using the principle that “the brain learns best when it does, rather than when it absorbs.” ([106] p. 54), the first phase of the enrichment programme was designed to have several activities for the participants to learn mathematics in an inquiry-based learning environment.

Five worksheets (see Appendixes A1, A2, A3, A4 and A5) were designed for completion by the participating students. The students’ participation in Phase 1 consisted of the five worksheets to be completed, labelled activities. The activities had no hard deadline, but students were

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encouraged to complete them as soon as possible as the main goal was to complete all five activities before the topic of complex numbers was taught in the next semester.

Once a student decided that s/he wanted to be part of the study, s/he was given a hard copy of Activity 1 to complete and to return before receiving the next activity. The timeframe for Activity 1 was from week 9 of the first semester until the end of the first semester. The timeframe for Activity 2 was during the break between the first and the second semester. The timeframe for Activity 3 was during week 1 and week 2 of the second semester. The timeframe for Activity 4 was during week 3, week 4 and week 5 of the second semester. The timeframe for Activity 5 was during week 6, week 7 and week 8 of the second semester, represented in Table 7.3.

Work	Activity 1				Activity 2	Activity 3		Activity 4			Activity 5		
Week	9	10	11	12		1	2	3	4	5	6	7	8
When	First Semester				Break	Second Semester							

Figure 7.3: A timeframe of the five mathematical activities

Each activity consisted of a worksheet consisting of problems for the student to investigate, explore and solve with blank spaces to fill in their final answers. Once the student had answered all the questions or did as many as he or she could, s/he had to return the completed activity to me in my office. When the student returned her/his activity, I had a brief look at the completed activity. Occasionally some relevant concepts or student concerns were discussed, after which a hard copy of the next activity was handed to the student. Thus the students dictated their own pace. However, on a few occasions I did send out e-mail reminders to caution slow students that they were taking too much time on a specific activity.

Each activity focused on a particular topic. Activity 1 was the warm-up activity. Activity 1 was meant to get the students to interact with real polynomials using technology. This activity intended students to make their own observations from experimenting with the graphs of polynomials. The students were given the opportunity to play with a *Geogebra* applet that could sketch quadratic, cubic and quartic polynomials by allowing them to change one coefficient at a time. The purpose of this activity was to get students to experiment and to relook at real polynomials and their roots.

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Activity 2 took part during the break between the first and the second semester. This activity had two parts. In part 1, students became acquainted with *Geogebra*. *Geogebra* is an interactive geometry, algebra, statistics and calculus software intended for the learning and teaching of mathematics from primary school to university level. In part 1 they had to create functions of their own choice with parameters which they could change and see the effects. In part 2 they were given notes on complex numbers which they had to read by themselves and then answer some basic questions on complex numbers, mainly involving manipulation. The purpose of this activity was to get students acquainted with technology such as *Geogebra* and to get them to do some self-reading on complex numbers before this was actually taught in class.

Activity 3 took place during weeks 1 and 2 of the second semester. This activity was based solely on complex numbers. In this activity the concept of polynomials with real or complex roots was explored. Some of the questions in Activity 3 also hinted at the idea of sibling curves. The purpose of this activity was to get students thinking about roots while subtly introducing the idea of sibling curves.

Activity 4 took place during weeks 3, 4 and 5 of the second semester. In Activity 4 the concept of sibling curves was exposed and students had to find sibling curves of two specific polynomial examples by themselves. In Activity 4 they also had to sketch the sibling curves without the use of technology. The purpose of this activity was to expose students properly for the first time to sibling curves.

Activity 5 took place during weeks 6, 7 and 8 of the second semester. In Activity 5 students came to explore sibling curves using technology. They were provided with two *Geogebra* routines. One routine enabled them to sketch sibling curves for any real quadratic function and using the other routine, they could sketch sibling curves for any complex quadratic function. The purpose of this activity was to get students to experiment with sibling curves and to see the connection between roots and sibling curves of a polynomial and to make their own observations.

7.3.3 Facebook

I created a *Facebook* group to connect all the participants in the study. The purpose of the *Facebook* group was to have a platform for general feedback and to create a network for all the participants through which they could communicate, allowing members to share ideas or to direct questions to me or to fellow students, especially over weekends or when I was not in my office.

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7.3.4 Journal

To assist with documentation during Phase 1 I decided to keep a personal journal. The purpose of the journal was to keep track of my own thoughts and of discussions I had had with the participants in the study. Keeping the journal helped me with reflection and observation at the end of Phase 1. These journal remarks also guided me in creating questions to be asked in Phase 2 of this study.

In this journal, I kept track of each student's visits and some of the issues that were discussed. As each visit was informal and none of the conversations was recorded, the journal entries served as documentation of the visits. Some students made appointments to visit me and others visited me during consultation hours or sporadically visited hoping to find me in my office.

7.4 Phase 2: Evaluation phase

In the second phase of the study there was an opportunity to capture students' perspectives on the enrichment project. This was done in two ways, firstly by means of a survey and secondly by comparing performance in a tutorial test on complex numbers between participants and the rest of the Linear Algebra 1 group. The first phase ended shortly before the participants were going to be taught complex numbers in the eighth week of the second semester. This meant that at the time some participants had not completed all five activities. This phase occurred towards the end of the second semester and close to the examination period, so the intended evaluation phases had to be less time consuming for students. Neither the survey nor comparison of marks required much time from the students.

7.4.1 Survey questionnaire

The objective of the survey was to ascertain students' impressions regarding student enrichment. The questionnaire for the survey was constructed after the completion of Phase 1. It consisted of three types of questions: Likert scale questions, questions with two options and open-ended questions. Below is a small sample of the questionnaire. A complete version of the questionnaire can be found in Appendix A6. Question 3 is an example of a Likert scale question. Question 11 is an example of a question with two options. Question 15 is an example of an open-ended question.

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3. I enjoy solving new problems.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

11. Would you do an enrichment project in holiday time to prepare for 2nd year Mathematics?

Yes	No

15. What did you learn in this project?

Figure 7.4: A sample from the survey

The questions in the questionnaire were to probe student's experiences regarding the study. For example in Question 3 I wanted to investigate how students felt about solving new problems. In Question 11 I wanted to investigate students' willingness to participate in future enrichment projects. In Question 15 I wanted to determine what students feel they have learned from doing this project.

The questionnaire was e-mailed to all 22 participants. The students had the option of collecting a hard copy of the survey from my office if they wanted to complete it in writing. Participants were instructed to complete it either online and e-mail it back or print it out, complete it and return it to my office. Nineteen of the participants completed the survey. The full survey can be found in Appendix A6.

7.4.2 Tutorial test marks comparison

During the 11th week of the Linear Algebra 1 course students wrote a tutorial test on complex numbers. Performance in this tutorial test was analysed quantitatively to investigate the

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impact of the acquired enrichment knowledge and skills. The goal here was to see how the technical skills of the participants in the project compared with those of the rest of class by looking at the results of the tutorial test on complex numbers.

The scope of the tutorial test was on complex numbers, including De Moivre's theorem. See Appendix A7 for a version of the tutorial test that the students wrote. The test was slightly modified for the other days, so that students on different days did not write identical versions of the test. The duration of this tutorial test was 30 minutes.

7.5 Interviews

After completion of Phase 2 - the evaluation of the research study, I conducted individual interviews with some of the participants. The purpose of the interviews was to probe students' experiences and attitudes towards the mathematics enrichment programme qualitatively. The objective with the interviews was to focus on obtaining student responses on experiences and how the study influenced their thinking and mathematical understanding.

The fact that I was conducting the interviews may have had an observer or Hawthorne effect [20], although I tried my best to keep the students comfortable and relaxed during the interviews.

7.5.1 Format of the interviews

From the group of students that participated in this study, four were selected for interviews. For the sake of diversity, the selection was based on the survey results, gender and on the degrees they were studying. Students were interviewed individually over a period of three weeks during November and December 2014, after they had completed their examinations. Each interview took place in my office and was tape recorded. The duration of each interview was approximately 30 minutes. The interviews were semi-structured and consisted of a set of benchmark questions that each participant was requested to answer. The interviewer had the freedom to ask more questions and/or probe participants on the responses or clarity of answers. The interviewer could also request a participant to expand on a specific topic that was relevant to the study and that a student brought up during the interview.

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7.5.2 Interview questions

The interviews were semi-structured. A set of benchmark questions was compiled. These questions can be found in Appendix A8. The benchmark questions were established after the completion of Phase 2 of the study. At the start of the interview I reminded each student of the purpose of the study and that I wanted to probe their experiences and attitudes towards the enrichment programme in which they participated. Students were also reminded that the interviews would be reported on anonymously and that they were requested to answer as honestly as possible. I did my best to put the students at ease. After asking these questions and some follow up questions, I concluded the interview by asking if the student had anything else they wanted to add, or any remarks.

7.6 Ethics

In this study data were collected from human participants. In this section, I discuss the ethics or protocols that were taken in the study. The researcher of the study was responsible for the ethical standards of the study. I took the following steps:

- Permission to conduct research on the first year Mathematics students was sought and granted by the Ethics Committee of the Natural and Agricultural Science Faculty of the University of Pretoria. Reference number EC141029-094 (see Appendix A9).
- Participants in the study were informed that participation in this study would be voluntary.
- I assured all the participants that all data collected from the survey would be reported on as anonymous.
- In the interviews all interviewees were assured of confidentiality. Permission was obtained from each candidate to tape-record the interviews. Candidates were also assured of the anonymity of their responses.
- The research report would be made available to the University of Pretoria.
- In the data analysis student names and student numbers were not used. Thus, confidentiality was ensured by making certain that the data could not be linked to individual subjects by name.
- As researcher I will make every effort to communicate the results of my study in such a way that misunderstanding and misuse of the research is minimised.

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- The methodology section of my study shows how the data were collected in sufficient detail that will allow other researchers to extend this study if they desire to.
- In my roles as co-ordinator, lecturer and researcher, I was aware of my ethical responsibilities that accompanied the gathering and reporting of the data. The methods of my research were described to all participants in this research study.

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Chapter 8

Research findings

8.1 Introduction

This chapter presents the research findings of the study. Here I report on the findings from the two phases, implementation and evaluation, of the enrichment part of the study. In particular I discuss the results from the activities assigned to the participants, *Facebook* participation, tutorial test results, survey questionnaire responses and the interviews conducted.

8.2 My perspectives and experiences

In this section I share some of my personal perspectives and experiences while conducting this study. In Section 8.2.1 I focus on my experiences during Phase 1. In Section 8.2.2 I focus on my experiences during Phase 2 and in Section 8.2.3 I will focus on my experiences in the period following Phase 2.

8.2.1 Phase 1 experiences

Phase 1 was the most time consuming but also the most rewarding part of the project for me. In this phase I had to devise a strategy for student enrichment for a selected group of students that involved a substantial degree of planning. This included student selection, designing activities and consultation.

The purpose of the project was to expose academically strong students to sibling curves in a well-structured and educationally sound manner. Keeping this in mind I had to carefully select

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possible candidates for this study. As a course-coordinator of the first year mathematics main stream, I had access to their semester test marks and could easily identify a small group of academically strong students that might be interested in taking part in this project. I was a little nervous that these students might not choose to take part in this project, because it was voluntary and required extra work from the students which they would not get credited for at university. However I was quite excited that out of the 30 invited participants, 22 students had decided to take part in this study (see Appendix A10).

As the five activities constituted the backbone of Phase 1, I had to set time aside to design the activities. I wanted to expose these students to "new mathematics" and to sharpen their problem solving skills using the content of complex numbers which they would benefit from academically. Each activity had a purpose and the main goal was to end up having two activities in which students get to explore the world of complex numbers through sibling curves. It was an enjoyable task setting the five activities, incorporating *Geogebra* routines that I coded.

Once a student had completed an activity s/he had to submit that activity and would then be handed a new one. I took this opportunity to interact with the students and superficially browsed their answers. Occasionally the students had quick questions which I could answer on the spot. Conversations were informative. During Activity 1, student M said, for example, "It is quite exciting to see Geogebra drawing maths". Student P remarked that she wished her high school teacher had this software to sketch polynomials.

It sometimes happened that when entering my office, I found that some students had slipped their activities under my door. It was convenient for me, because I could study their answers, before students visited me for the next activity. This enabled me to have a sensible discussion with students. The process of returning an activity and receiving a new activity was a smooth and satisfying process for both the students and I.

Students were free to consult me during my official consultation times or alternatively make an appointment to visit me in my office to discuss issues related to the project. Handling this group of 22 students was quite demanding sometimes. After the first two activities, I felt it was necessary to form a *Facebook* group to give general feedback and allowing students to ask questions which possibly other members from the group could benefit from and help them in completing the activities.

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The consultations and daily checking of *Facebook* were time consuming but also rewarding because it was a small group and I came to notice the enjoyment students had from taking part in this project. Students visited me frequently during the last two activities, which were on sibling curves, the harder part of the project. I personally enjoyed this interaction and found it satisfying to recognise students for making interesting observations or solving unseen questions.

Dealing with these students added a new dimension to my teaching that most lecturing staff rarely experience. Leading students in a mathematical discovery process that they voluntarily partake in is ultimately rewarding.

8.2.2 Phase 2 experiences

Phase 2 was less time consuming compared to Phase 1. In this phase, I implemented qualitative and quantitative methods for evaluating students' experiences regarding their involvement in the enrichment programme. As a course-coordinator I was relieved that this part of the project was less time consuming, because Phase 2 was near the end of the semester and I had important duties to attend to, for example compiling semester marks, setting exams and supplementary exams.

Conducting the survey was exciting to me as the designer of this enrichment case study. I was interested in capturing the participant's feedback from this study. I was pleased that the majority of the group, 19 of the 22 participants, had completed the survey questionnaires. These students wanted to share their experiences from the study.

It was enjoyable reading the students' comments for the open questions and how they experienced the project. It was heart-warming to read that students enjoyed being challenged and viewed mathematics in a new light.

> *I enjoy the project because it was challenging and I get to know how hard research is.* (Student O)

Another component of Phase 2 was to compare the marks of this group of selected students with the marks of the remainder of the class in a tutorial test. I was pleased to see that the group of participants performed well on routine questions on complex numbers as expected. However that was not the main aim of the study. I wanted to conduct this study to capture student experiences of enrichment.

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To complete Phase 2 I wanted to delve deeper into student's perspectives of what they felt regarding the project and their views on student enrichment. Having individual interviews with four of the participants was an appropriate qualitative method for achieving this goal. Before conducting the interviews, I again had to set time apart to plan. This included selecting students, acquiring a recording device and creating a few benchmark questions to be asked during the interview.

Conducting the interviews was another role reversal for me. In class I was their teacher, during the activities I was a collaborator/consultant, but during the interviews I felt like a reporter who had to ask the right questions and probe their experiences during the project.

I felt at ease interviewing the students, because during Phase 1 of the study I interacted with the students when they submitted their activities, which allowed me to establish a rapport with the students, which in turn made me feel at ease during the interviews. I found it fascinating listening to their responses and being alerted to their emotions. The students were openly telling me stories about what other teachers/lecturers do in their lectures.

In summary the four interviews were eye-openers for me, as I obtained a new perspective on how students think and how they experienced the project. In a sense this was an enrichment experience for myself as I had the valuable opportunity of discussing with students, mathematics removed from the "work for marks" paradigm.

8.2.3 Post Phase 2 experiences

After the completion of Phase 2 of the project, I reflected on the enrichment programme that I conducted with this small group of first year students. The survey questionnaire and interviewees gave me immediate feedback and related experiences of the participants, but I still wondered about the long term impact of this enrichment programme on the students and whether it was possible to find a less time-consuming method to challenge mathematically strong students.

In 2015 I was teaching WTW218, the Calculus 2A course at the University of Pretoria. Out of the 22 participants of the project, 18 of them were enrolled for the Calculus 2A course. I decided for the duration of the course to have a problem of the week which I posted on the course website. This was either a challenging, non-routine or interesting problem pertaining to the work done that week. The students had a week to solve the question and submit their answers to me.

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Out of a class of 450 students, I received several submissions for every problem. Not all the submissions were correct, so I decided to post the solutions to old problems of the week so that those who submitted could see where they had gone wrong or see an alternative solution. In total ten different students submitted at least one correct solution to a problem of the week. Five of these students were participants in the enrichment project and they submitted regular correct answers.

This idea of having a problem of the week was satisfying for a few reasons. It required far less planning and it could reach the whole class. Any student who was interested and wanted to solve a challenging problem could try it. As an extra bonus it was rewarding to see the growth of the problem solving skills of the participants in the project.

Even though the initial project was time-consuming I found this enrichment project on first years valuable on a personal level. It opened my eyes to the psyche of students and I think I have grown as a mathematics lecturer because of this project. This project also exposed me to the needs of mathematically strong students when teaching a diverse class. Running this project was satisfying and I would like to do another enrichment case study in future.

8.3 Facebook group

To get the participants to network and interact with each other I created a *Facebook* group. The title of the group was *Cardano*, named after the 17th century mathematician who is sometimes controversially credited with being first to have introduced complex numbers. Of the 22 students who participated in this study 18 students joined the *Facebook* group. The majority of these, 15 of the group of 22, joined the *Facebook* group as well as completed the survey, 3 joined the *Facebook* group but did not complete the survey and 4 completed the survey but did not join the *Facebook* group. Of the 18 students who joined *Facebook*, 5 completed only 3 activities, 2 completed only 4 activities and 11 completed all 5 activities, again the majority.

Although most of the posts were viewed, only 5 students posted comments during the study on the *Facebook* page. Nine of the posts related to the study and two posts were of a comical nature. During participation in Activity 4, the first activity on sibling curves when students were first confronted with the concept of a sibling curve, a sense of insecurity was picked up and one student even asked to clarify the concept of sibling curves as quoted below. Quotes in this section are given exactly as they appeared on *Facebook*.

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> *can someone pls give me a definition of a sibling curve* (Student T)

Using the *Facebook* platform, I redefined the concept of sibling curves for the group members. Almost everybody saw this post (17 out of 18 students who joined the *Facebook* group saw the post). At the end of the project one student commented as follows on the *Facebook* page:

> *it was an outstanding project that requires a lot of thinking and personally i hv learn a lot.* (Student O)

Although the *Facebook* group proved to be a good idea for connecting participants in the study, it is regrettable that not everybody joined in. A few students did not have a *Facebook* profile and others were not interested in joining as they preferred to work alone or would rather visit me in my office for consultation. It was a good platform to clear up misconceptions as almost all students viewed my posts. The *Facebook* group facility was not greatly utilised by the students as only 5 students posted comments on the *Facebook* page during the study.

8.4 Survey questionnaire

For the second phase of the study students' perspectives on the student enrichment project were captured. A survey questionnaire (see Appendix A6) was an effective way to obtain feedback from the participants in this study. Out of the 22 participants in the study, 19 (86%) students completed the survey.

Section 8.4.1-8.4.12 contains the results from the Likert Scale and binary option questions. Section 8.4.13-8.4.16 contains the results of the open-ended questions from the survey. It should be noted that English was not their mother-tongue or home language for some of the students who completed the questionnaire. All the quotes in this section 8.4.13-8.4.16 are directly from the answers written by the students. Grammatical errors and spelling mistakes were not corrected.

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8.4.1 Question 1 results: Enjoyment of the project

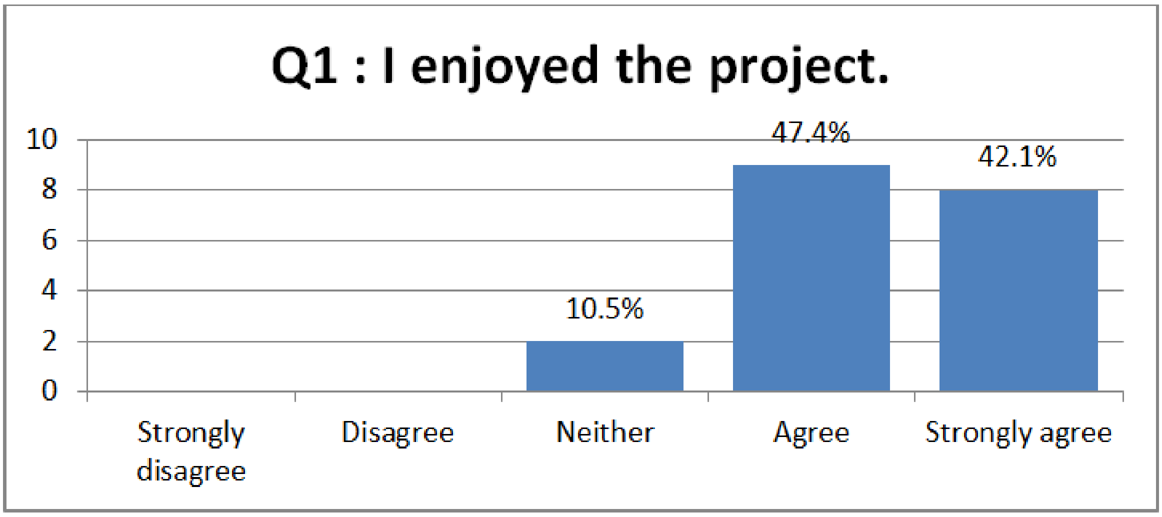

Figure 8.1: Response distribution for Q1

Responses show (Figure 8.1) that 17 (89.4%) students agreed or strongly agreed that they liked the project, compared with only 2 (10.5%) students that felt neutral about the project. These results indicate that the majority of students, by far, had appreciation for the enrichment programme and found joy in participating. No students disagreed with the statement, a meaningful result in itself. The results are testimony to the deserved place of enrichment in an already full timetable.

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8.4.2 Question 2 results: Reading on their own

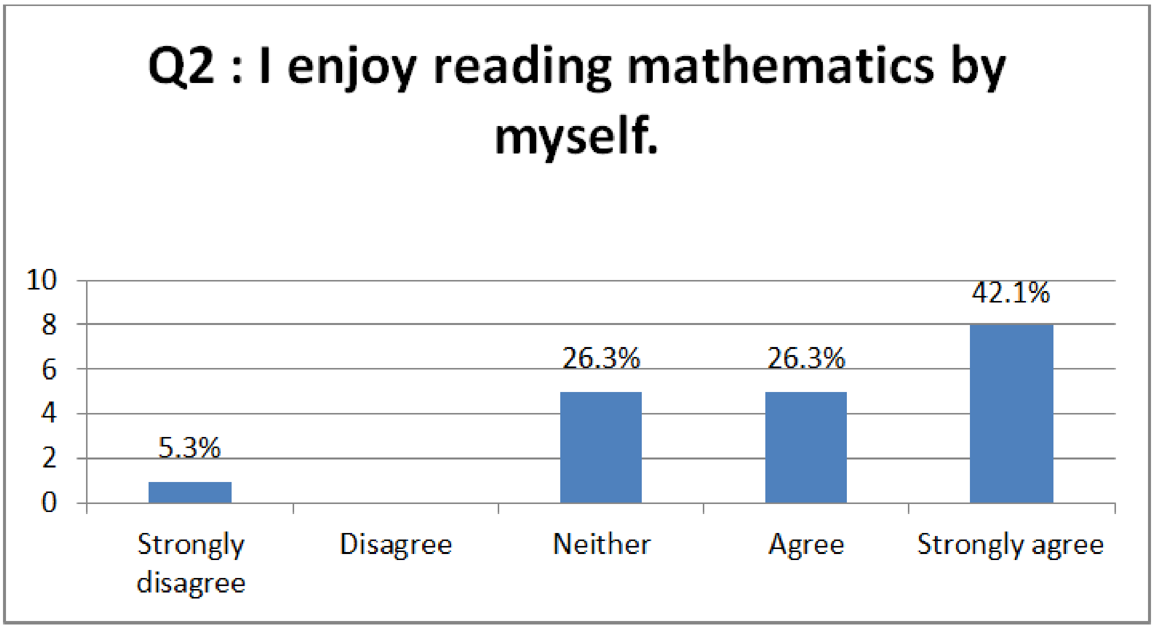

Figure 8.2: Response distribution for Q2

Responses (Figure 8.2) show that 13 students (68.4%) agreed or strongly agreed that they enjoyed reading mathematics by themselves, compared with 5 students (26.3%) that felt neutral about reading mathematics. These results indicate that students can find enjoyment from reading mathematics themselves but the group of students that strongly disagree prefer being taught.

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8.4.3 Question 3 results: Enjoyment of solving new problems

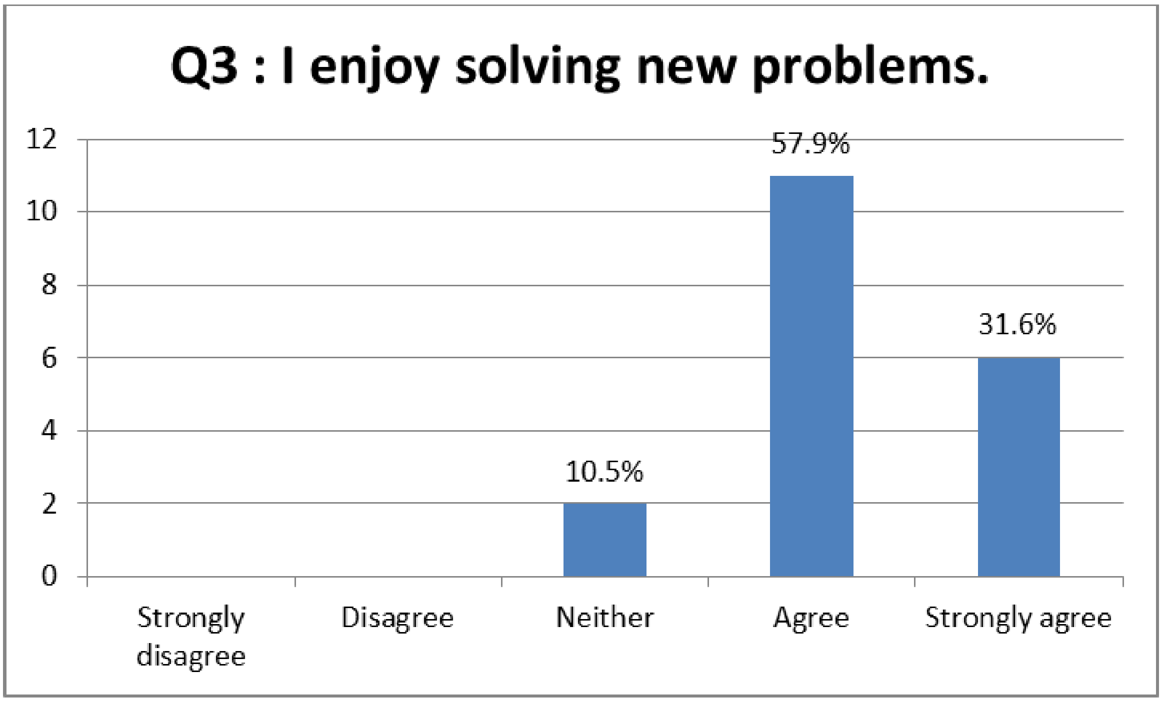

Figure 8.3: Response distribution for Q3

Responses show (Figure 8.3) that 17 students (89.5%) agreed or strongly agreed that they like solving new problems, compared with only 2 students (10.5%) who felt neutral about solving new problems. This is a strong indication that bright students wanted to see different and interesting problems and not only the typical run-of-the-mill questions. No students disagreed or strongly disagreed.

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8.4.4 Question 4 results: Coping with project and university work

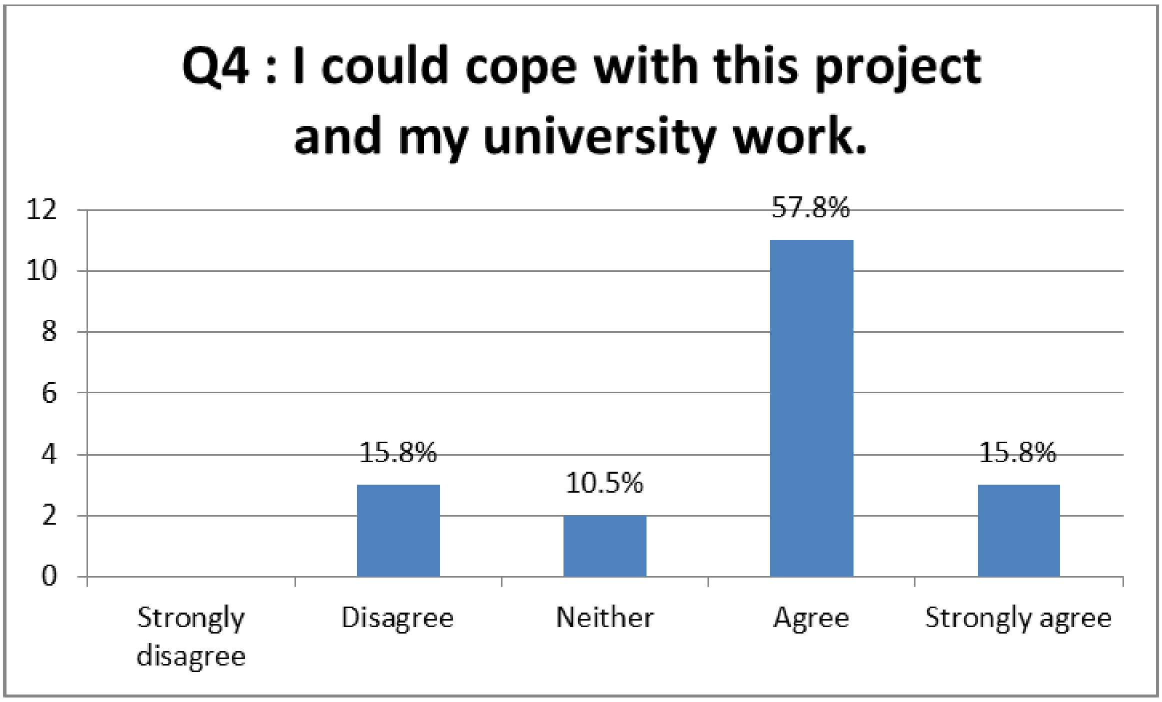

Figure 8.4: Response distribution for Q4

Responses show (Figure 8.4) that 14 students (73.6%) agreed or strongly agreed that they could cope with the project and university work simultaneously, compared with only 2 students (10.5%) who felt neutral and 3 students (15.8%) who struggled to cope with the project and their other university work simultaneously. Two of them were studying computer science and the third student was taking actuarial science. Their difficulties in coping could be because of work pressure in these courses. However the majority of the participants felt that they could handle the project whilst not neglecting their university work.

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8.4.5 Question 5 results: Working alone

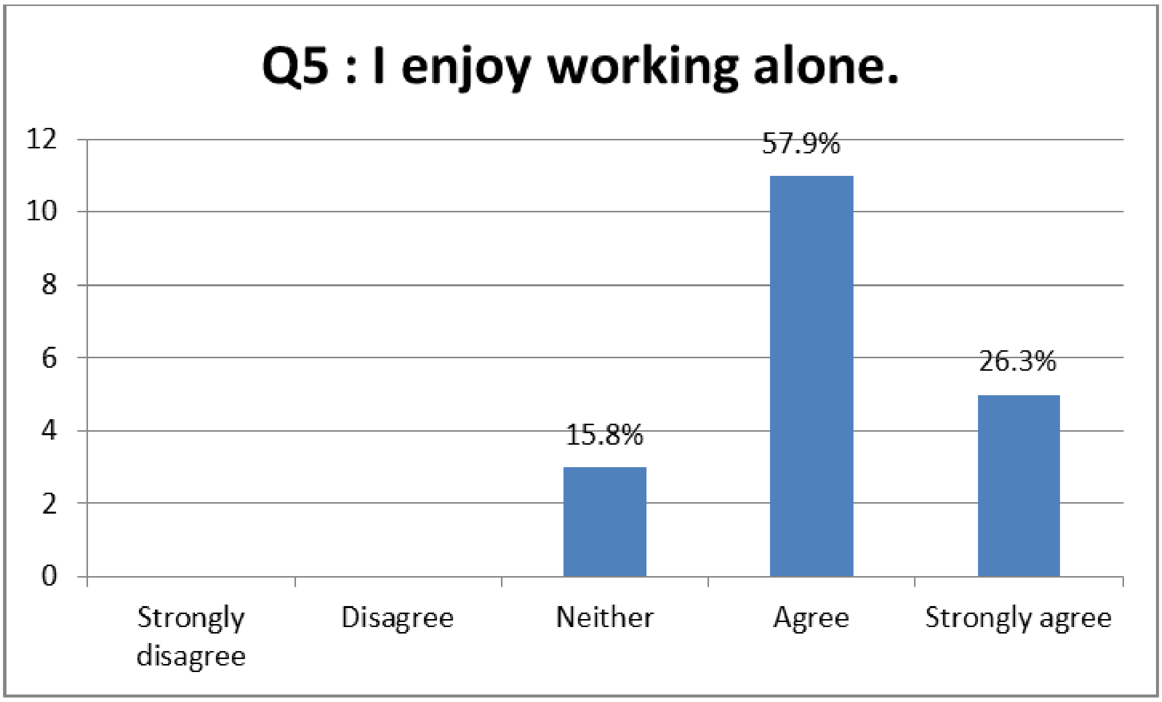

Figure 8.5: Response distribution for Q5

Responses show (Figure 8.5) that 16 students (84.2%) agreed or strongly agreed that they enjoyed working alone compared with the 3 students (15.8%) who felt neutral about working alone. These results indicate that when studying mathematics, most students prefer to solve problems by themselves and do not view mathematics as a collaborative exercise.

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8.4.6 Question 6 results: Enjoyment of technology

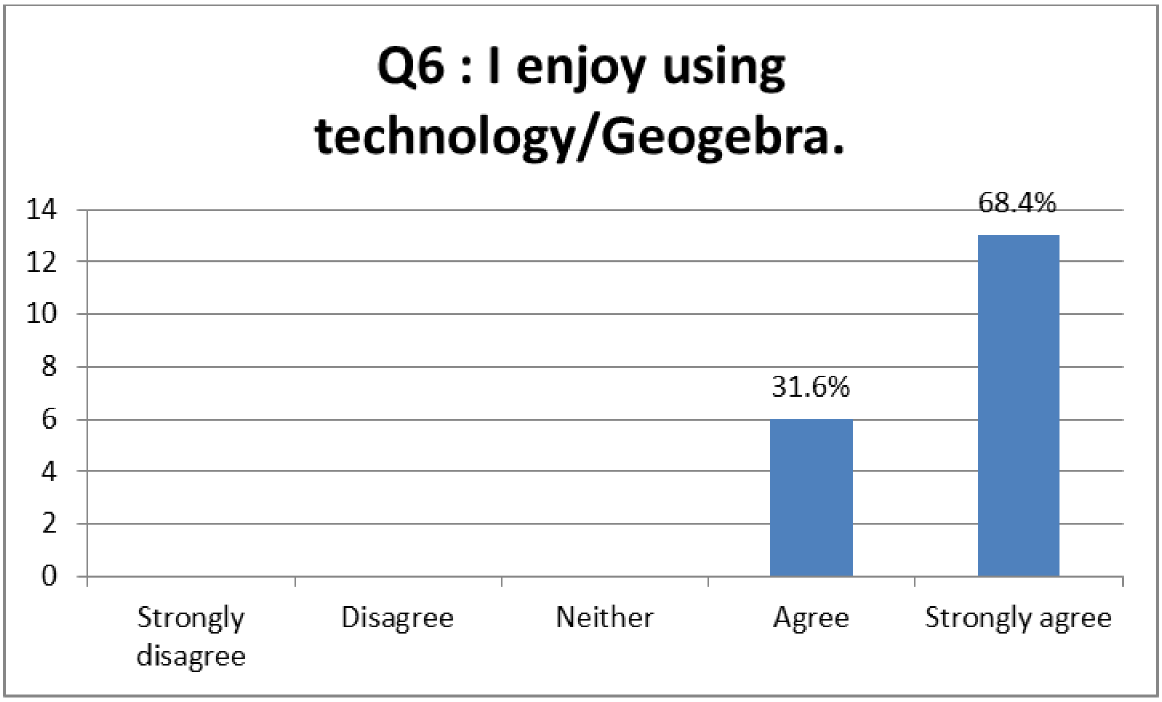

Figure 8.6: Response distribution for Q6

Responses show (Figure 8.6) that all 19 students (100%) agreed or strongly agreed that they enjoyed using technology/*Geogebra* in this project. These results could indicate that students found the project interesting because of the use of technology, amongst other factors. No students disagreed with the statement, which indicates that students were happy that technology was utilised in this project.

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8.4.7 Question 7 results: Usage of technology

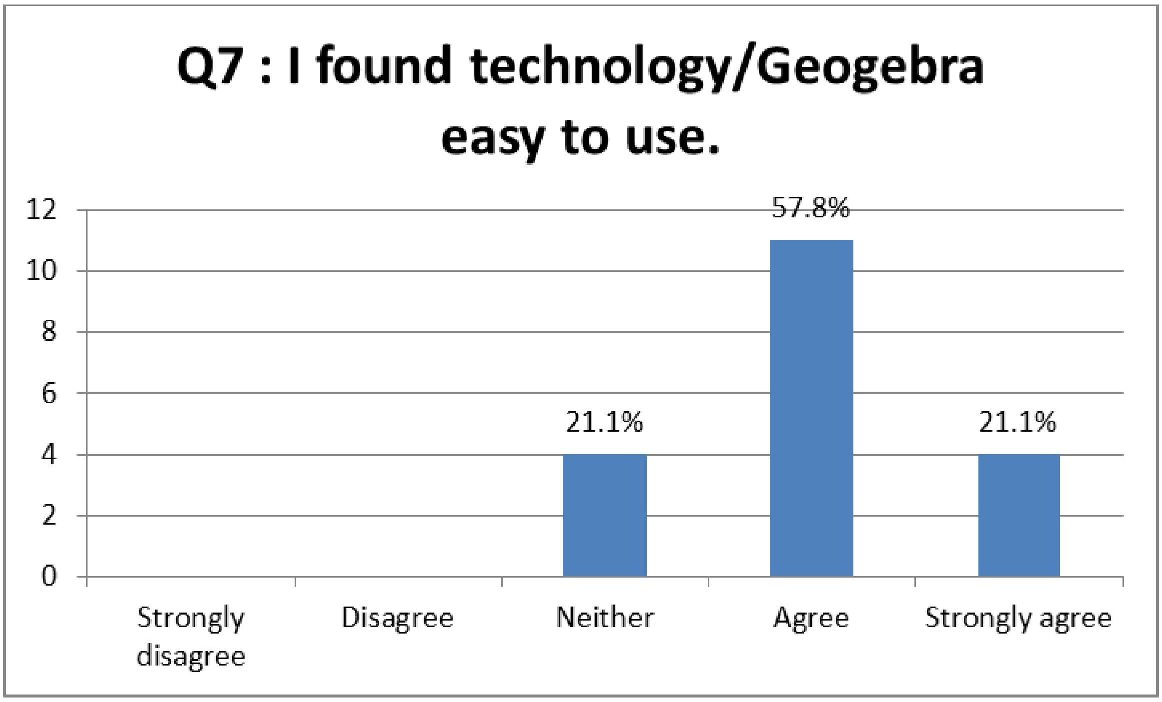

Figure 8.7: Response distribution for Q7

Responses show (Figure 8.7) that all 15 students (78.9%) agreed or strongly agreed that they found technology/Geogebra easy to use in this project, compared with the 4 students (21.0%) that did not. At the start of Activity 2, students were given a short demonstration on how to use Geogebra and these results show that most of the students could manage to use this software by themselves.

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8.4.8 Question 8 results: Recommendation of technology

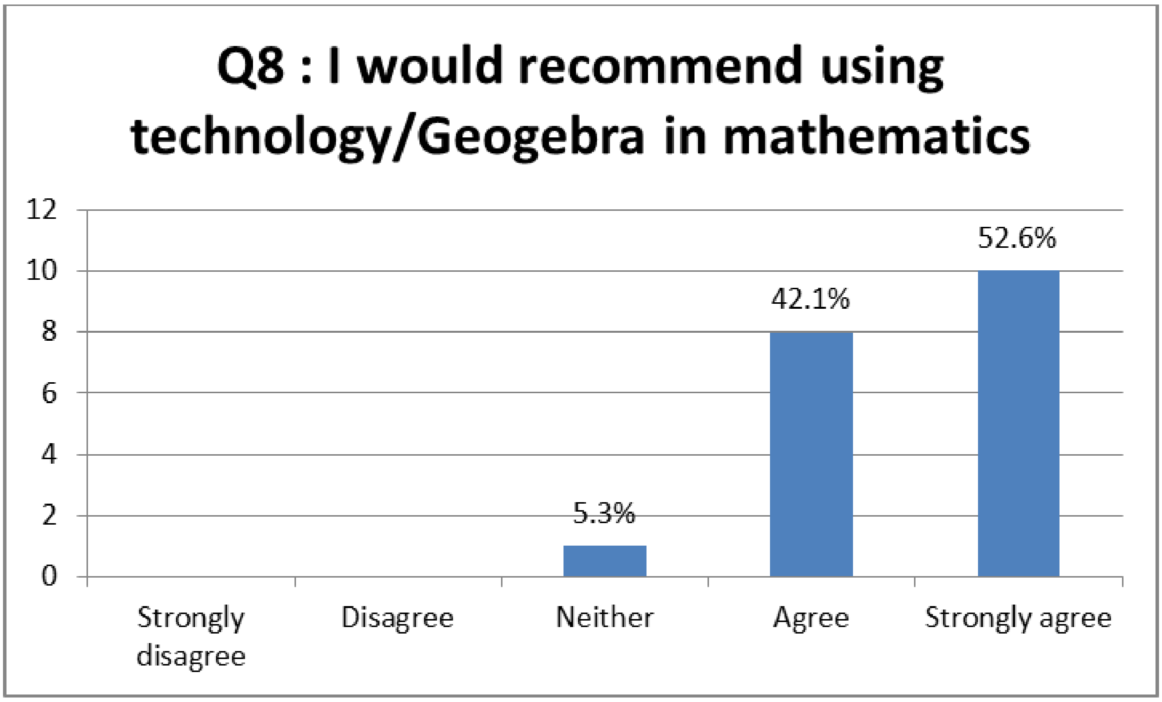

Figure 8.8: Response distribution for Q8

Responses show (Figure 8.8) that 18 students (94.7%) agreed or strongly agreed technology/Geogebra should be used in mathematics compared with the one student that felt neutral about the issue. These results indicate that the vast majority realise that the ability to sketch functions in 2D and 3D was not only useful in this project, but could be useful in the broader classroom setting.

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8.4.9 Question 9 results: Collaboration

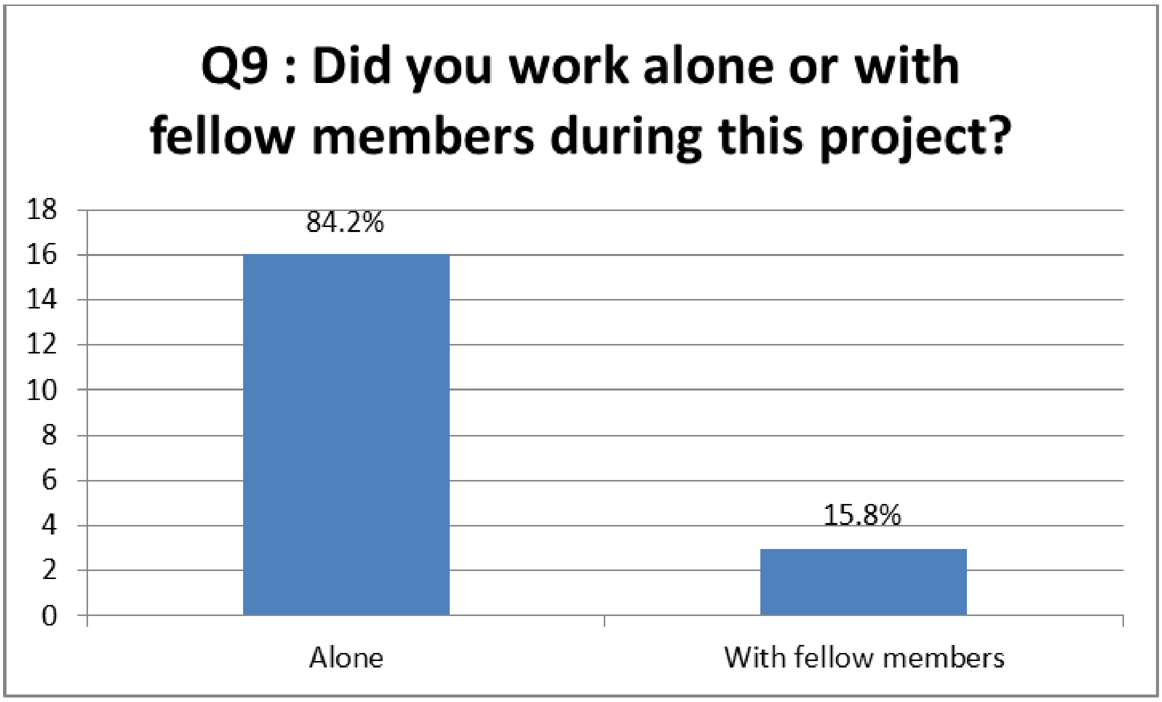

Figure 8.9: Response distribution for Q9

Responses show (Figure 8.9) that 16 students (84.2%) worked alone in this project compared with the 3 students (15.8%) who worked with fellow members of this project. These results strongly indicate that students prefer to rather work by themselves than in a group setting even if the material is new and possibly difficult.

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8.4.10 Question 10 results: Prior knowledge

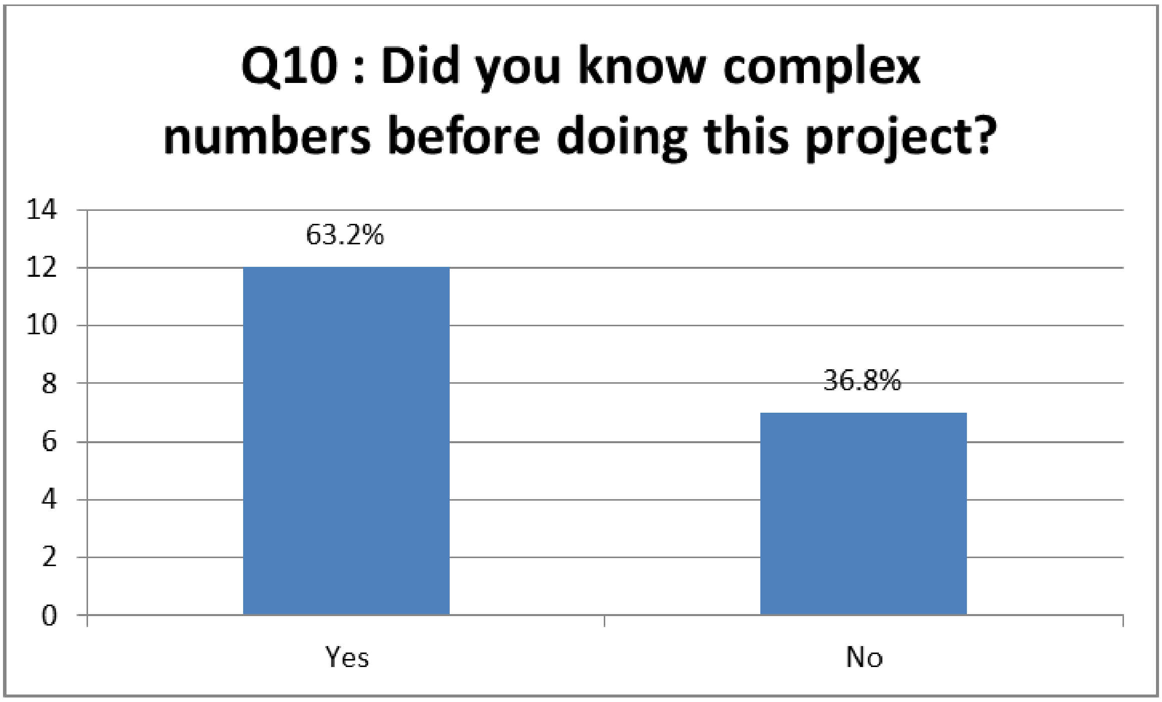

Figure 8.10: Response distribution for Q10

Responses show (Figure 8.10) that 12 students (63.1%) had seen complex numbers before the start of the project, whereas 7 students (36.8%) had not. These 12 students (63.1%) found Activity 2 and Activity 3 (the two activities introducing complex numbers) quite easy, but found Activity 4 and Activity 5 (the two activities on sibling curves) both interesting and challenging. Only 3 of the 7 students who had not seen complex numbers before doing this project completed all five activities. This indicates that while some students experienced difficulties with the project, some enjoyed the activities and persevered.

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8.4.11 Question 11 and 12 results: Interest in future enrichment projects

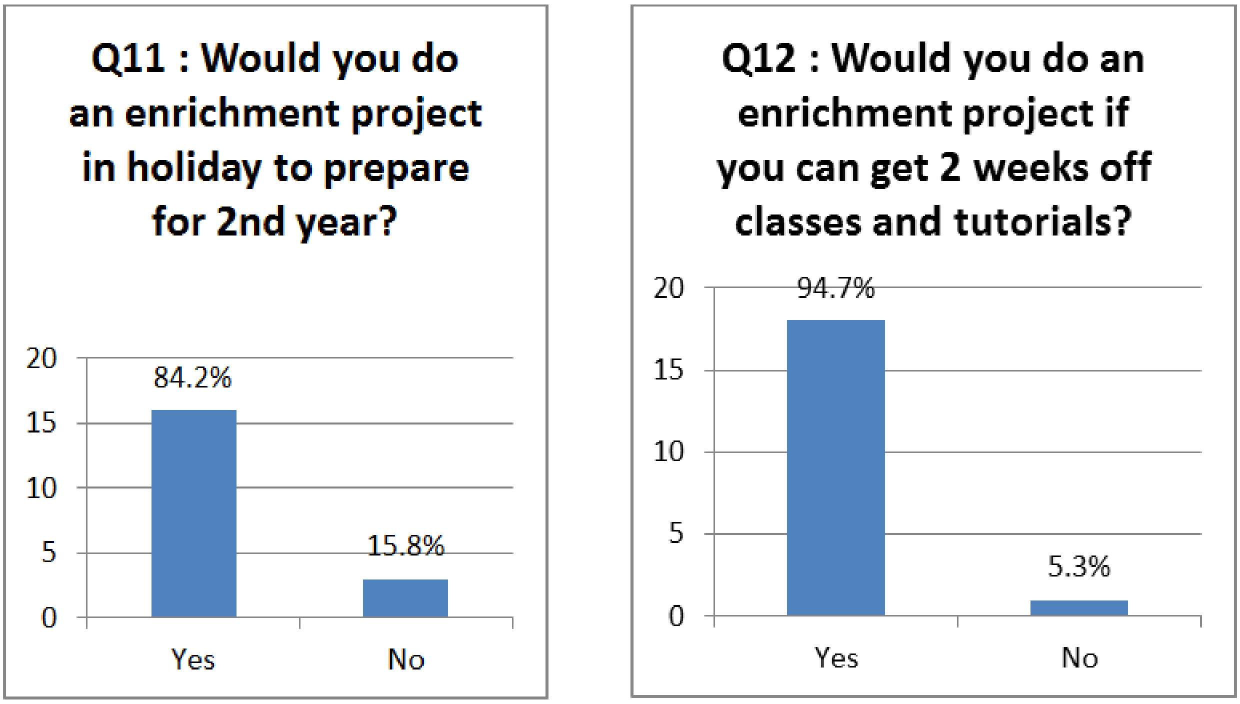

Figure 8.11: Response distribution for Q11 and Q12

Responses (see Figure 8.11) from Question 11 show that 16 students (84.2%) were willing to do an enrichment project in the holidays to prepare for 2nd year compared with the 3 students who were not keen to do that. These results are indicative that the majority of students enjoy being challenged and learning new material that would benefit their studies. Responses from Question 12 show that 18 (94.7%) students were willing to do an enrichment project instead of attending classes compared with 1 student that was not keen. These results again indicate that the majority of students enjoy learning new material in an exploring/experimenting manner, again indicating a deserved place for enrichment.

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8.4.12 Question 13: View of mathematics

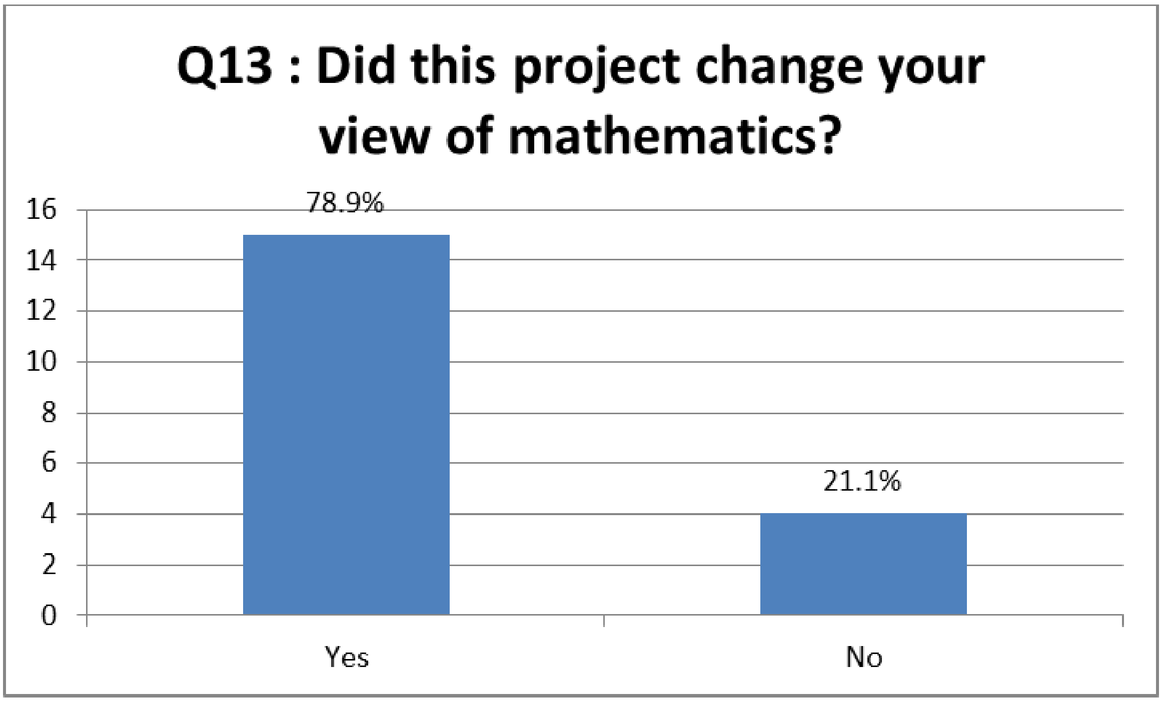

Figure 8.12: Response distribution for Q13

Responses show (Figure 8.12) that 15 students (78.9%) felt that the project had changed their view of mathematics while 4 students (21.0%) did not experience a change. In light of other finding the assumption is made that their views were changed positively. This result then bodes well for mathematics as a subject as the project seems to have favourably swayed students' predisposition towards mathematics . The results indicate that the majority's view of mathematics was changed positively and this could be due to exposure to new questions (see questionnaire Question 3) or the use of technology (see questionnaire Question 6).

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8.4.13 Question 14 results: Enjoyment of the project

Why did you enjoy the project? Or what did you not enjoy about the project?

For this open-ended question, the following noteworthy responses were given:

> *I enjoy the project because it was challenging and I get to know how hard research is.* (Student O)
>
> *Mostly because I learnt new, cool things and was able to visualise it nicely in Geogebra.* (Student G)
>
> *I enjoyed the project as it makes the complex numbers section of the Linear Algebra 1 course easier and I learnt new things.* (Student U)
>
> *The project offered short, fun activities that showed me further ideas in maths and gave me a broader perspective on what can be done in maths.* (Student B)
>
> *I enjoyed using Geogebra. It was amazing how I could draw certain graphs I never thought I could draw and seeing the graphs in 3-D was a great experience.* (Student R)
>
> *I expanded my knowledge of complex numbers, by also learning about sibling curves and other things related to this field.* (Student D)

From the comments and the results of the questionnaire, it appears that most students found the project interesting, also useful for the Linear Algebra 1 course and they learned from it, although one student M commented he did not enjoy the project as much as he had too much university work to do at the same time. This is in contrast to the responses in Question 1 of the questionnaire where student K felt neutral on whether he enjoyed the project. Student K did indicate in Question 4 that he could not cope with the project and his university work simultaneously. This student was studying actuarial science.

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8.4.14 Question 15 results: What was learnt from the project

What did you learn in this project?

Some of the interesting responses include:

> *Mathematics can be very visual. I am used to having everything on paper, with 'flat' graphs. Once you see everything in more dimensions, it almost comes alive.* (Student A)
>
> *Using the software to sketch 3D graphs was very nice and insightful.* (Student D)
>
> *How technology can be utilised in solving mathematical problems and representing solutions.* (Student F)
>
> *Besides learning about complex numbers, I learned that sitting down and struggling with a problem and eventually solving it is key in mastering a topic.* (Student L)

To this question, 11 students commented that the project improved their understanding of complex numbers. Nine students commented that it was exciting to use software to sketch curves in 2D and 3D. Six students commented that the project tapped into their problem solving skills.

From the student responses it follows that students felt that the project improved their understanding of complex numbers and problem solving skills. Students also enjoyed using Geogebra to experiment and make observations and found it an insightful experience.

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8.4.15 Question 16 results: Collaboration

Did you work alone or with fellow students during this project?

For this open-ended question, the following responses were notable:

> *I work alone because in the future I will be doing research and have to figure them out alone.* (Student O)

> *I worked alone almost exclusively, although I did talk to other members about the project and sometimes got a hint from them.* (Student T)

> *Alone. Most of the time I had to do the project over weekends at home.* (Student Q)

In response to this question, 16 students answered that they worked alone for various reasons as seen from the comments above. Some students wanted to see what they were capable of by themselves. Two students commented they used a part of both compared with only one student that commented he worked with a friend.

> *I worked alone in some activities but other activities I worked with my friend.* (Student J)

These data support the data from Question 9, whereas the same 16 students also commented why they prefer working alone. Some reasons included it is more satisfying solving themself or or it is more convenient to work alone.

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8.4.16 Question 17 results: Ideas for improvement

Any suggestions on how to improve the enrichment project or ideas to implement in an enrichment project.

To this open-ended question I had the following responses:

> *No improvements were needed. Everything in the project was self-explanatory and easy to understand.* (Student C)
>
> *None. I thoroughly enjoyed it. Maybe somehow incorporate Wolfram Alpha.* (Student Q)
>
> *No, not really but I will mention though I was not involved with it (as I do not have nor currently want a Facebook account) the Facebook group is probably a good idea – it makes one more aware of others involved with the project as well as making interaction/feedback convenient/easier.* (Student U)
>
> *The students involved should meet up as a group so that they can discuss what they are learning. You never know, it could help someone else in some way.* (Student R)

From the comments provided by the students the overall impression was that students thought the project was well organised. Student U commented that the *Facebook* group was a good idea, but he was not a member of this group. Student R was also not part of the *Facebook* group and thus had no way of interacting with other members of the group and points out that meeting up to share ideas and discoveries would be a way to improve the project. Another suggestion was to further extend the use of technology by incorporation *Wolfram Alpha*.

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8.5 Tutorial test

The tutorial test on complex numbers, written by all students in the Linear Algebra 1 course, covered polar form and De Moivre's theorem. Performance results were analysed to draw a comparison between students who took part in the project and others in the group. It should be noted that one of the project students was absent for this test. The statistics calculated for this tutorial test are in Table 8.3.1:

Group	Size (n)	Average	Standard deviation	Lowest mark	Highest mark
Non-project students	677	5.65	2.58	0	10
Completed activities 1-3	21	6.53	2.80	1	10
Completed activities 1-4	14	6.32	2.82	1	10
Completed activities 1-5	9	6.22	2.83	1	10

Table 8.3.1: Performances of various groups on a complex numbers tutorial test

The average of the project students was 6.53 and the average of the non-project students was 5.65 (see Table 8.3.1). Thus the project students performed better on average than the class a a whole, as was expected, since these students as a group were initially selected from the group of stronger than average students in the class, and they had more exposure to the topic.

The lowest mark of 1 out of 10 for the project group was attained by only one student (Student I), a student who completed all five activities. If we remove student I from the list, then the average for those who completed activities 1-3 increases to 6.81, for those who completed activities 1-4 it increases to 6.69 and for those who completed activities 1-5 it increases to 6.88. These marks are much higher than those of the non-project students.

It needs to be emphasised that the main aim of the project was not to improve the marks of participants in normal curricular activities. Enrichment provides wider exposure and depth of understanding and improved performance as a by-product is foreseeable but not the motivating factor.

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8.6 Interviews

English was not their mother-tongue or home language for some of the students that I interviewed. All the quotes in this section are taken directly from the answers given by the interviewees. Grammatical errors were not corrected. All interviews were conducted in English.

8.6.1 Interview 1 – Peter (Student O)

Peter is a first year black male student. His home language is SeTswana, which is one of the eleven official languages of South Africa. Peter enrolled for a BSc degree in applied mathematics. Peter completed all five activities with much enthusiasm.

Project

Peter greatly enjoyed doing the project and he mentions that he acquired valuable skills from it. He further commented that this project broadened his mathematical experience:

> *It opens your mind to think a little bit outside the box like when you are given a problem.*

Peter found the project challenging and he enjoyed challenging himself, which is why he enjoys mathematics. Peter remembers an example from an applied mathematics course in which he was given a challenging problem that required some mathematics and applied mathematics knowledge that he had to combine to solve the problem. It was an initial value problem that was not trivial and it took him a while to solve it, but he persevered and eventually did. Peter further adds that, in his opinion, not more than 10% of the exam papers should consist of challenging and previously unseen problems.

Peter further adds that he benefited from the project in terms of doing the Linear Algebra 1 course. It gave him the confidence to assist a fellow female student. He helped her when she did not understand subject content from lectures, for example how to multiply and divide complex numbers.

Peter's fondness for mathematics originated from secondary school - that is the reason why he is studying mathematics and applied mathematics. Peter is eager to learn more and he says that this project further stimulated his interest in mathematics.

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> *You don't always rely on a lecturer, just do your own stuff and so that you see other problems outside the box.*

Technology

Peter feels positive about the use of technology in mathematics. He particularly enjoyed using Geogebra in this project. He gained insight by using it for drawing three dimensional sketches. He commented that:

> *When you rotate it, you can see it from different angles, you can look at it from the top ... you can get lot of properties of the curve you are dealing with.*

During Activity 4 (this was the first activity on sibling curves) he consulted the Internet to get more information regarding sibling curves. Peter came across a paper by Harding and Engelbrecht [51] that he found useful during this activity, but he still found it challenging to draw the sibling curves by hand as the assignment required. Peter found Activity 5 (this was the second activity on sibling curves which also incorporated Geogebra) more enjoyable as he could use Geogebra routines (provided by me) to sketch the sibling curves instead of trying to sketch it himself.

Peter ended the interview by suggesting that lecturers should use technology in the Linear Algebra 1 course and Calculus 1B course to complement the material being taught.

Problem solving

Peter confirmed that he enjoyed having been confronted with new problems - this is what he wants to learn from. Peter wants to develop his problem solving skills in order to solve new problems as he is planning to become a researcher. He made the following comment regarding his experience of unseen problems in this project:

> *Its very difficult, I won't lie, but it helps you to understand mathematics very well.*

Peter explains that his strategy for problem solving is to first identify what kind of problem he is trying to solve and then try to solve it with known techniques. After exhausting the known methods, he would consult books and/or the internet to find out how similar questions

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were approached. He applied this strategy during Activity 4 of the project (the first activity on sibling curves) when he tried to find more information on sibling curves on the Internet. Peter's strategy was successful because his answers were commendable and he did this work independently without consulting me.

Peter worked alone during this study - he prefers to work alone. Peter is wary to work in a team, because he is aware of the danger that other team members might not contribute equally to the project. He is of the opinion that most research is done individually and would prefer to work alone again in future enrichment projects. Peter did join the *Facebook* group and was one of the few regular contributors to the group.

Peter enjoys both attending lectures and reading mathematics which is "useful to expand your knowledge and problem solving skills" and that increases your knowledge. Peter did some reading on mathematics before attending university and is also planning to do more mathematics reading before starting his second year.

> *I already understood a little bit of integration, when I was coming here, coz I found a book on integration.*

Summary

Peter indicated that the project was interesting to him. The project stimulated his interest in mathematics and sharpened his problem solving skills. It opened his eyes to think differently and deeper about problems. Peter further comments that this project made him realise how difficult research is, but that the use of technology could be a useful tool.

8.6.2 Interview 2 – Sara (Student T)

Sara is a first year white female student. Her home language is Afrikaans, which is one of the more prominent of the eleven official languages of South Africa. Sara is enrolled for a BSc degree in mathematical statistics. Sara worked hard during the project and completed all five the activities.

Project

Sara felt the project was well-designed and the scope of the project was good. The project provided a good experience in doing mathematics without being formally taught by a lecturer.

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Sara says the project was a good introduction to complex numbers and she benefited from it. She found the idea of sibling curves interesting and comments further:

> *When they started with complex numbers in the Linear Algebra 1 course, it was not new for me whilst the other students sat there and did not know what was going on.*

Sara adds that she found the first few activities quite enjoyable. She had more time to work on them, as the activities in the first semester and during the holiday. Sara confesses that she rushed the last few activities on sibling curves as it was in the second semester and she had limited time to spend on the project and consequently did not learn as much from them as she might have.

The project stimulated her interest in mathematics and made her question the ideas of multiplicity of roots and when sibling curves meet? She made the following profound comment :

> *Even when you are thinking nonsense, you are thinking.*

Solving unseen problems excites her. Sara recalls a challenging problem in one of her statistics semester tests. She stared at the problem and did not have any idea on how to solve it and suddenly she had a "light bulb moment". She realised that you had to read the problem carefully as it was unseen and then you had to think hard to solve it. Sara further comments that there should be more than 15% of these kinds of problems in tests and exams.

Technology

Sara says that using technology such as *Matlab* and *Geogebra* is very valuable in mathematics as the work is becoming more complex as you progress in your studies. She comments that drawing sibling curves by hand in the quadratic case is do-able, but much more efficient with *Geogebra*. The idea of sibling curves was made more beautiful by seeing and rotating them in *Geogebra*.

Sara likes the idea that courses such as the Calculus 1A course and the Calculus 1B course do not permit the use of calculators in tests and exams. Sara rarely uses her calculator. She made the following significant analogy:

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> *Using a calculator is like learning a language where you are allowed to use a dictionary. This will not allow you to speak the language, because you are constantly looking up words.*

Problem solving

According to Sara, her problem solving strategy depends on the type of unseen problem in question. She will start to determine whether it connects with material she has already acquired and whether she has sufficient background knowledge to solve the problem. If her background is not adequate, she will read up more to broaden her knowledge. Sara will then continue to use this new knowledge to solve the problem. If she gets stuck, she might *Google* or consult with others who think differently and get ideas on how to solve the problem. She comments:

> *Maybe somebody got it wrong, but they think so completely differently to you and you can use those ideas to maybe get to the right answer.*

Sara employed some of these strategies in the project. She also made use of *Google* and connected with a fellow Afrikaans-speaking student, who also participated in the project, and often discussed her ideas with him.

Sara prefers attending lectures to reading mathematics by herself, but it depends on who is presenting the class. Sara uses the example of a lecturer who worked too fast for her and in that situation it was more helpful to read the material from the textbook. She finds the exposure in class still valuable, but a good lecturer would explain the concepts/ideas before actually solving the problem. Sara really enjoys it if a lecture goes broader and wider than the textbook. She recalls one of her statistics lecturers who used examples in practical situations which she found quite interesting and made her think. She comments:

> *I feel very honoured if they [lecturers] share their passion with students. It makes me enjoy the subject and classes more.*

Sara is not sure if she would like or dislike doing a future enrichment project in a group. It depends on who the group members are and how they think. She prefers to first solve the problems by herself and then to discuss them with other students in a group setting, but also adds that she has not been in a group setting where everybody benefited from the group dynamics. Sara comments the following about working alone:

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Individually is much better in mathematics, as it forces you to think by yourself.

Summary

Sara indicated that she had experienced the project as a good exercise in learning mathematics without formally being taught, but it did not change her view of the subject. Sara is well aware mathematics is more than just numbers. She really enjoyed the use of technology and being challenged by unseen problems. She feels that is it necessary for students to be challenged in tests and exams but these should not be too many of them.

8.6.3 Interview 3 – Jane (Student P)

Jane is a first year white female student. Her home language is English, the more commonly used language at South African universities. Jane is enrolled for a BSc degree in education. Jane does private mathematics tutoring in her spare time and eagerly participated in this project.

Project

Jane thought the project was fun and she also liked how complex numbers were incorporated into the project. Not only did her mathematical thinking expand, but she had the opportunity to learn a new number system by herself and experiment with it.

Complex numbers were very nice to me.

Jane enjoyed the whole project, but struggled during Activity 4, which was the first activity that dealt with sibling curves. She commented :

I didn't know what type of maths this was.

Jane says that this project forced her to think, which she liked a lot. She benefited also from this exposure to complex numbers when they started with the topic of complex numbers in Linear Algebra 1 course.

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Technology

Jane really enjoyed the use of technology in this project. Her favourite activity was drawing her own graphs and having the ability to change a parameter and see how the shape of the graph changes using *Geogebra* during Activity 1 and Activity 2. This was new to her and she enjoyed having the ability to view functions visually. During Activity 5 she enjoyed being given *Geogebra* routines in which she could choose the parameters and see what the sibling curves looked like. Jane comments that having a visual aid makes an impressive difference.

> *A lot of people struggle with maths because they don't see visual pictures and things.*

The use of *Geogebra* stimulated her interest in mathematics in that it added visual information to any algebraic expression that she could come up with. She appreciated that the use of technology makes things more accurate and faster.

> *I cannot draw. They are always skew.*

Jane also feels that lecturers should incorporate technology in lectures when they can. It can be very helpful. She does comment that it depends on the content. Technology may not be useful during proving theorems but can be used for illustrating a result.

Problem solving

Jane gets tired of doing the same thing over and over again. So she enjoys seeing new problems as this extends her knowledge because it creates an opportunity to learn. She commented that this project on sibling curves improved her skills to solve non-routine problems.

Jane's strategy to solve problems is to break a new problem down into pieces first. She will continue to apply her current knowledge to attack the problem. If that does not work, she would Google for relevant articles and consult with people.

Jane definitely prefers being taught mathematics than reading mathematics by herself. She feels being taught is easier. During a lecture Jane actively listens and tries to follow the steps. She learns by listening, hearing and writing down. She comments:

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When somebody is explaining to you, then you know you can ask that person questions if you don't understand . . . . if you read you can often get confused and have nobody to ask.

Jane comments that during Activity 2 she had a little difficulty in reading the notes on complex numbers by herself and had to consult the lecturer to clear up a few concepts. She considers the notes as well written and the examples useful.

Regarding future enrichment projects, she feels that if the topic is complicated it is better to work in a group, otherwise if it is less complicated she would do it by herself. At university she enjoys doing tutorials by herself. Jane was quite surprised that study groups are more useful at university than secondary school and attributes this to the fact that students prepare more for study groups than they did at school.

I like working by myself a lot. I am not a team player.

Jane finds challenging questions a good way to test understanding. She found Calculus 1B challenging. "In integration and conic sections you can ask so many variations which requires you to practise more so that you can identify the technique." After solving a previously unseen or challenging question she "feels great, smart."

Jane feels that it is fair to challenge students in exams, but the contribution of these challenging questions should be between 5% and 7% of the exam as students should be asked what they studied for. Jane cautions that there should not be too many questions that require complex use of the material studied, but adds:

I don't see the point of you coming to study if you don't want to challenge yourself to become better.

Summary

Jane enjoyed doing the project as it was something different and the knowledge was helpful for the Linear Algebra 1 course. She feels positive about the use of technology in mathematics and specifically enjoyed that component in this project the most as it allows her to experiment with functions and see them visually. Jane further feels that challenging problems should not only be in projects but also in exams, but they should contribute between 5% and 7% to the exam paper. She gets excited when she solves a challenging problem all by herself.

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8.6.4 Interview 4 – Tim (Student L)

Tim is a first year black male student. His home language is isiZulu. Tim enrolled for a BSc degree in computer science. He enjoys using computers to solve problems. Tim completed only 4 of the 5 activities in this study.

Project

Tim enjoyed the project as a whole. Tim's favourite parts of the project were struggling with questions in this project and doing research by himself or consulting the lecturer. Tim says:

> *If we struggle with something, it helps with the aspects of understanding.*

Tim enjoyed the four activities he did. Tim further comments that he likes mathematics and had fun participating in this project, although university work in the end prevented him from completing all five activities.

Tim confesses that the work on sibling curves was hard for him, but this definitely benefited his understanding of the Linear Algebra 1 course.

> *It exposed me to a bit more and made it [complex numbers] easier to comprehend.*

Tim comments that his interest in mathematics grew because of this project and he is saddened that he will not be doing many mathematics courses in his second year. He is contemplating taking some maths modules as electives in the second year.

Technology

Tim feels that technology saves time. It allows you to accomplish things quicker, for example calculating turning points and tangent lines. He feels that by hand it is much slower. Tim further points out if you do calculations by hand you can use technology to verify that your calculation is correct. He comments:

> *Technology help with the understanding of maths or if you are on the right track.*

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Tim easily adapted to using *Geogebra*. He initially started with basic curves to become familiar with the software. His favourite feature of *Geogebra* was the instant results and the fact that he could change a parameter and visually see the graphs of functions.

> *I feel technology is good, but sometimes I feel like if we push technology to a certain scale we gonna get lazy per se and wont want to sketch our own graphs, so we will lack a bit of understanding.*

Problem solving

Tim enjoys the challenge of solving unseen questions. He finds the process of research and consulting with friends or lecturers satisfying and enjoyed this part of the project Tim feels that unseen and new problems expand his horizon and knowledge. He comments:

> *New problems exposes me to different knowledge.*

Tim's problem solving strategy is to start with the knowledge he has. He tries to solve the problem with the current skills. If that fails, he would do a search on the internet for useful knowledge and if that failed, he would consult with people.

Tim prefers reading mathematics by himself instead of being taught. He likes to figure out material by himself, but enjoys having lecturers which he can consult on material he could not figure out by himself. Tim sometimes reads ahead on upcoming material and goes to lectures to see how well he understood the work. He says that he enjoys lecturers using technology, interacting with the students and explaining the work more or examples from the workplace.

> *I prefer knowing the big picture, ultimately what I will be painting.*

According to Tim, doing enrichment projects in a group or by himself depends on the subject. For mathematics, he would prefer to do it by himself.

> *Coz it's a challenge to me and see if I can grasp it myself.*

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Tim says that if he struggles with a problem then a group may be useful, but by struggling he gets a better understanding instead of somebody showing him a solution.

Tim is of the opinion that no more than 15% to 20% of test and exam papers should be challenging. A big portion of the paper should be easy to moderate questions. Tim finds it satisfying to solve a tough question in a text or exam. He explains that he does the easy questions as quickly as possible so that he has sufficient time to think about the challenging and unseen questions.

Summary

Tim found the project satisfying and worthwhile. He enjoyed the exposure to Geogebra for sketching functions quickly and having the ability to change a parameter and instantly seeing the change. Tim feels that this project was challenging and it made him think about the subject. He also looks forward to challenging and unseen questions in an exam and feels they should not be more than 15% to 20%. The rest of the paper should be do-able if you have studied and know your work.

8.6.5 Thematic analysis

The aim of this section is to create a thematic analysis of the interviewees. Common themes have emerged that warrant further comment relating to the research questions.

Engagement

This project allowed me to share some of my research on sibling curves with a small group of students. Peter commented in his interview that the project opened up his mind and he interacted with mathematics in a new way. Sara, who did not know complex numbers before the start of the project, found the project stimulating. Jane commented in her interview that she found the mathematics from the project different to the mathematics done in class.

All the interviewees commented that they enjoyed engaging in the mathematics. Three of the four interviewees completed all five activities, except for Tim who did not finish Activity 5 due to university pressure, but would have liked to finish it.

The students also commented that they had benefited from the project by having developed a deeper understanding of complex numbers. Sara, who had never seen complex numbers before

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the start of the project, found complex numbers interesting and when they officially started with complex numbers in Linear Algebra 1 course, she found the content familiar.

Technology

The use of technology/*Geogebra* routines in Activity 1, Activity 2 and Activity 5 allowed the participants to experiment and explore functions and sibling curves by themselves. This type of activity rarely occurs in a classroom setting. All four interviewees felt positive about the use of technology in mathematics and they enjoyed this experience.

Peter further comments in his interview that lecturers should try to use technology to enhance lectures. Jane comments that *Geogebra* further stimulated her interest in mathematics because she now has the ability to draw functions without pen and paper. I believe that Jane, who is studying to be a mathematics teacher, will add technology to her teaching skillset. This study exposed Jane to technology and she now has first-hand experience of how exciting it is to have the ability to sketch functions quickly.

Incorporating technology in this study definitely had a positive effect on students as stated by all four interviewees. From Activity 5 feedback and answers they all enjoyed having the ability to view sibling curves in three dimensions instead of drawing them. It appears that the use of technology that allowed for experimentation further stimulated their interest in mathematics.

Problem solving

It is clear that mathematically strong students would like to be challenged, as supported by the interviews. In her interview Jane comments that she gets tired of doing the same thing over and over and would like to see new challenges. Peter, who is studying applied mathematics and wants to be a researcher, also likes being challenged.

All four students enjoy doing mathematics by themselves. Sara states that doing mathematics individually is much better as it forces you to think for yourself. All the interviewees felt that challenging questions should appear in a test and exam. Peter said not more than 10%, compared to Sara who said it should be less than 15%. Jane felt it should be between 5% and 7% while Tim felt it should be 15% to 20%.

It is clear that all the interviewees agree that a test or exam should have challenging questions where their understanding of the subject is being tested. The interviewees have different opin-

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ions on the quantity, but they all experience a sense of joy and pride when they solve these kind of questions.

Personal growth

It is clear from the interviews that these participants did not only benefit academically by having a broader understanding of complex numbers, but they experienced personal growth from doing this project.

For example after completing this enrichment case study, Peter had the confidence to assist a fellow female student. He helped her when she did not understand subject content from lectures, e.g. how to multiply and divide complex numbers. Tim mentioned that he was braver to struggle with difficult problems and enjoyed solving interesting problems. Jane felt that she now had the confidence to read new mathematics by herself, which is something she never did before in mathematics. Sara realised that it was not just good enough to be able to solve problems, but also useful to question the work you are doing and ask your own questions.

From the interviews it is clear that the personal growth of the participants is due to the experiences that came from stepping outside their comfort zone and having to experiment, explore mathematics in a guided fashion by themselves. Their confidence was boosted by them seeing their own problem solving potential.

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Chapter 9

Discussion and conclusions

9.1 Introduction

In this chapter I answer the research questions of the study in Section 9.2-9.4. In Section 9.5 and Section 9.6 I discuss limitations of and recommendations stemming from this study. Section 9.7 discusses further research. Section 9.8 considers the value of the study and this chapter is concluded in Section 9.9 by a few final remarks.

Drawing from the mathematical research on sibling curves and the student enrichment case study conducted, I will endeavour to answer the following research questions, as stated in Chapter 1:

> **Research question 1: How can the undergraduate mathematical content be enhanced through research?**
>
> **Research question 2: How can the enhanced undergraduate mathematical content be used for enrichment by means of inquiry-based learning?**
>
> **Research question 3: What are the educational experiences of this particular student enrichment programme?**

9.2 Research question 1

How can the undergraduate mathematical content be enhanced through research?

Undergraduate mathematics is wide-ranging and includes various topics ranging from Calculus, Linear Algebra, Real Analysis, Complex Analysis, Group Theory and others. Even though

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single courses may have similar names, undergraduate curricula at different universities differ. Yet there is a common body of material available for enhancement.

In this study I provide an example on enriching undergraduate mathematics content. I studied a problem stemming from complex numbers: How can you represent or visualize the complex roots of an equation? This led to the theory of sibling curves which I expanded through research. This new theory is accessible to undergraduate mathematics students and should give them a deeper understanding of roots and complex numbers.

It is often the case in undergraduate teaching that students ask questions on results that we do not have the time to prove in class. Occasionally we also teach results without proving them to students for example:

(a) It is impossible to find $\int e^{x^2} dx$ in terms of elementary functions.

(b) For any square matrices A, B, we have det(AB) = det(A).det(B).

(c) It is not possible to solve all quintic equations using radicals.

(d) Mathematical constants, π and e are irrational numbers.

Several more unproven results can be found on undergraduate level that we teach without showing students why these results are true. Pesic [81] actually used problem (c) and enriched the mathematical content needed to understand this result so that bright high school students or academically strong university students could appreciate the mathematics behind it.

In this thesis I was fortunate to have been able to use a high-school problem on how to represent the roots of a quadratic function and through research enriched the content of the topic of complex numbers. The enhancement of the content extended much further than first year mathematics. This new knowledge results in a richer understanding of complex numbers and the roots of polynomials and allowed me to create an enrichment programme to give participants a first taste of doing research themselves. However, not all undergraduate content is easily extended through research. It clearly depends on the topic and the content. Some undergraduate topics could be more difficult to enhance, but there is still room to create educational material to guide students through a series of investigations and questions to help them to rediscover important mathematical results themselves that may contribute towards strengthening their appreciation of mathematics, even if the mathematics are already known.

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In summary, my research on sibling curves is an example on how it is possible to enhance undergraduate content in mathematics for student enrichment by choosing appropriate problems in the relevant subject area. Here one undergraduate topic was expanded, but it is by no means a common occurrence to expand undergraduate topics to the extent that was done in Chapters 2-5.

This enhanced content has educational value, because it lends itself to enrichment. Most of the new knowledge is accessible to first year students and it can be visually demonstrated with the use of technology to further entice students as it addresses the problem on how to visualize the roots of a quadratic equation if the roots are complex.

The study also presents a short history that is useful for classroom use and as reading material for students. Presenting the history surrounding a mathematical topic or concept is in itself a form of enrichment as it positions the mathematics concerned on a timeline and provides context to the topic.

9.3 Research question 2

How can the enhanced undergraduate mathematical content be used for enrichment by means of inquiry-based learning?

In this study I provide an example of how to use enhanced undergraduate content for student enrichment at university level. The process involved two stages. Firstly I developed new content through research by enhancing complex numbers content, which is a first year mathematical topic. The second stage involved developing and conducting an enrichment programme targeting a select group of academically strong and willing students. Using the goals of inquiry-based learning [7] I created an environment for the participants to explore mathematics differently to what is typically done in the classroom setting.

The enhanced content was tailored to suit the level of understanding of the students and so that it had educational value for the students. Through five guided activities the participants were given the opportunity to explore mathematics instead of being taught mathematics conventionally. The students also benefited academically by obtaining early exposure to complex numbers, before being taught, and broadening their knowledge of complex numbers.

According to Anderson [1] an important factor of acquiring new information is to activate prior information. Using this principle and desiring all the participants to have a similar background,

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I designed the first three activities to tap into this prior knowledge of polynomials and roots.

To end up exposing the participants to the theory of sibling curves, which was my goal, I also had to implant background knowledge of complex numbers. This was achieved through Activity 2 and Activity 3. Desiring the participants to have the same complex numbers background highlights the issue of implementing inquiry-based learning. It is important for the participants to have the necessary background to fully participate [26]. After creating a common background of complex numbers for the participants, I concluded this enrichment programme by creating two activities on sibling curves.

To implement an enrichment programme using inquiry-based learning requires planning, organisation and time. I found it was important to have a clear goal of what content you planned to share with your students when using inquiry-based learning. If it happens that the participants do not have the necessary background, it is then important to implant the relevant background knowledge. Guidance on the actual activities by the designer is also important so that at the end of the programme the students will have gained knowledge and expanded their mathematical thinking.

It should be noted that enhanced undergraduate content as exposed in this study, is not an essential component to implementing an enrichment programme using inquiry-based learning. The designer could use existing content that is either part of the syllabus or supplements it in a fitting way. The content should be malleable to transform it into guided activities using inquiry-based learning where students can explore and learn by themselves with minimal guidance.

9.4 Research question 3

What are the educational experiences of this particular student enrichment programme?

In the chapter exposing the findings of this study. I shared my own and students' experiences regarding this enrichment case study. In this section I shall discuss the educational value this enrichment programme had for me and for the students.

Engagement: It was an eye-opening experience to interact with a small group of students closely as a lecturer. In this study, my role was not to convey knowledge, but rather to guide students in acquiring the knowledge. It was a pleasant task to be a consultant and to collaborate with students instead of being a formal teacher. This was a new form of engagement with

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students for me and thus served as enrichment for me personally in my teaching activities.

It is clear from the surveys that this project stimulated students' interest in mathematics at the onset of their university studies. I experienced this interest during the activities where students were more prepared to ask questions and engage with the material than in a classroom situation. Some students even collaborated with other participants or consulted other resources such as the internet. This interest and the students' feedback during the activities indicated a far more active engagement in the mathematics, which is one of the goals stated by Freeman ([38], [39]).

Technology: In this study the participants were provided with *Geogebra* routines to sketch polynomials of various degrees. In Activity 5 the participants could sketch sibling curves of real and quadratic polynomials by means of these routines. From the feedback on the activities, survey and interviews, it is clear that the students enjoyed having this ability to turn functions into graphs and being able to explore and investigate various functions by merely changing a parameter. This excited students and made mathematics more alive for them. This opportunity is not provided in the normal teaching activities and thus enriched these students beyond what is standard.

Participants could join a *Facebook* group dedicated to this project. This allowed me to give general feedback and answer questions that students had regarding the study or interacting with other participants. From the low level of *Facebook* activity by the participants, it appeared that this was not a popular medium for this group to discuss mathematics. The lesson learned from this is that in this case students did not experience the need for a social media platform and that other means should be explored for creating coherence, if at all necessary.

Depth: As a lecturer teaching a large class, you rarely get the time to explore a certain topic or problem in more depth with students. In this study I could explore the idea of representing the zeroes using sibling curves together with the students, which was a satisfying experience. Through guided activities, I could share this enhanced content with a group of selected first year mathematics students who appreciated and developed a deeper understanding of the particular mathematical topic, as indicated by students in the survey.

This enrichment programme also allows lecturers and students to avoid the trap that Freudenthal [37] calls the *anti-didactical inversion*, that is only teaching the final polished mathematical approach without showing its discovery or evolution over time. This "clean" approach hides the adventure of doing mathematics from students, which was unhidden in this enrichment

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programme.

Problem solving: In the survey and interviews the need that academically strong students or those planning to become researchers have to do non-routine or high-order thinking problems was highlighted. Our evidence clearly shows that students have a definite desire to be challenged and to sharpen their problem solving skills. This desire is sometimes neglected when merely posing run-of-the-mill questions. From the data it is evident that the project changed the student's view of mathematics, in that mathematics is not just a set of formulas and algorithms to learn, but it is also about problem solving skills. Mathematically strong students want to see interesting and new problems where they can apply newly acquired content knowledge as indicated by the data collected.

Personal growth: As a lecturer I have grown from doing this project. It added a new dimension to my teaching. I do not just focus on doing justice to the syllabus, but am now actively looking for ways to stimulate mathematically strong students in the class. It could range from doing an interesting example in class, inserting a challenging question in a test or creating resources to keep academically strong students motivated.

From the activities and survey it is clear that the participants have grown academically from doing this project. To illustrate, as indicated in an interview, a student mentioned that he became more confident in sharing his mathematical knowledge with a fellow student in the class. This growth in confidence could be attributed to the enrichment study. In the interviews the participants indicated they had acquired valuable problem solving skills that would benefit them in their later years of study.

In summary, this study demonstrated the willingness and enjoyment of academically strong students to participate in an enrichment programme. It highlights the fact that there is a need to stimulate these keen and interested students, in spite of their busy academic schedules.

Conducting an enrichment programme is time-consuming and you need to have suitable content for such a programme, although not necessarily content that you developed yourself. However, it is worth the time and resources, as this effort makes an impressionable impact on the students. Academically strong students' interest is ignited by enrichment and they have the potential of becoming world-class researchers. Enrichment programmes have the potential to add significantly in terms of education to the normal lecturing programme.

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9.5 Limitations of the study

This enrichment case study was conducted on a small group of academically strong first year students at the University of Pretoria enrolled for the mainstream Calculus 1A course. Due to the workload involved it would be difficult to extend it to more students.

Another limitation is with respect to the content of the enrichment case study - my research in complex numbers allowed me to enrich the first year syllabus of complex numbers. This in turn, allowed to me to develop an enrichment project on complex numbers to broaden academically strong student's knowledge and interest in mathematics. It should be pointed out that not all subject matter can easily be enhanced and thus the enrichment topic will have to be chosen carefully.

As seen from this study, the implementation for an enrichment programme will require three elements: a dedicated lecturer, suitably enhanced content and keen students to participate in the enrichment exercise. Such a combination is not always readily available.

Only a small portion of the enhanced content was used for enrichment purposes. No data have been gathered on how students would experience enrichment involving other aspects of the enhanced content.

9.6 Recommendations

As a result of experiences in conducting this study, the following recommendations can be considered.

Content: This enrichment programme only exposed the participants to a portion of the research I did on sibling curves. Activity 4 and Activity 5 focussed on the sibling curves of quadratic functions. Further enrichment activities can be constructed to look at the hyperbolic paraboloid on which the sibling curves of quadratic functions lie. More enrichment activities could investigate the sibling curves of higher order polynomials or other interesting functions such as rational functions.

Further enrichment activities could use existing content which is not covered in class, instead of enhanced content. This material can be transformed into enrichment activities for a group of students by creating inquiry-based activities guiding students, as in this study, to investigate

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a suitable topic in mathematics.

Group meetings: This enrichment programme can be modified by having group meetings, in which students get the opportunity to ask questions, discuss material or even explain their results to the rest of the group. This will create better opportunity for collaboration, meeting like-minded students and improving confidence in mathematics. These group meetings can be with or without the programme designer, depending on the need of the participants.

Adjusting the programme: Another modification to this enrichment programme is to have fewer activities. One of the pitfalls that I experienced in my model, was that due to the time required for doing five activities, I did lose some of the students along the way. They either lost interest or, due to academic pressures, they had to prioritise and stop taking part in this enrichment programme. A possible solution to this could be to have enrichment activities during the holidays when there is less academic pressure on students. Holidays can also be used to expose students to enhanced content for upcoming courses or to have inquiry-based activities that will expose students in a more interactive engagement of the class material before it is actually being taught.

Similar projects: I would recommend that other lecturers create and implement their own enrichment programmes using inquiry-based learning. It is gratifying working with a small group of keen students willing to broaden their mathematical knowledge and skills. Creating an inquiry-based environment allows students to engage with the material and give the lecturer an avenue to share knowledge s/he might not have the time for in class.

Furthermore it would be fortunate if other tertiary mathematics departments have the drive to create and support enrichment programmes, similar to this one or others that were reviewed, for undergraduate students who could potentially become postgraduate students in the department. These enrichment programmes would need resources to stimulate these young minds, such as venues and human resources with respect to funding, people and time. A possible target group for implementing an enrichment programme for undergraduate students, could be first year students planning to major in mathematics or considering doing this. An enrichment programme would give them a taste of what to expect in later years to come.

9.7 Future research

In Section 9.7.1 I discuss possibilities for further mathematical research stemming from the work from Chapters 2-5. In Section 9.7.2 I discuss possibilities for further educational research

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on enrichment stemming from the work from Chapters 6-8.

9.7.1 Mathematical research

This research focused on enhancement of one mathematical topic in undergraduate mathematics. I investigated the topic of complex numbers and explored the question of how to represent the complex roots of a function. This led to the idea of sibling curves. Further research can be done on sibling curves to make the theory even richer. In Chapter 4 the quadratic case was considered. Further research could investigate the cubic case. A second research topic could be to explore other geometric properties of sibling curves and to use this theory to create a new proof of the Abel-Ruffini theorem.

Further research could also include looking at other problems taken from undergraduate mathematics and enhancing it for students. These problems could be from other subject areas like Calculus, Linear Algebra, Analysis or Group Theory, all from undergraduate mathematics. With careful investigation and appropriate research it is possible that material can be generated that is suitable for student enrichment. This extended knowledge can then be turned into material for enrichment to broaden the material taught in class.

9.7.2 Educational research

One possible education research question could be to find other effective ways to use enhanced content for enrichment for students. Research can be done on creating efficient enrichment programmes for undergraduate students with this new extended knowledge. Possible topics can explore the use of technology to create software for students to explore at their own pace without the guidance of lecturers.

Another research topic could be to measure the success by investigating the long-term effects of such an enrichment programme on a group of students. For example how did these enrichment activities influence them personally and academically in the long run? Did the enrichment programme improve their problem solving skills for later years at university?

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9.8 Value of the study

A novel way of visualizing complex roots, which should be enlightening to the mathematics community: In Chapters 2 and 3, I demonstrated how sibling curves offer a very elegant and geometric way to visualize the roots of any complex function, including polynomials. The roots turn out to be the points where the sibling curves cut the horizontal plane. Furthermore in Chapter 4 we proved that a polynomial of degree n has n sibling curves which contain the n roots of the polynomial, which provide us a "bigger picture" of the complex roots of polynomials.

Hyperbolic paraboloid – all quadratic sibling curves lie on it: In Chapter 5 I focused on complex quadratic functions and investigated their sibling curves. In this study it was noticed in the real quadratic case that we always get two sibling curves which are both parabolas and intersect. However if we allow the coefficients to be complex, then most often we end up with two sibling curves that do not intersect. Further investigation shows that in the quadratic case the sibling curves are always congruent and always lie on a hyperbolic paraboloid determined by the coefficients of the quadratic function.

Highlighting enrichment at university level: While doing this study, I found limited resources and case studies for enrichment at university level. A major part of enrichment research is done at primary and high-school level to stimulate the interest of academically strong and curious minds and to broaden their knowledge. I trust my enrichment case study will put the spotlight on enrichment at university level. Enrichment at undergraduate level is a perfect vehicle to sharpen problem solving skills and show students their own potential.

Useful for a wider audience, all teaching staff at universities: The enrichment case study was on a group of first year mathematics student at the University of Pretoria. This content can be modified for a wider audience and be adapted to conduct similar enrichment programmes for other undergraduate mathematical students studying complex numbers, possibly at other universities. The educational principles noted in Chapter 6 show that enrichment can also be applied to other subjects besides mathematics as were shown in this study.

Set an example to encourage research at undergraduate level: This enrichment project exposed first year mathematical students to non-routine problems using inquiry-based learning. This study showed that an enrichment case study is an opportunity for undergraduate students to explore areas they are keen and interested in investigating. Participating in an enrichment programme will encourage curious minds to ask their own questions and get a taste of research.

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This opportunity of guided exploration is rarely given to undergraduate students to deepen their knowledge and problem solving skills.

9.9 Concluding remarks

> *Teaching is like conducting an orchestra. Students, like different instruments, differ in learning style and contribute to the classroom in different ways. An orchestra requires collaboration among different instruments, which parallels to the need for collaboration in the classroom. Students and teachers must work together to achieve the greatest learning possible.* (Laura Mcbride)

The goal of a teacher or lecturer is to make learning engaging and interesting. It is a well-documented fact that that not all students will find the same lesson equally engaging. One student might find a lesson quite challenging, whereas another could find it extremely boring. As a teacher or lecturer you need to "conduct this orchestra of students" sitting in front of you, creating an environment for students to learn and reach their own potential. Occasionally academically strong students are neglected instruments that do not get used to their fullest extent due to resources or pressures of finishing an overloaded syllabus.

In this study we demonstrated how enrichment can be used like "sheet music" to pay attention to academically strong students and make these instruments come alive. Enrichment promotes out-of-classroom learning for keen and willing students. These students get the opportunity to test and sharpen their problem solving skills which lead to personal development. Moreover these students get the opportunity to delve deeper in the subject material allowing them to explore their own curiosity and to ask their own questions or make their own discoveries instead of being given the results.

All universities want to produce capable problem solvers and thinkers. One of the aims of tertiary education in mathematics should be to empower students with knowledge and to develop problem solving skills. These skills are important if students want to become researchers and make "their own music." So I strongly believe it is ultimately important for undergraduate lecturers to guide their students and create opportunities in which academically strong students, especially through enrichment, will get a small taste of research. This will create an environment for exploring and investigating work related to their studies, before being thrown into the deep end when they do research for honours or masters degrees.

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Appendix A1

Activity 1 - Polynomials

Name and Surname :
Student number :
E-mail address :

A. Visit http://cardanogroup.wordpress.com/

B. You are a mathematician and have to answer the questions below. Experiment with the polynomials on the website and use your observations to answer the questions below.

C. Return your answers when you are done. Next time bring a flash stick to copy some files.

1. Do all real quadratic polynomials have at least one real root?

2. Do all real cubic polynomials have at least one real root?

3. Do all real quartic polynomials have at least one real root?

4. Do all real quintic polynomials have at least one real root?

5. What effect does coefficient a have on real quadratics?

6. What effect does coefficient b have on real quadratics?

7. What effect does coefficient c have on real quadratics?

8. Can you find a cubic polynomial with roots -2, 0 and 2?

a = 1 b = 0 c = ? d = ?

9. Can you find a quartic polynomial with a local minimum at point P(1;0)?

a=2, b=0, c=-4, d=?, e=?

10. Any other observations/remarks?

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Appendix A2
Activity 2 - Other functions

Name and Surname :
Student number :
E-mail address :

PART A – other functions

1. Please install Geogebra 3d.

2 : Type in x^2 + k * x * y + y^3 = 0. It creates a slider for k. Change k. What do you notice? (if possible, switch on animation for k).

3 : There is a formula for a quadratic equation, cubic equation and quartic equation. Galois proved there is no formula for a quintic equation.

In algebra, the **Abel–Ruffini theorem** (also known as **Abel's impossibility theorem**) states that there is no *general* algebraic solution—that is, solution in radicals— to polynomial equations of degree five or higher.

How many real roots does x^5 – 3x^3 + 3x – 1 =0 have? (use geogebra to sketch it)

True or false : Every real polynomial of degree 6 has at least one real root.

True or false : Every real polynomial of degree 7 has at least one real root.

4. Use Geogebra to sketch y = e^(-k x^2). This is known as the bell curve and is used often in Statistics. Change the slider k. What do you notice?

5 : Sketch your own curves. List at least 2 examples. Also explain why they are interesting.

Eg. Try y = sin (kx) + r*x + 1. Which has a bigger effect? k or r?

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Part B - Complex numbers

1. Read the pdf provided.

2. What is $i \times i$?

3. Calculate the following

(a) $(1 + 2i) + (3+4i) =$

(b) $(1 + 2i) - (3- 4i) =$

(c) $(1 + 2i) \times (3 - 4i) =$

(d) $(1 + 2i) \div (3- 4i) =$

(e) $(3-4i) \div (1 + 2i) =$

4. How can you show z^2 + 3z + 5 = 0 has no real roots?

5. Find the complex roots of z^2 + 3z + 5 = 0 in the form a + bi.

6. What are the complex roots of z^2 + 1 = 0?

7. What do you think of complex numbers?

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Appendix A3
Activity 3 – Complex numbers

Name and Surname :
Student number :
E-mail address :

1. Prior to this year have you worked with complex numbers before?

2. For each equation, state whether the roots are real or not.

(a) Z^2 + 5Z + 1 = 0
(b) Z^2 – 5Z + 1 = 0
(c) 2Z^2 + Z + 10 = 0
(d) -3Z^2 + Z + 22 =0
(e) 2Z^2 + 3Z – 5 =0

3. Which line is wrong in this FALSE proof that shows 1 is -1?

-1 = sqrt(-1) times sqrt(-1)	line 1.
So -1 = sqrt(-1) times sqrt(-1) = sqrt((-1)(-1))	line 2.
So -1 = sqrt(-1) times sqrt(-1) = sqrt((-1)(-1)) = sqrt(1)	line 3.
So -1 = sqrt(-1) times sqrt(-1) = sqrt((-1)(-1)) = sqrt(1)=1	line 4.

4. Consider f : C to C, where C is the set of complex numbers where f(z) = z^2 – z.
   Calculate the following:
   (a) f(5)
   (b) f(1+i)
   (c) f(i)
   (d) f(1-i)
   (e) If x is a real number, is f(x) a real number?

5. Consider f : C to C, where C is the set of complex numbers where f(z) = z^2 + 1.
   (a) Are the roots of f(z) real or complex? What are they?

   (b) For which values of z will f(z) be a real number?

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Appendix A4
Activity 4 – Sibling curves

Name and Surname :
Student number :
E-mail address :

1. Let us look at f(z) = z^2 + 1 again.
   (a) What is f(1+i)?
   (b) If x is real is f(x) real?
   (c) If x=it for some real number t is f(x) real?
   (d) If r is a root of f(z), is f(r) real?

2. DEFINITION. Suppose f is a complex valued function. If a is a complex number such that f(a) is a real number, then plot the point (real(a), imaginary(a),f(a)). All these points lie on curves known as sibling curves. So we are looking for **all input** whose out is REAL.
   (a) Do you see how this allows us to give a 3D representation of complex functions?

   (b) Can you give me at least two points on these sibling curves of f(z)=z^2+1? Remember the points are in 3D. I.e. (1,2,3) does not work since f(1+2i) is not 3!

   (c) What do you think the 3D curves look like for f(z)=z^2 + 1?

3. Consider f(z) = z^2 – 2z.
   (a) Let z = x+iy where x and y are real numbers. Fill in below.
   Then f(x+iy) = (x+iy)^2 – 2(x+iy) =
   (b) A complex number is real when the imaginary part is ………………….
   (c) So when is f(z) a real number?

   (d) Can you give me 2 examples when f(z) is real? Do not give real numbers.

   (e) Can you sketch all the points on these sibling curves? What shapes do you get?

   Bring a flash stick next time to get software that will allow you to sketch sibling curves for quadratics.

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Appendix A5
Activity 5 – Geogebra

Name and Surname :
Student number :
E-mail address :

Part A – file A

1. Open File A. Let a=1, b=-3,c=2. What do you see?

2. (a) What are the roots for z^2-3z+2=0?

   (b) On which sibling curve do the roots lie?

   (c) What do the sibling curves look like from above? (Rotate them.)

3. (a) What are the roots for z^2+z+1=0?

   (b) Change the coefficient to draw these sibling curves. On which one do the roots lie?

   (c) What do the sibling curves look like from above? (Rotate them.)

4. (a) What does the blue curve represent?

   (b) What does the red curve represent?

   (c) What shape are the curves?

   (d) Any other comments? Rotate and look at the curves!!

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Part B – file B

Now open file B. We consider quadratic equations where the coefficients can be complex.

Hence 6 parameters.

1. What kind of polynomial is (a1+a2i)z^2 + (b1+b2i)z + (c1+c2i) if a2,b2,c2 are all zero?

2. (a) What are the roots of z^2 + (1+i)z? (hint factorize)

   (b) To draw these sibling curves you let

a_1 =	a_2 =
b_1 =	b_2 =
c_1=	c_2=

   (c) What do the sibling curves look like from above?

   (d) What shape are the sibling curves?

   (e) If you chance c_1 can the sibling curves meet(give value if exist)?

   (f) If you change c_2 can the sibling curves meet(give value if exist)?

   (g) What effect does changing a_2 have?

3. Construct other quadratic equations.
   (a) what is a_1?
       what is a_2?
       What is b_1?
       What is b_2?
       What is c_1?
       What is c_2?
   (b) What happened to the sibling curves in this situation?

4. (a) Is it possible for the sibling curves to meet twice or more?

   (b) Any other comments?

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Appendix A6

Complex numbers project survey

For each statement select the most appropriate box. For example, I enjoy eating an apple. If you pick strongly disagree, then it means you absolutely do not like eating apples.

1. I enjoyed the project.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

2. I enjoy reading mathematics by myself.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

3. I enjoy solving new problems.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

4. I could cope with this project and my university work.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

5. I enjoy working alone.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

6. I enjoyed using technology/Geogebra.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

7. I found technology/Geogebra easy to use.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

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8. I would recommend using technology/Geogebra in Mathematics.

Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree

For these questions select one of the two options.

9. Did you work alone or with fellow students during the project?

Alone	With fellow students

10. Did you know complex numbers before doing this project?

Yes	No

11. Would you do an enrichment project in holiday time to prepare for 2nd year Mathematics?

Yes	No

12. Would you do an enrichment project if you can get 2 weeks off classes and tutorials?

Yes	No

13. Did this project change your view of mathematics?

Yes	No

For these questions please fill in a sentence or two as an answer.

14. Why did you enjoy the project? Or what did you not enjoy about the project?

15. What did you learn in this project?

16. Did you work alone or with fellow members during this project?

17. Any suggestions on how to improve the enrichment project or ideas to implement in an enrichment project.

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Appendix A7

WTW126 Tutorial Test 5

Total : 10

1. Write the following complex numbers in polar form with main argument :

(a) $z = 1 + \sqrt{3}i$ (1.5 marks)

(b) $w = -2 - 2i$ (1.5 marks)

2. Referring to question 1, calculate the following complex numbers in polar form with main argument :

(a) zw (2.5 marks)

(b) $\frac{z}{w}$ (2.5 marks)

(c) z^5 (2 marks)

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Appendix A8

Interview benchmark questions

1. What are you studying at the moment?

2. Which parts of the project did you enjoy?

3. Which parts of this project did you not enjoy?

4. Do you think the study of sibling curves improved your understanding of complex numbers for WTW126?

5. Did this project stimulate your interest in mathematics?

6. What did you think of the use of technology in mathematics?

7. Which do you prefer more and why: being taught mathematics or reading mathematics on your own?

8. Would you prefer student enrichment to be done individually or in a group?

9. How would you solve a new and unseen problem?

10. What would you say is a fair percentage of tough/challenging questions that can be asked in a test or exam?

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Appendix A9
Ethics permission

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ETHICS COMMITTEE

Faculty of Natural and Agricultural Sciences

2 March 2015

Mr H Wiggins

Department of Maths and Applied Maths

University of Pretoria

Pretoria

0002

Dear Mr Wiggins

EC141029-094 First year mathematics enrichment case study

Your application conforms to the requirements of the NAS Ethics Committee

Kind regards

Prof NH Casey

Chairman: Ethics Committee

Agriculture Building 10-20
University of Pretoria
Private bag X20, Hatfield 0028
Republic of South Africa

Tel: 012 420 4107
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Appendix A10
Participants data for activities, facebook and survey

STUDENT ID	STUDY	ST1	ST2	A1	A2	A3	A4	A5	FACEBOOK	SURVEY
A	BSc - Mathematical Statistics	91	91	1	1	1	1	1	1	1
B	BSc - Physics	87	94	1	1	1	1	1		1
C	BSc - Computer Science	77	77	1	1	1			1	1
D	BSc - Computer Science	87	90	1	1	1	1	1	1	1
E	BSc - Actuarial and Financial Maths	100	99	1	1	1	1			1
F	BSc - Physics	88	91	1	1	1	1	1	1	1
G	BSc - Computer Science	74	91	1	1	1	1	1	1	1
H	BSc - Mathematical Statistics	90	90	1	1	1	1	1	1	1
I	BSc - Physics	91	81	1	1	1	1	1	1	
J	BEd - FET: General	75	67	1	1	1			1	1
K	BSc - Actuarial and Financial Maths	92	98	1	1	1	1	1	1	1
L	BSc - Computer Science	71	78	1	1	1			1	1
M	BSc - Mathematical Statistics	74	70		1	1	1		1	1
N	BEd - FET: General	73	73	1	1	1			1	1
O	BSc - Applied Mathematics	71	71	1	1	1	1	1	1	1
P	BEd - FET: General	74	76	1	1	1	1	1	1	1
Q	BSc - Computer Science	77	88	1	1	1	1		1	1
R	BSc - Mathematical Statistics	78	87	1	1	1	1			1
S	BSc - Actuarial and Financial Maths	91	94	1	1	1	1		1	
T	BSc - Mathematical Statistics	93	82	1	1	1	1	1	1	1
U	BSc - Physics	79	90	1	1	1	1	1		1
V	BSc - Actuarial and Financial Maths	91	87	1	1	1			1	
Totals				21	22	22	17	12	18	18

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