African Journal of Research in SMT Education, Volume 8(1), 2004, pp. 53-64

Realistic mathematics education: Eliciting alternative mathematical conceptions of learners

Hayley Barnes
University of Pretoria, hbarnes@gk.up.ac.za

Abstract
This paper focuses on a Grade 8 mathematics intervention that has been developed and implemented in a local high school in South Africa. The theory of Realistic Mathematics Education (RME) has been applied in the design of the intervention, which aims to assist low attaining learners in a remedial[1] programme that is run during school hours. Based on the premise that eliciting and addressing learners' alternative conceptions in mathematics is beneficial in assisting them to improve their understanding, the paper seeks to explore the role that RME plays pertaining to this particular supposition. The paper presents and discusses examples of learners' responses to contextual problems given to them during the course of the intervention.

Introduction
Literature that has been generated from studies done in a number of countries (including other developing countries such as Indonesia) has shown that the Realistic Mathematics Education (RME) theory is a promising direction to improve and enhance learners' understanding in mathematics (Armanto, 2002; Fauzan, 2002). To explore the potential of using RME for teaching remedial classes, a case study of twelve Grade 8 low attaining learners in mathematics is being conducted at a high school in South Africa. The theory of RME has been employed in the design and implementation of an intervention in an attempt to address perceived deficits in the learners' conceptual understanding of basic mathematical concepts. Within this framework, this paper seeks to the explore the following question:

> Can the theory of Realistic Mathematics Education (RME) play a role in eliciting the alternative mathematical conceptions of learners in order to assist them in overcoming these?

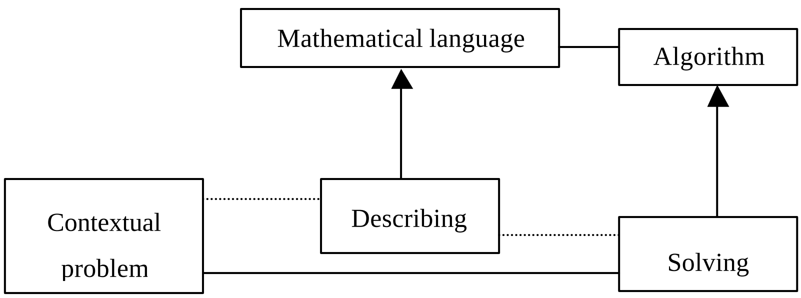
Although a large percentage of learners at the school do take extra mathematics classes of some type after school, there remain a growing number of learners who continue to perform far below the expected level. A few years ago the school took the decision that certain learners in Grades 8 and 9 were achieving such low results in Mathematics and English that it would be in the best interests of these learners to allow them to attend additional lessons in these two subjects during school. In order to accommodate these lessons within the school day, and to reduce the workload of these learners, they are not required to take a third language, which is usually compulsory for all learners in Grades 8 and 9. The Mathematics and English teachers respectively of these learners give the remedial Mathematics and English lessons and each teacher decides what instructional methods, content and material to use for their own classes. There is therefore not any policy or specific methodology for these remedial lessons. The time is created on the timetable and certain learners are identified (by their poor academic performance) as needing the remedial assistance and the task of the teacher is to provide this. From the personal experience of the author, as well as validation from the teachers at the school, it appears that the remedial mathematics lessons have mostly been a repeat of the normal mathematics lessons rather than an intervention to diagnose and address the

[1] The word remedial is used here as these classes are referred to as remedial classes by the school

mathematical difficulties experienced by the learner. This paper describes and analyses a case study being carried out with two grade 8 remedial classes.

Theoretical underpinnings of RME

Realistic Mathematics Education has its roots in Hans Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1973; Gravemeijer, 1994). To this end, Freudenthal accentuated the actual activity of doing mathematics; an activity, which he envisaged should predominantly consist of organising or mathematising subject matter, taken from reality. Learners should therefore learn mathematics by mathematising subject matter from real contexts and their own mathematical activity rather than from the traditional view of presenting mathematics to them as a ready-made system with general applicability (Gravemeijer, 1994). These real situations can include contextual problems or mathematically authentic contexts for learners where they experience the problem presented as relevant and real. The verb "mathematising" or the noun thereof "mathematisation" implies activities in which one engages for the purposes of generality, certainty, exactness and brevity (Treffers, 1987; Gravemeijer, Cobb, Bowers & Whiteneack, as cited in Rasmussen & King, 2000). Through a process of progressive mathematisation, learners are given the opportunity to reinvent mathematical insights, knowledge and procedures. In doing so learners go through stages referred to in RME as horizontal and then vertical mathematisation (see Figure 1). Horizontal mathematisation is when learners use their informal strategies to describe and solve a contextual problem and vertical mathematisation occurs when the learners' informal strategies lead them to solve the problem using mathematical language or to find a suitable algorithm (Treffers, 1987; Gravemeijer, 1994). For example, in what we would typically refer to as a "word sum", the process of extracting the important information required and using an informal strategy such as trial and error to solve the problem, would be the horizontal mathematising. Translating the problem into mathematical language through using symbols and later progressing to selecting an algorithm such as an equation could be considered vertical mathematisation, as it involves working with the problem on different levels.



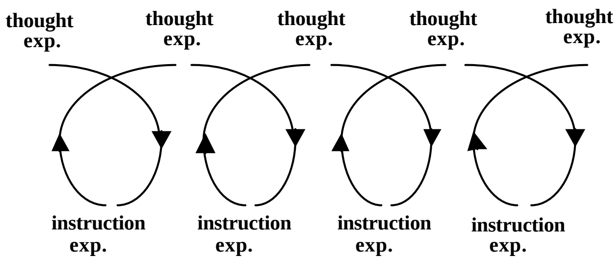
(Adapted from Gravemeijer, 1994.)

Figure 1: Horizontal mathematisation (.............); Vertical mathematisation (———▶)

The traditional formal and authoritarian approach to teaching mathematics that has dominated in South African classrooms for a number of years has not afforded learners many opportunities to make use of horizontal mathematisation. Mathematics lessons are often presented in such a way that the learners are introduced to the mathematical language relevant to a particular section of work and then shown a few examples of using the correct algorithms to solve problems pertaining to the topic before being given an exercise or worksheet to complete. The exercises or worksheets are usually intended to allow learners to put the algorithms they have been taught into practice and may even contain some contextual problems that require the use of these algorithms. According to the RME model depicted in Figure 1, this type of approach places learners immediately in the more formal vertical mathematisation process. The danger in this approach is that when learners have entered that process without first having gone through a process of horizontal mathematisation, a strong possibility exists that if they forget the algorithms they were taught; they do not have a strategy in place to assist them in solving the problem. This experience can be equated to someone being shown and told what is on the other side of a river and being expected to use what is there for their own benefit. However, they are not given or shown the bridge that assists one in crossing to the other side in order to make proper use of what is there. The horizontal mathematisation process provides this bridge.

Developmental research

The RME theory is one that is constantly "under construction", being developed and refined in an ongoing cycle of designing, experimenting, analysing and reflecting (Gravemeijer, 1999). Developmental research plays a central role in this process and, in contrast to traditional instructional design models, focuses on the teaching-learning process, zooming in specifically on the mental processes of learners (Rasmussen & King, 2000). Cyclic processes of thought experiments and instructional experiments form the crux of the method of developmental research and serve a dual function (see Figure 2 where exp. serves as an abbreviation for experiment). They both clarify researchers' learning about learners' thinking and address the pragmatic affairs of revising instructional sequences (Gravemeijer, 1999). Instructional sequences are designed by the curriculum developer who starts off with a thought experiment (abbreviated to "thought exp." in Figure 2) that imagines a route that learners could have invented for themselves. The lesson is implemented and the actual process of learning that takes place in relation to the anticipated trajectory is analysed. This analysis can then provide valuable information in order to revise the instructional activities. It was during this type of analysis that the potential value of using RME to elicit alternative conceptions was first identified.



(Gravemeijer & Cobb, 2002)

Figure 2: Developmental research, a cumulative cyclic process

RME instructional design principles

Gravemeijer (1994, 1999) identifies three key heuristic principles of RME for this process of instructional design, namely:

- Guided reinvention through progressive mathematisation
- Didactical phenomenology
- Self developed or emergent models

The dominant principle being explored and applied throughout this study is the principle of *guided reinvention through progressive mathematisation*. During the first and second cycles the author focused exclusively on this principle as this was the initial attempt to try and design and implement instructional tasks and activities, based on the RME theory but in a remedial context for low attaining Grade 8 learners. Examples of these activities are included in the boxes further on.

The principle of guided reinvention requires that well-chosen contextual problems be presented to learners that offer them opportunities to develop informal, highly context-specific solution strategies (Doorman, 2001). These informal solution procedures may then function as foothold inventions for formalisation and generalisation, a process referred to as "progressive mathematising" (Gravemeijer, 1994). The reinvention process is set in motion when learners use their everyday language (informal description) to structure contextual problems into informal or more formal mathematical forms (Armanto, 2002). The instructional designer therefore tries to compile a set of problems that can lead to a series of processes that together result in the reinvention of the intended mathematics (Doorman, 2001).

The idea is not that learners are expected to reinvent everything on their own, but that Freudenthal's concept of "guided reinvention" should apply (Freudenthal, 1973). This strategy should in turn allow learners to regard the knowledge they acquire as knowledge for which they have been responsible and which belongs to them. With guidance, the learners are afforded the opportunity to construct their own mathematical knowledge store on this basis. The word realistic in the RME theory does not indicate however that everyday contexts need to be continuously sought or used to motivate learners to reinvent the mathematics. Rather, the contexts selected for use in the process of instructional design should be experientially real for learners in order to act as a catalyst for progressive mathematisation (Gravemeijer, 1999).

Setting the scene

The case providing the context for the study discussed in this paper is a public, all girls secondary school (Grades 8-12) with approximately 1300 learners, in an urban area. The school has an excellent academic record regarding Grade 12 results and has a boarding facility and large grounds with an extensive array of extra mural activities on offer to learners in the afternoons. The school has a number of optional systems in place to provide learners with opportunities for extra mathematics lessons both during school as well as after school.

Extra mathematics support available after school

As part of the case study, a front end analysis (in the form of a survey) was carried out at the school to determine how many learners attend extra mathematics classes outside of school hours, their reasons for doing so and what options were available and attended by learners. The survey was completed by a total of 1026 learners (79%) and it was found that total of 41% of these learners who take mathematics indicated that they attend some form of extra mathematics classes. The breakdown per grade of learners attending extra mathematics classes did not vary significantly between the grades with the lowest frequency being Grade 8's (36%) and the highest percentage being Grade 10's (45%). Figure 3 indicates the reasons mentioned by learners for attending the classes, the most frequent reason cited being "to improve" followed by "to pass" and "because I don't understand the work in class."

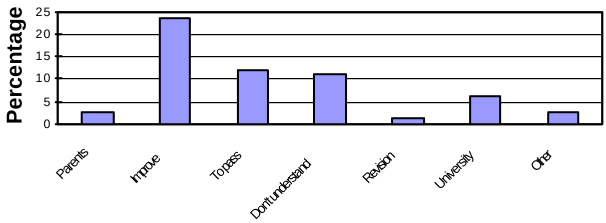


Figure 3: Reasons indicated by learners for attending the lessons

The study

Based on the frequency of learners attending extra mathematics classes, the pending implementation of mathematical literacy in accordance with the new curriculum (which has the potential to increase these frequencies) and the increasing number of learners joining the remedial classes, the decision was taken to conceptualise a study aimed at developing and implementing an intervention.

The immediate intended outcomes of the intervention are that it will improve the attitude and confidence of the participants to and in mathematics, and address perceived deficits in their conceptual understanding of place value, fractions and decimals, and basic algebra. It is then expected that if these immediate outcomes are achieved, that the long-term outcome of academic improvement related to these topics should in turn be demonstrated by the learners. In order to realise these outcomes, the following assumption was made:

> Alternative mathematical conceptions that may be hampering learners' understanding and performance in mathematics need to be elicited and addressed in order to improve their conceptual understanding.

Method

The research design is a case study, which Adelman refers to as "… the study of an instance in action" (Adelman, 1980 as cited in Cohen, Manion & Morrison, 2000). There are in fact 12 individual cases within this case study in the form of 12 learners who are being studied as they take part in the remedial intervention. At the school, there are ten Grade 8 mathematics classes, each yielding a small number of remedial learners. However, for practical reasons and due to time constraints, the intervention was only implemented with two of these small groups, which together currently yield a total of 12 learners. Two of the teachers at the school indicated a desire for their classes to be used, so these groups were "selected". The first class will be referred to as 8A and consists of 5 learners (originally 4) whereas the second class (8B) consists of 7 learners. All learners were identified by their teachers as needing remedial assistance in mathematics on the basis of their final mark for mathematics in Grade 7, their mathematics mark for their first term of Grade 8 and the results of a baseline assessment administered to all Grade 8 learners.

At the time of writing this paper (October, 2003), the first two cycles of fieldwork within the case study had been completed. The learners have been through a number of lessons (approximately 16 lessons of 45 minutes per class) pertaining to the topics of place value in whole numbers and decimals, and fractions. The third stage is currently underway and continues to deal with the concepts of fractions and decimals while also including some work on basic algebra.

Data collection

Both qualitative as well as quantitative data have and are still in the process of being collected. Attitude questionnaires as well as concept tests (to ascertain learners' conceptual understanding in terms of place value and calculation of whole numbers, fractions and decimals, and basic algebra) were given to all the remedial learners in Grade 8 prior to the start of the intervention. The intervention has been implemented in three cycles in accordance with the second, third and fourth academic school terms. Only the data for the first two cycles are included in the results. At the end of cycles one and two, learners wrote short post-tests which included the relevant items from the original pre-test, relating to the concepts dealt with during that cycle, as well as some additional items to obtain diagnostic information for the following cycle. Observation schedules from the classroom teacher, as well as from a research assistant (for some of the lessons) were collected approximately every third lesson, and the author also kept logs and observation schedules of the lessons. Samples of the learners' work completed during the lessons were often also reserved for data collection and analysis. The data discussed in this paper are limited to these latter mentioned samples.

The application of RME in the context of the case study

As suggested by the developmental research component of RME, cyclic processes of thought and instructional experiments (as presented in Figure 2), based on a combination of the author's experience and related literature on how learners learn mathematics (Gravemeijer & Cobb, 2002), were designed and implemented by the author in order to observe and analyse the actual process of learning that takes place by the remedial learners during the course of these lessons. Observations from each lesson (by the author as well as the class teacher and on occasions by an assistant researcher) provided valuable information that assisted the author in revising and adapting the instructional activities for the next class or lesson.

Although the RME approach is widely applied in "regular" mathematics classes (most commonly in The Netherlands and the United States), this study involves a remedial mathematics intervention being implemented with Grade 8 learners, as has already been mentioned. One of the challenges is that the learners have previously been taught the concepts that the intervention is focussing on, but still demonstrate a lack of conceptual understanding in this regard. This situation presents two main problems. Firstly they have some formal procedures in place for solving problems but are not sure when or why to use them with regard to the mathematical problems they encounter. Secondly they know they have been taught this work in previous grades and one has to avoid worsening their self-confidence by making them feel like they are continuously having to "relearn" work already done in several grades. The formal procedures that they have previously been taught and are aware of (and that should be a set of tools that they can choose from in order to solve the problem at hand) are often in fact disempowering these particular learners as they spend unnecessary time meaninglessly trying to search for the correct algorithm to apply to solve the problem. They know there is one algorithm or formula that can assist them in solving the problem but often when they select one, they cannot explain or justify their reasons for doing so. They find the horizontal mathematisation difficult. This has increased the challenge for the author in selecting appropriate contextual problems that encourage them to rather first try out their own informal strategies rather than immediately jumping into the vertical mathematisation process in search of an algorithm or procedure with little understanding of these processes. And it is here where their alternative conceptions become most evident. Examples of such alternative conceptions are discussed below in the learners' responses to the problems presented in the boxes.

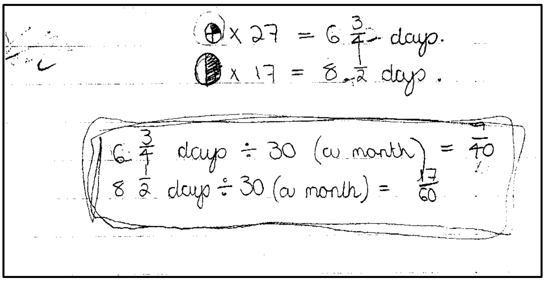
Results and discussion

The contextual problems and examples of learners' work portrayed in this discussion are all taken from a series of lessons administered during the second cycle of the intervention. During the lessons, the contextual problems were usually given to learners to explore, preceded by a brief discussion on the context and followed by a classroom discussion where various solutions offered by the learners were shared and discussed.

Box 1

My cat needs to take two types of pills and an insulin injection twice a day to control its diabetes. The cat takes half a big pill in the morning and again in the evening and a quarter of a small pill also in the morning and again in the evening. Firstly, the vet has given me 17 big pills (8A were told 18) and 27 small pills (8A were told 10) to start off with, how many days will these pills last me for before I have to go back to get more? Secondly, how many of each pill should I buy each month so that they last me for a whole month?

(Part A)



(Part B)

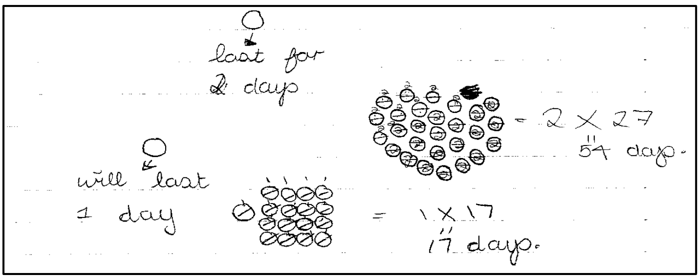


Figure 4: Example of Jackie's work

Note that in Jackie's first attempt, she is aware that she needs to multiply. The alternative conception is that she says: ¼ x 27and gets an answer of 6¾. This mistake in itself is evidence of the danger of learners focussing too much on vertical mathematisation. The concern is that this learner went straight to multiplying without considering that 27 pills, of which not even a whole one is taken per day, should at least last more than 27 days. When questioned about her answer and thought process and understanding of how many quarters there are in a whole, the learner realised her mistake and was left to try another strategy. When the author returned, the learner presented her reworked solution, which is demonstrated as part B of her displayed work above and which shows a more positive attempt at horizontal mathematisation.

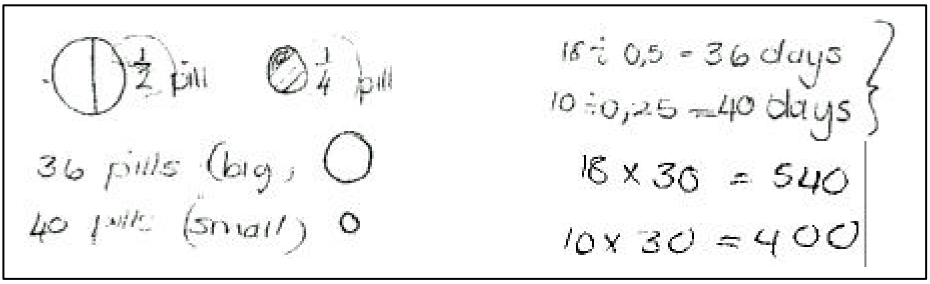


Figure 5: An example of Jennifer's work

In Jennifer's work in Figure 5, there are also examples of alternative conceptions coming to the fore. Firstly, if one looks at Jennifer's representation of ¼ in the diagram she drew, while she has divided the pill into four parts, these parts are not necessarily equal. The second alternative conception, which is not evident from the example as it now stands (as Jennifer erased and corrected this after discussion), is the following. Initially she wrote:

> 18 ÷ ½ = 15 (8A were told that there were 18 instead of 17 big pills from the vet). When questioned on how she got 15 as her answer, Jennifer demonstrated on her calculator how she had said: 18 ÷ 1,2 = 15. She explained how she thought that ½ and 1,2 were equal. When questioned on what she thought ½ meant, she replied that it meant 1 ÷2 and was asked by the author to try this on her calculator. She obviously then got 0,5 and not 1,2 and realised her mistake which she rectified. Jennifer also appears not to have a real grasp of the problem in that she repeatedly mixes up the pills and the number of days. Although she does reach the correct answer regarding the number of days the pills will last, when required to work out how many pills are needed for one month, she appears to resort to an algorithm without considering the context. She multiplies the number of pills initially handed out by 30, instead of multiplying the number of pills needed per day by 30.

Box 2

A recipe that you find for making apple tarts states that you need ¾ of an apple to make one apple tart. You want to make 10 apple tarts. How many apples do you need?

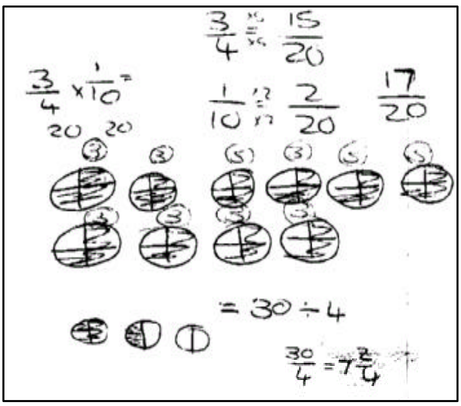
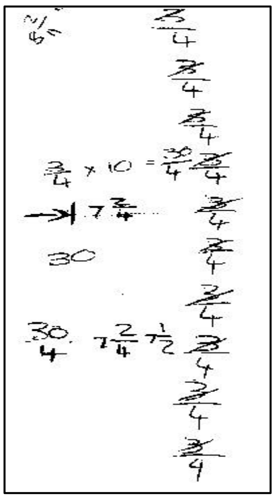


Figure 6: An example of Tembi's work



The solution (Fig. 6) belongs to Thembi. Academically she has scored the highest throughout the year on all the assessments done in their regular mathematics classes in relation to her other classmates taking part in the intervention. What is interesting about Thembi's solution here is how she takes herself through a process of trying quite an abstract level of vertical mathematisation, then resorting to horizontal mathematisation before moving onto a more basic attempt at vertical mathematisation. She starts of by applying an "algorithm" to try and get the answer. She realises she must make use of multiplication but multiplies by one tenth instead of ten. She then proceeds to make use of equivalent fractions to obtain a common denominator in order to add the fractions. She is not satisfied with her answer of $^{17}/_{20}$ and resorts to horizontal mathematisation in order to check this. She does this by drawing the number of apple tarts and counting the number of quarters required (a total of 30). She then correctly translates this into 7 whole apples and another two quarters of an apple being necessary. Immediately then she goes on to use a basic form of vertical mathematisation to verify this solution and by doing this corrects her own first mistake of multiplying by one tenth instead of ten. This in itself an explicit example of how a learner has used the process of horizontal mathematisation to check a solution and to then correct the answer initially reached. Resorting back to vertical mathematisation has then allowed the learner to identify and correct the source of the error.

In the example below (Fig. 7), Gabi correctly makes use of horizontal mathematisation by drawing the ten apple tarts. She then correctly establishes that there will be one whole additional (extra) apple resulting from every four tarts (as only ¾ of an apple is used in a tart leaving ¼ over from each of the four tarts) and a half an apple over from two tarts. For ten tarts she therefore has two and a half additional apples if she has ten apples. Her only mistake is then to incorrectly add this value onto 10 instead of taking it away. The benefit of this solution for both the learner and the teacher (or in this case the researcher) was that it was immediately obvious to ascertain where the mistake had been made and to then deal with this conception. Solutions of two other learners in the class yielded a similar error with slightly different visual representations.

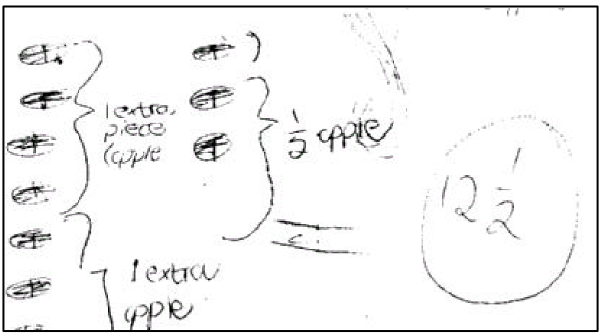


Figure 7: An example of Gabi's work

Box 3

You decide to start making banana bread to sell in order to earn some extra money. To start off with, you decide to make 5 loaves of banana bread. According to the recipe, each loaf requires 4 ½ bananas. How many bananas will you need to make the 5 loaves of banana bread? Show your working out in the space provided below.

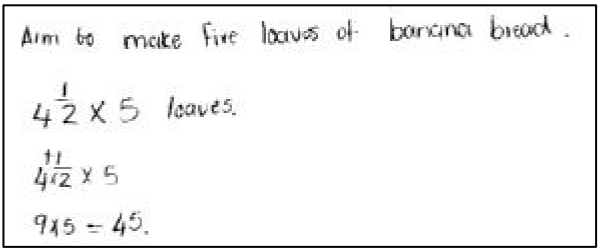
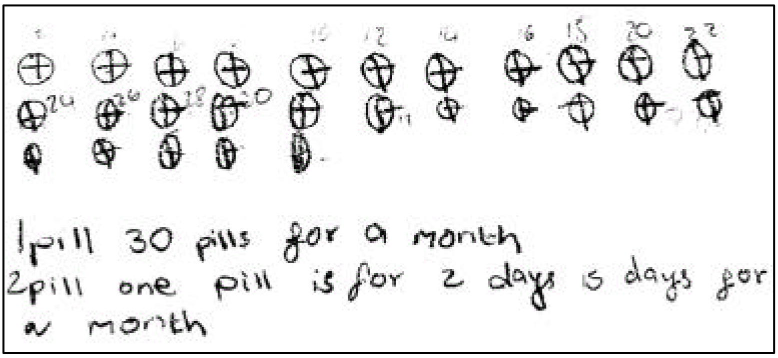
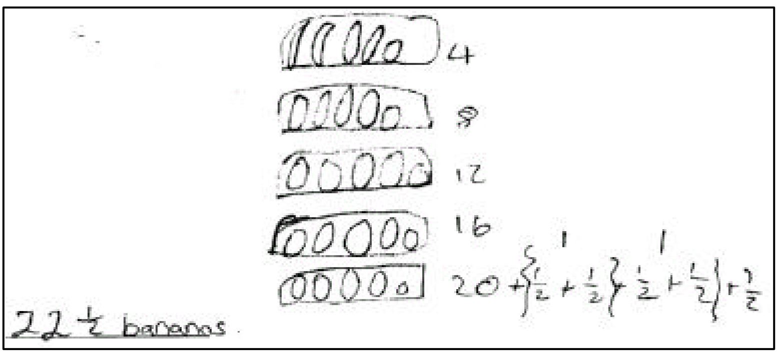


Figure 8: Jennifer's response to the "banana bread" problem.

As can be seen in Jennifer's solution in Figure 8, she correctly selects multiplication as her strategy. However, when she carries out the actual multiplication, she tries to change the mixed number into an improper fraction and in doing so she "loses" the denominator and gets 9 instead of $^{9}/_{2}$, rendering her final answer incorrect. In contrast to Jennifer's more formal solutions, examples of solutions provided by Phillipa in response to the problems in Boxes 1 and 3 are shown in Figure 9. The use of horizontal mathematisation is more evident in these responses of Phillipa whereas Jennifer more often resorts immediately to vertical mathematisation.



Solution to Box 1problem



Solution to Box 3 problem

Figure 9: Phillipa's responses to the problems

Phillipa and Jennifer both often demonstrated the same incorrect conceptions regarding fractions in the concept tests (see Figure 10 for an example). However, Phillipa's use of horizontal mathematisation appears to assist her more in overcoming this lack of understanding in order to still arrive at the correct solution when solving contextual problems involving fractions. Jennifer on the other hand has demonstrated a dependency on algorithms and an aversion to trying to understand and represent the problems in her own way throughout the intervention. Her biggest obstacle in fact has been the number of alternative conceptions in mathematics that she has brought with her. It is as though she has collected a "mental toolbox" of mathematical instruments over the years, without fully understanding how they really work or what they do. As soon as she attempts a mathematical problem, she immediately starts searching through her toolbox trying out the tools she thinks might be most suitable. This method is often done on a trial and error basis, and failing to find the right tool and not knowing which one to choose results in much frustration for her. She often found the correct tool though and then proceeded to operate it incorrectly, hence the alternative conceptions coming to the fore in any case, allowing for these to then be addressed.

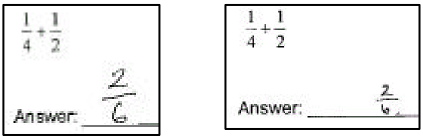


Figure 10: Phillipa and Jennifer's respective responses to an item from one of the concept tests

Conclusion

This paper has sought to investigate whether or not RME can play a role in eliciting and overcoming alternative conceptions held by low attaining learners in mathematics. In doing so the assumption is made that alternative mathematical conceptions, which may be hampering learners' understanding and performance in mathematics, need to be elicited and addressed in order to improve their conceptual understanding. By examining and comparing some examples of learners' responses to the contextual problems presented, emerging alternative conceptions and strategies to possibly overcome these through the use of RME were discussed, especially with reference to horizontal and vertical mathematisation. In the examples, learners who were able to make use of horizontal mathematisation during the process of reaching the solution appeared more capable of either identifying or avoiding errors that arose due to incorrect conceptions. On the other hand the vertical mathematisation also played a role in bringing alternative conceptions to the fore and assisting learners in identifying their errors so that they could correct them. It can therefore be concluded that RME has played a role in eliciting and addressing alternative conceptions of learners in this intervention. This has been done firstly through the application of the principle of guided reinvention in the design of contextual problems. These problems in turn initiated a process that saw learners engaging in horizontal and/or vertical mathematisation, which then resulted in alternative conceptions coming to the fore in order to be discussed and dealt with.

Acknowledgements

Many thanks to Tjeerd Plomp and Chris Breen for their valuable input and to Elsie Venter for her patient assistance with the quantitative data.

References

Armanto, D. (2002). *Teaching multiplication and division realistically in Indonesian primary schools: A prototype of local instruction theory.* Doctoral dissertation. The Netherlands, Enschede: University of Twente.

Cohen, L., Manion, L. & Morrison, K. (2000). *Research methods in education, 5th Edition.* London, UK:

RoutledgeFalmer.

Doorman, M. (2001). How to guide students? A reinvention course on modeling movement. *Paper presented at 'The Netherlands and Taiwan conference on common sense in mathematics education', Taipei, Taiwan*. Retrieved from the World Wide Web on 22/04/03 from: (http://www.fi.ruu.nl/en/publications.shtml)

Fauzan, A. (2002). *Applying Realistic Mathematics Education (RME) in teaching geometry in Indonesian primary schools.* Doctoral dissertation. The Netherlands, Enschede: University of Twente.

Freudenthal, H. (1973). *Mathematics as an educational task.* The Netherlands, Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education.* The Netherlands, Dordrecht: Kluwer Academic.

Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education.* The Netherlands, Utrecht: Freudenthal Institute.

Gravemeijer, K.P.E. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155-177.

Gravemeijer, K & Cobb, P. (2002). Designing classroom-learning environments that support mathematical learning. Presented at the *American Educational Research Association* conference in April 2001.

Rasmussen, C.L. and King, K.D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*, 161-173.

Treffers, A. (1987). *Three Dimensions. A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project.* The Netherlands, Dordrecht: Reidel.