On Clifford $\mathcal{A}$-algebras and a framework for localization of $\mathcal{A}$-modules

by

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Declaration

I, the undersigned, declare that the thesis, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

Signature:

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Dedication

This thesis is dedicated to *Mekidy*, *Abigo*, and *Amy*.

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Contents

Abstract

Nowadays, Clifford $\mathcal{A}$-algebras are hot areas of research due to their *applicability to different disciplines; capacity to create relationship between quadratic and linear $\mathcal{A}$-morphisms; and relation to tensor $\mathcal{A}$-algebras*. In this work, we investigate *the commutative property of the Clifford functor on sheaves of Clifford algebras, the natural filtration of Clifford $\mathcal{A}$-algebras, and localization of vector sheaves*; out of which two papers are extracted for publication [43], [44]. To present the thesis in a coherent way, we organize the thesis in five chapters.

Chapter 1 is a part where relevant classical results are reviewed. Chapter 2 covers basic results on Clifford $\mathcal{A}$-algebras of quadratic $\mathcal{A}$-modules (which are of course results obtained by Prof. PP Ntumba [42]).

In Chapter 3, we discuss the commutativity of the *Clifford functor $\mathcal{C}l$* and *the algebra extension functor (through the tensor product)* of the ground algebra sheaf $\mathcal{A}$ of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$. We also observe the existence of an isomorphism between the functors $\mathcal{S}^{-1}$ and $(\mathcal{S}^{-1}\mathcal{A}) \otimes \_$. As a particular case, we show the commutativity of the Clifford functor $\mathcal{C}l$ and the localization functor $\mathcal{S}^{-1}$. A discussion about the localization of $\mathcal{A}$-modules at *prime ideal subsheaves* and at *subsheaves induced by maximal ideals* is also included.

In Chapter 4, we study two main $\mathcal{A}$-isomorphisms of Clifford $\mathcal{A}$-algebras: *the main involution* and *the anti-involution $\mathcal{A}$-isomorphisms,* which split Clifford $\mathcal{A}$-algebras into *even* sub-$\mathcal{A}$-algebras and sub-$\mathcal{A}$-modules of *odd* products. Next, we give a definition for *the natural filtration of Clifford $\mathcal{A}$-algebras* and show that for every $\mathcal{A}$-algebra sheaf $\mathcal{E}$, endowed with a regular filtration, one obtains a new graded $\mathcal{A}$-algebra sheaf, denoted $Gr(\mathcal{E})$, which turns out to be $\mathcal{A}$-isomorphic to $\mathcal{E}$.

A conclusive remark and list of research topics that can be addressed in connection with this research do appear in Chapter 5.

Chapter 1

Preliminary Concepts and Results

1.1 Introduction

In this chapter we review classical concepts and results regarding different algebras (tensor, symmetric, exterior, and Clifford algebras), *quadratic modules*, and *localization of rings* and modules. The results raised in this chapter are ground work for the results obtained in the next chapters in the setting of sheaves of algebraic structures. The notions of *gradation*, *filtration*, *opposite and twisted algebras*, and *graded isomorphisms* are also addressed here so as to lay down a sound background to the central concept, construction and fundamental properties of *Clifford algebras*. Definitions and results stated in the form of remarks and propositions are adopted from the materials cited. We don't include the proofs of most of the results in this chapter except the proofs of the results related to localization.

1.2 Algebras

Algebras are rings with a compatible vector space or module structure. This section uses a natural and intuitive way to introduce algebras and reviews algebras with universal properties such as tensor, symmetric, and exterior algebras which are, of course, related to Clifford algebras.

1.2.1 Definitions and Basic Properties

The points in this section are adopted from [22, pp. 477-479], [50, pp. 28-30], and [53, pp. 27-29]. More detailed information about the proofs of the results can be found from these materials.
Let K be a commutative ring with unity. An abelian group A which has a structure of both an associative ring and a K-module where the property

$$\lambda(xy) = (\lambda x)y = x(\lambda y)$$

is satisfied for all $\lambda \in K$ and $x, y \in A$ is called an *associative algebra*. We say that A is unital if it contains an element 1 such that $1.x = x = x.1$ for all $x \in A$. A set $B \subset A$ is said to *generate* A if every element of A can be represented as a linear combination of products of elements of B. The *center* of an algebra A is the set $\mathbb{Z}(A) = \{a \in A | \ ab = ba \textit{ for all } b \in A\}$. The center is a commutative *subalgebra containing* K. An algebra over K is said to be *central* if its center coincides with K.

Remark 1.1

i) One can also define an algebra A over K as follows: It is an associative ring A together with a non-zero ring homomorphism $\phi : K \longrightarrow A$ such that *a)* $\phi(K) \subseteq \mathbb{Z}(A)$, i.e., $\phi(k).a = a.\phi(k)$, for all $k \in K$, $a \in A$.

b) The map $K \times A \longrightarrow A : (k, a) \longmapsto \phi(k).a$ turns A into a K-vector space when K is a field and A has a unity.

ii) Since ϕ (from the above item) is non-zero, it follows that ϕ is injective when K is a field, i.e., $K \cong Im(\phi)$. Thus, K may be considered as a subalgebra of A.

iii) An algebra over K is called finitely generated if it is finitely generated as a module over K.

iv) A K-subspace I of A is called a left ideal (resp., a right ideal) of A if for any $a \in A$ and $x \in I$ one has $a.x \in I$ $(resp., x.a \in I)$. I is called a two-sided ideal if it is both a left and right ideal.

v) If I is a two-sided ideal of A, the factor ring (quotient ring) A/I has a K-algebra structure induced by A. This algebra is called a factor (quotient) algebra.

vi) A K-algebra A is called left-Noetherian if it satisfies the ascending chain condition on the left ideals, that is, for any increasing sequence of left ideals $I_1 \subseteq I_2 \subseteq \cdots$, there exists a number n such that $I_k = I_n$ for all $k \geq n$. An analogous definition works for a right-Noetherian algebras. A Noetherian algebra is an algebra which is both left and right Noetherian.

1.2.2 Tensor, Symmetric and Exterior Algebras

Tensor algebra is useful in giving a uniform description of all objects of linear algebra and even in arranging them in algebraic structure [53, pp. 295]. The tensor algebra of an R-module M, denoted $T(M)$, is the algebra of tensors on M (of any rank) with multiplication being the tensor product.

Definition 1.1 Let R be a commutative ring and M be an R-module. For each $k \geq 1$, let $T^k(M) = \underbrace{M \otimes_R \cdots \otimes_R M}_{ktimes}$ and set $T^0(M) = R$. Define

$$T(M) = R \oplus T^1(M) \oplus T^2(M) \oplus \cdots = \oplus_{k=0}^{\infty} T^K(M).$$

$T(M)$ is an R-module and if the ring multiplication is defined as

$$(m_1 \otimes m_2 \otimes \cdots \otimes m_i)(m_1^{'} \otimes m_2^{'} \otimes \cdots \otimes m_j^{'}) = m_1 \otimes m_2 \otimes \cdots \otimes m_i \otimes m_1^{'} \otimes m_2^{'} \otimes \cdots \otimes m_j^{'},$$

$T(M)$ becomes an R-algebra. The algebra $T(M)$ is called the *tensor algebra of M* .

Explicitly, the tensor algebra satisfies the following universal property, which formally expresses the statement that it is the most general algebra containing M.

Proposition 1.1 Universal property of tensor algebras [50, pp. 93-94, Prop. 5.2.2] *Let M be an R-module and A an R-algebra. Then for any R-module homomorphism $\varphi : M \longrightarrow A$ there exists a unique R-algebra homomorphism $\psi : T(M) \longrightarrow A$ such that $\psi|_M = \varphi$. That is the following diagram commutes.*

$$\begin{array}{ccc} M & \longrightarrow & T(M) \\ & \searrow^{\varphi} & \downarrow^{\psi} \\ & & A \end{array}$$

The above universal property shows that the construction of the tensor algebra is functorial in nature. That is, T is a functor from the category of R-modules to the category of R-algebras. The functoriality of T means that any R-module homomorphism from M to N extends uniquely to an algebra homomorphism from $T(M)$ to $T(N)$.

Proposition 1.2 [22, p. 483, Proposition 16.2.5] *Let M be a free R-module of rank n with basis $e_1, \cdots, e_n$. Then, $T(M)$ is a free R-module with a basis which consists of all tensors $e_1 \otimes \cdots \otimes e_n$.*

Remark 1.2 Since every element of $T(M)$ is a finite sum of elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_n$, as an algebra, it is generated by $T^0(M) = R$ and $T^1(M) = M$.

From the universal property of the tensor algebra, if we consider only symmetric or skew-symmetric multilinear maps, we will arrive at the notions of the symmetric or the exterior algebra of M, respectively, which are discussed in the next sections.

The *symmetric algebra* $S(M)$ of an R-module M is an algebra that is constructed from the tensor algebra $T(M)$ by taking the quotient algebra of $T(M)$ with its two-sided ideal $\mathfrak{I}(M)$ generated by all differences of products $m_1 \otimes m_2 - m_2 \otimes m_1$ where m_1 and m_2 are in M.

Definition 1.2 Let M be an R-module, $T(M)$ its tensor algebra and $\mathfrak{I}(M)$ the ideal of $T(M)$ generated by elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$ for $m_1, m_2 \in M$. The quotient algebra $S(M) = T(M)/\mathfrak{I}(M)$ is called the *symmetric algebra* of M.

Remark 1.3 *The following results follow from the definition.*

i) [54, p. 66, Proposition 2.14] $S(M)$ is a commutative R-algebra. $S(M)$ is *generated as a ring* by $S^0(M)$ and $S^1(M)$, which are the respective images of $T^0(M)$ and $T^1(M)$. Note that $S^0(M)$ lies in the center of $S(M)$ (which actually follows from the fact that $T^0(M)$ lies in the center of $T(M)$) and, by construction, any two elements of $S^1(M)$ commute. Thus, $S(M)$ is generated by a set of pairwise commuting elements, and therefore $S(M)$ is commutative.

ii) [50, p. 97, Corollary 5.3.6] Let M be a free R-module of rank n with basis $e_1, \cdots, e_n$. Then $S(M)$ is isomorphic to the polynomial algebra in n commuting variables over R, which is $R[x_1, \cdots, x_n]$.

The Grassmann or the exterior algebra is constructed like the symmetric algebra where the symmetric property of the corresponding multilinear map is now replaced by the skew-symmetric property.

Definition 1.3 Let M be an R-module, $T(M)$ its tensor algebra and $\mathfrak{J}(M)$ the ideal of $T(M)$ generated by elements of the form $m \otimes m$ for any $m \in M$. The quotient algebra

$$\bigwedge(M) := T(M)/\mathfrak{J}(M).$$

is called the *exterior algebra* of M.

Remark 1.4 [50, pp. 98-99, Properties 5.4.3]

i) The exterior product $m_1 \wedge \cdots \wedge m_n$ is the image of $m_1 \otimes \cdots \otimes m_n$ under the natural surjection $T^n(M) \longrightarrow \bigwedge^n(M)$.

ii) The is alternating on elements of M, which means that $m \wedge m = 0$ for all $m \in M$.

iii) The exterior product is anticommutative on the elements of M. Indeed, if $m, n \in M$, then

$$0 = (m + n) \wedge (m + n) = m \wedge m + m \wedge n + n \wedge m + n \wedge n = m \wedge n + n \wedge m.$$

Therefore,

$m \wedge n =$ - $(n \wedge m)$.

Proposition 1.3 *Let M be a free R-module of rank n with basis $e_1, \cdots, e_n$. Then the following hold:*

i) [22, p. 487, Proposion 16.4.4] *Let $k \geq 1$. $\bigwedge^k(M)$ is a free R-module with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ where $i_1 < \cdots < i_k$. In particular, its rank is given by $rk(\bigwedge^k(M)) = \begin{pmatrix} n \\ k \end{pmatrix}$.*

ii) [54, p. 80, Theorem 3.1] *$\bigwedge(M)$ is a free R-module of rank 2^n.*

1.2.3 Grading and Filtration in Algebras

The definitions and results mentioned here can be found in [22] and [50].

Definition 1.4 Let R be a ring. A *gradation* on R is a decomposition

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots = \bigoplus_{n=0}^{\infty} R_n$$

of R as a direct sum of subgroups R_n such that $R_i R_j \subseteq R_{i+j}$ for all i and j. A ring with a gradation is called a *graded ring*.

Remark 1.5

i) The group R_n is called the *homogeneous component* of R of degree n, and elements of R_n are called *homogeneous elements* of degree n. Every $r \in R$ has a unique expression $r = \sum_{n \geq 0} r_n$ with $r_n \in R_n$ for every n and $r_n = 0$ for almost all n. This decomposition is called the *homogeneous decomposition* of r, and r_n is called the *homogeneous component* of r of degree n.

ii) [50, p. 36, Lemma 2.8.3] For an ideal $\mathfrak{a}$ of a graded ring R, the following three conditions are equivalent:

1) For every $a \in \mathfrak{a}$, all homogeneous components of a belong to $\mathfrak{a}$.

2) $\mathfrak{a} = \bigoplus_{n \geq 0}(\mathfrak{a} \cap R_n)$

3) $\mathfrak{a}$ is generated (as an ideal) by homogeneous elements.

Definition 1.5 A *left module* M over a graded ring R such that $M = \bigoplus_{n=0}^{\infty} M_n$, which is a direct sum of subgroups M_n, and $R_i M_j \subseteq M_{i+j}$ is called a *graded module*. A *graded module* that is also a *graded ring* is called a *graded algebra*. A *graded ring* could also be viewed as a *graded $\mathbb{Z}$-algebra*.

Examples

The following are the most common examples of graded algebras.

i) *Polynomial rings:- The homogeneous elements of degree n are exactly the homogeneous polynomials of degree n.*

ii) *The tensor algebra $T(M)$ of a module M:- The homogeneous elements of degree n are the tensors of rank n, $T^n(M)$.*

iii) *The exterior algebra $\bigwedge(M)$ and symmetric algebra $S(M)$ are also graded algebras.*

Remark 1.6 *The following results follow as a natural consequence of the definition.*

i) A graded ring is a graded module over itself. A submodule of a module which is graded is called a graded submodule. An ideal in a graded ring is homogeneous if and only if it is a graded submodule. A subring is, by definition, a graded subring if it is a graded submodule. The annihilator of a graded module is a homogeneous ideal.

ii) Any (non-graded) ring R can be given a gradation by letting $R_0 = R$ and $R_i = 0$ for $i > 0$. This is called the trivial gradation on R.

iii) Let K be a fixed commutative ring. A *superalgebra* over K is a K-module A with a direct sum decomposition $A = A_0 \bigoplus A_1$ together with a bilinear multiplication $A \times A \longrightarrow A$ such that $A_iA_j \subseteq A_{i+j}$ where the subscripts are read modulo 2. A superring, or $\mathbb{Z}_2$-graded ring, is a superalgebra over the ring of integers $\mathbb{Z}$. The elements of A_i are said to be homogeneous. The parity of a homogeneous element x, denoted by $|x|$, is 0 or 1 according to whether it is in A_0 or A_1. Elements of parity 0 are said to be even and those of parity 1 to be odd. If x and

y are both homogeneous then so is the product xy and $|xy| = |x| + |y|$. The identity element in a unital superalgebra is necessarily even. A commutative superalgebra is one which satisfies a graded version of commutativity. Specifically, A is commutative if $yx = (-1)^{|x||y|}xy$ for all homogeneous elements x and y of A.

Definition 1.6 The *parity grading* of a module M (resp. an algebra A) is merely a decomposition of M (resp. an algebra A) into a direct sum of submodules M_0 and M_1 (resp. submodules A_0 and A_1), the elements of which are respectively called *even* or *odd*.

Remark 1.7 Let A be a superalgebra over a commutative ring K. The submodule A_0, consisting of all even elements, is closed under multiplication and contains the identity of A and therefore forms a subalgebra of A, naturally called the *even subalgebra*. It forms an ordinary algebra over K.

In algebra, *filtration* refers to a family $\{S_i : i \in I\}$ of subobjects of a given algebraic structure S, with the index i running over some index set I that is a totally ordered set and subject to the condition that if $i \leq j$ in I then $S_i \subseteq S_j$.

Definition 1.7 A *filtered algebra* over the field K is an algebra A over K which has an *increasing sequence* $\{0\} \subset A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots \subset A$ of subspaces of A such that

$$A = \bigcup_{i \in \mathbb{N}} A_i$$

and is *compatible* with the multiplication in the following sense

$$\forall m, n \in \mathbb{N},\ A_m \cdot A_n \subset A_{n+m}.$$

Examples

i) Any graded algebra A graded by $\mathbb{N}$, for example $A = A_0' \oplus A_1' \oplus A_2' \oplus \cdots = \bigoplus_{n=0}^{\infty} A_n'$, has a filtration given by $A_n = \bigoplus_{i=0}^{n} A_i'$.

ii) One can construct a graded algebra from a filtered algebra. Indeed if A is a filtered algebra, with filtration $\{0\} \subset A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots \subset A$, the associated graded algebra, say $\mathcal{G}(A)$, can be constructed as follows: Define $\mathcal{G}(A) = \bigoplus_{n\in\mathbb{N}} G_n$, where, $G_0 = A_0$, and $\forall n > 0, \quad G_n = A_n/A_{n-1}$, and product in $\mathcal{G}(A)$ is defined by $(x + A_{n-1})(y + A_{m-1}) = x \cdot y + A_{n+m-1}$ for all $x \in A_n$ and $y \in A_m$. It is easy to see that the new multiplication is well defined and endows $\mathcal{G}(A)$ with the structure of a graded algebra, with gradation $\{G_n\}_{n\in\mathbb{N}}$. Furthermore if A is associative then so is $\mathcal{G}(A)$ and if A is unital $\mathcal{G}(A)$ will be unital as well.

The following definition is found in [25].

Definition 1.8 Let A be an algebra and $(A^{\leq k})_{k\in\mathbb{Z}}$ be a family of submodules of A.

1. The family is called an *increasing filtration* of A if

i) Every $A^{\leq k}$ is contained in $A^{\leq k+1}$;

ii) $A^{\leq j}A^{\leq k} \subseteq A^{\leq j+k}$; and

iii) $1_A \in A^{\leq 0}$.

2. An increasing filtration $(A^{\leq k})_{k\in\mathbb{Z}}$ is said to be *regular* if $\bigcap_{k\in\mathbb{Z}} A^{\leq k} = 0$ and $\bigcup_{k\in\mathbb{Z}} A^{\leq k} = A$.

3. An increasing filtration $(A^{\leq k})_{k\in\mathbb{Z}}$ is said to be *natural* if $A^{\leq k} = \{0\}$ whenever k is a negative integer.

In the chapters to come, especially in Chapter 4, we are going to explore different results related to filtration in the framework of Clifford $\mathcal{A}$-algebras.

1.2.4 Opposite and Twisted Algebras

Our main references for this section are [2] and [25].

Definition 1.9 The *opposite algebra* A^o or A^{op} of an algebra A is the algebra with the same set of elements and the same addition but with multiplication $*$, given by $a * b = ba$ for a and b in A.

Remark 1.8

i) If K is the base ring for the algebra A then the K-dual H$om_K(M, K)$ for a right module M over A is a right module over the opposite algebra A^o of A.

ii) Every algebra morphism $\varphi : A \longrightarrow B$ gives rise to an algebra morphism $\varphi^o : A^o \longrightarrow B^o$ defined by $\varphi^o(x^o) := \varphi(x)^o$.

Definition 1.10 The *twisted algebra* A^t of a graded algebra A is the algebra defined by the same set of elements and the same operations except that multiplication is defined as $x^t y^t = (-1)^{|x||y|}(xy)^t$, where $|x|$ and $|y|$ stand for the parity of x and y, respectively.

Remark 1.9

i) A *twisted opposite algebra* A^{t_o} of an algebra A is constructed in such a way that both the properties of opposite and twisted are satisfied. That is A^{t_o} is defined by the same set of elements and the same operations except that multiplication is defined as $x^{t_o}y^{t_o} = (-1)^{|x||y|}(yx)^{t_o}$, where $|x|$ and $|y|$ stand for the parity of x and y, respectively.

ii) $(x^o)^o$ and $(x^t)^t$ is identified with x.

iii) If A is graded, bothA^o, A^t, and A^{t_o} are also graded.

iv) If $\varphi : A \longrightarrow B$ is a graded algebra morphism, then

a) $\varphi^o : A^o \longrightarrow B^o$ defined by $\varphi^o(x^o) := \varphi(x)^o$;

b) $\varphi^t : A^t \longrightarrow B^t$ defined by $\varphi^t(x^t) := \varphi(x)^t$; and

c) $\varphi^{t_o} : A^{t_o} \longrightarrow B^{t_o}$ defined by $\varphi^{t_o}(x^{t_o}) := \varphi(x)^{t_o}$.

are also graded algebra morphisms of the respective algebras.

v) The set $\{A, A^o, A^t, A^{t_o}\}$ forms a group of order four under the binary operation $\star$ defined by the following table

$\star$	A	A^t	A^o	A^{t_o}
A	A	A^t	A^o	A^{t_o}
A^t	A^t	A	A^{t_o}	A^o
A^o	A^o	A^{t_o}	A	A^t
A^{t_o}	A^{t_o}	A^o	A^t	A

Definition 1.11 Let A and B be two graded algebras. The *twisted tensor product* (or *graded tensor product*) of A and B, denoted by $A\hat{\otimes}B$, is the algebra defined by the same set of elements and the same operations as $A \otimes B$ except that multiplication is defined as

$$(x \otimes y)(x' \otimes y') = (-1)^{|x'||y|}(xx' \otimes yy').$$

Remark 1.10 The identity element of the associative algebra $A\hat{\otimes}B$ is $1_A \otimes 1_B$. When the characteristic of K is 2, $A\hat{\otimes}B= A \otimes B$.

1.2.5 Graded Isomorphisms

Definition 1.12 Let $A = \bigoplus_{n=0}^{\infty} A_n$ and $B = \bigoplus_{n=0}^{\infty} B_n$ be graded R-algebras and $\eta : A \longrightarrow B$ be a map between them. η is called a *graded R-algebra homomorphism* if it is an R-algebra homomorphism that *respects grading*, i.e., $\eta(A_n) \subseteq B_n$ for each n in the indexing set.

Remark 1.11 Every graded algebra with *parity grading* $A = A_0 \oplus A_1$ admits a graded automorphism φ such that $\varphi(x) = (-1)^{|x|}x$ for every homogeneous x.

Definition 1.13 An *automorphism* of an algebra A is a *linear isomorphism* $\varphi : A \longrightarrow A$ such that $\varphi(ab) = \varphi(a)\varphi(b)$. An *antiautomorphism* of an algebra A is a linear isomorphism $\alpha : A \longrightarrow A$ such that $\alpha(ab) = \alpha(b)\alpha(a)$ for all $a, b \in A$. An (anti)automorphism α is an *involution* if $\alpha^2 = id$.

Examples

1) Any algebra A over a commutative ring K may be regarded as a purely even superalgebra over K; that is, by taking A_1 to be trivial.

2) Any $\mathbb{Z}$ or $\mathbb{N}$-graded algebra may be regarded as superalgebra by taking the grading modulo 2. This includes examples such as tensor algebras and polynomial rings over K.

3) In particular, any exterior algebra over K is a superalgebra. The exterior algebra is the standard example of a supercommutative algebra.

1.3 Clifford Algebras

1.3.1 Introduction

Clifford algebras that developed to generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems. They are intimately connected with quadratic forms and orthogonal transformations.Indeed, the motive for their creation is to address problems that require finding a K-algebra that contains a quadratic module (M, q) in which q looks like a square, i.e., problems that seek a K-linear map $f : M \longrightarrow A$, where A is a unital associative K-algebra, such that $f(m)^2 = q(m)1_A$ for each $m \in M$. The most familiar Clifford algebra that is developed so far is the *Orthogonal Clifford algebra*, which is most commonly referred as *Riemannian Clifford algebra*. In this section we shall present a short review of the historical development and applications of Clifford algebras, define the Clifford algebra of a quadratic module, and see some of the basic results related to the Clifford algebra of a quadratic module. The references mostly used for this section are [34, pp.237-238], [18, pp.61-94], [2, pp.5-6], [17], [25], [20, pp.6-8], [47, pp.22-26], [28] and [49, pp.10-12].

1.3.2 Historical Remarks on Clifford Algebras

Clifford algebras were discovered by William Kingdon Clifford $(in\ 1878)$ as part of his search for a generalization of the quaternions. He considered an algebra generated by $V = \mathbb{R}^n$ subject to the relation $v^2 = -\|v\|^2$ for all $v \in V$. Lipschitz $(in\ 1886)$ was the first to define groups constructed from the numbers introduced by Clifford and use them to represent rotations in an Euclidean space. Dirac $(in\ 1928)$, in his work on the relativistic wave equation of the electron, introduced matrices that provide a representation of the Clifford algebra of Minkowski space. Brauer and Weyl $(in\ 1935)$ connected the Clifford and Dirac ideas with the representations developed for Lie algebras to find the spinorial and projective representations of the orthogonal groups in any number of dimensions.

1.3.3 Applications of Clifford Algebras

Clifford algebras have important applications in a variety of fields such as differential geometry, theoretical physics, cybernetics, robotics, image processing, engineering, computer aided design (CAD), computer aided manufacturing (CAM), computer graphics, etc. This is so because geometric algebra (a Clifford algebra over $K = \mathbb{R}$) is a powerful mathematical tool that offers a natural and direct way to model geometric objects and their transformations. It makes geometric objects (points, lines and planes) into basic elements of computation and defines few universal operators that are applicable to all types of geometric elements. If we trace back the historical development of the geometrical modeling (via algebra) of the external world so as to process using computers, it was linear algebra that was applied for the said purpose. But this approach ends up with a limitation that all geometric objects have to be represented either by vectors and/or matrices. This assumption had created a separation between geometric reasoning and matrix-based algorithms, which in turn, had led to implementation errors of the programs developed at that time. To fill this limitation researchers suggested geometric algebra (Clifford Algebra) as an alternative for linear algebra because of its natural and intuitive way to model and manipulate the geometric objects by way of unified treatment of subspaces of any dimensionality which makes many operators of the algebra universally applicable to all types of elements. Furthermore, several mathematical theories, such as projective geometry, complex numbers and the quaternions, are naturally integrated in geometric algebra providing a unified framework.

As stated in [26], the applications of Clifford algebra have an enormous range. In the same paper an overview has been given on its applications in areas of neural computing, image and signal processing, computer and robot version, control problems and other areas that have been developed over the past 15 years. Interested readers are referred to [26] for detailed information.

1.3.4 Quadratic Modules

The reader is referred to [25] for most of the sequel.

Definition 1.14 *1)* Let K be a commutative ring with unity and M a K-module. A mapping $q : M \longrightarrow K$ is called a *quadratic form*on M if

i) $q(km) = k^2 q(m)$ for all $m \in M$ and $k \in K$; and

ii) $b_q : M \times M \longrightarrow K$ defined by $b_q(m, n) := q(m + n) - q(m) - q(n)$ is K-bilinear.

$2)$ M together with q, denoted by (M, q), is called a *quadratic module* and b_q is called the *associated bilinear mapping*.

Remark 1.12

i) b_q is *symmetric*.

ii) Every vector space over K becomes a quadratic space w.r.t. the trivial quadratic form $q \equiv 0$.

iii) Condition (ii) of the definition of a quadratic form defines an *inner product* on $M \times M$.

iv) If $Char(K) \neq 2$, (M, q) be a free quadratic module and (e_i) a basis for M. Then

$$q(m) = \tfrac{1}{2} \textstyle\sum_{i,j} b_q(e_i, e_j) m_i m_j, \quad \text{where } m = \textstyle\sum_j m_j e_j$$

and if there is a basis which is b_q-orthogonal, the expression for $q(m)$ reduces to diagonal form

$$q(m) = \textstyle\sum_i q(e_i) m_i^2, \quad \text{where } m = \textstyle\sum_i m_i e_i.$$

v) As long as $Char(K) \neq 2$ we can construct q from b_q since the equality $b_q(m, m) = q(2m) - 2q(m) = 2q(m)$ yields $q(m) = \frac{1}{2} b_q(m, m)$. It is easy to check that if $Char(K) \neq 2$ and $b_q : M \times M \longrightarrow K$ is any symmetric K-bilinear form, then $q : m \longmapsto \frac{1}{2} b_q(m, m)$ is a quadratic form whose associated symmetric bilinear form is b_q.

vi) With every symmetric bilinear mapping $b_q : M \times M \longrightarrow K$ we can associate a linear mapping b_q^* from M into $\mathrm{H}om_K(M, K)$ defined by $b_q^*(m)(m') := b_q(m, m')$. And with every quadratic form $q : M \longrightarrow K$ we can associate $q^* : M \longrightarrow \mathrm{H}om_K(M, K)$ defined by $q^*(m)(m') := b_q(m, m')$. We say that q (resp. b_q) is nondegenerate if q^* (resp. b_q^*) is bijective, and we say that q (resp. b_q) is weakly nondegenerate if q^* (resp. b_q^*) is injective. When K is a field and M is a finite dimensional vector space, every weakly nondegenerate quadratic form is nondegenerate; but if $dim(M)$ is infinite, no quadratic form on M is nondegenerate even if it is weakly nondegenerate.

Recall that a sequence of two linear mappings φ and ψ:

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \ ;$$

is called an exact sequence if $Im(\varphi) = Ker(\psi)$. A sequence of several mappings, or an infinite sequence of mappings, is said to be exact if all the subsequences of two consecutive mappings are exact. A functor F on the category of K-modules Mod_K into itself is exact if it transforms every exact sequence into an exact sequence. The functors Hom and $\bigotimes$ are not exact for all rings K. Instead the former is left exact and the latter is right exact. This means that for all modules P and all exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \ ,$$

$$N' \longrightarrow N \longrightarrow N'' \longrightarrow 0 \ ,$$

we get these exact sequences;

$$0 \longrightarrow \mathrm{H}om(N'', P) \longrightarrow \mathrm{H}om(N, P) \longrightarrow \mathrm{H}om(N', P) \ ,$$

$$0 \longrightarrow \mathrm{H}om(P, M') \longrightarrow \mathrm{H}om(P, M) \longrightarrow \mathrm{H}om(P, M'') \ ,$$

$$P \bigotimes N' \longrightarrow P \bigotimes N \longrightarrow P \bigotimes N'' \longrightarrow 0 \ .$$

Definition 1.15 A K-module P is called *injective* if the functor H$om(..., P)$ is exact; it is called *projective* if the functor H$om(P, ...)$ is exact; and is called *flat* if the functor $P \bigotimes ...$ is exact.

Remark 1.13 Because of the left exactness of the functor Hom and the right exactness of the functor $\bigotimes$, we get at once the following statements:

i) P is injective iff the mapping H$om(N, P) \longrightarrow$ H$om(N', P)$ is surjective whenever $N' \longrightarrow N$ is injective.

ii) P is projective iff the mapping $\mathrm{H}om(P, M) \longrightarrow \mathrm{H}om(P, M'')$ is surjective whenever $M \longrightarrow M''$ is surjective.

iii) P is flat iff the mapping $P \bigotimes N' \longrightarrow P \bigotimes N$ is injective whenever $N' \longrightarrow N$ is injective.

Definition 1.16 A module M is called *finitely presented* if it is finitely generated and if there exists a surjective morphism $\varphi : P \longrightarrow M$ such that P is projective and $Ker(\varphi)$ is finitely generated.

Definition 1.17 Let (M, q) be a quadratic module where b_q is the associated bilinear mapping. Then

1) Two elements x and y of M are said to be *orthogonal* (with respect to q or b_q) if $b_q(x, y) = 0$.

2) For every subset P of M, the set $P^{\perp} := \{x \in M : x$ is orthogonal to all the elements of $P\}$ is called the *submodule of M orthogonal to P*.

3) If (M', q') is another quadratic K-module, their *orthogonal sum* $(M, q) \perp (M', q')$ is the couple $(M \oplus M', q \perp q')$, where $q \perp q'$ is the quadratic mapping on $M \oplus M'$ defined in such a way that:

$$(q \perp q')(x, x') = q(x) + q'(x')$$

Remark 1.14 *i)* Let P be a submodule of M. If the restriction of b_q is nondegenerate, then $M = P \oplus P^{\perp}$.

ii) It follows from the definition that $b_{q \perp q'}((x, x'), (y, y')) = b_q(x, y) + b_{q'}(x', y')$.

iii) The quadratic mapping $q \perp q'$ is nondegenerate if and only if both q and q' are nondegenerate.

1.3.5 The Clifford Algebra of a Quadratic Module

There are several ways to understand the *Clifford algebra* $Cl(M,q)$ of a quadratic module (M,q): from the very abstract to the very concrete. The latter is good for computations, whereas the former is good to prove theorems which may free us from computations. Here, we will look at $Cl(M,q)$ by starting with the categorical definition.

Definition 1.18 Let (M,q) be a quadratic module and $A_K(M,q)$ be the associated category whose objects are the linear mappings $f : M \longrightarrow A$, where A is an associative algebra, such that $f(m)^2 = q(m)1_A$ (where 1_A is the unit of A) for all $m \in M$ and whose morphism from $f : M \longrightarrow A$ to $g : M \longrightarrow B$ is an algebra morphism $h : A \longrightarrow B$ such that $g = hof$. If $A_K(M,q)$ contains an *initial universal object* f (which is unique up to isomorphism), its target is called the *Clifford algebra associated with* (M,q).

Remark 1.15 *i)* We use $Cl_K(M,q)$ or $Cl(M,q)$ to denote the Clifford algebra that is associated to (M,q). The unit element of $A_K(M,q)$ is usually denoted by 1_q.

ii) $f : M \longrightarrow A$ is an initial universal object in $A_K(M,q)$ entails that there is a unique $h : A \longrightarrow B$ for each $g \in A_K(M,q)$ which is $g : M \longrightarrow B$ satisfying $g = hof$.

iii) The objects of $A_K(M,q)$ are sometimes called Clifford mappings. Clearly, for all $m,n \in M$ and $f \in A_K(M,q)$, $f(m)f(n) + f(n)f(m) = b_q(m,n)1_q$. Consequently, $f(m)$ and $f(n)$ anticommute if m and n are orthogonal in M.

iv) From the definition of a Clifford algebra, we deduce that every Clifford map factors uniquely via the Clifford algebra.

v) The Clifford algebra $Cl(M, q)$, if it exists, defines a *covariant functor* from the category of quadratic K-modules to the category of associative unital K-algebras.

vi) [25, p. 106, Lemma 3.1.1] If $I(M, q)$ is the *two sided ideal* of $T(M)$ generated by all elements $m \otimes m - q(m)$ where m runs through M, then

$$Cl(M, q) := T(M)/I(M, q).$$

From this construction, it is easy to see that the product in $Cl(M, q)$, usually called the Clifford product, of two elements $m \equiv m + I(M, q)$ and $n \equiv n + I(M, q)$ is defined as:

$$mn = (m + I(M, q))(n + I(M, q)) := m \otimes n + I(M, q).$$

Note that this product is associative and bilinear.

vii) The fact that Clifford algebras are a generalization of real numbers, complex numbers, and quaternions can be verified as follows:

a) If $M = K = \mathbb{R}$ and $q(x) := -x^2$, then $Cl(M, q) = \mathbb{C}$.

b) If $M = \mathbb{R}^2$ and $q(x_1, x_2) := -x_1^2 - x_2^2$, then $Cl(M, q) = \mathbb{H}$.

viii) If $(m_j)_{j \in J}$ is a family of generators of M indexed by a totally ordered set J and f is a Clifford mapping, the products

$$f(m_{j_1})f(m_{j_2}) \cdots f(m_{j_n}) \; with \; n \geq 0 \; and \; j_1 < j_2 < \cdots < j_n$$

constitute a family of generators of $Cl(M, q)$. If $dim\ M = n$, then $dim\ Cl(M, q) = 2^n$.

ix) Since the ideal $I(M, q)$ (in vi above) is not homogeneous, $Cl(M, q)$ doesn't inherit a grading from $T(M)$. But since the ideal has even parity, $Cl(M, q)$ does inherit a $\mathbb{Z}_2$-grading. Moreover $Cl(M, q)$ inherits a filtration from the canonical filtration of $T(M)$.

Definition 1.19 For the Clifford algebra $Cl(M,q)$ and a Clifford mapping f, one can consider a filtration defined as follows:

1. $Cl^{\leq k}(M,q) = 0$ if k is negative integer;

2. $Cl^{\leq k}(M,q) = K1_q$ if $k = 0$;

3. $Cl^{\leq k}(M,q) = K1_q \bigotimes f(M)$ if $k = 1$; and

4. $Cl^{\leq k}(M,q)$ = The submodule of $Cl(M,q)$ generated by all products $f(m_1)f(m_2)...f(m_j)$, $0 \leq j \leq k$ if $k \geq 2$.

This filtration is called the *natural filtration* of the Clifford algebra $Cl(M,q)$.

1.4 Localization of Rings and Modules

The technique of localization reduces many problems in commutative algebra to problems about local rings. That is, the importance of localization of a ring is to partition the ring, infer results on those partitions, and try to generalize the results on those partitions(localizations) on the ring. This often turns out to be extremely useful in a sense that most problems with which commutative algebra has been successful are those that can be reduced to a local case [16, pp.57], [31, pp.294-308], [45, p.97]. Localization is a way of introducing "denominators" to a given commutative ring with unity or a module. That is, it introduces a new ring/module out of an existing one so that it consists of fractions $\frac{x}{s}$ where the denominators s range over a multiplicative subset S of the ring R. The prototype is the construction of the field of fractions for an integral domain. The basic example for localization of a ring is the construction of the ring $\mathbb{Q}$ of rational numbers from the ring $\mathbb{Z}$ of rational integers. In this section we shall explain how to construct new rings/modules by inverting more general sets of elements.

1.4.1 Localization of Rings

A ring R is called a *local ring* if it is commutative and has a unique maximal ideal. Localization of a ring expands a ring into a local ring by adjoining inverses of some of its elements which are contained in a set called a *multiplicative subset*.

Definition 1.20 *Let R be a ring and S a subset R. We say that S is multiplicative if*

i) $1 \in S$

ii) $st \in S$ *for all* $s, t \in S$

Remark 1.16 *The most common examples of a multiplicative set are* $< r > := \{r^n | r \in R \text{ and } n \geq 0\}$ *and* $R \setminus \mathfrak{p}$ *where* $\mathfrak{p}$ *is a prime ideal of a commutative ring* R.

Construction of the ring of fractions Let R be a commutative ring and S a multiplicative subset of R. Consider the equivalence relation $\sim$ on $R \times S$ defined by $(r, s) \sim (n, t)$ iff $(ns - rt)u = 0$ for some $u \in S$. Denote by $\frac{r}{s}$ the equivalence class of (r, s) and the set $R \times S / \sim$ by $S^{-1}R$. $S^{-1}R$ is a commutative ring with unity with respect to addition and multiplication defined as follows:

$$\frac{r}{s} + \frac{n}{t} := \frac{rt + ns}{st}$$

$$\frac{r}{s}\frac{n}{t} := \frac{rn}{st}$$

There is a ring homomorphism from R to $S^{-1}R$, say $\psi : R \longrightarrow S^{-1}R$, defined by $r \longmapsto \frac{r}{1}$, such that the image of S consists of units (invertible elements) in $S^{-1}R$ and every $q \in S^{-1}R$ is expressible in the form $q = \psi(t)^{-1}\psi(r)$ for some $r \in R$ and $t \in S$. In such a construction, $S^{-1}R$ satisfies the following universal property [22, p. 347, Proposition 11.1.2]. Indeed, given any commutative ring R^* and a ring homomorphism $\phi : R \longrightarrow R^*$ such that $\phi(s)$ is a unit of R^* for all $s \in S$, then there exists a unique homomorphism $\eta : S^{-1}R \longrightarrow R^*$ such that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & R^* \\ {\scriptstyle\psi}\downarrow & \nearrow{\scriptstyle\eta} & \\ S^{-1}R & & \end{array} .$$

Definition 1.21 Let R be a commutative ring with unity and S a multiplicative subset of R. $S^{-1}R$ together with ψ is called a *ring of fractions (localization)* of R with respect to S if

i) $\psi(t)$ is a unit element of $S^{-1}R$ for all $t \in S$.

ii) Every $q \in S^{-1}R$ is expressible in the form $q = \psi(t)^{-1}\psi(r)$ for some $r \in R$ and $t \in S$.

If R is a commutative local ring and $\mathfrak{m}$ its maximal ideal, and $x \in R \setminus \mathfrak{m}$, then x is a unit (otherwise x generates a proper ideal, not contained in $\mathfrak{m}$, which is impossible) [32, p. 110]. These local rings form local objects in commutative algebra since for any commutative ring R and any prime ideal $\mathfrak{p}$ of R, one can localize R at $\mathfrak{p}$ to get a local ring $R_{\mathfrak{p}}$(see Part 2 in the next example). Recall that an ideal $\mathfrak{p}$ of a commutative ring R is a prime ideal if 1 is not in $\mathfrak{p}$ and for every $s, t \in R$ we have the implication: $st \in \mathfrak{p} \Rightarrow s \in \mathfrak{p}$ or $t \in \mathfrak{p}$. This is equivalent to saying that $R - \mathfrak{p}$ is a multiplicative set.

Examples

1) If R is a commutative ring with unity and $S = \{1\}$, then $S^{-1}R = R$.

2) Let R be a commutative ring with unity, $\mathfrak{p}$ a prime ideal of R, and S the set-theoretic complement of $\mathfrak{p}$. $S^{-1}R$ is called the *localization of R at $\mathfrak{p}$*, which is often denoted by $R_{\mathfrak{p}}$.

Let R be an integral domain with field of fractions K. Then its localization $R_{\mathfrak{p}}$ at a prime ideal $\mathfrak{p}$ can be viewed as a subring of K. Moreover,

$$R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}} = \bigcap_{m} R_{\mathfrak{m}}$$

where the first intersection is over all prime ideals and the second over the maximal ideals.

Remark 1.17

i) $S^{-1}R = \{0\}$ iff S contains 0 and the ring homomorphism $R \longrightarrow S^{-1}R$ is injective if and only if S does not contain any zero divisors.

ii) It is essential to be clear about the distinction between $R/\mathfrak{p}$ and $R_{\mathfrak{p}}$. In rough terms we may think of $R/\mathfrak{p}$ as being formed from R by 'putting the elements in $\mathfrak{p}$ equal to 0', while $R_{\mathfrak{p}}$ is formed by 'making the elements outside $\mathfrak{p}$ invertible'.

iii) [30, pp.428-429] If I is an ideal of R, then $S^{-1}I = \{s^{-1}i | s \in S, \ i \in I\}$ is an ideal of $S^{-1}R$. If J is an ideal of $S^{-1}R$, then $R \bigcap J$, i.e., the inverse image of J under the canonical homomorphism ψ, is an ideal of R.

iv) [30, pp.429-430, Corollary 8.48] Let R be a commutative ring with identity.

a) If R is Noetherian, then $S^{-1}R$ is Noetherian.

b) If every nonzero prime ideal in R is maximal, then the same is true for $S^{-1}R$

c) If I is an ideal of R, then the ideal $S^{-1}I$ of $S^{-1}R$ is proper if and only if $I \cap S = \emptyset$.

.

v) [30, p.432,] Let R be a commutative ring with identity and I is an ideal R contained in all maximal ideals. If M is a finitely generated unital R-module and $IM = M$, then $M = 0$.

vi) [13, p.357, Corollary 10.3.3] There is a bijection between the set of prime ideals of $S^{-1}R$ and the set of prime ideals of R which do not intersect S. This bijection is induced by the given homomorphism $R \longrightarrow S^{-1}R$.

vii) Another way to describe the localization of a ring R at a multiplicative subset S is via category theory. Consider all R-algebras A, so that, under the canonical homomorphism $R \longrightarrow A$, every element of S is mapped into a unit. These algebras are objects of the category, with algebra homomorphisms as morphisms. Then, the localization of R by S is the initial universal object of this category.

1.4.2 Localization of Modules

One can apply the construction used in the previous section to a left module M over a ring R so as to obtain its module of fractions, denoted by $S^{-1}M$, with denominators in a multiplicative subset S of R. Here, instead of multiplication, scalar multiplication shall be defined by

$$\frac{r}{t}.\frac{m}{s} := \frac{rm}{ts}, \text{ where } \frac{r}{t} \in S^{-1}R \text{ and } \frac{m}{s} \in S^{-1}M.$$

Then $S^{-1}M$ is a left $S^{-1}R$-module and there is a canonical module homomorphism $\eta : M \longrightarrow S^{-1}M$ defined by $\eta(m) := \frac{m}{1}$.

Proposition 1.4 *[25, p. 22] For every R-linear mapping $\varphi : M \longrightarrow N$ of the R-module M into $S^{-1}R$-module N, there exists a unique $S^{-1}R$-linear mapping $\bar{\varphi} : S^{-1}M \longrightarrow N$ making the following diagram commutative*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\eta}\downarrow & \nearrow{\scriptstyle\bar{\varphi}} & \\ S^{-1}M & & \end{array}.$$

Proof. For the proof of the proposition, define $\bar{\varphi}(\frac{m}{1}) = \eta(m)$ for each $m \in M$. This clearly gives the required unique $S^{-1}R$-linear mapping. ■

Corollary 1.1 *Given a multiplicative set $T \subseteq R$ containing S, we have a unique morphism of R-modules $\eta_S^T : S^{-1}M \longrightarrow T^{-1}M$ making the following diagram commutative*

$$\begin{array}{ccc} M & \xrightarrow{\eta^T} & T^{-1}M \; . \\ {\scriptstyle \eta^S}\downarrow & \nearrow{\scriptstyle \eta_S^T} & \\ S^{-1}M & & \end{array}$$

This follows immediately from Proposition 1.4. The morphism η_S^T sends a fraction $\frac{m}{s}$ to the same fraction considered as an element of $T^{-1}M$.

Proposition 1.5 *Let $S \subseteq T \subseteq R$ be multiplicative subsets. Assume that for every $t \in T$ there exists $r \in R$ such that $tr \in S$. Then for any R-module M, the canonical morphism $\eta_S^T : S^{-1}M \longrightarrow T^{-1}M$ is invertible.*

Proof. Notice that η_S^T is surjective. Indeed, given $\frac{y}{t} \in T^{-1}M$ and $r \in R$ such that $tr \in S$, we have $\frac{y}{t} = \frac{ry}{rt}$ and the latter is in the image of η_S^T. On the other hand, $\eta_S^T(\frac{x}{u}) = 0$ implies that there exists $t \in T$ such that $tx = 0$. If $r \in R$ is such that $tr \in S$, we get also $(rt)x = 0$. This proves that $\frac{x}{u} = 0$ in $S^{-1}M$. ■

Note that for any morphism of R-modules $\varphi : M \longrightarrow N$, there is a unique morphism of R-modules $S^{-1}\varphi : S^{-1}M \longrightarrow S^{-1}N$ such that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle \eta^S}\downarrow & & \downarrow{\scriptstyle \eta^S} \\ S^{-1}M & \xrightarrow[S^{-1}\varphi]{} & S^{-1}N \end{array}$$

If φ is surjective/injective, so is $S^{-1}\varphi$. This actually follows from the following proposition:

Proposition 1.6 *Let M be an R-module. The kernel of $\eta^S : M \longrightarrow S^{-1}M$ consists of the elements $x \in M$ such that $sx = 0$ for some $s \in S$. Moreover, the natural morphism*

$$S^{-1}M \longrightarrow S^{-1}(M/ker(\eta^S)) \tag{1.1}$$

is an isomorphism.

Proof. The kernel η^S consists of $x \in M$ such that $\frac{x}{1} = 0$. This means that $(1, x) \sim (1, 0)$ which is equivalent to the existence of $s \in S$ with $0 = s(1.x - 1.0) = sx$. Let us show that (1.1) is invertible. Surjectivity is clear. For injectivity, we consider a fraction $\frac{x}{s}$ in the kernel of (1.1). As multiplication by s is invertible in $S^{-1}(M/ker(\eta^S))$, we know also that $\frac{x}{1}$ is in the kernel of (1.1). By the first part of the proposition, there exists $s' \in S$ such that $s'x \in ker(\eta^S)$. Thus, there is $t \in S$ such that $ts'x = 0$. This implies that $\frac{x}{s} = 0$. ■

Proposition 1.7 *Let $I \subset S^{-1}R$ be an ideal. Then, $I = S^{-1}I_o$ with $I_o = (\eta^S)^{-1}(I)$.*

Proof. We know that $S^{-1}I_o \longrightarrow S^{-1}R$ is injective and its image is contained in I. To show that $S^{-1}I_o = I$, take $\frac{r}{s} \in I$. Then, $\frac{r}{1} = s\frac{r}{s} \in I$. It follows that $r \in I_o$ so that $\frac{r}{s} \in S^{-1}I_o$. This proves the claim. ■

Remark 1.18

i) [25, p.23, Theorem 1.10.3] Localization of a module is tightly linked the tensor product

$$S^{-1}M \simeq S^{-1}R \otimes_R M$$

with in an R-isomorphism. This way of thinking about localizing is often referred to as extension of scalars. As a tensor product, the localization satisfies the usual universal property.

ii) [50, p.31, Property 2.7.3] Localization of modules is a functor from the category of R-modules to the category of $S^{-1}R$-modules.

iii) The localization functor (usually) preserves Hom and tensor products in the following sense: The natural map

$$S^{-1}(M \otimes_R N) \longrightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

is an isomorphism [25, p.24, Corollary 1.10.5] and if M is finitely presented, the natural map

$$S^{-1}\mathrm{H}om_R(M, N) \longrightarrow \mathrm{H}om_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

is an isomorphism*[25, p.25, Proposition 1.10.8].*

iv) [25, p. 61, Proposition 2.3.3] Let $q : M \longrightarrow N$ be a quadratic mapping defined on a finitely presented module M. The following three assertions are equivalent:

a) q is nondegenerate;

b) for every prime ideal $\mathfrak{p}$ the quadratic mapping $q_{\mathfrak{p}} : M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ is nondegenerate; and

c) for every maximal ideal $\mathfrak{m}$ the quadratic mapping $q_{\mathfrak{m}} : M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ is nondegenerate.

Chapter 2

Sheaves of Clifford Algebras

This chapter is concerned with the description of *sheaves of Clifford algebras* (or C*lifford $\mathcal{A}$-algebras* in short), which are the natural counterparts of Clifford algebras of quadratic vector spaces in the sheaf-theoretic context. Points which we feel are vital for their description such as tensor, symmetric, exterior *$\mathcal{A}$-algebras*, *$\mathcal{A}$-quadratic morphisms*, Clifford $\mathcal{A}$-algebras, ideal sheaf of sheaves of rings, and the parity grading of Clifford $\mathcal{A}$-algebras are included. The case of Clifford $\mathcal{A}$-algebras of Riemannian quadratic free $\mathcal{A}$-modules of finite rank has also been addressed so as to obtain the rank of the associated Clifford free $\mathcal{A}$-algebra, which is stated in Theorem 2.3.

In the discussions to come, the pair $(X, \mathcal{A})$, or just $\mathcal{A}$, will denote a fixed $\mathbb{C}$-algebraized space, i.e., $\mathcal{A}$ is a sheaf of unital and commutative $\mathbb{C}$-algebras over a topological space X. We will assume that all sheaves encountered herein are defined over the topological space X. On the other hand, we will also mainly use the notation of [35]; thus, for instance, $\mathcal{A}$-$\mathcal{M}od_X$ will stand for the category of $\mathcal{A}$-modules with their respective $\mathcal{A}$-morphisms.

2.1 Preliminary on sheaves and presheaves

Definition 2.1 By a *sheaf (of sets)* one means a triple $(\mathcal{S}, \pi, X)$, where $\mathcal{S}$ and X are topological spaces and $\pi : \mathcal{S} \to X$ is a (surjective) *local homeomorphism*,i.e., for every $z \in \mathcal{S}$, there exists an open neighborhood V of z in $\mathcal{S}$ such that $\pi(V)$ is an open neighborhood of $\pi(z)$ in X and the restriction of π to V is a homeomorphism.

Remark 2.1 *i)* Instead of the triple, it is common to refer just to $\mathcal{S}$, by simply saying $\mathcal{S}$ is a sheaf over X (or $\mathcal{S}$ is a sheaf space).

ii) For any $x \in X$, the set $\mathcal{S}_x := \pi^{-1}(\{x\}) \equiv \pi^{-1}(x)$ is called the of $\mathcal{S}$ over $x \in X$, or the of $\mathcal{S}$ at x.

iii) By a of a given sheaf $\mathcal{S}$ over X, one means any open subset $\mathcal{E}$ of $\mathcal{S}$ such that $(\mathcal{E}, \pi|_{\mathcal{E}}, X)$ is a sheaf over X.

iv) Let $(\mathcal{S}, \pi, X)$ be a sheaf and U be an open subset of X. A in $\mathcal{S}$ over U is a continuous map $\mathrm{s} : U \to \mathcal{S}$ such that $\pi \circ \mathrm{s} = id_U$.

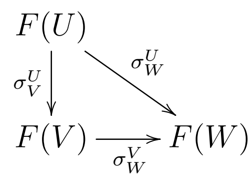
Definition 2.2 A F of sets on X is an assignment (correspondence) that associates a set $F(U)$ to every open subset U of X in such a way that the following conditions are satisfied:

a) For any open sets U, V of X, with $V \subseteq U$, there exists a restriction map

$$\sigma_V^U : F(U) \to F(V).$$

b) For every open set U of X, $\sigma_U^U = id_{F(U)}$.

c) For any open sets U, V, W in X, with $W \subseteq V \subseteq U$, $\sigma_W^U = \sigma_W^V \circ \sigma_V^U$.



Notation *If $\mathcal{S}$ is a sheaf on a topological space X, then $\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S})$, stands for the set of local sections of $\mathcal{S}$ on U, and $\Gamma(\mathcal{S}) := \Gamma_{\mathcal{S}} \equiv (\Gamma(U, \mathcal{S}), \sigma_V^U)$, where σ_V^U is the restriction map, stands for the presheaf of sections of $\mathcal{S}$.*

Definition 2.3 Let $F \equiv (F(U), \sigma_V^U)$ be a presheaf (of sets) on a topological space X. Then, we say that F is a *complete presheaf* if the following conditions are satisfied:

i) If U is an open subset of X and $\mathcal{U} \equiv \{U_i\}_{i \in I}$ is an open covering of U, and $\mathrm{s}_1, \mathrm{s}_2 \in F(U)$ such that $\sigma_{U_i}^U(\mathrm{s}_1) = \sigma_{U_i}^U(\mathrm{s}_2)$, for every $i \in I$, then $\mathrm{s}_1 = \mathrm{s}_2$ (the converse is certainly true).

ii) Let U and $\mathcal{U}$ be as in (1); moreover let $(\mathrm{s}_i) \in \prod_i F(U_i)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$ in $\mathcal{U}$ one has

$$\sigma_{U_{ij}}^{U_i}(\mathrm{s}_i) \equiv \mathrm{s}_i|_{U_{ij}} = \mathrm{s}_j|_{U_{ij}} \equiv \sigma_{U_{ij}}^{U_j}(\mathrm{s}_j).$$

Then, there exists an element $\mathrm{s} \in F(U)$ such that $\sigma_{U_i}^U(\mathrm{s}) \equiv \mathrm{s}|_{U_i} = \mathrm{s}_i$, for all $i \in I$.

Remark 2.2 $\Gamma_{\mathcal{S}}$ is also a complete presheaf.

Definition 2.4 Let $\mathcal{S}h_X$ and $\mathcal{C}o\mathcal{P}\mathcal{S}h_X$ be the category of sheaves (of sets) and the category of complete presheaves on X, respectively. The mapping

$$\Gamma : \mathcal{S}h_X \longrightarrow \mathcal{C}o\mathcal{P}\mathcal{S}h_X$$

defined by $\mathcal{S} \longmapsto \Gamma(\mathcal{S})$ is called the *section functor*.

Remark 2.3

i) A *presheaf of an algebraic structure* (groups, rings, modules, etc) on a topological space X is a contravariant functor of the category of open subsets of X into a particular category whose objects are sets with the type of algebraic structure under consideration. For example, if $F \equiv (F(U), \sigma_V^U)$ is a presheaf of groups on X, then each one of the sets $F(U)$, $U \subseteq X$, is a group and for each pair of open sets V, U in X, with $V \subseteq U$, the restriction map $\sigma_V^U : F(U) \to F(V)$ is a group morphism and the corresponding stalk of F at $x \in X$, $F_x = \varinjlim_{x \in U} F(U)$ is a group.

ii) [35, p. 99, Definition 1.6] Let $A \equiv (A(U), \sigma_V^U)$ be a presheaf of $\mathbb{C}$-algebras and $E \equiv (E(U), \rho_V^U)$ be a presheaf of abelian groups, on a topological space X, such that

a) $E(U)$ is a (left) $A(U)$-module for every open subset U of X; and

b) For every open sets V, U in X, with $V \subseteq U$, one has $\rho_V^U(a.s) = \sigma_V^U(a).\rho_V^U(s)$, for any $a \in A(U)$ and $s \in E(U)$.

Then, E is called a *presheaf of modules* (more precisely, of $A(U)$-*modules*) on X. Yet, it is customary to say, simply, that E is an A-*presheaf* on X.

Definition 2.5 Let $E \equiv (E(U), \rho_V^U)$ and $F \equiv (F(U), \lambda_V^U)$ be two presheaves on a topological space X.

1. By a *morphism of presheaves* E into F, say ϕ, denoted by $\phi : E \to F$, one means a family of maps, indexed by τ (the topology of X), say, $\phi \equiv (\phi_U)_{U \in \tau}$, in such a manner that

$\phi_U : E(U) \to F(U), U \in \tau$, while, for any U, V in τ, $V \subseteq U$, the following diagram of maps

$$\begin{array}{ccc} E(U) & \xrightarrow{\phi_U} & F(U) \\ {\scriptstyle \rho_V^U}\downarrow & & \downarrow{\scriptstyle \lambda_V^U} \\ \mathcal{E}(V) & \xrightarrow{\phi_V} & F(V) \end{array}$$

commutes.

2. If $A \equiv (A(U), \sigma_V^U)$ is a presheaf of algebras and E and F are A-presheaves, $\phi \equiv (\phi_U)_{U\in\tau}$ is called a *morphism of A-presheaves (A-morphism)* if ϕ is a morphism of E into F and ϕ_U is an $A(U)$-morphism of the $A(U)$-modules concerned.

3. For a presheaf $E \equiv (E(U), \rho_V^U)$ there is an associated sheaf $(\mathcal{E}, \pi, X)$ where the individual members of E and the corresponding local sections of $\mathcal{E}$ (over the same subset of X) are related by a canonical map

$$\rho_U : E(U) \longrightarrow \Gamma(U, \mathcal{E}) \equiv \mathcal{E}(U)$$

defined by

$$\mathrm{s} \longmapsto \tilde{\mathrm{s}}$$

where s is a section in $E(U)$ and $\tilde{\mathrm{s}}$ is the class of local sections of $\mathcal{E}$ over U that have the same germ at each $x \in U$. This sheaf $(\mathcal{E}, \pi, X)$ is called the *sheafification* of $E \equiv (E(U), \rho_V^U)$. It is written as $\mathcal{E} = \mathbf{S}(E)$.

4. Let $\mathcal{PS}h_X$ and $\mathcal{S}h_X$ be the category of presheaves (of sets) and the category of sheaves (of sets) on X, respectively. The mapping

$$\mathbf{S} : \mathcal{PS}h_X \longrightarrow \mathcal{S}h_X$$

where $\mathbf{S}$ is defined as in (3) is called the *sheafification functor*.

Remark 2.4

i) The sheafification of a given presheaf endowed with an algebraic structure yields a sheaf with simillar structure. i.e., the sheafification functor preserves algebraic structure. More generally, a given sheaf of sets $\mathcal{S}$ on a topological space X is endowed with some particular algebraic structure if and only if this is the case for its complete presheaf of sections $\Gamma(\mathcal{S})$ on X.

ii) If $\phi : E \to F$ is an A-morphism in the category of A-presheaves on X (denoted by $A{-}\mathcal{PS}h_X$), then $\tilde{\phi} \equiv \mathbf{S}(\phi)$ is an $\mathcal{A}$-morphism in the category of $\mathcal{A}$-modules.

iii) Let $\mathcal{F}$ be an $\mathcal{A}$-module on a topological space X and $\mathcal{E}$ a sub-$\mathcal{A}$-module of $\mathcal{F}$. Then for every short exact $\mathcal{A}$-sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0$$

one has $\mathcal{S} = \mathcal{F}/\mathcal{E}$ within $\mathcal{A}$-isomorphism.

iv) In this thesis, $\mathcal{A}$-modules (resp. $\mathcal{A}$-algebras) on a topological space X will also be referred to as $\mathcal{A}_X$-*modules* (resp. $\mathcal{A}_X$-*algebras*) or simply as $\mathcal{A}$-*modules* (resp. $\mathcal{A}$-*algebras*) if the context is understood.

Definition 2.6 Let $\mathcal{E}$ be an $\mathcal{A}$-module.

1) $\mathcal{E}$ is called a *free* $\mathcal{A}$-*module* of rank n, $(n \in \mathbb{N})$, whenever one has $\mathcal{E} = \mathcal{A}^n$, within $\mathcal{A}$-isomorphism.

2) $\mathcal{E}$ is called a *vector sheaf* (locally free $\mathcal{A}$-module of finite rank n) $(n \in \mathbb{N})$ on X, if for every $x \in X$, there exists an open neighborhood U of x in X such that

$$\mathcal{E}|_U = \mathcal{A}^n|_U$$

within $\mathcal{A}|_U$-isomorphism.

3) $\mathcal{E}$ is called a *line sheaf* if it is a vector sheaf of rank one.

4) Given a locally free $\mathcal{A}$-module of finite rank n. An open covering of X, say, $\mathcal{U} = (\mathrm{U}_\alpha)_{\alpha\in\mathrm{I}}$, satisfying $\mathcal{E}|_{\mathrm{U}_\alpha} = \mathcal{A}^n|_{\mathrm{U}_\alpha}$ within an $\mathcal{A}|_{\mathrm{U}_\alpha}$-isomorphism, is called *a local frame* of $\mathcal{E}$. Moreover, any open set U in X for which $\mathcal{E}|_{\mathrm{U}} = \mathcal{A}^n|_{\mathrm{U}}$ holds (within an $\mathcal{A}|_{\mathrm{U}}$-isomorphism) is called a *local gauge* of $\mathcal{E}$.

5) 5.1) An *$\mathcal{A}$-valued inner product* ρ on $\mathcal{E}$ is a sheaf morphism $\rho : \mathcal{E}\oplus\mathcal{E} \longrightarrow \mathcal{A}$ which is *$\mathcal{A}$-bilinear, positive definite, and symmetric.*

5.2) If ρ is an $\mathcal{A}$-valued inner product on $\mathcal{E}$, then the pair $(\mathcal{E}, \rho)$ is called an *inner product $\mathcal{A}$-module* on X.

5.3) An $\mathcal{A}$-valued inner product ρ on $\mathcal{E}$ is said to be *strongly non-degenerate* whenever the induced $\mathcal{A}$-morphism $\bar{\rho} : \mathcal{E} \longrightarrow \mathcal{E}^* \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is an *$\mathcal{A}$-isomorphism* of the $\mathcal{A}$-modules involved.

5.4) A *Riemannian $\mathcal{A}$-module* $\mathcal{E}$ on X is an $\mathcal{A}$-module $\mathcal{E}$ on X endowed with a *strongly non-degenerate $\mathcal{A}$-valued* inner product.

6) A sheaf morphism $q : \mathcal{E} \longrightarrow \mathcal{A}$ of the underlying sheaves of sets of $\mathcal{E}$ and $\mathcal{A}$ is called a *$\mathcal{A}$-quadratic morphism* if for every open $U \subseteq X$, the set map $q_U : \mathcal{E}(U) \longrightarrow \mathcal{A}(U)$ is *quadratic*. i.e., for any open $U \subseteq X$ and sections $\lambda \in \mathcal{A}(U)$ and $s, t \in \mathcal{E}(U)$ one has

i) $q_U(\lambda s) = \lambda^2 q_U(s)$; and

ii) $b_{q_u} : \mathcal{E}(U) \times \mathcal{E}(U) \longrightarrow \mathcal{A}(U)$ defined by $b_{q_u}(s,t) := q_U(s+t) - q_U(s) - q_U(t)$ is $\mathcal{A}(U)$-bilinear.

The pair $(\mathcal{E}, q)$, where q is $\mathcal{A}$-quadratic, is called a quadratic $\mathcal{A}$-module.

7) A quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$ is called *Riemannian quadratic $\mathcal{A}$-module* if the *q-induced $\mathcal{A}$-bilinear morphism* b is a *Riemannian $\mathcal{A}$-metric,* i.e., a *strongly non-degenerate, symmetric and positive definite $\mathcal{A}$-valued inner product*.

It is also possible to define the notions of an $\mathcal{A}$-quadratic morphism and quadratic $\mathcal{A}$-module as follows.

Definition 2.7 Let $\mathcal{E}$ be an $\mathcal{A}$-module and $F : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{S}h\mathcal{S}et_X$ the forgetful functor of the category of $\mathcal{A}$-modules into the category of sheaves of sets. A morphism $q \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}), F(\mathcal{A}))$ is called *$\mathcal{A}$-quadratic* on $\mathcal{E}$ if the following are satisfied:

(1) Given any open subset U of X and scalar $\lambda \in \mathcal{A}(U)$, define $\lambda \in \mathrm{H}om_{\mathcal{A}(U)}(\mathcal{A}(U), \mathcal{A}(U)) \equiv \mathrm{E}nd_{\mathcal{A}(U)}\mathcal{A}(U) \simeq \mathcal{A}(U)$ by

$$\lambda(s) := \lambda s,$$

for every $s \in \mathcal{A}(U)$. Then,

$$q_U \circ \lambda \equiv q \circ \lambda := ev(\lambda^2, q(-)) \equiv ev_U(\lambda^2, q_U(-)),$$

where $ev \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}nd_{\mathcal{A}}\mathcal{A}) \oplus F(\mathcal{A}), F(\mathcal{A}))$ (ev is called the *evaluation morphism*) is given by

$$ev_U(\psi, \alpha) \equiv ev(\psi, \alpha) := \psi_U(\alpha) \equiv \psi_U \cdot \alpha$$

for any open $U \subseteq X$ and sections $\alpha \in \mathcal{A}(U)$ and $\psi \in (\mathcal{E}nd_{\mathcal{A}}\mathcal{A})(U)$.

(2) The *morphism* $B_q \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(F(\mathcal{E}) \oplus F(\mathcal{E}), F(\mathcal{A}))$, given by

$$B_q := (q \circ +) - (q \circ pr_1) - (q \circ pr_2), \tag{2.1}$$

where $pr_i, + : F(\mathcal{E}) \oplus F(\mathcal{E}) \longrightarrow F(\mathcal{E})$ are the i-th projection and addition morphisms, respectively, *is $\mathcal{A}$-bilinear.*

The pair $(\mathcal{E}, q)$ is called a *quadratic $\mathcal{A}$-module.*

We shall denote by

$$\mathsf{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) \tag{2.2}$$

the group of $\mathcal{A}|_U$-quadratic morphisms on $\mathcal{E}|_U$. The set (2.2) is an $\mathcal{A}(U)$-module. In fact, for any $\alpha \in \mathcal{A}(U)$ and $q \equiv (q_V)_{U \supseteq V,\ open} \in \mathsf{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$, one sets the following:

$$(\alpha \cdot q)_V := \alpha|_V \cdot q_V \equiv \alpha \cdot q_V,$$

which thus provides the $\mathcal{A}(U)$-module structure of (2.2). On the other hand, it is readily verified that the *collection* $(\mathsf{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U), \sigma^U_V)$ *is a complete presheaf of modules* (the restriction maps are defined as follows: if $q \in \mathsf{Q}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$, then $\sigma^U_V(q) := (q_W)_{V \supseteq W,\ open})$. The sheaf generated by this complete presheaf is called the *sheaf of quadratic morphisms* of $\mathcal{E}$ and is denoted

$$\mathcal{Q}(\mathcal{E}, \mathcal{A}) \equiv \mathcal{Q}(\mathcal{E}).$$

Given an arbitrary $\mathcal{A}$-bilinear form b on $\mathcal{E}$, the *morphism*

$$q_b := b \circ \Delta, \tag{2.3}$$

where Δ is the diagonal $\mathcal{A}$-morphism of $\mathcal{E}$ (that is, for every open U in X and section s in $\mathcal{E}(U)$, $\Delta_U(s) \equiv \Delta(s) := (s, s)$), *is clearly a quadratic $\mathcal{A}$-morphism on $\mathcal{E}$.*

Let $\mathcal{B}(\mathcal{E}) \equiv \mathcal{L}^2_{\mathcal{A}}(\mathcal{E}, \mathcal{E}; \mathcal{A})$ be the $\mathcal{A}$-module of $\mathcal{A}$-bilinear forms (cf. [39]), the $\mathcal{S}h\mathcal{S}et_X$- *morphism* $\Xi : \mathcal{B}(\mathcal{E}) \longrightarrow \mathcal{Q}(\mathcal{E})$ such that

$$\Xi_U(b) := q_b, \tag{2.4}$$

for any open $U \subseteq X$ and section $b \in \mathcal{B}(\mathcal{E})(U) := \mathsf{L}^2_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U; \mathcal{A}|_U)$, where q_b is given as in (2.3), *is clearly an* $\mathcal{A}$*-morphism, with the sub-*$\mathcal{A}$*-module* $\mathcal{A}(\mathcal{E})$ *of skew-symmetric* $\mathcal{A}$*-bilinear forms being the kernel of* Ξ. On the other hand, it is immediate that $\Theta : \mathcal{Q}(\mathcal{E}) \longrightarrow \mathcal{B}(\mathcal{E})$, such that, for every open $U \subseteq X$ and section $q \in \mathcal{Q}(\mathcal{E})(U)$,

$$\Theta_U(q) := B_q, \tag{2.5}$$

where B_q is as given in 2.1, *is an* $\mathcal{A}$*-morphism of* $\mathcal{Q}(\mathcal{E})$ *into the sub-*$\mathcal{A}$*-module* $\mathcal{S}(\mathcal{E})$ *of* $\mathcal{B}(\mathcal{E})$ *of symmetric* $\mathcal{A}$*-bilinear forms.*

Suppose now that the *characteristic* of $\mathcal{A}$ is not 2, that is the characteristic of every *individual* algebra $\mathcal{A}(U)$, where U is an open subset of X, is not 2. In the above $\mathcal{A}$-morphism Θ, let's replace B_q in (2.5) by the symmetric $\mathcal{A}|_U$-bilinear form

$$b_q := \frac{1}{2}B_q \tag{2.6}$$

for every quadratic $\mathcal{A}|_U$-form $q \in \mathcal{Q}(\mathcal{E})(U)$. So, one has

$$b_q = \frac{1}{2}\{(q \circ +) - (q \circ pr_1) - (q \circ pr_2)\}, \tag{2.7}$$

where $pr_i : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$ $(i = 1, 2)$ is the i-th projection and, as expected, $+ : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$ is the addition $\mathcal{A}|_U$-morphism. Clearly,

$$b_q \circ \Delta = q, \tag{2.8}$$

with Δ the diagonal $\mathcal{A}|_U$-morphism on $\mathcal{E}|_U$.

Setting

$$\widetilde{\Theta} = \frac{1}{2}\Theta,$$

one has

$$\Xi \circ \widetilde{\Theta} = \mathsf{Id}_{\mathcal{Q}(\mathcal{E})},$$

which implies that $\widetilde{\Theta}$ is injective and Ξ surjective. Clearly, $\mathsf{Im}\widetilde{\Theta} \subseteq \mathcal{S}(\mathcal{E})$. Conversely, for any symmetric $b \in \mathcal{B}(\mathcal{E})(U)$, one has that $\Xi_U(b) := q_b = b \circ \Delta$ and

$$b_q = b.$$

Thus, if we consider $\Xi|_{\mathcal{S}(\mathcal{E})}$, it is clear that

$$\widetilde{\Theta} \circ \Xi_{\mathcal{S}=}\mathsf{Id}_{\mathcal{S}(\mathcal{E})}.$$

Hence, we have proved:

Proposition 2.1 *Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}$-module, with $\mathcal{A}$ a sheaf of algebras of characteristic other than 2. Then,*

$$\mathcal{Q}(\mathcal{E}) = \mathcal{S}(\mathcal{E}), \tag{2.9}$$

within an $\mathcal{A}$-isomorphism.

We come now to the following crucial notion.

Definition 2.8 Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}$-module, and $\mathcal{K}$ an *associative* and *unital $\mathcal{A}$-algebra.* A sheaf morphism $\varphi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ is called a *Clifford sheaf morphism* if

$$\varphi^2 = \mathit{ev}(q, -) \cdot 1, \tag{2.10}$$

where: (a) $\mathit{ev} : \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \oplus \mathcal{E} \longrightarrow \mathcal{A}$ is the *evaluation $\mathcal{A}$-morphism.* (b) $1 \in \mathrm{H}om_{\mathcal{S}h\mathcal{S}et_X}(\mathcal{K}, \mathcal{K})$ is the constant $\mathcal{S}h\mathcal{S}et_X$-morphism $1_W(t) = 1_{\mathcal{K}(W)}$ for every open $W \subseteq X$ and section $t \in \mathcal{K}(W)$.

For every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$, (2.10) becomes

$$\varphi_U(s)^2 \equiv \varphi(s)^2 := q(s) \cdot 1 \equiv q_U(s) \cdot 1_{\mathcal{K}(U)}.$$

On the other hand, let us consider another section $t \in \mathcal{E}(U)$; then, we have

$$\varphi_U(s)\varphi_U(t) + \varphi_U(t)\varphi_U(s) = 2b_U(s,t) \cdot 1_{\mathcal{K}(U)}, \tag{2.11}$$

where

$$b := \widetilde{\Theta}_X(q).$$

We will call b the *$\mathcal{A}$-bilinear morphism induced by* the quadratic $\mathcal{A}$-morphism q.

2.2 Tensor, Symmetric, and Exterior Algebra Sheaves of $\mathcal{A}$-modules

In this section we briefly discuss tensor, symmetric and exterior algebra sheaves since they relate to Clifford $\mathcal{A}$-algebras.

Definition 2.9 Let $\mathcal{A} \equiv (\mathcal{A}(U), \pi_V^U)$ be a commutative and unital algebra sheaf and $\mathcal{E} \equiv (\mathcal{E}(U), \rho_V^U)$ be an $\mathcal{A}$-module. One defines the *tensor algebra sheaf* of $\mathcal{E}$, denoted $\mathcal{T}(\mathcal{E})$, as the *sheafification* of the $\Gamma(\mathcal{A})$-presheaf

$$T(\Gamma(\mathcal{E})) \equiv (T(\Gamma(\mathcal{E}))(U) := T(\mathcal{E}(U)), \sigma_V^U), \tag{2.12}$$

where $T(\mathcal{E}(U))$ is the tensor algebra of the $\mathcal{A}(U)$-module $\mathcal{E}(U)$, viz.,

$$T(\mathcal{E}(U)) = \oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)),$$

where

$$T^n(\mathcal{E}(U)) := \underbrace{\mathcal{E}(U) \otimes_{\mathcal{A}(U)} \cdots \otimes_{\mathcal{A}(U)} \mathcal{E}(U)}_{n \text{ times}},$$

with

$$T^0(\mathcal{E}(U)) := \mathcal{A}(U) \quad \text{and} \quad T^1(\mathcal{E}(U)) := \mathcal{E}(U).$$

Remark 2.5

i) By taking the respective sheafifications of the presheaves $(T^0(\mathcal{E}(U)), \pi_V^U)$ and $(T^1(\mathcal{E}(U)), \rho_V^U)$, one obtains

$$\mathcal{T}^0(\mathcal{E}) = \mathcal{A} \quad \text{and} \quad \mathcal{T}^1(\mathcal{E}) = \mathcal{E}$$

within $\mathcal{A}$-isomorphisms. Moreover, for any finite sequence $(s_1, \cdots, s_n)$, $n \geqslant 2$, of sections in $\mathcal{E}(U)$, where $U \in \tau_X$, and any open $V \subseteq U$,

$$\sigma_V^U(s_1 \otimes \cdots \otimes s_n) = \rho_V^U(s_1) \otimes \cdots \otimes \rho_V^U(s_n) \equiv s_1|_V \otimes \cdots \otimes s_n|_V.$$

.

ii) Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules on X and $U \in \tau_X$. Then for any $\mathcal{A}(U)$-morphism $\varphi_U : \mathcal{E}(U) \longrightarrow \mathcal{F}(U)$, there is a unique morphism of $\mathcal{A}(U)$-algebras $T(\varphi_U) : T(\mathcal{E}(U)) \longrightarrow T(\mathcal{F}(U))$ making the following diagram of $\mathcal{A}(U)$-modules commute.

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\varphi_U} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ T(\mathcal{E}(U)) & \xrightarrow{T(\varphi_U)} & T(\mathcal{F}(U)) \end{array}$$

This defines a functor $T : A - \mathcal{PS}h_X \longrightarrow A - Alg_X$ from the category of $\Gamma(\mathcal{A})$-presheaves of modules over the topological space X into the category of $\Gamma(\mathcal{A})$-presheaves of algebras over X (see [35, p. 111]).

Lemma 2.1 *Let $(X, \mathcal{A})$ be an algebraized space, $\mathcal{E}$ and $\mathcal{K}$, respectively, an $\mathcal{A}_X$-module and an $\mathcal{A}_X$-algebra, and $\varphi : \mathcal{E} \longrightarrow \mathcal{K}$ an $\mathcal{A}$-morphism. Then, φ extends uniquely to an $\mathcal{A}$-morphism $\psi : \mathcal{T}(\mathcal{E}) \longrightarrow \mathcal{K}$ of $\mathcal{A}_X$-algebras.*

Proof. We consider the generating presheaf

$$T(\Gamma(\mathcal{E})) := \big(T(\Gamma(\mathcal{E}))(U) \equiv T(\mathcal{E}(U)), \sigma_V^U\big)$$

of the $\mathcal{A}_X$-algebra $\mathcal{T}(\mathcal{E})$. For any sections $s_1, s_2, \ldots, s_k$ in $\mathcal{E}(U)$, where $U \subseteq X$ is open, let

$$(\Gamma(\psi))_U(s_1 \otimes \cdots \otimes s_k) := \varphi_U(s_1) \cdots \varphi_U(s_k)$$

where $(\Gamma(\psi))_U : T(\mathcal{E}(U)) \longrightarrow \mathcal{K}(U)$ is the (unique) extension of the $\mathcal{A}(U)$-morphism $\varphi_U : \mathcal{E}(U) \longrightarrow \mathcal{K}(U)$ on $T(\mathcal{E}(U))$, (cf. [25, p. 8, Proposition 1.4.1]). It is clear that the family $(\Gamma(\psi))_{U \in \tau_X}$ yields a unique $\Gamma(\mathcal{A})$-morphism

$$\Gamma(\psi) : T(\Gamma(\mathcal{E})) \longrightarrow \Gamma(\mathcal{K})$$

extending the $\Gamma(\mathcal{A})$-morphism $\Gamma(\varphi) \equiv \varphi : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{K})$. Thus, *the sheafification* $\mathbf{S}(\Gamma(\psi)) \equiv \psi$ *extends uniquely the $\mathcal{A}$-morphism φ.* ■

Definition 2.10 Let $\mathcal{E}$ be an $\mathcal{A}$-module and $\mathcal{T}(\mathcal{E})$ the tensor algebra sheaf of $\mathcal{E}$. For each open subset U of the topological space X, let $I(U)$ be the two sided ideal of $T(\mathcal{E}(U))$ generated by elements of the form $s_1 \otimes s_2 - s_2 \otimes s_1$ where $s_1, s_2 \in \mathcal{E}(U)$. The quotient algebra sheaf $\mathcal{S}(\mathcal{E})$ generated by the $\Gamma(\mathcal{A})$-presheaf $U \longmapsto T(\mathcal{E}(U))/I(U)$ is called the *symmetric algebra sheaf* of $\mathcal{E}$.

Definition 2.11 Let X be a topological space and $\mathcal{R} \equiv (\mathcal{R}, \pi, X)$ a sheaf of rings on X. A *left ideal sheaf $\mathcal{J}$ in $\mathcal{R}$* is a subsheaf $\mathcal{J}$ of $\mathcal{R}$ satisfying the following conditions:

(1) $\mathcal{J}$ is a sheaf of abelian groups in $\mathcal{R}$,

(2) For every $x \in X$ and $z \in \mathcal{R}_x$, $z\mathcal{J}_x \subseteq \mathcal{J}_x$.

Equivalently (see [35, p. 104, (1. 67)]), an ideal sheaf $\mathcal{J}$ in a sheaf of rings $\mathcal{R}$ is a subsheaf of $\mathcal{R}$ such that $\mathcal{J}$ is a sheaf of abelian groups in $\mathcal{R}$ and

$$\mathcal{R}(U)\mathcal{J}(U) \subseteq \mathcal{J}(U),$$

for every open U in X.

Remark 2.6

i) The ideal sheaf $\mathcal{I}$, obtained by sheafifying the presheaf $\mathcal{I} \equiv (\mathcal{I}(U))_{U\in\tau_X}$ (cf. Definition 2.10) is a graded subsheaf of the tensor algebra sheaf $\mathcal{T}(\mathcal{E})$ since its corresponding presheaf is generated by homogeneous elements. Therefore $\mathcal{S}(\mathcal{E})$ is also a graded algebra sheaf with grading

$$\mathcal{S}(\mathcal{E}) = \oplus_{n=0}^{\infty}\mathcal{S}^n(\mathcal{E}).$$

ii) $\mathcal{S}(\mathcal{E})$ is a commutative $\mathcal{A}$-algebra. Indeed, $\mathcal{T}(\mathcal{E})$ is generated by $\mathcal{T}^0(\mathcal{E}) = \mathcal{A}$ and $\mathcal{T}^1(\mathcal{E}) = \mathcal{E}$, and therefore $\mathcal{S}(\mathcal{E})$ is generated by $\mathcal{S}^0(\mathcal{E})$ and $\mathcal{S}^1(\mathcal{E})$. Note that, for each open $U \subseteq X$, $S^0(\mathcal{E}(U))$ lies in the center of $S(\mathcal{E}(U))$ (for $T^0(\mathcal{E}(U))$ lies in the center of $T(\mathcal{E}(U))$), and by construction any two elements of $S^1(\mathcal{E}(U))$ commute. Thus, $S(\mathcal{E}(U))$ is generated by a set of pairwise commuting elements, and therefore $\mathcal{S}(\mathcal{E})$ is commutative.

Definition 2.12 *Let $\mathcal{E}$ be an $\mathcal{A}$-module and $\mathcal{T}(\mathcal{E})$ the tensor algebra sheaf of $\mathcal{E}$. For each open subset U of the topological space X, let $I(U)$ be the two sided ideal of $T(\mathcal{E}(U))$ generated by elements of the form $s \otimes s$ where $s \in \mathcal{E}(U)$. The quotient algebra sheaf $\bigwedge(\mathcal{E})$ generated by the $\Gamma(\mathcal{A})$-presheaf $U \longmapsto T(\mathcal{E}(U))/I(U)$ is called the exterior algebra sheaf of $\mathcal{E}$.*

As is in ([35, pp.307-315]), the *exterior algebra of* $\mathcal{E}$, which is denoted by $\bigwedge \mathcal{E}$, is given by

$$\bigwedge \mathcal{E} := \oplus_{n=0}^{\infty} \bigwedge{}^n \mathcal{E},$$

where $\bigwedge^n \mathcal{E}$, is the *sheafification of the presheaf of* $\mathcal{A}(U)$*-algebras*

$$\begin{cases} U \longmapsto \bigwedge^n(\mathcal{E}(U)) \equiv \bigwedge^n_{\mathcal{A}(U)}(\mathcal{E}(U)), & n \geq 2 \\ U \longmapsto \bigwedge^n(\mathcal{E}(U)) := \mathcal{E}(U), & n = 1 \\ U \longmapsto \bigwedge^n(\mathcal{E}(U)) := \mathcal{A}(U), & n = 0 \end{cases}$$

where U ranges over the open subsets of X.

From the definition it follows that

$$\bigwedge{}^0 \mathcal{E} = \mathcal{A} \quad \text{and} \quad \bigwedge{}^1 \mathcal{E} = \mathcal{E},$$

within $\mathcal{A}$-isomorphisms. Moreover, the exterior algebra $\bigwedge \mathcal{E}$ can be constructed (ibid.) as the sheaf generated by the presheaf of $\mathcal{A}(U)$-algebras, which to each open set $U \subseteq X$, assigns the corresponding exterior algebra

$$\bigwedge \mathcal{E}(U) \equiv \oplus_{n=0}^{\infty} \bigwedge{}^n(\mathcal{E}(U)).$$

Remark 2.7

i) Similar to the case of symmetric algebra sheaves, the exterior algebra sheaf of an $\mathcal{A}$-module $\mathcal{E}$ has a natural grading. That is, $\bigwedge(\mathcal{E})$ is also a graded algebra sheaf with grading

$$\bigwedge(\mathcal{E}) = \oplus_{n=0}^{\infty} \bigwedge{}^n(\mathcal{E}).$$

ii) We will show in the next section that, given an $\mathcal{A}_X$-module $\mathcal{E}$, the exterior $\mathcal{A}$-algebra $\bigwedge(\mathcal{E})$ is isomorphic to the Clifford $\mathcal{A}$-algebra of the quadratic $\mathcal{A}$-module $(\mathcal{E}, q \equiv 0)$.

Definition 2.13 Let $\mathcal{R}$ be a sheaf of rings on a topological space X and $\mathcal{S} \subseteq \mathcal{R}$ a sheaf of nonempty sets in $\mathcal{R}$. By a *subring sheaf in* $\mathcal{R}$, *generated by* $\mathcal{S}$, we mean the smallest subring sheaf in $\mathcal{R}$ containing $\mathcal{S}$. Likewise, the *ideal sheaf in* $\mathcal{R}$ *generated by* $\mathcal{S}$ is the smallest ideal sheaf in $\mathcal{R}$ containing $\mathcal{S}$.

As in the classical case, we have the following.

Lemma 2.2 *Let* $\mathcal{S} \subseteq \mathcal{R}$ *be a sheaf of nonempty sets in a sheaf of unital rings* $\mathcal{R}$, *defined on a topological space* X. *Then, the ideal sheaf in* $\mathcal{R}$, *generated by the presheaf of sets* $\mathcal{S}$, *is the sheaf obtained from the presheaf of sets* $J(\mathcal{S})$ *such that, for every open* U *in* X,

$$J(\mathcal{S})(U) := \{\sum_{i=1}^{n} \alpha_i s_i \beta_i : \ \alpha_i, \ \beta_i \in \mathcal{R}(U), \ s_i \in \mathcal{S}(U), \ n \geq 1\}.$$

Moreover, if $\mathcal{R}$ *is commutative, then*

$$J(\mathcal{S})(U) = \{\sum_{i=1}^{n} \alpha_i s_i : \ \alpha_i \in \mathcal{R}(U), \ s_i \in \mathcal{S}(U), \ n \geq 1\}.$$

Proof. Let τ be the topology considered on X. For any $U, \ V \in \tau$ with $V \subseteq U$, let

$$\lambda_V^U(\sum_{i=1}^{n} \alpha_i s_i \beta_i) := \sum_{i=1}^{n} \rho_V^U(\alpha_i)\sigma_V^U(s_i)\rho_V^U(\beta_i) \equiv \sum_{i=1}^{n} \alpha_i|_V s_i|_V \beta_i|_V,$$

where the $(\rho_V^U)_{U, \ V \in T}$ and $(\sigma_V^U)_{U, \ V \in \tau}$ are the restriction maps for the (complete) presheaves of sections of the sheaves $\mathcal{R}$ and $\mathcal{S}$, respectively. It clearly follows that the collection $J(\mathcal{S}) \equiv (J(\mathcal{S})(U), \lambda_V^U)_{U,V \in \tau}$ is a subpresheaf of ideals of the (complete) presheaf of sections $\Gamma(\mathcal{R}) \equiv (\mathcal{R}(U), \rho_V^U)_{U, \ V \in \tau}$. Hence, $\mathrm{S}(J(\mathcal{S}))$ is an ideal sheaf in $\mathcal{R}$.

The second statement of the lemma is simply a particular case of the first statement. ∎

2.3 Clifford $\mathcal{A}$-algebras of Quadratic $\mathcal{A}$-modules

Roughly speaking, a *sheaf of Clifford algebras* or a *Clifford $\mathcal{A}$-algebra* of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$ on a topological space X is a *universal $\mathcal{A}$-algebra* into which we can embed $\mathcal{E}$, and such that the square of an $\mathcal{A}$-morphism in the sought $\mathcal{A}$-algebra corresponds to the quadratic $\mathcal{A}$-morphism on $\mathcal{E}$. This loose definition of a Clifford $\mathcal{A}$-algebra may be traced back to [32, p. 749].

Definition 2.14 By a *Clifford $\mathcal{A}$-algebra* of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$, we mean any pair $(\mathcal{C}, \varphi_{\mathcal{C}})$, where $\mathcal{C}$ is an *associative* and *unital $\mathcal{A}$-algebra* and $\varphi_{\mathcal{C}} \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$ is a , which satisfies the following conditions:

(1) $\mathcal{C}$ is generated by the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$ and the unital line sub-$\mathcal{A}$-algebra $1_{\mathcal{C}}$ of $\mathcal{C}$.

(2) Every Clifford $\mathcal{A}$-morphism $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is an associative and unital $\mathcal{A}$-algebra, factors through the Clifford $\mathcal{A}$-morphism $\varphi_{\mathcal{C}}$, i.e., there is a 1-respecting $\mathcal{A}$-algebra morphism $\Phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ such that

$$\varphi = \Phi \circ \varphi_{\mathcal{C}}.$$

Since $\Phi(\varphi_{\mathcal{C}}(\mathcal{E})) = \varphi(\mathcal{E})$, $\mathcal{C}$ is generated by its unital line sub-$\mathcal{A}$-algebra and the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$, and Φ is 1-respecting, it follows that Φ is *uniquely determined* by the Clifford $\mathcal{A}$-morphism φ. If we denote by

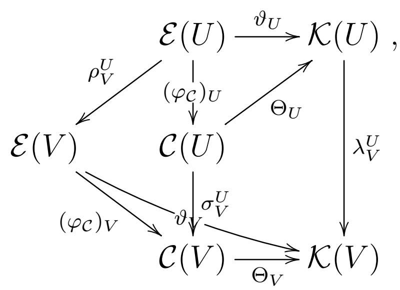
$$\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$$

the *sheaf of Clifford maps*, then $\mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ *is isomorphic to a subsheaf of* $\mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$. In fact, given any open subset U of X, let $\vartheta \in \mathcal{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})(U)$, that is, $\vartheta \in Hom^{Cl}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{K}|_U)$. Since $\mathcal{C} \equiv (\mathcal{C}, \varphi_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$, for any open $V \subseteq U$, there is a $\Theta_V \in Hom_{\mathcal{A}(V)}(\mathcal{C}(V), \mathcal{K}(V))$ such that $\Theta_V(1_{\mathcal{C}(V)}) = 1_{\mathcal{K}(V)}$ and $\vartheta_V = \Theta_V \circ (\varphi_{\mathcal{C}})_V$. We contend that the family $\Theta \equiv (\Theta_V)_{\ open V \subseteq U}$

defines an $\mathcal{A}|_U$-morphism $\Theta \in \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$. Since, for any open $V \subseteq U$, $\Theta_V \in \mathrm{H}om_{\mathcal{A}(V)}(\mathcal{C}(V), \mathcal{K}(V))$ and $\Theta_V(1_{\mathcal{C}(V)}) = 1_{\mathcal{K}(V)}$, we need only show that if (λ^U_V), (ρ^U_V) and (σ^U_V) are the families of restriction maps of the sheaves $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$, respectively, then

$$\lambda^U_V \circ \Theta_U = \Theta_V \circ \sigma^U_V,$$

for any open sets U, V in X with $V \subseteq U$. With no loss of generality, let $s \in \mathcal{C}(U)$, with $s = (\varphi_{\mathcal{C}})_U(e)$ for some $e \in \mathcal{E}(U)$. Then, based on the diagram below



clearly, one has

$$\begin{aligned}(\lambda^U_V \circ \Theta_U)((\varphi_{\mathcal{C}})_U(e)) &= (\lambda^U_V \circ \vartheta_U)(e) = (\vartheta_V \circ \rho^U_V)(e) \\ &= (\Theta_V \circ (\varphi_{\mathcal{C}})_V \circ \rho^U_V)(e) = (\Theta_V \circ \sigma^U_V)((\varphi_{\mathcal{C}})_U(e)).\end{aligned}$$

Next, for every open U in X, we denote by $\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ the $\mathcal{A}(U)$-module consisting of $\mathcal{A}|_U$-morphisms Θ, uniquely determined by Clifford $\mathcal{A}|_U$-morphisms $\vartheta \in \mathrm{H}om^{Cl}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{K}|_U)$. Furthermore, let (α^U_V) be the collection of restriction maps for the $\mathcal{A}$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$. The collection

$$(\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U), \alpha^U_V) \tag{2.13}$$

clearly determines a presheaf. Moreover, it is a complete presheaf. Indeed, if $U = \cup_{i\in I} U_i$ and $\Theta_1,\ \Theta_2 \in \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ with

$$\alpha^U_{U_i}(\Theta_1) \equiv \Theta_1|_{U_i} = \Theta_2|_{U_i} \equiv \alpha^U_{U_i}(\Theta_2)$$

for every $i \in I$, then, clearly, $\Theta_1 = \Theta_2$. Now, let $(\Theta_i) \in \prod_{i \in I} \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$ in $\mathcal{U} \equiv \{U_i,\ i \in I\}$, one has

$$\Theta_i|_{U_{ij}} = \Theta_j|_{U_{ij}}.$$

Then, since $\underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) \subseteq \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$, for any open U in X, there is $\Theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})(U)$ such that

$$\Theta|_{U_i} = \Theta_i,$$

for every $i \in I$. It follows that Θ is 1-respecting; in addition, since it is linear, $\Theta \in \underline{\mathrm{H}}om_{\mathcal{A}|_U}(\mathcal{C}|_U, \mathcal{K}|_U)$. Hence,

$$\mathcal{H}om_{\mathcal{A}}^{Cl}(\mathcal{E}, \mathcal{K}) \simeq \underline{\mathcal{H}}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K}) \subseteq \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K}),$$

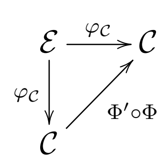
where $\underline{\mathcal{H}}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ is the sheafification of the complete presheaf (2.13).

Condition (2) of Definition 2.14 could therefore be restated as follows:

$(2')$ For every associative and unital $\mathcal{A}$-algebra $\mathcal{K}$, $\mathcal{H}om_{\mathcal{A}}^{Cl}(\mathcal{E}, \mathcal{K})$ is isomorphic to a subsheaf of $\mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$.

Lemma 2.3 *Let $(\mathcal{C}, \varphi_{\mathcal{C}})$ be a Clifford $\mathcal{A}$-algebra of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$. Then, $(\mathcal{C}', \varphi_{\mathcal{C}'})$ is also a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$ if and only if there is an $\mathcal{A}$-isomorphism $\Phi : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $\Phi \circ \varphi_{\mathcal{C}} = \varphi_{\mathcal{C}'}$.*

Proof. Suppose $(\mathcal{C}', \varphi_{\mathcal{C}'})$ is also a Clifford $\mathcal{A}$-algebra. Then, there exist unique $\mathcal{A}$-morphisms $\Phi : \mathcal{C} \longrightarrow \mathcal{C}'$ and $\Phi' : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $\Phi \circ \varphi_{\mathcal{C}} = \varphi_{\mathcal{C}'}$ and $\Phi \circ \varphi_{\mathcal{C}'} = \varphi_{\mathcal{C}}$. Since $\Phi' \circ \Phi \circ \varphi_{\mathcal{C}} = \Phi' \circ \varphi_{\mathcal{C}'} = \varphi_{\mathcal{C}}$, the diagram



is commutative. But, only one $\mathcal{A}$-morphism exists making the above diagram commutative; and clearly $\mathrm{Id}_{\mathcal{C}}$ does just that. As $\mathcal{C}$ is generated by $\varphi_{\mathcal{C}}(\mathcal{E})$ and its unital line sub-$\mathcal{A}$-algebra, $\Phi' \circ \Phi = 1_{\mathcal{C}}$. In a similar way, one shows that $\Phi \circ \Phi' = 1_{\mathcal{C}'}$, whence we see that Φ is an $\mathcal{A}$-isomorphism with $\Phi^{-1} = \Phi'$. ■

Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}_X$-module, and denote by $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q) \equiv \mathcal{A}_{\mathcal{C}l}(\mathcal{E})$ the *category*, described as follows (see [25] for *classical notations*) : **(i)** its objects are the *Clifford $\mathcal{A}$-morphisms*

$$\varphi \in \mathrm{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K}) \subseteq \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K}) \tag{2.14}$$

where $\mathcal{K}$ is any associative and unital $\mathcal{A}_X$-algebra.

(ii) Given objects $\varphi \in \mathrm{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ and $\psi \in \mathrm{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{L})$ ($\mathcal{K}$ and $\mathcal{L}$ are associative and unital $\mathcal{A}_X$-algebras), a morphism $u : \varphi \longrightarrow \psi$ is a 1-*respecting* $\mathcal{A}$-morphism of $\mathcal{A}_X$-algebras $\mathcal{K}$ and $\mathcal{L}$ such that

$$\psi = u \circ \varphi.$$

If the category $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q)$ contains an initial universal object ρ (which is unique up to $\mathcal{A}$-isomorphism), its target is called the *Clifford $\mathcal{A}_X$-algebra sheaf associated with the quadratic $\mathcal{A}_X$-module* $(\mathcal{E}, q)$; we shall denote it by $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q) \equiv \mathcal{C}l(\mathcal{E}, q) \equiv \mathcal{C}l(\mathcal{E})$.

The universality of ρ means that, for every object $\varphi \in \mathrm{H}om^{Cl}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$, there exists a unique 1-respecting $\mathcal{A}$-morphism u of $\mathcal{A}$-algebra sheaves $Cl_{\mathcal{A}}(\mathcal{E}, q)$ and $\mathcal{K}$ such that $\varphi = u \circ \rho$. It is clear that $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q) \simeq \mathcal{A}$ when $\mathcal{E}$ is reduced to *the zero sheaf* on X.

For the purpose of Theorem 2.1 below, we recall (see, for instance, [35, p. 129, (5.5)]) that given two presheaves of sections $\Gamma(\mathcal{E}) \equiv (\mathcal{E}(U), \rho^U_V)$ and $\Gamma(\mathcal{F}) \equiv (\mathcal{F}(U), \sigma^U_V)$ on a topological space X, the correspondence

$$U \longrightarrow \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U),$$

with U running over the open sets in X, along with the obvious restriction morphisms, yields a presheaf of $\Gamma(\mathcal{A})$-modules on X, which we denote by

$$\Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F}).$$

In the same vein, we denote by $T(\Gamma(\mathcal{E}))$ the tensor algebra presheaf

$$\Gamma(\mathcal{A}) \oplus \Gamma(\mathcal{E}) \oplus \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E}) \oplus \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E}) \oplus \cdots$$

on X.

Theorem 2.1 *Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}_X$-module and $I(\mathcal{E}, q)$ the two-sided ideal presheaf in the tensor algebra presheaf $T(\Gamma(\mathcal{E}))$ such that, for every open U in X, $I(\mathcal{E}, q)(U)$ is a two-sided ideal of the algebra $T(\Gamma(\mathcal{E}))(U) \equiv T(\mathcal{E}(U))$ generated by elements of the form*

$$s \otimes s - q_U(s) \equiv s \otimes s - q(s),$$

with s running through $\mathcal{E}(U)$. Moreover, let $\mathcal{T}(\mathcal{E})$ and $\mathcal{I}(\mathcal{E}, q)$ denote the tensor algebra sheaf and the two-sided ideal sheaf generated by the presheaves $T(\Gamma(\mathcal{E}))$ and $I(\mathcal{E}, q)$, respectively. Furthermore, set

$$\mathcal{C}l(\mathcal{E}, q) := \mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E}, q); \tag{2.15}$$

the $\mathcal{A}$-morphism

$$\rho : \mathcal{E} \longrightarrow \mathcal{T}(\mathcal{E}) \longrightarrow \mathcal{C}l(\mathcal{E}, q) \tag{2.16}$$

is an initial universal object in the category $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q)$.

Proof. Let $\varphi : \mathcal{E} \longrightarrow \mathcal{K}$ be an object of $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q)$. By Lemma 2.1, φ extends to a 1-respecting $\mathcal{A}$-algebra morphism $\overline{\varphi} : \mathcal{T}(\mathcal{E}) \longrightarrow \mathcal{K}$, obtained by sheafifying the $\Gamma(\mathcal{A})$-morphism $\phi : T(\Gamma(\mathcal{E})) \longrightarrow$

$\Gamma(\mathcal{K})$, where, for each open U in X, the morphism ϕ_U, which sends $T(\Gamma(\mathcal{E}))(U) \equiv T(\mathcal{E}(U))$ into $\mathcal{K}(U)$, is the extension of φ_U. Since, for any open $U \subseteq X$ and section $s \in \mathcal{E}(U)$,

$$\varphi_U(s)^2 = q_U(s) \cdot 1_{\mathcal{K}(U)},$$

it follows that

$$\phi_U(s \otimes s - q_U(s)) = \varphi_U(s)^2 - q_U(s) \cdot 1_{\mathcal{K}(U)} = 0,$$

that is,

$$\phi_U|_{I(\mathcal{E},q)(U)} = 0. \tag{2.17}$$

Thus,

$$\overline{\varphi}|_{\mathcal{I}(\mathcal{E},q)} = 0.$$

On the other hand, $I(\mathcal{E}, q)(U)$ being an ideal of the algebra $T(\mathcal{E}(U))$, Equation (2.17) implies that there is an $\mathcal{A}(U)$-morphism

$$\psi_U : Cl(\mathcal{E}(U), q_U) \longrightarrow \mathcal{K}(U)$$

such that

$$\varphi_U = \psi_U \circ \rho_U,$$

where ρ_U is the Clifford $\mathcal{A}(U)$-morphism $\mathcal{E}(U) \longrightarrow Cl(\mathcal{E}(U), q_U)$. Clearly, the family $\psi \equiv (\psi_U)_{\ open U \supseteq X}$ yields a $\Gamma(\mathcal{A})$-morphism such that

$$\varphi = \psi \circ \rho; \tag{2.18}$$

ψ is the only $\mathcal{A}$-morphism satisfying the equality (2.18). By sheafification, one has

$$\varphi = \mathbf{S}(\varphi) = \mathbf{S}(\psi) \circ \mathbf{S}(\rho) \equiv \psi \circ \rho.$$

■

Since *inductive limits commute with quotients and tensor products*, it follows from (2.15) that, for any $x \in X$,

$$\mathcal{C}l_{\mathcal{A}}(\mathcal{E},q)_x = \big(\mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E},q)\big)_x \simeq \mathcal{T}(\mathcal{E})_x/\mathcal{I}(\mathcal{E},q)_x \simeq T(\mathcal{E}_x)/I(\mathcal{E}_x,q_x)$$

$$= \mathrm{C}l_{\mathcal{A}_x}(\mathcal{E}_x,q_x) \equiv \mathrm{C}l_{\mathcal{A}_x}(\mathcal{E}_x) \equiv \mathrm{C}l(\mathcal{E}_x).$$

Here, we will assume that, for any quadratic $\mathcal{A}_X$-module $\mathcal{E}$, the canonical mappings $\mathcal{A} \longrightarrow \mathcal{C}l_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{E} \longrightarrow \mathcal{C}l_{\mathcal{A}}(\mathcal{E})$ are injective; thus, by [35, pp. 60- 62, Lemma 12.1], for any $x \in X$, $1_{\mathcal{A}_x}$ is identified with the unit element 1_{q_x} of $\mathcal{C}l_{\mathcal{A}}(\mathcal{E},q)_x$, and every $z \in \mathcal{E}_x$ is identified with its image $\rho_x(z) \equiv \rho(z)$ in $\mathcal{C}l_{\mathcal{A}}(\mathcal{E},q)_x$. Whenever these identifications are guaranteed, the pair $(\mathcal{E},q)$ is called a quadratic $\mathcal{A}_X$-module. From now on, all quadratic $\mathcal{A}_X$-modules are understood to be Cliffordian.

If $(\mathcal{E},q_{\mathcal{E}})$ and $(\mathcal{F},q_{\mathcal{F}})$ are quadratic $\mathcal{A}_X$-modules; by definition, an $\mathcal{A}$-morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ is called an $\mathcal{A}_X$-$\mathcal{M}od^q$ morphism if

$$q_{\mathcal{F}} \circ \varphi = q_{\mathcal{E}}.$$

Now, consider an $\mathcal{A}_X$-$\mathcal{M}od^q$ morphism $\varphi : (\mathcal{E},q_{\mathcal{E}}) \longrightarrow (\mathcal{F},q_{\mathcal{F}})$ and let $\mathcal{C}l_{\mathcal{A}}(\mathcal{E},q_{\mathcal{E}})$ and $\mathcal{C}l_{\mathcal{A}}(\mathcal{F},q_{\mathcal{F}})$ be Clifford $\mathcal{A}$-algebras of quadratic $\mathcal{A}$-modules $(\mathcal{E},q_{\mathcal{E}})$ and $(\mathcal{F},q_{\mathcal{F}})$, respectively. If $\rho_{\mathcal{E}}$ and $\rho_{\mathcal{F}}$ are the universal objects in $\mathcal{A}_{\mathcal{C}l}(\mathcal{E},q_{\mathcal{E}})$ and $\mathcal{A}_{\mathcal{C}l}(\mathcal{F},q_{\mathcal{F}})$, respectively, corresponding to the Clifford $\mathcal{A}$-algebras $\mathcal{C}l_{\mathcal{A}}(\mathcal{E},q_{\mathcal{E}})$ and $\mathcal{C}l_{\mathcal{A}}(\mathcal{F},q_{\mathcal{F}})$, one has

$$(\rho_{\mathcal{F}} \circ \varphi)^2 = q_{\mathcal{E}} \cdot 1_{\mathcal{A}},$$

that is,

$$\rho_{\mathcal{F}} \circ \varphi \in \mathrm{H}om^{\mathcal{C}l}_{\mathcal{A}}(\mathcal{E},\mathcal{C}l_{\mathcal{A}}(\mathcal{F},q_{\mathcal{F}})).$$

By the universal property of $\rho_{\mathcal{E}}$, there exists a unique $\mathcal{A}_X$-$\mathcal{A}lg$ morphism, denoted by $\mathcal{C}l_{\mathcal{A}}(\varphi)$, such that

$$\mathcal{C}l_{\mathcal{A}}(\varphi) \circ \rho_{\mathcal{E}} = \rho_{\mathcal{F}} \circ \varphi,$$

that is, such that the diagram

$$\begin{array}{ccc} (\mathcal{E}, q_{\mathcal{E}}) & \xrightarrow{\rho_{\mathcal{E}}} & \mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q_{\mathcal{E}}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle \mathcal{C}l_{\mathcal{A}}(\varphi)} \\ (\mathcal{F}, q_{\mathcal{F}}) & \xrightarrow[\rho_{\mathcal{F}}]{} & \mathcal{C}l_{\mathcal{A}}(\mathcal{F}, q_{\mathcal{F}}) \end{array}$$

commutes. It is easily seen that the *map* $\mathcal{C}l_{\mathcal{A}} : \mathcal{A}_X\text{-}\mathcal{M}od^q \longrightarrow \mathcal{A}_X\text{-}\mathcal{A}lg$ *induces a covariant functor between these two categories.*

Example 2.1 Let $\mathcal{E} \equiv (\mathcal{E}(U), \rho_V^U)$ be a line $\mathcal{A}$-module on a topological space X (i.e., $\mathcal{E} \simeq \mathcal{A}$), with a family of nowhere-zero sections $(s_U)_{\ openU\subseteq X}$ for its *generators*; that is, s_U is the generator of the corresponding $\mathcal{E}(U)$ and

$$\rho_U^X(s_X) = s_U,$$

for any U. Next, we note that the *tensor algebra sheaf*

$$\mathcal{T}(\mathcal{E}) = \mathcal{A} \oplus \mathcal{E} \oplus \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \oplus \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \oplus \cdots$$

is $\mathcal{A}$-isomorphic to the sheaf $\mathcal{P}$ of $\mathcal{A}$-algebras of polynomials in indeterminate s_U. In fact, consider the $\mathcal{A}$-morphism $\varphi : \mathcal{T}(\mathcal{E}) \longrightarrow \mathcal{P}$ such that, for any $a \in \mathcal{A}(X)$,

$$\varphi_X(a + s + s \otimes s + \cdots + \underbrace{s \otimes \cdots \otimes s}_{n}) = a + s + s^2 + \cdots + s^n,$$

where $s := s_X$. Clearly, φ is an $\mathcal{A}$-isomorphism. By this $\mathcal{A}$-isomorphism and since the presheaf $T(\Gamma(\mathcal{E}))$ is complete (one has, for every open $U \subseteq X$, $\mathcal{T}(\mathcal{E})(U) \simeq T(\Gamma(\mathcal{E}))(U) \equiv T(\Gamma(\mathcal{E})(U))$), $\mathcal{I}(\mathcal{E}, q)$ is the ideal sheaf in $\mathcal{T}(\mathcal{E})$ such that, for every open $U \subseteq X$, $\mathcal{I}(\mathcal{E}, q)(U)$ is generated by $s_U^2 - q_U(s_U)$. Thus, $\mathcal{C}l(\mathcal{E}, q)(U)$ is a free $\mathcal{A}(U)$-module with basis $(1, s_U)$; whence

$$\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q) = \mathcal{C}l_{\mathcal{A}}(\mathcal{A}, q) = \mathcal{A}^2,$$

with the equality signs being actually $\mathcal{A}$-isomorphisms.

2.4 Clifford Algebras for Riemannian Quadratic Free $\mathcal{A}$-modules of Finite Rank

Making use of techniques underlying the proof in [15, pp 294, 295, Theorem VIII.2.B] that every quadratic vector space of finite dimension admits a Clifford algebra, we show in Theorem 2.4, deemed to be the main result of this section, that *with every Riemannian quadratic free $\mathcal{A}$-module of finite rank is associated up to $\mathcal{A}$-isomorphism a Clifford free $\mathcal{A}$-algebra of rank 2^n if the rank of the Riemannian quadratic free $\mathcal{A}$-module is n.*

Remark 2.8 We recall (see [35, pp. 335- 340]) that an *ordered algebraized space* $(X, \mathcal{A})$ *satisfies the inverse-closed section condition* ([41])if every nowhere-zero section of $\mathcal{A}$ is invertible and is *enriched with square root* if every nonnegative section of $\mathcal{A}$ has a square root. For the remainder of this section, unless otherwise mentioned, any pair $(\mathcal{E}, q)$ will denote a *Riemannian quadratic free $\mathcal{A}$-module of finite rank,* where the *sheaf $\mathcal{A}$ of algebras satisfies the inverse-closed section condition* and is *enriched with square root.* In this context, *if φ is a Clifford $\mathcal{A}$-morphism of $(\mathcal{E}, q)$ into $\mathcal{K}$, then, for any orthogonal gauge $(e_1, \ldots, e_n) \subseteq \mathcal{E}(U)^n$ of $\mathcal{E}$ on an open $U \subseteq X$, $\varphi_U(e_i)$, for any $i = 1, \ldots, n$, is nowhere zero.* Indeed, if b is the Riemannian $\mathcal{A}$-metric associated with q, then $q_U(e_i) = b_U(e_i, e_i)$; since b is Riemannian and e_i is nowhere zero, therefore $\varphi_U(e_i)$ is nowhere zero.

Proposition 2.2 *Let $(\mathcal{E}, q)$ be a Riemannian free $\mathcal{A}$-module of rank n. For every open U in X, let $B(U)$ be the set consisting of all the orthogonal bases of $\mathcal{E}(U)$. If, for every U, $V \in \tau_X$ with $V \subseteq U$,*

$$\rho^U_V : B(U) \longrightarrow B(V)$$

denotes the natural restriction, the collection $B := (B(U), \rho^U_V)$ determines a complete presheaf of sets (of orthogonal bases).

Proof. That B is a presheaf is immediate. Now, let U be an open subset of X and $\mathcal{U} \equiv (U_i)_{i\in I}$ a covering of U. Next, let $s \equiv (s^1, \dots, s^n)$ and $t \equiv (t^1, \dots, t^n)$ be orthogonal bases of $\mathcal{E}(U)$, i.e. $s, t \in B(U)$, such that $s|_{U_i} = t|_{U_i}$ for every $i \in I$. More explicitly, $s^j|_{U_i} = t^j|_{U_i}$ for every $i \in I$ and $j = 1, \dots, n$. Since $s^j, t^j \in \mathcal{E}(U)$ $(j = 1, \dots, n)$, it follows that $s^j = t^j$; thus, $s = t$. Hence, axiom (i) Definition 2.3 is fulfilled.

For axiom (ii) Definition 2.3, let $s_i \in B(U_i)$ such that, for every $U_i \cap U_j \equiv U_{ij} \neq \emptyset$ in $\mathcal{U}$,

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}.$$

Again, using the fact that $\Gamma(\mathcal{E})$ is complete, one has that there exists $t^k \in \mathcal{E}(U)$ such that $t^k|_{U_i} = s_i^k$, $k = 1, \dots, n$. Therefore, $t \equiv (t^1, \dots, t^n)$ is such that $t|_{U_i} = s_i$, $i \in I$. Clearly, t is orthogonal. ■

Keeping with the notation of Proposition 2.2, we will call the sheaf generated by B the *sheaf of orthogonal bases* of $\mathcal{E}$, and will denote it by $\mathcal{B}$, i.e. $\mathcal{B} = \mathbf{S}B$.

Theorem 2.2 *Let $(\mathcal{E}, q)$ be a Riemannian quadratic free $\mathcal{A}$-module of rank n, $\mathcal{C}$ an associative and unital $\mathcal{A}$-algebra, and $\varphi_{\mathcal{C}} \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$ a Clifford $\mathcal{A}$-morphism such that, given the sheaf of orthogonal bases $e_U := (e_{U,1}, \dots, e_{U,n})$ of $\mathcal{E}$, the sheaf of sets, consisting of elements of the form*

$$(\varphi_{\mathcal{C}})_U(e_{U,J}) := (\varphi_{\mathcal{C}})_U(e_{U,j_1})(\varphi_{\mathcal{C}})_U(e_{U,j_2}) \cdots (\varphi_{\mathcal{C}})_U(e_{U,j_m}),$$

where $J = (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$, assuming that $(\varphi_{\mathcal{C}})_U(e_{U,\emptyset}) = 1_{\mathcal{C}(U)}$, is a sheaf of bases for the underlying free $\mathcal{A}$-module of $\mathcal{C}$. Then, the pair $(\mathcal{C}, \varphi_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$.

Proof. In fact, let φ be a Clifford $\mathcal{A}$-morphism of $(\mathcal{E}, q)$ into some associative and unital $\mathcal{A}$-algebra $\mathcal{K}$. Moreover, let Φ be an $\mathcal{A}$-morphism of $\mathcal{C}$ into $\mathcal{K}$, given by:

$$\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J})) := \varphi_U(e_{U,J}),$$

where

$$\varphi_U(e_{U,J}) := \varphi_U(e_{U,j_1})\varphi_U(e_{U,j_2})\cdots\varphi_U(e_{U,j_m})$$

with $J := (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$.

We claim that Φ is multiplicative and 1-respecting, hence an $\mathcal{A}$-morphism of the $\mathcal{A}$-algebras $\mathcal{C}$ and $\mathcal{K}$. To this end, it suffices to show that every Φ_U is multiplicative on $((\varphi_{\mathcal{C}})_U(e_{U,J}))_{J\in\mathcal{P}(I_n)}$, where $I_n = \{1,\ldots,n\}$.

Let us consider the product in $\mathcal{C}(U)$:

$$(\varphi_{\mathcal{C}})_U(e_{U,J}) \cdot (\varphi_{\mathcal{C}})_U(e_{U,J'}) = \\ (\varphi_{\mathcal{C}})_U(e_{U,j_1})\cdots(\varphi_{\mathcal{C}})_U(e_{U,j_m})(\varphi_{\mathcal{C}})_U(e_{U,j'_1})\cdots(\varphi_{\mathcal{C}})_U(e_{U,j'_p}), \tag{2.19}$$

and the following product in $\mathcal{K}(U)$:

$$(\varphi_U)(e_{U,J}) \cdot (\varphi_U)(e_{U,J'}) = \\ (\varphi_U)(e_{U,j_1})\cdots(\varphi_U)(e_{U,j_m})(\varphi_U)(e_{U,j'_1})\cdots(\varphi_U)(e_{U,j'_p}). \tag{2.20}$$

The right-hand sides of (2.19) and (2.20) reduce to

$$\begin{aligned} (\varphi_{\mathcal{C}})_U(e_{U,J}) \cdot (\varphi_{\mathcal{C}})_U(e_{U,J'}) &= \lambda(\varphi_{\mathcal{C}})_U(e_{U,L}), where \;\; \lambda \in \mathcal{A}(U) \\ \varphi_U(e_{U,J}) \cdot \varphi_U(e_{U,J'}) &= \lambda\varphi_U(e_{U,L}), \end{aligned}$$

where $L = (1 \leq l_1 < l_2 < \cdots < l_r \leq n)$. Therefore, given J and J':

$$\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J})(\varphi_{\mathcal{C}})_U(e_{U,J'})) = \Phi_U(\lambda(\varphi_{\mathcal{C}})_U(e_{U,L})) = \lambda\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,L}))$$

$$= \lambda\varphi_U(e_{U,L}) = \varphi_U(e_{U,J})\varphi_U(e_{U,J'}) = \Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J}))\Phi_U((\varphi_{\mathcal{C}})_U(e_{U,J'}));$$

furthermore, since $\Phi_U(1_{\mathcal{C}(U)}) = \Phi_U((\varphi_{\mathcal{C}})_U(e_{U,\emptyset})) = \varphi_U(e_{U,\emptyset}) = 1_{\mathcal{K}(U)}$, Φ_U is an $\mathcal{A}(U)$-morphism, taking $\mathcal{C}(U)$ into $\mathcal{K}(U)$. Hence, $\Phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{C},\mathcal{K})$ and is 1-respecting. As $\mathcal{C}$ is generated by the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$ and the unital line sub-$\mathcal{A}$-algebra $1_{\mathcal{C}}$ of $\mathcal{C}$, then $\mathcal{C}$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$. ■

We also recall (see [35, pp. 335- 340]) that for any *ordered algebraized space* $(X, \mathcal{A})$ *satisfying the inverse-closed section condition* ([41]) and *enriched with square root, if* $(\mathcal{E}, \rho)$ *is a free Riemannian* $\mathcal{A}$*-module of finite rank* $n \in \mathbb{N}$ and

$$(s_1, \ldots, s_n) \subseteq \mathcal{E}(U)^n \simeq \mathcal{E}^n(U),$$

where U is open in X, is a (local) gauge of $\mathcal{E}$([41]), *then there exists an orthonormal gauge of* $\mathcal{E}$, obtained from $(s_1, \ldots, s_n)$, say,

$$(t_1, \ldots, t_n) \subseteq \mathcal{E}(U)^n;$$

more accurately, $t_1, \ldots, t_n$ are such that

$$\rho_U(t_i, t_j) = \delta_{ij}$$

where δ_{ij} is the Kroneecker sections of $\mathcal{A}$ over U for all $1 \leq i, j \leq n$, and

$$[t_1, \ldots, t_m] = [s_1, \ldots, s_m],$$

for every $1 \leq m \leq n$.

Hence, we have

Proposition 2.3 *Let* $(X, \mathcal{A})$ *be an ordered algebraized space, enriched with square root, and satisfying the inverse-closed section condition. Moreover, let* $(\mathcal{E}, q)$ *be a Riemannian quadratic free* $\mathcal{A}$*-module of rank* n*,* $\mathcal{K}$ *an associative and unital* $\mathcal{A}$*-algebra, and* φ *an* $\mathcal{A}$*-morphism of* $\mathcal{E}$ *into* $\mathcal{K}$*. Then,* φ *is Clifford if and only if*

$$\varphi_U(e_i)^2 = q_U(e_i) \cdot 1_{\mathcal{K}(U)}, \quad i = 1, \ldots, n \tag{2.21}$$

and

$$\varphi_U(e_i)\varphi_U(e_j) + \varphi_U(e_j)\varphi_U(e_i) = 0, \quad 1 \leq i \neq j \leq n, \tag{2.22}$$

for any open $U \subseteq X$ *and orthogonal gauge* $(e_1, \ldots, e_n)$ *of* $(\mathcal{E}(U), q_U) \equiv (\mathcal{E}(U), b_U)$, *where* $b \equiv (b_U)_{\ openU\subseteq X}$ *is the* q*-induced Riemannian* $\mathcal{A}$*-metric.*

Proof. The condition is obviously necessary. Indeed, let us consider an open subset U of X, and an orthogonal basis $(e_1, \ldots, e_n)$ of $(\mathcal{E}(U), q_U)$. Clearly, for any $i = 1, \ldots, n$,

$$\varphi_U(e_i)^2 = q_U(e_i) \cdot 1_{\mathcal{K}(U)}.$$

As for (2.22), one easily applies (2.17) and the fact that $(e_1, \ldots, e_n)$ is orthogonal.

Conversely, for any open $U \subseteq X$ and section $s \in \mathcal{E}(U)$, with $s = \sum_{i=1}^{n} \alpha^i e_i$, we have

$$\begin{aligned} \varphi_U(s)^2 &= \left[\sum_{i=1}^{n} \alpha^i \varphi_U(e_i)\right]^2 = \sum_{i=1}^{n} (\alpha^i)^2 \varphi_U(e_i)^2 \\ &= \left[\sum_{i=1}^{n} (\alpha^i)^2 q_U(e_i)\right] 1_{\mathcal{K}(U)} = \left[\sum_{i=1}^{n} q_U(\alpha^i e_i)\right] 1_{\mathcal{K}(U)} = q_U(s) \cdot 1_{\mathcal{K}(U)}. \end{aligned}$$

■

In the same vein, we observe the following. As in [15, p. 288], we reduce the number of terms in products over $\mathcal{K}(U)$ as follows: For a product

$$a \equiv \varphi_U(e_{i_1})\varphi_U(e_{i_2}) \cdots \varphi_U(e_{i_p}), \quad 1 \leq p \leq n,$$

$i)$ if $i_k > i_{k+1}$, we interchange $\varphi_U(e_{i_k})$ and $\varphi_U(e_{i_{k+1}})$ and multiply by (-1): since

$$\varphi_U(e_{i_k})\varphi_U(e_{i_{k+1}}) + \varphi_U(e_{i_{k+1}})\varphi_U(e_{i_k}) = 0.$$

$ii)$ if $i_k = i_{k+1}$, we replace $\varphi_U(e_{i_k})\varphi_U(e_{i_{k+1}}) = \varphi_U(e_{i_k})^2$ by $q_U(e_{i_k}) \cdot 1_{\mathcal{K}(U)}$.

This process will ultimately yield the following expression:

$$a = \lambda\varphi_U(e_{j_1})\varphi_U(e_{j_2})\cdots\varphi_U(e_{j_m}),$$

where $\lambda \in \mathcal{A}(U)$, and $J \equiv (1 \le j_1 < j_2 < \cdots < j_m \le n)$ an increasing sequence of indices.

As a convention, we let

$$\varphi_U(e_J) := \varphi_U(e_{j_1})\varphi_U(e_{j_2})\cdots\varphi_U(e_{j_m}),$$

where $J \equiv (1 \le j_1 < j_2 < \cdots < j_m \le n)$, and

$$\varphi_U(e_\emptyset) := 1_{\mathcal{K}(U)}$$

for the empty sequence $\emptyset$. Clearly, the 2^n elements $\varphi_U(e_J)$ of $\mathcal{K}(U)$ linearly span the sub-$\mathcal{A}(U)$-algebra $\mathcal{L}(U)$ of $\mathcal{K}(U)$, generated by $1_{\mathcal{K}(U)}$ and $\varphi_U(\mathcal{E}(U)) \equiv \varphi(\mathcal{E})(U)$. Thus, we have proved the following.

Theorem 2.3 *Let φ be a Clifford $\mathcal{A}$-morphism of a Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$ of rank n into an associative and unital $\mathcal{A}$-algebra $\mathcal{K}$. Then, $\mathcal{K}$ contains a generalized locally free $\mathcal{A}$-module with maximum rank $\le 2^n$, and containing the unital line sub-$\mathcal{A}$-module and the sub-$\mathcal{A}$-module $\varphi(\mathcal{E})$.*

Theorem 2.4 *With every Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$, there is an associated Clifford free $\mathcal{A}$-algebra $\mathcal{C} \equiv \mathcal{C}(\mathcal{E}, q)$; moreover, rank $\mathcal{C} = 2^n$ if $n =$ rank $\mathcal{E}$.*

Proof. Let $\mathcal{B}$ be a sheaf of orthogonal bases of $\mathcal{E} \equiv (\mathcal{E}, q)$, and $\mathcal{P}$ the sheaf of algebras of anticommutative polynomials over $\mathcal{A}$, such that if $p \in \mathcal{P}(U)$, for some open U in X, then p is an anticommutative polynomial in $e_1, e_2, \ldots, e_n$, where $(e_1, e_2, \ldots, e_n)$ is a fixed orthogonal basis in $\mathcal{B}(U)$. If U and V are open subsets of X with $V \subseteq U$, we fix orthogonal bases $(e_1, \ldots, e_n)$

and $(f_1, \ldots, f_n)$ in $\mathcal{E}(U)$ and $\mathcal{E}(V)$, respectively, in such a way that $\rho^U_V(e_i) \equiv e_i|_V = f_i$, for every $i = 1, \ldots, n$, where the (ρ^U_V) are restriction maps for the (complete) presheaf of sections of $\mathcal{E}$. Furthermore, we denote by $1_{\mathcal{P}(U)} \equiv 1$ the polynomial $e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n}$, where $m_i = 0$, $i = 1, \ldots, n$. On every open $U \subseteq X$, define the product in $\mathcal{P}(U)$ as follows:

$$(e_1^{p_1} e_2^{p_2} \ldots e_n^{p_n}) \cdot (e_1^{q_1} e_2^{q_2} \ldots e_n^{q_n}) = (-1)^{\sum_{i<j} q_i p_j} e_1^{p_1+q_1} \ldots e_n^{p_n+q_n}. \tag{2.23}$$

Moreover, still under the assumption that $(e_1, \ldots, e_n)$ is the fixed orthogonal basis of $\mathcal{E}(U)$, the section $e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n}$ of $\mathcal{P}$ over U such that

$$m_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

is denoted e_j. This notation ensures an identification of $\mathcal{E}$ with a sub-$\mathcal{A}$-module of $\mathcal{P}$. On the other hand, in every $\mathcal{P}(U)$, one has

$$e_i e_j = -e_j e_i \quad i \neq j.$$

The product thus defined on every $\mathcal{P}(U)$ is associative, for one easily shows that, by multiplying both members of (2.23) on the right by a polynomial $e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n}$, one obtains the following equality:

$$\sum_{i<j} q_i p_j + \sum_{i<j} r_i (p_j + q_j) = \sum_{i<j} (q_i + r_i) p_j + \sum_{i<j} r_i q_i.$$

For every i, $1 \leq i \leq n$, let $q_U(e_i) := a_i \in \mathcal{A}(U)$. Next, consider the correspondence

$$U \longmapsto \mathcal{C}(U) \subseteq \mathcal{P}(U), \tag{2.24}$$

where $\mathcal{C}(U)$ is the free $\mathcal{A}(U)$-module, with a basis consisting of the 2^n sections

$$e_1^{m_1} e_2^{m_2} \ldots e_n^{m_n},$$

where $0 \leq m_i \leq 1$ for every i. It is clear that the correspondence (2.24) together with the restriction maps restricted to the $\mathcal{C}(U)$, where U runs over the open subsets of X, yield a free $\mathcal{A}$-module of rank 2^n. We will denote the free $\mathcal{A}$-module thus obtained by $\mathcal{C} \equiv (\mathcal{C}(U), \lambda_V^U)$. Let's also consider the *projection $\mathcal{A}$-morphism* $\pi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{C})$, defined by:

$$\pi_U(e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}) := a_1^{\lfloor \frac{p_1}{2} \rfloor} \cdots a_n^{\lfloor \frac{p_n}{2} \rfloor} e_1^{\overline{p_1}} e_2^{\overline{p_2}} \cdots e_n^{\overline{p_n}},$$

where $(e_1, e_2, \cdots, e_n)$ is the fixed orthogonal basis of $\mathcal{E}(U)$,

$$\lfloor \frac{p_i}{2} \rfloor = \max\{x \in \mathbb{Z} : \ x \leq \frac{p_i}{2}\}$$

for every $1 \leq i \leq n$, and

$$p_i = 2l_i + \overline{p_i}, \quad l_i \in \mathbb{Z}, \overline{p_i} \in \mathbb{Z},$$

viz. $\overline{p_i}$ is the remainder of p_i modulo 2.

Given sections f and g of the $\mathcal{A}$-algebra $\mathcal{P}$ over an open subset U of X, one has

$$\pi_U(f \cdot g) = \pi_U(\pi_U(f) \cdot \pi_U(g)),$$

which is easily verified by taking $f = e_1^{p_1} \cdots e_n^{p_n}$ and $g = e_1^{q_1} \cdots e_n^{q_n}$.

Finally, we define on the free $\mathcal{A}$-module $\mathcal{C}$ the following multiplication: if $s, t \in \mathcal{C}(U)$, where U is open in X, then

$$s * t := \pi_U(s \cdot t).$$

$*$ is associative; the proof of this fact may be found in [15, p. 295]. Hence, $\mathcal{C}$ is an associative and unital free $\mathcal{A}$-algebra, which contains $\mathcal{E}$. Let's denote by $\iota_{\mathcal{C}}$ the inclusion $\mathcal{E} \subseteq \mathcal{C}$. Since, for every open $U \subseteq X$ and orthogonal basis $(e_1, \cdots, e_n)$ of $\mathcal{E}(U)$,

$$(\iota_{\mathcal{C}})_U(e_i)^2 = a_i \cdot 1_{\mathcal{C}(U)}$$

and

$$(\iota_{\mathcal{C}})_U(e_i)(\iota_{\mathcal{C}})_U(e_j) + (\iota_{\mathcal{C}})_U(e_j)(\iota_{\mathcal{C}})_U(e_i) = 0, \quad 1 \leq i \neq j \leq n,$$

the pair $\mathcal{C} \equiv (\mathcal{C}, \iota_{\mathcal{C}})$ is a Clifford $\mathcal{A}$-algebra of $(\mathcal{E}, q)$, by virtue of Theorem 2.2. ■

Chapter 3

The Commutative Property of the Clifford and Localization Functors

In this chapter we try to investigate the commutative property that the Clifford functor has with the algebra extension (mainly through the tensor product) functor of the ground algebra sheaf of a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$. As a particular case, we also show that the Clifford functor commutes with the localization functor. With regard to the organization of the chapter, we start with brief remarks on extension of the scalars of $\mathcal{A}$-modules, which is dealt with in Section 3.1. Section 3.2 is a discussion about sheaves of $\mathcal{A}$-modules of fractions (with denominator a monoid-subsheaf $\mathcal{S}$ of $\mathcal{A}$) and change of the algebra sheaf $\mathcal{A}$ of scalars, on a topological space X. The main results in this section include: Theorem 3.4, which stipulates that *given a sheaf $\mathcal{A}$ of unital and commutative algebras on a topological space X and $\mathcal{S}$ a sheaf of submonoids in $\mathcal{A}$, the sheaf $\mathcal{S}^{-1}\mathcal{A}$ is an algebra sheaf on X*; and Theorem 3.6, which shows that *the functor $\mathcal{S}^{-1} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$ is exact and equivalent to the functor $\mathcal{S}^{-1}\mathcal{A} \otimes \_$, i.e., $\mathcal{S}^{-1}\mathcal{E} \simeq \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$.*

Section 3.3 deals with localization of $\mathcal{A}$-modules at prime ideal subsheaves of $\mathcal{A}$ and at *locally maximal-idealized subsheaves* of $\mathcal{A}$. For this purpose, a *Nakayama's lemma* for this sheaf-theoretic setting is investigated. Finally, in Section 3.4, we establish *the commutativity between the Clifford and the extension of the algebra sheaf $\mathcal{A}$ of scalars of an $\mathcal{A}$-module $\mathcal{E}$ functors*, which, in turn, gives rise to the isomorphism depicted by the diagram of Corollary 3.8.

3.1 Extension of Scalars of $\mathcal{A}$-modules

Given a sheaf morphism $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ of *unital* and *commutative* algebra sheaves $\mathcal{A} \equiv (\mathcal{A}, \tau_{\mathcal{A}}, X)$ and $\mathcal{B} \equiv (\mathcal{B}, \tau_{\mathcal{B}}, X)$ and an $\mathcal{A}$-module $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$, $\mathcal{E}$ may be made into a $\mathcal{B}$-module as follows: for all $x \in X$, $b \in \mathcal{B}_x$ and $e \in \mathcal{E}_x$, the product be is defined to be $\varphi_x(b)e$, i.e., $\mathcal{E}_x$ is also a $\mathcal{B}_x$-module. This follows from the fact that φ is stalk preserving,($\varphi_x(\mathcal{B}_x) \subseteq \mathcal{A}_x$) and $\mathcal{E}_x$ is an $\mathcal{A}_x$-module for every $x \in X$. And clearly, the "*exterior module multiplication in $\mathcal{E}$*", viz. the map

$$\mathcal{B} \circ \mathcal{E} \longrightarrow \mathcal{E} : \quad (b, e) \longmapsto be \equiv \varphi_x(b)e \in \mathcal{E}_x \subseteq \mathcal{E},$$

with $\tau_{\mathcal{B}}(b) = \pi(e) = x \in X$, *is continuous.* Actually, it is a consequence of the fact that $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ is a continuous map and the exterior module multiplication in $\mathcal{E}$ is continuous with respect to $\mathcal{A}$. (For the sake of convenience, we have used the notation $\mathcal{B} \circ \mathcal{E} := \{(b, e) \in \mathcal{B} \times \mathcal{E} : \ \tau_{\mathcal{B}}(b) = \pi(e)\}$ in conformity with [35, p. 87, (1.1)].) Such an algebra sheaf morphism $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ is called an $\mathcal{B}$, even though φ is not necessarily injective. Our terminology is different from the terminology of Mallios, [35, p. 260ff], which states the following: given two $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ on X, any short exact $\mathcal{A}$-sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow 0$$

is called an $\mathcal{A}$-*extension* of $\mathcal{E}$ by $\mathcal{F}$.

Now, let us suppose that $\mathcal{E}$ is a free $\mathcal{B}$-module of finite rank n on X; we may derive from $\mathcal{E}$ two free $\mathcal{A}$-modules, called the *extensions* of $\mathcal{E}$, which are $\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})$; in this vein, see [38] for *complexification* of $\mathcal{A}$-modules, which is defined to be the process of obtaining new free $\mathcal{A}$-modules by *enlarging the* $\mathbb{R}$-*algebra sheaf* $\mathcal{A}$ *to a* $\mathbb{C}$-*algebra sheaf,* denoted $\mathcal{A}_{\mathbb{C}}$. Indeed, for any $x\in X$, an element $a\in\mathcal{A}_x$ multiplies with an element $a'\otimes e\in(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E})_x=\mathcal{A}_x\otimes_{\mathcal{B}_x}\mathcal{E}_x=\mathcal{A}_x\otimes_{\mathcal{B}_x}(\mathcal{B}_x)^n=(\mathcal{A}_x)^n$ (the last three equalities actually stand for $\mathcal{B}_x$-isomorphisms; to corroborate this fact, see [35, p.123, (3.18); p.130, (5.9); p.131, (5.18)]) or an element $z\in\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})_x=\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{B}^n)_x=(\mathcal{A}^*)^n_x=(\mathcal{A}^n_x)^*=\mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x,\mathcal{B}^n_x)$, with these equalities being valid within $\mathcal{B}_x$-isomorphisms, in the following way:

$$a(a'\otimes e)=(aa')\otimes e \quad \text{and} \quad (az)(a')=z(aa').$$

Yet, still under the assumption that $\mathcal{E}$ is a free $\mathcal{B}$-module of finite rank on a topological space X, and $\varphi:\mathcal{B}\longrightarrow\mathcal{A}$ a sheaf morphism of unital and commutative algebra sheaves $\mathcal{A}$ and $\mathcal{B}$, the next lemma is related to the canonical $\mathcal{B}$-morphisms, $\mathcal{E}\longrightarrow\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})\longrightarrow\mathcal{E}$, given, for any $x\in X$, $e\in\mathcal{E}_x$ and $z\in\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})_x\simeq\mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x,\mathcal{E}_x)$, by $e\longmapsto 1_{\mathcal{A}_x}\otimes e$ and $z\longmapsto z(1_{\mathcal{A}_x})$, respectively; the former is not always surjective, whereas the latter is not always injective. When $\mathcal{A}=\mathcal{B}$, these $\mathcal{B}$-morphisms are bijective for *any* given $\mathcal{B}$-module $\mathcal{E}$, *not necessarily free*, and both $\mathcal{B}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{B},\mathcal{E})$ are $\mathcal{B}$-isomorphic to $\mathcal{E}$.

Lemma 3.1 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves on a topological space X, $\varphi:\mathcal{B}\longrightarrow\mathcal{A}$ a surjective sheaf morphism, and $\mathcal{E}\equiv(\mathcal{E},\pi,X)$ a locally free $\mathcal{B}$-module of rank n (i.e., a vector sheaf). Then, $\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ is canonically $\mathcal{B}$-isomorphic to the quotient $\mathcal{E}/(\ker\varphi)\mathcal{E}$, and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})$ is $\mathcal{B}$-isomorphic to the sub-$\mathcal{B}$-module of $\mathcal{E}$, whose stalks consist of elements $z\in\mathcal{E}_x$ $(\pi(z)=x\in X)$ such that $(\ker\varphi)_xz=(\ker(\varphi_x))z=0_x$.*

Proof. Let $\iota : \ker\varphi \longrightarrow \mathcal{B}$ be the natural injection, then, clearly,

$$0 \longrightarrow \ker\varphi \xrightarrow{\iota} \mathcal{B} \xrightarrow{\varphi} \mathcal{A} \longrightarrow 0 \tag{3.1}$$

is exact. Tensoring (3.1) with the vector sheaf $\mathcal{E}$ yields an exact $\mathcal{B}$-sequence (see [35, p.131, Theorem 5.1]), viz.

$$0 \longrightarrow \ker\varphi \otimes_{\mathcal{B}} \mathcal{E} \longrightarrow \mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} \longrightarrow \mathcal{A} \otimes_{\mathcal{B}} \mathcal{E} \longrightarrow 0. \tag{3.2}$$

Note that, for $x \in X$,

$$(\ker\varphi \otimes_{\mathcal{B}} \mathcal{E})_x = \ker\varphi_x \otimes_{\mathcal{B}_x} \mathcal{E}_x = (\ker\varphi_x)\mathcal{E}_x = ((\ker\varphi)\mathcal{E})_x, \tag{3.3}$$

within $\mathcal{B}_x$-isomorphisms (*the second $\mathcal{B}_x$-isomorphism in* (3.3) *is a classical result*; see, for instance, the proof of [25, p.18, Lemma 1.9.1]). $(\ker\varphi)\mathcal{E}$ is the $\mathcal{B}$-module, obtained by sheafifying the presheaf

$$U \longmapsto \langle(\ker\varphi_U)\mathcal{E}(U)\rangle,$$

where $\langle(\ker\varphi_U)\mathcal{E}(U)\rangle$ is the $\mathcal{B}(U)$-module generated by the set $(\ker\varphi_U)\mathcal{E}(U)$, that is, the set of $t \in \mathcal{E}(U)$ such that $t = \alpha \cdot s$, with $\alpha \in \ker\varphi_U$ and $s \in \mathcal{E}(U)$. The restriction maps for this presheaf are obvious. It follows from (3.3) that

$$\ker\varphi \otimes_{\mathcal{B}} \mathcal{E} = (\ker\varphi)\mathcal{E}, \tag{3.4}$$

within $\mathcal{B}$-isomorphism. Since $\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} = \mathcal{E}$ within $\mathcal{B}$-isomorphism, it follows, taking also account of (3.4), that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E} = \mathcal{E}/(\ker\varphi)\mathcal{E}$ within $\mathcal{B}$-isomorphism.

For any $x \in X$, the following sequence of $\mathcal{B}_x$-modules, namely

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x) \xrightarrow{\mu} \mathrm{Hom}_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x) \xrightarrow{\nu} \mathrm{Hom}_{\mathcal{B}_x}(\ker(\varphi_x), \mathcal{E}_x)\ , \tag{3.5}$$

where $\mu := \mathrm{Hom}_{\mathcal{B}_x}(\varphi_x^*, \mathcal{E}_x)$ and $\nu := \mathrm{Hom}_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)$, is exact. ($\mu$ and ν are given by: for $f \in \mathrm{Hom}_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$, $\mu(f) = \mathrm{Hom}_{\mathcal{B}_x}(\varphi_x^*, \mathcal{E}_x)(f) := f \circ \varphi_x$; similarly, for $g \in \mathrm{Hom}_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x)$, $\nu(g) =$

$\text{H}om_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)(g) := g \circ \iota_x$.) For (3.5), see, for instance, [8, p.227, Theorem 1]. The exactness of (3.5) implies that $\text{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$ is $\mathcal{B}_x$-isomorphic to the sub-$\mathcal{B}_x$-module of $\text{H}om_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x) \simeq \mathcal{E}_x$ consisting of z such that $\text{H}om_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)(z) = 0_x$, i.e., $\ker(\varphi_x)z = 0_x$. Obviously, if $\mathcal{U} \equiv (U_\alpha)_{\alpha \in I}$ is a *local frame* of $\mathcal{E}$, i.e., for all $\alpha \in I$, $\mathcal{E}|_{U_\alpha} = \mathcal{B}^n|_{U_\alpha}$, within $\mathcal{B}|_{U_\alpha}$-isomorphism, then $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})(U_\alpha) = (\mathcal{A}^*|_{U_\alpha})^n$, within $\mathcal{B}|_{U_\alpha}$-isomorphism; consequently, for any $x \in X$, $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x = (\mathcal{A}_x^*)^n$ within $\mathcal{B}_x$-isomorphism. On the other hand, $\text{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x) = \text{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{B}_x^n) = (\mathcal{A}_x^*)^n$, within $\mathcal{B}_x$-isomorphism (cf. [35, p.299, (5.8)]). Thus, $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x = \text{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$, within $\mathcal{B}_x$-isomorphism. Hence, for all $x \in X$, the corresponding stalk $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x$ is $\mathcal{B}_x$-isomorphic to the sub-$\mathcal{B}_x$-module of $\mathcal{E}_x$ consisting of z such that $\ker(\varphi_x)z = 0_x$. ■

Using the results discussed above, it is possible to obtain the following $\mathcal{A}$-isomorphisms:

$$\mathcal{T}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}) = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{T}_{\mathcal{B}}(\mathcal{E}) \tag{3.6}$$

and

$$\mathcal{S}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}) = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{S}_{\mathcal{B}}(\mathcal{E}), \tag{3.7}$$

where $\mathcal{T}_{\mathcal{B}}(\mathcal{E})$ and $\mathcal{S}_{\mathcal{B}}(\mathcal{E})$ are the *tensor algebra* and *symmetric algebra sheaves of $\mathcal{E}$ on X*, respectively; they both are *sheaves of $\mathcal{B}$-algebras* (or *$\mathcal{B}$-algebras*, for short) on X. From Section 2.2 of this thesis, we recall that the $\mathcal{B}$-algebra $\mathcal{T}_{\mathcal{B}}(\mathcal{E})$ may be constructed *equivalently* as the sheaf generated by the presheaf $T\mathcal{E} \equiv ((T\mathcal{E})(U) := T(\mathcal{E}(U)))_{\text{open } U \subseteq X}$ of $\mathcal{B}(U)$-algebras, given by

$$U \longmapsto \oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)) \equiv T(\mathcal{E}(U)),$$

where $U \subseteq X$ is open, along with the obvious restriction maps. Indeed, with every open $U \subseteq X$, one associates the following (canonical) $\mathcal{B}(U)$-morphism

$$T\mathcal{E}(U) \equiv \oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)) \xrightarrow{\phi_U} (\mathcal{T}_{\mathcal{B}}\mathcal{E})(U) := \oplus_{n=0}^{\infty} (\mathcal{T}_{\mathcal{B}}^n \mathcal{E})(U); \tag{3.8}$$

therefore one obtains a morphism $\overline{\phi}$ of the sheaves, generated by the presheaves of $\mathcal{B}(U)$-algebras, which appear in the two members of (3.8). It suffices to prove that $\overline{\phi}$ is a fiber-wise $\mathcal{B}$-isomorphism. To this end, we observe the following $\mathcal{B}_x$-isomorphisms

$$\begin{aligned}(\mathbf{S}(T\mathcal{E}))_x &= (\oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)))_x = \varinjlim_{x\in U}(\oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)))\\ &\equiv \varinjlim_{x\in U}(T(\mathcal{E}(U))) = T(\varinjlim_{x\in U}\mathcal{E}(U)) = T(\mathcal{E}_x) = \oplus_{n=0}^{\infty}(T^n(\mathcal{E}(U)))_x\\ &= \oplus_{n=0}^{\infty}(\varinjlim_{x\in U} T^n(\mathcal{E}(U))) = (\oplus_{n=0}^{\infty}\mathcal{T}^n\mathcal{E})_x \equiv (\mathcal{T}\mathcal{E})_x,\end{aligned}$$

for every $x \in X$.

Since, by virtue of [35, p.130, (5.9)],

$$\mathcal{T}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E})_x = (\mathcal{T}_{\mathcal{A}})_x(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E})_x = \mathcal{T}_{\mathcal{A}_x}(\mathcal{A}_x\otimes_{\mathcal{B}_x}\mathcal{E}_x) = \mathcal{A}_x\otimes_{\mathcal{B}_x}\mathcal{T}_{\mathcal{B}_x}(\mathcal{E}_x),$$

within $\mathcal{A}_x$-isomorphisms (see [25, p.18] for the last $\mathcal{A}_x$-isomorphism), it follows, on the basis of [35, p.68, Theorem 12.1], that (3.6) is fulfilled. Likewise, one obtains (3.7).

Now, the next theorem is a connection between the functor $\mathcal{H}om$ and the bifunctor $\otimes$, which is classically called the *adjoint associativity of $\mathcal{H}om$ and tensor product.* This result is also established by Kashiwara and Schapira [27, p.439, Proposition 18.2.3(ii)] for presheaves and sheaves constructed on a *site X with values in a certain category with suitable properties.* We recall that a *site X is a small category $\mathcal{C}_X$ endowed with a Grothendieck topology.* The adjunction associativity formula suggests the following: *Let $\mathcal{R}$ be a sheaf of rings on a site X, and k_X a sheaf of k-algebras on X, where k denotes a commutative unital ring. If we denote by PSh($\mathcal{R}$) the category of presheaves of $\mathcal{R}$-modules, then, given $F \in$ PSh($\mathcal{R}^{op}$), $G \in$ PSh($\mathcal{R}$) and $H \in$ PSh(k_X), there is an isomorphism*

$$\mathcal{H}om_{k_X}(F\otimes_{\mathcal{R}} G, H) \simeq \mathcal{H}om_{\mathcal{R}}(G, \mathcal{H}om_{k_X}(F, H)), \tag{3.9}$$

functorial in F, G and H. (The notations used are those of [27, p.439, Proposition 18.2.3(ii)].) For the purpose of *a version of the adjunction associativity formula* in our setting, let us notice that *given algebra sheaves $\mathcal{K}$ and $\mathcal{L}$ on a given topological space X, a $(\mathcal{K}, \mathcal{L})$-bimodule $\mathcal{E}$ on X and a left $\mathcal{K}$-module $\mathcal{F}$ on X, the sheaf $\mathcal{H}om_{\mathcal{K}}(\mathcal{E}, \mathcal{F})$ is a left $\mathcal{L}$-module.* We assume that all the sheaves involved in Theorem 3.1 are defined on a given topological space X.

Theorem 3.1 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves, $\mathcal{E}$ a locally free left $\mathcal{B}$-module of rank m, $\mathcal{G}$ a left $\mathcal{A}$-module. Moreover, let $\mathcal{F}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule such that as a left $\mathcal{A}$-module, $\mathcal{F}$ is locally free and of rank n. Then,*

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G}) \tag{3.10}$$

within isomorphism of group sheaves.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be local frames of $\mathcal{E}$ and $\mathcal{F}$, respectively. That $\mathcal{W} \equiv \mathcal{U} \cap \mathcal{V} := \{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V}\}$ is a common local frame of $\mathcal{E}$ and $\mathcal{F}$ is clear. So, if $U \in \mathcal{W}$, then, applying [35, p. 137, (6.22), (6.23), (6.24′)], one has the following $\mathcal{B}|_U$-isomorphisms:

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))|_U = \mathcal{H}om_{\mathcal{B}|_U}(\mathcal{B}^m|_U, \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{A}^n, \mathcal{G}|_U)),$$

that is,

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))|_U = \mathcal{G}^{mn}|_U. \tag{3.11}$$

In the same way, one shows that

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})|_U = \mathcal{G}^{mn}|_U \tag{3.12}$$

within an $\mathcal{A}|_U$-isomorphism. On the other hand, for any open subset W of X, one has the following morphism

$$\mathrm{H}om_{\mathcal{B}|_W}(\mathcal{E}|_W, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})|_W) \xrightarrow{\varphi_W} \mathrm{H}om_{\mathcal{A}|_W}(\mathcal{E}|_W \otimes_{\mathcal{B}|_W} \mathcal{F}|_W, \mathcal{G}|_W),$$

which is given by

$$\varphi_W(\alpha)(s \otimes t) := (\alpha_Z(s))_Z(t) \equiv \alpha(s)(t),$$

where $\alpha \in \mathrm{H}om_{\mathcal{B}|_W}(\mathcal{E}|_W, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})|_W)$, $s \in (\mathcal{E}|_W)(Z) = \mathcal{E}(Z)$, $t \in \mathcal{F}(Z)$, with Z an open subset of W. Clearly, the family $\varphi \equiv (\varphi_W)_{\ open\ W \subseteq X}$ yields an $\mathcal{A}$-morphism. We shall indeed show that the sheafification $\mathbf{S}(\varphi) \equiv \widetilde{\varphi}$ of φ is an $\mathcal{A}$-isomorphism. For this purpose, we notice that, by virtue of (3.11) and (3.12),

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x = \mathcal{G}_x^{mn} = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})_x, \tag{3.13}$$

for any $x \in X$. The equalities in (3.13) are valid up to group isomorphisms. Furthermore, as

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x)$$

and

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})_x = \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x, \mathcal{G}_x),$$

for any $x \in X$, φ_x is a $\mathcal{B}_x$-isomorphism (see [5, p. 198, Theorem 15.6]). Whence, by [35, p. 68, Theorem 12.1], φ is an $\mathcal{A}$-isomorphism, and the proof is complete. ■

When $\mathcal{E}$ is a locally free $\mathcal{B}$-module of finite rank as in Theorem 3.1, and $\mathcal{F}$ an $\mathcal{A}$-module, the following canonical $\mathcal{A}$-isomorphisms follow from (3.10)

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{A}, \mathcal{F}) = \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}). \tag{3.14}$$

Moreover, by [35, p.130, (5.14) and (5.15)], one has

$$\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} = (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{F} \tag{3.15}$$

within $\mathcal{A}$-isomorphism.

The fact that the category $\mathcal{A}$-$\mathcal{M}od_X$ of sheaves of $\mathcal{A}$-modules, where $\mathcal{A}$ is commutative and unital, on a fixed topological space X is an *abelian category* [35, p.158] heralds the following.

Definition 3.1 (i) An object $\mathcal{P} \in \mathcal{A}\text{-}\mathcal{M}od_X$ is *projective object* if the functor

$$\mathrm{H}om_{\mathcal{A}}(\mathcal{P}, \cdot) : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{A}(X)\text{-}\mathrm{M}od, \tag{3.16}$$

where $\mathcal{A}(X)$-Mod is the category of modules over the algebra $\mathcal{A}(X)$, is exact.

(ii) An object $\mathcal{F} \in \mathcal{A}\text{-}\mathcal{M}od_X$ is *flat object* if the functor $\mathcal{E} \longmapsto \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$ is exact.

(iii) Given a unital and commutative algebra sheaf $\mathcal{B}$ on X, an *extension* $\mathcal{B} \longrightarrow \mathcal{A}$ is called *flat extension of $\mathcal{A}$-algebras* if $\mathcal{A}$ is a flat $\mathcal{B}$-module.

Lemma 3.2 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves on a given topological space X, and $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a unity-preserving sheaf morphism. For any locally free $\mathcal{B}$-module $\mathcal{E}$ of finite rank on X, $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}$ is $\mathcal{A}$-projective if $\mathcal{E}$ is $\mathcal{B}$-projective. On the other hand, for any $\mathcal{B}$-module $\mathcal{F}$ on X, the $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}$ is $\mathcal{A}$-flat if $\mathcal{F}$ is $\mathcal{B}$-flat.*

Proof. By the $\mathcal{A}$-isomorphism (3.14), one has

$$\begin{aligned}\mathrm{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F}) &= \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F})(X) \\ &= \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})(X) = \mathrm{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}).\end{aligned}$$

Therefore, if $\mathrm{H}om_{\mathcal{B}}(\mathcal{E}, \cdot)$ is exact, then $\mathrm{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \cdot)$ is exact, which means that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}$ is $\mathcal{A}$-projective whenever $\mathcal{E}$ is $\mathcal{B}$-projective. The remaining assertion is also easy to prove. ■

Now, as in Lemma 3.2, we assume that $\mathcal{E}$ is a locally free $\mathcal{B}$-module of rank n, and $\mathcal{F}$ any $\mathcal{B}$-module, both on the same topological space X. Moreover, let us consider the following canonical $\mathcal{A}$-morphism

$$\Phi \equiv (\Phi_x)_{x \in X} : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \tag{3.17}$$

such that, for all $x \in X$,

$$\Phi_x(a \otimes z)(a' \otimes e) := aa' \otimes z(e), \tag{3.18}$$

where $a,\ a' \in \mathcal{A}_x$, $z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x = \varinjlim_{x \in U} \mathrm{Hom}_{\mathcal{B}|_U}((\mathcal{B}|_U)^n, \mathcal{F}|_U) = \mathcal{F}_x^n = \mathrm{Hom}_{\mathcal{B}_x}(\mathcal{B}_x^n, \mathcal{F}_x)$ (with U a *local gauge* of $\mathcal{E}$) and $e \in \mathcal{E}_x$. Observe the following $\mathcal{A}_x$-isomorphisms

$$\begin{aligned}
\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})_x &= \varinjlim_{x \in U} \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{E}|_U, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U) \\
&= \varinjlim_{x \in U} \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{A}|_U \otimes_{\mathcal{B}|U} (\mathcal{B}|_U)^n, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U) \\
&= \varinjlim_{x \in U} \mathrm{Hom}_{\mathcal{A}|_U}((\mathcal{A}|_U)^n, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U) \\
&= \varinjlim_{x \in U} (\mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U)^n \\
&= (\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x)^n
\end{aligned}$$

(U is a local gauge of $\mathcal{E}$); therefore, Φ is well-defined.

Lemma 3.3 *Let $\mathcal{E}$ be a locally free $\mathcal{B}$-module of rank n on a topological space X such that every stalk $\mathcal{E}_x$ is $\mathcal{B}_x$-projective. Then, $\mathcal{E}$ is $\mathcal{B}$-projective.*

Proof. In fact, let

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$$

be a $\mathcal{B}$-exact sequence. Since exactness is transferred to stalks of sheaves (cf. [35, p.113, (2.34)]) and, for any $x \in X$ and $\mathcal{B}$-module $\mathcal{F}$ on X, $\mathrm{Hom}_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{F}_x) = \mathcal{F}_x^n = \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x$, one has

$$0 \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S}')_x \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S})_x \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S}'')_x \longrightarrow 0. \tag{3.19}$$

The exactness of (3.19) follows from the fact that $\mathcal{E}_x$, $x \in X$, is $\mathcal{B}_x$-projective (see, for instance, [8, p.231, Proposition 4]). On the other hand, since *any complex* $\mathcal{G}' \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{G}''$ *of sheaves* *is exact if and only if, for any $x \in X$, the induced complex* $\mathcal{G}'_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{G}''_x$ *is exact* (cf. [48, p.99, Proposition 5.3.4], [35, p. 113, (2.34)]), one obtains the following exact $\mathcal{B}$-sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S}') \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S}) \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{S}'') \longrightarrow 0;$$

whence the vector sheaf $\mathcal{E}$ is $\mathcal{B}$-projective. ■

Keeping with the notations of Lemma 3.2, we have.

Theorem 3.2 *Suppose that every stalk $\mathcal{E}_x$ of the vector sheaf $\mathcal{E}$ is $\mathcal{B}_x$-projective, then the canonical sheaf morphism* (3.17) *is $\mathcal{A}$-isomorphic.*

Proof. Since every $\mathcal{E}_x$ is projective, it follows, by means of [25, p.19, Proposition 1.9.7], that

$$\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x = \mathcal{H}om_{\mathcal{A}_x}(\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{E}_x, \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x)$$

within $\mathcal{A}_x$-isomorphism; whence the $\mathcal{A}$-morphism Φ is isomorphic (see, for instance, [35, p.68, Theorem 12.1]). ■

Definition 3.2 [27, p.446, Definition 18.5.1] An $\mathcal{A}$-module $\mathcal{E}$ on a topological space X is said to be *locally finitely presented* if there is an open covering $\mathcal{U} \equiv (U_\alpha)_{\alpha \in I}$ of X such that, for every $\alpha \in I$, the $\mathcal{A}$-sequence

$$(\mathcal{A}|_{U_\alpha})^m = \mathcal{A}^m|_{U_\alpha} \longrightarrow \mathcal{A}^n|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha} \longrightarrow 0, \tag{3.20}$$

where m, $n \in \mathbb{N}$, is exact.

We shall state a *useful property of flat $\mathcal{A}$-extensions,* which stipulates that under certain conditions the functors $\otimes$ and $\mathcal{H}om$ *commute.*

First, let us recall the following result.

Lemma 3.4 [51, p.114, Exercise 12] *Let $(X, \mathcal{A})$ be an algebraized space and $\mathcal{E}$ a finitely presented $\mathcal{A}$-module on X. Then, for any $\mathcal{A}$-module $\mathcal{F}$ on X and $x \in X$, the natural morphism*

$$(\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}))_x \longrightarrow \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x) \tag{3.21}$$

is an isomorphism.

Theorem 3.3 *Let $\mathcal{A}$, $\mathcal{B}$ be unital commutative algebra sheaves on a topological space X, and $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a flat extension. For any locally finitely presented $\mathcal{B}$-module $\mathcal{E}$ on X,*

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \tag{3.22}$$

within $\mathcal{A}$-isomorphism, for any $\mathcal{B}$-module $\mathcal{F}$ on X.

Proof. That (3.22) holds for free $\mathcal{B}$-modules of finite rank is obvious. In fact, suppose that $\mathcal{E} \simeq \mathcal{B}^n$ $(n \in \mathbb{N})$, then, one has,

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}^n = (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})^n, \tag{3.23}$$

with the preceding equalities being valid within $\mathcal{A}$-isomorphisms. On the other hand,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \simeq (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})^n. \tag{3.24}$$

Fix $x \in X$; if $a \in \mathcal{A}_x$, $z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x = \mathcal{F}_x^n$, and

$$\Phi_x(a \otimes z)(a' \otimes e) := aa' \otimes z(e) = 0$$

for all $a' \in \mathcal{A}_x$ and $e \in \mathcal{E}_x$, then $a \otimes z = 0$; this implies that Φ_x is injective. Moreover, since both $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})$ are free as finite direct sums of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}$ (cf. (3.23) and (3.24)), it follows that Φ_x is bijective. Hence, Φ is an isomorphism (see, for instance, [35, p.68, Theorem 12.1]).

Now, let us suppose that $\mathcal{E}$ is properly locally finitely presented; so, for every $x \in X$, there are an open set $U \subseteq X$ and locally free $\mathcal{B}$-modules $\mathcal{E}_1$ and $\mathcal{E}_0$ of finite rank such that

$$\mathcal{E}_1|_U \simeq \mathcal{B}^m|_U \longrightarrow \mathcal{E}_0|_U \simeq \mathcal{B}^n|_U \longrightarrow \mathcal{E}|_U \longrightarrow 0$$

is right exact. One thus obtains, by virtue of Lemma 3.4, the following diagram, for every $x \in X$,

$$0 \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_x, \mathcal{F}_x) \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{0x}, \mathcal{F}_x) \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{1x}, \mathcal{F}_x),$$

with $\mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{0x}, \mathcal{F}_x) = \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{0x}, \mathcal{A}_x \otimes \mathcal{F}_x)$ and $\mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{1x}, \mathcal{F}_x) = \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{1x}, \mathcal{A}_x \otimes \mathcal{F}_x)$ within $\mathcal{A}_x$-isomorphisms. On the other hand, since

$$0 \longrightarrow \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_x, \mathcal{A}_x \otimes \mathcal{F}_x) \longrightarrow \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{0x}, \mathcal{A}_x \otimes \mathcal{F}_x),$$

it follows that

$$\begin{aligned}\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x &= \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathrm{H}om_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{F}_x) \\ &= \mathrm{H}om_{\mathcal{A}_x}(\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{E}_x, \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})_x\end{aligned}$$

within $\mathcal{A}_x$-isomorphisms. Since the last $\mathcal{A}_x$-isomorphisms hold for any $x \in X$,

$$\Phi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}),$$

where Φ_x is given by (3.18), is an $\mathcal{A}$-isomorphism. ■

3.2 $\mathcal{A}$-modules of Fractions

As part of the required generalities on $\mathcal{A}$-quadratic morphisms for the attainment of the goal set for our ongoing project are *sheaves of $\mathcal{A}$-modules of fractions* on a topological space X, which are also simply called $\mathcal{A}$-*modules of fractions* on X. Indeed, given a sheaf $\mathcal{A}$ of unital algebras, a *subsheaf of multiplicative sets* (also called a *sheaf of multiplicatively closed subsets*) is a subsheaf $\mathcal{S}$ of submonoids of $\mathcal{A}$. In other words, for every open U in X, $\mathcal{S}(U)$ is a multiplicative subset of the unital algebra $\mathcal{A}(U)$ (cf. [8, pp. 17-20], [25, p. 21], [32, pp. 107- 109]), that is, a subset containing $1_{\mathcal{A}(U)}$, and such that, if $s, t \in \mathcal{S}(U)$, then $st, ts \in \mathcal{S}(U)$. Let $\mathfrak{C}$ be the category of all sheaf morphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\varphi(\mathcal{S}) \subseteq \mathcal{B}^{\bullet}$, where $\mathcal{B}^{\bullet}$ is the subsheaf of groups of units of $\mathcal{B}$; so for any open U in X and section $s \in \mathcal{S}(U) \subseteq \mathcal{A}(U)$, $\varphi_U(s)$ is invertible in $\mathcal{B}(U)$. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\psi : \mathcal{A} \longrightarrow \mathcal{C}$ are two objects of $\mathfrak{C}$, a morphism u of φ into ψ is a sheaf morphism $u : \mathcal{B} \longrightarrow \mathcal{C}$ making the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ & \searrow^{\psi} & \downarrow u \\ & & \mathcal{C} \end{array}$$

commute. The sheaf morphism $\mathcal{A} \longrightarrow 0$ is a *final universal object* in $\mathfrak{C}$, and in this case, if every $\mathcal{S}_x$ contains 0_x, then the morphism $\mathcal{A} \longrightarrow 0$ is the unique object of $\mathfrak{C}$. If $\mathfrak{C}$ contains an *initial universal object* $\varphi : \mathcal{A} \longrightarrow \mathcal{K}$, it is unique up to sheaf isomorphism, and $\mathcal{K}$ is called the *sheaf of algebras of fractions of $\mathcal{A}$ with denominator in $\mathcal{S}$*, and is denoted $\mathcal{S}^{-1}\mathcal{A}$.

Definition 3.3 Let X be a topological space, $\mathcal{A} \equiv (\mathcal{A}, \pi, X)$ a sheaf of unital and commutative algebras, and $\mathcal{S} \equiv (\mathcal{S}, \pi|_{\mathcal{S}}, X)$ a sheaf of submonoids in $\mathcal{A}$. A *sheaf of algebras of fractions of $\mathcal{A}$ by $\mathcal{S}$* is a sheaf of algebras, denoted $\mathcal{S}^{-1}\mathcal{A}$, such that, for every point $x \in X$, the corresponding stalk $(\mathcal{S}^{-1}\mathcal{A})_x$ is an *algebra of fractions of $\mathcal{A}_x$ by $\mathcal{S}_x$*; in other words,

$$(\mathcal{S}^{-1}\mathcal{A})_x = \mathcal{S}_x^{-1}\mathcal{A}_x \tag{3.25}$$

for all $x \in X$.

Explicitly, fix x in X; the stalk $\mathcal{S}_x$ is a submonoid of the unital and commutative algebra $\mathcal{A}_x$. The algebra of fractions of $\mathcal{A}_x$ by $\mathcal{S}_x$ is defined (see [32, pp. 107- 111]) by considering the equivalence relation, on the set $\mathcal{A}_x \times \mathcal{S}_x$:

$$(r, s) \sim (r', s')$$

provided there exists an element $t \in \mathcal{S}_x$ such that

$$t(s'r - sr') = 0.$$

The equivalence class containing a pair (r, s) is denoted by $\frac{r}{s}$, and the set of all equivalence classes is denoted by $\mathcal{S}_x^{-1}\mathcal{A}_x$. The set $\mathcal{S}_x^{-1}\mathcal{A}_x$ becomes an algebra by virtue of the operations

$$\frac{r}{s} + \frac{r'}{s'} := \frac{s'r + sr'}{ss'}$$

and

$$\frac{r}{s}\frac{r'}{s'} := \frac{rr'}{ss'}.$$

Theorem 3.4 *$\mathcal{S}^{-1}\mathcal{A}$ is an algebra sheaf on X.*

Proof. Let us consider the projection map

$$q : \mathcal{A} \circ \mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \tag{3.26}$$

given by

$$q_x(r, s) := \frac{r}{s}, \tag{3.27}$$

for every $x \in X$, $r \in \mathcal{A}_x$ and $s \in \mathcal{S}_x$. ($\mathcal{A} \circ \mathcal{S}$ is the subsheaf of the sheaf $\mathcal{A} \times \mathcal{S}$, given by $\mathcal{A} \circ \mathcal{S} := \{(a, s) \in \mathcal{A} \times \mathcal{S} : \pi(a) = \pi|_{\mathcal{S}}(s)\}$.) By considering the topology coinduced by q on $\mathcal{S}^{-1}\mathcal{A}$, that is, $U \subseteq \mathcal{S}^{-1}\mathcal{A}$ is open if and only if $q^{-1}(U)$ is open in $\mathcal{A} \circ \mathcal{S}$, with $\mathcal{A} \circ \mathcal{S}$ carrying the relative topology from $\mathcal{A} \times \mathcal{S}$, we quickly show that the *map*

$$\sigma : \mathcal{S}^{-1}\mathcal{A} \longrightarrow X$$

such that

$$\begin{array}{ccc} \mathcal{A} \circ \mathcal{S} & \xrightarrow{\quad q \quad} & \mathcal{S}^{-1}\mathcal{A}\,, \\ & {\scriptstyle \tau}\searrow \qquad \swarrow{\scriptstyle \sigma} & \\ & X & \end{array}$$

where τ is the obvious projection, *is a local homeomorphism*; hence $\mathcal{S}^{-1}\mathcal{A} \equiv (\mathcal{S}^{-1}\mathcal{A}, \sigma, X)$ *is a sheaf of algebras on* X*.* Indeed, for any open U in X, we clearly have that $\sigma^{-1}(U)$ is open in $\mathcal{S}^{-1}\mathcal{A}$, which implies that σ is continuous. To show that σ is a local homeomorphism, consider a point $z \in \mathcal{S}^{-1}\mathcal{A}$ and let V be an open neighborhood of z in $\mathcal{S}^{-1}\mathcal{A}$. Then, $q^{-1}(z) \subseteq q^{-1}(V)$, with $q^{-1}(V)$ open in $\mathcal{A} \circ \mathcal{S}$. Next, let $u \in q^{-1}(z)$ and W an open neighborhood of u such that $\tau|_W$ is a homeomorphism. (The projection τ is a local homeomorphism for the following reason: *Given two sheaves of $\mathcal{A}$-modules* $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ *and* $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ *on* X*, the triple* $(\mathcal{E} \oplus \mathcal{E}', \sigma, X)$*, where* $\mathcal{E} \oplus \mathcal{E}' := \{(z, z') \in \mathcal{E} \times \mathcal{E}' : \ \pi(z) = \pi'(z')\}$ *and* $\sigma : \mathcal{E} \oplus \mathcal{E}' \longrightarrow X : (z, z') \longmapsto \sigma(z, z') := \pi(z) = \pi'(z')$*, is a sheaf on* X*, viz.* σ *is a local homeomorphism.* See [35, p. 120].) Since $\sigma(z) \in X$ and σ is continuous, there exists an open neighborhood O of $\sigma(z)$ in X such that $z \in \sigma^{-1}(O)$. That $\sigma|_{\sigma^{-1}(O \cap \tau(W))}$ is homeomorphic is clear. Indeed, let us first show that σ is bijective on $\sigma^{-1}(O \cap \tau(W))$. To this end, consider $z_1 \neq z_2$ in $\sigma^{-1}(O \cap \tau(W))$; so $q^{-1}(z_1) \cap q^{-1}(z_2) = \emptyset$, whence $\tau(q^{-1}(z_1) \cap W) \cap \tau(q^{-1}(z_2) \cap W) = \emptyset$. Consequently, $\sigma(z_1) \neq \sigma(z_2)$; hence, $\sigma|_{\sigma^{-1}(O \cap \tau(W))}$ is injective. For surjectiveness, let $\alpha \in O \cap \tau(W)$. Then, $\sigma(q(\tau^{-1}(\alpha))) = \alpha$, with $q(\tau^{-1}(\alpha)) \in \sigma^{-1}(O \cap \tau(W))$. Finally, let V be open in $\sigma^{-1}(O \cap \tau(W))$. It follows that $q^{-1}(V)$ is open, and since

$$q^{-1}(V) \subseteq q^{-1}(\sigma^{-1}(O \cap \tau(W))) = \tau^{-1}(O \cap \tau(W)) \subseteq W,$$

$\tau(q^{-1}(V))$ is open. But

$$\tau(q^{-1}(V)) = (\sigma \circ q)(q^{-1}(V)) = \sigma(V),$$

therefore $\sigma(V)$ is open in $O \cap \tau(W)$. Thus, as required, σ is a homeomorphism on $\sigma^{-1}(O \cap \tau(W))$.

■

Theorem 3.5 *Every sheaf $\mathcal{A}$ of unital and commutative algebras admits a sheaf $\mathcal{S}^{-1}\mathcal{A}$ of algebras of fractions of $\mathcal{A}$ with denominator in $\mathcal{S}$.*

Proof. Fix an open set U in X. We consider pairs (a, s) and (a', s') of $(\mathcal{A} \times \mathcal{S})(U) = \mathcal{A}(U) \times \mathcal{S}(U)$; they are said to be equivalent if there exists $t \in \mathcal{S}(U)$ such that

$$t(s'a - sa') = 0;$$

it is easy to verify that the above condition defines an equivalence relation on $\mathcal{A}(U) \times \mathcal{S}(U)$. We denote by $\mathcal{S}(U)^{-1}\mathcal{A}(U)$ the set of equivalence classes; the equivalence class containing a pair (a, s) is denoted by $\frac{a}{s}$. By defining a multiplication and an addition on $\mathcal{S}(U)^{-1}\mathcal{A}(U)$ by the rules

$$\Big(\frac{a}{s}\Big)\Big(\frac{a'}{s'}\Big) := \frac{aa'}{ss'} \tag{3.28}$$

and

$$\frac{a}{s} + \frac{a'}{s'} = \frac{s'a + sa'}{ss'}, \tag{3.29}$$

respectively, $\mathcal{S}(U)^{-1}\mathcal{A}(U)$ acquires a ring structure. One can easily verify that (3.28) and (3.29) are well-defined. The zero and unity sections are the fractions $\frac{0}{1}$ and $\frac{1}{1}$, respectively.

Next, let us show that the collection $(\mathcal{S}(U)^{-1}\mathcal{A}(U))_{\ open\ U \subseteq X}$ induces a *complete* presheaf on X. First, it is clear that the collection in question yields a presheaf. Now, let us show that this presheaf is complete. Indeed, let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of an open subset U of X, and let $\frac{a}{s}$, $\frac{a'}{s'}$ be two elements ("sections") of $\mathcal{S}(U)^{-1}\mathcal{A}(U)$ such that

$$\rho^U_{U_i}(\frac{a}{s}) = \rho^U_{U_i}(\frac{a'}{s'}),$$

where the $(\rho^U_{U_i})_{(U,U_i)\in\tau^*}$ are the restriction maps of the aforementioned presheaf. But, assuming that the $(\alpha^U_{U_i})_{(U,U_i)\in\tau^*}$ are the restriction maps of the underlying presheaf of $\mathcal{A}$, if (a,s) and (a',s') are pairs representing the classes $\frac{a}{s}$ and $\frac{a'}{s'}$, respectively, then

$$\rho^U_{U_i}(\tfrac{a}{s}) = \frac{\alpha^U_{U_i}(a)}{\alpha^U_{U_i}(s)}, \quad \rho^U_{U_i}(\tfrac{a'}{s'}) = \frac{\alpha^U_{U_i}(a')}{\alpha^U_{U_i}(s')};$$

since $\mathcal{A}$ is a sheaf and $\mathcal{S}$ a subsheaf of $\mathcal{A}$,

$$\frac{a}{s} = \frac{a'}{s'},$$

which means that axiom (i) of Definition 3.2 is satisfied.

For axiom (ii) of Definition 3.2, let $(\frac{a_i}{s_i}) \in \prod_{i\in I}\mathcal{S}(U_i)^{-1}\mathcal{A}(U_i)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$ in $\mathcal{U}$, one has

$$\rho^{U_i}_{U_{ij}}(\frac{a_i}{s_i}) = \rho^{U_j}_{U_{ij}}(\frac{a_j}{s_j}). \tag{3.30}$$

For any $i \in I$, let (a_i, s_i) represent the corresponding equivalence class $\frac{a_i}{s_i}$. From Equation (3.30), it follows that

$$\alpha^{U_i}_{U_{ij}}(a_i) = \alpha^{U_j}_{U_{ij}}(a_j)$$

and

$$\alpha^{U_i}_{U_{ij}}(s_i) = \alpha^{U_j}_{U_{ij}}(s_j).$$

Clearly, there are $a \in \mathcal{A}(U)$ and $s \in \mathcal{S}(U)$ such that

$$\alpha^U_{U_i}(a) = a_i, \quad \alpha^U_{U_i}(s) = s_i forall i \in I;$$

so

$$\rho^U_{U_i}(\frac{a}{s}) = \frac{a_i}{s_i}.$$

By considering the prescription

$$\mathcal{A}(U) \times \mathcal{S}(U)^{-1}\mathcal{A}(U) \longrightarrow \mathcal{S}(U)^{-1}\mathcal{A}(U)$$

such that

$$(\lambda, \frac{a}{s}) \longmapsto (\frac{\lambda}{1})(\frac{a}{s}) = \frac{\lambda a}{s}$$

defines an $\mathcal{A}(U)$-algebra structure on $\mathcal{S}(U)^{-1}\mathcal{A}(U)$. Thus, the sheafification $\mathcal{S}^{-1}\mathcal{A}$ of the presheaf $(\mathcal{S}(U)^{-1}\mathcal{A}(U), \rho^U_V)_{(U,V)\in\tau^*}$ is an $\mathcal{A}$-algebra sheaf.

Finally, let us show that the sheaf morphism $\varphi_\mathcal{S} : \mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A}$ such that, for any open $U \subseteq X$ and section $a \in \mathcal{A}(U)$,

$$(\varphi_\mathcal{S})_U(a) := \frac{a}{1}$$

is an initial universal object in the category $\mathcal{C}$; in other words, $\mathcal{S}^{-1}\mathcal{A}$ is a sheaf of algebras of fractions of $\mathcal{A}$ with denominator in $\mathcal{S}$. To this end, suppose that $\frac{a}{s} = \frac{a'}{s'} \in (\mathcal{S}^{-1}\mathcal{A})(U)$. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be an object of $\mathfrak{C}$. By an easy calculation, one sees that

$$f_U(a)f_U(s)^{-1} = f_U(a')f_U(s')^{-1};$$

the map

$$h_U : (\mathcal{S}^{-1}\mathcal{A})(U) \longrightarrow \mathcal{B}(U)$$

given by $h_U(\frac{a}{s}) := f_U(a)f_U(s)^{-1}$, for all $\frac{a}{s} \in (\mathcal{S}^{-1}\mathcal{A})(U)$ is thus well-defined and is such that

$$h_U \circ (\varphi_\mathcal{S})_U = f_U$$

which is unique. It is trivially verified that

$$\rho_V^U \circ (\varphi_{\mathcal{S}})_U = (\varphi_{\mathcal{S}})_V \circ \alpha_V^U,$$

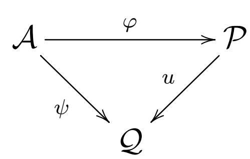
for any pair $(U, V) \in \tau^*$, corroborating the fact that $\varphi_{\mathcal{S}}$ is a sheaf morphism and the required initial universal object. ■

Corollary 3.1 *For any open $U \subseteq X$,*

$$(\mathcal{S}^{-1}\mathcal{A})(U) = \mathcal{S}(U)^{-1}\mathcal{A}(U)$$

within an $\mathcal{A}(U)$-bijection.

Given a sheaf $\mathcal{A}$ of unital and commutative algebras on a given topological space X, let $\mathfrak{C}$ be the category of morphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ of sheaves of *unital* algebras such that $\varphi(\mathcal{S}) \subseteq \mathcal{P}^\bullet$, where $\mathcal{P}^\bullet$ is the subsheaf of units of $\mathcal{P}$; so, for any point $x \in X$ and element $z \in \mathcal{S}_x$, $\varphi_x(z)$ is invertible in $\mathcal{P}_x$. If $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ and $\psi : \mathcal{A} \longrightarrow \mathcal{Q}$ are two objects in $\mathfrak{C}$, a *morphism* u of φ to ψ is a sheaf morphism $u : \mathcal{P} \longrightarrow \mathcal{Q}$ making the diagram



commute. The sheaf morphism $\mathcal{A} \longrightarrow 0$ is a in $\mathfrak{C}$, and in this case, if every $\mathcal{S}_x$ contains 0_x, then the morphism $\mathcal{A} \longrightarrow 0$ is the unique object of $\mathfrak{C}$.

Lemma 3.5 *The sheaf morphism*

$$\varphi_{\mathcal{S}} : \mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \tag{3.31}$$

such that $(\varphi_{\mathcal{S}})_x(r) := \frac{r}{1_x} \equiv \frac{r}{1}$, for every $x \in X$ and $r \in \mathcal{A}_x$, is a initial universal object in $\mathfrak{C}$.

Proof. Clearly, $\varphi_{\mathcal{S}}$ can be decomposed as

$$\varphi_{\mathcal{S}} = q \circ \iota, \tag{3.32}$$

where $\iota : \mathcal{A} \longrightarrow \mathcal{A} \circ \mathcal{S}$ is the injection given by

$$\iota_x(r) = (r, 1),$$

for any $r \in \mathcal{A}_x \subseteq \mathcal{A}$ ($x \in X$), and q is the projection (3.26). Thus, $\varphi_{\mathcal{S}}$ is continuous. Since, in addition, $\varphi_{\mathcal{S}}$ is "*fiber preserving*", it is a morphism of sheaves of algebras $\mathcal{A}$ and $\mathcal{S}^{-1}\mathcal{A}$.

Now, let $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ be an object of $\mathfrak{C}$. It is clear that if $r, r' \in \mathcal{A}_x$, $s, s' \in \mathcal{S}_x$, where $x \in X$, and $\frac{r}{s} = \frac{r'}{s'}$,

$$\varphi_x(r)\varphi_x(s)^{-1} = \varphi_x(r')\varphi_x(s')^{-1};$$

so that we can define a map

$$\psi : \mathcal{S}^{-1}\mathcal{A} \longrightarrow \mathcal{P}$$

such that

$$\psi_x(\frac{r}{s}) = \varphi_x(r)\varphi_x(s)^{-1},$$

for all $\frac{r}{s} \in (\mathcal{S}^{-1}\mathcal{A})_x$. It is trivially verified that, for every $x \in X$, ψ_x is the unique algebra homomorphism such that $\psi_x \circ (\varphi_{\mathcal{S}})_x = \varphi_x$. By virtue of (3.32), we have that, for every open U in $\mathcal{P}$,

$$\varphi^{-1}(U) = (\varphi_{\mathcal{S}})^{-1}(\psi^{-1}(U)) = \iota^{-1}q^{-1}(\psi^{-1}(U)),$$

with $\varphi^{-1}(U)$ open in $\mathcal{A}$. But $\iota(\mathcal{A})$ is open in $\mathcal{A} \circ \mathcal{S}$, therefore $q^{-1}(\psi^{-1}(U))$ is open in $\mathcal{A} \circ \mathcal{S}$; so $\psi^{-1}(U)$ is open in $\mathcal{S}^{-1}\mathcal{A}$, whence ψ is continuous. We deduce that ψ is the unique sheaf morphism such that $\psi \circ \varphi_{\mathcal{S}} = \varphi$, which means that $\varphi_{\mathcal{S}}$ is the required initial universal object. ■

In keeping with the above notations, let, now, $\mathcal{E} \equiv (\mathcal{E}, \rho, X)$ be an $\mathcal{A}$-module on a topological space X, and $\mathfrak{D}$ the category whose objects are the $\mathcal{A}$-morphisms $\varphi : \mathcal{E} \longrightarrow \mathcal{P}$ from $\mathcal{E}$ into any $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{P}$; given an $\mathcal{A}$-morphism $\varphi' : \mathcal{E} \longrightarrow \mathcal{P}'$, a morphism from φ to φ' is an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $u : \mathcal{P} \longrightarrow \mathcal{P}'$ such that $\varphi' = u \circ \varphi$. If $\mathfrak{D}$ contains an initial universal object $\varphi^{\mathcal{S}}_{\mathcal{E}} \equiv \varphi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{M}$, then $\mathcal{M}$ is called the *sheaf of* $(\mathcal{S}^{-1}\mathcal{A})$*-modules of fractions of* $\mathcal{E}$ *with denominator in* $\mathcal{S}$ and is denoted by $\mathcal{S}^{-1}\mathcal{E}$.

We need to show that $\varphi_{\mathcal{E}}$ exists in the category $\mathfrak{D}$. For this purpose, we define, on every stalk $\mathcal{E}_x \times \mathcal{S}_x$, the following equivalence relation: Two elements (e, s) and (e', s') of $\mathcal{E}_x \times \mathcal{S}_x$, $x \in X$, are said to be equivalent if there exists an element $t \in \mathcal{S}_x$ such that

$$t(s'e - se') = 0;$$

the set of all equivalence classes in $\mathcal{E}_x \times \mathcal{S}_x$ is called the *module of fractions of the module* $\mathcal{E}_x$ *with denominator in* $\mathcal{S}_x$ (see, for instance, [9, pp. 60-70], [25, pp. 21-25]), and is denoted by $\mathcal{S}_x^{-1}\mathcal{E}_x$. The equivalence class containing the pair (e, s) in $\mathcal{S}_x^{-1}\mathcal{E}_x$ is denoted by $\frac{e}{s}$. It is easy to see that every $\mathcal{S}_x^{-1}\mathcal{E}_x$ becomes an $\mathcal{S}_x^{-1}\mathcal{A}_x$-module under the operations

$$\frac{e_1}{s_1} + \frac{e_2}{s_2} := \frac{s_2 e_1 + s_1 e_2}{s_1 s_2}$$

and

$$\frac{p}{q}\frac{e}{s} := \frac{pe}{qs},$$

where $\frac{e}{s}$, $\frac{e_1}{s_1}$, $\frac{e_2}{s_2} \in \mathcal{S}_x^{-1}\mathcal{E}_x$ and $\frac{p}{q} \in \mathcal{S}_x^{-1}\mathcal{A}_x$.

As was the case for the sheaf of algebras of fractions $\mathcal{S}^{-1}\mathcal{A}$ above, one shows that the space

$$\mathcal{S}^{-1}\mathcal{E} := \sum_{x \in X} \mathcal{S}_x^{-1}\mathcal{E}_x,$$

endowed with the final topology determined by the natural map

$$q : \mathcal{E} \circ \mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{E}$$

is a sheaf of $(\mathcal{S}^{-1}\mathcal{A})$-modules. (Again we have assumed the notation $\mathcal{E} \circ \mathcal{S} := \{(e, s) \in \mathcal{E} \times \mathcal{S} : \rho(e) = \pi(s)\}$.) Moreover, the mapping

$$\varphi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{E} \tag{3.33}$$

such that

$$(\varphi_{\mathcal{E}})_x(e) := \frac{e}{1_x} = \frac{e}{1},$$

is an $\mathcal{A}$-morphism; similar to the proof of Lemma 3.5, one shows that $\varphi_{\mathcal{E}}$ *is an initial universal object in* $\mathfrak{D}$.

Every sheaf $\mathcal{S}$ of submonoids in a sheaf $\mathcal{A}$ of unital and commutative algebras over a topological space X yields a functor from the category $\mathcal{A}$-$\mathcal{M}od_X$ of $\mathcal{A}$-modules into the category $(\mathcal{S}^{-1}\mathcal{A})$-$\mathcal{M}od_X$ of $(\mathcal{S}^{-1}\mathcal{A})$-modules; more accurately, with every $(\mathcal{A}$-$\mathcal{M}od_X)$-object $\mathcal{E}$, we associate the $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{S}^{-1}\mathcal{E}$, and with every $\mathcal{A}$-morphism $\psi : \mathcal{E} \longrightarrow \mathcal{F}$ we associate the $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$, which is obtained in the following way: because of the universal property of $\mathcal{S}^{-1}\mathcal{E}$, the $\mathcal{A}$-morphism

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

can be factorized in a unique way through $\mathcal{S}^{-1}\mathcal{E}$, that is,

$$\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

is unique with respect to satisfying the equation

$$\mathcal{S}^{-1}\psi \circ \varphi^{\mathcal{S}}_{\mathcal{E}} \equiv \mathcal{S}^{-1}\psi \circ \varphi_{\mathcal{E}} = \varphi_{\mathcal{F}} \circ \psi \equiv \varphi^{\mathcal{S}}_{\mathcal{F}} \circ \psi. \tag{3.34}$$

In particular, (3.34) implies that, for any $x \in X$ and element $\frac{e}{s} \in (\mathcal{S}^{-1}\mathcal{E})_x$,

$$(\mathcal{S}^{-1}\psi)_x(\frac{e}{s}) := \frac{\psi_x(e)}{s}. \tag{3.35}$$

Theorem 3.6 *The functor $\mathcal{S}^{-1} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$, $\mathcal{S}^{-1}(\mathcal{E}) := \mathcal{S}^{-1}\mathcal{E}$, $\mathcal{S}^{-1}(\psi) := \mathcal{S}^{-1}\psi$, for all $(\mathcal{A}\text{-}\mathcal{M}od_X)$-objects $\mathcal{E}$ and $\mathcal{A}$-morphisms ψ, is exact. Moreover, there is a one-to-one correspondence between functors $\mathcal{S}^{-1}$ and $\mathcal{S}^{-1}\mathcal{A} \otimes \_ : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$ such that $\mathcal{E} \longmapsto \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$, and $\psi \longmapsto \mathcal{S}^{-1}\mathcal{A} \otimes \psi := 1_{\mathcal{S}^{-1}\mathcal{A}} \otimes \psi$, for all $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{A}$-morphisms ψ.*

Proof. Let us consider an exact sequence $\mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}''$ in the category $\mathcal{A}\text{-}\mathcal{M}od_X$. Since $\psi \circ \varphi = 0$ and, for any $x \in X$ and element $\frac{e'}{s'} \in (\mathcal{S}^{-1}\mathcal{E}')_x$, on applying (3.35), one has

$$(\mathcal{S}^{-1}\psi \circ \mathcal{S}^{-1}\varphi)_x(\frac{e'}{s'}) = (\mathcal{S}^{-1}\psi)_x(\frac{\varphi_x(e')}{s'}) = \frac{\psi_x(\varphi_x(e'))}{s'} = \frac{(\psi \circ \varphi)_x(e')}{s'} = 0;$$

it follows that

$$\mathcal{S}^{-1}\psi \circ \mathcal{S}^{-1}\varphi = 0,$$

i.e., im $\mathcal{S}^{-1}\varphi \subseteq \ker \mathcal{S}^{-1}\psi$. Now, let us show, for every $x \in X$, the inclusion

$$\ker(\mathcal{S}^{-1}\psi)_x \simeq (\ker \mathcal{S}^{-1}\psi)_x \subseteq (\text{im } \mathcal{S}^{-1}\varphi)_x \simeq \text{im}(\mathcal{S}^{-1}\varphi)_x$$

(cf. [35, pp. 108, 109; (2.11), (2.13)]); in other words, we must prove that every fraction $\frac{e}{s} \in \ker(\mathcal{S}^{-1}\psi)_x$ is contained in $\text{im}(\mathcal{S}^{-1}\varphi)_x$. Since $\mathcal{S}^{-1}\mathcal{E}$ is an $(\mathcal{S}^{-1}\mathcal{A})$-module, claiming that $(\mathcal{S}^{-1}\psi)_x(\frac{e}{s}) = 0$ implies that $(\mathcal{S}^{-1}\psi)_x(\frac{e}{1}) = 0$; whence $\psi_x(e) \in \ker(\varphi_{\mathcal{E}''})_x$, where $\varphi_{\mathcal{E}''}$ is the canonical mapping $\mathcal{E}'' \longrightarrow \mathcal{S}^{-1}\mathcal{E}''$. By the classical result (cf. [25, p. 22]), which states that, *given a unital and commutative ring K, a multiplicative subset S of K, and a K-module M, an element $x \in M$ belongs to the kernel of the canonical morphism $M \longrightarrow S^{-1}M$, $x \longmapsto \frac{x}{1}$, if and only if there exist $t \in S$ such that $tx = 0$*, we have that $\psi_x(e) \in \ker(\varphi_{\mathcal{E}''})_x$ if and only if there exists $t \in \mathcal{S}_x$

such that $t\psi_x(e) = \psi_x(te) = 0_x \equiv 0$. Therefore there exists $e' \in \mathcal{E}'_x$ such that $\varphi_x(e') = te$, whence $\frac{e}{s} = (\mathcal{S}^{-1}\varphi)_x(\frac{e'}{st})$ as required.

The proof of the second part is just as straightforward. In fact, there is an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ resulting from the universal property of $\mathcal{S}^{-1}\mathcal{E}$; more precisely, for every $x \in X$ and $e \in \mathcal{E}_x$, we have the following commutative diagram

$$\begin{array}{ccc} e & \longrightarrow & \frac{e}{1} \\ & \searrow & \downarrow \\ & & 1_x \otimes e. \end{array} \tag{3.36}$$

On the other hand, the tensor product $\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ yields an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{E}$ such that, for any $x \in X$ and elements $\frac{r}{s} \in (\mathcal{S}^{-1}\mathcal{A})_x$ and $e \in \mathcal{E}_x$, $\frac{r}{s} \otimes e$ is mapped onto $\frac{re}{s}$. The vertical arrow in (3.36) yields an $(\mathcal{S}^{-1}\mathcal{A})$-morphism which maps $\frac{re}{s} \in (\mathcal{S}^{-1}\mathcal{E})_x$ onto $\frac{r}{s} \otimes e \in (\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E})_x = (\mathcal{S}^{-1}\mathcal{A})_x \otimes \mathcal{E}_x$ (the preceding equality actually stands for an $\mathcal{A}_x$-isomorphism, cf. [35, p. 130, (5.9)]). Clearly, the $(\mathcal{S}^{-1}\mathcal{A})_x$-morphisms $\frac{re}{s} \longmapsto \frac{r}{s} \otimes e$ and $\frac{r}{s} \otimes e \longmapsto \frac{re}{s}$ are inverse isomorphisms. By [35, p. 68, Theorem 12.1], $\mathcal{S}^{-1}\mathcal{E} = \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism, and if we denote this isomorphism by $\mathcal{S}^{-1}_{\mathcal{E}}$, clearly, it follows that $\mathcal{S}^{-1}_{\mathcal{E}''} \circ \psi = (1_{\mathcal{S}^{-1}\mathcal{A}} \otimes \psi) \circ \mathcal{S}^{-1}_{\mathcal{E}}$, i.e., the $\mathcal{S}^{-1}_{\mathcal{E}}$'s form an equivalence transformation. ■

Corollary 3.2 *The algebra sheaf extension $\mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A}$ is flat.*

Proof. Indeed, the exactness of the functor $\mathcal{S}^{-1}\mathcal{A} \otimes \_$ follows immediately from the exactness of the functor $\mathcal{S}^{-1}$ (cf. Theorem 3.6). ■

Corollary 3.3 *For all $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, one has*

$$\mathcal{S}^{-1}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) = \mathcal{S}^{-1}\mathcal{E} \otimes_{\mathcal{S}^{-1}\mathcal{A}} \mathcal{S}^{-1}\mathcal{F} \tag{3.37}$$

within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism.

Proof. Indeed, by an easy calculation, one has:

$$\begin{aligned}(\mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{E})\otimes_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{F}) &= \left[(\mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{E})\otimes_{\mathcal{S}^{-1}\mathcal{A}}\mathcal{S}^{-1}\mathcal{A}\right]\otimes_{\mathcal{A}}\mathcal{F}\\ &= \left[\mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}(\mathcal{E}\otimes_{\mathcal{S}^{-1}\mathcal{A}}\mathcal{S}^{-1}\mathcal{A})\right]\otimes_{\mathcal{A}}\mathcal{F}\\ &= (\mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{E})\otimes_{\mathcal{A}}\mathcal{F}\\ &= \mathcal{S}^{-1}\mathcal{A}\otimes_{\mathcal{A}}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F})\\ &= \mathcal{S}^{-1}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F}),\end{aligned}$$

valid within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphisms. ■

Relation (3.37) shows that the *functors* $\mathcal{S}^{-1}$ *and* $\otimes$ *commute.* In Theorem 3.7 below, we show that the *functor* $\mathcal{S}^{-1}$ *commutes with the functor* $\mathcal{H}om$ under certain conditions. See, for instance, [25, p. 19, Proposition 1.9.7] and [9, p. 76, Proposition 19] for the classical case.

Theorem 3.7 *For all* $\mathcal{A}$*-modules* $\mathcal{E}$ *and* $\mathcal{F}$ *on a topological space* X*, the* $(\mathcal{S}^{-1}\mathcal{A})$*-morphism*

$$\vartheta:\mathcal{S}^{-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})\longrightarrow\mathcal{H}om_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E},\mathcal{S}^{-1}\mathcal{F}),\tag{3.38}$$

given by

$$\vartheta_x(f/s)(e/t):=f(e)/st,\tag{3.39}$$

where $x\in X$*,* $s,t\in\mathcal{S}_x$*,* $e\in(\mathcal{S}^{-1}\mathcal{E})_x$*,* $f\in\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})_x$*, is an* $(\mathcal{S}^{-1}\mathcal{A})$*-isomorphism whenever* $\mathcal{E}$ *is a locally finitely presented* $\mathcal{A}$*-module.*

Proof. On the basis of Lemma 3.4, since $\mathcal{E}$ is a locally finitely presented $\mathcal{A}$-module, one has

$$\begin{aligned}\left(\mathcal{S}^{-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})\right)_x &= (\mathcal{S}^{-1}\mathcal{A})_x\otimes_{\mathcal{A}_x}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})_x\\ &= \mathcal{S}_x^{-1}\mathcal{A}_x\otimes_{\mathcal{A}_x}\mathrm{Hom}_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{F}_x)\\ &= \mathcal{S}_x^{-1}\mathrm{Hom}_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{F}_x),\end{aligned}$$

therefore elements of the $(\mathcal{S}_x^{-1}\mathcal{A}_x)$-module $\mathcal{S}_x^{-1}\mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{F}_x)$ are of the form f/s, with $f \in \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{F}_x)$ and $s \in \mathcal{S}_x \subseteq \mathcal{A}_x$. By Theorem 3.6, the functor $\mathcal{S}^{-1}$ is exact, therefore, $\mathcal{S}^{-1}\mathcal{E}$ is locally finitely presented, so that

$$\mathcal{H}om_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E},\mathcal{S}^{-1}\mathcal{F})_x = \mathrm{H}om_{(\mathcal{S}^{-1}\mathcal{A})_x}\big((\mathcal{S}^{-1}\mathcal{E})_x,(\mathcal{S}^{-1}\mathcal{F})_x\big);$$

hence, (3.39) is well-defined, and ϑ is clearly an $(\mathcal{S}^{-1}\mathcal{A})$-morphism. By virtue of Theorem 3.3 and Corollary 3.2, (3.38) is an $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism whenever $\mathcal{E}$ is a locally finitely presented $\mathcal{A}$-module on X. ■

3.3 Localization of Vector Sheaves

In Section 3.2 of this thesis, we saw the introduction of localization of $\mathcal{A}$-modules. For easy referencing, we herewith recall results that are useful for the sequel. Given a sheaf $\mathcal{A}_X := \mathcal{A} \equiv (\mathcal{A},\pi,X)$ of unital and commutative algebras and a sheaf $\mathcal{S}_X := \mathcal{S} \equiv (\mathcal{S},\pi|_{\mathcal{S}},X)$ of submonoids in $\mathcal{A}$, a *sheaf of algebras of fractions of* $\mathcal{A}$ *by* $\mathcal{S}$ is a sheaf of algebras, denoted $\mathcal{S}^{-1}\mathcal{A}$ or $\mathcal{A}_{\mathcal{S}}$, such that, for every point $x \in X$, the corresponding stalk $(\mathcal{S}^{-1}\mathcal{A})_x$ is an algebra of fractions of $\mathcal{A}_x$ by $\mathcal{S}_x$; in other words,

$$(\mathcal{S}^{-1}\mathcal{A})_x := \mathcal{S}_x^{-1}\mathcal{A}_x \equiv \mathcal{A}_{x\mathcal{S}_x} =: (\mathcal{A}_{\mathcal{S}})_x, \qquad \text{for all } x \in X, \tag{3.40}$$

that is, the *localization functor* $\mathcal{S}^{-1}$ *commutes with direct limits.* For a fixed sheaf $\mathcal{A}$ of unital and commutative algebras and a fixed subsheaf $\mathcal{S} \subseteq \mathcal{A}$ of submonoids, we obtain a category, denoted $\mathfrak{C}$, whose objects are morphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ of sheaves of unital and commutative algebras such that $\varphi(\mathcal{S}) \subseteq \mathcal{P}^\bullet$, where $\mathcal{P}^\bullet$ is the subsheaf of units of $\mathcal{P}$; so, for any point $x \in X$ and element $z \in \mathcal{S}_x$, $\varphi_x(z)$ is invertible in $\mathcal{P}_x$. If $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ and $\psi : \mathcal{A} \longrightarrow \mathcal{Q}$ are two objects in $\mathfrak{C}$, a morphism u

of φ to ψ is a sheaf morphism $u : \mathcal{P} \longrightarrow \mathcal{Q}$ such that $u \circ \varphi = \psi$. Next, we recall that the sheaf morphism $\varphi_{\mathcal{S}} : \mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A}$ such that $(\varphi_{\mathcal{S}})_x(r) := \frac{r}{1}$, for every $x \in X$ and $r \in \mathcal{A}_x$, is a universal object in $\mathfrak{C}$.

Again assuming that $\mathcal{A}_X \equiv \mathcal{A}$ and $\mathcal{S} \subseteq \mathcal{A}$ are as above, let $\mathcal{E}_X \equiv \mathcal{E}$ be an $\mathcal{A}$-module, and $\mathfrak{D}$ the category whose objects are the $\mathcal{A}$-morphisms $\varphi : \mathcal{E} \longrightarrow \mathcal{P}$ from $\mathcal{E}$ into an $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{P}$. Given an $\mathcal{A}$-morphism $\varphi' : \mathcal{E} \longrightarrow \mathcal{P}'$, a morphism from φ to φ' is an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $u : \mathcal{P} \longrightarrow \mathcal{P}'$ such that $\varphi' = u \circ \varphi$. If $\mathfrak{D}$ contains an initial universal object $\varphi_{\mathcal{E}}^{\mathcal{S}} \equiv \varphi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{M}$, then $\mathcal{M}$, an $(\mathcal{S}^{-1}\mathcal{A})$-module, is called the *sheaf of* $(\mathcal{S}^{-1}\mathcal{A})$*-modules of fractions of* $\mathcal{E}$ *with denominator in* $\mathcal{S}$ and is denoted by $\mathcal{S}^{-1}\mathcal{E}$ or $\mathcal{E}_{\mathcal{S}}$. Every sheaf $\mathcal{S}$ of submonoids in $\mathcal{A}$ yields a functor, called *the localization functor*, from the category $\mathcal{A}$-$\mathcal{M}od_X$ of $\mathcal{A}$-modules into the category $(\mathcal{S}^{-1}\mathcal{A})$-$\mathcal{M}od_X$ of $(\mathcal{S}^{-1}\mathcal{A})$-modules; precisely, it is the functor that sends an object $\mathcal{E}$ of the category $\mathcal{A}$-$\mathcal{M}od_X$ to the $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{S}^{-1}\mathcal{E}$, and as for morphisms, it associates with every $\mathcal{A}$-morphism $\psi : \mathcal{E} \longrightarrow \mathcal{F}$ the $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$, described as follows: because of the universal property of $\mathcal{S}^{-1}\mathcal{E}$, the $\mathcal{A}$-morphism

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

can be factorized in a unique way through $\mathcal{S}^{-1}\mathcal{E}$, that is,

$$\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

is unique and satisfies the equation

$$\mathcal{S}^{-1}\psi \circ \varphi_{\mathcal{E}}^{\mathcal{S}} \equiv \mathcal{S}^{-1}\psi \circ \varphi_{\mathcal{E}} = \varphi_{\mathcal{F}} \circ \psi \equiv \varphi_{\mathcal{F}}^{\mathcal{S}} \circ \psi. \tag{3.41}$$

More explicitly, (3.41) means that, for any $x \in X$ and element $\frac{e}{s} \in (\mathcal{S}^{-1}\mathcal{E})_x$,

$$(\mathcal{S}^{-1}\psi)_x(\frac{e}{s}) := \frac{\psi_x(e)}{s}.$$

Now, we recall (see [24, p. 109]) that, given an algebraized space $(X, \mathcal{A})$, a *sheaf of ideals* on X is a subsheaf $\mathfrak{a}$ of $\mathcal{A}$ such that, for every open set U, $\mathfrak{a}(U)$ is an ideal in $\mathcal{A}(U)$. Fibre-wise, we have that, for all $x \in X$, $\mathfrak{a}_x$ is an ideal in $\mathcal{A}_x$. For any $\mathcal{A}$-module $\mathcal{E}$, if $\mathfrak{a}$ is a sheaf of ideals in $\mathcal{A}$, the notation $\mathfrak{a}\mathcal{E}$ means that, for all open $U \subseteq X$, $(\mathfrak{a}\mathcal{E})(U) = \mathfrak{a}(U)\mathcal{E}(U)$; in other words, for all $x \in X$, $(\mathfrak{a}\mathcal{E})_x = \mathfrak{a}_x\mathcal{E}_x$.

Let us now consider a *sheaf of prime ideals* $\mathfrak{p}$ in $\mathcal{A}_X$; in other words, for every open set U, $\mathfrak{p}(U)$ is a prime ideal in $\mathcal{A}(U)$. Clearly, if $\mathcal{A} - \mathfrak{p}$ is the sheaf obtained by sheafifying the presheaf

$$U \longmapsto \mathcal{A}(U) - \mathfrak{p}(U),$$

$\mathcal{A} - \mathfrak{p}$ is a subsheaf of $\mathcal{A}$ of submonoids. Let $\mathcal{S} \equiv \mathcal{S}_{\mathfrak{p}} := \mathcal{A} - \mathfrak{p}$; for any $\mathcal{A}_X$-module $\mathcal{E}$, the sheaves $\mathcal{S}^{-1}\mathcal{A}$ and $\mathcal{S}^{-1}\mathcal{E}$ are denoted $\mathcal{A}_{\mathfrak{p}}$ and $\mathcal{E}_{\mathfrak{p}}$, respectively, and are called the For all $x \in X$,

$$(\mathcal{A}_{\mathfrak{p}})_x = (\mathcal{A}_x)_{\mathfrak{p}_x} \quad \text{and} \quad (\mathcal{E}_{\mathfrak{p}})_x = (\mathcal{E}_x)_{\mathfrak{p}_x}.$$

In the sequel, we will adopt the following abuse of language: by *a sheaf of rings* on a topological space X, we shall mean *a sheaf of unital and commutative rings.* We recall that a is a commutative ringed space $(X, \mathcal{R})$ such that, for all $x \in X$, the stalk $\mathcal{R}_x$ is a *local* ring. See [23, p. 92, (4.1.9)]. Analogously, a is an algebraized space $(X, \mathcal{A})$ in which every stalk $\mathcal{A}_x$ is a local algebra.

We now give different *fibre-wise* characterizations of locally algebraized spaces. For the proof of the following theorem, see [6, p. 18, Proposition 2].

Theorem 3.8 *Let $\mathcal{A}_X$ be a sheaf of algebras and, for all $x \in X$, $\mathfrak{m}_x \subsetneq \mathcal{A}_x$ a proper ideal. At every $x \in X$, the following properties are equivalent:*

(i) *$\mathcal{A}_x$ is a local algebra with $\mathfrak{m}_x$ the maximal ideal in $\mathcal{A}_x$.*

(ii) *Every element of* $\mathcal{A}_x - \mathfrak{m}_x$ *is a unit in* $\mathcal{A}_x$.

(iii) $\mathfrak{m}_x$ *is a maximal ideal and every element of the form* $1_x + m_x \equiv 1 + m$, *with* $m \in \mathfrak{m}_x$, *is a unit in* $\mathcal{A}_x$.

Let $\mathcal{A} \equiv \mathcal{A}_X$ be a sheaf of algebras and $\mathcal{S} \subseteq \mathcal{A}$ a sheaf of submonoids in $\mathcal{A}$. For any sheaf of ideals $\mathfrak{a} \subseteq \mathcal{A}$, by the *extension of* $\mathfrak{a}$ *to* $\mathcal{S}^{-1}\mathcal{A} \equiv \mathcal{A}_{\mathcal{S}}$, we shall mean the sheaf of ideals, denoted $\mathfrak{a}\mathcal{A}_{\mathcal{S}}$, in $\mathcal{A}_{\mathcal{S}}$, generated by $\tau(\mathfrak{a})$, where τ is the canonical mapping $\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{S}}$. Clearly, for any $x \in X$,

$$(\mathfrak{a}\mathcal{A}_{\mathcal{S}})_x = \mathfrak{a}_x(\mathcal{A}_{\mathcal{S}})_x = \{\frac{a}{s};\ a \in \mathfrak{a}_x,\ s \in \mathcal{S}_x\}.$$

On the other hand, given a sheaf of ideals $\mathfrak{b}$ in $\mathcal{A}_{\mathcal{S}}$, its pre-image $\tau^{-1}(\mathfrak{b})$ is called the *restriction of* $\mathfrak{b}$ *to* $\mathcal{A}$. For all $x \in X$, if we let $\tau_x(z) := z$ for any $z \in \mathcal{A}_x$, then

$$\tau^{-1}(\mathfrak{b}) = \mathfrak{b} \cap \mathcal{A}.$$

Now, consider a subsheaf $\mathfrak{p}$ of prime ideals of $\mathcal{A}_X$. We denote by $\mathcal{S} := \mathcal{A} - \mathfrak{p}$ the sheaf whose stalk at a point $x \in X$ is the multiplicative system $\mathcal{S}_x = \mathcal{A}_x - \mathfrak{p}_x$; we call $\mathcal{A}_{\mathcal{S}} := \mathcal{A}_{\mathcal{A}-\mathfrak{p}}$ the *localization of* $\mathcal{A}$ *at* $\mathfrak{p}$. By an abuse of language, we will write $\mathcal{A}_{\mathfrak{p}}$ instead of the more accurate but cumbersome notation $\mathcal{A}_{\mathcal{A}-\mathfrak{p}}$.

Lemma 3.6 *For any subsheaf* $\mathfrak{p}$ *of prime ideals of a sheaf* $\mathcal{A}_X$ *of algebras, the localization* $\mathcal{A}_{\mathfrak{p}}$ *is a locally ringed space with* $\mathfrak{p}_x(\mathcal{A}_{\mathfrak{p}})_x$ *being the maximal ideal of* $(\mathcal{A}_{\mathfrak{p}})_x$, *where* $x \in X$.

Proof. For all $x \in X$, it is easy to see that $(\mathcal{A}_{\mathfrak{p}})_x - \mathfrak{p}_x(\mathcal{A}_{\mathfrak{p}})_x$ consists of units in $(\mathcal{A}_{\mathfrak{p}})_x$. By virtue of Theorem 3.8, $\mathcal{A}_{\mathfrak{p}}$ is a sheaf of local algebras, with, for all $x \in X$, $\mathfrak{p}_x(\mathcal{A}_{\mathfrak{p}})_x$ being the maximal ideal for the corresponding ring $(\mathcal{A}_{\mathfrak{p}})_x$. ■

Let us now introduce a version of .

Lemma 3.7 (**Nakayama**) *Let $\mathcal{E}$ be a locally finitely generated $\mathcal{A}_X$-module and $\mathfrak{a} \subseteq \mathcal{A}_X \equiv \mathcal{A}$ a subsheaf of ideals such that, for all $x \in X$, $\mathfrak{a}_x$ is contained in every maximal ideal of $\mathcal{A}_x$. If $\mathfrak{a}\mathcal{E} = \mathcal{E}$, then $\mathcal{E} = 0$.*

Proof. Since $\mathcal{E}$ is locally finitely generated, there locally exists an exact sequence

$$\mathcal{L}_0 \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{L}_0$ is locally free of finite rank over $\mathcal{A}$, i.e., for all $x \in X$, there is an open neighborhood U of $x \in X$ such that

$$\mathcal{A}^n|_U \simeq \mathcal{L}_0|_U \longrightarrow \mathcal{E}|_U \longrightarrow 0.$$

Since *inductive limits preserve exactness of sequences* (see, for instance, [7, p. 206, Corollary 2, Proposition 7]),

$$\mathcal{A}_x^n \simeq \mathcal{L}_{0,x} \longrightarrow \mathcal{E}_x \longrightarrow 0$$

for all $x \in X$. Moreover, for all $x \in X$, $\mathfrak{a}_x\mathcal{E}_x = (\mathfrak{a}\mathcal{E})_x = \mathcal{E}_x$, where $\mathfrak{a}_x$ is an ideal of $\mathcal{A}_x$, contained in every maximal ideal of $\mathcal{A}_x$. Applying Nakayama's lemma on the stalks of $\mathcal{E}$, we have that $\mathcal{E}_x = 0$, for all $x \in X$. Thus, $\mathcal{E} = 0$. ■

Under the same conditions of Lemma 3.11, we have:

Corollary 3.4 *Let $\mathcal{F} \subseteq \mathcal{E}$ be a sub-$\mathcal{A}_X$-module such that $\mathcal{E} = \mathcal{F} + \mathfrak{a}\mathcal{E}$. Then, $\mathcal{E} = \mathcal{F}$.*

Proof. For all $x \in X$, $(\mathcal{E}/\mathcal{F})_x = (\mathcal{F}_x + (\mathfrak{a}\mathcal{E})_x)/\mathcal{F}_x = (\mathcal{F}_x + \mathfrak{a}_x\mathcal{E}_x)/\mathcal{F}_x = \mathfrak{a}_x(\mathcal{E}_x/\mathcal{F}_x)$. Since $\mathcal{E}_x/\mathcal{F}_x$, just as $\mathcal{E}_x$, is finitely generated, by the classical Nakayama's lemma, $\mathcal{E}_x/\mathcal{F}_x = 0$. Thus, $\mathcal{E}_x = \mathcal{F}_x$, for all $x \in X$; hence, $\mathcal{E} = \mathcal{F}$. ■

As in the classical case, see, for instance, [25, pp. 30, 31, Corollary(1.12.3)], one shows the following.

Corollary 3.5 *Let $\mathcal{E}$ be a locally finitely generated $\mathcal{A}_X$-module and $\mathfrak{a}$ an ideal sheaf in $\mathcal{A}_X$, where $\mathcal{A}_X \equiv \mathcal{A}$ is a sheaf of local algebras such that, given any open $U \subseteq X$, $\mathfrak{a}(U)$ is the maximal ideal of $\mathcal{A}(U)$. Moreover, let $s_1, s_2, \cdots, s_n$ be n (local) sections of $\mathcal{E}$ over some open $U \subseteq X$. Then, the following conditions are equivalent:*

(a) $(s_1, s_2, \cdots, s_n)$ *is a minimal family of generators of* $\mathcal{E}(U)$*;*

(b) *the image* $(\overline{s_1}, \overline{s_2}, \cdots, \overline{s_n})$ *in* $\mathcal{E}(U)/\mathfrak{a}(U)\mathcal{E}(U)$ *is a basis of* $\mathcal{E}(U)/\mathfrak{a}(U)\mathcal{E}(U)$ *as a module over the residue field* $\mathcal{A}(U)/\mathfrak{a}(U)$.

Clearly, given $\mathcal{A}$ as in Corollary 3.8, we deduce that if $\mathcal{E}$ is a locally free $\mathcal{A}$-module of finite rank, then $\mathcal{E}/\mathfrak{a}\mathcal{E}$ is locally free of the same rank as $\mathcal{E}$.

The next theorem, patterned after [25, p. 31, Theorem 1.12.4], too derives from Nakayama's lemma. First, let us recall the following notions which were discussed in Section 3.1: Given an algebraized space $(X, \mathcal{A})$, an $\mathcal{A}$-module $\mathcal{E}$ is called *projective* if the functor $\mathcal{H}om(\mathcal{E}, \_)$ is exact; it is called *flat* if the functor $\mathcal{E} \otimes \_$ is exact. *Equivalently,* $\mathcal{E}$ is projective if and only if the mapping $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \to \mathcal{H}om(\mathcal{E}, \mathcal{F}'')$ is surjective whenever $\mathcal{F} \to \mathcal{F}''$ is surjective, and $\mathcal{E}$ is flat if and only if the mapping $\mathcal{E} \otimes \mathcal{F}' \to \mathcal{E} \otimes \mathcal{F}$ is injective whenever $\mathcal{F}' \to \mathcal{F}$ is injective. Flatness is a local property and one can easily verify that $\mathcal{E}$ is flat if and only if its stalk $\mathcal{E}_x$, at any point $x \in X$, is a flat module over the algebra $\mathcal{A}_x$ (see [19, p. 112] or [27, p. 446, Proposition 18.5.2]).

Let us recall a classical notion. An $\mathcal{A}$-module $\mathcal{E}$ is called if there exists an exact sequence

$$\mathcal{L}_1 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{E} \longrightarrow 0, \tag{3.42}$$

where $\mathcal{L}_1$ and $\mathcal{L}_0$ are free $\mathcal{A}$-modules of finite rank. This is equivalent to saying that there exists an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{N} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{3.43}$$

where $\mathcal{N}$ is free of finite rank and $\mathcal{K}$ is finitely generated.

Theorem 3.9 *Let $(X, \mathcal{A})$ be an algebraized space, where $\mathcal{A}$ is a locally algebraized space on X, with, for all $x \in X$, $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{A}_x$. Moreover, let $\mathcal{E}$ be a finitely presented $\mathcal{A}$-module. Then, the following assertions are equivalent:*

(a) *$\mathcal{E}$ is free;*

(b) *$\mathcal{E}$ is projective;*

(c) *$\mathcal{E}$ is flat;*

(d) *The morphism* $\mathfrak{m} \otimes \mathcal{E} \xrightarrow{\varphi} \mathcal{E}$, $\varphi_x(m \otimes e) := me$ $(m \in \mathfrak{m}_x,\ e \in \mathcal{E}_x,\ x \in X)$, *is injective.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are evident, due to the fact that, for any open $U \subseteq X$, $\mathcal{E}(U)$ is a finitely presented module over a local algebra $\mathcal{A}(U)$. See, for instance, [25, p. 31, Theorem 1.12.4]. Therefore, for any open $U \subseteq X$, $\mathcal{E}(U)$ is free if and only if it is projective if and only if it is flat. Hence, $\mathcal{E}$ is free if and only if it is projective if and only if it flat. Let us show that $(c) \Rightarrow (d)$. Since $\mathcal{A} \otimes \mathcal{E} \simeq \mathcal{E}$ and $\mathfrak{m} \longrightarrow \mathcal{A}$ is injective, one has that $\mathfrak{m} \otimes \mathcal{E} \longrightarrow \mathcal{A} \otimes \mathcal{E} \simeq \mathcal{E}$ is injective. It remains to show that $(d) \Rightarrow (a)$. Since, for all $x \in X$, $\mathcal{A}_x$ is a local algebra, it is known that the injectivity of the morphism $\varphi_x : \mathfrak{m}_x \otimes \mathcal{E}_x \longrightarrow \mathcal{E}_x$, $m \otimes e \longmapsto me$, implies that $\mathcal{E}_x$ is free, that is to say there exists $n \in \mathbb{N}$ such that $\mathcal{E}_x \simeq \mathcal{A}_x^n$ ($n \leq$ rank $\mathcal{L}_0$, where $\mathcal{L}_0$ is as in the sequence (3.42)). As n is the same for all x, $\mathcal{E} \simeq \mathcal{A}^n$, which ends the proof. ∎

For the purpose of the sequel, let us recall (cf. [43]) the following: Given a sheaf $\mathcal{A}_X := \mathcal{A} \equiv (\mathcal{A}, \pi, X)$ of unital and commutative algebras and a sheaf $\mathcal{S}_X := \mathcal{S} \equiv (\mathcal{S}, \pi|_{\mathcal{S}}, X)$ of submonoids in $\mathcal{A}$, the localization functor $\mathcal{S}^{-1} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$, $\mathcal{S}^{-1}(\mathcal{E}) := \mathcal{S}^{-1}\mathcal{E}$, satisfies the property that, *given any locally finitely presented $\mathcal{A}$-module $\mathcal{E}$ and any $\mathcal{A}$-module $\mathcal{F}$, the $(\mathcal{S}^{-1}\mathcal{A})$-morphism*

$$\vartheta : \mathcal{S}^{-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E},\mathcal{S}^{-1}\mathcal{F})$$

such that, for any $x \in X$,

$$\vartheta_x\Big(\frac{f}{s}\Big)\Big(\frac{e}{t}\Big) := \frac{f(e)}{st},$$

where $s,t \in \mathcal{S}_x$, $e \in (\mathcal{S}^{-1}\mathcal{E})_x$, $f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})_x$, is an $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism. This result is based on the fact that, for such $\mathcal{E}$ and $\mathcal{F}$, the natural morphism $\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})\big)_x \longrightarrow \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{F}_x)$ is an $\mathcal{A}_x$-isomorphism (see, Theorem 3.7 or [51, p. 114, (*) and subsequent remarks] or [23, p. 110, (5.2.6)]).

We shall now show that flatness of $\mathcal{A}$-modules is a local property. We begin with two auxiliary results.

Lemma 3.8 *Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be algebra sheaves on a topological space X and $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$, $\varphi' : \mathcal{A}' \longrightarrow \mathcal{A}''$ be morphisms. Moreover, let $\mathcal{E}$ be an $\mathcal{A}$-module and $\mathcal{E}'$ an $\mathcal{A}'$-module. Then,*

$$\Phi : (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}') \otimes_{\mathcal{A}'} \mathcal{E}' \simeq \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}', \tag{3.44}$$

such that, for any $x \in X$, $z \in \mathcal{E}_x$, $a' \in \mathcal{A}'_x$ and $z' \in \mathcal{E}'_x$,

$$\Phi_x\big((z \otimes a') \otimes z'\big) = z \otimes a'z'. \tag{3.45}$$

Proof. Since, for any $x \in X$, Φ_x is an $\mathcal{A}_x$-isomorphism (cf. [6, p. 124, Remark 2]), Φ is an $\mathcal{A}$-isomorphism. ■

Lemma 3.9 *Let $(X,\mathcal{A})$ be an algebraized space and $\mathcal{E}$ an $\mathcal{A}$-module on X. For every $x \in X$, the canonical map*

$$\mathcal{E}_x \longrightarrow \prod_{\mathfrak{m}_x \in \mathrm{Spm}\mathcal{A}_x} (\mathcal{E}_x)_{\mathfrak{m}_x},$$

where $\mathsf{Spm}\mathcal{A}_x$ *is the maximal spectrum of* $\mathcal{A}_x$*, is injective.*

Proof. See [6, p. 126, Lemma 4]. ■

Now, before we proceed to show that flatness of $\mathcal{A}$-modules is a local property, let us construct a *sheaf with maximal ideals as stalks.* In fact, let $(X, \mathcal{A})$ be a locally algebraized space; for every $x \in X$, let $\mathfrak{m}_x$ be the unique maximal ideal in the corresponding local ring $\mathcal{A}_x$. Furthermore, we assume that we have a family of sections of $\mathcal{A}$, say $\mathfrak{G} = \{s\}$, that is, the generic element of $\mathfrak{G}$, which is a function $s : U \longrightarrow \mathfrak{m} \subseteq \mathcal{A}$, is also given as a triple $(U, s, \mathfrak{m})$, where $U \equiv \mathrm{D}om(s)$ is open in X and $\mathfrak{m} := \bigcup_{x\in X} \mathfrak{m}_x = \sum_{x\in X} \mathfrak{m}_x$. Next, suppose that the family $\mathfrak{G}$ satisfies the following three conditions: (i) For every $x \in U \equiv \mathrm{D}om(s)$, $s(x) \in \mathfrak{m}_x$; equivalently, every s is identified with a family $s \in \prod_{x\in U} \mathfrak{m}_x$; (ii) For each $x \in X$, $\mathfrak{m}_x \subseteq \bigcup_{s\in\mathfrak{G}} im(s)$, i.e., $\mathfrak{m} = \bigcup_{s\in\mathfrak{G}} s(U)$; (iii) Let s, $t \in \mathfrak{G}$, with $U \equiv \mathrm{D}om(s)$ and $V \equiv \mathrm{D}om(t)$. If $z \in s(U) \bigcap t(V)$, there exists an open set W in X such that $x \in W \subseteq U \bigcap V$ (where $z = s(x) = t(x)$) and $s|_W = t|_W$.

With the projection map $\pi : \mathfrak{m} \longrightarrow X$, $\pi(\mathfrak{m}_x) = \{x\}$, $x \in X$, one shows (cf. [35, pp. 12-14, Theorem 3.1]) that the triple $(\mathfrak{m}, \pi, X)$ turns out to be a sheaf. Indeed, the family

$$\mathcal{B} := \{s(V) : \ s \in \mathfrak{G} \text{ and } V \text{ open in } X \text{ with } V \subseteq \mathrm{D}om(s)\}$$

is a basis for the topology of $\mathfrak{m}$ with respect to which the map $\pi : \mathfrak{m} \longrightarrow X$ is a local homeomorphism. We shall call the sheaf $\mathfrak{m}$, thus obtained, the *sheaf induced by the maximal ideals* $\mathfrak{m}_x$, $x \in X$. It is clear that $\mathfrak{m}$ *is a sheaf of prime ideals.*

Theorem 3.10 *Let* $(X, \mathcal{A})$ *be an algebraized space and* $(X, \mathcal{A}')$ *a locally algebraized space. For any morphism* $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$ *and any* $\mathcal{A}'$*-module* $\mathcal{E}$*, the following conditions are equivalent:*

(i) *As an* $\mathcal{A}$*-module,* $\mathcal{E}$ *is flat.*

(ii) *The $\mathcal{A}'$-module $\mathcal{E}_{\mathfrak{m}'}$, where $\mathfrak{m}'$ is the sheaf induced by the maximal ideals $\mathfrak{m}'_x \subset \mathcal{A}_x$, is a flat $\mathcal{A}$-module.* (As a set, $\mathcal{E}_{\mathfrak{m}'} = \sum_{x \in X} (\mathcal{E}_x)_{\mathfrak{m}'_x}$.)

(iii) *The $\mathcal{A}'$-module $\mathcal{E}_{\mathfrak{m}'}$ of (ii) is a flat $\mathcal{A}_{\mathfrak{m}}$-module, where $\mathfrak{m} := \varphi^{-1}(\mathfrak{m}')$.*

Proof. Suppose that (i) holds; since *small filtrant inductive limits of flat $\mathcal{A}$-modules are flat* (cf. [27, p. 446, Proposition 18.5.2(iv)]), for every $x \in X$, $\mathcal{E}_x$ is $\mathcal{A}_x$-flat. As in the classical case (cf. [6, p. 127, Proposition 5]) one shows that, for any maximal ideal $\mathfrak{m}'_x \subset \mathcal{A}'_x$, the localization $(\mathcal{E}_x)_{\mathfrak{m}'_x}$ is a flat $\mathcal{A}_x$-module. Now, let $\mathcal{F}' \longrightarrow \mathcal{F}$ be an $\mathcal{A}$-monomorphism. Since, for every $x \in X$, $\mathcal{F}'_x \otimes_{\mathcal{A}_x} (\mathcal{E}_x)_{\mathfrak{m}'_x} \longrightarrow \mathcal{F}_x \otimes_{\mathcal{A}_x} (\mathcal{E}_x)_{\mathfrak{m}'_x}$ is an $\mathcal{A}_x$-monomorphism, it follows (cf. [35, pp. 60, 61, Lemma 12.1]) that $\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'} \longrightarrow \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'}$ is an $\mathcal{A}$-monomorphism. Thus, (i) implies (ii).

Next, let us show that $(ii) \Leftrightarrow (iii)$. Assume first that $\mathcal{E}_{\mathfrak{m}'}$ is a flat $\mathcal{A}$-module. As $\mathfrak{m} := \varphi^{-1}(\mathfrak{m}')$ is a subsheaf of ideals of $\mathcal{A}$, we can derive from φ by means of the localization functor associated with $\mathfrak{m}'$ a morphism $\varphi_{\mathfrak{m}'} : \mathcal{A}_{\mathfrak{m}} \longrightarrow \mathcal{A}'_{\mathfrak{m}'}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{A}' \\ {\scriptstyle \sigma_{\mathfrak{m}}} \downarrow & & \downarrow {\scriptstyle \sigma'_{\mathfrak{m}'}} \\ \mathcal{A}_{\mathfrak{m}} & \xrightarrow[\varphi_{\mathfrak{m}'}]{} & \mathcal{A}'_{\mathfrak{m}'}, \end{array}$$

where $\sigma_{\mathfrak{m}}$ and $\sigma'_{\mathfrak{m}'}$ are canonical, commutes; it is clear that $\mathcal{E}_{\mathfrak{m}'}$ can be regarded as an $\mathcal{A}_{\mathfrak{m}}$-module. Now, if $\mathcal{F}' \longrightarrow \mathcal{F}$ is a monomorphism of $\mathcal{A}_{\mathfrak{m}}$-modules, the flatness of $\mathcal{E}_{\mathfrak{m}'}$ over $\mathcal{A}$ induces an $\mathcal{A}$-monomorphism:

$$\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'} \longrightarrow \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'}.$$

Using the $\mathcal{A}_{\mathfrak{m}}$-isomorphisms $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}} \simeq \mathcal{F}$ and $\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}} \simeq \mathcal{F}'$, one has

$$\mathcal{F}' \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \simeq (\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}}) \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \longrightarrow (\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}}) \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}} \simeq \mathcal{F} \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}};$$

whence $\mathcal{E}_{\mathfrak{m}'}$ is a flat $\mathcal{A}_{\mathfrak{m}}$-module. Conversely, let $\mathcal{F}' \longrightarrow \mathcal{F}$ be any monomorphism of $\mathcal{A}$-modules. As localization functors are exact (See Theorem 3.6), $\mathcal{F}'_{\mathfrak{m}} \longrightarrow \mathcal{F}_{\mathfrak{m}}$ is a monomorphism of $\mathcal{A}_{\mathfrak{m}}$-modules; and since $\mathcal{E}_{\mathfrak{m}'}$ is $\mathcal{A}_{\mathfrak{m}}$-flat,

$$\mathcal{F}'_{\mathfrak{m}} \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \longrightarrow \mathcal{F}_{\mathfrak{m}} \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'}$$

is an $\mathcal{A}_{\mathfrak{m}}$-monomorphism. By Lemma 3.8,

$$\mathcal{F}'_{\mathfrak{m}} \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \simeq (\mathcal{F}'_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}}) \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \simeq \mathcal{F}'_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'}$$

and

$$\mathcal{F}_{\mathfrak{m}} \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \simeq (\mathcal{F}_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}}) \otimes_{\mathcal{A}_{\mathfrak{m}}} \mathcal{E}_{\mathfrak{m}'} \simeq \mathcal{F}_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'},$$

therefore

$$\mathcal{F}'_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'} \longrightarrow \mathcal{F}_{\mathfrak{m}} \otimes_{\mathcal{A}} \mathcal{E}_{\mathfrak{m}'}$$

is injective; whence $\mathcal{E}_{\mathfrak{m}'}$ is $\mathcal{A}$-flat.

Finally, let us show that $(ii) \Rightarrow (i)$ in order to obtain all the remaining equivalences. Suppose that $\mathcal{E}_{\mathfrak{m}'}$ is $\mathcal{A}$-flat, where $\mathfrak{m}'$ is a sheaf of ideals such that, for all $x \in X$, $\mathfrak{m}'_x$ is maximal in $\mathcal{A}_x$. Let $\mathcal{F}' \longrightarrow \mathcal{F}$ be a monomorphism of $\mathcal{A}$-modules. For all $x \in X$, if $(\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E})_x \simeq \mathcal{F}'_x \otimes_{\mathcal{A}_x} \mathcal{E}_x$ and $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{E})_x \simeq \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x$ are considered as $\mathcal{A}'_x$-modules, in the commutative diagram (cf. [6, pp. 127, 128, Proposition 5])

$$\begin{array}{ccc}
(\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E})_x & \longrightarrow & (\mathcal{F} \otimes_{\mathcal{A}} \mathcal{E})_x \\
\downarrow & & \downarrow \\
\prod_{\mathfrak{m}'_x \in \mathsf{Spm}(\mathcal{A}'_x)} \big((\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E})_x\big)_{\mathfrak{m}'_x} & \longrightarrow & \prod_{\mathfrak{m}'_x \in \mathsf{Spm}(\mathcal{A}'_x)} \big((\mathcal{F} \otimes_{\mathcal{A}} \mathcal{E})_x\big)_{\mathfrak{m}'_x},
\end{array}$$

the upper horizontal arrow is injective. Hence, $\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$ is injective, and, consequently, $\mathcal{E}$ is $\mathcal{A}$-flat. ■

In classical theory, see for instance [25, p. 30, Corollary 1.11.8], *localization of modules with respect to prime ideals is equivalent to localization with respect to maximal ideals.* Though, we have reservation as to whether sheaves of maximal ideals do, indeed, exist, we nevertheless show, in the corollary below, that localization of $\mathcal{A}$-modules with respect to prime ideal subsheaves is equivalent to localization with respect to sheaves induced (in the sense of the paragraph just before Theorem 3.10) by maximal ideals. To do this, we consider $\mathcal{A}$-modules, with $\mathcal{A}$ admitting prime ideal subsheaves.

Corollary 3.6 *For any $\mathcal{A}$-module $\mathcal{E}$, the following conditions are equivalent:*

(i) *$\mathcal{E}$ is $\mathcal{A}$-flat;*

(ii) *$\mathcal{E}_{\mathfrak{p}}$ is $\mathcal{A}_{\mathfrak{p}}$-flat for every sheaf $\mathfrak{p}$ of prime ideals in $\mathcal{A}$;*

(iii) *$\mathcal{E}_{\mathfrak{m}}$ is $\mathcal{A}_{\mathfrak{m}}$-flat for every sheaf induced by the maximal ideals $\mathfrak{m}_x$, $x \in X$.*

Proof. The $\mathcal{A}$-module $\mathcal{E}$ is flat if and only if, given any injective $\mathcal{A}$-morphism $\mathcal{F}' \longrightarrow \mathcal{F}$, the induced $\mathcal{A}$-morphism $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}' \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ is injective. Since localization functors commute with tensor products (See Corollary 3.3), $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})_{\mathfrak{p}} \simeq \mathcal{E}_{\mathfrak{p}} \otimes_{\mathcal{A}_{\mathfrak{p}}} \mathcal{F}_{\mathfrak{p}}$ and $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}')_{\mathfrak{p}} \simeq \mathcal{E}_{\mathfrak{p}} \otimes_{\mathcal{A}_{\mathfrak{p}}} \mathcal{F}'_{\mathfrak{p}}$, the implication $(i) \Rightarrow (ii)$ follows from the exactness of localization functors (See Theorem 3.6) and from the fact that $\mathcal{F}_{\mathfrak{p}} = \mathcal{F}$ (resp. $\mathcal{F}'_{\mathfrak{p}} = \mathcal{F}'$) if $\mathcal{F}$ (resp. $\mathcal{F}'$) is an $\mathcal{A}_{\mathfrak{p}}$-module. The implication $(ii) \Rightarrow (ii)$ is trivial, and the implication $(iii) \Rightarrow (i)$ is taken care of by Theorem 3.10. ■

3.4 Change of the Algebra Sheaf of Scalars in Clifford Algebras

We are now in the position to use the definition of *sheaves of Clifford $\mathcal{A}$-algebras* (*Clifford $\mathcal{A}$-algebras* in short) *of quadratic $\mathcal{A}$-modules*, set over arbitrary topological spaces as quotient sheaves of tensor algebra sheaves over certain ideal sheaves to show that direct limits commute with the *Clifford functor* $\mathcal{C}l : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{A}\text{-}\mathcal{A}lg_X$, where $\mathcal{A}\text{-}\mathcal{M}od_X$ and $\mathcal{A}\text{-}\mathcal{A}lg_X$ stand for the categories of sheaves of $\mathcal{A}$-modules and $\mathcal{A}$-algebras on X, respectively. Theorem 3.12, which is concerned with the *commutativity of the Clifford functor $\mathcal{C}l$ with the extension functor through tensor product*, is proved by means of Lemma 3.10. [25, p. 54, Lemma 2.1.3] is a classical counterpart of Lemma 3.10.

Lemma 3.10 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module on a topological space X, and $\mathcal{F}$ any $\mathcal{A}$-module, also on X. For any open subset U of X, let $(e_i^U \equiv e_i)_{i\in I}$ be a basis of $\mathcal{E}(U)$ and $(t_{i,j}^U \equiv t_{i,j})_{i,j\in I}$ be a family of sections in $\mathcal{F}(U)$ such that $t_{i,j} = t_{j,i}$. Then, there exists a unique $\mathcal{A}|_U$-quadratic morphism $q \in \mathrm{Q}uad_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U) \equiv \mathcal{Q}uad_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(U)$ such that*

$$q_V(e_i|_V) = t_{i,i}|_V, \quad i \in I, \tag{3.46}$$

and

$$(B_q)_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i \neq j \text{ in } I, \tag{3.47}$$

where B_q is the associated $\mathcal{A}|_U$-bilinear morphism of q.

Proof. If g is an $\mathcal{A}$-bilinear morphism $\mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{F}$, the sheaf morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ such that

$$\varphi_V(s) = g_V(s, s),$$

for any open $V \subseteq X$ and section $s \in \mathcal{E}(V)$, is $\mathcal{A}$-quadratic; clearly, the associated $\mathcal{A}$-bilinear morphism is the sheaf morphism $B_\varphi \equiv (B_{\varphi,V})_{\,openV\subseteq X}$ such that

$$B_{\varphi,V}(s,t) = g_V(s,t) + g_V(t,s),$$

for all $s, t \in \mathcal{E}(V)$. Next, define a total order on the indexing set I and let $g : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ be the $\mathcal{A}|_U$-bilinear morphism such that

$$g_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i, j \in I \text{ with } i \leq j,$$

and

$$g_V(e_i|_V, e_j|_V) = 0, \quad i, j \in I \text{ with } i > j.$$

The $\mathcal{A}|_U$-quadratic morphism $q : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ such that

$$q_V(s) = g_V(s,s),$$

for any open set $V \subseteq X$ and section $s \in (\mathcal{E}|_U)(V)$, satisfies the conditions of the lemma.

Now, let us prove the uniqueness of q. To this end, suppose that there is another $\mathcal{A}|_U$-quadratic $\overline{q} : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ satisfying (3.40) and (3.41), that is,

$$(\overline{q})_V(e_i|_V) = t_{i,i}|_V, \quad i \in I,$$

and

$$(B_{\overline{q}})_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i \neq j \text{ in } I,$$

for any open $V \subseteq U$. It follows that, for any $i, j \in I$ and open $V \subseteq U$,

$$\big((q)_V - (\overline{q})_V\big)(e_i|_V) = 0$$

and

$$\big(B_{q-\overline{q}}\big)_V(e_i|_V, e_j|_V) = B_q\big((e_i|_V, e_j|_V)\big) - B_{\overline{q}}\big((e_i|_V, e_j|_V)\big) = 0.$$

By an easy calculation, one shows that, for any $s \in \mathcal{E}(U)$

$$\left(B_{q-\overline{q}}\right)_V (s|_V, s|_V) = 2(q-\overline{q})|_V(s|_V) = 0,$$

whence

$$q = \overline{q}.$$

■

Here is a very useful result of this section.

Theorem 3.11 *Let $\mathcal{A}$, $\mathcal{A}'$ be unital algebra sheaves on a topological space X, $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$ a sheaf morphism, $\mathcal{E}$ and $\mathcal{F}$ two $\mathcal{A}$-modules on X, and $q : \mathcal{E} \longrightarrow \mathcal{F}$ an $\mathcal{A}$-quadratic sheaf morphism. Then, there exists a unique $\mathcal{A}'$-quadratic morphism $q' : \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{F}$ such that*

$$q' \circ (1 \otimes \mathrm{id}_{\mathcal{E}}) = 1 \otimes (q \circ \mathrm{id}_{\mathcal{E}}), \tag{3.48}$$

where $1 \in \mathrm{E}nd_{\mathcal{A}'}\mathcal{A}'$ is the constant endomorphism of the underlying sheaf of sets of $\mathcal{A}'$ such that, for every $s \in \mathcal{A}'(U)$, where U is open in X, $1_U(s) := 1_{\mathcal{A}'(U)} \equiv 1 \in \mathcal{A}'(U)$.

Section-wise, (3.48) means that, for every open set U in X and sections $r \in \mathcal{A}'(U)$, $s \in \mathcal{E}(U)$,

$$\begin{aligned}[q'_U \circ (1_U \otimes (\mathrm{id}_{\mathcal{E}})_U)](s) &:= q'_U(1 \otimes s) \\ &= 1 \otimes q_U(s) := [1_U \otimes (q_U \circ (\mathrm{id}_{\mathcal{E}}))_U](r \otimes s).\end{aligned} \tag{3.49}$$

Proof. It is clear that q' is unique; we therefore simply need to prove its existence. Suppose that $\mathcal{E}$ is free. For a fixed open set U in X, we let $(s_i)_{i\in I}$ be a basis of $\mathcal{E}(U)$ and set $t_{i,i} = q_U(s_i) \equiv q(s_i)$ for all $i \in I$, $t_{i,j} = (B_q)_U(s_i, s_j)$ for all $i, j \in I$ such that $i \neq j$. Since $\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U)$ is a free $\mathcal{A}'(U)$-module with basis $(1 \otimes s_i)_{i\in I}$, according to Lemma 3.10, there exists a unique $\mathcal{A}'(U)$-quadratic mapping $q'_U : \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)$ such that $q'_U(1 \otimes s_i) = 1 \otimes t_{i,i}$ for all $i \in I$ and $(B_{q'_U})(1 \otimes s_i, 1 \otimes s_j) = 1 \otimes t_{i,j}$ for all $i, j \in I$ with $i \neq j$. Obviously, q'_U satisfies (3.49). Since q'_U is unique and U is arbitrary, the family $(q'_U)_{\ openU\subseteq X}$ yields an $\mathcal{A}'$-quadratic morphism $q' : \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{F}$ satisfying the required condition (3.49).

Suppose now that $\mathcal{E}$ is not free. As above, fix an open set U in X; $\mathcal{E}(U)$ being an $\mathcal{A}(U)$-module is isomorphic to a quotient $\mathcal{A}(U)$-module of a free $\mathcal{A}(U)$-module. By [25, p. 57, Theorem 2.2.3], one shows that there exists a unique $\mathcal{A}'(U)$-quadratic morphism

$$q'_U : \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)$$

such that

$$q'_U(1' \otimes s) = 1' \otimes q_U(s)$$

for any $s \in \mathcal{E}(U)$. Furthermore, we note that the collections $(\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U))_{\ open\ U\subseteq X}$ and $(\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)_{\ open\ U\subseteq X}$ induce the presheaves of modules $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ and $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$, respectively; if τ^* denotes the set of pairs (U, V), where both U and V are open in X and such that $V \subseteq U$, we shall let $(\mu^U_V)_{(U,V)\in\tau^*}$ and $(\sigma^U_V)_{(U,V)\in\tau^*}$ denote the restriction maps of the preceding presheaves, that is, $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ and $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$, respectively. Since, for any $(U, V) \in \tau^*$,

$$\sigma^U_V \circ q'_U = q'_V \circ \mu^U_V,$$

it follows that the family $(q'_U)_{\ openU\subseteq X}$ yields the sought presheaf morphism of $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ into $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$. It is obvious that $(q'_U)_{\ open\ U\subseteq X}$ is unique. ■

The relevance of Lemma 3.11 will be evident in the proof of Theorem 3.12, below.

Lemma 3.11 *If $(\mathcal{E}, q)$ is a quadratic $\mathcal{A}$-module on a topological space X, then, for every $x \in X$,*

$$\mathcal{C}l(\mathcal{E})_x = Cl(\mathcal{E}_x) \tag{3.50}$$

within $\mathcal{A}_x$-isomorphism, where $Cl(\mathcal{E}_x)$ is the usual Clifford algebra associated with the quadratic $\mathcal{A}_x$-module $(\mathcal{E}_x, q_x)$.

Proof. If $\mathcal{I}(\mathcal{E}, q) \equiv \mathcal{I}(\mathcal{E})$ is the two-sided ideal sheaf in the tensor algebra (sheaf) $T(\mathcal{E})$ determined by the presheaf $J(\mathcal{E}, q)$, where, for any open set U in X, $J(\mathcal{E}, q)(U)$ is a two-sided ideal of the tensor algebra $T(\mathcal{E}(U))$ generated by elements of the form

$$s \otimes s - q_U(s) \equiv s \otimes s - q(s),$$

with s running through $\mathcal{E}(U)$, then, by Theorem 2.1, the Clifford $\mathcal{A}$-algebra of $\mathcal{E}$, denoted by $\mathcal{C}l(\mathcal{E}) \equiv \mathcal{C}l(\mathcal{E}, q) \equiv \mathcal{C}l_{\mathcal{A}}(\mathcal{E})$, is given by

$$\mathcal{C}l(\mathcal{E}) := \mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E}).$$

On account of an equivalent result stated in the proof of Theorem 2.1[see p.62], for every $x \in X$,

$$\mathcal{C}l(\mathcal{E})_x = \mathcal{T}(\mathcal{E})_x/\mathcal{I}(\mathcal{E})_x = T(\mathcal{E}_x)/I(\mathcal{E}_x) = Cl(\mathcal{E}_x),$$

where the preceding equalities actually stand for $\mathcal{A}_x$-isomorphisms. ■

As is known (cf. [25, p.110, Proposition 3.1.9]), let K and K' be unital commutative algebras, $f : K \longrightarrow K'$ an algebra morphism, which respects 1, and (E, q) an object in the category ${}_K\widetilde{Mod}$ of quadratic K-modules. Moreover, let $S : {}_K\widetilde{Mod} \longrightarrow {}_{K'}\widetilde{Mod}$ be such that

$$S(M, q) \equiv S(M) = K' \otimes_K M \equiv K' \otimes_K (M, q),$$

then

$$Cl_{K'} \circ S \simeq S \circ Cl_K; \tag{3.51}$$

that is, for every quadratic K-module (M, q),

$$Cl_{K'}(K' \otimes_K (M, q)) = K' \otimes_K Cl_K(M, q) \tag{3.52}$$

within K'-isomorphism. We shall see in Theorem 3.12 that the isomorphism (3.51) also holds for categories of sheaves of quadratic modules. For the classical case, see, for instance, [14, p. 46, Proposition 3.1.4].

Theorem 3.12 *Let $\mathcal{A}$, $\mathcal{B}$ be unital commutative algebra sheaves on a topological space X, $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ a morphism of algebra sheaves, and $(\mathcal{E}, q)$ a quadratic $\mathcal{A}$-module on X. The Clifford algebra sheaf $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})$ of the quadratic $\mathcal{B}$-module $\mathcal{B} \otimes_{\mathcal{A}} (\mathcal{E}, q) \equiv \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E} \equiv \mathcal{B} \otimes \mathcal{E}$, obtained by extending $\mathcal{A}$ to $\mathcal{B}$ via φ, is canonically isomorphic to the $\mathcal{B}$-algebra $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})$, that is,*

$$\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}) = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l_{\mathcal{A}}(\mathcal{E}) \tag{3.53}$$

within $\mathcal{B}$-isomorphism.

Proof. First, let's observe the following fact. As in Lemma 3.11, let $\mathcal{I}(\mathcal{E}, q)$ be the two-sided ideal sheaf in the tensor $\mathcal{A}$-algebra $\mathcal{T}(\mathcal{E})$, generated by the presheaf $(\otimes \circ \Delta - q)(\mathcal{E})$ of sets, which is such that, for any open U in X, and section $s \in \mathcal{E}(U)$,

$$(\otimes \circ \Delta - q)_U(s) := s \otimes s - q_U(s) \equiv s \otimes s - q(s).$$

(Δ is the diagonal $\mathcal{A}$-morphism $\mathcal{E} \longrightarrow \mathcal{E} \oplus \mathcal{E}$ with $\Delta_U(s) := (s, s)$.) The Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E})$ is generated by the presheaf $(Cl(\mathcal{E}(U)), \rho^U_V)_{(U,V) \in \tau^*}$, where

$$Cl(\mathcal{E}(U)) = T(\mathcal{E}(U))/I(\mathcal{E}(U)) \tag{3.54}$$

and

$$\rho^U_V(s + I(\mathcal{E}(U))) = \lambda^U_V(s) + I(\mathcal{E}(V));$$

assuming that $(T(\mathcal{E}(U)), \lambda^U_V)_{(U,V) \in \tau^*}$ is a generating presheaf of the tensor $\mathcal{A}$-algebra $\mathcal{T}(\mathcal{E})$. Indeed, (3.54) is guaranteed by the definition of quotient $\mathcal{A}$-modules and the completeness of the presheaves

$(T(\mathcal{E}(U)), \lambda_V^U)_{(U,V)\in\tau^*}$ and $(I(\mathcal{E}(U)), \lambda_V^U)_{(U,V)\in\tau^*}$. Now, consider a point $x \in X$; by Lemma 3.11,

$$\big(\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})\big)_x = Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x)$$

and

$$\big(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})\big)_x = \mathcal{B}_x \otimes_{\mathcal{A}_x} Cl(\mathcal{E}_x),$$

within $\mathcal{B}_x$-isomorphisms. By [25, p.110, Proposition 3.1.9], which is summarized by the isomorphism (3.52),

$$Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x) = \mathcal{B}_x \otimes_{\mathcal{A}_x} Cl(\mathcal{E}_x) \tag{3.55}$$

within $\mathcal{B}_x$-isomorphism. We denote the $\mathcal{B}_x$-isomorphism by φ_x. Next, let $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ be the Clifford maps that make $\mathcal{C}l_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{C}l_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{E})$, respectively, into sheaves of Clifford algebras; then $(\rho_{\mathcal{A}})_x$ and $(\rho_{\mathcal{B}})_x$ are Clifford morphisms associated with the Clifford algebras $Cl_{\mathcal{A}_x}(\mathcal{E}_x)$ and $Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x)$, respectively. The $\mathcal{B}_x$-isomorphism maps every $(\rho_{\mathcal{B}})_x(\lambda \otimes z)$ ($\lambda \in \mathcal{B}_x$ and $z \in \mathcal{E}_x$) onto $\lambda \otimes (\rho_{\mathcal{A}})_x(z)$. Since the presheaf, written loosely as $(\mathcal{B}(U) \otimes_{\mathcal{A}(U)} Cl(\mathcal{E}(U)))$ because its restriction maps are obvious, is a monopresheaf, it follows, from [35, p.68, Theorem 12.1], that the family $(\varphi_x)_{x\in X}$ of $\mathcal{B}_x$-isomorphisms yields the required $\mathcal{B}$-isomorphism of $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})$ onto $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})$.

■

Corollary 3.7 *Let $\mathcal{A}$ be a unital commutative algebra sheaf on a topological space X, and $\mathcal{S}$ a sheaf of submonoids in $\mathcal{A}$. Then, for any quadratic $\mathcal{A}$-module $\mathcal{E}$ on X, one has*

$$\mathcal{C}l_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E}) = \mathcal{C}l_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}) = \mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{C}l_{\mathcal{A}}(\mathcal{E}) = \mathcal{S}^{-1}\mathcal{C}l_{\mathcal{A}}(\mathcal{E}),$$

valid within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphisms.

One more corollary, indeed, follows from Theorem 3.12. Recall that the *$\mathcal{A}$-quadratic morphisms into a fixed target $\mathcal{A}$-module $\mathcal{F}$ constitute the objects of a category, denoted $\mathcal{C}_{\mathcal{A}}(\mathcal{F})$*; given two

$\mathcal{C}_{\mathcal{A}}(\mathcal{F})$-objects $q \equiv (\mathcal{E}, q, \mathcal{F})$ and $q' \equiv (\mathcal{E}', q', \mathcal{F})$, a morphism between them is an $\mathcal{A}$-morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{E}'$ such that $q' \circ \varphi = q$.

Let's assume the notations of Theorem 3.12. Then, we have

Corollary 3.8 *The sheaf isomorphism of Theorem 3.12 yields an isomorphism of functors* $\mathcal{C}l_{\mathcal{B}}(\mathcal{B}\otimes\_)$, $\mathcal{B}\otimes\mathcal{C}l(\_) : \mathcal{C}_{\mathcal{A}}(\mathcal{A}) \longrightarrow \mathcal{C}_{\mathcal{A}}(\mathcal{A})$ *of the category* $\mathcal{C}_{\mathcal{A}}(\mathcal{A})$ *of* $\mathcal{A}$*-quadratic morphisms into* $\mathcal{A}$. *Specifically, for any morphism* $\varphi : (\mathcal{E}, q, \mathcal{A}) \equiv (\mathcal{E}, q) \longrightarrow (\mathcal{E}', q') \equiv (\mathcal{E}', q', \mathcal{A})$, *the diagram*

$$\begin{array}{ccc} \mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes (\mathcal{E}, q)) & \longrightarrow & \mathcal{B} \otimes \mathcal{C}l(\mathcal{E}, q) \\ {\scriptstyle \mathcal{C}l(\mathcal{B}\otimes\varphi)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}\otimes\mathcal{C}l(\varphi)} \\ \mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes (\mathcal{E}', q')) & \longrightarrow & \mathcal{B} \otimes \mathcal{C}l(\mathcal{E}', q') \end{array}$$

commutes. (The horizontal arrows in the diagram are isomorphisms.)

Chapter 4

Filtration and Gradation of Clifford $\mathcal{A}$-algebras

Let $(X, \mathcal{A})$ be an algebraized space. Earlier in this thesis(see Section 2.2), we introduced sheaves of Clifford $\mathcal{A}$-algebras on X associated with arbitrary quadratic $\mathcal{A}$-modules as quotient sheaves of tensor algebra sheaves over certain ideal sheaves. In this chapter, we will need to resort to some category theory and we study two main $\mathcal{A}$-isomorphisms of Clifford $\mathcal{A}$-algebras: *the main involution* and *the anti-involution $\mathcal{A}$-isomorphisms*, which split each Clifford $\mathcal{A}$-algebra into an *even* sub-$\mathcal{A}$-algebra and a sub-$\mathcal{A}$-module of *odd* products. Next, we give a definition for *the natural filtration of Clifford $\mathcal{A}$-algebras* and show that for every $\mathcal{A}$-algebra sheaf $\mathcal{E}$, endowed with a regular filtration, one obtains a new *graded $\mathcal{A}$-algebra sheaf*, denoted $Gr(\mathcal{E})$, which turns out to be $\mathcal{A}$-isomorphic to $\mathcal{E}$. We conclude the chapter by discussing the *parity grading of Clifford $\mathcal{A}$-algebras*.

The chapter is divided into three sections. In §1, we establish that Clifford $\mathcal{A}$-algebras are $\mathbb{Z}_2$-graded $\mathcal{A}$-algebras. §2 is devoted to the natural filtration of Clifford $\mathcal{A}$-algebras, which is modeled after the classical case (cf. [25]). For any quadratic $\mathcal{A}_X$-module $(\mathcal{E}, q)$, the canonical $\mathcal{A}$-morphism $\mathcal{E} \longrightarrow \mathcal{C}l^{\leq 1}(\mathcal{E}, q) \longrightarrow \mathcal{G}r^1(\mathcal{C}l(\mathcal{E}, q))$ induces a surjective $\mathcal{A}$-morphism $\bigwedge(\mathcal{E}) \longrightarrow \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$ of $\mathcal{A}_X$-algebras. This foregoing result is strengthened whenever $(\mathcal{E}, q)$ is locally free, for one has that $\bigwedge(\mathcal{E}) \simeq \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$ within $\mathcal{A}$-isomorphism. Another result of this section is that, given a quadratic locally free $\mathcal{A}_X$-module of rank n, endowed with q-orthogonal bases, its Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E}, q)$ is locally free of rank 2^n. $\S 3$ addresses the parity grading of Clifford $\mathcal{A}$-algebras.

4.1 Main Involution and Anti-Involution

Let $(X, \mathcal{A})$ be an algebraized space, $(\mathcal{E}, q)$ a quadratic $\mathcal{A}_X$-module; and, for any $n \in \mathbb{N}$, let $\mathcal{T}^n(\mathcal{E})$ be the $\mathcal{A}$-module on X, generated by the $\Gamma(\mathcal{A})$-presheaf, defined by the correspondence

$$U \longmapsto T^n(\mathcal{E}(U)), \tag{4.1}$$

where U is open in X and $T^n(\mathcal{E}(U))$ the $\mathcal{A}(U)$-module of homogeneous elements (sections) of degree n of the tensor algebra $T(\mathcal{E}(U))$. (See Section 2.2. Elements of $T^n(\mathcal{E}(U))$ are called *homogeneous of degree* n.) The restriction maps for the presheaf induced by correspondences of the form (4.1) are obvious. For every open set U in X, set

$$T(\mathcal{E}(U))_+ := \oplus_{m=0}^{\infty} T^{2m}(\mathcal{E}(U)) \tag{4.2}$$

and

$$T(\mathcal{E}(U))_- := \oplus_{m=0}^{\infty} T^{2m+1}(\mathcal{E}(U)). \tag{4.3}$$

Clearly, the families $(T(\mathcal{E}(U))_+)_{\text{open } U \subseteq X}$ and $(T(\mathcal{E}(U))_-)_{\text{open } U \subseteq X}$ yield two $\Gamma(\mathcal{A})$-presheaves, the sheafifications of which are denoted by $\mathcal{T}(\mathcal{E})_+$ and $\mathcal{T}(\mathcal{E})_-$; so

$$\mathcal{T}(\mathcal{E}) = \mathcal{T}(\mathcal{E})_+ \oplus \mathcal{T}(\mathcal{E})_- \tag{4.4}$$

within $\mathcal{A}$-isomorphism. For every open $U \subseteq X$, it is easy to see that

$$T(\mathcal{E}(U))_+T(\mathcal{E}(U))_+ \subseteq T(\mathcal{E}(U))_+, \quad T(\mathcal{E}(U))_+T(\mathcal{E}(U))_- \subseteq T(\mathcal{E}(U))_-, \tag{4.5}$$

and

$$T(\mathcal{E}(U))_-T(\mathcal{E}(U))_+ \subseteq T(\mathcal{E}(U))_-, \quad T(\mathcal{E}(U))_-T(\mathcal{E}(U))_- \subseteq T(\mathcal{E}(U))_+. \tag{4.6}$$

From (4.5) and (4.6), we deduce that

$$T(\mathcal{E})_+T(\mathcal{E})_+ \subseteq T(\mathcal{E})_+, \quad T(\mathcal{E})_+T(\mathcal{E})_- \subseteq T(\mathcal{E})_- \tag{4.7}$$

and

$$T(\mathcal{E})_-T(\mathcal{E})_+ \subseteq T(\mathcal{E})_-, \quad T(\mathcal{E})_-T(\mathcal{E})_- \subseteq T(\mathcal{E})_+. \tag{4.8}$$

For every open $U \subseteq X$, the ideal $I(\mathcal{E}, q)(U) \equiv I(\mathcal{E}(U))$ is generated by elements of $T(\mathcal{E}(U))_+$; since $T(\mathcal{E}(U)) = T(\mathcal{E}(U))_+ \oplus T(\mathcal{E}(U))_-$,

$$I(\mathcal{E}(U)) = \big(I(\mathcal{E}(U)) \cap T(\mathcal{E}(U))\big)_+ \oplus \big(I(\mathcal{E}(U))) \cap T(\mathcal{E}(U)\big)_-. \tag{4.9}$$

Set

$$C(\mathcal{E}(U))_+ := T(\mathcal{E}(U))_+/\big(I(\mathcal{E}(U)) \cap T(\mathcal{E}(U))_+\big) \tag{4.10}$$

and

$$C(\mathcal{E}(U))_- := T(\mathcal{E}(U))_-/\big(I(\mathcal{E}(U)) \cap T(\mathcal{E}(U))_-\big), \tag{4.11}$$

for every open $U \subseteq X$. If $\mathcal{C}(\mathcal{E})_+$ and $\mathcal{C}(\mathcal{E})_-$ denote the $\mathcal{A}$-modules generated by the presheaves obtained by means of the families $(C(\mathcal{E}(U))_+)_{\ open\ U\subseteq X}$ and $(C(\mathcal{E}(U))_-)_{\ open\ U\subseteq X}$, respectively, one has

$$\mathcal{C}l(\mathcal{E}, q) = \mathcal{C}(\mathcal{E})_+ \oplus \mathcal{C}(\mathcal{E})_- \tag{4.12}$$

within an $\mathcal{A}$-isomorphism,

$$\mathcal{C}(\mathcal{E})_+\mathcal{C}(\mathcal{E})_+ \subseteq \mathcal{C}(\mathcal{E})_+; \;\; \mathcal{C}(\mathcal{E})_+\mathcal{C}(\mathcal{E})_- \subseteq \mathcal{C}(\mathcal{E})_- \tag{4.13}$$

and

$$\mathcal{C}(\mathcal{E})_-\mathcal{C}(\mathcal{E})_+ \subseteq \mathcal{C}(\mathcal{E})_-; \;\; \mathcal{C}(\mathcal{E})_-\mathcal{C}(\mathcal{E})_- \subseteq \mathcal{C}(\mathcal{E})_+. \tag{4.14}$$

The sections of $\mathcal{C}(\mathcal{E})_+$ are called *even section*, those of $\mathcal{C}(\mathcal{E})_-$ *odd section*; it is clear that $\mathcal{C}(\mathcal{E})_+$ is a sub-$\mathcal{A}$-algebra of the Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E}, q)$. An $\mathcal{A}$-algebra with a decomposition (4.12), satisfying (4.13) and (4.14), is called, as in the classical case (see for instance [33, p. 9]), a $\mathbb{Z}_2$-*graded* $\mathcal{A}$-*algebra*. The $\mathcal{A}$-morphism $\Pi : \mathcal{C}l(\mathcal{E}, q) \longrightarrow \mathcal{C}l(\mathcal{E}, q)$ such that

$$\Pi_U(s) := s \quad \text{if } s \in \mathcal{C}(\mathcal{E})(U)_+$$
$$\Pi_U(s) := -s \quad \text{if } s \in \mathcal{C}(\mathcal{E})(U)_-$$

is an $\mathcal{A}$-automorphism of $\mathcal{C}l(\mathcal{E}, q)$, called the *main involution*. Clearly, if $\mathcal{A}$ is of characteristic 2, Π is the identity on $\mathcal{C}l(\mathcal{E}, q)$.

The $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E}, q)$ also inherits a canonical $\mathcal{A}$-antiautomorphism from the tensor algebra $\mathcal{T}(\mathcal{E})$. More accurately, for any open $U \subseteq X$ and sections $(s_1, \cdots, s_n) \in \mathcal{E}^n(U) = \mathcal{E}(U)^n$, the map ${}^T\!\alpha^n_U : T^n(\mathcal{E}(U)) \longrightarrow T^n(\mathcal{E}(U))$, given by

$${}^T\!\alpha^n_U(s_1 \otimes \cdots \otimes s_n) := s_n \otimes \cdots \otimes s_1,$$

clearly, defines an $\mathcal{A}(U)$-antiautomorphism of $T^n(\mathcal{E}(U))$. Let ${}^T\!\alpha_U$ be the $\mathcal{A}(U)$-endomorphism of $T(\mathcal{E}(U))$, given by

$${}^T\!\alpha_U = \oplus_{n=0}^{\infty} {}^T\!\alpha^n_U,$$

that is, an extension of all the $\mathcal{A}(U)$-endomorphisms ${}^T\!\alpha^n_U$. Clearly, one has

$$({}^T\!\alpha_U)^2 = \text{id}_{T(\mathcal{E}(U))}.$$

The family $({}^{T}\!\alpha_U)_{\textit{open } U\subseteq X}$ yields an $\mathcal{A}$-antiautomorphism of the tensor algebra $\mathcal{T}(\mathcal{E})$. For every open U in X,

$$({}^{T}\!\alpha_U)(s) = s, \quad \text{if } s \in T^0(\mathcal{E}(U)) := \mathcal{A}(U)$$
$$({}^{T}\!\alpha_U)(s) = s, \quad \text{if } s \in T^1(\mathcal{E}(U)) := \mathcal{E}(U),$$

that is, the elements of $T^0(\mathcal{E}(U)) \oplus T^1(\mathcal{E}(U))$ are preserved by ${}^{T}\!\alpha_U$. Moreover, ${}^{T}\!\alpha_U$ preserves the two-sided ideal $I(\mathcal{E}(U))$ because

$$({}^{T}\!\alpha_U)(s \otimes s - q_U(s) \cdot 1) = s \otimes s - q_U(s) \cdot 1,$$

for every section $s \in \mathcal{E}(U)$. Hence, the family $({}^{T}\!\alpha_U)_{\textit{open } U\subseteq X}$ defines an $\mathcal{A}$-antiautomorphism on the Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E}, q)$, whose square is the identity; we will call it the *main $\mathcal{A}$-antiautomorphism* of $Cl(\mathcal{E}, q)$, and by an abuse of notation, also denote it by α.

4.2 The Natural Filtration of a Clifford $\mathcal{A}$-algebra

In this section, we set up the definition of *graded $\mathcal{A}_X$-algebras over $\mathbb{Z}$*, or simply *$\mathbb{Z}$-graded $\mathcal{A}_X$-algebras*, following rather closely the classical notion of $\mathbb{Z}$-graded vector spaces, as laid down in [15, 25, 32]. Thus, we start first with *filtration of $\mathcal{A}_X$-algebras*: Let $\mathcal{E}$ be a sheaf of unital $\mathcal{A}_X$-algebras (for short, a *unital $\mathcal{A}_X$-algebra*). A family $(\mathcal{E}^{\leq k})_{k\in\mathbb{Z}}$ of sub-$\mathcal{A}_X$-modules of $\mathcal{E}$ is called an *increasing filtration* of $\mathcal{E}$ if it satisfies the following conditions: (i) $\mathcal{E}^{\leq k} \subseteq \mathcal{E}^{\leq k+1}$, (ii) $\mathcal{E}^{\leq j}\mathcal{E}^{\leq k} \subseteq \mathcal{E}^{\leq j+k}$, (iii) If $\mathcal{L}$ is the *unital line* sub-$\mathcal{A}$-algebra of $\mathcal{E}$, then $\mathcal{L} \subseteq \mathcal{E}^{\leq 0}$.

An increasing filtration $(\mathcal{E}^{\leq k})_{k\in\mathbb{Z}}$ is said to be *regular filtration* if (iv) $\bigcap_{k\in\mathbb{Z}} \mathcal{E}^{\leq k} = 0$, and (v) $\bigcup_{k\in\mathbb{Z}} \mathcal{E}^{\leq k} = \mathcal{E}$.

In the classical theory of Clifford algebras, Clifford algebras of quadratic modules admit several filtrations (see Definition 1.19); for instance, as given in Definition 1.19, given a quadratic module

(M, q), where M is a module over a unital and associative ring K, the following filtration of the Clifford algebra $Cl(M, q)$ is called *natural filtration*: For every negative integer k, set $Cl^{\leq k}(M, q) = 0$, and for every $k \geq 0$, $Cl^{\leq k}(M, q)$ is the submodule of $Cl(M, q)$ generated by products of the form

$$\rho(a_1)\rho(a_2)\cdots\rho(a_j), \qquad 0 \leq j \leq k, \tag{4.15}$$

where $a_1, \ldots, a_j \in M$, and $\rho : M \longrightarrow Cl(M, q)$ the natural linear map making $Cl(M, q)$ into a Clifford algebra. When $j = 0$, the product in (4.15) means 1_K. Now, *for the Clifford $\mathcal{A}$-algebra $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$ of a quadratic $\mathcal{A}_X$-module $\mathcal{E}$, the natural filtration is defined in a similar way*. In fact, for $k < 0$, $\mathcal{C}l^{\leq k}_{\mathcal{A}}(\mathcal{E}, q) = 0$, i.e., the constant zero sheaf; for $k \geq 0$, $\mathcal{C}l^{\leq k}_{\mathcal{A}}(\mathcal{E}, q)$ is the sub-$\mathcal{A}$-module of $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$ obtained by sheafifying the presheaf induced by assignments of the form

$$U \longmapsto Cl^{\leq k}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U),$$

where $Cl^{\leq k}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)$ is the sub-$\mathcal{A}(U)$-module of $Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)$ generated by products

$$\rho_U(s_1)\rho_U(s_2)\cdots\rho_U(s_j), \qquad 0 \leq j \leq k, \tag{4.16}$$

where $s_1, \ldots, s_j \in \mathcal{E}(U)$, and $\rho \equiv (\rho_U)_{\ open\ U \subseteq X} : (\mathcal{E}, q) \longrightarrow \mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$ is the defining $\mathcal{A}$-morphism for the Clifford $\mathcal{A}$-algebra $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$. As in (4.15), when $j = 0$, the product in (4.16) means 1_{q_U}. Clearly,

$$\mathcal{C}l^{\leq 0}_{\mathcal{A}}(\mathcal{E}, q) = \mathcal{I},$$

where $\mathcal{I}$ is the unital line sub-$\mathcal{A}$-algebra of $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$ (*section-wise*, one has that $\mathcal{I}(U) = \mathcal{A}(U)1_{q_U}$), and

$$\mathcal{C}l^{\leq k}_{\mathcal{A}}(\mathcal{E}, q) = \mathcal{I} \oplus \rho(\mathcal{E}) \oplus \cdots \oplus \underbrace{\rho(\mathcal{E})\cdots\rho(\mathcal{E})}_{k},$$

where $\underbrace{\rho(\mathcal{E})\cdots\rho(\mathcal{E})}_{k}$ is the sheafification of the presheaf, given by correspondences

$$U \longmapsto \rho_U(s_1)\cdots\rho_U(s_k)$$

along with restrictions

$$\big(\rho_U(s_1)\cdots\rho_U(s_k)\big)|_V := \rho_V(s_1|_V)\cdots\rho_V(s_k|_V),$$

with $s_1, \ldots, s_k$ sections of $\mathcal{E}$ over an open subset U of X.

It is clear that the family $(\mathcal{C}l^{\leq k}_{\mathcal{A}}(\mathcal{E}, q))_{k\in\mathbb{Z}}$ *is a regular increasing filtration of the Clifford $\mathcal{A}$-algebra* $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$.

Definition 4.1 Let $(X, \mathcal{A})$ be an algebraized space and $\mathcal{F}$ an $\mathcal{A}$-algebra on X. A *regular increasing filtration* $(\mathcal{F}^{\leq k})_{k\in\mathbb{Z}}$ of $\mathcal{F}$ is said *to derive from a grading* if, for every $k \in \mathbb{Z}$, there exists a sub-$\mathcal{A}$-module $\mathcal{F}^k$ of $\mathcal{F}$ such that: (i) $\mathcal{F}^{\leq k-1} \oplus \mathcal{F}^k = \mathcal{F}^{\leq k}$, (ii) $(\mathcal{F}^k)_{k\in\mathbb{Z}}$ is a grading of $\mathcal{F}$.

The next lemma is useful for deriving isomorphic graded $\mathcal{A}$-algebras from given ones; its proof is classical.

Lemma 4.1 *Let $(X, \mathcal{A})$ be an algebraized space; the exactness of the two $\mathcal{A}$-sequences*

$$\mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0 \quad \textit{and} \quad \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

implies the exactness of the $\mathcal{A}$-sequence

$$(\mathcal{E}' \otimes_{\mathcal{A}} \mathcal{F}) \oplus (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}') \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \longrightarrow \mathcal{E}'' \otimes_{\mathcal{A}} \mathcal{F}'' \longrightarrow 0. \tag{4.17}$$

Proof. Since $\mathcal{E}'' \otimes_{\mathcal{A}} \mathcal{F}'' \longrightarrow 0$ is surjective if and only if $\mathcal{E}''_x \otimes_{\mathcal{A}_x} \mathcal{F}''_x \longrightarrow 0$ is surjective for any $x \in X$, it suffices to show that

$$(\mathcal{E}' \otimes_{\mathcal{A}} \mathcal{F})_x \oplus (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}')_x \longrightarrow (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})_x \longrightarrow (\mathcal{E}'' \otimes_{\mathcal{A}} \mathcal{F}'')_x \longrightarrow 0 \tag{4.18}$$

is exact for any point $x \in X$. (For this sufficient condition, see, for instance, [51, p. 50, Theorem 6.5].) But, for $x \in X$, the $\mathcal{A}_x$-sequence (4.18) is equivalent to

$$(\mathcal{E}'_x \otimes_{\mathcal{A}_x} \mathcal{F}_x) \oplus (\mathcal{E}_x \otimes_{\mathcal{A}_x} \mathcal{F}'_x) \longrightarrow \mathcal{E}_x \otimes_{\mathcal{A}_x} \mathcal{F}_x \longrightarrow \mathcal{E}''_x \otimes_{\mathcal{A}_x} \mathcal{F}''_x \longrightarrow 0,$$

which is exact (see, for instance, [25, p. 13, (1.6.3)]). ■

Given a quadratic $\mathcal{A}$-module on a topological space X, there is a natural filtration

$$\widetilde{\mathcal{F}}^0 \subseteq \widetilde{\mathcal{F}}^1 \subseteq \widetilde{\mathcal{F}}^2 \subseteq \ldots \subseteq \mathcal{T}(\mathcal{E}) \tag{4.19}$$

of the tensor $\mathcal{A}$-algebra $\mathcal{T}(\mathcal{E})$, where

$$\widetilde{\mathcal{F}}^j \equiv \mathbf{S}\big(\sum_{i\leq j} \otimes^i \Gamma(\mathcal{E})\big) \simeq \sum_{i\leq j} \mathbf{S}(\otimes^i \Gamma(\mathcal{E})), \tag{4.20}$$

for any integer $j \geq 0$, and

$$\widetilde{\mathcal{F}}^i \otimes \widetilde{\mathcal{F}}^j \subseteq \widetilde{\mathcal{F}}^{i+j}. \tag{4.21}$$

Set

$$\mathcal{F}^i := \rho(\widetilde{\mathcal{F}}^i) = \widetilde{\mathcal{F}}^i/(\mathcal{I}(\mathcal{E},q) \cap \widetilde{\mathcal{F}}^i) \tag{4.22}$$

so as to obtain a filtration

$$\mathcal{F}^0 \subseteq \mathcal{F}^1 \subseteq \mathcal{F}^2 \subseteq \ldots \subseteq \mathcal{C}l(\mathcal{E},q) \tag{4.23}$$

of the Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E},q)$. Clearly, for any open $U \subseteq X$ and sections $s \in \widetilde{\mathcal{F}}^i(U)$ and $t \in \widetilde{\mathcal{F}}^i(U)$, one has

$$\big[s + \big(\widetilde{\mathcal{F}}^i(U) \cap \mathcal{I}_q(\mathcal{E})(U)\big)\big] \cdot \big[t + \big(\widetilde{\mathcal{F}}^j(U) \cap \mathcal{I}_q(\mathcal{E})(U)\big)\big]$$

$$:= (s \otimes t) + \big[(\widetilde{\mathcal{F}}^i \otimes \widetilde{\mathcal{F}}^j)(U) \cap \mathcal{I}_q(\mathcal{E})(U)\big],$$

from which we deduce, by property (4.21), that

$$\mathcal{F}^i \cdot \mathcal{F}^j \subseteq \mathcal{F}^{i+j} \tag{4.24}$$

for all i, j. Thus, $\mathcal{C}l(\mathcal{E}, q)$ is a *filtered $\mathcal{A}$-algebra* over X. Since $\mathcal{F}^{i-1} \subseteq \mathcal{F}^i$, for all $i \geq 0$, (4.24) yields the $\mathcal{A}$-morphism

$$(\mathcal{F}^i/\mathcal{F}^{i-1}) \cdot (\mathcal{F}^j/\mathcal{F}^{j-1}) \longrightarrow \mathcal{F}^{i+j}/\mathcal{F}^{i+j-1} \tag{4.25}$$

for all i, j. For all $i \geq 0$, on setting

$$\mathcal{G}^i := \mathcal{F}^i/\mathcal{F}^{i-1}, \tag{4.26}$$

we thus obtain the *associated graded $\mathcal{A}$-algebra*

$$\mathcal{G}^* := \bigoplus_{i \geq 0} \mathcal{G}^i. \tag{4.27}$$

Let $\mathcal{F}$ be a $\mathbb{Z}$-graded $\mathcal{A}$-algebra, i.e., there is a family $(\mathcal{F}^k)_{k\in\mathbb{Z}}$ of sub-$\mathcal{A}$-modules of $\mathcal{F}$ such that $\mathcal{F} = \oplus_{k\in\mathbb{Z}}\mathcal{F}^k$ and $\mathcal{F}^j\mathcal{F}^k \subseteq \mathcal{F}^{j+k}$ for all $j, k \in \mathbb{Z}$. It is obvious that, by setting

$$\mathcal{F}^{\leq k} := \bigoplus_{i \leq k} \mathcal{F}^i,$$

the family $(\mathcal{F}^{\leq k})_{k\in\mathbb{Z}}$ is a regular increasing filtration of $\mathcal{F}$. Indeed, the only condition that needs checking is the requirement whether $\cap_{k\in\mathbb{Z}}\mathcal{F}^{\leq k} = 0$; in fact, suppose that there exists an open set U in X such that $\cap_{k\in\mathbb{Z}}\mathcal{F}^{\leq k}(U) \neq 0$, so there is a non-zero section $s \in \cap_{k\in\mathbb{Z}}\mathcal{F}^{\leq k}(U)$. But $\mathcal{F}(U) = \oplus_{k\in\mathbb{Z}}\mathcal{F}^k(U)$, therefore $s = \oplus_{k\in\mathbb{Z}} s$, which is absurd. On the other hand, a regular filtration $(\mathcal{E}^{\leq k})_{k\in\mathbb{Z}}$ of an $\mathcal{A}$-algebra $\mathcal{E}$ is said *to derive from a grading* if, for every $k \in \mathbb{Z}$, there exists a sub-$\mathcal{A}$-module $\mathcal{E}^k$ such that $\mathcal{E}^{\leq k-1} \oplus \mathcal{E}^k = \mathcal{E}^{\leq k}$ with the family $(\mathcal{E}^k)_{k\in\mathbb{Z}}$ being a grading of $\mathcal{E}$, i.e., $\mathcal{E} = \bigoplus_{k\in\mathbb{Z}} \mathcal{E}^k$.

Lemma 4.2 *Let $(X, \mathcal{A})$ be an algebraized space and $\mathcal{F}$ an $\mathcal{A}_X$-algebra, endowed with a regular filtration $(\mathcal{F}^{\leq k})_{k\in\mathbb{Z}}$. Then,*

$$\mathcal{G}r(\mathcal{F}) := \bigoplus_{k\in\mathbb{Z}} \mathcal{G}r^k(\mathcal{F}), \tag{4.28}$$

where

$$\mathcal{G}r^k(\mathcal{F}) := \mathcal{F}^{\leq k}/\mathcal{F}^{\leq k-1}, \quad k \in \mathbb{Z}, \tag{4.29}$$

is a graded $\mathcal{A}_X$-algebra. Moreover, if $(\mathcal{F}^{\leq k})_{k\in\mathbb{Z}}$ is derived from a grading, then $\mathcal{F}$ is canonically isomorphic to $\mathcal{G}r(\mathcal{F})$.

Proof. We define multiplication fiber-wise on $\mathcal{G}r(\mathcal{F})$ as follows: For every $x \in X$ and elements $z \in \mathcal{F}_x^{\leq k}$, $z' \in \mathcal{F}_x^{\leq l}$,

$$(z + \mathcal{F}_x^{\leq k-1})(z' + \mathcal{F}_x^{\leq l-1}) := zz' + \mathcal{F}_x^{\leq k+l-1}.$$

It is easy to check that the above multiplication, which, in fact, is induced by that of $\mathcal{F}$, is well-defined. Next, let us consider the diagram

$$\begin{array}{ccc} \mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j} & \xrightarrow{\varphi_{\leq i,\leq j}} & \mathcal{G}r^i(\mathcal{F}) \otimes \mathcal{G}r^j(\mathcal{F}) \\ {\scriptstyle \lambda_{i,j}} \downarrow & & \downarrow {\scriptstyle \mathcal{G}r(\lambda_{i,j})} \\ \mathcal{F}^{\leq i+j} & \xrightarrow[\varphi_{\leq i+j}]{} & \mathcal{G}r^{i+j}(\mathcal{F}), \end{array} \tag{4.30}$$

where $\varphi_{\leq i,\leq j} := \varphi_{\leq i} \otimes \varphi_{\leq j}$, with $\varphi_{\leq i}$ the canonical $\mathcal{A}$-morphism $\mathcal{F}^{\leq i} \longrightarrow \mathcal{G}r^i(\mathcal{F})$. As for the $\mathcal{A}$-morphism $\lambda_{i,j}$, we notice that this is the $\mathcal{A}$-morphism corresponding to the $\mathcal{A}$-bilinear morphism $\mathcal{F}^{\leq i} \times \mathcal{F}^{\leq j} \longrightarrow \mathcal{F}^{\leq i+j}$ in the commutative diagram

$$\begin{array}{ccc} \mathcal{F}^{\leq i} \times \mathcal{F}^{\leq j} & \longrightarrow & \mathcal{F}^{\leq i+j} \\ \downarrow & \nearrow_{\lambda_{i,j}} & \\ \mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j} & & \end{array}.$$

We now show that the diagram (4.30) commutes. Indeed, first, we easily observe that since

$$\mathcal{F}^{\leq i-1} \longrightarrow \mathcal{F}^{\leq i} \longrightarrow \mathcal{G}r^i(\mathcal{F}) \longrightarrow 0, \qquad i \in \mathbb{Z} \tag{4.31}$$

is exact, the same property is transferred to the stalks of the corresponding presheaves of sections, i.e., for every $x \in X$,

$$\mathcal{F}_x^{\leq i-1} \longrightarrow \mathcal{F}_x^{\leq i} \longrightarrow \mathcal{G}r^i(\mathcal{F})_x \longrightarrow 0, \qquad i \in \mathbb{Z}\,, \tag{4.32}$$

is exact, which implies, by virtue of Lemma 4.1, the exactness of the following $\mathcal{A}_x$-sequence

$$(\mathcal{F}_x^{\leq i-1} \otimes \mathcal{F}_x^{\leq j}) \oplus (\mathcal{F}_x^{\leq i} \otimes \mathcal{F}_x^{\leq j-1}) \longrightarrow \mathcal{F}_x^{\leq i} \otimes \mathcal{F}_x^{\leq j}$$

$$\longrightarrow \mathcal{G}r^i(\mathcal{F})_x \otimes \mathcal{G}r^j(\mathcal{F})_x \longrightarrow 0, \qquad i, j \in \mathbb{Z}, \tag{4.33}$$

and subsequently the exactness of

$$(\mathcal{F}^{\leq i-1} \otimes \mathcal{F}^{\leq j}) \oplus (\mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j-1}) \longrightarrow \mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j}$$

$$\longrightarrow \mathcal{G}r^i(\mathcal{F}) \otimes \mathcal{G}r^j(\mathcal{F}) \longrightarrow 0, \qquad i, j \in \mathbb{Z}. \tag{4.34}$$

Then, we observe that

$$\ker(\varphi_{\leq i, \leq j}) = \big\langle (\mathcal{F}^{\leq i-1} \otimes \mathcal{F}^{\leq j}) \oplus (\mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j-1}) \big\rangle,$$

that is, $\ker(\varphi_{\leq i, \leq j})$ is generated by $(\mathcal{F}^{\leq i-1} \otimes \mathcal{F}^{\leq j})$ and $(\mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j-1})$. Thus,

$$(\varphi_{\leq i+j} \circ \lambda_{i,j})\big((\mathcal{F}^{\leq i-1} \otimes \mathcal{F}^{\leq j}) \oplus (\mathcal{F}^{\leq i} \otimes \mathcal{F}^{\leq j-1})\big) = 0;$$

hence, there exists an $\mathcal{A}$-morphism

$$\mathcal{G}r(\lambda_{i,j}) : \mathcal{G}r^i(\mathcal{F}) \otimes \mathcal{G}r^j(\mathcal{F}) \longrightarrow \mathcal{G}r^{i+j}(\mathcal{F})$$

making the diagram (4.30) commute. It is easy to see that the $\mathcal{A}$-morphisms $\mathcal{G}r(\lambda_{i,j})_{i,j\in\mathbb{Z}}$ make $\mathcal{G}r(\mathcal{F})$ into a unital associative $\mathcal{A}$-algebra. Finally, if the filtration of $\mathcal{F}$ is derived from a grading, i.e., $\mathcal{F} = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}^k$, then $\mathcal{F}^k = \mathcal{G}r^k(\mathcal{F})$ within an $\mathcal{A}$-isomorphism; hence,

$$\mathcal{F} = \mathcal{G}r(\mathcal{F}),$$

within an $\mathcal{A}$-isomorphism. ■

The *graded* $\mathcal{A}_X$*-algebra* $\mathcal{G}r(\mathcal{F})$ (4.28) is called the graded $\mathcal{A}_X$-algebra *associated* with $\mathcal{F}$. Clearly, for any $x \in X$, if $(\mathcal{F}_x^{\leq k})_{k\in\mathbb{Z}}$ is an increasing filtration of the stalk $\mathcal{F}_x$, and $(\mathcal{F}^{\leq k})_x = \mathcal{F}_x^{\leq k}$ within $\mathcal{A}_x$-isomorphism, then

$$(\mathcal{G}r(\mathcal{F}))_x \simeq \bigoplus_{k\in\mathbb{Z}}(\mathcal{G}r^k(\mathcal{F}))_x = \bigoplus_{k\in\mathbb{Z}}(\mathcal{F}^{\leq k}/\mathcal{F}^{\leq k-1})_x$$

$$= \bigoplus_{k\in\mathbb{Z}}\left((\mathcal{F}^{\leq k})_x/(\mathcal{F}^{\leq k-1})_x\right) = \bigoplus_{k\in\mathbb{Z}}(\mathcal{F}_x^{\leq k}/\mathcal{F}_x^{\leq k-1}) \equiv \bigoplus_{k\in\mathbb{Z}} Gr^k(\mathcal{F}_x)$$

$$=: Gr(\mathcal{F}_x).$$

($\mathsf{G}r(\mathcal{F}_x)$ is the associated graded $\mathcal{A}_x$-algebra of $\mathcal{F}_x$, see [25, p. 110].) Next, we apply the construction of Lemma 4.2 to the *natural filtration* of the Clifford $\mathcal{A}_X$-algebra of a quadratic $\mathcal{A}_X$-module $(\mathcal{E}, q)$ to derive a surjective $\mathcal{A}$-morphism of the exterior $\mathcal{A}_X$-algebra $\bigwedge(\mathcal{E})$ onto the associated graded $\mathcal{A}_X$-algebra $\mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$.

Theorem 4.1 *Let* $(\mathcal{E}, q)$ *be a quadratic* $\mathcal{A}_X$*-module. Then, the canonical* $\mathcal{A}$*-morphism*

$$\mathcal{E} \longrightarrow \mathcal{C}l^{\leq 1}(\mathcal{E}, q) \longrightarrow \mathcal{G}r^1(\mathcal{C}l(\mathcal{E}, q))$$

induces a surjective $\mathcal{A}$*-morphism*

$$\bigwedge(\mathcal{E}) \longrightarrow \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q)) \tag{4.35}$$

of $\mathcal{A}_X$*-algebras.*

Proof. Let us consider the presheaf of sections $(\mathcal{E}(U), \rho_V^U) \equiv ((\mathcal{E}(U), q_U), \rho_V^U)$ of the $\mathcal{A}_X$-module $(\mathcal{E}, q)$ and generating presheaves $(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U), \sigma_V^U)$ and $(Gr^1(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)), \mu_V^U)$ of the sheaves $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q)$ and $\mathcal{G}r^1(\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q))$, respectively. (We recall the following:

$$Gr^k(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)) := Cl^{\leq k}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)/Cl^{\leq k-1}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U),$$

for all $k \in \mathbb{Z}$.) If $s \in \mathcal{E}(U)$, with U an open set in X, we denote by $\widetilde{s}$ the image of s in $Gr^1(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U))$ by the $\mathcal{A}(U)$-morphism

$$\mathcal{E}(U) \longrightarrow Cl^{\leq 1}(\mathcal{E}(U)) \longrightarrow Gr^1(Cl(\mathcal{E}(U))).$$

Now, let us consider the universal $\mathcal{A}$-morphism $\rho : \mathcal{E} \longrightarrow \mathcal{C}l(\mathcal{E}, q)$ (see (2.16)); since $\rho_U(s)^2 = q_U(s) \cdot 1 \in Cl^{\leq 0}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U) \subseteq Cl^{\leq 1}_{\mathcal{A}(U)}(\mathcal{E}(U), q_U)$, and $\widetilde{s}^2$ is the image of $\rho_U(s)^2$ in $Gr^2(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U))$, it follows $\widetilde{s}^2 = 0$. Thus, there is an algebra $\mathcal{A}(U)$-morphism ϑ_U of $\bigwedge(\mathcal{E}(U))$ into $Gr(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U))$ sending s onto $\widetilde{s}$. The mapping ϑ_U is surjective as $Gr(Cl_{\mathcal{A}(U)}(\mathcal{E}(U), q_U))$ is generated by the $\widetilde{s}$. By the sheafification process, from the presheaf morphism $(\vartheta_U)_{\ open U \subseteq X}$ one obtains the desired surjective $\mathcal{A}$-morphism $\bigwedge(\mathcal{E}) \longrightarrow \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$ of $\mathcal{A}_X$-algebras. ■

For the particular case where the pair $(\mathcal{E}, q)$ is a quadratic locally free $\mathcal{A}_X$-module, let us consider the natural filtration $\left(\widetilde{\mathcal{E}}^{\leq k}\right)_{k \in \mathbb{Z}}$ of the tensor $\mathcal{A}_X$-algebra $\mathcal{T}(\mathcal{E})$, which is given by

$$\widetilde{\mathcal{E}}^{\leq k} := \begin{cases} 0, & \text{if } k < 0 \\ \sum_{i \leq k} \otimes^i \mathcal{E}, & \text{if } k \geq 0. \end{cases}$$

Clearly, we have an *ascending filtration*

$$\cdots = 0 \subseteq \mathcal{A} \subseteq \widetilde{\mathcal{E}}^{\leq 1} \subseteq \widetilde{\mathcal{E}}^{\leq 2} \subseteq \cdots \subseteq \mathcal{T}(\mathcal{E}),$$

with

$$\widetilde{\mathcal{E}}^{\leq i} \otimes \widetilde{\mathcal{E}}^{\leq j} \subseteq \widetilde{\mathcal{E}}^{\leq i+j},$$

for all i, j. Next, let us set

$$\mathcal{E}^{\leq i} := \rho(\widetilde{\mathcal{E}}^{\leq i}),$$

where ρ is the natural $\mathcal{A}$-morphism $\rho : \mathcal{E} \longrightarrow \mathcal{C}l(\mathcal{E}, q)$; we thus obtain a filtration

$$\cdots = 0 \subseteq \mathcal{E}^{\leq 0} \subseteq \mathcal{E}^{\leq 1} \subseteq \mathcal{E}^{\leq 2} \subseteq \cdots \subseteq \mathcal{C}l(\mathcal{E}, q), \tag{4.36}$$

which also satisfies the property that

$$\mathcal{E}^{\leq i} \cdot \mathcal{E}^{\leq j} \subseteq \mathcal{E}^{\leq i+j},$$

for all i, j. We now show, for this particular case, that (4.35) becomes an $\mathcal{A}$-isomorphism.

Theorem 4.2 *For any quadratic locally free $\mathcal{A}_X$-module $(\mathcal{E}, q)$,*

$$\bigwedge(\mathcal{E}) = \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$$

within $\mathcal{A}$-isomorphism.

Proof. By Theorem 4.1, the canonical $\mathcal{A}$-morphism $\bigwedge(\mathcal{E}) \longrightarrow \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$ is surjective. But this map is in addition injective for any quadratic locally free $\mathcal{A}_X$-module $(\mathcal{E}, q)$. In fact, since $(\bigwedge \mathcal{E})_x = \bigwedge(\mathcal{E}_x)$ for all $x \in X$ (cf. [35, p. 310, (7.14′)]), and $\mathcal{G}r(\mathcal{C}l(\mathcal{E}, q)_x) \simeq Gr(\mathcal{C}l(\mathcal{E}, q)_x) = Gr(\mathcal{T}(\mathcal{E}, q)_x/\mathcal{I}(\mathcal{E}, q)_x) = Gr(T(\mathcal{E}_x, q_x)/I(\mathcal{E}_x, q_x)) = Gr(Cl(\mathcal{E}_x, q_x))$, and since for any quadratic form q on a free R-module M, $Gr(Cl(M, q))$ is naturally isomorphic to the exterior algebra $\bigwedge M$ (see [29, p. 197, (1.6)] or [33, p. 10, Proposition 1.2]), it follows that

$$(\bigwedge \mathcal{E})_x \simeq \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))_x,$$

for all $x \in X$. On applying [35, p. 68, Theorem 12.1], one obtains an $\mathcal{A}$-isomorphism $\bigwedge \mathcal{E} \simeq \mathcal{G}r(\mathcal{C}l(\mathcal{E}, q))$. ■

The Clifford $\mathcal{A}_X$-algebra of an orthogonal sum of two $\mathcal{A}_X$-modules requires, as in the classical case, the notion of a *twisted tensor product*. In fact, let $\mathcal{E}$ and $\mathcal{F}$ be $\mathbb{Z}_2$-graded $\mathcal{A}_X$-modules; explicitly, one has

$$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1, \quad \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1,$$

where $\mathcal{E}_i$, $\mathcal{F}_i$, $i = 0, 1$, are sub-$\mathcal{A}_X$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. The tensor product $\mathcal{E} \otimes \mathcal{F}$ is also a graded $\mathcal{A}_X$-module, its grading is as follows:

$$\mathcal{E} \otimes \mathcal{F} = (\mathcal{E} \otimes \mathcal{F})_0 \oplus (\mathcal{E} \otimes \mathcal{F})_1,$$

where

$$(\mathcal{E} \otimes \mathcal{F})_0 = (\mathcal{E}_0 \otimes \mathcal{F}_0) \oplus (\mathcal{E}_1 \otimes \mathcal{F}_1),$$

$$(\mathcal{E} \otimes \mathcal{F})_1 = (\mathcal{E}_0 \otimes \mathcal{F}_1) \oplus (\mathcal{E}_1 \otimes \mathcal{F}_0).$$

The twisted tensor product $\mathcal{E}\widehat{\otimes}\mathcal{F}$ of $\mathcal{A}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ is defined as follows: As an $\mathcal{A}_X$-module $\mathcal{E}\widehat{\otimes}\mathcal{F}$ is the usual tensor product $\mathcal{E} \otimes \mathcal{F}$, but where the product is induced by the $\Gamma(\mathcal{A})$-morphism $\pi \equiv (\pi_U)_{X \supseteq U,\ open}$, where, for each open U in X,

$$\pi_U : (\mathcal{E}(U) \otimes \mathcal{F}(U)) \times (\mathcal{E}(U) \otimes \mathcal{F}(U)) \longrightarrow \mathcal{E}(U) \otimes \mathcal{F}(U)$$

is given by

$$\pi_U(s \otimes t, s' \otimes t') = (-1)^{\partial s' \partial t} ss' \otimes tt',$$

where $\partial s'$ (∂t, resp.) is the order of s' (t, resp.) with respect to the parity grading of $\mathcal{E}(U)$ ($\mathcal{F}(U)$, resp.).

Theorem 4.3 *Let* $(\mathcal{E}, q) = (\mathcal{E}_1, q_1) \perp (\mathcal{E}_2, q_2)$ *be a* q*-orthogonal decomposition of the quadratic* $\mathcal{A}_X$*-module* $\mathcal{E}$ $(q_i = q|_{\mathcal{E}_i},\ i = 1, 2)$. *Then,*

$$\mathcal{C}l(\mathcal{E}, q) \simeq \mathcal{C}l(\mathcal{E}_1, q_1) \widehat{\otimes}\ \mathcal{C}l(\mathcal{E}_2, q_2). \tag{4.37}$$

Proof. This result clearly derives from its classical counterpart; see, for instance, [25, p. 115, Theorem 3.2.4] or [29, p. 195, Theorem 1.3.1]. In fact, since $\big(Cl(\mathcal{E}(U), q_U)\big)_{X \supseteq U,\ open}$ and

$\big(Cl(\mathcal{E}_i(U), q_{iU})\big)_{open\ U\subseteq X}$ are generating presheaves of $\mathcal{C}l(\mathcal{E}, q)$ and $\mathcal{C}l(\mathcal{E}_i, q_i)$, respectively, and since, for each open $U \subseteq X$,

$$Cl(\mathcal{E}(U), q_U) \simeq Cl(\mathcal{E}_1(U), q_U)\widehat{\otimes}Cl(\mathcal{E}_2(U), q_U),$$

one applies sheafification on the corresponding presheaves of algebras to conclude that $\mathcal{A}_X$-algebras $\mathcal{C}l(\mathcal{E}, q)$ and $\mathcal{C}l(\mathcal{E}_1, q_1)\widehat{\otimes}\ \mathcal{C}l(\mathcal{E}_2, q_2)$ are isomorphic. ■

Let us assume that the pair $(X, \mathcal{A})$ is an *ordered algebraized space* endowed with *square root,* i.e., every positive section of $\mathcal{A}$ is invertible and has a square root. For further background on ordered algebraized spaces, cf. [35, p. 96ff]. Next, recall that a subsheaf $\mathcal{P} \subseteq \mathcal{A}$ defines an order if (i) $\lambda\mathcal{P} \subseteq \mathcal{P}$ for any $\lambda \in \mathbb{R}^+ \subseteq \mathcal{A}$, (ii) $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$, and (iii) $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$. See, for instance, [35, pp. 316, 317]. The elements of $\mathcal{P} \subseteq \mathcal{A}$ are called *positive*; more precisely, a (local) section of $\mathcal{A}$ is called positive if it takes on values in $\mathcal{P}$, i.e., if s is a section over U, for any $x \in U$, $s(x) \neq 0$ and $s(x) \in \mathcal{P}_x$. On the other hand, we assume that $(\mathcal{E}, q)$ is a quadratic locally free $\mathcal{A}_X$-module of rank n, with q being such that its associated $\mathcal{A}$-bilinear morphism $b \equiv b_q$ satisfies the following condition:

$$b(s, s)(x) \neq 0,$$

for any nowhere-zero $s \in \mathcal{E}(U)$ and $x \in U$. We claim that for any local gauge U of $\mathcal{E}$, we can apply the standard Gram-Schmidt orthogonalization process to transform any given basis of $\mathcal{E}(U)$ into an orthonormal one. The procedure is similar to the one in [35, pp. 337- 340]. Indeed, let $(s_i)_{1\leq i\leq n} \subseteq \mathcal{E}(U) \simeq \mathcal{A}^n(U) \simeq \mathcal{A}(U)^n$ be a local gauge of $\mathcal{E}$. Moreover, let

$$|s_i| := \sqrt{b(s_i, s_i)}, \qquad 1 \leq i \leq n,$$

so that if we set

$$\widetilde{s_i} := |s_i|^{-1} \cdot s_i \in \mathcal{E}(U), \qquad 1 \leq i \leq n,$$

one has

$$|\widetilde{s_i}| = 1 \in \mathcal{A}(U), \qquad 1 \leq i \leq n.$$

Thus, let

$$(\widetilde{s_i})_{1 \leq i \leq m} \subseteq \mathcal{E}(U), \qquad m \leq n-1,$$

such that

$$b(\widetilde{s_i}, \widetilde{s_j}) = \delta_{ij}, \qquad 1 \leq i, j \leq m.$$

Setting

$$t := s_{m+1} - \sum_{i=1}^{m} b(s_{m+1}, \widetilde{s_i})(\widetilde{s_i}),$$

we observe that, for every $x \in U$, $t(x) \neq 0$ and $b(t,t)(x) \neq 0$. We have obtained a new section of $\mathcal{A}$, which is

$$\widetilde{s_{m+1}} = |t|^{-1} \cdot t \in \mathcal{E}(U).$$

Hence, step by step, one obtains a family

$$(\widetilde{s_i})_{1 \leq i \leq n} \subseteq \mathcal{E}(U)$$

such that

$$b(\widetilde{s_i}, \widetilde{s_j}) = \delta_{ij}, \qquad 1 \leq i, j \leq n.$$

We shall call such $\mathcal{A}_X$-modules $(\mathcal{E}, q)$ *quadratic locally free $\mathcal{A}_X$-modules endowed with q-orthogonal bases*.

Theorem 4.4 *Let $(\mathcal{E}, q)$ be a quadratic locally free $\mathcal{A}_X$-module of rank n, endowed with q-orthogonal bases. Then, $\mathcal{C}l(\mathcal{E}, q)$ is a locally free $\mathcal{A}_X$-module of rank 2^n.*

Proof. Let $U \subseteq X$ be a local gauge of $\mathcal{E}$, that is, U is open in X and such that

$$\mathcal{E} \simeq \mathcal{A}^n|_U. \tag{4.38}$$

As (4.38) is equivalent to

$$\mathcal{E}_x \simeq (\mathcal{A}^n)_x \simeq (\mathcal{A}_x)^n, \qquad x \in U$$

(see, for instance, [35, p. 123, Definition 3.2]), it follows that, since, for all $x \in U$, $\mathcal{E}_x$ has an orthogonal basis, $\mathcal{C}l(\mathcal{E}, q)_x \simeq Cl(\mathcal{E}_x, q_x)$ is a free module of rank 2^n (see [25, p. 116, Corollary]). The latter implies that

$$\mathcal{C}l(\mathcal{E}, q)|_U \simeq \mathcal{A}^{2n}|_U \simeq (\mathcal{A}|_U)^{2n},$$

where U is as above, that is a local gauge of $\mathcal{E}$. Since U is arbitrary, $\mathcal{C}l(\mathcal{E}, q)$ is locally free of rank 2^n. ■

4.3 The Parity Grading of Clifford $\mathcal{A}$-algebras

The parity grading of a module is merely a decomposition of it into a direct sum of two submodules, whose elements are either even or odd homogeneous elements. $\mathcal{C}l(\mathcal{E}, q)$ can inherit a parity grading from $\mathcal{T}(\mathcal{E})$ over $\mathbb{Z}_2$. Indeed we can set

$$\mathcal{T}_0(\mathcal{E}) := \oplus_{m=0}^{\infty} \mathcal{T}^{2m}(\mathcal{E}) \quad and \quad \mathcal{T}_1(\mathcal{E}) := \oplus_{m=0}^{\infty} \mathcal{T}^{2m+1}(\mathcal{E}). \tag{4.39}$$

where the lower indices 0 and 1 of $\mathcal{T}$ are elements of $\mathbb{Z}_2$. Consider the two-sided ideal sheaf $\mathcal{I}(\mathcal{E}, q)$ of the tensor algebra (sheaf) $\mathcal{T}(\mathcal{E})$ generated by the presheaf $J(\mathcal{E}, q) \equiv (J(\mathcal{E}, q)(U), \mu_V^U)$, where $J(\mathcal{E}, q)(U)$ is a two-sided ideal of the tensor algebra $T(\Gamma(\mathcal{E}))(U) \equiv T(\mathcal{E}(U))$ generated by elements of the form

$$s \otimes s - q_U(s),$$

with s running through $\mathcal{E}(U)$.

As a submodule, $\mathcal{I}(\mathcal{E}, q)$ is graded since it is a direct sum of $\mathcal{I}_0(\mathcal{E}, q) := \mathcal{T}_0(\mathcal{E}) \cap \mathcal{I}(\mathcal{E}, q)$ and $\mathcal{I}_1(\mathcal{E}, q) := \mathcal{T}_1(\mathcal{E}) \cap \mathcal{I}(\mathcal{E}, q)$. Consequently $Cl(\mathcal{E}, q)$ is the direct sum of two submodules $Cl_0(\mathcal{E}, q)$ and $Cl_1(\mathcal{E}, q)$ respectively isomorphic to $T_i(\mathcal{E})/I_i(\mathcal{E}, q)$ with $i = 0, 1$. The elements of $Cl_0(\mathcal{E}, q)$ and $Cl_1(\mathcal{E}, q)$ are said to be *even* and *odd*, respectively.

Now, if $\varphi_{\mathcal{C}}$ is the Clifford $\mathcal{A}$-morphism of a Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$ into its Clifford $\mathcal{A}$-algebra $\mathcal{C} \equiv \mathcal{C}(\mathcal{E}, q)$, the $\mathcal{A}$-morphism φ', such that $\varphi' := -\varphi_{\mathcal{C}}$, is another Clifford $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{C}$; thus there exists an $\mathcal{A}$-endomorphism Π of (the unital $\mathcal{A}$-algebra) $\mathcal{C}$ such that

$$\Pi(1_{\mathcal{C}}) = 1_{\mathcal{C}}, \quad \Pi \circ \varphi_{\mathcal{C}} = \varphi' = -\varphi_{\mathcal{C}}. \tag{4.40}$$

Clearly,

$$\Pi^2(1_{\mathcal{C}}) = 1_{\mathcal{C}}, \quad \Pi^2 \circ \varphi_{\mathcal{C}} = \varphi_{\mathcal{C}}; \tag{4.41}$$

it follows that Π is an *$\mathcal{A}$-involutive automorphism* of $\mathcal{C}$, called the *principal $\mathcal{A}$-automorphism* of the Clifford $\mathcal{A}$-algebra $\mathcal{C}(\mathcal{E}, q)$.

For every open $U \subseteq X$, let $C_+(U)$ denote the sub-$\mathcal{A}(U)$-module of the $\mathcal{A}(U)$-algebra $\Gamma(U, \mathcal{C}) \equiv \mathcal{C}(U)$ consisting of the *eigenvector sections* of Π_U for the *eigenvalue section* $+1$ (cf. [39]). It is evident that $C_+(U)$ *is a sub-$\mathcal{A}(U)$-algebra of $\mathcal{C}(U)$, containing any product of any even number of nowhere-zero sections in* $(\varphi_{\mathcal{C}})_U(\mathcal{E}(U))$:

$$(\varphi_{\mathcal{C}})_U(s_1)(\varphi_{\mathcal{C}})_U(s_2)\cdots(\varphi_{\mathcal{C}})_U(s_{2p}).$$

Conversely, if $(e_1, e_2, \ldots, e_n)$ is an orthogonal basis of $\mathcal{E}(U)$, reducing the number of terms in any product

$$(\varphi_{\mathcal{C}})_U(e_{i_1})(\varphi_{\mathcal{C}})_U(e_{i_2})\cdots(\varphi_{\mathcal{C}})_U(e_{i_p}),$$

does not change the parity of the number of terms involved. Thus, $C_+(U)$ is linearly generated by the elements $(\varphi_{\mathcal{C}})_U(e_J)$, with $J = (1 \leq j_1 < \cdots < j_m \leq n)$ for an even m.

By letting U vary over the open subsets of X, the family $(C_+(U), {}_+\lambda_V^U)$, where ${}_+\lambda_V^U := \sigma_V^U|_{C_+(U)}$, with the (σ_V^U) being the restriction maps for the (complete) presheaf of sections $\Gamma(\mathcal{C})$ of the $\mathcal{A}$-algebra $\mathcal{C}$, forms a *complete presheaf of algebras on* X. Indeed, let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open covering of U, and let $s, t \in C_+(U)$ such that

$$
{}_+\lambda_{U_\alpha}^U(s) \equiv s_\alpha = t_\alpha \equiv {}_+\lambda_{U_\alpha}^U(t)
$$

for every $\alpha \in I$. Since $C_+(U) \subseteq \mathcal{C}(U)$, and $\mathcal{C}$ is an $\mathcal{A}$-algebra, it follows that $s = t$. Thus, axiom (i) Definition 2.3 is fulfilled. For axiom $(ii)$of Definition 2.3, let $s_\alpha \in C_+(U_\alpha)$, $\alpha \in I$, such that for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ in $\mathcal{U}$, one has

$$
{}_+\lambda_{U_{\alpha\beta}}^{U_\alpha}(s_\alpha) \equiv s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}} \equiv {}_+\lambda_{U_{\alpha\beta}}^{U_\beta}(s_\beta).
$$

Without loss of generality, suppose that

$$
s_\alpha = (\varphi_{\mathcal{C}})_{U_\alpha}(s_{\alpha,1}) \cdots (\varphi_{\mathcal{C}})_{U_\alpha}(s_{\alpha,2p})
$$

and

$$
s_\beta = (\varphi_{\mathcal{C}})_{U_\beta}(s_{\beta,1}) \cdots (\varphi_{\mathcal{C}})_{U_\beta}(s_{\beta,2q})
$$

with $s_{\alpha,1}, \ldots, s_{\alpha,2p} \in \mathcal{E}(U_\alpha)$ and $s_{\beta,1}, \ldots, s_{\beta,2q} \in \mathcal{E}(U_\beta)$. It is evident that there exists an $s \in \mathcal{C}(U)$ such that

$$
\sigma_{U_\alpha}^U(s) \equiv s|_{U_\alpha} = s_\alpha,
$$

for every $\alpha \in I$. Clearly, s is of the form

$$
s = (\varphi_{\mathcal{C}})_U(s_1^1) \cdots (\varphi_{\mathcal{C}})_U(s_{2p_1}^1) + \cdots + (\varphi_{\mathcal{C}})_U(s_1^k) \cdots (\varphi_{\mathcal{C}})_U(s_{2p_k}^k),
$$

where for any $i = 1, \ldots, k$, p_i is an integer $\leq n$, and every $s^i_1, \ldots, s^i_{2p_i} \in \mathcal{E}(U)$. Indeed, if s contains a product of an odd number of terms, then, for any $U_\alpha \in \mathcal{U}$, $s|_{U_\alpha} := s_\alpha \notin C_+(U)$. Thus, $C_+ \equiv (C_+(U), {}_+\lambda^U_V)$ is a complete presheaf of algebras. The sheafification of the presheaf C_+, denoted $\mathcal{C}_+ \equiv \mathbf{S}C_+$, is called the $\mathcal{C}$.

Now, let $C_-(U)$ be the eigen sub-$\mathcal{A}(U)$-module of $\mathcal{C}(U)$ for the eigenvalue section -1. Clearly, *elements of* $C_-(U) \subseteq \mathcal{C}(U)$ *are linearly generated by products of an odd number of terms of* $(\varphi_\mathcal{C})_U(\mathcal{E}(U))$. One proceeds as above to show that pairs $(C_-(U), {}_-\lambda^U_V)$, where ${}_-\lambda^U_V = \sigma^U_V|_{C_-(U)}$, *yield a complete presheaf.* However, we notice that every $C_-(U)$ *is not an algebra;* so the presheaf $(C_-(U), {}_-\lambda^U_V)$ is not a presheaf of algebras, but a presheaf of modules instead. Its sheafification, denoted $\mathcal{C}_-$, is called the .

Definition 4.2 Let $\mathcal{C}$ be an $\mathcal{A}$-algebra. The $\mathcal{A}$-algebra $\mathcal{C}^*$, in which products are defined to be products in $\mathcal{C}$ but in the reverse order, is called the *opposite $\mathcal{A}$-algebra of $\mathcal{C}$*.

Specifically, let U be open in X and $s, t \in \mathcal{C}(U)$; then, if $*$ denotes the product in $\mathcal{C}^*(U)$, one has

$$s * t := ts.$$

Now, considering $\varphi_\mathcal{C}$ as a Clifford $\mathcal{A}$-morphism of the Riemannian quadratic free $\mathcal{A}$-module $(\mathcal{E}, q)$ into its Clifford $\mathcal{A}$-algebra $\mathcal{C}$, $\varphi_\mathcal{C}$, which we denote by $\varphi^*_\mathcal{C}$, as an $\mathcal{A}$-morphism from $\mathcal{E}$ into $\mathcal{C}^*$, is again a Clifford $\mathcal{A}$-morphism. Thus, there exists a 1-respecting $\mathcal{A}$-morphism $\tau : \mathcal{C} \longrightarrow \mathcal{C}^*$ such that

$$\tau \circ \varphi_\mathcal{C} = \varphi^*_\mathcal{C}.$$

But, $\tau_U((\varphi_\mathcal{C})_U(s)) = (\varphi^*_\mathcal{C})_U(s) = (\varphi_\mathcal{C})_U(s)$ for any open set U in X and section $s \in \mathcal{E}(U)$, and since $1_\mathcal{C}$ and $\varphi_\mathcal{C}(\mathcal{E}) = \varphi^*_\mathcal{C}(\mathcal{E})$ generate both $\mathcal{C}$ and $\mathcal{C}^*$, it follows that τ is bijective, hence, a 1-respecting $\mathcal{A}$-isomorphism of $\mathcal{C}$ into $\mathcal{C}^*$. We conclude that τ is the only $\mathcal{A}$-*antiautomorphism*, fixing the sections of $\varphi_{\mathcal{C}}(\mathcal{E})$. For any open U in X and sections $s_1, \ldots, s_k \in \mathcal{E}(U)$, $\tau_U((\varphi_{\mathcal{C}})_U(s_1) \cdots (\varphi_{\mathcal{C}})_U(s_k)) = (\varphi_{\mathcal{C}})_U(s_k) \cdots (\varphi_{\mathcal{C}})_U(s_1)$, it follows that $\tau^2 = 1$, i.e., τ is an $\mathcal{A}$-*involution*.

Using sections, one easily sees that $\Pi \circ \tau = \tau \circ \Pi$, which is the only $\mathcal{A}$-antiautomorphism of $\mathcal{C}$ sending sections of $\varphi_{\mathcal{C}}(\mathcal{E})$ to their opposites. On the other hand, $\Pi \circ \tau$ is an $\mathcal{A}$-involution and is called the *conjugate* of $\mathcal{C}$.

Chapter 5

Conclusion and Future Work

The main foci of our investigation in this thesis were: the commutative property of the Clifford functor on sheaves of Clifford algebras, the natural filtration of Clifford $\mathcal{A}$-algebras, and localization of vector sheaves. But there are classical results that need to be researched in connection to sheaves of Clifford algebras. These include:

1. $\mathcal{A}$-modules of fractions of non-commutative sheaves of rings

2. Globalization of $\mathcal{A}$-modules

3. Graded quadratic extensions of scalars of $\mathcal{A}$-modules

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