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Variable annuity guarantees pricing under the Variance-Gamma framework

by

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Submitted in partial fulfillment of the requirements for the degree

Magister Scientae

to the Department of Mathematics and Applied Mathematics

in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

October 2014

Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Abstract

The purpose of this study is to investigate the pricing of variable annuity embedded derivatives in a Lévy process setting. This is one of the practical issues that continues to face life insurers in the management of derivatives embedded within these products. It also addresses how such providers can protect themselves against adverse scenarios through a hedging framework built from the pricing framework.

The aim is to comparatively consider the price differentials of a life insurer that prices its variable annuity guarantees under the more actuarially accepted regime-switching framework versus the use of a Lévy framework. The framework should address the inadequacies of conventional deterministic pricing approaches used by life insurers given the increasing complexity of the option-like products sold. The study applies finance models in the insurance context given the similarities in payoff structure of the products offered while taking into account the differences that may exist.

The underlying Lévy process used in this study is the Variance-Gamma (VG) process. This process is useful in option pricing given its ability to model higher moments, skewness and kurtosis, and also incorporate stochastic volatility.

The research results compare well with the regime-switching framework besides the added merit in the use of a more refined model for the underlying that captures most of the observed market dynamics.

Keywords: Embedded option, variable annuity, Variance-Gamma

Acknowledgements

To my parents, David Waweru and Jacinta Ngugi.

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List of abbreviations and notation

ALSI	All Share Index
APN	Advisory Practice Note
ARCH	Autoregressive Conditional Heteroskedasticity
ASSA	Actuarial Society of South Africa
CGMY	Carr, Geman, Madan and Yor
CIR	Cox-Ingersoll-Ross
ES	Expected Shortfall
FSB	Financial Services Board
IFRS	International Financial Reporting Standard
JSE	Johannesburg Securities Exchange
MC	Monte Carlo
MEMM	Minimal-Entropy Martingale Measure
MMCT	Mean Martingale Correcting Term
NIG	Normal Inverse Gaussian
RSLN	Regime-Switching Log-Normal model
SAM	Solvency Assessment and Management
SAP	Standard of Actuarial Practice
SV	Stochastic Volatility
VaR	Value-at-Risk
VG	Variance-Gamma
VGSV	Variance-Gamma Stochastic Volatility
B_t	Bank account at time t
$M(t)$	Markov process
$p_{i,j}$	Transition probability from state i to state j

$\mathbb{P}$	Real-world probability measure
$\Phi(\cdot)$	Standard normal cumulative distribution function
q	Dividend yield
$\mathbb{Q}$	Risk-neutral probability measure
${}_T q_x$	Probability of a life aged x dying within the next T years
r	Continuously compounded risk-free rate
σ	Volatility parameter
${}_T s_x$	Probability of a life aged x surrendering within the next T years
T	Maturity time of an option
X	Strike price of an option
W_t	Wiener process

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Summary and conclusions

1.1. Introduction

The financial industry has, in the past century, been a haven of periods of market stability such as the 1970s and early 1980s followed by seasons of instability with market declines as in the late 1980s. The early 1990s were also relatively stable years in the markets with the latter part of the decade experiencing huge volatility. This trend has seen market variables such as interest rates and exchange rates fluctuating wildly at such times resulting in huge crises. Financial solutions have been sought for these market vagaries (see, for example, Zenios (1995), Schofield (2007), Sadr (2009), Cummins and Weiss (2009)) with hedging as a tool being widely used to protect market participants.

In the same period, the pool of market participants has grown with different industries seeking solutions from the capital markets. This has been through various initiatives such as raising finance, moving from the traditional share capital as a source of finance and creating products that protect the market participants in times of adverse exposure. The insurance industry has been one of such.

The industry strongly adopted actuarial principles in the asset-liability management (ALM) realm since its early days. The first actuaries sought to provide a solid basis in the management of assets and liabilities by solely using actuarial models. William Morgan the first actuary at the '*Society for Equitable Assurances on Lives and Survivorships*' adopted a scientific actuarial basis in undertaking the first valuation in 1775. These principles would guide the asset-liability management of such insurers in the decades that followed. They were rooted in mathematical formulae on mortality and compound interest in an attempt to reduce uncertainty on life insurance surpluses/deficits (see Storr-Best (1970), Lewin (1998) and Dennett (2004)). During that time, ALM was just as important, failure to which Cornelius Walford as quoted in Storr-Best (1970) would note, "Companies sprang up like gnats on a summer's evening and disappeared as quickly."

Refinements were gradually made to these formulae and actuarial bases became sounder. Ideas such as assessmentism, where incomes and outgos are matched annually without an attempt to smooth the policyholder premiums over the years, were employed but with their own challenges. The technological state of the time meant that the focus was to manually determine the expected present value of cashflows normally under more theoretical than practical assumptions. Reserving which is defined as, "… the setting aside of assets to cover negative expected future cashflows …" (Dickson, Hardy & Waters 2009) was used by insurers as a safe way of managing the asset-liability mismatch that may occur from time to time. In a recollection of the actuarial history, Skerman (1998) notes that asset returns remained a source of volatility in this ALM world with actuaries being little in control and resulting to setting up investment reserve funds.

It was not until 1950s when actuaries began to more practically consider assets and liabilities together following on from the novel research by Frank Redington on the theory of immunization. Using this theory he noted that, "… a term can be found for the asset proceeds such that changes in the rate of interest change the value of the liability outgo and the asset proceeds by an equal amount…asset proceeds and liability outgo form two cash flows which must be treated consistently in valuation." (Skerman 1998).

Not much later life insurers started investing in overseas assets, which meant exposure to currency fluctuations and further necessitated the need for a change in the way ALM was done.

It is possibly the last three decades that have however brought the most radical changes in the operations of the insurance industry. In this period, the insurers have moved to combine their models of mortality with finance models in an endeavour to build a framework that will price their products fairly while ensuring sound risk management. The finance and actuarial worlds have become more interconnected than ever, a subject discussed in Embrechts (2000).

Business competitiveness and awareness has resulted in insurers moving from their traditional assurance products to combined assurance and savings products. In the same period, policyholders have become more sophisticated in terms of their needs and these needs coupled with the advent of powerful computational facilities have led to increasingly complex products.

The products have necessitated a new approach to managing risk or at the very least a modification of the old techniques and methods used. Hedging, defined by Stephens (2000) as the endeavour to try counteract or where possible offset the risks that an entity faces by holding an offsetting position, a concept hitherto left for the financial world, has slowly started to find its way in the insurance industry. Restructurings have been undertaken in some life insurance companies where treasury type operations have arisen in asset-liability management. These companies now have an asset management function and an intermediary that links up the insurer's liabilities with its assets. This approach results in a link between policyholders and shareholders if the insurer is not a mutual and a possible conflict between covering the liabilities well with the assets available versus maximizing profits, which is what the latter want.

In a scenario where there is such a shift in the management of assets and liabilities, inadvertently, new possibilities and risks arise. The begging question is whether the benefits of the new structures can be appropriately priced and the risks inherent in them better managed with efficacy being achieved. The central theme of this study is the variable annuity product and the different structural presentations of the product to the insureds. In particular, embedded derivatives in variable annuities will be considered and techniques to price and hedge them in view of the increasing complexity and competitive importance of these guarantees in the life insurance industry.

1.2. Dissertation objectives and structure

A variable annuity is an annuity where the premiums are mainly invested in the financial markets with the receipts by the annuitant being dependent, in some way, on the performance of the premiums invested.

The need for differentiation in the life insurance industry has meant that the insurers will normally provide guarantees of some form to the annuitant. There are four main guarantees in variable annuities namely: Guaranteed Minimum Maturity Benefit (GMMB), Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB) and Guaranteed Minimum Withdrawal Benefit (GMWB).

The GMDB benefit guarantees a lump-sum on death regardless of the underlying asset's account performance, the GMMB benefit gives a minimum guarantee to the policyholder at maturity, the GMIB benefit gives a guaranteed income until the policyholder dies and the GMAB benefit offers a guaranteed amount regardless of the performance of the underlying asset account this differing with the GMMB in that for the latter, the guarantee is only applicable at maturity. They are normally written as GMxB with 'x' representing the respective benefit concerned (Olivieri & Pitacco 2011). The report seeks to consider the first two with a perspective of how financial engineering can provide solutions to price and hedge against risks brought by the sale of these annuities.

To this end, this study has five main objectives:

1. To provide familiarity on practical issues, facing life insurers, which arise from embedded options in GMxB products world over.
2. To implement two embedded option pricing frameworks: the regime-switching framework and the Variance-Gamma framework, the latter being a more refined model for movement of the underlying asset.
3. To investigate which methods are suitable with the more conventional and actuarially accepted regime switching framework acting as the comparative base.
4. To investigate how sensitivity to the underlying can be mitigated in a more refined model setting using financial engineering tools available in the capital markets and the continuing relevance of these tools in the insurance context.
5. To provide recommendations on the use of these pricing and hedging strategies based on an appropriate risk measure calculation.

Objective	Objective 1	Objective 2	Objective 3	Objective 4	Objective 5
Chapter(s) achieved	Chapters 1, 2, 3 & 6	Chapter 4	Chapter 4	Chapter 5	Chapter 5
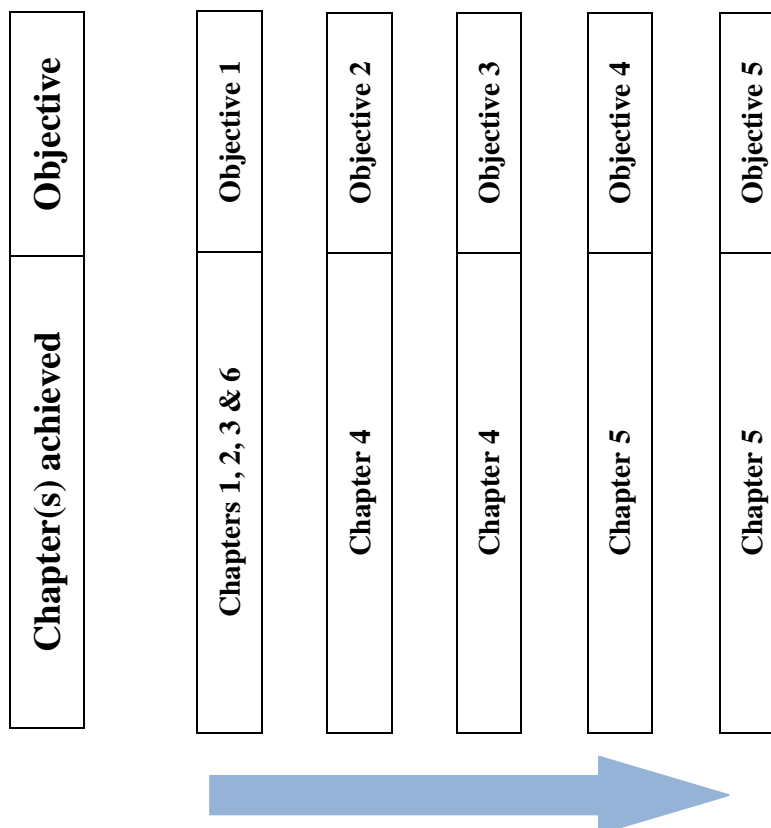
Figure 1.1: Objectives achievement flow

The research questions for this study are thus:

a. Can the guarantees embedded in the GMxBs be effectively priced and hedged using available jump model financial engineering tools and is there a best practice?
b. What impact would this have on the life insurer's capital management and does this present itself as a solution?

The research builds on propositions laid by Milevsky and Salisbury (2006), Bauer, Kling and Russ (2008), Bacinello et al. (2011) and Feng and Volkmer (2012) who after considering different aspects of variable annuities in a Black-Scholes setting, propose the research theme as an area for further research; in particular research question (a). Wu (2009) after considering the pricing of embedded options noted the importance of hedging these options in the wake of the market crises and recommended it as an area of further research. The study conducted by Wilkie, Waters and Yang (2003), Jaimungal (2004) and Ballotta (2010) in this direction will also be of relevance to this research forming a basis from which some of the objectives are addressed.

The value of this study will be implementing the non-conventional embedded option pricing models in a South African (SA) life insurance setting using the JSE ALSI, analysing the results obtained and trying to find the best practice for the long-term sustainability of the industry. The author is not aware of such a study in an SA setting.

Research in this path has been done by Foroughi, Jones and Dardis (2003) who discuss the topic of investment guarantees in an SA context but do not venture into hedging. Maitland (2001) discussed the immunization of nominal liabilities in SA and uses principal component analysis in his research. Thomson (2011) discusses the pricing of liabilities in an incomplete market with a specific application to the SA retirement fund whereas Raubenheimer and Kruger (2010) propose a dynamic stochastic programming model for use in ALM and note the necessity of further research on this area in SA.

Literature Review

The previous chapter discussed briefly the state of the life insurance industry and the high-level but significant changes that have occurred in the past. This chapter will delve into these issues in more detail. The experiences of some well-known life insurance companies are discussed and the need for a proactive undertaking in the endeavour to solve the challenges that face the industry identified.

The story of *Equitable Life*, the first mutual company established as the '*Society for Equitable Assurances on Lives and Survivorships*', is one of a solid structure for over two centuries when the actuarial and finance worlds were not very interconnected and a collapse partly attributable to the close interdependence of the two worlds in the latter years. *The Equitable* sold to policyholders deferred annuities with the option of having either a Guaranteed Annuity Rate (GAR) or the Current Annuity Rate (CAR). The exercise of these options was mostly reliant on the prevailing interest rate and although CAR was the more generous annuity, market volatility could lead to the GAR option being exercised. The insurer did not hedge itself against the exercise of these guarantees and was exposed to the possibility of heavy losses in case of adverse market movements.

In the early 1990s, CAR fell below GAR due to interest rate movements and the option was exercised by the policyholders. In a sworn affidavit during his litigation, Christopher Headdon the ex-CEO of *The Equitable* noted that the cost of the guarantees to the company was in excess of £ 1 billion but, "… at no time did Equitable ever hedge or reinsure adequately against the GAR risk to counteract it …" (Hodson 2007). In the Penrose report submitted in 2004, the committee noted that the *Equitable Life* management could have taken measures to avoid the failure that occurred presumably by instituting a risk management framework that would protect it from the guarantees that lay embedded in its product offerings if they were to mature (Penrose 2004).

As the *Equitable* case was waning, in the midst of the 2008 crises, life insurers that were not so well prepared were facing a crisis not so dissimilar. In the United States where a huge proportion of variable annuities with guarantees are offered, the risk exposure if the guarantees matured was huge. In a report delivered to the *Actuarial Society of South Africa* (ASSA) post the crises, the total losses from these annuities was in excess of $ 4 billion (approximately R 40 billion) as shown in Table 2.1 below.

Table 2.1: Variable annuity losses during the 2007/2008 crises

Company	Loss
Manulife:	**$1.5bn**
The Hartford:	**$834m**
ING:	**$700m**
Axa:	**$520m**
Old Mutual (Bermuda):	**$500m**
Philadelphia Lincoln Financial:	**$145m**

Source: Addae (2010)

Though all these losses were not solely due guarantees, a decent proportion could be attributed to poor risk management systems that did not totally understand the complex guarantees in question and never had structures in place to hedge against adverse exposure.

In the case of *Old Mutual-Bermuda* (OMB), the insurer sold poorly priced policies with accumulation and death benefit guarantees. There were weaknesses in the overall risk management framework with the guarantees being underpriced and the hedging approach adopted being inadequate. None of these was picked up as the policy sales grew exponentially until the credit spreads started widening and global equity markets fell resulting in the maturity of the guarantees, huge losses and the need for capital injection. As *Old Mutual* noted in 2009 and 2010, the need for products with guarantees continued to grow but the challenge remained getting enough capital to sustain them (Old Mutual 2010).

The situation in Asia was no different and though the demographics of the region favours the sale of variable annuities, post the 2008 crises, major industry players *Hartford Life*, *ING Life* and *Mitsui Life* decided to discontinue from this market. In a report on the Japanese variable annuity industry, Asada (2009) notes that deficiencies in risk management were a major factor that led to the financial losses arising from the maturity of the guarantees. Though the use of reinsurance helped mitigate the risk in some instances, the author alludes to hedging as an important tool that could have assisted in the circumstances.

The above are but a few of the many insurers that continue to face huge difficulties in pricing and hedging their guarantees and it is from this context that the study will be conducted. The dissertation will try to explore this issue further in the eyes of business agility which is defined as, " … the ability to identify, anticipate and respond to relevant changes in operating conditions-those changes that directly impact an insurer's ability to achieve sustained performance …" (Capgemini 2011).

The issue of how to price for these guarantees has remained a challenge for many life insurance companies. The work done in the early 1980s by David Wilkie on the reserving of the guarantees in unit-linked products resulted in insurers shying away from such features but as the derivatives market continued to evolve, the guarantees started re-emerging. Hedging the guarantees has however presented itself as an even bigger challenge in the wake of huge market volatility. Wu (2009) notes that the 2008 financial crises has made it an imperative task for insurance companies to not only be concerned about the correct valuation of the embedded options but also the containment of risks arising from holding the options through hedging.

Statutory and professional bodies have continued to give guidance on the issue from the understanding that such financial guarantees, if not well checked, can lead to the collapse of some industry players and due to contagion effects greatly affect the industry.

The *Actuarial Society of South Africa* (ASSA) under Advisory Practice Note (APN) 110 has recently sought to give actuaries clarity on the matter this replacing its predecessor Professional Guidance Note (PGN) 110. In the first PGN issued in 2003 and subsequent versions, ASSA has required that the nature and extent of risk inherent in the guarantees be appropriately recognized with a balance being struck between the practicality and complexity of the methodology adopted. The emphasis of the APN remains the calculation of reserves that serve as a buffer to the guarantees if they do mature.

The APN notes that the guarantees above are closely linked to the derivatives traded in the financial markets with the only difference being that they are embedded in life insurance policies. It recommends, "…the use of market consistent stochastic models to quantify reserves required to finance shortfalls in respect of embedded investment derivatives." (Actuarial Society of South Africa 2012). The variables that affect the future liabilities should be simulated stochastically and the future liabilities due to the guarantees projected to the maturity date. The present value of any shortfalls arising thereof is then treated as the reserve. The spirit of the guidance is that the reserve should be the expected present value of any shortfalls. Though the guidance alludes to a link between the embedded investment derivatives and the financial markets, it leaves it for the actuary to ensure that the insurer is well protected at all times.

Hill, Visser and Trachtman (2008) through a research commissioned by the *Society of Actuaries* (SOA) discuss the stochastic pricing of these embedded derivatives. The necessity of such a proactive step is noted by the authors given changing market factors such as the low interest rates that have led to the maturity of some of the guarantees and competitive pressures that have led to the increasing incorporation of guarantees in life insurance products.

The authors use the Black-Scholes (BS) model to assist in pricing the GMAB embedded derivative, which has a payoff pattern almost similar to a put option. As the markets become more unpredictable and guarantees more complex, the ability to use the BS method becomes harder and stochastic techniques become all the more important. The scenarios used in the stochastic techniques must be justifiable and Hill, Visser and Trachtman (2008) discuss three categories of scenarios that can be adopted: historic, long-term risk neutral and market consistent. Of the three, the last approach is considered a viable approach for calculating the price and hedging costs of the derivative given that the value is expected to be linked, directly or indirectly, with other financial instruments trading in the financial markets.

In the use of stochastic simulation, it is crucial that the scenarios used be large enough if a more comprehensive and accurate picture on the embedded derivative is to be attained. This can be done with more ease when the underlying distribution is well tabulated with desirable statistical properties but in other instances such as long-tailed distributions, " … the number of scenarios, even focusing on the average may need to be much larger." (Hill, Visser & Trachtman 2008). The authors then discuss the use of the Greeks in assessing the sensitivity of the option to certain market factors.

The adoption of a stochastic approach in the valuation is justified by the unpredictability of most of the factors that affect the option value including interest rates, equity returns, and policyholder behaviour. These need to be given a stochastic consideration in both the pricing and hedging of the option with the authors being of the opinion that the expectations on pricing actuaries are changing in this regard.

Against the common convention of insurance companies, the spirit of the report is the use of financial market tools to assist in the pricing and hedging of these options. The use of stochastic reserving in the recent past by the insurance and actuarial worlds is seen as a step in this direction.

This step is not in vain since in financial parlance, the insurance contracts can be seen as non-linear instruments with the underlying that leads to the exercise of the option, usually mortality, being non-linear. The insurer thus holds what are short 'option' positions on selling a product with guarantees. Reserving has been the past practice in 'hedging' such positions but looking at financial engineering techniques to try contain the risk is a promising recent development.

Czernicki and Maloof (2008) discuss the topic of variable annuity (VA) guarantee hedging from the premise that volatility in the markets has made it a necessity. They note that some insurers have used first-order Greeks to try hedge themselves but this protection is only helpful in times of small market swings. There is always a fine line between offering competitive products versus hedgeable products with most insurers tending to offer competitive but complex products that are a challenge to hedge. The authors note the use of control-variate techniques and fund mapping as new but encouraging developments in meeting this challenge. The importance of adopting hedging as a key consideration during the VA product design phase is emphasized instead of having it as a post-product design consideration. Such an approach demands a thorough understanding of the guarantees and hence methodologies to price and hedge them.

In a research conducted in 2011 by Langley et al. for *Barclays Capital*, the trend on VA is an upward one. The financial crises has however meant that hedging costs have increased dramatically and the question of whether hedging is still justifiable in this context is a necessary one. There are however companies that have displayed the necessity and benefits of hedging in the same period, *Prudential Insurance US* life company being a case in point.

During the 2008/2009 crises, the insurer was able to reduce losses from its guarantees through efficient hedging. Langley et al. (2011) notes, "… Jackson's hedging program is one example of how effective hedging can minimize the impact of changes in the value of GMWB guarantees."

In the same period however, other insurers offering VA have had to retreat or totally exit the market due to the guarantees biting with one-time US industry leaders *AXA* and *ING* being victims. The necessity of pricing and hedging the guarantees appropriately is vital and, in particular, a thorough analysis of market volatility and interest rates which can prohibitively raise the cost of VA hedging as noted by Langley and Preston (2012). The authors note the importance of identifying the key factors affecting the guarantee, devising plausible future scenarios and using available derivatives, from simple to complex, to hedge the adverse scenarios.

The fall in interest rates and/or equity price levels and increase in the volatility of the equity, forex and treasury markets are identified as factors leading to increasing hedge costs for the guarantees and thus the need to price using a framework that caters for these possibilities. In a paper delivered at the Actuaries' Club of Harford/Springfield meeting, Heurtelou (2012) notes examples of capital market products that can be used to hedge against adverse eventualities if they were to occur.

Though risks do exist that cannot be hedged using the financial markets, the author notes that most of the risk sources in the VA guarantees can be hedged using tools such as equity futures, treasury futures and interest rate/equity variance swaps. The goal of this research is to delve further into such propositions in a South African context.

The hedging strategy that should be adopted in finding a solution for the guarantees is a tractable and robust hedge as in the work of, for example, Branger and Mahayni (2006). A hedging strategy is then said to be *robust*, "… if it dominates the claim to be hedged whenever the realized volatility path stays within some given deterministic volatility interval…" and *tractable*, "… if it can be written as the sum of Black-Scholes (BS) strategies." (Branger & Mahayni 2006).

A measure of the effectiveness of a hedge can then be the formula noted in Richards (2012):

$$Hedge\ effectiveness = \left(1 - \frac{C_W}{C_{WO}}\right) \times 100\%,\qquad(2.1)$$

where:

C_W is the capital requirement with hedging assets,
C_{WO} is the capital requirement without hedging assets.

The desire is that the percent of hedge effectiveness be as high as possible and certainly non-negative where the strategy adopted would be worse than not having a strategy at all.

Coughlan et al. (2011) describe basis risk as, "… whenever there are differences, or mismatches, between the underlying hedged item and the hedging instrument." They then discuss a five-step framework for assessing hedge effectiveness in the context of longevity risk with their measure calculating the extent of risk reduction:

$$Relative\ risk\ reduction = 1 - \frac{R_{LH}}{R_L},\qquad(2.2)$$

where:

R_{LH} is the risk measure value when the liability and hedge are considered together,
R_L is the risk measure value when the liability is considered alone.

Life Contingencies and Embedded Options

This chapter focuses on the fundamental mathematical concepts associated with embedded options for completeness purposes. Any reader who is unfamiliar with embedded options will find this brief chapter useful and for the familiar reader, the chapter provides a quick but useful reminder for later chapters.

3.1. Embedded options/derivatives

The *Financial Accounting Standards Board* (FASB) defines embedded derivatives as, "... components of contractual arrangements that, by themselves (i.e. on a stand-alone basis), would satisfy the criteria in the definition of a derivative." (PricewaterhouseCoopers 2013) The original contractual arrangement is referred to as the host contract and the combination of the embedded derivative and the host contract becomes a hybrid instrument.

In many jurisdictions such as in the FASB, *International Accounting Standards Board* (IASB) standards and in particular under International Financial Reporting Standard (IFRS) 9, the requirement is that the embedded derivative be seen as a distinct financial instrument from the host contract and thus be accounted for separately though this has to be read in conjunction with IFRS 4.

The distinction between embedded derivatives and embedded options underlies the fact that derivatives is a more encompassing word that includes but is not limited to options. In this research the two words are used interchangeably and assumed not to be significantly different.

As noted in Chapters 1 and 2, purposefully or unpurposefully, life insurance companies find themselves selling products that contain embedded derivatives. The question of how to identify and value these options becomes all the more important more so in the wake of the recent financial crises.

3.2. Embedded options in GMxB products

The options that lie embedded in the GMxB contracts discussed above have a cost to the company usually referred to as the cost of guarantee. This can be a maturity, surrender or death guarantee where the cost becomes borne on contract expiry, surrender or policyholder death respectively.

3.2.1. Maturity guarantee

The contract in this case guarantees a payoff that is at least:

$$(1 + g_M)^T \times P, \tag{3.1}$$

where:

g_M is the per time unit guarantee rate,
P is the initial single premium.

If at maturity the investment by the insurance company, I_T, is less than this then the cost to the company at maturity time T is given by:

$$\{(1 + g_M)^T \times P - I_T\}. \tag{3.2}$$

.

If the investment is greater then the cost to the company is zero since the investment is more than sufficient to pay the guarantee (i.e. the cost to the company is the excess amount the insurer has to pay over the account value if the latter falls short).

The expected value of the cost to the company at maturity is then given by the expression:

$$Expected\ cost\ to\ company = (1 - {_T}s_x) \times (1 - {_T}q_x) \times MAX\{0, (1 + g_M)^T \times P - I_T\}, \tag{3.3}$$

where:

> ${_T}q_x$ is the probability of a life aged x dying within the next T years,
> ${_T}s_x$ is the probability of a life aged x surrendering within the next T years.

3.2.2. Surrender guarantee

The surrender guarantee undertakes to pay the policyholder a proportion of the original premium P:

$$Amount = (1 + g_S) \times P. \tag{3.4}$$

with $g_S < 0$ to dissuade policyholders from surrendering too soon.

If at any time the policy is surrendered and the value of the investment, I_t, is less than the surrender amount then the extra payout by the company is given by:

$$\{(1 + g_S) \times P - I_t\}. \tag{3.5}$$

The fair charge by the insurer today for a surrender that occurs in year $t \in \mathbb{N}$ is then given by:

$$Fair\ charge = {_1}s_{x+t-1} \times (1 - {_t}q_x) \times MAX\{0, (1 + g_S) \times P - I_t\} \times e^{-\int_0^t r_u du}, \tag{3.6}$$

where:

> r_u is the risk-free rate of interest.

3.2.3. Death guarantee

The death guarantee amount that is paid to the estate for death that occurs at a random time t is at least:

$$Amount = (1 + g_D)^t \times P, \tag{3.7}$$

where:

g_D is the per time unit guarantee rate,
P is the initial single premium.

If death occurs at such a time t when the value of the investment, I_t, is less than this amount then the company has to make an extra payout of:

$$\{(1 + g_D)^t \times P - I_t\}. \tag{3.8}$$

The fair charge by the insurer today for a guarantee that matures at year $t \in \mathbb{N}$ is then given by:

$$Fair\ charge = \left(1 - {}_{t-1}s_x\right) \times \left(1 - {}_{t-1}q_x\right) \times q_{x+t-1}$$
$$\times MAX\{0, (1 + g_D)^t \times P - I_t\} \times e^{-\int_0^t r_u du}. \tag{3.9}$$

The maturity or death amounts can be expressed in a continuous compounding sense as:

$$Amount = e^{gT} \times P, \tag{3.10}$$

where:

g is the per time unit guarantee rate for the respective guarantee.

and the above results still hold in this setting.

The guarantees have payoffs very similar to the vanilla and exotic options that are widely traded in the financial markets. In particular, the GMxBs are put options whose maturity date is known for the maturity guarantee but random for the surrender and death guarantees. The payoff at such a maturity time, T, for the holder of the option with strike price X and underlying S_t is expressed as:

$$(X - S_T)_+ = MAX\{0, X - S_T\}. \tag{3.11}$$

If we consider the payout to be made at maturity time, T, and a guarantee rate of g per unit time for the GMxB in question, the policyholder receives:

$$Y_T = MAX(e^{gT}P, I_T) = e^{gT} MAX(P - e^{-gT} I_T, 0) + I_T. \tag{3.12}$$

The value of the guarantee at T is the account value I_T added to a put option with an exercise of P and the underlying being the discounted account value.

Further, the guarantees can be a return of premiums, roll-up or the rising-floor guarantee. In this report, it is assumed that the guaranteed benefit does not change as the fund value changes or if it does it is a compound growth through a constant interest rate, essentially an interest guarantee. The policyholder also normally has the choice regarding which product to choose and in some instances the investment strategy to be followed. The insurance company should therefore be able to protect itself from adverse exercise of any of the above products.

Modelling the Dynamics of GMxB Embedded Options

This chapter is a core chapter in the dissertation. It first introduces the notions of stochastic calculus and measure theory so as to ensure that the research is comprehensive enough, rigorous where needs be and self-contained. It starts by explaining the more commonly known Black-Scholes framework to set the background. The three main frameworks in this research namely the regime-switching, VG and VGSV frameworks are then presented from Section 4.3.

The choice in this chapter is the adoption of a parallel approach where a framework is theoretically discussed and then applied to the JSE ALSI index data in the pricing of the guarantees. The leading motive in such an approach is so as to ensure that all the results are presented as closely as possible to the theoretical discussion of the framework and in so doing shed some immediate practical light on the theory while achieving continuity of structure.

4.1. The Black-Scholes framework

The theory of stochastic calculus has been heavily used to develop option pricing frameworks. This calculus is a generalisation of the ordinary calculus with the introduction of a random component from Brownian motion; an important ideal in the theory of finance given the random behaviour of financial markets. The theory of probability measures has equally been heavily applied in expressing option prices as expectations (see Björk (1998) and Shreve (2004)). The two alternative approaches are referred to as the partial differential equation (PDE) approach and the martingale approach.

Inherently important in the PDE formulation is Itô calculus and Itô's lemma which are applied a lot in the world of financial mathematics. Itô's lemma states that if an asset X_t has the dynamics:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

(4.1)

then

$$df(X_t, t) = \frac{\partial f}{\partial t}(X_t, t)dt + \frac{\partial f}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)dX_t^2.$$

(4.2)

where we use the Itô multiplication table:

Table 4.1: Itô multiplication table

	dt	dW_{1t}	dW_{2t}
dt	0	0	0
dW_{1t}	0	dt	0
dW_{2t}	0	0	dt

with W_{1t} and W_{2t} being two independent Weiner processes.

4.1.1. The Black-Scholes model

A partial differential equation (PDE) that gives the price of an option can then be derived and Fischer Black and Myron Scholes in their seminal paper (Black & Scholes 1973) derived such an equation for an option on an underlying X_t with expiry time T.

The Black-Scholes equation can alternatively be derived using the concept of a martingale measure (see Harrison and Kreps (1979), Kreps (1981) and Harrison and Pliska (1981)). This approach is built from the premise that in order to avoid arbitrage in any trading strategy, the probabilities used must be such that, "...if at any trading time and state of the world we take the expectation of any asset's future value, then it will be equal to the value it has in that state of the world at that trading time." (Joshi 2003).

Closely interlinked and important is the existence of risk-neutral measures where option prices 'define' probability measures. These measures always imply a 'risk-neutral' evolution where the asset grows at the risk-free rate but do not assert a probability distribution of such an asset's future price behaviour.

Björk (1998) and Joshi (2003) discuss the seminal work of the authors above starting from the fundamentals and arriving at the same PDE for an option. The less technical reader is referred there.

Proposition 4.1 (Joshi 2003)

If a contingent claim has a payoff function $G(\{X_u, 0 \le u \le T\})$ and $G(X_T)$ is a sufficiently integrable function, then the price V_t of the contingent claim is given by $V_t = f(X_t, t)$ where f solves the Black-Scholes partial differential equation,

$$\frac{\partial}{\partial t} f(x,t) + (r-q)x \frac{\partial}{\partial x} f(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x,t) - rf(x,t) = 0, \qquad (4.3)$$

where:

> *r is the continuously compounded risk-free rate,*
> *σ is the volatility parameter,*
> *q is the dividend yield.*

Proof: (See Björk (1998)).

Remark: A consequence of the PDE derivation is the obtainment of a hedging strategy that involves continuous trading in the underlying. This is referred to as delta-hedging and for an option with price $f(X, t)$ one should hold $\frac{\partial f}{\partial X}$ units of the underlying at any time.

Further, though the r, σ and q parameters can be variables, in subsequent sections they are assumed to be constants.

4.1.2.   Imperfections of the Black-Scholes model

The Black-Scholes model is based on certain assumptions which have become "suspect" over time (see for example Cont and Tankov (2004)).

Schoutens (2003) and Buesser (2013) note some of the Black-Scholes model assumptions as:

- There is no market friction arising from, inter alia, taxes and transaction costs.
- The underlying asset's returns process is a logarithmic diffusion.
- The volatility parameter is the main risk determinant.

Empirical evidence (see for example Cont (2001)) has however noted that:

- The log returns are not normally distributed.
- The volatilities (or parameters of uncertainty) have a stochastic term structure and are clustered, that is, volatility clusters.

These imperfections have motivated the extension of the Black-Scholes model into new models that can explain these empirical observations well. Achdou and Pironneau (2005) note some of the extensions that would allow for more flexibility and they include:

- Modelling the volatility as a stochastic process.
- Using a regime switching model to characterize the performance over the so called different regimes.
- Generalising the Black-Scholes model by assuming that the spot price is a Lévy process.

This consideration would however need a recap of some more fundamental principles in quantitative finance which follow in the next sections.

4.2. Change of measure and Girsanov's theorem

The theory of pricing has long been viewed from a real-world context and using real-world probabilities. However, in a quantitative finance context and in the pricing of contingent claims, the pricing is no longer done using the real-world probability measure rather the risk-neutral probability measure. This is key if the prices obtained are to be arbitrage-free and separates the statistical framework from the no-arbitrage pricing framework.

The key theorem that links these two measures is the Cameron-Martin-Girsanov theorem (normally referred as Girsanov's theorem) which is stated here below:

Theorem 4.1 (Björk 1998)

If W_t is a $\mathbb{P}$-Brownian motion and γ_t is an $\mathcal{F}$-previsible process satisfying the boundedness condition $\mathbb{E}_{\mathbb{P}} exp\left(\frac{1}{2}\int_0^T \gamma_t^2 \, dt\right) < \infty$, then there exists a measure $\mathbb{Q}$ such that:

    i.   *$\mathbb{Q}$ is equivalent to $\mathbb{P}$ (i.e. if A is any event on the same sample space that $\mathbb{Q}$ and $\mathbb{P}$ operate, then $\mathbb{Q}(A) > 0 \iff \mathbb{P}(A) > 0$),*

    ii.   *$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left(-\int_0^T \gamma_t \, dW_t - \frac{1}{2}\int_0^T \gamma_t^2 \, dt\right)$,*

    iii.   *$\widetilde{W}_t = W_t + \int_0^t \gamma_s \, ds$ is a $\mathbb{Q}$-Brownian motion.*

Proof: (See Björk (1998)).

Remark: This result, with certain constraints, allows us to change between the two measures depending on the context in question.

4.3. Incomplete markets

The Black-Scholes model and some variations such as the binomial tree model (see Cox, Ross and Rubinstein (1979)) assume that it is possible to completely replicate a security using other securities available in the market hence the assumption of the existence of a replicating portfolio. It is thus possible, in such a setting, to wipe away all the risk undertaken by an options/derivatives writer by holding the replicating portfolio and the market is referred to as a *complete market*. Björk (1998) defines a complete market as one in which every contingent claim can be replicated.

The theory of completeness is heavily interconnected with the existence of a unique martingale measure in the martingale approach to no-arbitrage pricing.

Definition 4.1 (Björk 1998)

A probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ is called an equivalent martingale measure (or more commonly as a martingale measure) for a market model, the numeraire and the time interval $[0, T]$ if it has the following properties:

- *$\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$,*
- *All price processes of the market model and the numeraire are martingales under $\mathbb{Q}$ on the time interval $[0, T]$.*

Björk (1998) discusses arbitrage-free pricing in the martingale pricing framework and notes the Second Fundamental Theorem of Mathematical Finance as:

Theorem 4.2 (Björk 1998)

Assume that the market is arbitrage free. Then the market is complete if and only if the martingale measure is unique.

Proof: (See Björk (1998)).

The foregoing remarks result in a very ideal financial world where non-satiation and not risk-preferences influence the unique pricing of derivatives. The world of mathematical finance has often been built using this framework given its tractability and the work of, inter alia, Black, Scholes and Merton (see Black and Scholes (1973), Merton (1973)) is in this setting.

However, the very nature of the financial markets including heavy tails and volatility clustering disproves the existence of a unique martingale measure and issues such as risk-preferences and liquidity considerations do indeed influence pricing. This results in the concept of incomplete markets which, as Björk (1998) notes, has more random sources than there are traded assets.

In the incomplete market and as a consequence of the above, the ability to completely hedge and uniquely price a derivative is not just in doubt but largely impossible. A choice then has to be made for a reasonable arbitrage-free incomplete market model for the particular derivative(s) to be priced and/or hedged.

It is in this incomplete market context, and based on the justifications above, that the pricing and hedging of the embedded options discussed herein is considered. Where needs be, empirical methods are used in deciding how to calibrate the particular choice of model and given the proprietary nature of insurer information that would otherwise be used in parameter estimation, in some instances, the selection made is a logical assumption within a realistic range.

4.4. Regime-Switching Processes

The regime-switching theory hinges on the common observation that the price and return performance in the financial markets is affected by events such as financial crises and reserve bank policies some of them dramatically so. The market prices are influenced by dissimilar forces over different periods but over any one period there is a force that is more influential on the price thereby dominating the others. The periodic effect of these forces (which are referred to as regimes) on prices should then be captured by using a model that takes into consideration the different regime that market is in and naturally leads to the need for an estimation of the probabilities of switching between regimes.

Hardy (2003) notes that regime-switching models suppose that a discrete process changes between regimes arbitrarily with every regime being characterized by its own parameter/coefficient set. Markov processes are used to describe where the price process is at any time thereby a means to calculate the probabilities of switching between regimes.

Definition 4.2 (Parzen 1999)

A stochastic process, $M(t)$, is said to be a Markov process if it satisfies the Markov (past forgetting) property i.e. for any set of time points $t_1 < t_2 < \cdots < t_n$ in the index set of the process and for any real numbers $m_1 < m_2 < \cdots < m_n$

$$P[M(t_n) \leq m_n \mid M(t_1) = m_1, \cdots, M(t_{n-1}) = m_{n-1}]$$
$$= P[M(t_n) \leq m_n \mid M(t_{n-1}) = m_{n-1}].$$

Remark: (Parzen 1999) A real number m is said to be a possible value, or a state, of the stochastic process $\{M(t),\ t \in T\}$ if there exists a time t in T such that the probability $P[m - h < M(t) < m + h\,]$ is positive for every $h > 0$.

The research by Hamilton (1989) discussing a regime-switching model in an econometric setting is viewed as the base from which the application of regime-switching models in finance sprang. In his paper, he discusses an auto-regressive (AR) regime-switching model in the context of the United States Gross National Product time series. In subsequent research on this Hamilton and Susmel (1994) consider the autoregressive conditional heteroskedasticity (ARCH) model under different regimes whereas in Hamilton (1996) the subject area is considered in a specification testing context.

The novel discussion of the regime-switching models in a quantitative finance context was done by Bollen (1998) where the author discusses the valuation of European and American-style options in a regime-switching framework. The option values so obtained exemplify a volatility smile similar to what is empirically observed adding credence to the use of such models and following on from which the application in the wider finance context has developed. Ang and Bekaert (2002) discussed the theory in an interest rate modelling framework for the interest rates of the United States of America, Germany and the United Kingdom. Their results pointed to the existence of interest rate regimes and further added acceptance to the adoption of such models.

The regime-switching log-normal model (RSLN) is one of the main regime-switching models that have been applied in the finance and actuarial worlds. The process in this case switches at random between K log-normal processes and as Hardy (2003) notes; this enables the introduction of stochastic volatility in the modelling. The volatility will, at any one time, take on the values implied by one of the K regimes and it switches between these values randomly.

The popularity of the RSLN model has grown mainly due to its simplicity and tractability compared to other models in the same sphere yet with the ability to capture the randomness seen in the market through the switching of volatility between the different regimes. Hardy (2003) notes that the RSLN model when switching between two regimes fits the stock index data quite well more so in the context of insurance that has a link to equities. The author's discussion of the framework is used here below:

Assumption 4.1

Let $M_t \in \{1, 2, \cdots K\}$ be the regime applying in the time interval $[t, t + 1]$, let X_t be the total return index value at time t and Y_t be the log-return process. If $Y_t = ln\left(\frac{X_{t+1}}{X_t}\right)$ then,

$$Y_t \mid M_t \sim N\left(\mu_{M_t}, \sigma^2_{M_t}\right), \qquad (4.4)$$

where:

μ_{M_t} *is the mean of the applying regime,*
$\sigma^2_{M_t}$ *is the variance of the applying regime.*

A two-regime model has been found to be adequate in most cases (see for example Hamilton and Susmel (1994), Harris (1997), Harris (1999) and Hardy (2001)) and in the subsequent discussion and chapters for the pricing considerations, given the ability of the model to balance parsimony and accuracy, this will be the model used.

The matrix $\boldsymbol{\mathcal{P}}$, labelled (4.5) below, is the matrix of transition probabilities and contains the probabilities of transition from one regime to another:

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \tag{4.5}$$

It is assumed that transition from one state is at period end and therefore:

Definition 4.3

The probability defined as,

$$p_{i,j} = P[M_{t+1} = j \mid M_t = i], \qquad i \ \& \ j \ \in \{1,2\} \tag{4.6}$$

is the probability that the process is in regime j at the next time period given that it was in regime i in the previous time period and $p_{11} + p_{12} = 1$, $p_{21} + p_{22} = 1$.

Remark: Parameter estimation in this framework is done using the method of maximum likelihood with a total of six parameters to be estimated and the interested reader is referred to Hardy (2001).

4.4.1. Estimating the Markov Switching model

The following analysis uses daily and monthly data obtained on the *Johannesburg Securities Exchange* (JSE) All Share Index (ALSI) for the period from 1st July 1994 to 30th June 2013 to estimate a two-regime Markov-Switching (MS) model using the vector of log-returns on the ALSI. The transition probabilities so estimated are assumed to be constant with the possibility of extending this to estimate time-varying transition probabilities.

The Matlab platforms by Perlin (2012) and Kritzman, Page and Turkington (2012) are used with their accuracy as suitable estimation platforms having being counter-checked and confirmed using research done elsewhere (see for example Rathgeber, Stadler and Stöckl (2013), Møller and Sander (2013)). The particular Matlab codes are contained in Appendix A.

The computations in this section and throughout the rest of the report are done on a HP with a 1.8 GHz Intel Core i3 processor and 4 GB 1600 MHz DDR3 memory.

4.4.1.1. Hamilton filter approach

In this Markov model estimation, the Hamilton filter approach, an expectation maximisation algorithm discussed in Hamilton (1989, 1994), is used on the daily data and the parameter estimates obtained are:

Table 4.2: Hamilton filter parameter estimation on daily JSE/ALSI returns data

	Mean (μ)	Variance (σ^2)	Sigma (σ)
State 1	0.0009	0.000064	0.008
State 2	-0.0010	0.000395	0.019875

The transposed transition probability matrix is shown below:

```
---> Transition Probabilities Matrix (std. error, p-value) <---

    0.99 (0.08,0.00)    0.04 (0.09,0.70)
    0.01 ( NaN, NaN)    0.96 ( NaN, NaN)
```

Figure 4.1: Hamilton filter transposed transition probability matrix on daily JSE ALSI returns data

The various plots are shown here below with the third plot on the smoothed probabilities being of particular interest.
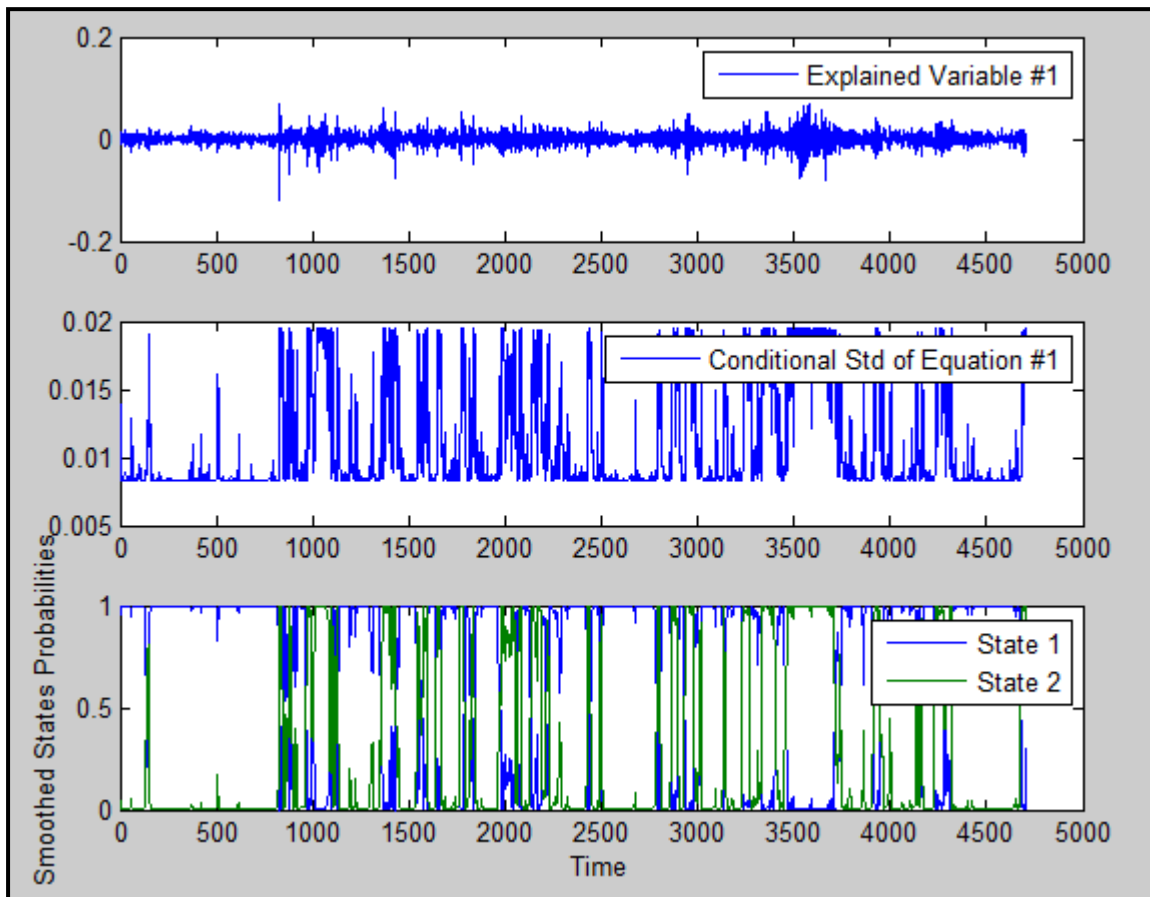
Figure 4.2: Hamilton filter RS plots for the daily JSE/ALSI returns data: July 1994-June 2013

The main advantage of the regime switching model is that though from the first time series plot it may be a challenge to identify when the regime does actually occur, the third plot clearly indicates the switching of regimes across time from July 1994 to June 2013. The plots signal a possible balance in the time spent between the two regimes.

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4.4.1.2. Baum-Welch algorithm fitting

The Baum-Welch algorithm is a Hidden Markov Model (HMM) estimation algorithm that uses the maximum likelihood approach in estimation. It is also a special case of the expectation maximisation approach and iteratively maximises a proxy to the log-likelihood to obtain the most optimal model (Bishop 2006). Its powerfulness and huge acceptance in the estimation of parameters in a probabilistic context made it a worthwhile consideration for the case at hand and for the daily data it yields the following parameters.

```
........Transition probability matrix........
    0.9839    0.0161
    0.0444    0.9556

..............Mean (mu) vector..............
    0.0010   -0.0011

...............Sigma vector................
    0.0081    0.0199
```
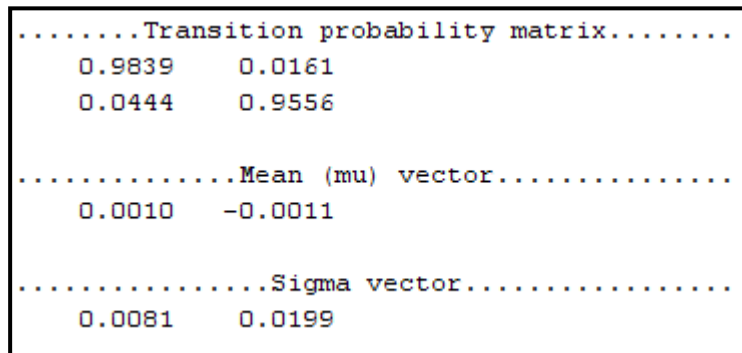
Figure 4.3: Baum-Welch transition probability matrix: Daily JSE/ALSI returns data
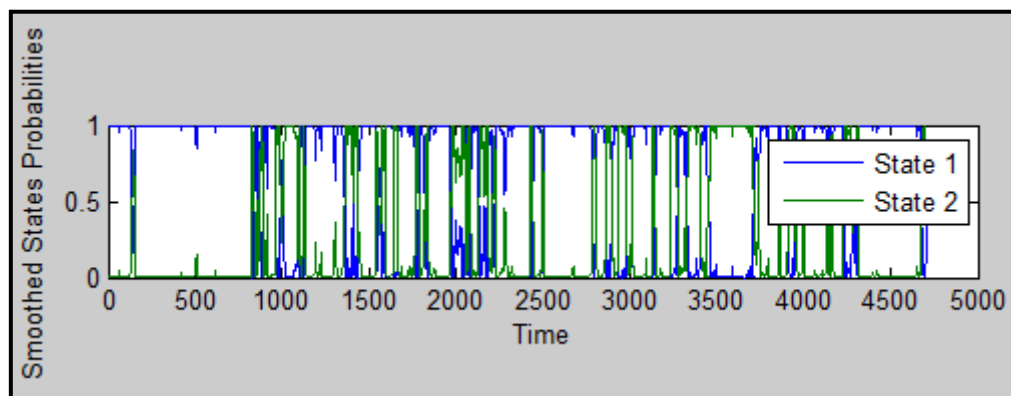
Figure 4.4: Baum-Welch RS plot for the daily JSE/ALSI returns data: July 1994-June 2013

In both the Hamilton and Baum-Welch approaches, the parameters so obtained do not show any material disparities and it is the parameters from the latter that are used forthwith given that it is a complete estimation method and has been shown (see Mitra and Date (2010), Mitra (2014)) to identify regimes that the Hamilton approach may fail to capture.

Further, it is worth pointing out that though the research focuses on a two regime setting, an attempt to fit a three-regime model on the data results in a third-regime that is not any more instructive over the two-regime model and hence for the case at hand, with this further justification, the two-regime model is used.

The parameters are re-estimated using the Baum-Welch approach on monthly data as shown in Figure 4.5 below. The use of monthly data over daily data is because the former is more effective in identifying regimes when the use is to capture long-term changes as is the case for embedded guarantees. Mitra and Date (2010) note that regime-switching models concentrate on long-term trends instead of continuous time dynamics and thus such models switch regimes over larger units of time, such as monthly, instead of continuously, as would be the proxy case with daily data.

```
........Transition probability matrix........
     0.9345     0.0655
     0.0935     0.9065


..............Mean (mu) vector..............
     0.0161    -0.0034


...............Sigma vector...............
     0.0371     0.0775
```
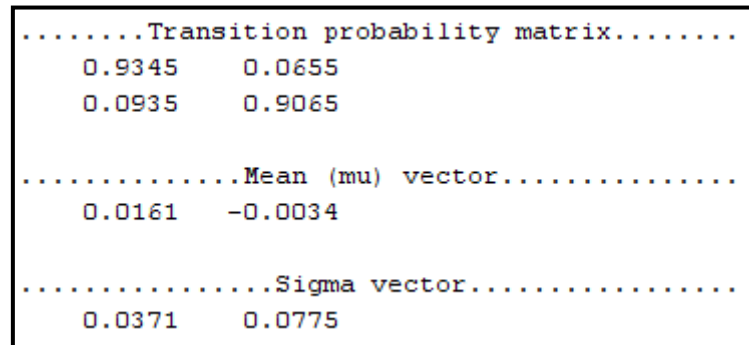
Figure 4.5: Baum-Welch parameters for the monthly JSE/ALSI returns data: July 1994-June 2012

We thus have the transition probability matrix given by:

$$\mathcal{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.9345 & 0.0655 \\ 0.0935 & 0.9065 \end{bmatrix}. \tag{4.7}$$

If we apply linearity of the mean and square root adjustment for volatility, Figure 4.5 above indicates that regime 1 is a low volatility regime (an annualized volatility of roughly 13%) coupled with high returns (an annualized return of 19%). Regime 2 is a high volatility regime (an annualized volatility of roughly 27%) with low (negative) returns (an annualized return of -4%). This intuitively makes sense since periods of high volatility have tended to be associated with falling markets and vice versa.

The probability of moving from a high-volatility regime to a low-volatility regime is 9.35% which is indicative of a high persistency rate of 90.65%. The smoothed probability plot for this period is shown below:
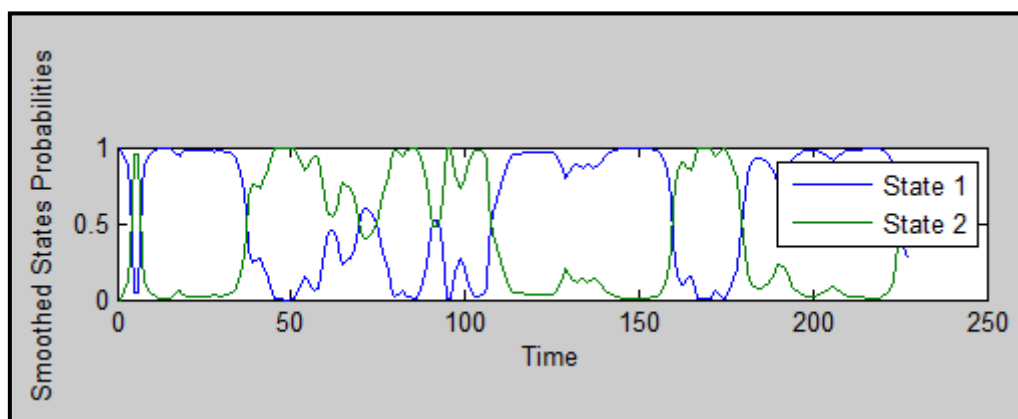
Figure 4.6: Baum-Welch smoothed probability plot for the monthly JSE/ALSI returns data: July 1994-June 2013

The expected duration in the high-volatility regime (regime/state 2), which is approximated by $\frac{1}{1-p_{22}}$, is 10.7 months and the expected duration in the low-volatility regime (regime/state 1) is approximately 15.3 months.

The steady-state probabilities yield:

$$\lim_{K \to \infty} \boldsymbol{\mathcal{P}}^K = \begin{bmatrix} 0.5881 & 0.4119 \\ 0.5881 & 0.4119 \end{bmatrix}. \tag{4.8}$$

Hence 58.81% of the months will be in the low-volatility regime and 41.19% of the months will be in the high-volatility regime.

The 1998 global crisis (around 48 months from July 1994) and the 2008 financial crisis (around 168 months from July 1994) present themselves as periods of high volatility which is a true reflection of the happenings at the time.

The next important step after the identification of the two regimes is the pricing of financial instruments using a methodology that is consistent with the RS framework which exists in an incomplete markets context (as discussed above). This necessitates the adoption of a strategy other than the replicating portfolio strategy in the pricing of options and in this particular case the embedded derivatives.

In an incomplete market, there is no unique equivalent martingale measure; in fact, there are infinitely many. The challenge becomes how to choose a reliable pricing measure from this infinite set. In the section that follows, the Esscher transform, which is used extensively by actuaries, is applied to obtain the martingale measure adopted for the pricing.

4.4.2. Esscher transform martingale measure

Gerber and Shiu (1994) introduced the Esscher transform in the pricing of options through their 1994 seminal paper. The authors noted that the risk-neutral Esscher transform provides, critically, an unambiguous solution in the choice of the equivalent martingale measure.

The model theory used below to identify an EMM for the two-regime switching lognormal (RSLN2) model is based on Elliott et al. (2005), Siu (2005), Xiao-nan et al. (2013) and Qiu (2013).

4.4.2.1. Model theory

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathcal{T}$ is the time index set $[0, T]$ and $\{W_t\}_{t \in \mathcal{T}}$ a standard Brownian motion on the probability space. Further, assume that the states of the market are modelled by a continuous-time hidden Markov Chain process $\{M_t\}_{t \in \mathcal{T}}$ on the probability space with a finite state space with $\{M_t\}_{t \in \mathcal{T}}$ and $\{W_t\}_{t \in \mathcal{T}}$ being independent.

We assume that the interest rate of the bank account, the expected growth rate of the asset and the volatility all depend on the state/regime that we are in, that is,

$$
\begin{aligned}
r_t &:= r(t, M_t) \\
\mu_t &:= \mu(t, M_t) \\
\sigma_t &:= \sigma(t, M_t).
\end{aligned}
\tag{4.9}
$$

Definition 4.4

The financial model defined below, which consists of a bond/bank and stock (which could be the reference insurer's asset portfolio, a company's share, market index etc), is used:

$$
(B_t, X_t)_{0 \le t \le T},
\tag{4.10}
$$

where:

B_t *is the bank account with dynamics* $dB_t = r_t B_t dt, \quad B_0 = 1,$

X_t *is the stock price with Markov-modulated dynamics,*
$dX_t = \mu_t X_t dt + \sigma_t X_t dW_t, \quad X_0 = x.$

If $Y_t = \ln\left(\frac{X_t}{X_0}\right)$ as in Assumption 4.1, then Elliott et al. (2005) note that the dynamics of X_t can be expressed as:

$$
X_t = X_u \, e^{(Y_t - Y_u)},
\tag{4.11}
$$

where:

$$
Y_t = \int_0^t \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^t \sigma_s \, dW_s.
\tag{4.12}
$$

Let $\{\mathcal{F}_t^M\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ be the $\mathbb{P}$-augmentation of the natural filtrations generated by $\{M_t\}_{t \in \mathcal{T}}$ and $\{Y_t\}_{t \in \mathcal{T}}$ respectively and for each $t \in \mathcal{T}$, let $\mathcal{G}_t$ be the σ-algebra $\mathcal{F}_t^M \vee \mathcal{F}_t^Y$. Further, define $\theta_t := \theta(t, M_t)$ as the regime switching Esscher transform. In order to be able to change measures, Elliott et al. (2005) show that the Radon-Nikodym derivative of this transform is expressed as:

$$
\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} \bigg|_{\mathcal{G}_t} = exp\left(\int_0^t \theta_s \sigma_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 \, ds \right).
\tag{4.13}
$$

Defining $\{\tilde{\theta}_t\}_{t \in \mathcal{T}}$ as the family of risk-neutral regime switching Esscher parameters, the authors invoke Bayes' rule and show that $\tilde{\theta}_t$ is actually related to the market price of risk (MPR) and can be determined uniquely as:

$$
\tilde{\theta}_t = \frac{r_t - \mu_t}{\sigma_t^2} = -\frac{MPR_t}{\sigma_t}, \quad t \in \mathcal{T}.
\tag{4.14}
$$

and for our two regime case we have:

$$\tilde{\theta} = \left(\frac{r_1 - \mu_1}{\sigma_1^2}, \frac{r_2 - \mu_2}{\sigma_1^2} \right). \tag{4.15}$$

The stock price dynamics under $\mathbb{Q}_{\tilde{\theta}}$ are then given by:

$$dX_t = r_t X_t dt + \sigma_t X_t dW_t, \tag{4.16}$$

from which it is then possible to price European options with payoff $G(X_t)$ as:

$$V(t, X_t) = \mathbb{E}^{\mathbb{Q}_{\tilde{\theta}}} \left[e^{-\int_t^T r_s ds} G(X_t) \middle| \mathcal{F}_t^M \vee \mathcal{F}_t^Y \right], \tag{4.17}$$

which is evaluated as an iterated expectation.

Definition 4.5 (Frittelli 2000)

Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. The relative entropy $\mathcal{I}(\mathbb{Q}, \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined by:

$$\mathcal{I}(\mathbb{Q}, \mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} ln \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] & if \ \mathbb{Q} \ll \mathbb{P} \\ + \infty & otherwise \end{cases}. \tag{4.18}$$

If M is a set of probability measures on $(\Omega, \mathcal{F})$, set

$$\mathcal{I}(M, \mathbb{P}) = inf\{\mathcal{I}(\mathbb{Q}, \mathbb{P}): \mathbb{Q} \in M\}. \tag{4.19}$$

Definition 4.6 (Frittelli 2000)

A probability measure $\mathbb{Q}_0 \in \mathcal{M}$ is called the minimal entropy martingale measure (MEMM) if it satisfies:

$$\mathcal{I}(\mathbb{Q}_0, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}} \mathcal{I}(\mathbb{Q}, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} ln \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right]. \tag{4.20}$$

where $\mathcal{M}$ is a set of martingale measures.

Remark: It is worth noting that the measure so chosen is the minimal entropy martingale measure (MEMM) which makes it a good choice and the interested reader is once again referred to Elliott et al. (2005) for the details and proofs. Benth and Karlsen (2005) and Benth and Meyer-Brandis (2005) discuss the fact that this measure is a theoretical one in some sense. Hubaleka and Sgarra (2006) show the close relationship between the MEMM and the Esscher transform for exponential Lévy models with Benth and Sgarra (2012) discussing how to use it in a more practical sense by considering the case of power markets. The interested reader is referred to the authors work.

The price of a call option is then calculated using the result derived by Xiao-nan et al. (2013) where:

a) The inner expectation, in the iterated expectation, evaluates to:

$$\mathbb{E}^{\mathbb{Q}_{\tilde{\sigma}}}\left[e^{-\int_t^T r_s ds}(S_T - K)^+ \mid \mathcal{F}_T^M\right] = S_t\, \Phi(d_1) - Ke^{-\int_t^T r_s ds}\Phi(d_2), \tag{4.21}$$

where:

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \int_t^T\left(r_s - \frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T \sigma_s^2\, ds}}, \qquad d_1 = d_2 + \sqrt{\int_t^T \sigma_s^2\, ds},$$

$\Phi(\cdot)$ is the standard normal cumulative distribution function.

b) To evaluate the outer expectation we use the probability density function of $O_i(t)$, the occupation time of $\{M_t\}$ in state i over $[t, T]$, $t \in \mathcal{T}$, which for a fixed time t and regime i is defined (Yoon, Jang & Roh 2011) as $f_i(t, u)$:

$$f_1(t, u) := e^{-\beta_2(t-u)-\beta_1 u}\left[\begin{array}{l}\left(\left(\frac{\beta_1\beta_2 u}{t-u}\right)^{\frac{1}{2}}\right)\Psi_1\left(2(\beta_1\beta_2 u(t-u))^{\frac{1}{2}}\right) + \\ \beta_1\Psi_0\left(2(\beta_1\beta_2 u(t-u))^{\frac{1}{2}}\right)\end{array}\right], \tag{4.22}$$

$$f_2(t, u) := e^{-\beta_2(t-u)-\beta_1 u}\left[\begin{array}{l}\left(\left(\frac{\beta_1\beta_2(t-u)}{u}\right)^{\frac{1}{2}}\right)\Psi_1\left(2(\beta_1\beta_2 u(t-u))^{\frac{1}{2}}\right) + \\ \beta_2\Psi_0\left(2(\beta_1\beta_2 u(t-u))^{\frac{1}{2}}\right)\end{array}\right]. \tag{4.23}$$

where:

$$0 < O_i(t) = u < t < \infty,$$

$\Psi_a(z) = \left(\frac{z}{2}\right)^a \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(a+n+1)}$ is the modified Bessel function,

$f_1(t, 0) = 0$, $f_1(t, t) = e^{-\beta_1 t}$, $f_2(t, 0) = e^{-\beta_2 t}$ and $f_2(t, t) = 0$.

Denote the conditional call price (CCP) evaluated using the inner expectation above as $C_i(S_i, K, T, O_i(t))$ (conditional on the occupation time $O_i(t)$). Without loss of generality, if we stand at time 0, as in Xiao-nan et al. (2013), let:

$$R_{[0,T]} = \int_0^T r_s\, ds = r_1 O_i(t) + r_2(T - O_i(t)). \tag{4.24}$$

$$S_{[0,T]} = \int_0^T \sigma_s^2\, ds = \sigma_1^2 O_i(t) + \sigma_2^2(T - O_i(t)). \tag{4.25}$$

Then

$$C_i(S_i, K, T, O_i(t)) = S_0 \, \Phi(d_1) - Ke^{-[(r_1-r_2)O_i(t)+r_2T]}\Phi(d_2), \qquad (4.26)$$

where:

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left((r_1 - r_2)O_i(t) + r_2T\right) - \frac{1}{2}\left((\sigma_1^2 - \sigma_2^2)O_i(t) + \sigma_2^2T\right)}{\sqrt{(\sigma_1^2 - \sigma_2^2)O_i(t) + \sigma_2^2T}},$$

$$d_1 = d_2 + \sqrt{(\sigma_1^2 - \sigma_2^2)O_i(t) + \sigma_2^2T}.$$

The risk-neutral price of a European call option at time 0 (now) with initial regime $i \in \{1, 2\}$ is then given by:

$$C_i(S_i, K, T) = \int_0^T C_i(S_i, K, T, u)f_i(T, u)du + \delta_2(i)e^{-\beta_2 T}C_i(S_i, K, T, 0)$$
$$+ \delta_1(i)e^{-\beta_1 T}C_i(S_i, K, T, T), \qquad (4.27)$$

where:

$$\delta_i(k) = Kronecker\ delta = \begin{cases} 1 & if\ \ i = k, \\ 0 & if\ \ i \neq k. \end{cases}$$

If the put-call parity relation is applied on the CCP, we can derive the conditional put price (CPP) as follows:

Theorem 4.3

The conditional put price, $P_i(S_i, K, T, O_i(t))$, (conditional on the occupation time $O_i(t)$) is given by:

$$P_i(S_i, K, T, O_i(t)) = Ke^{-[(r_1-r_2)O_i(t)+r_2T]}\Phi(-d_2) - S_0\Phi(-d_1). \qquad (4.28)$$

Proof:

$$P_i(S_i, K, T, O_i(t)) = CPP = CCP - S_0 + Ke^{-\int_0^T r_s ds},$$

$$= S_0 \, \Phi(d_1) - Ke^{-\int_0^T r_s ds}\Phi(d_2) - S_0 + Ke^{-\int_0^T r_s ds},$$

$$= -S_0\left(1 - \Phi(d_1)\right) + Ke^{-\int_0^T r_s ds}\left(1 - \Phi(d_2)\right),$$

$$= Ke^{-\int_0^T r_s ds}\Phi(-d_2) - S_0\Phi(-d_1),$$

$$= Ke^{-[(r_1-r_2)O_i(t)+r_2T]}\Phi(-d_2) - S_0\Phi(-d_1).$$

We then have the risk-neutral price of a European put option at time 0 with initial regime $i \in \{1, 2\}$ given by:

$$P_i(S_i, K, T) = \int_0^T P_i(S_i, K, T, u) f_i(T, u) du + \delta_2(i) e^{-\beta_2 T} P_i(S_i, K, T, 0)$$
$$+ \delta_1(i) e^{-\beta_1 T} P_i(S_i, K, T, T).$$

(4.29)

4.4.2.2. Model application to European type options

Using the regime parameters obtained above and the interquartile range of the SA 10-year government bond yields to derive a monthly estimate of the continuously compounded rate, r_i , we have:

Table 4.3: RSLN2 model parameters on the JSE/ALSI returns data

	μ_i	σ_i	r_i	$\widetilde{\theta}_i$
Regime 1	0.0161	0.0371	0.011	-3.70529
Regime 2	-0.0034	0.0775	0.0067	1.68158

It is worthwhile to make a comment about the risk free rate at this point. A simplifying assumption would be that the risk-free rate is regime independent but this is debatable considering that regimes represent economic cycles. Modelling the interest rate using a Markov model is however outside the scope of this research and an estimate consistent with the long-run average is used as in Fairbrother (2012).

The transition intensity (generator) matrix for the case at hand is estimated using an approximation derived from the Kolmogorov forward equation as follows:

Definition 4.7 (Stewart 2009)

If we have two matrices $\boldsymbol{P_t}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{P_t} = e^{Qt},$$

(4.30)

then

$$\boldsymbol{P_t'} = \boldsymbol{P_t Q}.$$

(4.31)

Remark: Equation (4.30) is the matrix exponential solution to the Kolmogorov forward equation which is represented in matrix form in (4.31).

In the foregoing we set:

$$\boldsymbol{P_t} = \begin{pmatrix} 1 - a(t) & a(t) \\ b(t) & 1 - b(t) \end{pmatrix} \text{ and } \boldsymbol{Q} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

(4.32)

Using this we derive two ordinary differential equations as follows:

$$a'(t) + (\alpha + \beta)a(t) = \alpha, \ a(0) = 0,$$
$$b'(t) + (\alpha + \beta)b(t) = \beta, \ b(0) = 0. \tag{4.33}$$

The solution to these two differential equations is then obtained as:

$$a(t) = \frac{\alpha}{\alpha + \beta}\left(1 - e^{-(\alpha+\beta)t}\right) \text{ and } b(t) = \frac{\beta}{\alpha + \beta}\left(1 - e^{-(\alpha+\beta)t}\right). \tag{4.34}$$

and

$$\boldsymbol{P_t} = \begin{pmatrix} \dfrac{\beta + \alpha e^{-(\alpha+\beta)t}}{\alpha + \beta} & \dfrac{\alpha}{\alpha + \beta}\left(1 - e^{-(\alpha+\beta)t}\right) \\ \dfrac{\beta}{\alpha + \beta}\left(1 - e^{-(\alpha+\beta)t}\right) & \dfrac{\alpha + \beta e^{-(\alpha+\beta)t}}{\alpha + \beta} \end{pmatrix}. \tag{4.35}$$

If we invoke homogeneity, that is, $\boldsymbol{Q}$ does not depend on $\boldsymbol{t}$, we note that the $\boldsymbol{P_t}$ matrix has only two variables namely $V_1 = \alpha t$ and $V_2 = \beta t$ thus,

$$\boldsymbol{P_t} = \begin{pmatrix} \dfrac{V_2 + V_1 e^{-V_1 - V_2}}{V_1 + V_2} & \dfrac{V_1(1 - e^{-V_1 - V_2})}{V_1 + V_2} \\ \dfrac{V_2(1 - e^{-V_1 - V_2})}{V_1 + V_2} & \dfrac{V_1 + V_2 e^{-V_1 - V_2}}{V_1 + V_2} \end{pmatrix}. \tag{4.36}$$

The estimated transition probability matrix (4.7) shown below,

$$\boldsymbol{\mathcal{P}_{\text{est}}} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.9345 & 0.0655 \\ 0.0935 & 0.9065 \end{bmatrix}$$

with the essential continuous time transition matrix constraint that $p_{12} + p_{21} < 1$, and the estimates,

$$\hat{V}_1 = -\ln(p_{11} - p_{21})\frac{p_{12}}{p_{12} + p_{21}} \text{ and } \hat{V}_2 = -\ln(p_{11} - p_{21})\frac{p_{21}}{p_{12} + p_{21}}, \tag{4.37}$$

yields the generator $\boldsymbol{Q}$ matrix as:

$$\boldsymbol{Q} = \begin{bmatrix} -\beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{bmatrix} = \begin{bmatrix} -0.071335 & 0.071335 \\ 0.101829 & -0.101829 \end{bmatrix}. \tag{4.38}$$

This is numerically implemented in Matlab to price the embedded options with the put price, which is representative of the guarantees' costs, being calculated.

We assume that the policyholder is aged x, pays a single premium, P, time is measured in months and the strike price (K) is initially taken to be fixed at R 1 000. The CPP over a range of S_0 and occupation time, with initial regime 1, $O_1(t)$ is plotted in Figures 4.7 and 4.8 below:
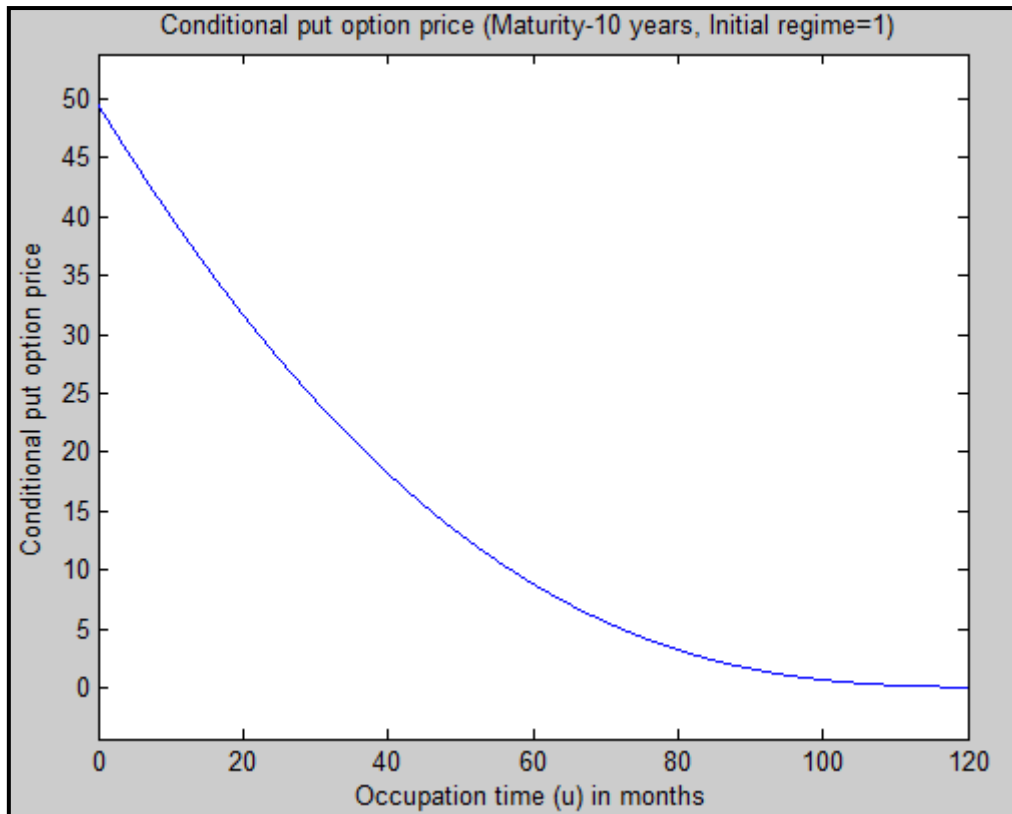
Figure 4.7: Conditional put option price, initial regime 1, maturity 10 years (2D).
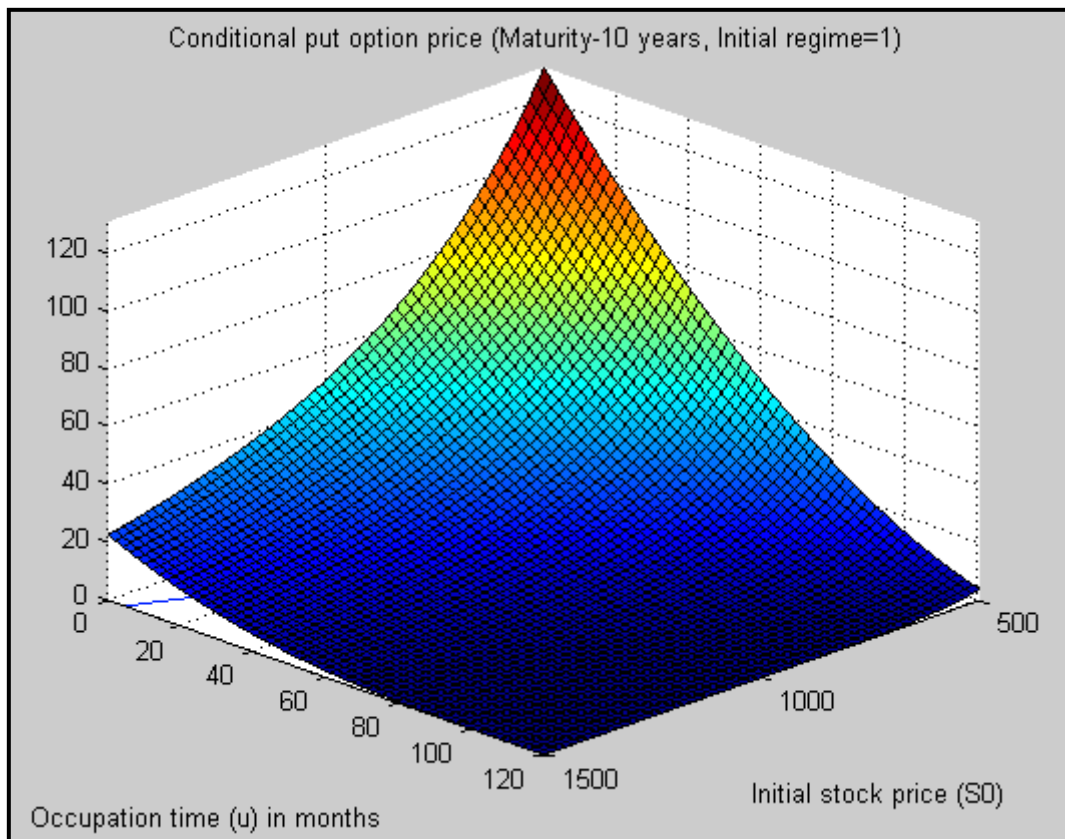
Figure 4.8: Conditional put option price, initial regime 1, maturity 10 years (3D).

The plot of the integrand used in the calculation of the risk-neutral price is shown in Figures 4.9 and 4.10 below:
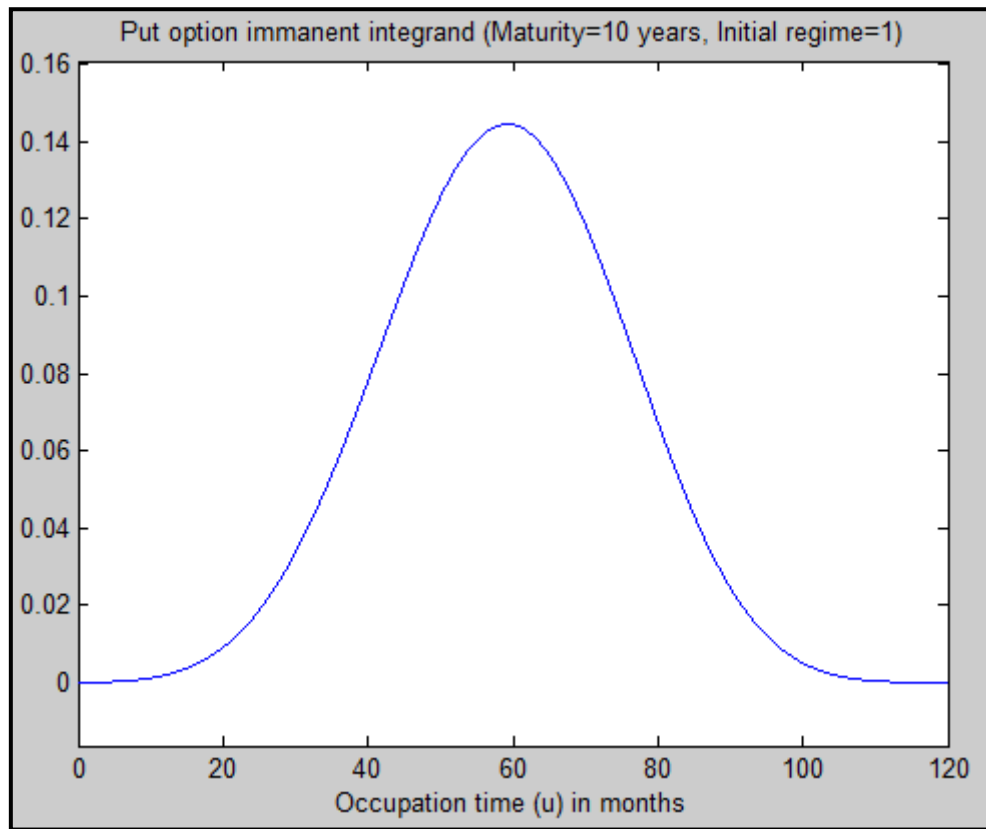
Figure 4.9: Put option immanent integrand, initial regime 1, maturity 10 years (2D).
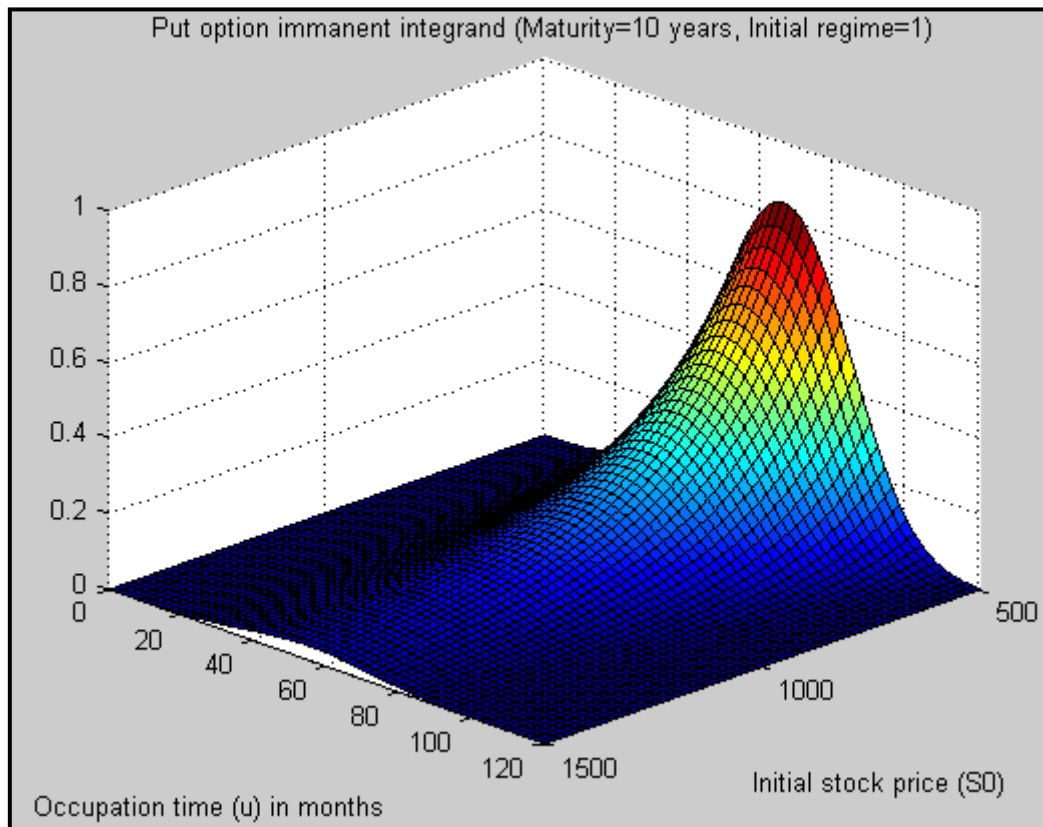
Figure 4.10: Put option immanent integrand, initial regime 1, maturity 10 years (3D).

The plots above give a pictorial idea of the behaviour of the option as we vary the initial stock price and occupation time. The price is a decreasing function of both the occupation time and the initial stock price. The former could be explained by more regime predictability the higher the occupation time and hence the lower the risk premium resulting in decreasing prices. The latter happens due to the higher likelihood of the option ending out-of-the-money as the initial stock price increases. The tables below gives a more comprehensive picture with the put option prices derived for a 1-year, 5-year and 10-year maturity for a fixed initial regime:

Numerical integration is done using both the adaptive Simpson quadrature which is a numerical integration technique that recursively refines the intervals if the desired error tolerance has not been met (Gil, Segura & Temme 2007) and the adaptive Gauss-Lobatto quadrature. The Matlab inbuilt functions *quad* and *quadl* are used and the full codes are in Appendix A.

Adaptive Simpson quadrature

The European put option prices using the adaptive Simpson quadrature are:

Table 4.4: Adaptive Simpson quadrature European put option prices

1-year European put option			5-year European put option			10-year European put option		
S_0	Initial regime	Estimated price	S_0	Initial regime	Estimated price	S_0	Initial regime	Estimated price
R 500	1	387.349	R 500	1	129.797	R 500	1	37.2609
R 750	1	148.962	R 750	1	45.5412	R 750	1	14.045
R 1000	1	23.5663	R 1000	1	16.6205	R 1000	1	6.0902
R 1250	1	3.0408	R 1250	1	6.5905	R 1250	1	2.9404
R 1500	1	0.4523	R 1500	1	2.8346	R 1500	1	1.5413
R 500	2	407.099	R 500	2	145.999	R 500	2	42.9114
R 750	2	178.025	R 750	2	56.9321	R 750	2	17.1509
R 1000	2	48.4509	R 1000	2	22.8966	R 1000	2	7.7966
R 1250	2	10.4553	R 1250	2	9.8498	R 1250	2	3.9136
R 1500	2	2.1227	R 1500	2	4.5374	R 1500	2	2.1209

Adaptive Gauss-Lobatto quadrature

The prices from the adaptive Gauss-Lobatto quadrature are illustrated below:

Table 4.5: Adaptive Gauss-Lobatto quadrature European put option prices

1-year European put option			5-year European put option			10-year European put option		
S_0	Initial regime	Estimated price	S_0	Initial regime	Estimated price	S_0	Initial regime	Estimated price
R 500	1	343.96	R 500	1	129.803	R 500	1	37.2604
R 750	1	132.701	R 750	1	45.5377	R 750	1	14.0441
R 1000	1	22.7882	R 1000	1	16.6197	R 1000	1	6.0887
R 1250	1	3.0416	R 1250	1	6.5874	R 1250	1	2.937
R 1500	1	0.4526	R 1500	1	2.8336	R 1500	1	1.54
R 500	2	381.351	R 500	2	146.005	R 500	2	42.9101
R 750	2	166.285	R 750	2	56.9254	R 750	2	17.1494
R 1000	2	46.3572	R 1000	2	22.8937	R 1000	2	7.7932
R 1250	2	10.071	R 1250	2	9.8468	R 1250	2	3.9097
R 1500	2	2.124	R 1500	2	4.5352	R 1500	2	2.1179

The price between the two integration methods is significantly different for the 1-year put option but as the tenor increases, the difference in price between the two methods falls. The adaptive Gauss-Lobatto quadrature is chosen henceforth on the basis that the integrand is a smooth function and its efficiency as noted in Mikhailov and Nögel (2003).

The difference in the European put option prices relative to the initial regime decreases as the tenor increases as captured in the table and plot below.

Table 4.6: European put option price differences: Regime 2 – Regime 1

Tenor S_0	Regime 2 price – Regime 1 price		
	1-year	5-year	10-year
R 500	37.3912	16.2022	5.6497
R 750	33.5845	11.3877	3.1053
R 1000	23.569	6.274	1.7045
R 1250	7.0294	3.2594	0.9727
R 1500	1.6714	1.7016	0.5779

The higher differences for short tenors could be explained by the possibility of only but a few regime changes before maturity and the price reflects this. In the longer term, the possibility of regime changes irrespective of the initial regime signals the reduced significance of the initial regime and hence the price difference falls as the tenor increases.
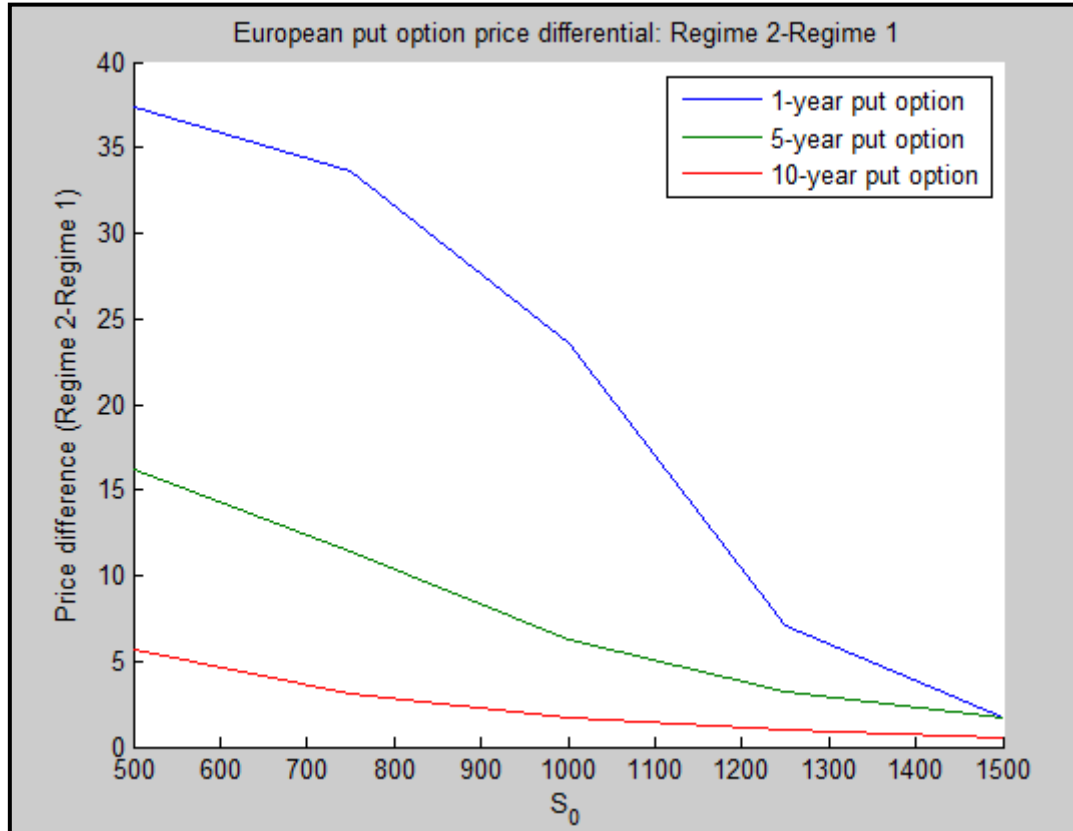
Figure 4.11: European put option price differential: Regime 2 – Regime 1.

4.4.3. Regime switching model applied to pricing the GMxBs

The above results on put option valuation reflect the prices in a general option pricing context. The discussion that follows narrows down to GMxBs and incorporates mortality in the pricing. It is assumed that the premium, of $R\ 1\ 000$ is paid up front, a term of 10 years, policyholder is aged x and an S_0 of $R\ 1\ 000$. The mortality rates used are derived from the ASSA2008 national population model for the year 2013, the SA85-90 Assured Lives Mortality table and the SAIFL 98 and SAIML 98 standard tables (Dorrington & Tootla 2007).

Table 4.7: SAIML98 and SAIFL98 standard tables mortality rates

	Males	**Females**
Age (x)	$_1q_x$	$_1q_x$
50	0.00628	0.00283
55	0.00979	0.00442
60	0.01536	0.00694
65	0.0234	0.01105
70	0.03319	0.0174

Table 4.8: ASSA2008 10-year survival and mortality rates

	Males		**Females**	
Age	$_{10}p_x$	$_{10}q_x = 1 - \ _{10}p_x$	$_{10}p_x$	$_{10}q_x = 1 - \ _{10}p_x$
45	0.67704	0.32296	0.74243	0.25757
50	0.58828	0.41172	0.63710	0.36290
55	0.50778	0.49222	0.54613	0.45387
60	0.45722	0.54278	0.54441	0.45559
65	0.40762	0.59238	0.54970	0.45030
70	0.30741	0.69259	0.43839	0.56161
75	0.26982	0.73018	0.43108	0.56892

4.4.3.1. Guaranteed Minimum Maturity Benefit

In this benefit, we make the assumption that the guarantee is a roll-up maturity guarantee rate of $g_M\ per\ annum$ and that the policyholder does not surrender nor withdraw from the account.

On the basis above and a 10-year contract, the maturity guarantee charges are as in Table 4.9 below:

Table 4.9: 10-year 5% GMMB charges under the RS economy

	Male		Female	
	Regime 1	**Regime 2**	**Regime 1**	**Regime 2**
Age (x)				
50	22.4458	26.5907	24.3084	28.7973
55	19.3743	22.9520	20.8376	24.6856
60	17.4453	20.6668	20.7720	24.6079
65	15.5526	18.4245	20.9739	24.8470
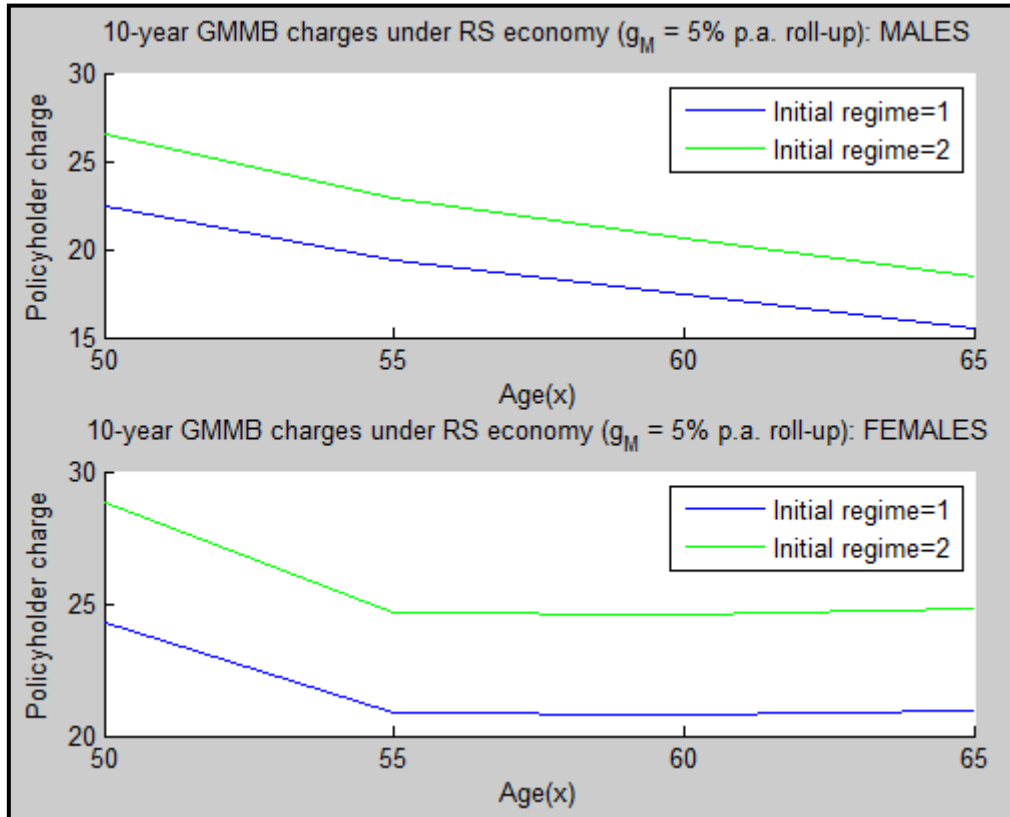
Figure 4.12: 10-year GMMB charges under RS economy.

The guarantee charges when the g_M is 10 % are:

Table 4.10: 10-year 10% GMMB charges under the RS economy

	Male		Female	
	Regime 1	**Regime 2**	**Regime 1**	**Regime 2**
Age (x)				
50	93.8580	104.7755	101.6465	113.4700
55	81.0144	90.4379	87.1334	97.2687
60	72.9484	81.4337	86.8591	96.9625
65	65.0336	72.5983	87.7032	97.9048

A higher maturity guarantee rate results in a higher guarantee charge as expected with females needing a higher guarantee charge than their male counterparts. This is because of their lower mortality rates irrespective of the starting regime.

It is evident that the guarantee charges, if we start from regime 2 are higher than if we start from regime 1. This makes intuitive sense since regime 2 is a high volatility regime hence a high risk regime. It thus requires a higher compensation premium for risk besides its high persistency rate. The charge differences are quite high, in the tune of approximately an extra 11 % and 18% for $g_M = 0.10$ and $g_M = 0.05$ respectively if we start from regime 2 over regime 1 which is quite significant. The need to take into account regime switching in the embedded guarantees pricing context is thus critical.

4.4.3.2. Guaranteed Minimum Death Benefit

In this case, for a 10-year contract it is assumed that the guarantee rate g_D is an annual roll-up and that the estate receives the higher of the rolled-up value and the account value at the end of the year of death, from which, using an expectation algorithm in Matlab and based on the mortality rates we have:

Table 4.11: 10-year 0% GMDB charges under the RS economy

Death (maturity) in:	$g_D = 0\%$							
	Female				Male			
	Age $(x) = 50$		Age $(x) = 60$		Age $(x) = 50$		Age $(x) = 60$	
	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
1st year	0.064491	**0.131191**	0.15815	**0.321719**	0.14311	**0.291123**	0.350027	**0.712047**
2nd year	0.068619	0.115443	0.168296	0.283138	0.151811	0.255403	0.369261	0.621238
3rd year	**0.077427**	0.116656	**0.189625**	0.285699	**0.170563**	0.256981	**0.410682**	0.618757
4th year	0.072687	0.103636	0.1777	0.253361	0.159266	0.227077	0.377224	0.537837
5th year	0.066275	0.091294	0.162057	0.223234	0.144664	0.199275	0.33553	0.462194
6th year	0.059683	0.080071	0.145486	0.195185	0.129462	0.173687	0.292809	0.392835
7th year	0.053291	0.07048	0.129628	0.171439	0.114994	0.152085	0.252549	0.334008
8th year	0.047347	0.061788	0.114669	0.149643	0.101592	0.132578	0.215912	0.281765
9th year	0.041973	0.054188	0.100975	0.130362	0.089424	0.115448	0.183355	0.236716
10th year	0.037125	0.047519	0.088372	0.113111	0.078493	0.100467	0.154859	0.198211
Maximum charge	**0.077427**	**0.131191**	**0.189625**	**0.321719**	**0.170563**	**0.291123**	**0.410682**	**0.712047**

If the insurer guarantees the R 1000 strike to the policyholder without any roll-up, the maximum charges are shown above across regimes, age and gender. If the insurer adopts the maximum charge in its pricing, then any other outcome can be withstood by the insurer.

If the policyholder buys the product during a high volatility regime, then the charge is higher than if the initial regime is a low volatility regime. Once again the high persistence of regime 2 is a major factor in this regard. Across age, an older policyholder has a higher charge for the same maturity after inception and this intuitively makes sense given that the older the age at inception then the higher the probability of dying and guarantee, in an option sense, been exercised. Closely linked to this are the charges across gender. Since males have a higher mortality than females, for the same inception age, the male charges are higher.

Overall, if there is no roll-up, the worst case for the insurer is if death occurs in the early years after policy inception and the charges should then be based on estimations from this early period.

If there is an annual guarantee roll-up rate, the rate chosen should be done so with caution since a high rate results in the maximum charge been in the 10th year and if this is not used and death occurs in the 10th year then the insurer will be heavily exposed if they are invested in the JSE ALSI irrespective of the initial regime and age as evidenced in the table below.

Table 4.12: 10-year 5% GMDB charges under the RS economy

Death (maturity) in:	g_D = 5%							
	Female				Male			
	Age (x) = 50		Age (x) = 60		Age (x) = 50		Age (x) = 60	
	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
1st year	0.064491	0.131191	0.15815	0.321719	0.14311	0.291123	0.350027	0.712047
2nd year	0.094401	0.153926	0.231532	0.377523	0.208852	0.340543	0.508007	0.828329
3rd year	0.132965	0.186259	0.325642	0.456164	0.292908	0.41031	0.705263	0.987943
4th year	0.150726	0.201295	0.368482	0.49211	0.330255	0.441058	0.782216	1.044654
5th year	0.163391	0.210355	0.399527	0.514366	0.356647	0.459161	0.827198	1.064966
6th year	0.173318	0.217798	0.422489	0.530919	0.375956	0.472442	0.850314	**1.068542**
7th year	0.180971	0.223335	0.440204	0.543252	0.390509	0.481923	**0.857633**	1.058398
8th year	0.187108	0.227738	0.453151	0.551553	0.401475	0.488656	0.853248	1.038532
9th year	0.192311	0.231519	0.462648	0.556973	0.409721	0.493256	0.840095	1.011375
10th year	0.196653	0.234639	0.468105	0.558524	0.415778	0.49609	0.820286	0.978734
Maximum charge	0.196653	0.234639	0.468105	0.558524	0.415778	0.49609	0.857633	1.068542

The optimal charge will be achieved if the chosen rate is guided by not just the historical JSE ALSI investment performance but the mortality rates as well and with the understanding that late inception ages (for instance a 60-year old in the case considered) mean a higher likelihood of the guarantees maturing given the high mortality rates at such ages. The necessity of considering the initial regime is once again evident given the fair guarantee charge differentials when the two regimes are compared even in the case where there is an annual roll-up rate.

4.5. Lévy Processes

The use of regime switching models becomes more complicated if we have to consider more than two regimes. In this case, the parameters to be estimated easily lead to a non-parsimonious model more so when it is obvious that the market has more than two regimes such as adding an additional average volatility regime for the case considered above. The research thus far has also been silent on the issue of the risk of regime change and how to price this. In a market with only two regimes, the switch presents itself as a jump and needs to be considered.

The foregoing discussion and the initial discussion on the limitations of the Black-Scholes framework necessitates the need of a refined model to be used in the pricing and hedging of derivative securities and critically so, that the probability distribution of the underlying be flexible enough in capturing the higher order moments usually observed.

Schoutens (2003) notes that what is needed is a flexible stochastic process that would generalize Brownian motion. The process should have independent and stationary increments based on a more generic distribution than the normal distribution. Such a distribution, inadvertently, has to be infinitely divisible and stochastic processes so defined are called *Lévy processes*.

A distribution is said to be infinitely divisible if for any integer k, it can be represented as the law of a sum $\sum_{n=1}^{k} x_n$ of independent identically distributed random variables x_n. This idea of infinite divisibility can equally be defined using the characteristic function $\phi(u)$ of a distribution. If for every $k \in \mathbb{N}$, $\phi(u)$ is the kth power of a characteristic function, then the distribution is said to be *infinitely divisible* (Schoutens 2003).

Definition 4.8 (Schoutens, Simons & Tistaert 2005)

A Lévy process $X = \{X_t, t \geq 0\}$ is a stochastic process defined on a probability space $(\Omega,\ \mathcal{F}, \mathbb{P})$ which satisfies the following properties:

   a) *The paths of X are right continuous with left limits almost surely,*
   b) *$X_0 = 0$ almost surely,*
   c) *X has independent increments; for $0 \leq s \leq t$, $X_t - X_s$ is independent of $\sigma(X_u : u \leq s)$,*
   d) *X has stationary increments; for $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.*

Remark: The process thus starts at zero and has independent and stationary increments such that the distribution of the increment is an infinitely divisible distribution. A non-negative, non-decreasing Lévy process is defined as a *subordinator*.

The departure from normality noted above means that the distribution's skewness, a measure of the degree to which a distribution is asymmetric, is non-zero and the kurtosis, a measure of the fatness of the tails, is different from three. This limits the choice of the distributions as the distribution so chosen must be able to capture skewness and excess kurtosis. They include, inter alia, the Variance-Gamma (VG), the Normal Inverse Gaussian (NIG) and the Carr, Geman, Madan and Yor (CGMY) distributions.

4.6. Variance-Gamma Lévy Process

The VG process is a Lévy process which was derived in Madan and Milne (1991) and discussed more generally in Madan et al. (1998). In the former case, the authors discussed the symmetric VG process whereas in the latter the general non-symmetric VG process is considered. The two processes meet the fundamental expectation of any market model, that it be arbitrage free, the log price is expressed as a Brownian motion considered at a random time, they incorporate long-tailedness and the general case further takes into account skewness.

The VG process has been given a number of representations and two representations from Schoutens (2003), that will be useful for the context under consideration, are discussed here below:

Definition 4.9 (Schoutens 2003)

The VG process $X^{(VG)} = \{X_t^{(VG)},\ t \geq 0\}$ is the process such that:
  i. $X_0^{(VG)} = 0$,
  ii. *The process has independent and stationary increments,*
  iii. *The increment $X_{s+t}^{(VG)} - X_s^{(VG)} \sim VG(\sigma\sqrt{t},\ \frac{v}{t},\ t\theta)$ i.e. the increment follows a VG distribution.*

Definition 4.10 (Schoutens 2003)

Let $G = \{G_t,\ t \geq 0\}$ be a Gamma process with parameters $a = \frac{t}{v},\ v > 0$ and $b = \frac{1}{v},\ v > 0$ and let $W = \{W_t,\ t \geq 0\}$ be a standard Brownian motion. Further, let $\sigma > 0$ and $\theta \in \mathbb{R}$, then the VG process is defined as:

$$X_t^{(VG)} = \theta G_t + \sigma W_{G_t}. \tag{4.39}$$

In this case, the parameters θ and v are useful in controlling for skewness and kurtosis respectively with θ accounting for skewness in the distribution and v accounting for the excess kurtosis relative to the normal distribution.

It is worth noting that with a different parametrisation, Madan et al. (1998) show the VG process as a difference of two independent Gamma processes. This parametrisation is quite useful more so in simulation contexts.

4.6.1. Simulating Variance-Gamma process

Monte-Carlo simulation is used in the simulation of the embedded guarantees in the GMxB products considered and hence a need to discuss a technique for simulating a VG process.

In Definition 4.10, it was noted that a VG process can be obtained by time-changing a standard Brownian motion with drift by a Gamma process. If one can sample these two processes then a sample path for a VG process can then be obtained.

4.6.1.1. Simulating a Gamma process

Schoutens (2003) discusses the simulation of a Gamma process noting that if a random variable $X \sim Gamma(a, b)$, then, for $c > 0$, $\frac{X}{c} \sim Gamma(a, bc)$. All that is needed is thus a good generator of a $Gamma(a, 1)$ for which the Berman's gamma generator is used as follows:

  i.   Generate two independent uniform random numbers u_1 and u_2,

  ii.   Set $x = u_1^{\frac{1}{a}}$ and $y = u_2^{\frac{1}{1-a}}$,

  iii.   If $x + y \leq 1$ then go to (iv) else go to (i),

  iv.   Return the number $-x log(u_1 u_2)$ as the $Gamma(a, 1)$ random number.

In simulating a sample path of a Gamma process $G = \{G_t, t \geq 0\}$ where $G_t \sim Gamma(at, b)$, we simulate the value of this process at time points $\{n\Delta t, n = 0, 1, \dots\}$ by:

  a.   Generate independent $Gamma(a\Delta t, b)$ random numbers $\{g_n, n \geq 1\}$ using Berman's generator. (For small Δt, $a\Delta t$ will be smaller than 1 hence we can use Berman's generator),

  b.   Then

$$G_0 = 0, \qquad G_{n\Delta t} = G_{(n-1)\Delta t} + g_n, \quad n \geq 1. \tag{4.40}$$

4.6.1.2. Simulating a standard Brownian motion

The simulation of a standard Brownian motion, $W = \{W_t, t \geq 0\}$ is simplified by the observation that the process has normally distributed independent increments.

  a.   Generate a series of standard normal random numbers $\{v_n, n = 1, 2, \dots\}$ using the Box-Muller or inverse transformation method and for very small Δt,

  b.   Then for time points $\{n\Delta t, n = 0, 1, \dots\}$ we have:

$$W_0 = 0, \qquad W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t}\, v_n, \qquad n \geq 1. \tag{4.41}$$

With the above two simulated processes, sample paths of a VG process can then be easily obtained.

4.6.2. Model theory

Madan et al. (1998) derive the risk-neutral stock/underlying price dynamics under a VG economy as:

$$X_t^{(VG)} = \theta_{VG}G_t + \sigma_{VG}W_{G_t},$$
$$S(t) = S(0)e^{(r-q+\omega)t+X_t^{(VG)}}, \tag{4.42}$$

where:

$$\omega = \frac{1}{v}\ln\left(1 - \theta_{VG}v - \frac{\sigma_{VG}^2 v}{2}\right),$$
θ_{VG} is the drift of the diffusion part,
σ_{VG} is the volatility of the diffusion part,
v is the variance rate of the gamma part.

It is worthwhile to make some comments about the change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$ applied in deriving the risk-neutral price. The Mean Martingale Correcting Term (MMCT) is the approach used in the change of measure. The MMCT has been shown to be a special case of the Esscher transform approach (see Miyahara (2004)). Using the characteristic function approach, the author shows that to obtain the MMCT change of measure, we shift the VG process X_t to $X_t + \omega t$ and in terms of the characteristic function, we have:

$$\phi_{X_t}^{\mathbb{Q}}(u) = \phi_{X_t}(u)e^{\omega t}. \tag{4.43}$$

If we assume that the dividend yield q is incorporated in the stock price, which is a reasonable assumption in an option pricing context, we have:

$$S(t) = S(0)e^{(r+\omega)t+X_t^{(VG)}}. \tag{4.44}$$

Denoting the risk neutral measure with $\mathbb{Q}$, the value of a European call option is then given by:

$$c_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S(T) - K)^+]. \tag{4.45}$$

After taking the expectation, Madan et al. (1998) arrive at the expression for the time 0 price of the option as:

$$c(S(0); K, T) = S(0)\Psi\left(d\sqrt{\frac{1-c_1}{v}}, (\alpha+s)\sqrt{\frac{v}{1-c_1}}, \frac{T}{v}\right) - Ke^{-rT}\Psi\left(d\sqrt{\frac{1-c_2}{v}}, \alpha s\sqrt{\frac{v}{1-c_2}}, \frac{T}{v}\right), \tag{4.46}$$

where:

$$d = \frac{1}{s}\left[\ln\left(\frac{S(0)}{K}\right) + rT + \frac{T}{v}\ln\left(\frac{1-c_1}{1-c_2}\right)\right],$$

$$\zeta = -\frac{\theta}{\sigma^2}, \quad s = \frac{\sigma}{\sqrt{1+\left(\frac{\theta}{\sigma}\right)^2 \frac{v}{2}}}, \quad \alpha = \zeta s,$$

$$c_1 = \frac{\nu(\alpha+s)^2}{2}, \, c_2 = \frac{\nu\alpha^2}{2},$$

Ψ involves the modified Bessel function of the second kind and a degenerate hypergeometric function.

The put-call parity relation is applied to the call option price in Hirsa and Madan (2001) and the resultant price of a European put option is:

$$p(S(0); K, T) = Ke^{-rT}\Psi\left(-d\sqrt{\frac{1-c_2}{\nu}}, \alpha s\sqrt{\frac{\nu}{1-c_2}}, \frac{T}{\nu}\right) - S(0)\Psi\left(-d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \frac{T}{\nu}\right). \quad (4.47)$$

The application of the VG process in the pricing of options has been done by many authors both in the context of American and European options such as Madan et al. (1998), Cont and Voltchkova (2005), Moosbrucker (2006) and Wang (2009).

The goal of this and subsequent sections is to consider the VG and related processes in the pricing of embedded derivatives in life insurance products, in particular the GMxB products, and equally important and intimately connected to pricing the hedging of the main risks inherent in them using these frameworks.

4.6.3. Fitting the Variance-Gamma model

The calibration of the model can be done using log-returns data (see Seneta (2004) and Cepni et al. (2013)) or implicitly using option prices where the option pricing formula is based on the characteristic function of the log price process of the stock. The latter is considered a more robust approach since it identifies parameters implied in the option prices which are more representative of future expectations.

However, the use of implied parameters should not come at the total disregard of the hugely valuable information that lies veiled in historical data. Implied parameters must be compared with historical parameters and best estimates of what the parameters are arrived at based on these two trays of information.

In the absence of long tenor options to estimate implied parameters in South Africa, as would be needed for embedded options in a life insurance context, two possibilities exist: first is to calibrate the VG model based on United States of America (US) index options such as the S&P500 options which would be useful for the long tenors due to the presence of a deeper market. However, this approach is heavily dependent on the assumption that there is a link in behaviour between the US and SA markets. The US is a developed market whereas SA is an emerging market hence there is no reason to believe that a relationship between the parameters exists. The second possibility is the use of historical data to calibrate the VG model and whereas this may not always be seen as ideal from the understanding that the future may not reflect the past, making any other assumption is probably faultier.

Hardy (2001) discusses the danger of ignoring historical data more so in the absence of any other information. The author notes that United Kingdom (UK) insurers in the 1980s and 1990s decided to make assumptions that ignored historical information and this led to the collapse of some of them. If the historical information had been used in the parametrisation, the parameters so obtained would have captured the extremities that led to the difficulties that the insurers found themselves in and thereby avoided the corporate survival challenges that ensued.

In the context of variable annuity contracts, which in most cases rely heavily on the stock market returns, Hardy (2001) concludes that we cannot let, "... modelling fall into disrepute because we 'cannot know' whether the past is an adequate representation of the future." Objective historical data should thus not be ignored in modelling on the basis that the long-term contracts may not follow a related process as preceding years and it is from this basis that the VG model is calibrated for the case at hand.

The use of such an approach is affirmed if there is reason to believe that what has been observed in the historical period considered is likely to manifest in the subsequent pricing period for which the results are used. This is the case in the research at hand given the complete cycles observed in the historical period used. Ulmer (2010) notes that this is comparable to making a choice on the risk-neutral distribution and does in some sense reflect the market's choice of measure. The actuarial guidance on the matter in South Africa is given by the Advisory Practice Note (APN) 110 which notes that in the absence of suitable instruments to fully calibrate the model, probable market values can be used provided the results don't deviate from market consistent values.

The hypothesis is that the VG framework can be used in the pricing and hedging of the options and the report will assess on acceptance or non-acceptance of this hypothesis for the SA context. A comparison is made with the RS model and a sensitivity analyses carried out to assess how sensitive the prices are to modest changes in the parameters so estimated.

Cepni et al. (2013) fitted the VG model to a number of developed and emerging markets in assessing its adequacy vis-à-vis the normal distribution, the Normal Inverse Gaussian (NIG) and the Heston model. In their research, which includes South Africa, they conclude that the VG model performs better than all the other models they researched on when daily and weekly returns are considered for the twenty countries in the study. The authors however do not discuss the estimation of parameters nor show the parameters derived from the log-returns for their models.

4.6.3.1. Maximum-likelihood estimation of VG model from log-returns data

The data used is once again based on the end-of-month JSE ALSI closing values. The log-returns are derived and the plot for the period from July 1994 to June 2013 is shown in Figure 4.13 below.

The parameters can be estimated using the method of maximum-likelihood where if we have a series of independent log-returns with $\pi = (\sigma, \nu, \theta)$ being the VG probability density function parameters, then we find the set of parameters π that will maximise the logarithm of the likelihood function $\mathcal{L}(\pi) = \prod_{i=1}^{x} f(x_i; \pi)$ where the x_i represents the series of the independent log-returns as in Brigo et al. (2007).
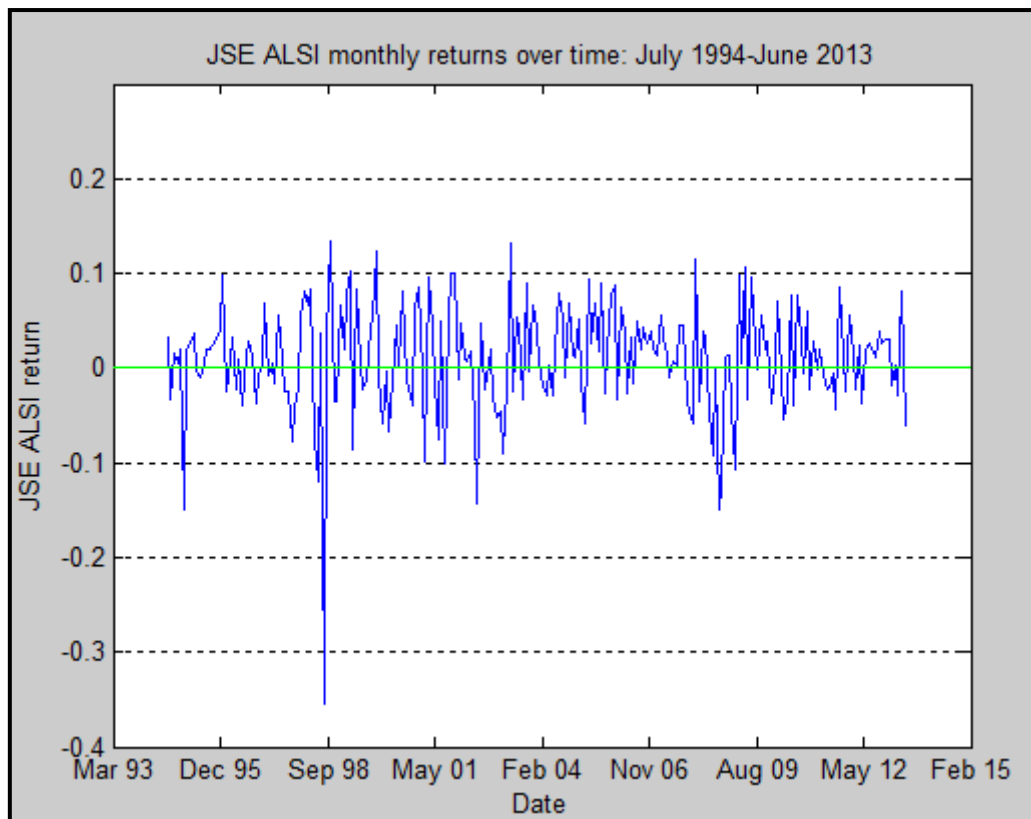
Figure 4.13: JSE ALSI monthly returns time series: July 1994 – June 2013

The authors discuss the central moments in a VG context and note that if we have sample estimates of the mean, variance, skewness and kurtosis of the log-returns denoted by M, V, S and K respectively then the parameters can be initially estimated, using the method of moments, as:

$$\bar{\sigma} = \sqrt{\frac{V}{\Delta t}},$$
$$\bar{v} = \left(\frac{K}{3} - 1\right)\Delta t,$$
$$\bar{\theta} = \frac{S\bar{\sigma}\sqrt{\Delta t}}{3v},$$
$$\bar{\mu} = \frac{\bar{X}}{\Delta t} - \bar{\theta}.$$

(4.48)

A comment regarding the all crucial independence of returns assumption noted above is necessary and is checked using the ARIMA procedure in the Statistical Analysis Software (SAS) where this is satisfied if there is no autocorrelation in the historical returns. The ARIMA procedure shows no significant lags in the autocorrelations and partial autocorrelations as shown in Figures 4.14 and 4.15 hence the independence assumption required in an MLE is satisfied.

```
                              The ARIMA Procedure

                      Name of Variable = Logarithmic_return

                      Mean of Working Series      0.008574
                      Standard Deviation          0.057106
                      Number of Observations           227


                              Autocorrelations

Lag   Covariance   Correlation   -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1   Std Error

 0    0.0032611     1.00000      |                    |********************|        0
 1   -0.0000765     -.02347      |                    .|  .                |   0.066372
 2    8.66319E-6    0.00266      |                    .|  .                |   0.066409
 3    0.00027537    0.08444      |                    .|**.                |   0.066409
 4   -0.0001546     -.04742      |                   . *|  .                |   0.066881
 5   -0.0003788     -.11616      |                   .**|  .                |   0.067029
 6    0.00013628    0.04179      |                    .|* .                |   0.067910
 7   -0.0002399     -.07356      |                   . *|  .                |   0.068023
 8   -0.0000661     -.02028      |                    .| .                 |   0.068372
 9    0.00030076    0.09223      |                    .|**.                |   0.068399
10   -0.0001762     -.05403      |                   . *|  .                |   0.068945
11    0.00002589    0.00794      |                    .|  .                |   0.069131
12   -0.0000470     -.01442      |                    .|  .                |   0.069135

                      "." marks two standard errors
```

Figure 4.14: JSE ALSI monthly returns autocorrelation plot

```
                        The ARIMA Procedure

                     Partial Autocorrelations

Lag    Correlation   -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

 1      -0.02347     |                    .|  .                |
 2       0.00211     |                    .|  .                |
 3       0.08460     |                    .|**.                |
 4      -0.04379     |                   . *|  .                |
 5      -0.11987     |                   .**|  .                |
 6       0.03048     |                    .|* .                |
 7      -0.06358     |                   . *|  .                |
 8      -0.00669     |                    .|  .                |
 9       0.07857     |                    .|**.                |
10      -0.05175     |                   . *|  .                |
11       0.00899     |                    .|  .                |
12      -0.04585     |                   . *|  .                |
```
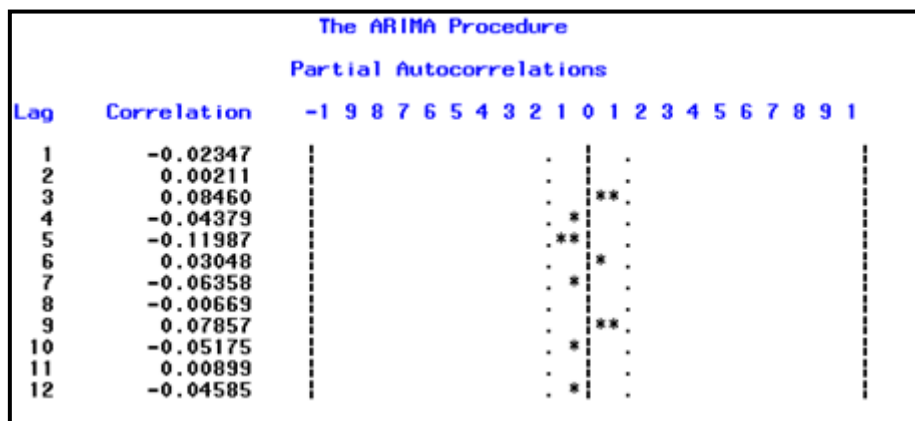
Figure 4.15: JSE ALSI monthly returns partial autocorrelation plot

The above also points to the logarithmic return for the index being a suitable choice for a market invariant and can thus be seen as an uncertainty source at the JSE.

This estimation procedure is applied to the data with an easily scalable Δt of 1 from which we obtain the initial estimates as follows:

Table 4.13: Initial MME VG parameter estimates

Parameter	Mean (M)	Variance (V)	Skewness (S)	Kurtosis (K)
Estimate	0.0085738	0.0032755	-1.3049	9.544
Parameter	$\bar{\sigma}$	$\bar{\nu}$	$\bar{\theta}$	$\bar{\mu}$
Estimate	0.057232	2.1813	-0.011413	0.019987

These estimates are then used as the initial estimates in the MLE to fit the JSE ALSI monthly log-returns and we obtain the final parameter estimates based on the monthly returns as:

Table 4.14: Final MLE VG parameter estimates

Parameter	θ	ν	σ	μ
Estimate	-0.0148	0.4461	0.0544	0.0234

The parameters θ and ν, that are useful in controlling the skewness and kurtosis respectively, are non-zero which is indicative of the need for the use of a framework other than the normal distribution and the VG presents itself as a credible alternative. The θ parameter is negative which is indicative of negative skewness for the JSE ALSI returns distribution.

The histogram plot below shows the extent of skewness and kurtosis with the super-imposed red curve being the normal distribution curve. In the South African context, the assumption of normality in returns for the index does not hold.
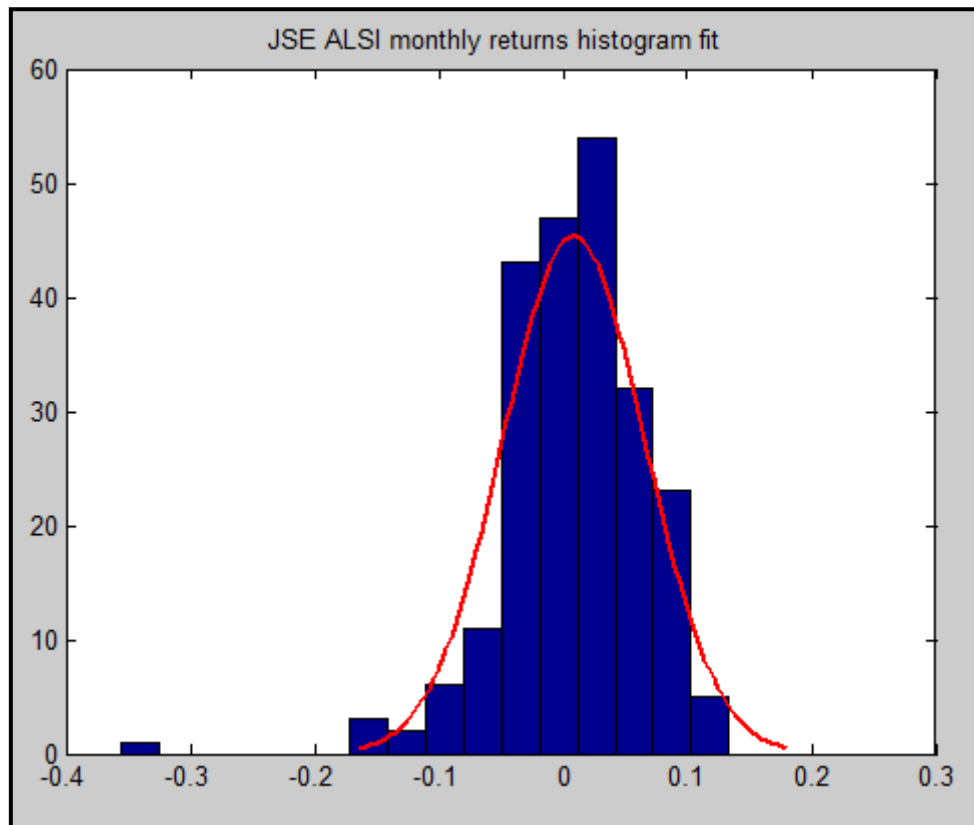
Figure 4.16: JSE ALSI monthly returns histogram fit

The QQ-plot in Figure 4.17 indicates the presence of tails that are thicker than in the normal distribution case hence the log-returns are heavily kurtotic.
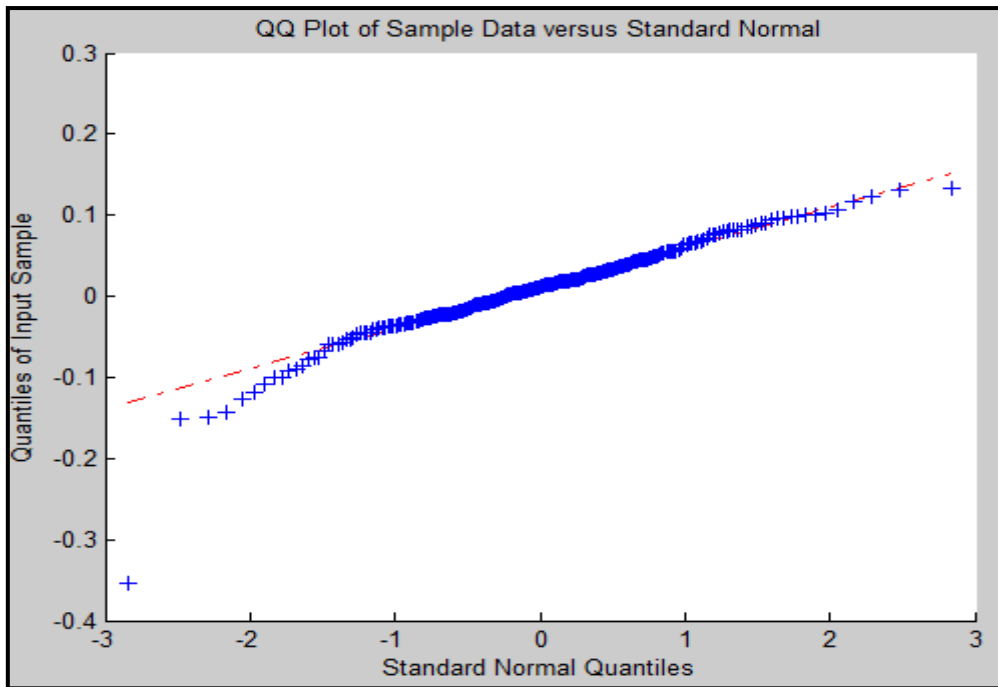
Figure 4.17: QQ plot of the JSE ALSI monthly returns

The goodness-of-fit tests for normality shown below all lead to a rejection of the normality assumption for the returns at a 1 % level of significance with the p-values for all the three tests being < 0.010.


```
      Goodness-of-Fit Tests for Normal Distribution

Test                     ----Statistic-----    ------p Value------

Kolmogorov-Smirnov    D     0.07112887      Pr > D       <0.010
Cramer-von Mises      W-Sq  0.21243629      Pr > W-Sq    <0.005
Anderson-Darling      A-Sq  1.50139543      Pr > A-Sq    <0.005
```

Figure 4.18: Goodness-of-fit tests for the JSE ALSI monthly returns

This further adds credence to the use of a VG economy in the pricing of options for the SA context for which Wang (2009) notes that for such a pricing (and hedging), the use of a probability distribution that correctly and accurately reflects the underlying asset's behavior is critical. The adoption of the VG framework in the particular context at hand, the embedded options, flows naturally from this given the investment of most of the life insurers funds in the stock markets.

The moments obtained are estimated from monthly log-returns data and for ease of interpretation are changed to annualized form in Table 4.15 below. Backus, Foresi and Wu (2004) note that if the one-period skewness and excess kurtosis is denoted by S and K respectively, then the T-period skewness and excess kurtosis are given by $S^* = \frac{S}{\sqrt{T}}$ and $K^* = \frac{K}{T}$, respectively. The standard deviation is adjusted using the square root of time adjustment whereas the mean is additive.

Table 4.15: Estimated statistical moments on the JSE ALSI returns

	Mean	Std dev.	Skewness	Excess kurtosis
Monthly rate	0.008574	0.057232	-1.3049	6.544
Annualized rate	0.102886	0.198257	-0.3767	0.5453

The pricing calculations are however done using a month as the time unit.

4.6.4. Arbitrage-freeness, parameter risk-neutrality, stability and robustness

The VG model belongs to the so-called exponential Lévy models. Cont and Tankov (2004) show that the VG model is arbitrage free since its trajectories, being an exponential Lévy model, are neither increasing nor decreasing with probability 1. It thus satisfies a key expectation in quantitative finance, that a pricing model should be arbitrage-free.

The pricing of options is done in the risk-neutral world and it is critical that the parameters used in the pricing be the risk-neutral parameters. Madan et al. (1998) note that unlike diffusion based price processes, for a VG model, the statistical parameter estimates need not be equal to the risk-neutral parameters. The danger with using statistical parameters in this setting is that the risk premium that is incorporated in the risk-neutral parameters partly due to market incompleteness may not be incorporated in the statistical parameter estimation approach. A parameter sensitivity analysis can however help in gauging the possible impact of this on the prices so obtained.

The authors do however note that if one takes the risk-neutral parameters as constant across time then pooled time series data can be used to jointly estimate the statistical and risk-neutral processes. This approach is necessary more so when sufficient data is not available but the possibility of price discrepancies between the real option prices and those calculated using the parameters so obtained still remains.

In the analyses done, further to the comments made above, the use of Monte Carlo simulation instead of analytical pricing formulas avoids the possibility of excessive price estimation error due to the parameter estimation methodology. The VG process is calibrated using the MLE approach and future JSE ALSI scenarios built. The options under consideration are vanilla hence the payoff is calculated at maturity and this discounted using the continuously compounded risk-free interest rate. The risk-free interest rate is readily available and thus does not introduce major discrepancies if any. The credibility of the approach is discussed in the enlightening paper by Embrechts (2000) who notes that in the context of insurance pricing, the method gives a more objective description of the underlying randomness.

In the process of calibration, great care has to be taken to ensure that the model parameters so obtained not only closely match the observed markets prices but that they are also stable.

The 95 % confidence intervals for the parameters are contained in Table 4.16 below together with the standard errors. The standard errors are all reasonably low with the highest standard error being on the estimated ν parameter hence 95% of the time, if we assume a VG economy for the JSE ALSI and fit using the log-returns, the parameters obtained will be within the confidence intervals shown in the table above.

Table 4.16: Statistical significance of the VG estimated parameters

Parameter	θ	ν	σ	μ
LCI	-0.0406	0.0210	0.0474	-0.0015
UCI	0.0111	0.8711	0.0613	0.0482
Standard error	*0.013197*	*0.216873*	*0.003565*	*0.012677*

4.6.5. Model application to European type options

Kling, Ruez and Rub (2010) discuss product valuation under the assumption of independence between the financial markets and demographic mortality which is a reasonable assumption that can be tested through a correlation test. Further, if the insurer is risk-neutral with respect to mortality risk then we are able to use the financial market risk neutral measure and the mortality measure in the GMxB valuation. Finally, the reasonable assumption that the underlying account is linked directly to a market index as discussed in Coleman et al. (2005) is made, in this case the JSE ALSI, and used to price the costs in an SA context.

The assumption of risk-neutrality to mortality results in a European type option which can be priced with the VG option price formula. This is based on the extra cost to the insurer above the investment value $(\max(f(g,P) - I_T)$ where $f(g,P)$ is the guaranteed amount which, if we ignore mortality, is a function of the guarantee rate and the initial premium) which should be used to calculate the charge for the embedded option.

4.6.5.1. Monte Carlo pricing under the Variance-Gamma model

The following procedure as discussed in Schoutens (2003) is followed where, as would be expected, the accuracy of the final estimate is a function of the sample paths chosen:

1. Fit/calibrate the model on the available market data based on some measure of fit.

2. Using (1),
    a. Simulate a significant number, m, of paths of the stock-price process $S = \{S_t,\ 0 \le t \le T\}$ by simulating the log price process via a simulation of the time-changing process:
        i. Simulate the rate of time change process $y = \{y_t, 0 \le t \le T\}$,
        ii. Calculate from (i) the time change $Y = \{Y_t = \int_0^t y_s ds\,, 0 \le t \le T\}$,
        iii. Simulate the VG process $X = \{X_t, 0 \le t \le Y_T\}$ sampling over the period $[0, Y_T]$,
        iv. Calculate the time-changed VG process $X_{Y_t}, t \in [0, T]$,
        v. Calculate the stock-price process $S = \{S_t, 0 \le t \le T\}$.

    b. For each path i, calculate the payoff function $\mathbb{g}_i = G(\{S_u, 0 \le u \le T\})$.

3. Calculate the mean of the sample payoffs to get an estimate of the expected payoff:

$$\widehat{\mathbb{g}} = \frac{1}{m}\sum_{i=1}^{m} \mathbb{g}_i. \tag{4.49}$$

4. Discount the estimated payoff at the risk-free rate to get an estimate of the value of the embedded option:

$$Value\ of\ derivative = e^{-rT}\widehat{g}. \qquad (4.50)$$

4.6.5.2. Applying the model to pricing

The application of the VG model to the European type options follows. The formula for the price is based on the nature of the payoff and the assumption that under the risk-neutral measure, the one-period account value log-returns are independent random variables that follow a VG distribution:

$$\ln\left(\frac{I_t}{I_{t-1}}\right) = r_t \sim VG(\sigma, v, \theta). \qquad (4.51)$$

This is applied for a European put option with a strike amount of $R\,1\,000$ using three different methods: analytical, integration and Monte Carlo simulation approach in Table 4.17 below.

Table 4.17: European put option prices under a VG economy

1-year European put option			5-year European put option			10-year European put option		
S_0	Method	Estimated option price	S_0	Method	Estimated option price	S_0	Method	Estimated option price
R 500	Analytical	399.8171	R 500	Analytical	143.2720	R 500	Monte Carlo	41.4081
R 750	Analytical	163.3511	R 750	Analytical	50.4111	R 750	Monte Carlo	14.7013
R 1000	Analytical	33.1087	R 1000	Analytical	17.2321	R 1000	Monte Carlo	5.7913
R 1250	Analytical	4.1009	R 1250	Analytical	6.0712	R 1250	Monte Carlo	2.5341
R 1500	Analytical	0.4288	R 1500	Analytical	2.2467	R 1500	Monte Carlo	1.1630
R 500	Monte Carlo	399.8015	R 500	Monte Carlo	143.2635			
R 750	Monte Carlo	163.1880	R 750	Monte Carlo	50.3417			
R 1000	Monte Carlo	33.1958	R 1000	Monte Carlo	17.2493			
R 1250	Monte Carlo	4.1272	R 1250	Monte Carlo	6.0300			
R 1500	Monte Carlo	0.4358	R 1500	Monte Carlo	2.2177			
R 500	Integration	399.8171	R 500	Integration	143.2721			
R 750	Integration	163.3511	R 750	Integration	50.4112			
R 1000	Integration	33.1087	R 1000	Integration	17.2323			
R 1250	Integration	4.1009	R 1250	Integration	6.0715			
R 1500	Integration	0.4288	R 1500	Integration	2.2470			

The analytical pricing approach is as discussed in Madan, Carr and Chang (1998) whereas the integration method involves brute force integration as discussed in Rebonato (2004). These two approaches require a numerical integration method and are quite sensitive to the method used. In the case at hand, though the integral is an infinite integral, a bound large enough is chosen to capture the 'infiniteness' of the upper bound but controlled in order to evaluate the integral to a finite value. The numerical evaluation is done using, *quadgk,* the Gauss-Kronrod quadrature in Matlab which is noted as an efficient and accurate approach more so for infinite intervals and cases of singularities at endpoints.

Binkowski (2008) discusses the errors introduced by such an integration cut-off and the choice of the method used in which it is noted that the errors are very minimal with the prices obtained being accurate to at least 3 decimal places. This is considered a sufficient level of accuracy for this research. The Monte Carlo simulation follows the discussion at the beginning of this sub-section.

It is observed that the analytical and brute force integration method prices are comparatively close to each other with the Monte Carlo (MC) method prices differing slightly from the two as shown in Table 4.17 above. The MC method requires a lot of simulations and computer time for the same answer for short tenor options under the VG framework and hence would not be the preferred method in such cases. However, for large tenors and due to the failure of numerical integration methods to converge, the first two methods fail. Since the embedded options tend to have long-maturities, it is for this reason that the Monte Carlo approach is used forthwith for their pricing. This does come at the cost of speed since for accurate MC pricing, a lot of simulations are needed but the further merit of the need for only the risk-free rate parameter estimation to obtain the price at time 0 noted in Section 4.6.4 makes the choice sensible.

In Table 4.17 above, a simulation of 10,000 stock price possibilities at maturity was done for 100 runs and the average taken. A Monte Carlo simulation of 10,000 possible stock index values after 10 years starting from R 1000 is shown in the plot and snapshot of the simulated stock price/index values below:
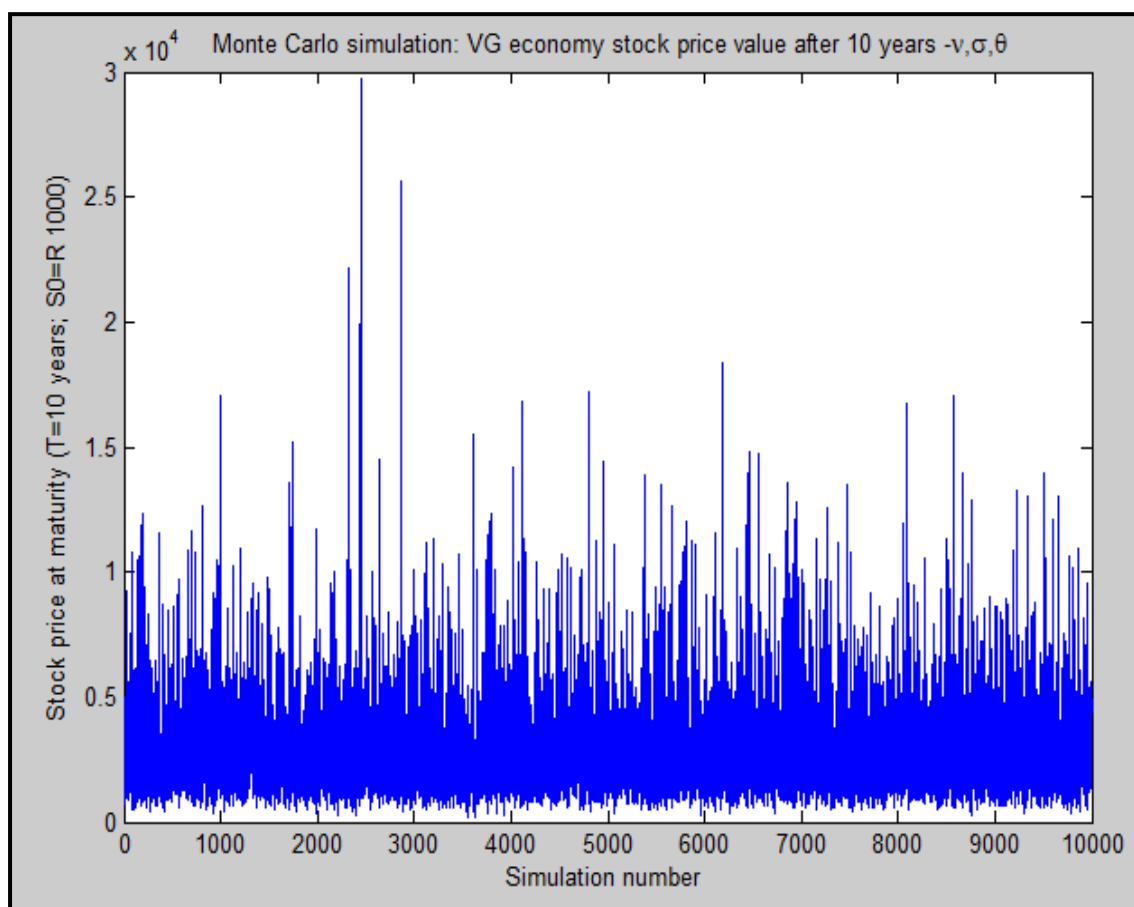
Figure 4.19: Simulated 10-year stock index values in a VG economy

```
2331.66638
2138.38232
932.230281
 524.61483
 3079.0495
1209.35685
2100.73155
2998.61262
 5791.4007
2340.16209
1108.13029
2737.07275
1008.83435
6419.32653
```

Figure 4.20: Snapshot of simulated 10-year VG stock index values

The simulation shows the ability of the MC approach to simulate stock price ranges across a wide spectrum in a VG economy. Though most of the simulated values lie within reasonable ranges expected in normal market times, the ability to simulate extremes and thereby capture the possibility of the markets performing exceptionally well or very poorly during periods of turbulence accommodates a desired quality of any market model (heavy tailedness) in current times.

4.6.5.3. VG parameter sensitivity analysis

The lack of long-tenor options from which the VG model would be calibrated in SA means that the parameters so used must be tested to assess the impact on the price if they change from the base case. This crucial sensitivity analyses on the 10-year European put option (with both S_0 and strike price R 1 000) to changes in the three VG parameters is shown in Table 4.18 below where one of the parameters is adjusted with the others being held constant.

Table 4.18: VG option price sensitivity as the parameters vary

	10-year European put option		
	θ	ν	σ
Base scenario + 5%	7.7436	5.8851	59.0012
Base scenario + 1%	*5.2948*	*5.8166*	*12.4292*
Base scenario	**5.7913**	**5.7913**	**5.7913**
Base scenario − 1%	*6.7502*	*5.7722*	*1.8476*
Base scenario − 5%	16.3129	5.7148	0

The European put options at long maturities under the VG economy appear to be quite sensitive to changes in the volatility parameter, modestly sensitive to the skewness parameter and least insensitive to the kurtosis parameter.

As volatility increases, the price increases, significantly so. This intuitively makes sense since in a more volatile economy (high volatility) the prices are expected to be higher as was noted in Section 4.4. The put option price appears to be an increasing function of the kurtosis parameter. As the parameter increases, the price increases.

If we consider the theta parameter, as it becomes more negative, which is indicative of greater negative skewness, the prices increase signalling a skewness premium in the prices. From the base scenario to a 1 % increase, the price decreases and when the tweak is by 5 % the price has increased over the base scenario. This makes intuitive sense since theta is a measure of skewness. As we move from the base scenario and add a positive tweak, the skewness is decreasing and this reflects itself in the price through a price decrease. A theta parameter of 0 (a tweak of about +1.5%) yields a price of 5.2124 in which case we are working with the symmetric VG case discussed in Madan and Seneta (1990) which yields the lowest price. After this, positive skewness sets in and the price starts to increase to adjust for the skewness premium.

It is also worthwhile to note that the analysis above, with an adjustment of $\pm$ 5 % on the parameters, will lie outside the 95 % confidence intervals in Table 4.17 other than for the ν parameter. However, the prices don't show alot of sensitivity to the ν parameter hence the overall statistical confidence from the above results is above the commonly accepted 95 %. The only concern will then be the correlation risk since in practice the parameters don't change independently and a change in one parameter could have a ripple effect on the rest. The issue of correlation is however outside the scope of this research.

4.6.6. Variance-Gamma model applied to pricing the GMxBs

The mortality rates from ASSA in Tables 4.7 and 4.8 are applied in this section to evaluate the GMxBs prices.

4.6.6.1. Guaranteed Minimum Maturity Benefit

As in the RS setting, we assume that the guarantee is a roll-up maturity guarantee with a rate of g_M *per annum* and that the policyholder neither surrenders nor withdraws from the account.

Table 4.19: 10-year 5% GMMB charges under the VG economy

Age (x)	g_M = 5 %	
	Male	**Female**
50	24.3212	26.3395
55	20.9931	22.5787
60	18.9030	22.5076
65	16.8520	22.7264

Table 4.20: 10-year 10% GMMB charges under the VG economy

Age (x)	g_M = 10 %	
	Male	**Female**
50	105.6165	114.3808
55	91.1639	98.0494
60	82.0873	97.7408
65	73.1810	98.6907

The figures below show price behaviour on a stand-alone and a comparative setting.
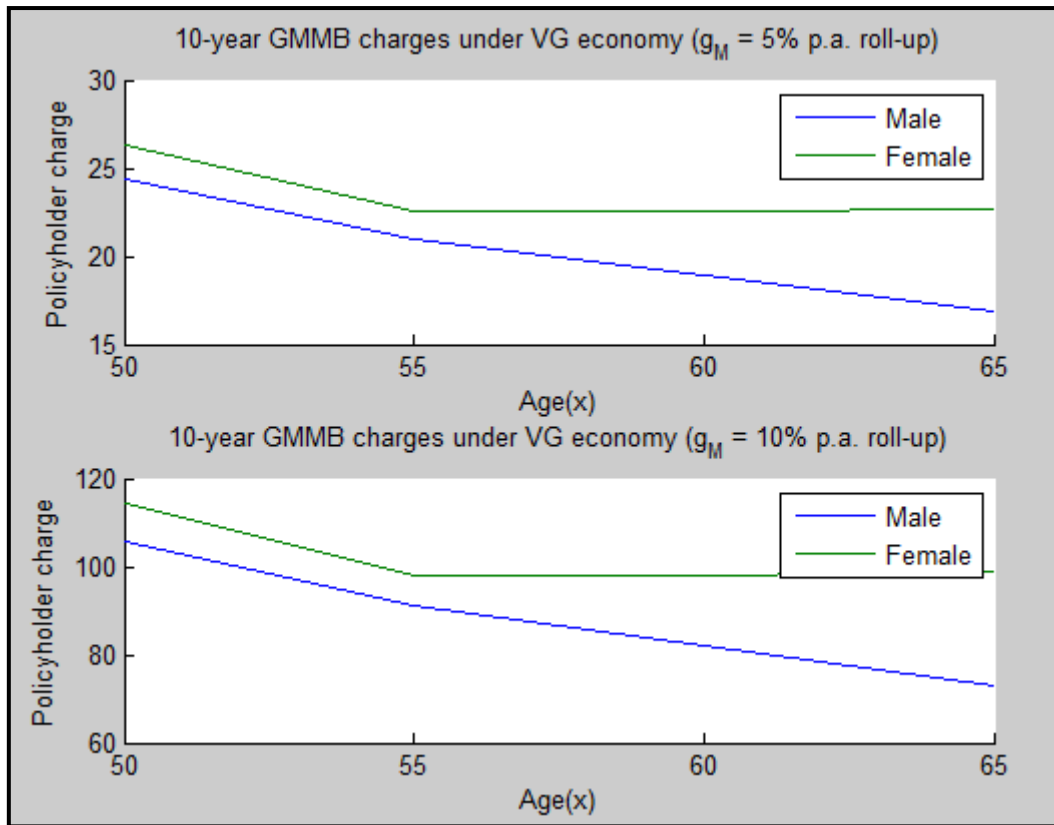
Figure 4.21: 10-year GMMB charges under the VG economy
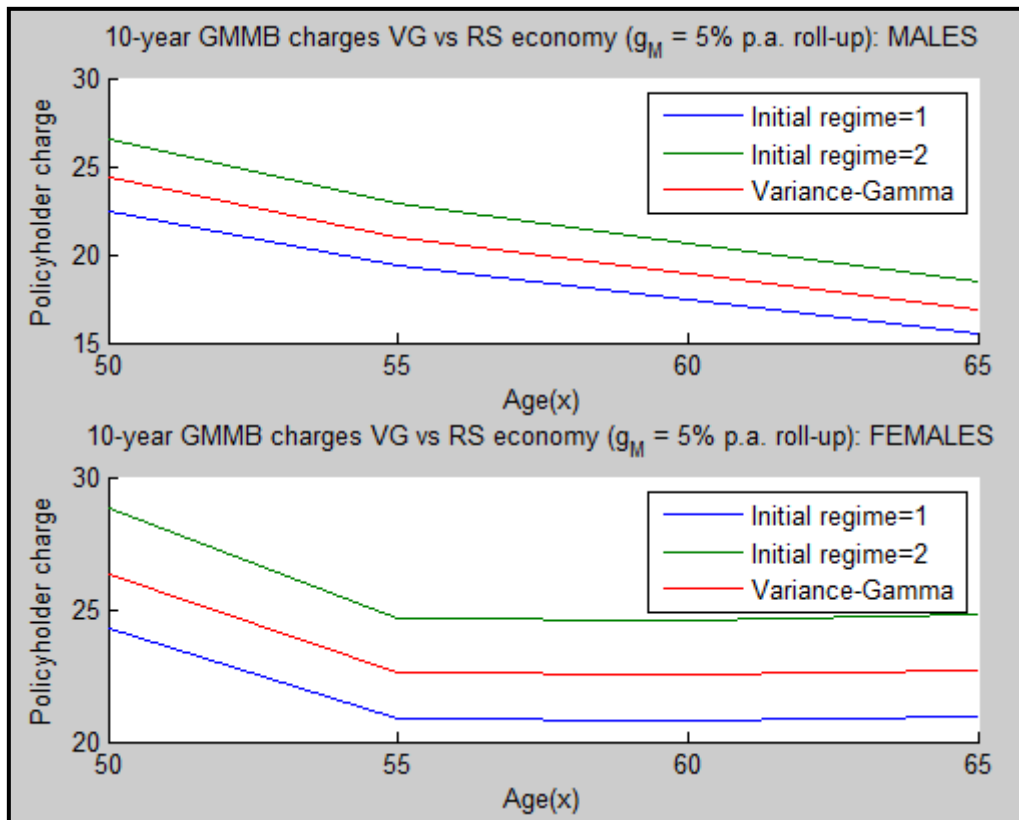
Figure 4.22: 10-year 5% GMMB charges: VG versus RS economy

In Figure 4.21, it is apparent that a higher guarantee rate results in a higher guarantee charge and owing to higher mortality rates for males over females, the charges for the latter are higher given the same starting age x across all the policy inception ages considered.

A comparative plot is shown in Figure 4.22 from which it is noteworthy that when the guarantee is 5 % the charges lie between the regime 1 and regime 2 charges obtained in the RS model.

If the guarantee rate is 10 % however, the charges obtained under the VG economy are slightly greater than the charges obtained in the RS setting as shown in the plots below.
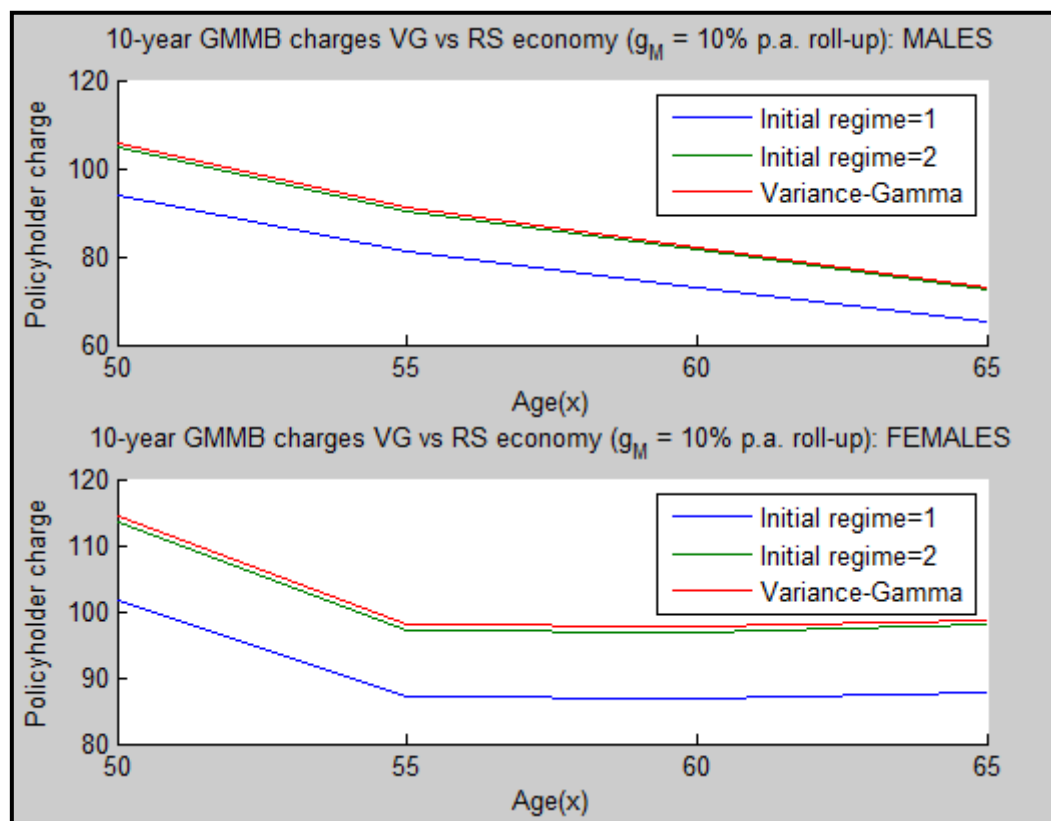
Figure 4.23: 10-year 10% GMMB charges: VG versus RS economy

This is explained by the fact that a higher guarantee rate increases the probability of the index value at maturity being less than the guaranteed amount. The chances of the option 'exercise' are thus higher and this should result in a higher charge. In the RS model, approximately 60% of the time, the economy is in regime 1 (low volatility, high return regime) and this could also have filtered through the index dynamics resulting in the RS model having higher values for S_T than the VG model. The VG model assumes the existence of one economy but captures extremes at the tails.

4.6.6.2. Guaranteed Minimum Death Benefit

The GMDB charges calculated using the ASSA basis are as shown in Table 4.21:

Table 4.21: 10-year 0% GMDB charges under the VG economy

Death (maturity) in:	$g_D = 0\%$			
	Female		Male	
	Age (x) = 50	Age (x) = 60	Age (x) = 50	Age (x) = 60
1st year	**0.093730**	**0.229853**	**0.207994**	**0.508723**
2nd year	0.093597	0.22956	0.207073	0.503681
3rd year	0.086108	0.210885	0.189687	0.456727
4th year	0.077493	0.189448	0.169795	0.402162
5th year	0.06854	0.167597	0.149609	0.346999
6th year	0.060654	0.147854	0.131569	0.297575
7th year	0.052923	0.128733	0.1142	0.250806
8th year	0.046384	0.112336	0.099525	0.21152
9th year	0.040585	0.097637	0.086467	0.177293
10th year	0.035261	0.083933	0.074551	0.147081
Maximum charge	**0.09373**	**0.229853**	**0.207994**	**0.508723**

In the case where the guarantee rate is 0 %, the charge is highest if death occurs in the first year with the charge falling across the years and being the least if death occurs in the 10th year. This is true for both males and females with the charges for the former being higher compared to their female counterparts for the same age at inception of a policy. This is attributable to the higher mortality rates of the males over the females and hence a higher likelihood of the guarantee maturing.

Table 4.22: 10-year 5% GMDB charges under the VG economy

Death (maturity) in:	$g_D = 5\%$			
	Female		Male	
	Age (x) = 50	Age (x) = 60	Age (x) = 50	Age (x) = 60
1st year	0.093521	0.229341	0.20753	0.50759
2nd year	0.129407	0.317387	0.286297	0.696383
3rd year	0.15095	0.369688	0.332527	0.800657
4th year	0.166954	0.408155	0.365813	0.866434
5th year	0.178552	0.436601	0.389741	0.903957
6th year	0.188185	0.45873	0.408205	0.923253
7th year	0.195456	0.475438	0.421765	**0.926278**
8th year	0.201666	0.488409	0.432712	0.919635
9th year	0.207301	0.49871	0.441658	0.905578
10th year	**0.21182**	**0.504207**	**0.447845**	0.88355
Maximum charge	**0.21182**	**0.504207**	**0.447845**	**0.926278**

If the insurer is in a VG economy with an annual guarantee rate of 5 %, the guarantee charge is highest if death occurs in the 10th year showing the impact of a guarantee rate. Death in any period before this will be less expensive to the company.

A comparison of the VG GMDB maximum charges with those obtained in the regime switching economy shows the VG charges sandwiched between regime 1 and regime 2 charges as per the bubble plot in Figure 4.24 below.
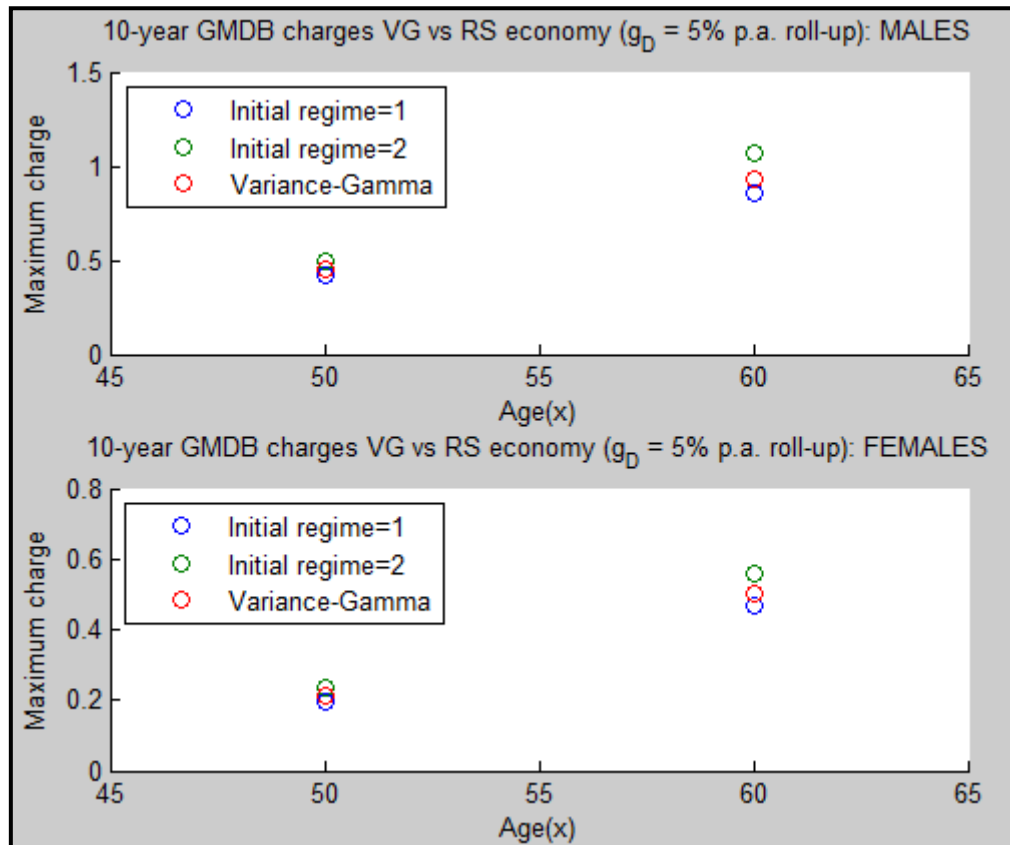
Figure 4.24: 10-year 5% GMDB charges: VG versus RS economy

4.7. Variance-Gamma Scaled Self-Decomposable process

The VG process presents itself as a credible tool in view of the kurtosis and skewness observed on analysing the returns from most markets (see for example Cepni et al. (2013)).

The results above show that the VG model, applied to embedded options, is consistent when compared with the regime switching model. The latter has been extensively applied in the actuarial world but given that the former explicitly accounts for skewness and kurtosis, it presents itself as a possible choice in the search for a more refined framework to be applied in the uncertain world of embedded option pricing.

However, the major problem with the VG model (and other Lévy processes) is the assumption of independent increments. This results in the VG density approaching the normal density as the time horizon increases. For example if the parameters are taken as constant and the time horizon increased on the VG density function, it approaches the normal density function over time which is captured in Brigo et al. (2007) and shown in the plot below.
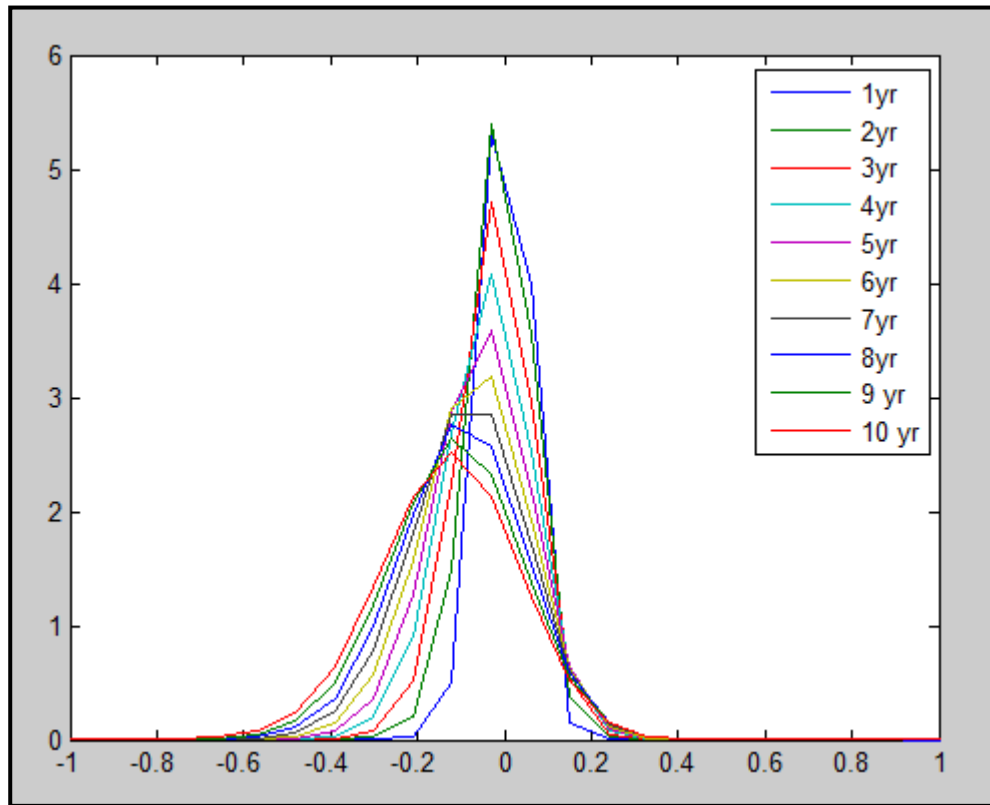
Figure 4.25: VG density behaviour as the time horizon increases

The accurate pricing of actuarial products which have long tenors thus needs a model that will not only capture the skewness and kurtosis, as VG does, but also preserve them given that these moments are independent of the time horizon.

Carr et al. (2007) discuss Self-Decomposable (SD) processes in option pricing of which the Variance-Gamma Scaled Self-Decomposable (VGSSD) process is one of them.

Definition 4.11 (Carr et al. 2007)

The distribution of a random variable X is said to be self-decomposable if for any constant c, $0 < c < 1$, there exists an independent random variable $X^{(c)}$ such that

$$X = cX + X^{(c)}.$$ (4.52)

i.e. the random variable has the same distribution as the sum of cX, a scaled down version of itself, and an independent residual random variable $X^{(c)}$.

The authors show that the VGSSD model is better able to capture the variation over time of option prices than does the VG model. O'Sullivan and Moloney (2010) consider the VGSSD model in an actuarial modelling context. In their paper, they note that the VGSSD model is able to preserve both the skewness and kurtosis which are independent of the time horizon.

The VGSSD process can be constructed from a VG process by defining the scaled stochastic process $X(t)$ such that it is equal in law $t^\gamma X_{VG}(1)$ where $X_{VG}(1)$ is a VG random variable at unit time as discussed in O'Sullivan and Moloney (2010).

In this context, the characteristic function of $X(t)$ is important and is given by:

$$\phi_{X(t)}(u) = \phi_{X_{VG}(1)}(ut^\gamma) = (1 - iut^\gamma v\theta + \frac{1}{2}u^2 t^{2\gamma} v\sigma^2)^{-\frac{1}{v}}. \qquad (4.53)$$

The moments of $X(t)$ are then derived as:

$$E[X(t)] = \theta t^\gamma,$$

$$Var[X(t)] = (\sigma^2 + \theta^2 v)t^{2\gamma},$$

$$Skew[X(t)] = \frac{(3\sigma^2\theta v + 2\theta^3 v^2)}{(\sigma^2 + \theta^2 v)^{\frac{3}{2}}}, \qquad (4.54)$$

$$Kurt[X(t)] = 3 + 3v\left[2 - \frac{\sigma^4}{(\sigma^2 + \theta^2 v)^2}\right].$$

and the stock index dynamics as:

$$S(t) = S(0)e^{(\mu+\omega)t + X_t^{(VGSSD)}} = S(0)e^{(\mu+\omega)t + X(t; \sigma, v, \theta, \gamma)}. \qquad (4.55)$$

However, even if the VGSSD model is applied in the pricing of the GMxBs, it does not capture stochastic volatility just like the VG model. The lack of stochastic volatility in a VGSSD model poses a need for a model that will deal successfully with the pricing of these options while incorporating stochastic volatility.

4.8. Stochastic Volatility Variance-Gamma process

The simultaneous pricing of long-term and short-term contracts, of which the GMxBs are a particular case, needs a model that has stochastic volatility and a jump component with the latter being modelled using the VG process. Indeed, Carr et al. (2003) have pointed out the need for stochastic volatility and jump models if option pricing is to consider variation in both the shorter and longer terms.

In the foregoing, the ideas discussed in Fiorani (2004) are used to describe the Variance-Gamma Stochastic Volatility (VGSV) framework in the endeavour to incorporate both jumps and stochastic volatility in the pricing.

4.8.1. Variance-Gamma Stochastic Volatility (VGSV) framework

Carr et al. (2003) introduced stochastic volatility into the VG framework using, inter alia, the well-known mean-reverting Cox-Ingersoll-Ross (CIR) framework. The introduction of stochastic volatility is done by randomly changing the business time of the VG process which is synchronous to changing the volatility randomly. As such, the CIR model so introduced is seen as a stochastic clock modelling the random business time and assumes that the flow is continuous but with a mean-reverting trend.

The process generated, VGCIR, is achieved by subordinating the VG to the time integral of a CIR process as discussed in Carr et al. (2003), Fiorani (2004) and Rachev et al. (2011). The authors' model theory follows here below.

We start with the VG model discussed in Section 4.6.2 where we have the VG process $X_t^{(VG)} = \theta_{VG} G_t + \sigma_{VG} W_{G_t}$. In this setting, we use an equivalent parameterisation of the VG model where we let:

$$C = \frac{1}{\nu},$$

$$G = \left(\sqrt{\frac{\theta^2 \nu^2}{4} + \frac{\sigma^2 \nu}{2}} - \frac{\theta \nu}{2} \right)^{-1},$$

$$M = \left(\sqrt{\frac{\theta^2 \nu^2}{4} + \frac{\sigma^2 \nu}{2}} + \frac{\theta \nu}{2} \right)^{-1}.$$

(4.56)

The log of the characteristic function of X_1 in this parameterisation is then given by:

$$\psi_{VG}(u; C, G, M) = C \log \left(\frac{GM}{GM + (M - G)iu + u^2} \right).$$

(4.57)

The intrinsic time process is subordinated in the VG process. We thus take the VG process $(X_t^{(VG)})_{t \geq 0}$ and then change the physical time t to the subordinator process $\upsilon = (\upsilon_t)_{t \geq 0}$.

Let $\upsilon(t)$ be the instantaneous rate of time change and solution to the stochastic differential equation (SDE):

$$d\upsilon = \kappa(\eta - \upsilon)dt + \lambda\sqrt{\upsilon}\, dW,$$

(4.58)

where:

$W(t)$ is a standard Brownian motion,
κ is the rate of mean reversion,
η is the long run rate of time change,
λ the volatility of the time change.

The process is stochastic and this captures stochastic volatility whereas the mean-reverting property reproduces the volatility clustering that has been empirically observed in most markets.

Rachev et al. (2011) note that there is no closed-form solution for the SDE but the characteristic function of the new clock, $\Upsilon(t)$, is known and thus we have,

$$\Upsilon(t) = \int_0^t v(u)du\,, \tag{4.59}$$

whose characteristic function is given by

$$\phi(u,t,v;\,\kappa,\eta,\lambda) = \frac{e^{\kappa^2\eta t/\lambda^2 + 2viu/\left(\kappa+\lambda\coth\left(\gamma t/2\right)\right)}}{\left(\cosh\left(\frac{\gamma t}{2}\right) + \frac{\kappa}{\gamma}\sinh\left(\frac{\gamma t}{2}\right)\right)^{2\kappa\eta/\lambda^2}}, \tag{4.60}$$

where:

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

The stock index price process $(S_t)_{t\geq 0}$ of the stochastic volatility VG process model described is then:

$$S_t = S_0 e^{\left(X(\Upsilon(t) - \Upsilon(t)\psi_X(-i)\,\right)}, \tag{4.61}$$

where:

$$X(\Upsilon(t)) \text{ is a VG process } X(t) \text{ time-changed with } \Upsilon(t).$$

This is then used to price options which can be done capably using the Fast Fourier transform technique or a Monte Carlo simulation approach.

The base parameters used in the initial fitting of this model are the unit parameters which are then tweaked to assess the sensitivity of the price to changes in these parameters:

Table 4.23: VGSV model estimation base parameters

Parameter	κ	η	λ
Base parameter	1	1	1

The Monte Carlo simulation approach discussed in Kienitz and Wetterau (2012) is used where we first generate asset paths through a discretisation of the VGCIR process. The paths are then used as inputs to price a put option with the additional parameters in Table 4.24. The authors show that the relative errors between this approach and the FFT technique are minimal.

Table 4.24: VGSV model option modelling parameters

Parameter	θ	ν	σ	r	S_0	K	T
Value	-0.0148	0.4461	0.0544	0.0088	R 1000	R 1000	10 years

In the VGCIR framework we then obtain:

Table 4.25: European put option prices under the VGCIR model for varying κ

	European put option price
$\kappa = 0.275$	27.4485
$\kappa = 1$	7.1820
$\kappa = 1.25$	6.6876

If we hold the κ and λ fixed, for varying η we obtain:

Table 4.26: European put option prices under the VGCIR model for varying η

	European put option price
$\eta = 0.5$	1.5012
$\eta = 1$	7.1820
$\eta = 2$	27.5972

Finally, if we fix κ and η and vary the λ parameter we obtain:

Table 4.27: European put option prices under the VGCIR model for varying λ

	European put option price
$\lambda = 0.5$	6.1506
$\lambda = 1$	7.1820
$\lambda = 1.5$	18.9073

The results above are then used to guide the final choice of the three parameter estimates with hindsight from RS and VG frameworks that the price so obtained should be in the vicinity of R 7 (ideally $\in$ (R 6, R 8)). This is reflective of the balance that has to be struck in real life between the results that are obtained from simulations and the more intuitive sense that is drawn from experience.

Table 4.28: VGCIR model final parameter estimates

Parameter	κ	η	λ
Parameter estimate	1.25	1	1

The price of a 10-year European call and put option is then obtained as:

```
............10-year VGCIR European call option price................
                    R 658.8432
............10-year VGCIR European put option price................
                    R 6.6876
```

Figure 4.26: 10-year VGCIR European style option prices

A Monte Carlo (MC) simulation is then done using these parameters to estimate the price of a 10-year European put option as a penultimate step in calculating the charge for the embedded guarantees.

Each MC run is done using 100 batches with each batch having 3000 stock price path simulations essentially 3000 option prices per batch and the put option prices so obtained are:

Table 4.29: Monte Carlo simulation of European put options under VGCIR model

Run	10-year European put option price
1st run	6.4005
2nd run	6.4024
3rd run	6.4688
4th run	6.4090
5th run	6.3702
6th run	6.3908

4.8.2. Variance-Gamma Cox-Ingersoll-Ross model applied to pricing the GMxBs

The pricing and analyses of the GMxB charges in the VGCIR economy follows with the mortality rates in Tables 4.7 and 4.8 once again being used.

4.8.2.1. Guaranteed Minimum Maturity Benefit

As in the other settings above, for the GMMB, we assume that the guarantee is a roll-up maturity guarantee with a rate of g_M per $annum$ and that the policyholder neither surrenders nor withdraws from the account.

Table 4.30: 10-year GMMB charges under the VGCIR economy

Age (x)	$g_M = 5\%$		$g_M = 10\%$	
	Male	Female	Male	Female
50	25.0355	27.1130	105.5694	114.3297
55	21.6096	23.2418	91.1232	98.0057
60	19.4581	23.1686	82.0507	97.6972
65	17.3469	23.3938	73.1484	98.6466

The charges generally decrease with increasing age and are once again higher for females over males given the same age at inception. As would naturally be expected, the higher guarantee rate of 10% results in higher guarantee charges.

A comparison of the charges so obtained vis-à-vis those of the regime switching economy show the VGCIR GMMB charges for the $g_M = 5\%$ contained between the high charges if the initial regime is regime 2 and the lower charges if the economy starts in regime 1.
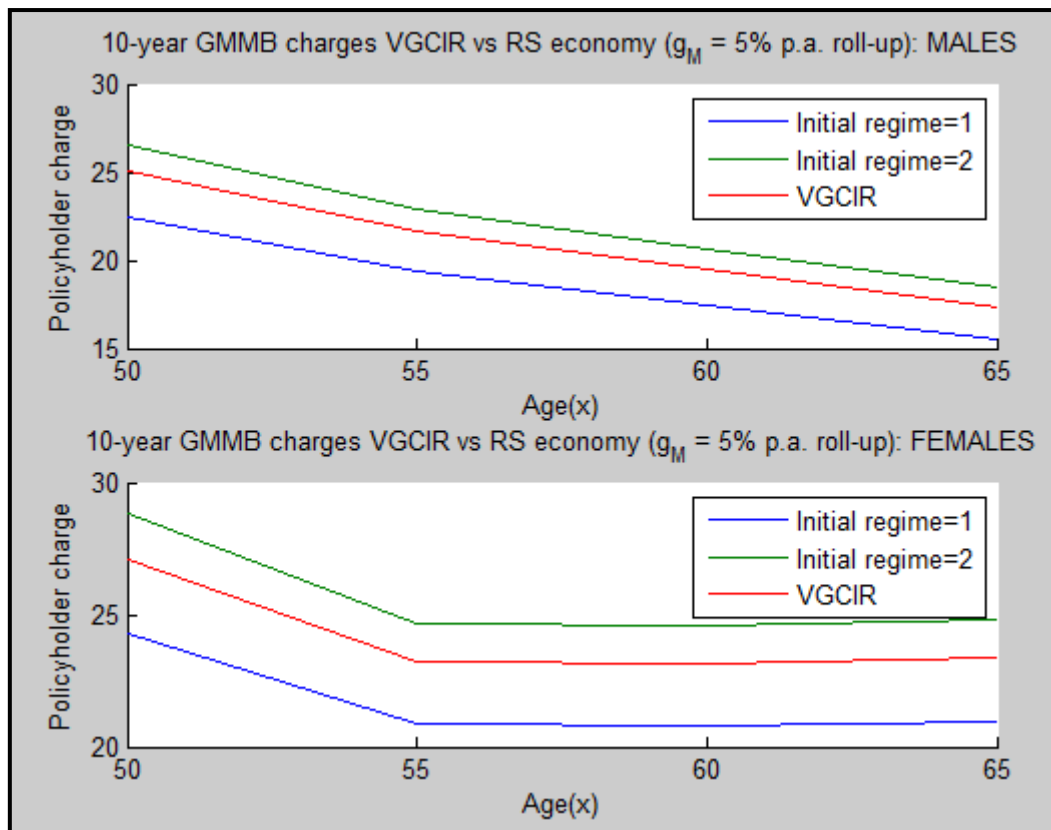
Figure 4.27: 10-year 5% GMMB charges: VGCIR versus RS economy

If we consider the $g_M = 10\%$ scenario, the charges are slightly higher in the VGCIR economy over the charges obtained in the regime switching economy.
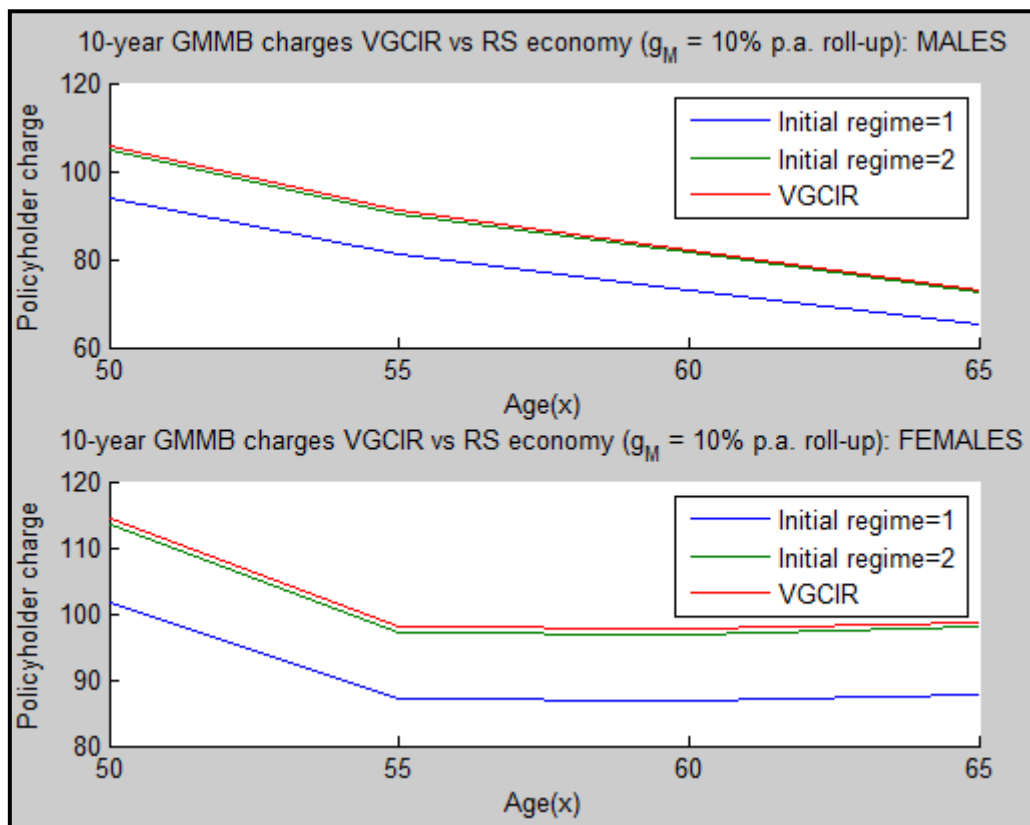
Figure 4.28: 10-year 10% GMMB charges: VGCIR versus RS economy

4.8.2.2. Guaranteed Minimum Death Benefit

The 10-year European put option prices obtained under the VGCIR setting are contained in the table below and show a downward trend across the years for the 0 % guarantee case and a humped trend for the 5 % guarantee case.

Table 4.31: Estimated GMDB charges under the VGCIR economy excluding mortality

Death (maturity) in:	Estimated put option price (0%)	Estimated put option price (5%)
1st year	29.6673	29.6155
2nd year	29.1460	40.1556
3rd year	25.4129	43.9194
4th year	21.3648	44.9162
5th year	17.6165	44.6743
6th year	14.6590	43.6037
7th year	11.7354	41.8569
8th year	9.6068	40.2069
9th year	7.8897	37.7359
10th year	6.3561	35.6068
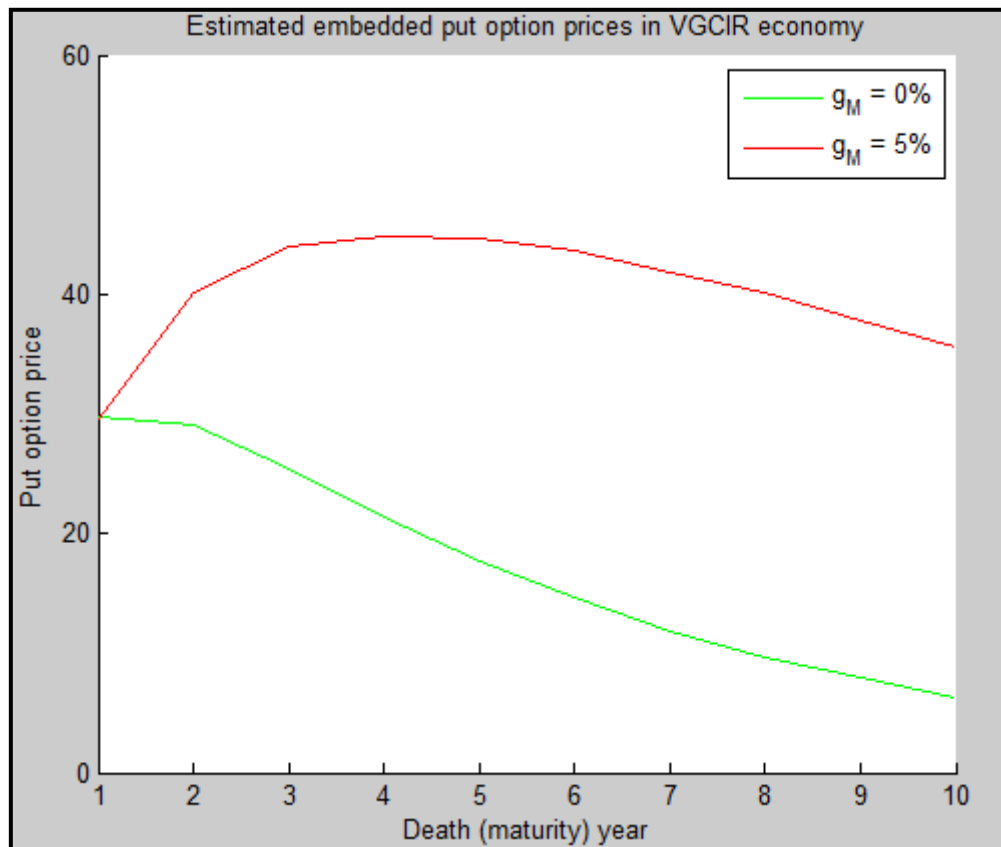
Figure 4.29: Estimated VGCIR economy embedded put option prices

The GMDB charges under the same framework calculated using the ASSA basis above are shown in the tables below:

Table 4.32: 10-year 0% GMDB charges under the VGCIR economy

Death (maturity) in:	$g_D = 0\%$			
	Female		Male	
	Age $(x) = 50$	Age $(x) = 60$	Age $(x) = 50$	Age $(x) = 60$
1st year	0.083958	0.205891	0.186311	0.45569
2nd year	**0.089806**	**0.220262**	**0.198686**	**0.48328**
3rd year	0.085388	0.209121	0.188101	0.452908
4th year	0.078317	0.191463	0.1716	0.406438
5th year	0.07025	0.171777	0.15334	0.355654
6th year	0.063696	0.15527	0.138168	0.3125
7th year	0.055476	0.134944	0.11971	0.262906
8th year	0.049405	0.119653	0.106008	0.225298
9th year	0.044182	0.106291	0.094131	0.193008
10th year	0.038756	0.092253	0.08194	0.16166
Maximum charge	**0.089806**	**0.220262**	**0.198686**	**0.48328**

In the case where the guarantee rate is 0 %, the charge is highest if death occurs in the second year with the charge falling across the years and being the least if death occurs in the 10th year. This is true for both males and females with the charges for the former being higher compared to their female counterparts for the same age at inception of a policy. As noted above, this is attributable to the higher mortality rates of the males over the females and hence a higher likelihood of the guarantee maturing.

If we assume that the guaranteed rate is 5 % p.a. then the charges obtained using the ASSA basis are:

Table 4.33: 10-year 5% GMDB charges under the VGCIR economy

Death (maturity) in:	$g_M = 5\%$			
	Female		Male	
	Age $(x) = 50$	Age $(x) = 60$	Age $(x) = 50$	Age $(x) = 60$
1st year	0.083812	0.205532	0.185985	0.454894
2nd year	0.12373	0.303463	0.273737	0.665834
3rd year	0.14757	0.36141	0.325081	0.78273
4th year	0.164649	0.402521	0.360763	0.854474
5th year	0.178149	0.435615	0.388861	0.901916
6th year	0.189466	0.461855	0.410985	0.929542
7th year	0.197869	0.481307	0.426971	0.937712
8th year	0.206773	0.500779	0.443672	**0.942928**
9th year	0.211321	0.508383	0.450224	0.923143
10th year	**0.21711**	**0.516799**	**0.45903**	0.905617
Maximum charge	**0.21711**	**0.516799**	**0.45903**	**0.942928**

If the guaranteed rate is 5 % p.a. the maximum charge occurs in year 10 and the charges are increasing across the years. This is attributable to the increasing guarantee amount as the per annum roll-up sets in.

If we consider how the GMDB charges in the VGCIR economy compare to those in the regime switching context at the 5 % level, we note the same trend as in the GMMB scenario where the maximum charges lie in between those of the regime-switching framework.
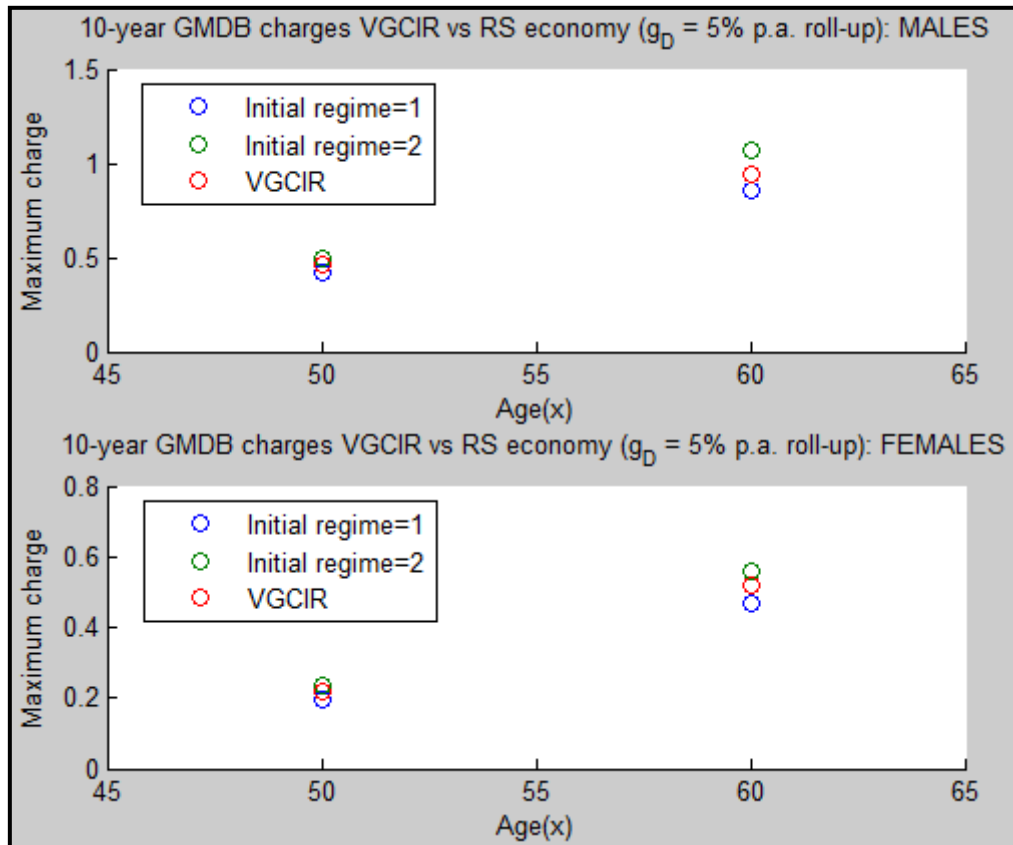
Figure 4.30: 10-year 5% GMDB charges: VGCIR versus RS economy

There has been a lot of research on the use of stochastic volatility models in the pricing of options and such have been shown to provide better and more realistic price processes for the underlying (see for example Hull and White (1987), Heston (1993), Ball and Roma (1994), Carr et al. (2001) and Ulmer (2010)). To the extent that such a framework does not deviate from regulatory, accounting and taxation guidelines, it is a plausible route to adopt in the pricing of the options. This will be another stride in walking across what Embrechts (2000) refers to as the financial bridge to actuarial pricing. The author notes that the correct pricing will be one that incorporates both financial theory and actuarial practices, a theme that this chapter has sought to present.

Risk Management of Life Contingent Embedded Options

Risk management is a complex issue in the financial world and much more so in a life insurance context. This chapter considers the hedging and risk measurement of the embedded options, which are key issues in risk management. The delta-hedging approach is used for the hedging and the Value-at-Risk and Expected Shortfall techniques implemented in Section 5.5 for the risk measurement.

5.1. Identifying GMxB risks

The GMxB products have risks embedded in them even if the approach used in the pricing is an almost perfect model. These include:

- Equity price risk
- Interest rate risk
- Credit risk
- Mortality/longevity risk
- Operational risk
- Basis risk
- Correlation risk

Equity price risk

This is the part of market risk that is caused by the equity price as a market risk factor. It arises from volatility in the equity prices and, "... refers to all assets and liabilities whose values are sensitive to changes in equity prices." (De Weert 2011).

Interest rate risk

The interest rate is a market risk factor whose risk is underpinned in, "... the concerns of the sensitivity of assets and liabilities to changes in the level of interest rate, the term structure of interest rates and interest rate volatility." (De Weert 2011).

Credit risk

This relates to those assets and activities that an insurance employs that have credit risk associated with them and do not form part of the trading activities of the insurance. If it does form part of the trading activities of the insurance, it is deemed market risk.

Mortality/longevity risk

Mortality risk is the risk that the policyholders die sooner than expected whereas longevity risk is the risk that they will live longer than expected. The Law of Large Numbers summarises to the conclusion that if an insurer sells a large enough number of policies then losses due to mortality will equal the expected loss. This is the assumption made in the subsequent analysis in this section; in particular that the insurer will diversify away this risk by selling a large enough number of policies.

Operational risk

This is the risk that arises from inadequate or failed internal controls and processes.

Basis risk

A hedge can contain risk only to the extent the available instruments allow but it is almost an impossibility to get a perfect hedge much more so in an incomplete market. The risk that remains after a hedge has been implemented is referred to as basis risk and is attributable to aspects such as the inability to do continuous time hedging due to infinite costs and differences between the chosen model and the exact model followed by the underlying's risk factors.

The above risks are interrelated in one way or another resulting in correlation risk.

5.2. Hedging the risks

The perfect pricing of an instrument in the financial world is, debatably, an exercise in futility if such a product cannot be hedged. The above analyses has considered the pricing of the GMxB products and the sensitivity of the benefits to tails of the account value distribution by using a jump model that accounts for skewness and kurtosis thereby being a realistic description of the real-world price dynamics.

The guarantees embedded in the products necessitate that the insurance company institute a risk management framework to manage the inherent risks. A hedging strategy is an imperative.

In this context, the hedging of instruments in a complete market is an easy task where the writer of the option, in this case the insurer, needs to purchase the replicating portfolio and they will be hedged. In an incomplete market setting such as the case at hand, additional considerations are necessary which results in fundamental hedge design considerations.

5.2.1.   Hedge design considerations

The choice of any hedging approach should result in a hedge that, inter alia:

- Takes into account appropriate charges.
- Reduces the profit and loss (P&L) volatility.
- Reduces the Capital-at-Risk in line with the Solvency Assessment and Management (SAM) and Solvency II guidelines.

Finkelstein and Holler (2009) discuss a methodology that can be used to derive a cash flow profile that will yield the hedge portfolio. This is shown pictorially in Figure 5.1 below:
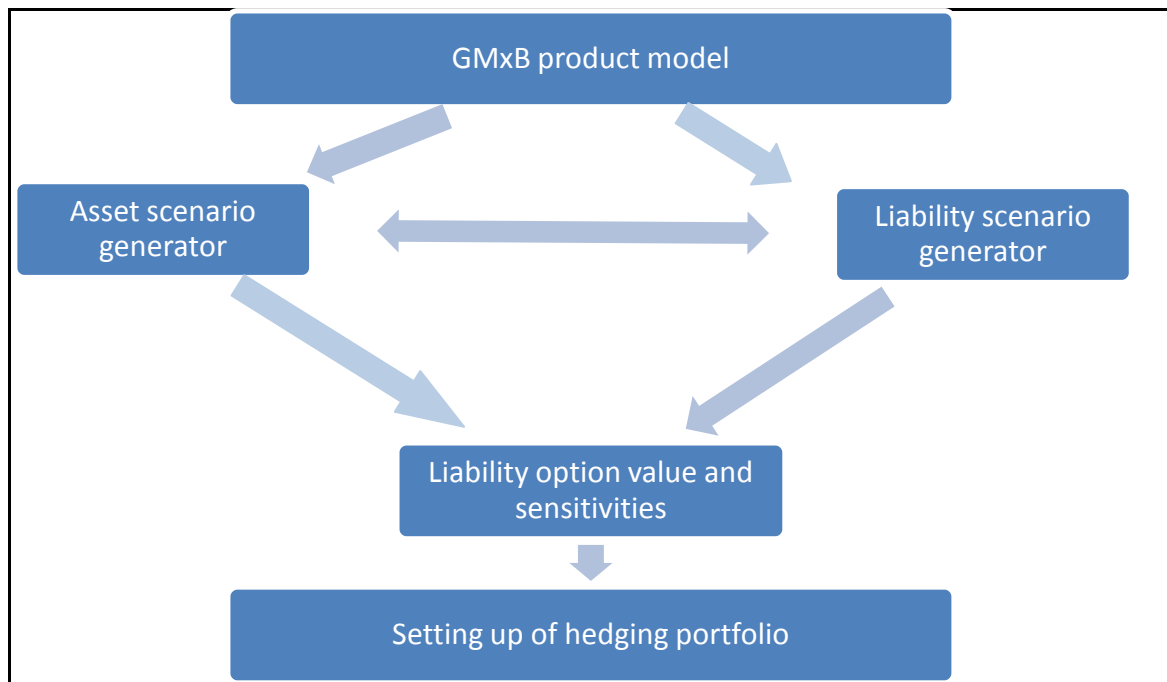
Figure 5.1: Methodology to setup a GMxB hedge portfolio

The research paper by Coleman et al. (2005) provides a basis from which the hedging, in a VGCIR economy, is considered in order to control the market risk embedded in the GMxBs.

5.2.2. Delta hedging

This is the hedging strategy that has been adopted by most in the financial world. It is a dynamic strategy that involves re-calculating the delta measure which is defined as the rate of change of the option price with respect to the underlying asset and is denoted by:

$$\Delta = \frac{\partial V}{\partial X}. \tag{5.1}$$

This immunises the portfolio against small changes in the underlying asset but requires continuous rebalancing with the resultant implication of high transaction costs, a potential inadequacy of this approach. This is applied in Section 5.3.

5.2.3. Risk minimisation hedging

The delta hedging approach is usually computed in a complete market setting. However, as noted above, markets are incomplete and in this setting the hedging strategy adopted should be as close as possible to the real-world price dynamics taking cognisance of the impossibility of continuous rebalancing, jump risk and volatility risk.

The consequence of the above remarks is that the choice of the hedging strategy is made in such a way as to reduce any commonly accepted measure of risk or any such measure of risk that is of most importance to the case at hand. Coleman et al. (2005) note that the goal of a risk minimisation hedging strategy is to compute an optimal hedging strategy that minimises the chosen measure of risk under a real-world price model.

The approach is a discrete hedging strategy and the authors note the challenges of using this method in a variable annuity setting, inter alia:

- The long maturity of insurance contracts with a possibility of model miscalibration and a danger of model risk,
- The modelling of stochastic implied volatilities,
- Sensitivity of the benefits to the tails of the account value distribution.

The model used should be able to accurately model the tails of the account value distribution hence the choice of the VGCIR model in this report.

5.3. Hedging using the Greeks

The Greeks are the conventional names given to the sensitivities of option prices to the various parameters that influence the price process including the underlying. These have formed the basis of any hedging carried out by many financial institutions in the past and present.

The Greeks hedging approach is the commonly used method in the SA insurance industry this consisting of the underlying and bonds in the hedging portfolio. Its main demerit is that it is a discrete time hedging framework therefore a need for rebalancing. Though in an ideal world a continuous time hedging strategy is desirable, this is not normally the carried out practice given that it is prohibitively costly thus the discrete approach still has industry relevance and is worth a consideration.

This approach however results in a discretisation bias since the hedge is considered at discrete time points. The use of finer discrete time points means a need for more frequent rebalancing and hence higher transaction costs for the hedging team. If the rebalancing time points are too distant from each other, the option writer is exposed to higher market risk due to changes in the underlying risk factors. A compromise has to be found between the need for a more economical approach and one that is frequent enough in rebalancing so as not to make the exercise redundant. The impact of this is assessed using varied rebalancing periods in Coleman et al. (2005) and has guided the choice of the 6-month rebalancing period adopted in the foregoing in the endeavour to achieve a compromise between rebalancing and transaction costs.

South Africa has the challenge that the markets are not liquid enough and the question of whether such a rebalancing is practical in this context persists. The SA markets contain just as many instruments as the rest of the world, more so the important ones, with the exception that it is in a less liquid setting. However, even in the more developed markets the long tenor options market is illiquid. The players adopt the discrete rebalancing approach using the short tenor options thus this approach still does have credence. Further, the existence of only a limited number of derivatives that would be used to hedge in the SA market does not materially affect the foregoing since the usual vanilla options suffice.

In the analysis that follows, the method used to estimate the Greeks is the perturbation approach discussed in McLeish (2005) where the parameter is perturbed by ϵ upwards and downwards under a Monte Carlo simulation and the relevant Greek calculated.

In the delta calculation we have:

$$\Delta = \frac{\partial V}{\partial X} \cong \frac{V^{up} - V^{down}}{2\epsilon},$$

(5.2)

and in the gamma calculation we have:

$$\Gamma = \frac{\partial^2 V}{\partial X^2} \cong \frac{V^{up} - 2\,V + V^{down}}{\epsilon^2}.$$

(5.3)

McLeish (2005) notes the potential demerits of using the perturbation approach in the analysis these being the use of more computer time given that there are three times the number of simulations and the potential bias introduced in the choice of ϵ. However, given that there is no derivable closed form solution in the VGCIR setting, any approach is likely to use just as much computer time as the perturbation approach and, if needs be, one can use common random numbers in the simulation of V^{up} and V^{down} to reduce the computer time. The longer-term nature of the embedded guarantees also means that any potential biases in the choice of ϵ can be ameliorated.

Using the VGCIR model considered in Chapter 4 for a 10-year European put option with a strike of R 1000 and with all the other parameters unchanged other than for a R 100 perturbation on the spot price, S_0, a Monte Carlo sample of this yields:

```
..........10-year VGCIR European put option price (S0 = 1000)...............
   6.1459    6.4729    5.9445    6.5219    6.2051

..........10-year VGCIR European put option price (Epsilon = +100)..........
   4.6648    4.2391    4.2296    4.8503    4.8162

..........10-year VGCIR European put option price (Epsilon = -100)..........
   9.0256    8.4039    9.1672    7.5731    8.2379

..........Monte Carlo Delta estimates...............
  -0.0218   -0.0208   -0.0247   -0.0136   -0.0171

..........Monte Carlo Gamma estimates...............
 1.0e-003 *

   0.1399   -0.0303    0.1508   -0.0620    0.0644
```

Figure 5.2: Monte Carlo estimation of the Greeks in a VGCIR economy

This is then repeated over an increased number of simulations and the average taken to obtain a credible estimate of both the delta and the gamma from which we obtain:

Table 5.1: Estimated Greeks in a VGCIR economy

Greek	Delta (Δ)	Gamma (Γ)
Estimate	-0.0215	0.00009118

If this is considered over a range of different spot prices for a fixed strike of R 1000, the behaviour of the delta and gamma is as shown in the plots below:
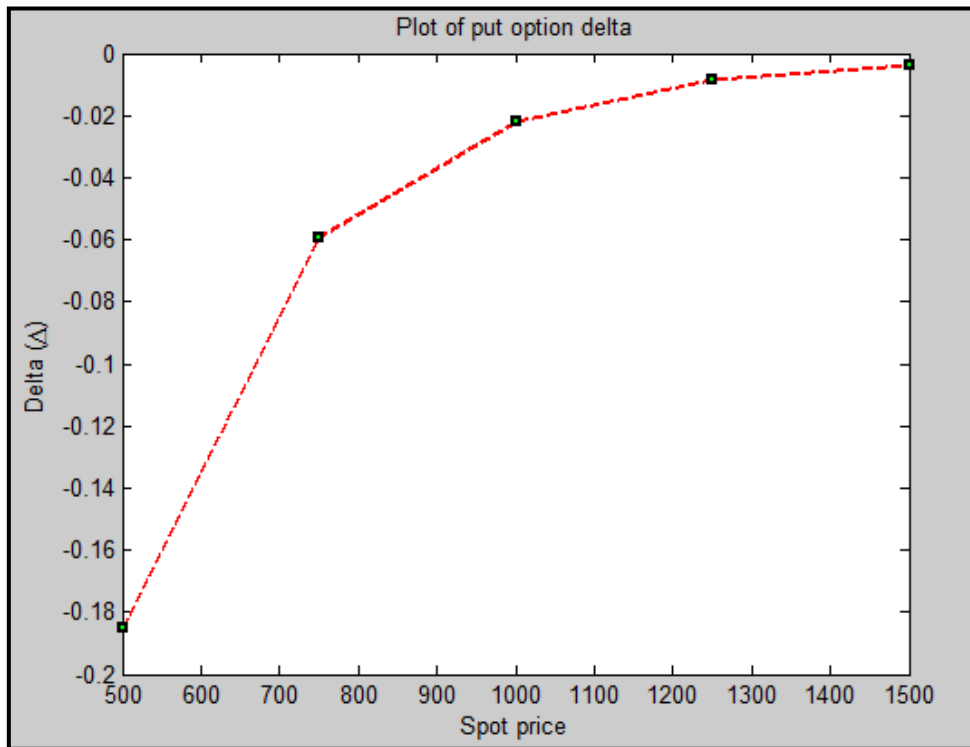
Figure 5.3: GMMB delta behaviour as the spot price varies in a VGCIR economy
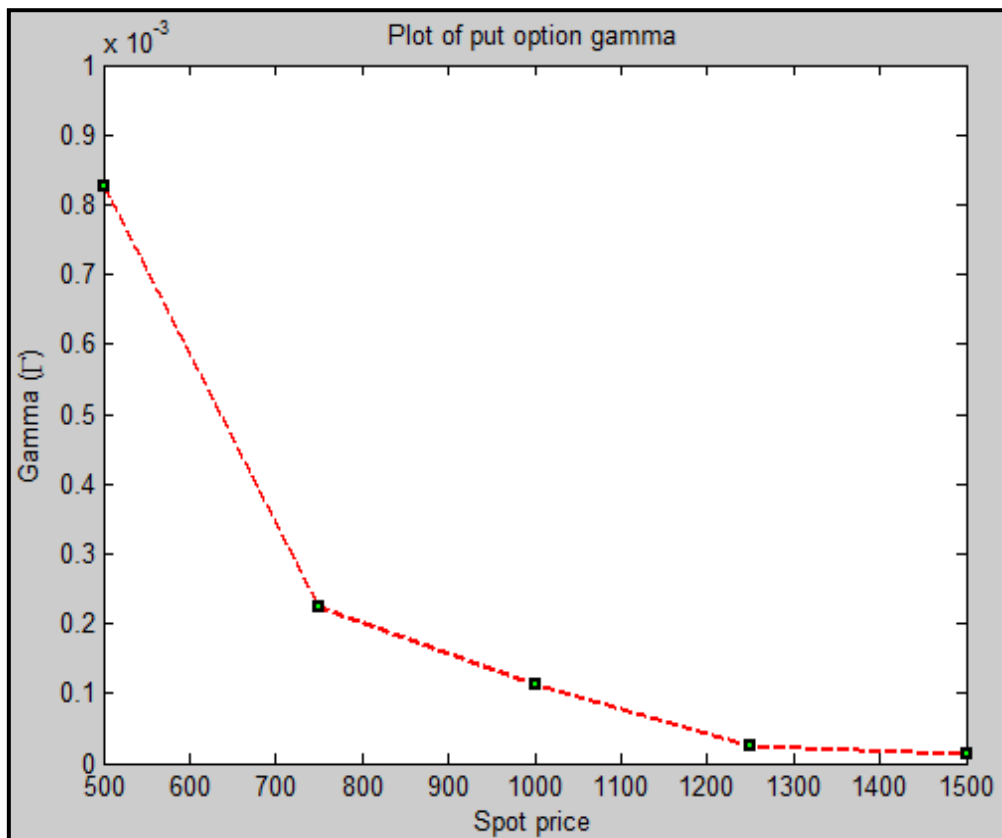
Figure 5.4: GMMB gamma behaviour as the spot price varies in a VGCIR economy

5.4. Constructing the hedge portfolio

The generic approach discussed in Coleman et al. (2005) is used in constructing the value of the hedging portfolio where at any time t_k, the value is denoted by $P\,(t_k)$:

$$P\,(t_k) = \; U\,(t_k).\varrho(t_k) + \; \eta\,(t_k), \tag{5.4}$$

where:

$U\,(t_k)$ is the value of the risky assets,
$(\varrho(t_k), \eta\,(t_k))$ is the optimal holding in risky assets $U\,(t_k)$
and riskless bonds respectively.

The optimal hedging strategy at time t_k, $(\varrho(t_k), \eta\,(t_k))$ is liquidated at time t_{k+1} and a new strategy $(\varrho(t_{k+1}), \eta\,(t_{k+1}))$ formed. The cumulative gain of the trading strategy at time t_k, $G\,(t_k)$ is given by:

$$G\,(t_k) = \sum_{j=0}^{k-1} \left(U_j(t_{j+1}) - U\,(t_j)\right).\varrho(t_j). \tag{5.5}$$

The cumulative cost of the trading strategy at time t_k, $C(t_k)$, is then given by:

$$C(t_k) = \; P(t_k) - G(t_k). \tag{5.6}$$

and this strategy will be self-financing if:

$$C(t_{k+1}) - C(t_k) = 0, \;\; k = 0,1, \dots, M-1. \tag{5.7}$$

In an incomplete market setting however, the strategies so adopted will not normally be self-financing and a criterion must be chosen in making the choice of the hedging strategy to be adopted. In the analyses that follows, if the strategy adopted with the rebalancing will meet the obligations of the insurer, this is considered adequate. This can be relaxed to show the impact of considerations such as taxes, transaction costs and restricted borrowing and lending at a variable interest rate but is beyond the scope of the current research.

5.4.1. Delta (Δ) – hedging portfolio

Using the results above, it is then possible to construct a hedge portfolio in a Δ - hedging context that consists of the underlying and bonds, be it the SA government bonds or the corporate bonds. This is arrived at using the relation:

$$V(t_i) = B(t_i) + \; \Delta(t_i)S(t_i), \tag{5.8}$$

where:

$B(t_i)$ is the number of units of the bonds,
$\varphi(t_i) = \Delta(t_i)$ is the units of the underlying asset.

The bonds are arrived at using the relation:

$$B(t_i) = V(t_i) - \Delta(t_i)S(t_i). \tag{5.9}$$

The hedging portfolio so created needs to be rebalanced at discrete times and in the foregoing it is assumed that the rebalancing is done semi-annually.

Coleman et al. (2005) and Kienitz and Wetterau (2012) discuss such a rebalancing where they note that at the rebalancing time point t_{i+1}, $t_{i+1} > t_i$, the gain is given by:

$$G(t_{i+1}) = \Delta(t_i)\big(S(t_{i+1}) - S(t_i)\big) + B(t_i)(D(t_{i+1})^{-1} - 1), \tag{5.10}$$

and the loss of the hedging strategy at time t_i is given by:

$$L(t_i) = \sum_{j=1}^{i} G(t_j) + \big(B(t_0) + \phi(t_0)S(t_0)\big) - V(t_i). \tag{5.11}$$

with the usual convention where the losses are denoted as negative.

Maré (2009) discusses the use of the delta-hedging approach in the SA equity market and concludes that it has an ongoing relevance. In establishing, rebalancing and maintaining the hedge, the author notes that tracking the hedge borrowing cost (HBC) is critical and this cost is given by:

$$HBC_{t_i} = HBC_{t_{i-1}}e^{r\delta t} + S(t_i)\big(\Delta(t_i) - \Delta(t_{i-1})\big). \tag{5.12}$$

Given a particular stock price path process, it is possible to consider such a rebalancing and we consider a GMMB setting with $g_M = 5\%$. In this case we first need to have a stock price path to be used. We have the tabular representation shown below where the choice of the stock price path is the average of the 10 stock price paths generated at random using the MC technique in a VGCIR economy.

Table 5.2: Simulated stock price paths in a VGCIR economy

Time (Years)	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$	$S(t_i)$
0	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00	1,000.00
0.5	1,071.70	1,033.30	830.10	1,239.00	1,039.20	1,039.70	1,181.20	1,085.10	1,178.70	1,042.30
1	1,034.60	1,385.60	994.70	1,402.20	1,115.50	816.30	960.30	1,114.20	1,620.60	1,337.30
1.5	1,071.70	1,424.30	892.60	1,631.00	1,184.70	869.50	1,144.00	1,375.10	1,510.10	1,749.90
2	1,186.10	1,020.30	908.20	1,821.20	1,230.90	721.50	1,113.80	1,096.60	1,639.80	1,814.30
2.5	1,272.20	1,199.90	938.80	1,975.60	1,201.90	635.10	1,277.50	1,310.40	1,714.20	1,626.70
3	1,722.70	1,301.70	1,275.20	2,478.50	1,163.20	634.00	1,537.30	1,485.10	2,211.80	1,443.50
3.5	1,762.40	1,300.70	1,181.40	2,491.20	1,303.80	822.30	1,471.40	1,491.10	1,835.50	1,522.10
4	1,862.80	1,599.40	962.30	2,818.60	1,469.70	621.40	1,484.90	1,468.00	2,009.50	1,724.60
4.5	2,186.80	1,766.70	833.90	3,568.00	1,134.00	626.90	1,916.40	1,585.30	2,202.00	1,600.80

5	1,965.10	1,907.00	782.60	3,741.50	1,173.90	633.30	2,039.90	1,878.70	2,376.40	2,061.20
5.5	2,369.20	2,197.20	875.40	4,171.70	1,079.00	709.30	1,992.10	2,496.90	2,740.80	2,098.30
6	3,242.40	2,342.00	966.10	3,781.20	1,248.60	778.20	1,850.40	2,407.20	3,482.90	2,355.80
6.5	3,005.90	2,519.90	1,148.40	3,058.10	1,370.30	866.00	2,034.50	2,795.00	3,901.70	2,816.60
7	3,155.80	2,755.60	1,217.70	3,042.80	1,360.50	603.60	1,489.70	3,532.70	4,232.30	2,503.20
7.5	3,279.60	3,154.00	1,221.90	2,846.50	1,022.40	656.20	1,642.30	4,365.20	4,467.90	3,282.60
8	3,690.40	2,523.90	918.20	2,848.20	1,255.10	632.40	2,162.30	3,486.30	4,868.90	3,422.10
8.5	4,574.10	3,025.10	1,000.90	3,577.10	1,404.60	681.20	2,360.60	4,030.10	5,360.00	3,947.10
9	3,664.00	3,104.40	1,489.30	4,157.00	1,645.50	754.60	2,653.50	3,724.80	6,799.60	4,524.60
9.5	3,793.00	3,097.20	1,602.50	4,401.40	1,729.30	670.00	2,560.70	3,614.00	5,353.90	4,797.60
10	4,156.60	3,613.60	1,482.70	4,232.60	1,455.70	709.00	2,891.10	3,185.30	7,964.50	5,265.30

This gives the averaged stock price path shown in the table below which is used in the rebalancing. In the delta calculation however, the stock price path is refined more for accuracy purposes. It is assumed that such a refinement would not significantly change the stock prices at the discrete rebalancing points chosen but is crucial in the delta calculation given its round-off sensitivities.

Table 5.3: Semi-annually rebalanced delta (Δ) – hedged GMMB portfolio

Time to maturity (Months)	$S(t_i)$	$V(t_i)$	$\Delta(t_i)$	$B(t_i)$	Gain from risky asset
120	1,000.00	35.8264	-0.0916	127.4264	
114	1,074.03	33.2065	-0.0868	126.4323	-6.78115
108	1,178.13	29.2269	-0.0727	114.877	-9.03588
102	1,285.29	24.8812	-0.0607	102.8983	-7.79053
96	1,255.27	30.2118	-0.0758	125.3613	1.82221
90	1,315.23	29.1564	-0.0743	126.878	-4.54497
84	1,525.30	19.5563	-0.0461	89.87263	-15.60820
78	1,518.19	22.0768	-0.0558	106.7918	0.32777
72	1,602.12	20.108	-0.0477	96.52912	-4.68329
66	1,742.08	15.5095	-0.0389	83.27641	-6.67609
60	1,855.96	12.4531	-0.0335	74.62776	-4.42993
54	2,072.99	7.5301	-0.0197	48.368	-7.27050
48	2,245.48	5.0249	-0.013	34.21614	-3.39805
42	2,351.64	3.6567	-0.0108	29.05441	-1.38008
36	2,389.39	3.0202	-0.0091	24.76365	-0.40770
30	2,593.86	1.443	-0.0045	13.11537	-1.86068
24	2,580.78	1.0623	-0.0041	11.6435	0.05886
18	2,996.08	0.12595	-0.00042265	1.392243	-1.70273
12	3,251.73	0.010874	-0.00002691	0.098362	-0.10805
6	3,161.96	0.000008170	0.00000248	-0.00784	0.00242
0	3,495.64				0.00083
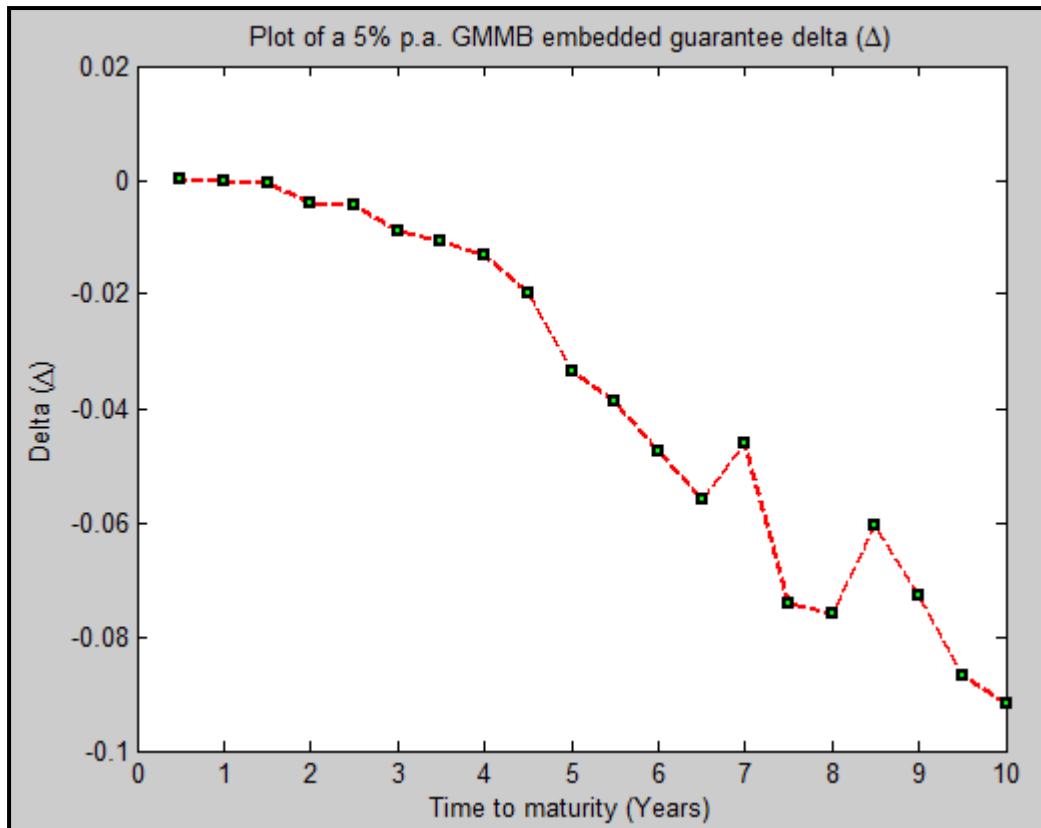
Figure 5.5: 5% GMMB delta behaviour as the time to maturity varies

The path chosen shows the behaviour of the hedging portfolio across the time to maturity if we assume a roll-up guarantee on the GMMB of 5 % per annum. The number of units in the underlying increases to zero as we approach maturity. This is explained by the performance of the index which outperforms the guarantee in this economy. The 5 % roll-up guarantee yields a maturity benefit after ten years of approximately R 1550 whereas the index would outperform this ending at R 3495 hence in financial parlance, the guarantee ends out-of-the money.

This means that the holding in the hedging portfolio should tend towards zero and is noticeable both from the table where $\Delta(t_i)$ and $B(t_i)$ are tending towards zero.

In the ten stock price paths used in getting the final price path, three of them (path 3, path 5 and path 6) would all result in the guarantee ending in-the-money and hence the holding in the hedging portfolio as we approach maturity would not tend to zero.

A higher roll-up guarantee rate would also result in more instances of the guarantee ending in-the-money but on the average, the guarantees are likely to end out-of-the money if we price and have the hedging portfolio's underlying as the ALSI index.

If we consider an averaged MC simulation of 20,000 paths, the stock price paths show a lot of stability in a MC sense as shown in Table 5.4 below and this would subsequently result in a smother hedging portfolio plot across the time to maturity.

Table 5.4: Averaged MC simulation stock price paths in a VGCIR economy

Time to maturity (Months)	Run 1 $S(t_i)$	Run 2 $S(t_i)$
120	1,000.00	1,000.00
114	1,054.90	1,054.20
108	1,112.40	1,111.20
102	1,170.90	1,169.90
96	1,234.80	1,234.00
90	1,300.50	1,302.10
84	1,371.60	1,369.90
78	1,444.90	1,442.20
72	1,523.40	1,519.50
66	1,608.10	1,600.50
60	1,694.80	1,686.10
54	1,784.80	1,777.40
48	1,880.80	1,873.50
42	1,984.80	1,975.10
36	2,091.30	2,082.30
30	2,206.60	2,199.10
24	2,327.40	2,315.00
18	2,455.10	2,437.00
12	2,587.80	2,569.90
6	2,726.50	2,706.90
0	2,874.00	2,857.50
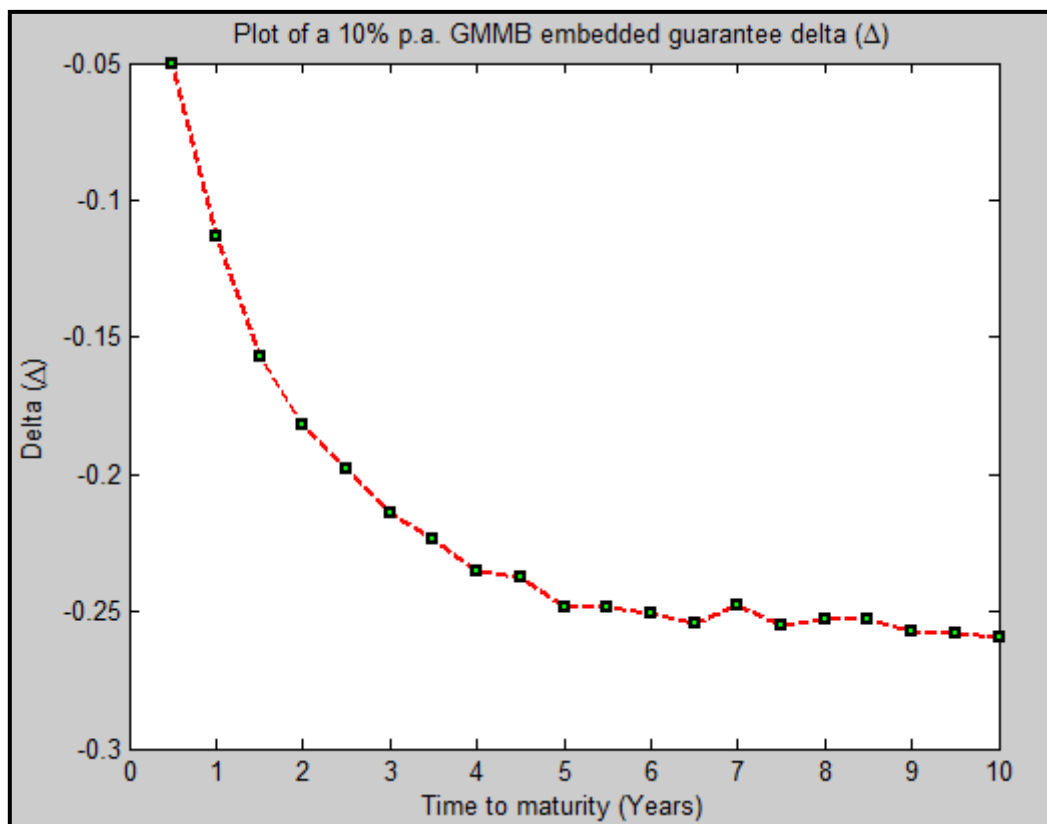
Figure 5.6: 10% GMMB delta behaviour as the time to maturity varies

A plot of the number of units of the underlying in the hedging portfolio (Δ) for an annual guarantee roll-up of 10 % shown in Figure 5.5 above depicts a smooth plot across time. It is clear from the plot that as we approach maturity, the Δ tends towards zero supported by the fact that the guarantee will end out-of-the money and the insurer will be able to meet its obligations with the investment in the ALSI even when the roll-up guarantee is 10 %.

The guaranteed amount across the roll-up guarantee rates is shown below where even with an unusually high guarantee rate of 10%, as noted above, the insurer is protected with the investment in the index yielding approximately $R\,2800$ compared to the guarantee value of $R\,2400$.

Table 5.5: VGCIR GMMB guaranteed amount trend as the guarantee rate varies

g_M	5%	6%	7%	8%	9%	10%	15%	20%
Value at maturity	1551.33	1689.48	1838.46	1999.01	2171.89	2357.95	3517.88	5159.78
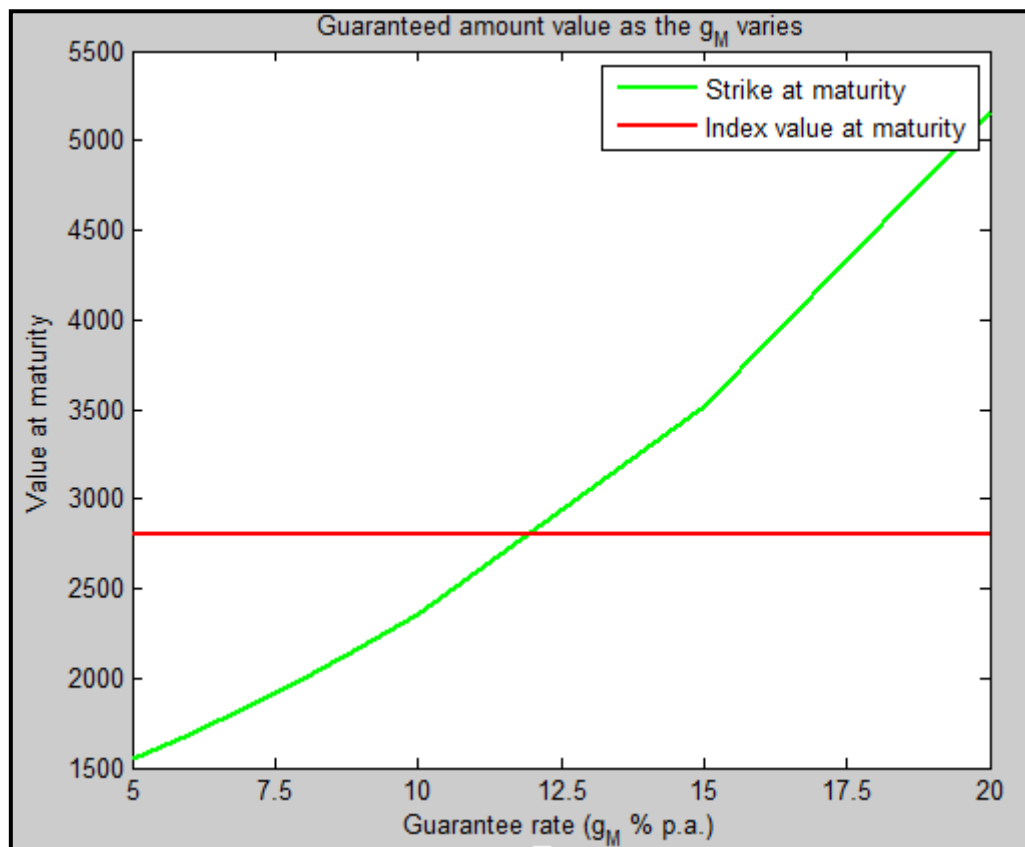
Figure 5.7: VGCIR GMMB guaranteed amount trend as the guarantee rate varies

If we consider the impact of the guaranteed rates from a Greeks perspective at time $T\,=\,0$, we have:

Table 5.6: VGCIR GMMB delta (Δ) and gamma (Γ) as the guarantee rate varies

	$g_M\,=\,0\,\%$	$g_M\,=\,5\,\%$	$g_M\,=\,7.5\,\%$	$g_M\,=\,10\,\%$
Delta (Δ)	-0.0215	-0.0916	-0.1677	-0.2580
Gamma (Γ)	0.00009118	0.00017441	0.0005157	0.00039617

As expected, the delta is a decreasing function of the guarantee rates whereas the gamma is first an increasing function then starts to decrease in the VGCIR economy. Intuitively, this makes sense because a higher guarantee signals a higher likelihood of the guarantee been in-the-money. Put another way, an increasing guarantee rate signals an increasing strike price at maturity and given that the guarantees can be likened to put options, this explains the trend. It is however worth noting that in the simulation runs that were conducted, the gamma is not very stable across the guarantee rates even with a Monte Carlo run of 100 batches each containing 2000 simulations and a different run may produce slightly different results.

The construction of a hedge portfolio in the GMDB context would follow the same structure with the noted trends expected if we assume that the insurer is risk-neutral towards mortality and is protected in the mortality sense by the natural hedging across ages and product sold.

5.4.2. Delta/Gamma (Δ/Γ) – hedging portfolio

The Δ-hedging approach considers only first order changes in the underlying asset. However, it is important to take into account higher order effects and this can be achieved using a Δ/Γ - hedging approach.

This approach necessitates the inclusion of other assets as hedging assets. Once again the construction discussed in Kienitz and Wetterau (2012) is used where if we wish to hedge an option with value $V_1(t)$ and bank account $B_1(t)$ using a Δ/Γ hedge, we use another option with value $V_2(t)$ to construct the hedging portfolio and solve the following system of equations:

$$V_1(t_i) = \phi_1(t_i)S(t_i) + \phi_2(t_i)V_2(t_i) + B_1(t_i),$$
$$\Delta_1(t_i) = \phi_1(t_i) + \phi_2(t_i)\Delta_2(t_i),$$
$$\Gamma_1(t_i) = \phi_2(t_i)\Gamma_2(t_i),$$

(5.13)

where:

$\phi_1(t_i)$ represents the number of units of the underlying,
$\phi_2(t_i)$ represents the number of units of the extra option used in the hedging portfolio.

Given the illiquidity observed in the SA option markets, the extra option so chosen in the hedging portfolio should have a short tenor of between 6 months to 1 year and will be changed with ease in the course of the periodic rebalancing.

5.4.3. Hedge efficiency, hedge effectiveness and basis risk

Kawaller (2009) notes the requirements of the *International Accounting Standards Board* (IASB) on how to assess the effectiveness of a hedge portfolio that has been implemented. The author notes that, "...effectiveness measures must relate the gains or losses of the derivative to those changes in the fair value of the hedged item that are due to the risk being hedged." This applies both when the hedge is implemented and at the appropriately chosen rebalancing time points. Further, there is a need to ensure that the measure of effectiveness so chosen is applied consistently.

The life insurer can adopt a number of measures in this regard which include:

- Immediate balance sheet shocks and stress tests,
- Multiple stress scenario analysis,
- Historic back testing,
- Mock and ongoing live testing.

Finkelstein and Holler (2009) note that the hedge asset portfolio should cover the GMxB liability value completely in the base scenario. The different parameters should then be tweaked both on a standalone and multiple parameter change basis. The capital losses from the hedged and unhedged scenarios are then analysed.

5.5. GMxB risk measures

Risk metrics have become an important tool in the risk assessment of financial institutions. This section considers risk measures in the embedded derivatives context.

5.5.1. Risk measures overview

Feng and Volkmer (2012) discuss the procedure adopted in the use of these risk measures and they note Monte Carlo simulation as an important tool in this regard.

The procedure is as follows:

i. *Monte Carlo simulation is used to generate a set of future scenarios as desired by the insurer to reflect the time frame and other economic factors that may prevail.*

ii. *Each scenario so generated is then used to calculate the variable of interest which is normally the profit or loss suffered by the insurer under that particular scenario.*

iii. *The scenarios are then used to give an empirical distribution of the variable and inferences can then be made on the basis of the risk metric chosen.*

The two commonly used and complementary risk metrics in the context at hand are the Value-at-Risk (VaR) and Expected Shortfall (ES) which both require that the loss measure be defined.

In the GMMB context, if we ignore the fees received by the insurer, the liability at maturity is given by:

$$\{(1 + g_M\%)^T \times P - I_T\}, \tag{5.14}$$

and at time 0 this is expressed as:

$$L_0 = e^{-rT}(G_M - I_T) \times \left(1 - {}_Tq_x\right), \tag{5.15}$$

where:

$$G_M = (1 + g_M\%)^T \times P \text{ is the value of the guaranteed amount at}$$
maturity,

${}_Tq_x$ is the probability of a life aged x dying within the next T years.

The expression for a GMDB benefit is the same with the only exception that the time T is now random and is the time of death. The liability in this case is given by:

$$L_0 = e^{-rT_x}\big(G_D - I_{T_x}\big) \times {}_{T_x}q_x.$$

(5.16)

Feng and Volkmer (2012) note that L_0 will normally be negative indicating that the guarantee will, in most instances, be out-of-the-money since the contracts sold should ideally be profitable. Even so, an analysis of the loss empirical distribution is worthwhile.

Definition 5.1 (Lambadiaris et al. 2003), (Feng 2014)

Value-at-Risk (VaR) is a quantile risk measure defined as the liability loss which is not expected to be exceeded with probability α. It is defined as the minimum capital required to ensure that there are sufficient funds to cover the future liability with probability of at least α, that is:

$$VaR_\alpha := inf\{x: \mathbb{P}[L_0 > x] < 1 - \alpha\}.$$

(5.17)

Definition 5.2, (Dowd 2005), (Feng 2014)

Expected Shortfall (ES) is on the other hand defined as the average of the worst $100(1 - \alpha)$ % of the losses, that is:

$$ES_\alpha := \mathbb{E}[L_0 \mid L_0 > VaR_\alpha].$$

(5.18)

5.5.2. GMMB risk measures

In this section, we consider the GMMB VaR and ES risk measure under the VGCIR economy using the order statistics approach and the mortality rates from Table 4.8 in Chapter 4. Feng (2014) notes that if we have ordered simulations then the VaR_α can be estimated using the $\alpha \times N - th$ order statistic $((1 - \alpha) \times N - th$ worst case) where N is the sample size of the simulations. This can then be used to calculate the ES_α.

The other parameters used are an $S_0 = R\ 1000$, the initial guaranteed amount is $R\ 1000$ with a roll-up guarantee rate of $g_M\ p.a.$ and the tenor, T, is 10 years.

If we use the order statistics approach on a 3000 simulations MC run with a p.a. roll-up g_M of 5 % we have:

Table 5.7: 5% GMMB Value-at-Risk risk measure

Age (x)	$g_M = 5\ \%$			
	Male		**Female**	
	$VaR_{0.05}$	$VaR_{0.01}$	$VaR_{0.05}$	$VaR_{0.01}$
50	143.7429	204.0343	158.3089	217.9468
55	124.2473	189.0002	131.815	193.2397
60	106.4980	158.2442	122.4994	182.6054
65	96.4723	141.1859	126.931	185.839

Table 5.8: 5% GMMB Expected Shortfall risk measure

	g_M = 5 %			
Age (x)	**Male**		**Female**	
	$ES_{0.05}$	$ES_{0.01}$	$ES_{0.05}$	$ES_{0.01}$
50	180.4982	228.7781	195.9967	237.1368
55	160.1589	208.4737	168.0458	209.7466
60	138.2776	175.4027	158.8591	205.4385
65	124.0301	156.3715	164.4567	206.3211

The VaR and ES are decreasing functions of age due to increasing mortality rates the higher the age at inception. A higher confidence level gives a higher VaR and ES figure at a given age which intuitively also makes sense.

The empirical loss distribution is skewed to the left which, as was noted, is a result of the need for product profitability during product design phase hence L_0 is normally negative.
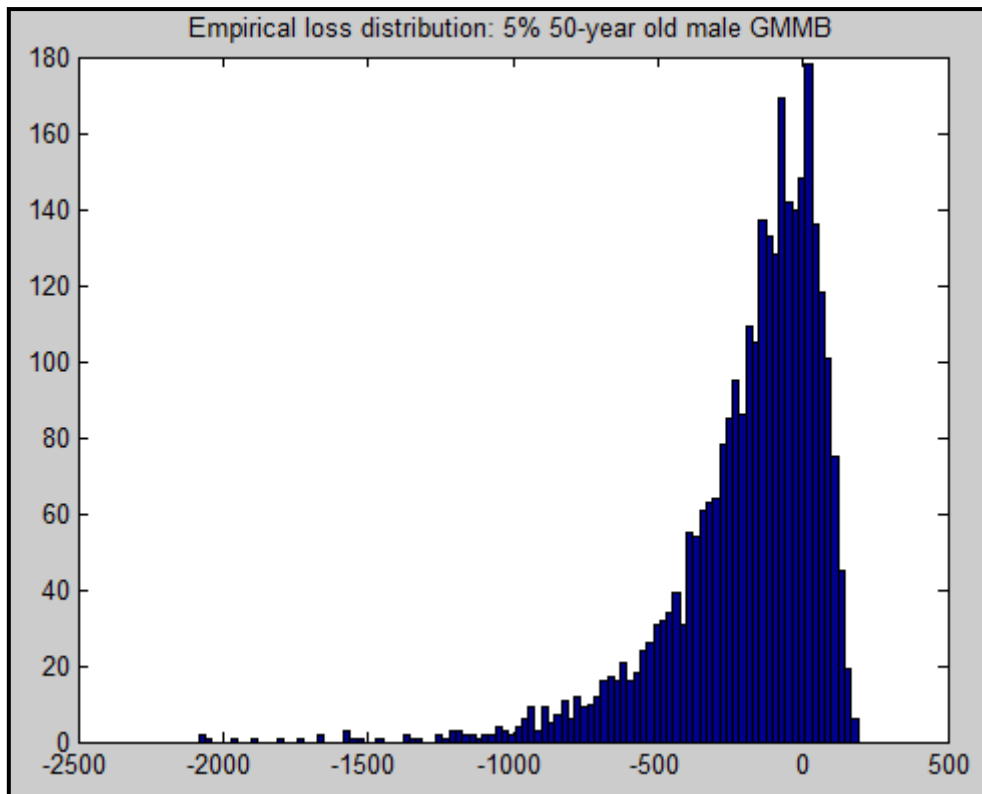
Figure 5.8: Empirical loss distribution: 5% 50-year old male GMMB

If the g_M is now 10 %, the numerics obtained are as follows:

Table 5.9: 10% GMMB Value-at-Risk risk measure

	g_M = 10 %			
Age (x)	**Male**		**Female**	
	$VaR_{0.05}$	$VaR_{0.01}$	$VaR_{0.05}$	$VaR_{0.01}$
50	304.9761	366.6765	341.0646	402.3733
55	267.7855	316.9249	287.0026	338.7351
60	238.5732	291.8451	122.4994	182.6056
65	214.2653	256.1725	283.1687	340.6562

Table 5.10: 10% GMMB Expected Shortfall risk measure

	g_M = 10 %			
Age (x)	**Male**		**Female**	
	$ES_{0.05}$	$ES_{0.01}$	$ES_{0.05}$	$ES_{0.01}$
50	343.4059	386.0683	379.6575	434.7905
55	299.3913	339.2701	320.3012	357.3411
60	269.391	304.6397	158.8591	205.4385
65	239.9221	269.2103	317.7235	359.6106

The VaR and ES figures are once again decreasing functions of the inception age x due to the increasing mortality rates. A higher guarantee rate gives higher risk measure figures signalling the need to hold higher capital as the guarantee increases as would be expected in any pricing framework.
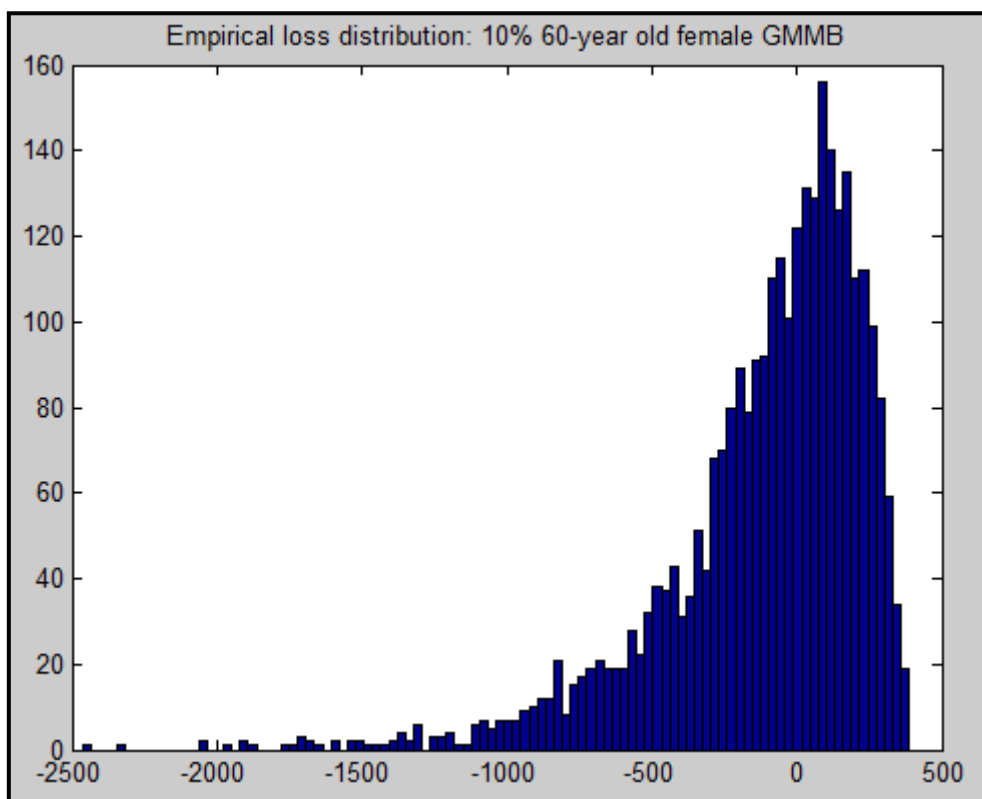
Figure 5.9: Empirical loss distribution: 10% 60-year old female GMMB

In the foregoing, Feng and Volkmer (2012) show that, irrespective of contract independence, the figures above are linearly proportional to the contract size and can thus be used to calculate the risk measures for different contract sizes.

The figures above can thus be viewed as a percentage of the initial account value, R 1000 as shown in the table below:

Table 5.11: 10% GMMB Value-at-Risk as a percentage risk measure

	$g_M = 10\%$			
Age (x)	**Male**		**Female**	
	$VaR_{0.05}$	$VaR_{0.01}$	$VaR_{0.05}$	$VaR_{0.01}$
50	30.4976 %	36.6677 %	34.1065 %	40.2373 %
55	26.7786 %	31.6925 %	28.7003 %	33.8735 %

and where needs be these proportions can be applied for different initial account values.

The insurer can be 95 % confident that the losses will not exceed 35 % of the initial account value and only in 1 % of the cases can the insurer expect the losses to exceed 40 % of the initial account value. It is worth mentioning that in practice, owing to the aggregation of risk measure calculations by most insurers across products and units, the proportions may be slightly less.

Embedded options and capital management

The regulatory framework in which banks and insurers operate has been one of the areas in finance that has undergone enormous transformation in the past years. This chapter considers the embedded options in this light; in particular, what are the regulatory guidelines in light of the results presented and what is the industry practice.

6.1. Insurance capital management

De Weert (2011) discusses the need for ongoing capital management in an insurance context whose key goals are:

- The optimization of the capital structure with an endeavour to achieve an optimal cost of capital,
- Performance optimization whose goal is to attain an optimal return on capital.

This is pictorially captured by the author below:

Figure 6.1: Insurance capital management framework (De Weert 2011)

In the bigger picture, it is important to consider how the embedded guarantees methodologies considered fit in from a capital management perspective to, in particular, the two key angles noted by De Weert (2011):

- Accounting perspective including the IFRS guidelines,
- Regulatory perspective including the Solvency Assessment and Management (SAM), Solvency II and the *Actuarial Society of South Africa* (ASSA) Advisory Practice Note (APN) 110 and Standard of Actuarial Practice (SAP) 104.

6.2. Accounting perspective

The balance sheet of a life insurer has more activity on the liability side since the insurer is contingently indebted to the policyholders. The liabilities are relatively illiquid and hence the liquidity mismatch risk is reduced.

The role of capital management, though heavily influenced and guided by the desire to ensure that the insurer holds the required capital in line with the risks taken, has this desire bent by the need to conform to the financial reporting standards. In particular, the guiding International Financial Reporting Standards (IFRSs) are IFRS 4 on Insurance Contracts and IFRS 9 on Financial Instruments.

In both these standards, there has been a move towards fair value accounting signaling the need for a valuation approach that comprehensively uses asset and liability values that are as close as possible to the market values. IFRS 4 in particular allows the insurer to use a different valuation methodology if such a move results in a move towards market-consistent valuation. This is allowed at a group level, business unit level and even at a portfolio level. Given the goal of increasing the reliability and prudence of the financial statements, the move towards the use of valuation methodologies that capture the market behavior would appear worthwhile provided the requisite comprehensive disclosures as required by the International Accounting Standards Board (IASB) are met.

6.3. Regulatory perspective

The life insurance embedded derivatives are governed, from a regulatory perspective, by the *Financial Services Board* (FSB) SAM guidelines which are comparable to the European Union equivalent of Solvency II and the ASSA requirements.

Actuarial Society of South Africa guidelines

Standard of Actuarial Practice (SAP) 104

This standard, issued by ASSA, deals with the calculation of the value of assets, liabilities and capital adequacy requirement of long-term insurers (Actuarial Society of South Africa 2012). It is an obligatory standard which long-term insurance statutory actuaries must comply with.

The Standard distinguishes between three forms of reporting which may dictate different valuation methodologies: valuation for financial reporting, statutory reporting and tax reporting purposes. Though no particular method is imposed, the financial soundness valuation principles are deemed as a key guide to this end with further guidance being the need to refer to the complementary APN 110:

.

Advisory Practice Note (APN) 110

This note as issued by ASSA deals with the allowance for embedded investment derivatives. It notes that a deterministic approach in valuing the embedded derivatives falls short of the real-world random behavior and that there is a need to adopt a stochastic method in the valuation.

APN 110 does not prescribe any particular stochastic model to aid in the valuation of the embedded derivatives. The overriding theme however is that the model so adopted by the actuary must be market-consistent and solid enough to assist in the quantification of reserves that will meet the costs of the guarantees at such a time that they mature.

The guidance notes that the long-term nature of life insurance makes calibration using tradable instruments a challenge. It however points that the actuary should use a historical analysis complemented with implied parameters for tenors where tradable derivatives exist. The guidance recommends a back-testing on the model adopted to ensure that it can reproduce the observed traded derivatives. Naturally, this adds to the robustness and credibility of the model so created. If the Monte Carlo approach is used, then a minimum of 2000 simulations is recommended unless the convergence is fast enough to justify a fewer number of simulations.

The need for precision, though a necessary consideration, must not be achieved at the expense of practicality. The note points that recognizing the nature and extent of risk in an embedded derivative is more vital than mathematical precision if the two are in conflict.

ASSA (2012) APN 110: 5.2

Some guarantees offered by life offices could be very complex instruments. As such, they may be very difficult to model precisely. Parameter estimation may often also be problematic. The actuary needs to bear in mind that the appropriate recognition of the nature and extent of risk involved in those guarantees is more important than surgical precision in the valuation models. For this reason, the actuary must use his/her judgement to strike an appropriate balance between complexity and practicality.

(Actuarial Society of South Africa 2012)

Finally, the APN notes that it is important to test the approach used to shock scenarios that can adversely affect the insurer.

Financial Services Board (FSB) and Solvency II guidelines

The *Financial Services Board* (FSB) in South Africa is tasked with the role of instituting regulatory capital requirements for, inter alia, life insurers. The embedded derivatives fall within this scope and currently the Solvency Assessment and Management (SAM) directive, which is in the last stages before full implementation, provides a guide. The European Union equivalent, Solvency II directive provides a good comparative base.

The directives are based on three pillars with Pillar 1 dealing with the quantitative requirements, Pillar 2 the governance and risk management frameworks and Pillar 3 on the disclosure and transparency, essentially market discipline. The valuation of embedded derivatives falls under Pillar 1 given that it is mainly a quantitative exercise.

Solvency II under Article 79 requires that the value of any embedded guarantees be taken into account in the valuation and that, *"... the assumptions used shall take account, either explicitly or implicitly, of the impact that future changes in financial and non-financial conditions may have on the exercise of those options."* (CEIOPS, 2009). The SAM directives have the same spirit.

SAM has a bias towards the use, in valuation, of risk-neutral frameworks though the directive does note that this may not be adequate for investment guarantees. Deviations from the risk-neutral setting are allowed provided such a move does not result in market-inconsistent values. Finally, the directive requires that all decrements and risk-drivers that materially affect the guarantees be considered in the valuation. The research has focused primarily on pricing and a consideration of the implications of decrements such as lapses and surrender was outside the scope of this research.

The consideration of correlation from a risk management perspective is discussed in Joubert and Langdell (2013) who note that the correlation matrix can provide a useful starting point in an endeavour to assess, pairwise, decrements and risk drivers. The authors provide a useful discussion on how to fix constructed matrices that mathematically fall short of the requirements for a correlation matrix. The use of such in the holistic assessment of life insurance capital management issues is likely to address any demerits of a stand-alone consideration of risk drivers in any setting.

It is worth pointing out the current corporate practice of three major life insurance operators in South Africa from a capital management perspective as contained in their annual reports. Given that their domicile is SA, the insurers largely follow the guidelines noted and discussed above with a considerable use of sensitivity analysis in their valuations to judge the exposure to economic and other unfavourable shocks.

This chapter has considered the qualitative aspects needed in the valuation of a life insurer's liabilities as per the various guidelines available. The goal was to gain an understanding of how these guidelines fit into the considered frameworks in the research. Overall, the frameworks considered do not fall short of the need for prudent, sound and market consistent models needed in valuation.

7.1. Summary

The variable annuity industry is one of the topical issues in the life insurance business at the time of writing this research and its importance undisputed. The report set out to investigate the state of the industry from a pricing and hedging perspective, the reasons, motivations and limitations for the as-is and the applicability or otherwise of a more refined model for the underlying given the link in performance between the annuities and the financial markets. The justification is that if the underlying can be thoroughly modelled then the annuities can be more accurately priced and hedged.

It is a well accepted premise in quantitative finance that the underlyings, more so asset prices, display many small jumps (see for example Kélani and Quittard-Pinon (2014)) thus the model so chosen should be aligned with this understanding. The published research, thus far, on the choice of such a model is however inconclusive and this is what has guided the discourse of this research, in particular, to answer (together with the associated objectives) whether:

    a. The guarantees embedded in variable annuities, more specifically GMxBs, can be effectively priced and hedged using a suitably chosen exponential jump model.
    b. The approach in (a) can concurrently meet the expectations of regulators and other stakeholders on the life insurer's approach to risk and capital management.

The Variance-Gamma (VG) model is a natural choice for the jump model given its ability to generate an infinite number of jumps within any finite interval and together with its related offshoots provided the frameworks to address these questions.

The analyses and empirical findings were developed and summarized in four steps: firstly the general state of the global variable annuity industry was discussed in Chapters 1 and 2, the products and associated guarantees in Chapter 3, the pricing frameworks and numerical illustrations in Chapter 4 and finally the risk and capital management considerations in Chapters 5 and 6.

Chapters 1, 2 and 3 served as a broad introduction to the subject matter. Chapter 1 was an introductory chapter that contained the objectives and structure from which the rest of the chapters are built. In Chapter 2, a more thorough literature review was presented from which the practical issues facing the industry were identified followed by a discussion of the core embedded derivatives concepts in Chapter 3. The discussion is model independent and concludes with guarantee representations using the more familiar notions of call and put options. This was important for the chapters that followed.

In Chapter 4 the discussion narrowed down to the pricing of the GMxBs. The standard Black-Scholes model was the starting point of the chapter given its huge acceptance and appeal in finance. The regime switching log normal model has however been shown to be better than the Black-Scholes model in the guarantees context by among others Hardy (2003) thus no pricing was done in the Black-Scholes setting.

The JSE ALSI was first fitted to a two regime Markov model using the Baum-Welch algorithm and two regimes for the period under consideration were clearly evident; a low-volatility and a high-volatility regime. The Esscher transform was used in making the choice of the martingale measure that was used to price the GMxB in an incomplete market setting.

The exponential jump models were then considered and applied in the pricing. These were the VG model and the VG stochastic volatility model. Monte Carlo simulation was used extensively to simulate the prices with a critical discussion on parameter estimation being done in Sections 4.6.3 and 4.6.4. The output was then analysed from a parameter sensitivity perspective. The prices showed statistical stability in this regard and the next step was the comparison of these prices with those of the two regime Markov model. The findings confirmed that the prices obtained from the VG model frameworks lay within the low volatility and high volatility regime prices of the Markov model in most cases. If the prices fell outside these price bounds then the prices so obtained were just slightly higher than those of a high volatility regime.

In considering the results from Chapter 4, theoretically more improved models can provide a pricing framework and conclusions not significantly different from current models that assume a normal distribution for the underlying returns. The former have the added advantage of incorporating stylized facts on financial markets behaviour and hence giver a truer picture over time. The use of this approach would thus provide benefits to the insurer more so from an asset-liability perspective.

The risk management discussion in Chapter 5 began with a hedging consideration of the contracts based on a VG model that incorporated stochastic volatility, the Variance-Gamma Cox-Ingersoll-Ross model, with the commonly used and more traditional approaches of delta and delta-gamma hedging being considered.

The delta hedging was based on the underlying asset and considered a semi-annually rebalanced hedge portfolio. The results showed that if the guarantee is a single digit percentage then the contract will most likely end out-of-the-money and the insurer is protected. The delta-gamma hedge discussion involved the use of both the underlying and any liquid options traded on the JSE for the hedging portfolio with the latter approach protecting the portfolio against changes in the delta. The hedge portfolio constructed in both cases is reasonably easy to apply in the context of the South African market and hence provides a platform to always ensure the insurer is protected.

The assessment of risk was done using two commonly accepted risk measures, Value-at-Risk (VaR) and Expected Shortfall (ES) using the Monte Carlo approach to risk measures. The risk measures showed that the insurer can expect to lose proportions greater than a quarter of the initial amount with considerably small probabilities. In other instances, the insurer will be protected and in the better instances, the insurer will not make a loss from the guarantees. The empirical loss distribution captured this with an extended negative skewness.

The life insurance industry is becoming a heavily regulated industry and Chapter 6 considered the foregoing in this light. The various guidelines at the time of this writing tilt to the use of a fair value approach in the pricing of guarantees with the need for a reliable, sound and consistent approach. The frameworks considered met the criteria and can thus be incorporated in the pricing toolbox of the life insurance industry.

The variable annuity industry is noted by many authors as one of the most complex in the life insurance industry not the least because of the riders offered, the link to the financial markets of these riders which adds to their complexity and the need for these riders due to increased competitiveness. Even in these circumstances, there is no dispute to the proposition that the pricing of options should take into account empirically observed features such as skewness and kurtosis.

This research has encouraged thinking in this regard. It does not suggest that the frameworks so applied are the perfect models, but rather frameworks more in tune with reality and the observed dynamics of the SA financial markets.

The scale of complexity and extent of debate on the subject matter correctly pre-empts the need for further studies on the topic in an endeavour to ensure that the variable annuities are appropriately priced and hedged. This research can be built on by exploring a number of issues which could not be looked at either due to the scope of the study or limitations which presented themselves as a result of the methodology adopted.

The use of a deterministic interest rate could be changed to assess the impact of stochastic interest rates on the prices. Kijima and Won (2007), Peng, Leung and Kwok (2009) and Tiong (2013) have considered research in this direction under different frameworks using some of the well known interest rate models. This approach is helpful in explicitly taking into account interest rate risk in the pricing and would be a further refinement to the framework.

The study has also made assumptions on the impact of some demographic risks such as mortality and policyholder behaviour. The mortality rates are based on the ASSA tables and though they give a representative picture, the use of a stochastic mortality model may be a viewpoint worth taking. Milevsky and Posner (2001) considered the use of exponential mortality models which could be incorporated to the case at hand or the application of the stochastic mortality models discussed in Cairns et al. (2008).

The sub-optimality of the policyholder (for instance, divorce and buying a house may link to withdrawal, start date of GMxBs may link to other personal event like spousal cancer treatment) could be much larger forces than a more refined model for movement of the underlying asset and its calibration in a deep / shallow market. A research from this angle could also provide interesting perspectives.

The assumption that the VA risk factors are independent may not always hold thus correlation risk arising from the relationship between the different risk drivers and decrements could be another perspective worth integrating into the current framework. The objective in this case will be to investigate whether the conclusions will be significantly different with regard to the pricing of the guarantees and their subsequent hedging.

Kienitz and Wetterau (2012) also note that Lévy processes may fail over different maturities due to the restrictions they impose on the shape of the future volatility surface. Another possible research extension could be in this direction with the recent work of Carr and Wu (2010), Homescu (2011) and Ma (2014) providing insightful direction on the subject.

7.2. Conclusion

The use of quantitative techniques in asset-liability management, though deemed sometimes complex, will continue to be of relevance in the insurance industry for many years to come. In recent years, senior management has moved towards simpler and more explainable models in this regard but interestingly, the products offered have continued to be complex.

The early adopters of any new model are normally able to gain competitive advantage and the VG together with its extended frameworks provide such a basis. The costs in this setting will not necessarily be too high for the insurer and the principal conclusion of this research is that the frameworks are tenable.

The younger generation which will evolve to buy life insurance in South Africa is likely to be a more demanding generation more so with regard to the innovativeness expected in this generation's products. This further adds to the complexity and dictates a move to complex yet realistic models. The issue of capital management on the insurer's balance sheet will however remain uncompromisable and whichever approach is adopted, the insurer must always endeavour to closely reflect, as much as possible, the market.

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Appendix A

The pricing of the embedded options was mainly done in Matlab given the software's efficiency in computational implementations.

The nature of the dissertation meant that the code is extensive and a selection of the code follows in this and subsequent appendices. The full codes will be available electronically on the author's Google page.

The code is interactive in most cases and allows the user to input the main parameters such as initial regime, initial index price, time to maturity and guaranteed amount or rate.

A.1 Hamilton filter approach

The code below uses the Hamilton filter approach to fit the JSE ALSI returns to a two regime Hidden Markov Model.

It is adapted from the Matlab code by Perlin (2012) which is available at:

https://sites.google.com/site/marceloperlin/matlab-code/ms_regress---a-package-formarkov-regime-switching-models-in-matlab

Main program

```matlab
%..............................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate: University of Pretoria
% Markov Switching estimation on the South African JSE ALSI log-returns
% Adapted from Perlin (2012)and amended accordingly for the case at hand
% Script MS_Regress_Fit.m
%..............................................................................%

clear;

logRet=importdata('JSE_ALSI_daily_returns.txt');  % load some data and
change depending on whether it is daily or monthly source file

dep=logRet(:,1);                    % Defining dependent variable from .txt
file
constVec=ones(length(dep),1);       % Defining a constant vector in mean
equation
indep=[constVec];                   % Defining some explanatory variables
k=2;                                % Number of States
S=[1 1];                            % Defining which ones from indep will
have switching effect
advOpt.distrib='Normal';            % The Distribution assumption
('Normal', 't' OR 'GED')
advOpt.std_method=1;                % Defining the method for calculation
of standard errors


[Spec_Out]=MS_Regress_Fit(dep,indep,k,S,advOpt); % Estimating the model
```

MS_Regress_Fit program

The main program above references the MS_Regress_Fit which is the code where the actual Markov model is fitted. It is shown herebelow.

```matlab
% Function for estimation of a general Markov Switching regression
%
%    Input:  dep     - Dependent Variable (vector (univariate model) or
matrix (multivariate) )
%            indep   - Independent variables (explanatory variables), should
%                      be cell array in the case of multivariate model (see
examples).
%            k       - Number of states (integer higher or equal to 2)
%            S       - This variable controls for where to include a Markov
Switching effect.
%                      See pdf file for details.
%            advOpt  - A structure with advanced options for algorithm.
%                      See pdf file for details.
%
%    Output: Spec_Output - A structure with all information regarding the
%                      model estimated from the data (see pdf for
details).
%
%    Author: Marcelo Perlin (UFRGS/BR)
%    Contact:  marceloperlin@gmail.com

function [Spec_Output]=MS_Regress_Fit(dep,indep,k,S,advOpt)

% Error checking lines

checkInputs(); % checking if inputs variables are OK

% building constCoeff for the cases when it is not specified

build_constCoeff();

% checking if all fields are specified and make sense

check_constCoeff();

% checking sizes of fields in constCoeff

checkSize_constCoeff();

% Pre calculations before calling the optimizer

preCalc_MSModel();

% Initialization of optimization algorithm

warning('off');

options=optimset('fmincon');
options=optimset(options,'display','off');

dispOut=advOpt.printIter;

% Defining linear contraints in model
```

```matlab
A=[];    % inequality constrain (not used)
b=[];    % inequality constrain (not used)

% equality constraint (each collum of Coeff.p must sum to 1)

beq=ones(k,1);
Aeq=zeros(k,numel(param0));

for i=1:k
    idx=Coeff_Tag.p(:,i);

    for j=1:numel(idx)
        if idx(j)==0
            continue;
        else
            Aeq(i,idx(j))=1;
        end
    end

end

for i=1:k
    if all(Aeq(i,:)==0)
        Aeq(i,:)=0;     % fixing equality restrictions for when using
contrained estimation in Coeff.p
        beq(i,:)=0;
    end
end

param0=param0'; % changing notation for param0

% Call to optimization function

switch advOpt.optimizer
    case 'fminsearch'
        options=optimset('fminsearch');
        options=optimset(options,'display','off');
        options=optimset(options,'MaxIter',500*numel(param0));
        options=optimset(options,'MaxFunEvals',500*numel(param0));


[param]=fminsearch(@(param)MS_Regress_Lik(dep,indep_nS,indep_S,param,k,S,ad
vOpt,dispOut),param0,options);

    case 'fminunc'
        options=optimset('fminunc');
        options=optimset(options,'display','off');

[param]=fminunc(@(param)MS_Regress_Lik(dep,indep_nS,indep_S,param,k,S,advOp
t,dispOut),param0,options);

    case 'fmincon'
        options=optimset('fmincon');
        options=optimset(options,'display','off');

[param]=fmincon(@(param)MS_Regress_Lik(dep,indep_nS,indep_S,param,k,S,advOp
t,dispOut),param0, ...
            A,b,Aeq,beq,lB,uB,[],options);
```

```matlab
end

% Calculation of Covariance Matrix

[V]=getvarMatrix_MS_Regress(dep,indep_nS,indep_S,param,k,S,std_method,advOp
t);
param_std=sqrt(diag((V)));

% Controls for covariance matrix. If found imaginary number for variance,
replace with
% Inf. This will then be showed at output

param_std(isinf(param_std))=0;
param_pvalues=2*(1-tcdf(abs(param./param_std),nr-numel(param)));

if ~isreal(param_std)
    for i=1:numel(param)
        if ~isreal(param_std(i))
            param_std(i)=Inf;
        end
    end
end

typeCall='se_calculation';

[Coeff_SE]=param2spec(param_std,Coeff_Tag,constCoeff,typeCall);
[Coeff_pValues]=param2spec(param_pvalues,Coeff_Tag,constCoeff,typeCall);

% After finding param, filter it to the data to get estimated output

[sumlik,Spec_Output]=MS_Regress_Lik(dep,indep_nS,indep_S,param,k,S,advOpt,0
);

% calculating smoothed probabilities

Prob_t_1=zeros(nr,k);
Prob_t_1(1,1:k)=1/k; % This is the matrix with probability of s(t)=j
conditional on the information in t-1

for i=2:nr
    Prob_t_1(i,1:k)=(Spec_Output.Coeff.p*Spec_Output.filtProb(i-1,1:k)')';
end

filtProb=Spec_Output.filtProb;

P=abs(Spec_Output.Coeff.p);

smoothProb=zeros(nr,k);
smoothProb(nr,1:k)=Spec_Output.filtProb(nr,:);  % last observation for
starting filter

for i=nr-1:-1:1     % work backwards in time for smoothed probs
    for j1=1:k
        for j2=1:k

smooth_value(1,j2)=smoothProb(i+1,j2)*filtProb(i,j1)*P(j2,j1)/Prob_t_1(i+1,
j2);
        end
```

```
        smoothProb(i,j1)=sum(smooth_value);
    end
end

% Calculating Expected Duration of regimes

stateDur=1./(1-diag(Spec_Output.Coeff.p));
Spec_Output.stateDur=stateDur;

% passing values to output structure

Spec_Output.smoothProb=smoothProb;
Spec_Output.nObs=size(Spec_Output.filtProb,1);
Spec_Output.nEq=nEq;
Spec_Output.Number_Parameters=numel(param);
Spec_Output.advOpt.distrib=distrib;
Spec_Output.advOpt.std_method=std_method;
Spec_Output.Coeff_SE=Coeff_SE;
Spec_Output.Coeff_pValues=Coeff_pValues;
Spec_Output.AIC=2*numel(param)-2*Spec_Output.LL;
Spec_Output.BIC=-2*Spec_Output.LL+numel(param)*log(Spec_Output.nObs*nEq);

% ploting probabilities

if advOpt.doPlots
    doPlots();
end

% Sending output to matlab's screen

disp(' ');
if advOpt.printOut
    doOutScreen()
end
```

Hamilton filter MLE output for JSE ALSI daily returns data

The fitted two state model from the Hamilton filter approach is contained in the tables below with the first table showing the model fit and the second the transition probability matrix.

```
***** Numerical Optimization Converged *****


Final log Likelihood: 14694.1548
Number of estimated parameters: 6
Number of Observations: 4704
Number of Equations: 1
Optimizer: fminsearch
Type of Switching Model: Univariate
Distribution Assumption -> Normal
Method SE calculation -> 1


***** Final Parameters for Equation #1 *****


---> Non Switching Parameters <---



--->   Switching Parameters (Distribution Parameters)  <---


State 1
    Model's Variance:      0.000064
    Std Error (p. value):  0.0000 (0.00)
State 2
    Model's Variance:      0.000395
    Std Error (p. value):  0.0000 (0.00)
```

```
--->   Switching Parameters (Regressors)  <---


Switching Parameters for Equation #1 - Indep column 1


State 1
   Value:                0.0009
   Std Error (p. value): 0.0001 (0.00)
State 2
   Value:                -0.0010
   Std Error (p. value): 0.0006 (0.10)

---> Transition Probabilities Matrix (std. error, p-value) <---


     0.99 (0.08,0.00)   0.04 (0.09,0.70)
     0.01 ( NaN, NaN)   0.96 ( NaN, NaN)

---> Expected Duration of Regimes <---


    Expected duration of Regime #1: 74.99 time periods
    Expected duration of Regime #2: 27.38 time periods
```

Hamilton filter MLE output for JSE ALSI monthly returns data

The Hamilton filter method is then applied to monthly data from which it is worthwhile to note that it understates the probabilities of transitioning from one state to the next and hence misses some of the instances of switches in regime.

```
***** Numerical Optimization Converged *****

Final log Likelihood: 342.9186
Number of estimated parameters: 6
Number of Observations: 227
Number of Equations: 1
Optimizer: fminsearch
Type of Switching Model: Univariate
Distribution Assumption -> Normal
Method SE calculation -> 1

***** Final Parameters for Equation #1 *****
```

```
--->    Switching Parameters (Distribution Parameters)   <---

State 1
    Model's Variance:       0.001409
    Std Error (p. value):  0.0003 (0.00)
State 2
    Model's Variance:       0.005908
    Std Error (p. value):  0.0012 (0.00)

--->    Switching Parameters (Regressors)   <---

Switching Parameters for Equation #1 - Indep column 1

State 1
   Value:               0.0155
   Std Error (p. value): 0.0035 (0.00)
State 2
   Value:               -0.0022
   Std Error (p. value): 0.0090 (0.81)

---> Transition Probabilities Matrix (std. error, p-value) <---

    0.96 (0.34,0.00)    0.06 (0.43,0.89)
    0.04 ( NaN, NaN)    0.94 ( NaN, NaN)

---> Expected Duration of Regimes <---

    Expected duration of Regime #1: 26.62 time periods
    Expected duration of Regime #2: 17.37 time periods
```

A.2 Baum-Welch approach

The Baum-Welch algorithm is an expectation maximisation approach to fitting the Markov model. It is preferred by some authors and hence considered for this research.

```matlab
%...............................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Hidden Markov Model estimation on the South African JSE ALSI log-returns
% Adapted from Kritzman, Page and Turkington (2012), Perlin (2012) and
amended accordingly for the case at hand
% Baum-Welch approach
%...............................................................%

function [] = MarkovSwitchingFitBaumWelchMethod(y)
T=length(y);

% Simple initial guesses for parameters - can be changed

mu=[mean(y),mean(y)]+randn(1,2)*std(y); sigma=[std(y),std(y)];
A=[.8,.2;.2,.8]; p=.5;
iteration=2;
likelihood(1)=-999; change_likelihood(1)=Inf;
tolerance=0.000001;
while change_likelihood(iteration-1) > tolerance
    for t=1:T % 0. probability of observing data, based on gaussian PDF
        B(t,1)=exp(-.5*((y(t)-mu(1))/sigma(1)).^2)/(sqrt(2*pi)*sigma(1));
        B(t,2)=exp(-.5*((y(t)-mu(2))/sigma(2)).^2)/(sqrt(2*pi)*sigma(2));
    end
    forward(1,:)=p.*B(1,:);
    scale(1,:)=sum(forward(1,:));
    forward(1,:)=forward(1,:)/sum(forward(1,:));
    for t=2:T % 1. probability of regimes given past data
        forward(t,:)=(forward(t-1,:)*A).*B(t,:);
        scale(t,:)=sum(forward(t,:));
        forward(t,:)=forward(t,:)/sum(forward(t,:));
    end
    backward(T,:)=B(T,:);
    backward(T,:)=backward(T,:)/sum(backward(T,:));
    for t=T-1:-1:1 % 2. probability of regime given future data
        backward(t,:)=(A*backward(t+1,:)')'.*B(t+1,:);
        backward(t,:)=backward(t,:)/sum(backward(t,:));
    end
    for t=1:T % 3-4. probability of regimes given all data
        smoothed(t,:)=forward(t,:).*backward(t,:);
        smoothed(t,:)=smoothed(t,:)/sum(smoothed(t,:));
    end
    for t=1:T-1 % 5. probability of each transition having occurred
        xi(:,:,t)=(A.*(forward(t,:)'*(backward(t+1,:).*B(t+1,:))));
        xi(:,:,t)=xi(:,:,t)/sum(sum(xi(:,:,t)));
    end
    p=smoothed(1,:);
    exp_num_transitions=sum(xi,3);
    A(1,:)=exp_num_transitions(1,:)/sum(sum(xi(1,:,:),2),3);
    A(2,:)=exp_num_transitions(2,:)/sum(sum(xi(2,:,:),2),3);
    mu(1)=(smoothed(:,1)'*y)'/sum(smoothed(:,1));
    mu(2)=(smoothed(:,2)'*y)'/sum(smoothed(:,2));
    sigma(1)=sqrt(sum(smoothed(:,1).*(y-mu(1)).^2/sum(smoothed(:,1))));
    sigma(2)=sqrt(sum(smoothed(:,2).*(y-mu(2)).^2/sum(smoothed(:,2))));
    likelihood(iteration+1)=sum(sum(log(scale)));
```

```matlab
        change_likelihood(iteration)=abs(likelihood(iteration+1)-
likelihood(iteration));
    iteration=iteration+1;
end

disp('...........Smoothed probabilities............');
disp(smoothed);
disp('........Transition probability matrix........');
disp(A);
disp('..............Mean (mu) vector...............');
disp(mu);
disp('................Sigma vector.................');
disp(sigma);

% Plotting time varying probabilities

smoothed(:,[1,2])=smoothed(:,[2,1]);

for i=1:iteration
    States{i}=['State ',num2str(i)];
end

for i=1:iteration
    States{i}=['State ',num2str(i)];
end

% figure(1);
% xlabel('Time');
% ylabel('Filtered States Probabilities');
% legend(States);

figure(1)

plot(smoothed);
xlabel('Time');
ylabel('Smoothed States Probabilities');
legend(States);

subplot(3,1,2);
plot(smoothed);
xlabel('Time');
ylabel('Smoothed States Probabilities');
legend(States);

end
```

The estimation from the Baum-Welch algorithm yields:

```
........Transition probability matrix........
    0.9345    0.0655
    0.0935    0.9065


..............Mean (mu) vector..............
    0.0161   -0.0034


................Sigma vector................
    0.0371    0.0775
```

A.3 Regime switching model pricing

<u>Conditional put option price function</u>

The function that follows below, `CondPutFun`, is a Matlab implementation of Equation 4.28 that is derived in Theorem 4.3

```matlab
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Conditional put option price function
%.......................................%
function y=CondPutFun(S0, K, T, Ot, r1, r2, sigma1, sigma2)
%Conditional put price; conditioned on the occupation time
d2=(log(S0/K)+((r1-r2)*Ot+r2*T)-0.5*((sigma1^2-
sigma2^2)*Ot+T*sigma2^2))/(sqrt((sigma1^2-sigma2^2)*Ot+T*sigma2^2));

d1=d2+sqrt((sigma1^2-sigma2^2)*Ot+T*sigma2^2);

CPP=K*exp(-((r1-r2)*Ot+r2*T))*normcdf(-d2)-S0*normcdf(-d1);

y=CPP;
end
```

<u>Regime switching option valuation: Call and put option</u>

The following Matlab code uses the conditional put price function above and Equation 4.29 to derive the call and put option price in a regime-switching framework.

The Matlab inbuilt functions *quad* and *quadl* are used to do the numerical integration representing the adaptive Simpson quadrature and the adaptive Gauss-Lobatto quadrature respectively.

```matlab
%...............................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Regime switching option valuation: Call and put option
%...............................................................................%
% Parameters

regime=input('The initial regime, i, is (1=Low volatility, 2=High
volatility): ');
S0=input('The initial stock price, S0, is: ');
T=input('The maturity time, T, in months is: ');
K=input('The guaranteed amount at maturity is: ');

%K=1000;  %The strike price is sometimes fixed at R 1000 since normally the
amount guaranteed is known ab initio


% These parameters are derived from monthly data on the JSE ALSI and
% 10-year SA government bond for the period from July 1994-June 2013
% beta1 and beta2 are the off diagonal elements in the generator matrix
r1=0.011; r2=0.0067;
sigma1=0.0371; sigma2=0.0775;
beta1=0.071335; beta2=0.101829;

%...............................................................................%
```

```matlab
% Call option function, numerical integration and pricing

funC = @(u) CondCallFun(S0, K, T, u, r1, r2, sigma1, sigma2).*myFfun(T, u,
regime, beta1, beta2);
q1 = quadl(funC,0,T);

%Plot of the function
%figure(1)
%ezplot(funC, [0,T])

C1=q1;

if regime==2
    Ot=0;
    C2=exp(-beta2*T)*CondCallFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    C2=0;
end

if regime==1
    Ot=T;
    C3=exp(-beta1*T)*CondCallFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    C3=0;
end

% Call option price from formula
CPrice=C1+C2+C3;

display('Call option price: '), disp(double(CPrice));

%.............................................................................%
% Put option function, numerical integration and pricing

funP = @(u) CondPutFun(S0, K, T, u, r1, r2, sigma1, sigma2).*myFfun(T, u,
regime, beta1, beta2);
q2 = quadl(funP,0,T);

% Plot of the function
%figure(2)

P1=q2;

if regime==2
    Ot=0;
    P2=exp(-beta2*T)*CondPutFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    P2=0;
end

if regime==1
    Ot=T;
    P3=exp(-beta1*T)*CondPutFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    P3=0;
end
% Put option price from formula
PPrice=P1+P2+P3;

display('Put option price: '), disp(double(PPrice));
```

```
figure(1)
funPP = @(S0, u) CondPutFun(S0, K, T, u, r1, r2, sigma1, sigma2);
ezsurfc(funPP, [500, 1500, 0, T]), view([135,30])
xlabel('Initial stock price (S0)'); ylabel('Occupation time (u) in
months');
title('Conditional put option price (Maturity-10 years, Initial regime=1)
');

figure(2)
ezplot(funP, [0,T])
xlabel('Occupation time (u) in months');
title('Put option immanent integrand (Maturity=10 years, Initial
regime=1)');

figure(3)
funPPP = @(S0, u) CondPutFun(S0, K, T, u, r1, r2, sigma1,
sigma2).*myFfun(T, u, regime, beta1, beta2);
ezsurfc(funPPP, [500, 1500, 0, T]), view([135,30])
xlabel('Initial stock price (S0)'); ylabel('Occupation time (u) in
months');
title('Put option immanent integrand (Maturity=10 years, Initial
regime=1)');
```

Guaranteed Minimum Death Benefit charge calculation

This Matlab code implements the regime switching model in a Guaranteed Minimum Death Benefit (GMDB) context by taking the year of death into consideration before multiplying with the applicable ASSA mortality rates.

This approach is important so as to assess the maximum cost over the contract period if death occurs at any time in point.

```
%.........................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Guaranteed Minimum Death Benefit (GMDB) charge calculation
%.........................................%
regime=input('The initial regime, i, is (1=Low volatility, 2=High
volatility): ');
S0=input('The initial stock price, S0, is: ');
D=input('The year of death is (first=1, second=2...): ');
T=D*12;
disp(['  The duration/time to maturity (in months) is: ' num2str(T)]);
g=input('The annual guarantee rate in decimal form is: ');
K=input('The initial guaranteed amount based on the premium is: ');
K=K*(1+g)^(D-1);
disp(['  The exercise amount used is: ' num2str(K)]);
%K=1000;  %The strike price is sometimes fixed at R 1000 since normally the
amount guaranteed is known ab initio

% These parameters are derived from monthly data on the JSE ALSI and
% 10-year SA government bond for the period from July 1994-June 2013
r1=0.011; r2=0.0067;
sigma1=0.0371; sigma2=0.0775;
beta1=0.071335; beta2=0.101829;

%.....................................................................%
```

```matlab
% Put option function, numerical integration and pricing

funP = @(u) CondPutFun(S0, K, T, u, r1, r2, sigma1, sigma2).*myFfun(T, u,
regime, beta1, beta2);
q2 = quadl(funP,0,T);

% Plot of the function
%figure(2)

P1=q2;

if regime==2
    Ot=0;
    P2=exp(-beta2*T)*CondPutFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    P2=0;
end

if regime==1
    Ot=T;
    P3=exp(-beta1*T)*CondPutFun(S0, K, T, Ot, r1, r2, sigma1, sigma2);
else
    P3=0;
end
% Put option price from formula
PPrice=P1+P2+P3;

display('Put option price: '), disp(double(PPrice));
```

Appendix B

B.1 Estimating the VG model using the MLE approach

This Matlab code is an implementation of the discussion in Section 4.6.3.1 where the Variance-Gamma model is fitted using the maximum likelihood estimation (MLE) approach.

```matlab
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Variance Gamma model estimation on the South African JSE ALSI log-returns
% Adapted from Brigo et al. (2007) and amended accordingly for the case at
hand
%......................................................................%

data = xlsread('JSEALSIDataAndLogarithmicReturns.xls','JSEALSILogReturns');

% Parameter estimation

dt=1;

figure(1)
hist(data);
figure(2)
histfit(data);
title('JSE ALSI monthly returns histogram fit ');
figure(3)
qqplot(data);
figure(4)
plot(data);
xlabel('Time (in months)'); ylabel('JSE ALSI returns');
title('JSE ALSI monthly returns over time: July 1994-June 2013 ');

M = mean(data);
display(['M: ', num2str(M)]);
V = var(data);
display(['V: ', num2str(V)]);
S = skewness(data);
display(['S: ', num2str(S)]);
K = kurtosis (data);
disp(['K: ', num2str(K)]);

% Initial VG parameter estimates
disp('............Initial MME parameter estimates................');
sigma = sqrt(V/dt);
disp(['Sigma: ',num2str(sigma)]);
nu = (K/3 -1)*dt;
disp(['Nu: ', num2str(nu)]);
theta = (S*sigma*sqrt(dt))/(3*nu);
disp(['Theta: ',num2str(theta)]);
mu = (M/dt)-theta;
disp(['Mu: ',num2str(mu)]);

% MLE VG parameter estimation
pdf_VG = @(data,theta, nu, sigma, mu) max(realmin, VGdensity(data, theta,
nu, sigma, mu, dt));
start = [theta, nu, sigma, mu];
lb = [-intmax 0 0 -intmax];
ub = [ intmax intmax intmax intmax ];
options = statset ('MaxIter', 1000 , 'MaxFunEvals', 100000);
```

```
[params, pci] = mle(data , 'pdf', pdf_VG , 'start', start, 'lower', lb,
'upper', ub, 'options', options);
disp('...........MLE VG final parameter estimates................');
disp('....Theta.......Nu......Sigma.......Mu.....');
disp(params);
disp(pci);
```

Initial MME estimates for use in the MLE

```
M: 0.0085738
V: 0.0032755
S: -1.3049
K: 9.544
...........Initial MME parameter estimates.........
Sigma: 0.057232
Nu: 2.1813
Theta: -0.011413
Mu: 0.019987
```

Final VG MLE parameter estimates

```
...........MLE VG final parameter estimates........
....Theta.......Nu......Sigma.......Mu.....
   -0.0148    0.4461    0.0544    0.0234
```

B.2 Autocorrelation and normality test on the JSE ALSI log returns

The procedures below, in the Statistical Analysis Software platform, test the hypothesis that the monthly returns are independent and normally distributed.

```
proc arima;
     identify var=Logarithmic_return nlag=12;
run;
proc univariate;
     var Logarithmic_return;
histogram Logarithmic_return/ normal(percents=1 5 20 40 60 80 95 99
midpercents)
                      name='MyPlot' ;
   inset n normal(ksdpval) / pos = ne format = 6.3;
run;
/* Return Analysis*/
/*Set graphic options & titles*/
goptions reset=all i=join;
axis1 label=(angle=90 'Return');
axis2 offset=(0 cm)label=('Year') minor=(number=98);
legend1 label=none cborder=black
        position=(bottom right inside)
        mode=share;
symbol1 i=join v=none c=red    w=1 l=1;
proc gplot;
   title1 "Return Analysis";
   plot Logarithmic_return*Date/ overlay legend=legend1 vaxis=axis1
haxis=axis2 vref= (0)
     cvref=(black);
run;
```

B.3 VG call and put option pricing

The following Matlab code is an implementation of Equations 4.44-4.47 whose results are presented in Section 4.6.5.2. The Monte Carlo simulation is based on the discussion in Section 4.6.1.

```matlab
%........................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate: University of Pretoria
% Variance Gamma call and put option pricing on the South African JSE ALSI
% log-returns using the theoretical framework in Madan, Carr and Chang
%(2008). Adapted from QuantCode Inc (2006) and amended accordingly for the
case at hand
%........................................................................%

clear all;


%%%%%%%%% parameters setting  %%%%%%%%%%%%%%%%
S0=input('The initial stock price, S0, is: ');
In=input('Are you pricing an 10-year option with a guarantee (Yes=1, No=0):
');
if In==1
    g=input('The annual guarantee rate in decimal form is: ');
    K=input('The initial guaranteed amount based on the premium is: ');
    K=K*(1+g)^10;
else
    K=input('The strike/guaranteed amount at maturity is: '); %K=1000;
end
T=input('The maturity time, T, in months is: ');



disp(['  The exercise amount used is: ' num2str(K)]);


sigma=0.0544; %volatility for VG model
r=0.0088; %risk free rate
VG_nu=0.4461; %nu for VG model
VG_theta=-0.0148; %theta of VG model
nsimulations=10000; % no. of MC simulations
nbatches=100; % Extra MC simulations for accuracy
S=S0;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Analytical pricing approach as discussed in Madan, Carr and Chang (1998)

v=VG_nu;
theta=-VG_theta;
zhi=-theta/(sigma*sigma);
s=sigma/(1+(theta/sigma)^2*(v/2))^0.5;
alpha=zhi*s;
c1=v*(alpha+s)^2/2;
c2=v*alpha*alpha/2;
tmp=log(S/K) + r*T + (T/v)*log( (1-c1)/(1-c2) );
d=(1/s)*tmp;


a=d*((1-c1)/v)^0.5;
b=(alpha+s)*(v/(1-c1))^0.5;
ga=T/v;


fun1 = @(u) PsiIntegrand(u, a, b, ga);
```

```matlab
tmp1=quadgk(fun1, 0, 200); % Use the same bounds as in the integrand below
otherwise inconsistencies will arise


a=d*((1-c2)/v)^0.5;
b=(alpha)*(v/(1-c2))^0.5;
ga=T/v;

fun1 = @(u) PsiIntegrand(u, a, b, ga);
tmp2=quadgk(fun1, 0, 200); % Use the same bounds as in the integrand above
otherwise inconsistencies will arise



VG_CallPriceA=S*tmp1-K*exp(-r*T)*tmp2;

%disp(VG_CallPriceA);

VG_PutPriceA=VG_CallPriceA-S+K*exp(-r*T);

display('Analytical formula put option price: '),
disp(double(VG_PutPriceA));
%----------------------------------------------------------
%Verify results using brute force integration as described in Rebonato
(2004).

fun1 = @(g) BruteForceIntegrand(g, r, T, K, S, VG_nu, VG_theta, sigma);
BFI=quadgk(fun1,0,200);

VG_CallPriceBFI=BFI;

%disp(VG_CallPriceBFI);

VG_PutPriceBFI=VG_CallPriceBFI-S+K*exp(-r*T);

display('Brute force integration put option price: '),
disp(double(VG_PutPriceBFI));
%----------------------------------------------------------
% MC simulation

omega=(1/VG_nu)*( log(1-VG_theta*VG_nu-sigma*sigma*VG_nu/2) );
StockProcess=zeros(nsimulations,1);
for l=1:nbatches
    for i=1:nsimulations
        g = gaminv(unifrnd(0,1), T/VG_nu, VG_nu);
        % The equivalent definition of a VG process
        h = VG_theta*g + sigma*sqrt(g)*norminv(unifrnd(0,1));
        StockProcess(i)=S*exp(r*T+omega*T+h);
    end
    %figure; plot(StockProcess); title(' Monte Carlo simulation: VG economy
stock price value after 10 years -\nu,\sigma,\theta'); xlabel ('Simulation
number ');  ylabel ('Stock price at maturity (T=10 years; S0=R 1000)');
    % disp(num2str(StockProcess));
    payoffvecC=max(StockProcess-K,0);

    payoffvecP=max(K-StockProcess,0);

    mc_callprice(l)=exp(-r*T)*mean(payoffvecC);

    mc_putprice(l)=exp(-r*T)*mean(payoffvecP);
```

```
end
mc_callprice_estimate=mean(mc_callprice);
%disp(mc_callprice_estimate);

%mc_putprice_estimate=mc_callprice_estimate-S+K*exp(-r*T);

mc_putprice_estimate=mean(mc_putprice);

display('Monte Carlo put option price: '),
disp(double(mc_putprice_estimate));
```

B.3 VG GMDB pricing algorithm

The Matlab code below is an implementation of the GMDB pricing discussed in Section 4.6.6.2.

```
%.............................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Variance Gamma call and put option pricing on the South African JSE ALSI
% log-returns using Madan, Carr and Chang (2008).
% Adapted from QuantCode Inc (2006) and amended accordingly for the case at
hand
%.............................................................................%

clear all;

%%%%%%%% parameters setting  %%%%%%%%%%%%%%%
S0=input('The initial stock price, S0, is: ');
D=input('The year of death is (first=1, second=2...): ');
T=D*12;
disp(['  The duration/time to maturity (in months) is: ' num2str(T)]);
g=input('The annual guarantee rate in decimal form is: ');
K=input('The initial guaranteed amount based on the premium is: ');
K=K*(1+g)^(D-1);
disp(['  The exercise amount used is: ' num2str(K)]);
%K=1000; input('The strike/guaranteed amount at maturity is: ');

sigma=0.0544; %volatility for VG model
r=0.0088; %risk free rate
VG_nu=0.4461; %nu for VG model
VG_theta=-0.0148; %theta of VG model
nsimulations=10000; % no. of MC simulations
nbatches=100; % Extra MC simulations for accuracy
S=S0;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%----------------------------------------------------------
% MC simulation

omega=(1/VG_nu)*( log(1-VG_theta*VG_nu-sigma*sigma*VG_nu/2) );
StockProcess=zeros(nsimulations,1);
for l=1:nbatches
    for i=1:nsimulations
        g = gaminv(unifrnd(0,1), T/VG_nu, VG_nu);
        % The equivalent definition of a VG process
        h = VG_theta*g + sigma*sqrt(g)*norminv(unifrnd(0,1));
        StockProcess(i)=S*exp(r*T+omega*T+h);
```

```matlab
    end
    %figure; plot(StockProcess); title(' Monte Carlo simulation: VG economy
stock price value after 10 years -\nu,\sigma,\theta'); xlabel ('Simulation
number ');  ylabel ('Stock price at maturity (T=10 years; S0=R 1000)');
    % disp(num2str(StockProcess));
    payoffvecC=max(StockProcess-K,0);

    payoffvecP=max(K-StockProcess,0);

    mc_callprice(l)=exp(-r*T)*mean(payoffvecC);

    mc_putprice(l)=exp(-r*T)*mean(payoffvecP);
end

mc_callprice_estimate=mean(mc_callprice);
%disp(mc_callprice_estimate);

%mc_putprice_estimate=mc_callprice_estimate-S+K*exp(-r*T);

mc_putprice_estimate=mean(mc_putprice);

display('Monte Carlo put option price: '),
disp(double(mc_putprice_estimate));
```

Appendix C

C.1 Monte Carlo VGCIR call and put option prices

This Matlab code is an implementation of Section 4.8.1 where the VG process is subordinated to the time integral of a CIR process. The discretisation approach is adopted in the generation of asset paths from the VGCIR process that are then used as inputs to estimate the respective option price.

```matlab
%.....................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate: University of Pretoria
% Levy models with Stochastic Volatility Variance Gamma-Cox Ingersoll Ross
(VGCIR)
% call and put option pricing on the South African JSE ALSI log-returns
using the theoretical framework discussed in Carr et al.(2003) and Fiorani
(2004).
% Adapted from Kienitz and  Wetterau (2012), Financial Modelling-Theory,
% Implementation and Practice with Matlab source and amended accordingly
% for the case at hand.
%.....................................................................%

clear all; clc;
%% Parameters


r = 0.0088;                       % Discount factor
d = 0;                            % Dividend yield


%%%%%%%% parameters setting  %%%%%%%%%%%%%%
S0=input('The initial stock price, S0, is: ');
In=input('Are you pricing an 10-year option with a guarantee (Yes=1, No=0):
');
if In==1
    g=input('The annual guarantee rate in decimal form is: ');
    K=input('The initial guaranteed amount based on the premium is: ');
    K=K*(1+g)^10;
else
    K=input('The strike/guaranteed amount at maturity is: '); %K=1000;
end
T=input('The maturity time, T, in months is: ');



disp(['  The exercise amount used is: ' num2str(K)]);



% VG model parameters: Estimated using method of maximum likelihood
theta=-0.0148;
nu=0.4461;
sigma=0.0544;


C = 1/nu;
G = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)-0.5*theta*nu)^(-1);
M = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)+0.5*theta*nu)^(-1);


kappa = 1.25;      % CIR parameter
eta = 1;           % CIR parameter
lambda = 1;        % CIR parameter
```

```matlab
%% Simulation parameters
NTime = 72; NSim = 3000; NBatches = 100;          % Time steps for the
overall process
NTime_clock = 1000;
intNt = 100;                  % Steps used for integration the CIR clock
allsteps = intNt * NTime;  % All Nts that have to be simulated
deltaT = T / NTime;
time = 0 : T/NTime : T;                           % Variable time for the
martingale correction factor


pathS = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);  % Stores the values of the log asset prices
lnS(:,1) = log(S0);


% precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));    % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                                 % Martingale correction
omegaT(1) = 0;


Y = zeros(NSim,NTime+1);          % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;


for l = 1 : NBatches
    % Generating time change
    deltaaT = T / allsteps;
    sdeltaaT = sqrt(deltaaT);
    W = randn(NSim,allsteps);
    for n = 1 : allsteps
        Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
        %Y1(Y1<0) = 0;                % absorbing
        Y1(Y1<0) = -Y1(Y1<0);         % reflecting
        y(:,n+1) = Y1;
    end
    y(:,1)=0;
    for m=1:NTime
        Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
    end

    Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
    DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
    diffomegaT = omegaT(2:end) - omegaT(1:end-1);     % Martingale
correction

    for m=2:NTime+1
        lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
    end
    pathS(:,:,1) = exp(lnS);
```

```
   payoffvecC=max(pathS(:, NTime+1)-K,0);

   payoffvecP=max(K-pathS(:, NTime+1),0);

   mc_callprice(l)=exp(-r*T)*mean(payoffvecC);

   mc_putprice(l)=exp(-r*T)*mean(payoffvecP);
end

PPrice=mean(mc_putprice);

disp('.............10-year VGCIR European put option
price................');
display(['                        R ', num2str(PPrice)]);
```

C.2 GMDB VGCIR Monte Carlo pricing

```
%..............................................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Levy models with Stochastic Volatility Variance Gamma-Cox Ingersoll Ross
(VGCIR)
% call and put option pricing on the South African JSE ALSI log-returns
using Carr et al.(2003) and Fiorani (2004).
% Adapted from Kienitz and  Wetterau (2012), Financial Modelling-Theory,
% Implementation and Practice with Matlab source and amended accordingly
% for the case at hand.
%..............................................................................%

clear all; clc;
%% Parameters

r = 0.0088;                      % Discount factor
d = 0;                           % Dividend yield

%%%%%%%% parameters setting  %%%%%%%%%%%%%%%
S0=input('The initial stock price, S0, is: ');
D=input('The year of death is (first=1, second=2...): ');
T=D*12;
disp(['  The duration/time to maturity (in months) is: ' num2str(T)]);
g=input('The annual guarantee rate in decimal form is: ');
K=input('The initial guaranteed amount based on the premium is: ');
K=K*(1+g)^(D-1);
disp(['  The exercise amount used is: ' num2str(K)]);
%K=1000; input('The strike/guaranteed amount at maturity is: ');



% VG model parameters: Estimated using method of maximum likelihood
theta=-0.0148;
nu=0.4461;
sigma=0.0544;

C = 1/nu;
G = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)-0.5*theta*nu)^(-1);
M = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)+0.5*theta*nu)^(-1);

kappa = 1.25;      % CIR parameter
eta = 1;          % CIR parameter
```

```matlab
lambda = 1;         % CIR parameter

%% Simulation parameters
NTime = 72; NSim = 3000; NBatches = 100;        % Time steps for the
overall process
NTime_clock = 1000;
intNt = 100;              % Steps used for integration the CIR clock
allsteps = intNt * NTime;  % All Nts that have to be simulated
deltaT = T / NTime;
time = 0 : T/NTime : T;                    % Variable time for the
martingale correction factor

pathS = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);   % Stores the values of the log asset prices
lnS(:,1) = log(S0);

% precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));    % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                              % Martingale correction
omegaT(1) = 0;

Y = zeros(NSim,NTime+1);           % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;

for l = 1 : NBatches
    % Generating time change
    deltaaT = T / allsteps;
    sdeltaaT = sqrt(deltaaT);
    W = randn(NSim,allsteps);
    for n = 1 : allsteps
        Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
        %Y1(Y1<0) = 0;                  % absorbing
        Y1(Y1<0) = -Y1(Y1<0);          % reflecting
        y(:,n+1) = Y1;
    end
    y(:,1)=0;
    for m=1:NTime
        Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
    end

    Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
    DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
    diffomegaT = omegaT(2:end) - omegaT(1:end-1);    % Martingale
correction

    for m=2:NTime+1
    lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
    end
```

```matlab
    pathS(:,:,1) = exp(lnS);

    payoffvecC=max(pathS(:, NTime+1)-K,0);

    payoffvecP=max(K-pathS(:, NTime+1),0);

    mc_callprice(l)=exp(-r*T)*mean(payoffvecC);

    mc_putprice(l)=exp(-r*T)*mean(payoffvecP);
end

PPrice=mean(mc_putprice);

disp('............10-year VGCIR European put option
price................');
display(['                        R ', num2str(PPrice)]);
```

Appendix D

D.1 Delta hedging under the VGCIR framework

The following code is an implementation of the results in Section 5.4.1 where the delta hedging approach is applied using the perturbation approach.

```matlab
%.............................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% Delta hedging levy models with Stochastic Volatility: Variance Gamma-Cox
Ingersoll Ross (VGCIR)
% Adapted from Kienitz and  Wetterau (2012), Financial Modelling-Theory,
% Implementation and Practice with Matlab source and amended accordingly
% for the case at hand.
%.............................................................%

clear all; clc;
%% Parameters

r = 0.0088;                       % Discount factor
d = 0;                            % Dividend yield

%%%%%%%% parameters setting  %%%%%%%%%%%%%%
S=input('The stock price, S(t), is: ');
S0=S;
eps=input('The perturbation used in delta and gamma estimation is: ');

D=input('The no. of years to maturity is (one=1, one and half=1.5,
two=2...): ');
T=D*12;
disp(['  The duration/time to maturity (in months) is: ' num2str(T)]);
g=input('The annual guarantee rate in decimal form is: ');
K=input('The initial guaranteed amount based on the premium is: ');
K=K*(1+g)^(10-1); % Assuming a 10-year option with a roll-up guarantee
disp(['  The exercise amount used is: ' num2str(K)]);

% VG model parameters: Estimated using method of maximum likelihood
theta=-0.0148;
nu=0.4461;
sigma=0.0544;

C = 1/nu;
G = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)-0.5*theta*nu)^(-1);
M = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)+0.5*theta*nu)^(-1);

kappa = 1.25;      % CIR parameter
eta = 1;           % CIR parameter
lambda = 1;        % CIR parameter

%% Simulation parameters
NTime = 72; NSim = 2000; NBatches = 100;        % Time steps for the
overall process
NTime_clock = 1000;
intNt = 100;            % Steps used for integration the CIR clock
allsteps = intNt * NTime;  % All Nts that have to be simulated
deltaT = T / NTime;
time = 0 : T/NTime : T;                        % Variable time for the
martingale correction factor
```

```matlab
pathS = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);  % Stores the values of the log asset prices
lnS(:,1) = log(S0);


% precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));   % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                                % Martingale correction
omegaT(1) = 0;


Y = zeros(NSim,NTime+1);            % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;

for l = 1 : NBatches
    % Generating time change
    deltaaT = T / allsteps;
    sdeltaaT = sqrt(deltaaT);
    W = randn(NSim,allsteps);
    for n = 1 : allsteps
        Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
        %Y1(Y1<0) = 0;                   % absorbing
        Y1(Y1<0) = -Y1(Y1<0);            % reflecting
        y(:,n+1) = Y1;
    end
    y(:,1)=0;
    for m=1:NTime
        Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
    end

    Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
    DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
    diffomegaT = omegaT(2:end) - omegaT(1:end-1);     % Martingale
correction

    for m=2:NTime+1
      lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
    end

    pathS(:,:,1) = exp(lnS);
    path=transpose(pathS(:,:,1));
    path=mean(path,2);
    %disp(path);
    payoffvecP=max(K-pathS(:, NTime+1),0);

    mc_putprice(l)=exp(-r*T)*mean(payoffvecP);

end
```

```matlab
PPrice=mean(mc_putprice);

disp('............10-year VGCIR European put option
price.................');
display(['                          R ', num2str(PPrice)]);

%% Upstate perturbation
S0=S+eps;

pathSU = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);  % Stores the values of the log asset prices
lnS(:,1) = log(S0);

% precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));   % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                           % Martingale correction
omegaT(1) = 0;

Y = zeros(NSim,NTime+1);          % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;

for l = 1 : NBatches
    % Generating time change
  deltaaT = T / allsteps;
  sdeltaaT = sqrt(deltaaT);
  W = randn(NSim,allsteps);
  for n = 1 : allsteps
      Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
      %Y1(Y1<0) = 0;                % absorbing
      Y1(Y1<0) = -Y1(Y1<0);         % reflecting
      y(:,n+1) = Y1;
  end
  y(:,1)=0;
  for m=1:NTime
       Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
  end

  Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
  DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
  diffomegaT = omegaT(2:end) - omegaT(1:end-1);     % Martingale
correction

  for m=2:NTime+1
    lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
  end

  pathSU(:,:,1) = exp(lnS);
```

```matlab
    payoffvecPU=max(K-pathSU(:, NTime+1),0);

    mc_putpriceU(l)=exp(-r*T)*mean(payoffvecPU);


end


PPriceU=mean(mc_putpriceU);


disp('............10-year VGCIR European put option
price.................');
display(['                              R ', num2str(PPriceU)]);


%% Downstate perturbation
S0=S-eps;


pathSD = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);  % Stores the values of the log asset prices
lnS(:,1) = log(S0);


% precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));    % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                              % Martingale correction
omegaT(1) = 0;


Y = zeros(NSim,NTime+1);             % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;


for l = 1 : NBatches
    % Generating time change
  deltaaT = T / allsteps;
  sdeltaaT = sqrt(deltaaT);
  W = randn(NSim,allsteps);
  for n = 1 : allsteps
      Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
      %Y1(Y1<0) = 0;                % absorbing
      Y1(Y1<0) = -Y1(Y1<0);        % reflecting
      y(:,n+1) = Y1;
  end
  y(:,1)=0;
  for m=1:NTime
        Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
  end

  Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
  DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
  diffomegaT = omegaT(2:end) - omegaT(1:end-1);     % Martingale
correction
```

```matlab
    for m=2:NTime+1
     lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
    end

    pathSD(:,:,1) = exp(lnS);

    payoffvecPD=max(K-pathSD(:, NTime+1),0);

    mc_putpriceD(l)=exp(-r*T)*mean(payoffvecPD);

end


PPriceD=mean(mc_putpriceD);

disp('............10-year VGCIR European put option
price.................');
display(['                           R ', num2str(PPriceD)]);
%disp('............10-year VGCIR European put option price (S0 =
1000).................');
%disp(mc_putprice);
%disp('............10-year VGCIR European put option price (Epsilon =
+100).................');
%disp(mc_putpriceU);
%disp('............10-year VGCIR European put option price (Epsilon = -
100).................');
%disp(mc_putpriceD);

%% Delta calculation
delta=(mc_putpriceU-mc_putpriceD)/(2*eps);
disp('............Monte Carlo Delta estimates.................');
%disp(delta);

mdelta=mean(delta);

disp(mdelta);

%% Gamma calculation

gamma=(mc_putpriceU-2*mc_putprice+mc_putpriceD)/(eps^2);
disp('............Monte Carlo Gamma estimates.................');
%disp(gamma);

mgamma=mean(gamma);

disp(mgamma);
```

D.2 GMxB risk measures and loss distribution

The loss distribution results used to calculate the GMxB risk measures in Section 5.5 are calculated using the Matlab implementation below.

```matlab
%.............................................................%
% Master's research: A M Ngugi (2014)
% MSc candidate University of Pretoria
% GMxB loss distribution with Stochastic Volatility: Variance Gamma-Cox
% Ingersoll Ross (VGCIR) dynamics. Adapted from Kienitz and  Wetterau
(2012),
% Financial Modelling-Theory, Implementation and Practice with Matlab
source
% and amended accordingly for the case at hand.
%.............................................................%

clear all; clc;
%% Parameters

r = 0.0088;                      % Discount factor
d = 0;                           % Dividend yield

%%%%%%%%% parameters setting  %%%%%%%%%%%%%%%%
S0=input('The initial stock price, S(0), is: ');
D=input('The no. of years to maturity is (one=1, one and half=1.5,
two=2...): ');
T=D*12;
disp(['  The duration/time to maturity (in months) is: ' num2str(T)]);
g=input('The annual guarantee rate in decimal form is: ');
K=input('The initial guaranteed amount based on the premium is: ');
K=K*(1+g)^(10-1); % Assuming a 10-year option with a roll-up guarantee
disp(['  The exercise amount used is: ' num2str(K)]);

tpx=input('The probability that a life aged x will NOT die within the next
T years: ');
% VG model parameters: Estimated using method of maximum likelihood
theta=-0.0148;
nu=0.4461;
sigma=0.0544;

C = 1/nu;
G = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)-0.5*theta*nu)^(-1);
M = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)+0.5*theta*nu)^(-1);

kappa = 1.25;      % CIR parameter
eta = 1;           % CIR parameter
lambda = 1;        % CIR parameter

%% Simulation parameters
NTime = 72; NSim = 3000; NBatches = 10;        % Time steps for the overall
process
NTime_clock = 1000;
intNt = 100;            % Steps used for integration the CIR clock
allsteps = intNt * NTime;  % All Nts that have to be simulated
deltaT = T / NTime;
time = 0 : T/NTime : T;                    % Variable time for the
martingale correction factor

pathS = zeros(NSim,NTime+1,NBatches);
lnS = ones(NSim,NTime+1);  % Stores the values of the log asset prices
lnS(:,1) = log(S0);
```

```matlab
 % precompute constants
psiVG = (-1i)*C*log(G*M/(G*M+(M-G)-1));   % Characteristic exponent
gamma = sqrt(kappa^2-2*lambda^2*1i*psiVG);  % CIR parameter
denom = ( cosh(0.5*gamma*time) ...
    + kappa*sinh(0.5*gamma*time)./gamma ).^(2*kappa*eta*lambda^(-2));
%denominator
% coth is inf at 0
phiCIR(time>0) = kappa^2*eta*time(time>0)*lambda^(-2) ...
        + 2*1i*psiVG./(kappa+gamma.*coth(gamma*time(time>0)/2)) ...
        - log(denom(time>0)); % Characteristic function
phiCIR(1) = log(denom(1));
omegaT = -phiCIR;                                % Martingale correction
omegaT(1) = 0;

Y = zeros(NSim,NTime+1);            % Integrated clock
y = zeros(NSim,allsteps+1);
y(:,1) = 1;
Y(:,1) = 0;

for l = 1 : NBatches
    % Generating time change
   deltaaT = T / allsteps;
   sdeltaaT = sqrt(deltaaT);
   W = randn(NSim,allsteps);
   for n = 1 : allsteps
       Y1 = y(:,n) + kappa * (eta - y(:,n)) * deltaaT + lambda *
sqrt(y(:,n))*sdeltaaT .* W(:,n); % Time change process
       %Y1(Y1<0) = 0;                % absorbing
       Y1(Y1<0) = -Y1(Y1<0);         % reflecting
       y(:,n+1) = Y1;
   end
   y(:,1)=0;
   for m=1:NTime
        Y(:,m+1) = Y(:,m) + sum(y(:,1+(m-1)*intNt:m*intNt),2)*deltaaT;
   end

   Intensity = C * (Y(:,2:end)-Y(:,1:end-1));
   DGam = gamrnd(Intensity,1/M) - gamrnd(Intensity,1/G);
   diffomegaT = omegaT(2:end) - omegaT(1:end-1);    % Martingale
correction
   for m=2:NTime+1
    lnS(:,m) = lnS(:,m-1) + (r-d)*deltaT + diffomegaT(m-1) + DGam(:,m-1);
%Simulate by difference
   end

   pathS(:,:,1) = exp(lnS);
   path=transpose(pathS(:,:,1));

   PnL=(K-pathS(:, NTime+1)).*tpx.*exp(-r*T);

end
filename = 'testdata.xlsx';
xlswrite(filename,PnL)

disp('...........Empirical distribution for GMMB...............');

disp(PnL);
```