**HEFAT2007**
**5th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics**
**Sun City, South Africa**
**Paper number: MM6**

# MULTI-FLUID STRATIFIED SHEAR FLOWS IN PIPES. PART 2.

# CRITICAL CONDITIONS IN THE DEVELOPMENT OF INTERFACIAL PROFILES

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## ABSTRACT
The focus of this paper is on mathematical formulation and computation of critical flow conditions in horizontal or nearly horizontal pipes. Continuity and momentum equations are derived considering an arbitrary number of fluids and then rearranged so as to yield a system of ordinary differential equations. It is shown that the matrix of the system needs to be invertible so as to compute interfacial profiles. Critical conditions are recognised as those for which the matrix becomes singular and the hypothesis of gradually varied flow fails. Some well known results of linear algebra are here used to define criteria capable of discerning between geometric and kinematic conditions of the flow which are certainly non-critical and others which may or may not be critical.

## INTRODUCTION
Under the hypothesis of stratified shear flows, uniform flow solutions give precious information about the final flow configuration which the fluids flowing in a prismatic pipe will eventually reach at a significant distance from the inlet or the outlet. Uniform flow solutions are therefore independent of possible boundary conditions. In long pipelines, the effects of boundary conditions cover limited lengths compared to the overall pipe length and so may be neglected during the first design step. Conversely, in short pipelines, boundary conditions may influence the overall length of the conduit, so uniform flow solutions are of little use. Actual flow depths developing in pipes due to particular inlet or outlet conditions may in fact inhibit the hydraulic performance of the system leading in turn to bad control of operations.

The focus of this paper is on critical conditions which can be met during the integration of multi-fluid flow profiles. The concept of critical flow depths, quite common in one-dimensional free-surface flows, will be extended to stratified flows with an arbitrary number of fluids.

The analysis will be limited to incompressible and isothermal flows. Continuity and momentum equations will be first derived for the steady state case and then rearranged, finally ending up with a system of ordinary differential equations. The system will be recast in matrix form and solved for the interfacial flow profiles moving either upstream or downstream according to the proper boundary conditions. This can be done until the matrix associated to the system of ordinary differential equations is non singular and therefore invertible. Mathematically speaking critical conditions will be defined as those for which the inverse of the above cited matrix does not exist. Physically speaking critical conditions will be instead classified as those for which the interfacial flow profiles explode, tending to asymptotes. In such situations gradually varied flows will not exist anymore.

The matrix coefficients will be derived in analytical form. Since the matrix singularity can be inferred from its spectrum, using Gershgorin's Circle Theorem on the localisation of matrix eigenvalues, critical conditions will be solved in closed form in some particular cases.

## NOMENCLATURE
Make reference to Part 1 of "Multi-fluid stratified shear flow in pipes" presented at HEFAT 2007.

## GOVERNING EQUATIONS
Consider the pipe geometry shown in **Figure 1**. At any time $t$ the flow of each phase is predominantly along the positive $x$ direction and at any position the pipe is inclined at an angle $q$ from the horizontal. Only incompressible and isothermal flows are here considered, therefore a multi-fluid flow system can be modelled using a combination of one-dimensional continuity and momentum equations in integral form.

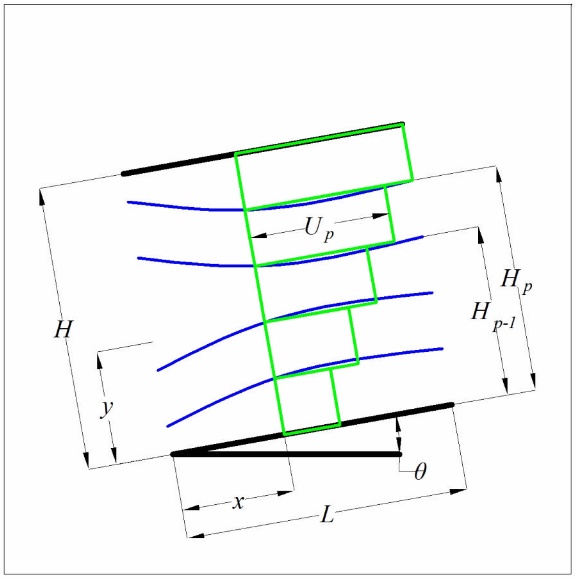


**Figure 1.** Steady non-uniform flow profiles.

The integral form of the continuity equations gives:

$$\frac{\partial}{\partial x}\left(r_p U_p A_p\right)=0 \tag{1}$$

while the integral form of the momentum equations gives:

$$\begin{aligned}&\frac{\partial}{\partial x}\left(r_p y_p U_p^2 A_p\right)=-\frac{\partial}{\partial x}\left(\int_{A_p} P_p \, dA_p\right)\\&-P_p^{\inf} P_{p-1}^j \frac{\partial H_{p-1}}{\partial x}+P_p^{\sup} P_p^j \frac{\partial H_p}{\partial x}\\&-T_p^i P_p^i-T_{p-1}^j P_{p-1}^j+T_p^j P_p^j-r_p A_p g \sin q\end{aligned} \tag{2}$$

Since the flow is incompressible, the continuity equations can be expressed in terms of the interfacial heights.

The pressure distribution is assumed to vary hydrostatically along the transverse section and thus, owing to the effects of interfacial tension $s_p$, the pressure acting on the upper side of each interface differs from the pressure acting on the lower side by a small amount proportional to the interfacial curvature $k_p$:

$$P_p^{\sup}-P_{p+1}^{\inf}+s_p k_p=0 \tag{3}$$

with:

$$k_p=\frac{\dfrac{\partial^2 H_p}{\partial x^2}}{\left(1+\left(\dfrac{\partial H_p}{\partial x}\right)^2\right)^{\frac{3}{2}}}\cong\frac{\partial^2 H_p}{\partial x^2} \tag{4}$$

By using Leibniz's theorem for differentiation of an integral, the momentum equations can be expressed quite straightforwardly as:

$$\begin{aligned}&r_p y_p U_p \frac{\partial U_p}{\partial x}+r_p U_p^2 \frac{\partial y_p}{\partial x}=-\frac{\partial \overline{P}^{\inf}(x,t)}{\partial x}\\&+\sum_{q=1}^{p-1}\left(\begin{array}{l}\left(r_q-r_{q+1}\right) g \cos q \dfrac{\partial H_q(x,t)}{\partial x}\\-s_q \dfrac{\partial k_q(x,t)}{\partial x}\end{array}\right)\\&-T_p^i \frac{P_p^i}{A_p}-T_{p-1}^j \frac{P_{p-1}^j}{A_p}+T_p^j \frac{P_p^j}{A_p}-r_p g \sin q\end{aligned} \tag{5}$$

or as:

$$\begin{aligned}&r_p y_p U_p \frac{\partial U_p}{\partial x}+r_p U_p^2 \frac{\partial y_p}{\partial x}=-\frac{\partial \overline{P}^{\sup}(x,t)}{\partial x}\\&-\sum_{q=1}^{n-p}\left(\begin{array}{l}\left(r_{n-q}-r_{n-q+1}\right) g \cos q \dfrac{\partial H_{n-q}(x,t)}{\partial x}\\-s_{n-q} \dfrac{\partial k_{n-q}(x,t)}{\partial x}\end{array}\right)\\&-T_p^i \frac{P_p^i}{A_p}-T_{p-1}^j \frac{P_{p-1}^j}{A_p}+T_p^j \frac{P_p^j}{A_p}-r_p g \sin q\end{aligned} \tag{6}$$

depending on whether the hydrostatic pressure distribution is calculated in terms of the pressure condition existing at the bottom or top part of the pipe.

Making use of continuity equations, subtracting the momentum equation of phase $p+1$ from the momentum equation of phase $p$ and rearranging, the following combined momentum equations are obtained:

$$\begin{aligned}&r_p y_p U_p \frac{\partial U_p}{\partial x}-r_{p+1} y_{p+1} U_{p+1} \frac{\partial U_{p+1}}{\partial x}\\&+r_p U_p^2 \frac{\partial y_p}{\partial x}-r_{p+1} U_{p+1}^2 \frac{\partial y_{p+1}}{\partial x}\\&+\left(r_p-r_{p+1}\right) g \cos q \frac{\partial H_p}{\partial x}-s_p \frac{\partial k_p}{\partial x}=F_p\end{aligned} \tag{7}$$

where the shear stress functions $F_p$ and the wall-fluid and fluid-fluid shear stresses are defined as in Part 1 of "Multi-fluid stratified shear flows in pipes".

The left hand sides of combined momentum equations take into account the inertial and gravitational terms, the effects of the interfacial tensions and do not depend on the choice of the shear stress closure laws in quasi-steady conditions.

The definitions of the shear stress relationships, instead, is fundamental to model the right hand side terms through which the steady uniform flow conditions are pursued.

## CRITICAL CONDITIONS IN THE INTEGRATION OF FLOW PROFILES

When variations of momentum coefficients and surface tension effects are neglected, continuity and combined momentum equations can be reduced to the form:

$$G_{pq}\frac{\partial H_q}{\partial x}=F_p \tag{8}$$

where:

$$\begin{aligned}G_{pq}=&\left(r_p y_p U_p^2\frac{P_{p-1}^j}{A_p}\right)d(p-1-q)\\&+\left(\begin{array}{l}(r_p-r_{p+1})g\cos q\\-r_p y_p U_p^2\dfrac{P_p^j}{A_p}\\-r_{p+1}y_{p+1}U_{p+1}^2\dfrac{P_p^j}{A_{p+1}}\end{array}\right)d(p-q)\\&+\left(r_{p+1}y_{p+1}U_{p+1}^2\frac{P_{p+1}^j}{A_{p+1}}\right)d(p+1-q)\end{aligned} \tag{9}$$

or, alternatively:

$$\begin{aligned}G_{pq}=&\left(r_p y_p Q_p^2\frac{P_{p-1}^j}{A_p^3}\right)d(p-1-q)\\&+\left(\begin{array}{l}(r_p-r_{p+1})g\cos q\\-r_p y_p Q_p^2\dfrac{P_p^j}{A_p^3}\\-r_{p+1}y_{p+1}Q_{p+1}^2\dfrac{P_p^j}{A_{p+1}^3}\end{array}\right)d(p-q)\\&+\left(r_{p+1}y_{p+1}Q_{p+1}^2\frac{P_{p+1}^j}{A_{p+1}^3}\right)d(p+1-q)\end{aligned} \tag{10}$$

$d$ being the Dirac's delta function.

The $G$ matrix inverse is rather important in computing steady non-uniform profiles of stratified shear flows. When $\det G\neq 0$, $G$ is said to be non singular and $G^{-1}$ exists, therefore interfacial slopes follow from:

$$\frac{\partial H_p}{\partial x}=G^{-1}{}_{pq}F_q \tag{11}$$

while flow profiles can be computed using for instance a standard step method and integrating either in the upstream or downstream direction according to the appropriate kinematic conditions. Conversely when $\det(G)=0$, $G$ is said to be singular and $G^{-1}$ does not exist implying that interfacial slopes reach a singular point tending to infinity. Singular points represent therefore critical physical situations in which the hypothesis of gradually varied flow is lost. The identification of such critical conditions or, less restrictively, of criteria discerning within geometric and kinematic conditions which are certainly non-critical and others which may or may not be critical is therefore of fundamental importance and is the objective of this work. Given that the $G$ matrix singularity can be inferred from its spectrum, Gershgorin's Circle Theorem [1,2] on the localisation of matrix eigenvalues is widely used to give sufficient conditions for the existence of its inverse. From the theorem , for instance, it follows that a strictly diagonally dominant matrix is non singular and its inverse can be computed. Since G is a diagonally dominant when:

$$\left|G_{pp}\right|\geq\sum_{\substack{q=1\\q\neq p}}^{n-1}\left|G_{pq}\right| \tag{12}$$

or:

$$\left|G_{pp}\right|\geq\sum_{\substack{q=1\\q\neq p}}^{n-1}\left|G_{qp}\right| \tag{13}$$

the elements on the principal diagonal are to be different from zero (i.e. $G_{pp}\neq 0$). From previous definitions it follows easily that $G_{pp}\neq 0$ when:

$$a_p Q_p^2+b_p Q_{p+1}^2\neq g_p \tag{14}$$

with:

$$a_p=r_p y_p\frac{P_p^j}{A_p^3}$$

$$b_p=r_{p+1}y_{p+1}\frac{P_p^j}{A_{p+1}^3}$$

$$g_p=(r_p-r_{p+1})g\cos q$$

When $G_{pp}<0$, conditions (12) may be expressed as:

$$-a_p Q_p^2+b_p Q_{p+1}^2>g_p \tag{15}$$

with:

$$a_p=r_p y_p\frac{P_{p-1}^j-P_p^j}{A_p^3}$$

$$b_p=r_{p+1}y_{p+1}\frac{P_p^j-P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

while conditions (13) may be expressed as:

$$-a_p Q_p^2 + b_p Q_{p+1}^2 > g_p \qquad (16)$$

with:

$$a_p = r_p y_p \frac{P_p^j - P_p^j}{A_p^3} = 0$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j - P_p^j}{A_{p+1}^3} = 0$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

When $G_{pp} > 0$, conditions (12) may be expressed as:

$$a_p Q_p^2 + b_p Q_{p+1}^2 < g_p \qquad (17)$$

with:

$$a_p = r_p y_p \frac{P_{p-1}^j + P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

while conditions (13) may be expressed as:

$$a_p Q_p^2 + b_p Q_{p+1}^2 < g_p \qquad (18)$$

with:

$$a_p = r_p y_p \frac{P_p^j + P_p^j}{A_p^3} = r_p y_p \frac{2P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_p^j}{A_{p+1}^3} = r_{p+1} y_{p+1} \frac{2P_p^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

## CRITICAL CONDITIONS FOR CO-CURRENT AND COUNTER-CURRENT FLOWS WITHOUT GRAVITATIONAL STRATIFICATION

In this paragraph, flow conditions which are critical in the sense of previous definitions are outlined for co-current and counter-current flows characterised by the absence of gravitational stratification. Two cases, each one to be further sub-divided in two sub-cases, are to be considered.

**Case 1**

$$G_{pp} < 0 \qquad (19)$$

**Case 1.1**

$G$ is a diagonally dominant matrix (i.e. $|G_{pp}| \geq |G_{pp-1}| + |G_{pp+1}|$) if:

$$-a_p Q_p^2 + b_p Q_{p+1}^2 \geq g_p \qquad (20)$$

with:

$$a_p = r_p y_p \frac{P_{p-1}^j - P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j - P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = 0$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by a couple of lines intersecting in the origin if $P_{p-1}^j < P_p^j$ and $P_p^j < P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are unbounded if $P_{p-1}^j < P_p^j$ and $P_p^j = P_{p+1}^j$.

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Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist if $P_{p-1}^j = P_p^j$ and $P_p^j < P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are indeterminate if $P_{p-1}^j = P_p^j$ and $P_p^j = P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are unbounded if $P_{p-1}^j = P_p^j$ and $P_p^j > P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist if $P_{p-1}^j > P_p^j$ and $P_p^j < P_{p+1}^j$.

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Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by a couple of lines intersecting in the origin if $P_{p-1}^j > P_p^j$ and $P_p^j > P_{p+1}^j$.

**Case 1.2**

$G^T$ is a diagonally dominant matrix (i.e. $|G_{pp}| \geq |G_{p-1p}| + |G_{p+1p}|$) if:

$$-a_p Q_p^2 + b_p Q_{p+1}^2 \geq g_p \qquad (21)$$

with:

$$a_p = r_p y_p \frac{P_p^j - P_p^j}{A_p^3} = 0$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j - P_p^j}{A_{p+1}^3} = 0$$

$$g_p = 0$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are unbounded.

**Case 2**

$$G_{pp} > 0 \tag{22}$$

**Case 2.1**

$G$ is a diagonally dominant matrix (i.e. $|G_{pp}| \ge |G_{pp-1}| + |G_{pp+1}|$) if:

$$a_p Q_p^2 + b_p Q_{p+1}^2 \le g_p \tag{23}$$

with:

$$a_p = r_p y_p \frac{P_{p-1}^j + P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = 0$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist.

**Case 2.2**

$G^T$ is a diagonally dominant matrix (i.e. $|G_{pp}| \ge |G_{p-1p}| + |G_{p+1p}|$) if:

$$a_p Q_p^2 + b_p Q_{p+1}^2 \le g_p \tag{24}$$

with:

$$a_p = r_p y_p \frac{P_p^j + P_p^j}{A_p^3} = r_p y_p \frac{2P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_p^j}{A_{p+1}^3} = r_{p+1} y_{p+1} \frac{2P_p^j}{A_{p+1}^3}$$

$$g_p = 0$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist.

## CRITICAL CONDITIONS FOR CO-CURRENT AND COUNTER-CURRENT FLOWS WITH GRAVITATIONAL STRATIFICATION

In this paragraph, flow conditions which are critical in the sense of previous definitions are outlined for co-current and counter-current flows characterised by the absence of gravitational stratification. Two cases, each one to be further sub-divided in two sub-cases, are to be considered.

**Case 1**

$$G_{pp} < 0 \tag{25}$$

**Case 1.1**

$G$ is a diagonally dominant matrix (i.e. $|G_{pp}| \ge |G_{pp-1}| + |G_{pp+1}|$) if:

$$-a_p Q_p^2 + b_p Q_{p+1}^2 \ge g_p \tag{26}$$

with:

$$a_p = r_p y_p \frac{P_{p-1}^j - P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j - P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by a couple of hyperbolas if $P_{p-1}^j < P_p^j$ and $P_p^j < P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by a couple of lines if $P_{p-1}^j < P_p^j$ and $P_p^j = P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by an ellipse if $P_{p-1}^j < P_p^j$ and $P_p^j > P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist if $P_{p-1}^j = P_p^j$ and $P_p^j < P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are indeterminate if $P_{p-1}^j = P_p^j$ and $P_p^j = P_{p+1}^j$.

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Solutions $(Q_p, Q_{p+1})$ satisfying inequalities do not exist if $P_{p-1}^j > P_p^j$ and $P_p^j = P_{p+1}^j$.

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by a couple of hyperbolas if $P_{p-1}^j > P_p^j$ and $P_p^j > P_{p+1}^j$.

**Case 1.2**

$G^T$ is a diagonally dominant matrix (i.e. $|G_{pp}| \geq |G_{p-1p}| + |G_{p+1p}|$) if:

$$-a_p Q_p^2 + b_p Q_{p+1}^2 \geq g_p \quad (27)$$

with:

$$a_p = r_p y_p \frac{P_p^j - P_p^j}{A_p^3} = 0$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j - P_p^j}{A_{p+1}^3} = 0$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are indeterminate.

**Case 2**

$$G_{pp} > 0 \quad (28)$$

**Case 2.1**

$G$ is a diagonally dominant matrix (i.e. $|G_{pp}| \geq |G_{pp-1}| + |G_{pp+1}|$) if:

$$a_p Q_p^2 + b_p Q_{p+1}^2 \leq g_p \quad (29)$$

with:

$$a_p = r_p y_p \frac{P_{p-1}^j + P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_{p+1}^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by an ellipse.

**Case 2.2**

$G^T$ is a diagonally dominant matrix (i.e. $|G_{pp}| \geq |G_{p-1p}| + |G_{p+1p}|$) if:

$$a_p Q_p^2 + b_p Q_{p+1}^2 \leq g_p \quad (30)$$

with:

$$a_p = r_p y_p \frac{P_p^j + P_p^j}{A_p^3} = r_p y_p \frac{2P_p^j}{A_p^3}$$

$$b_p = r_{p+1} y_{p+1} \frac{P_p^j + P_p^j}{A_{p+1}^3} = r_{p+1} y_{p+1} \frac{2P_p^j}{A_{p+1}^3}$$

$$g_p = (r_p - r_{p+1}) g \cos q$$

Solutions $(Q_p, Q_{p+1})$ satisfying inequalities are $\infty^2$ and are asymptotically bounded by an ellipse.

## CONCLUSIONS

In this paper a general mathematical model aiming at computation of critical flow conditions in horizontal or nearly horizontal pipes has been developed. The model stems from 1-D continuity and momentum equations in integral form treating an arbitrary number of fluids. It has been shown that these equations can be rearranged so as to yield a system of ordinary differential equations and that the matrix of the system needs to be invertible so as to compute interfacial profiles. Critical conditions have been recognised as those for which the matrix becomes singular and the hypothesis of gradually varied flow fails. Some well known results of linear algebra have been used to define criteria capable of discerning between geometric and kinematic conditions of the flow which are certainly non-critical and others which may or may not be critical.

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