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MULTI-FLUID STRATIFIED SHEAR FLOWS IN PIPES. PART 1.

UNIFORM FLOW SOLUTIONS

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ABSTRACT

The focus of this paper is on mathematical formulation and computation of uniform flow solutions in horizontal or nearly horizontal pipes. Continuity and momentum equations are derived considering an arbitrary number of fluids. Closure relationships are introduced and complete definitions of hydraulic diameters are given for co-current and counter-current flows. A numerical procedure to compute phase hold-ups and pressure gradients is outlined. The iterative and direct method is shown to be robust and not limited by the choice of the friction factor correlations needed to define shear stresses. Uniform flow equations are linearized so that Newton's methods can be applied in the search for solutions. It is given evidence that the Jacobian matrix of the shear stress functions can be used as a measure of the flow stability.

INTRODUCTION

Complex mixtures of immiscible fluids flowing together are frequently encountered in a variety of practical applications in civil, hydroelectric and nuclear plants as well as in chemical-process, petroleum, power and space industries. Classically they are referred to as multi-fluid or multi-phase flows.

Two fluid liquid-gas flows occur naturally in many hydraulic structures. In dam bottom outlets and side-spillways and in overflow structures commonly used in sewer systems such as baffled weirs and leaping weirs, significant water-air interactions may develop at the interfaces, so that incorrect results may be obtained when considering classical hydraulic theories. Two-fluid models should then be used in these cases to capture the actual flow behaviour. Two fluid liquid-gas and three-fluid liquid-liquid-gas flows have great importance in the oil and gas industry, given that all hydrocarbon reservoirs contain water, oil and gas in proportions which vary during the exploitation. When the production of a new petroleum reservoir starts, owing to the huge pressure existing in the accumulation, the quantity of water in the well flow rate is negligible, regardless of whether the field is bounded or not by an aquifer. In this first stage, two-fluid flows are far more prevalent than three-fluid flows. With ageing of the field, the productivity decreases as a result of the pressure depletion associated with fluid production. The negative effects of this inevitable pressure reduction are usually mitigated either by the water infiltrating naturally from a surrounding or underlying aquifer or by artificially injecting the water into the depleted reservoir. In both cases the quantity of water in the well flow rate grows significantly. In this second stage, three-fluid flows are far more frequent than two-fluid flows. On a world-wide basis, most petroleum companies produce already more water than oil. Furthermore most oil corporations have recognised the economical advantages of sending the well production directly to existing platforms, where the hydrocarbons may then be separated, instead of building new pricey platforms near the oil fields. Three-fluid flows therefore are becoming recurrent also in pipelines used for hydrocarbon recovery.

In this work, the analysis will be limited to incompressible and isothermal stratified shear flows, focusing on the definition and computation of uniform solutions in horizontal or nearly horizontal pipes. Continuity and momentum equations will be derived considering an arbitrary number of fluids. Closure relationships will be introduced and complete definitions of hydraulic diameters will be given for co-current and counter-current flows. A numerical procedure to compute phase hold-ups and pressure gradients will be outlined. The iterative and direct method will be shown to be robust and not limited by the choice of the friction factor correlations needed to define shear stresses. Uniform flow equations will be linearized thus making Newton's methods applicable in the search for solutions. It will be given evidence that the derived coefficients can be used as a measure of the flow stability.

NOMENCLATURE

The following list of symbols is common to Part 1, Part 2 and Part 3 of “Multi-fluid stratified shear flows in pipes”. All parts are presented at HEFAT 2007.

Roman symbols: upper-case letters

A	[m^2]	Area of the pipe
A_p	[m^2]	Area of the cross-section surface occupied by fluid p
D	[m]	Diameter of the pipe
D_p	[m]	Diameter of the equivalent hydraulic conduit wetted by fluid p
H_p	[m]	Height of the fluid-fluid interface between fluid p and fluid $p+1$
P	[m]	Perimeter of the pipe
P_p^i	[m]	Perimeter of the wall-fluid interface wetted by fluid p
P_p^j	[m]	Perimeter of the fluid-fluid interface between fluid p and fluid $p+1$
Q_p	[m^{3}s^{-1}]	Volumetric discharge of fluid p through its corresponding cross-section surface
U_p	[ms^{-1}]	Average velocity of fluid p through its corresponding cross-section surface

Roman symbols: lower-case letters

c	[ms^{-1}]	Wave celerity
f_p^i	[-]	Friction factor at the wall-fluid interface wetted by fluid p
f_p^j	[-]	Friction factor at the fluid-fluid interface between fluid p and fluid $p+1$
g	[ms^{-2}]	Acceleration due to gravity
i	[-]	Imaginary unit
k	[m^{-1}]	Wave number
m	[-]	Calibration parameter
n	[m]	Number of fluids
r	[m]	Absolute roughness of the pipe
t	[s]	Time
x		
y	[m]	Cartesian axis directions
z		

Greek symbols: upper-case letters

P_p	[kgm^{-1}s^{-2}]	Pressure acting on the cross-section surface occupied by fluid p
$P_p^{\inf}$	[kgm^{-1}s^{-2}]	Pressure acting on the lowermost part of the cross-section surface occupied by fluid p
$P_p^{\sup}$	[kgm^{-1}s^{-2}]	Pressure acting on the uppermost part of the cross-section surface occupied by fluid p
T_p^i	[kgm^{-1}s^{-2}]	Shear stress acting on the wall-fluid interface wetted by fluid p
T_p^j	[kgm^{-1}s^{-2}]	Shear stress acting on the fluid-fluid interface between fluid p and fluid $p+1$

Greek symbols: lower-case letters

a_p b_p g_p	[-]	Coefficients used to define critical conditions at the fluid-fluid interface between fluid p and fluid $p+1$
q	[°]	Angle of inclination of the pipe
k_p	[m^{-1}]	Curvature of the fluid-fluid interface between fluid p and fluid $p+1$
l	[m]	Wave length
m_p	[kgm^{-1}s^{-1}]	Dynamic viscosity of fluid p
n_p	[m^{2}s^{-1}]	Kinematic viscosity of fluid p
r_p	[kgm^{-3}]	Density of fluid p
s_p	[kgs^{-2}]	Surface tension at the fluid-fluid interface between fluid p and fluid $p+1$
y_p	[-]	Momentum coefficient of fluid p
w	[rads^{-1}]	Wave angular frequency

Superscripts

~	[-]	Dimensionless quantity
-	[-]	Steady state value
^	[-]	Perturbed value
i	[-]	Wall-fluid interface
j	[-]	Fluid-fluid interface
inf	[-]	Lowermost part of the cross-section surface
sup	[-]	Uppermost part of the cross-section surface

Subscripts

p, q	[-]	Indexes denoting either the fluid, the wall-fluid interface etc. $p = 1,..., n$ or the fluid-fluid interface $p = 1,..., n-1$

GOVERNING EQUATIONS

Consider the pipe geometry shown in **Figure 1**. At any time t the flow of each phase is predominantly along the positive x direction and at any position the pipe is inclined at an angle q from the horizontal. Only incompressible and isothermal flows are here considered, therefore a multi-fluid flow system can be modelled using a combination of one-dimensional continuity and momentum equations in integral form.

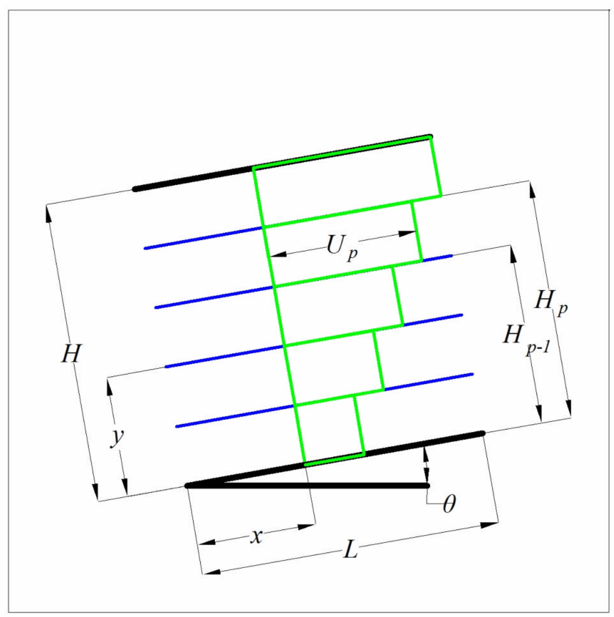

Figure 1. Steady uniform flow profiles.

By assuming a fully developed flow with non significant acceleration or change in properties (i.e. $\partial/\partial t(\ ) = \partial/\partial x(\ ) = 0$) continuity equations can be expressed in the following form:

$$U_p A_p = Q_p \tag{1}$$

while momentum equations result in:

$$0=-A_p\frac{dP_p}{dx}-T_p^iP_p^i$$
$$-T_{p-1}^jP_{p-1}^j+T_p^jP_p^j-\rho_pA_pg\sin\theta \qquad (2)$$

CLOSURE RELATIONSHIPS

Additional closure laws are required to solve the above set of equations because the number of variables to be determined exceeds the available number of equations. The additional relationships follow from the hypothesis of hydrostatic pressure distribution on the flow cross-section, as sketched in **Figure 2**, implying:

$$\frac{dP_p}{dx}=\frac{dP_{p+1}}{dx} \qquad (3)$$

and from definitions of wall-fluid and fluid-fluid shear stresses in terms of the kinematic flow field so that:

$$T_p^i=T_p^i\left(H_1,\ldots,H_{n-1},U_1,\ldots,U_n\right) \qquad (4)$$

$$T_p^j=T_p^j\left(H_1,\ldots,H_{n-1},U_1,\ldots,U_n\right) \qquad (5)$$

In particular, following the approach usually adopted in multi-phase flow calculations, the average wall-fluid shear stresses can be predicted in terms of the average velocities of the fluids as:

$$T_p^i=f_p^i\frac{\rho_p^iU_p^i\left|U_p^i\right|}{2} \qquad (6)$$

where:

$$U_p^i=U_p$$

and:

$$f_p^i=f_p^i\left(\mathrm{Re}_p,\frac{\varepsilon}{D_p}\right)\qquad \rho_p^i=\rho_p$$

with:

$$\mathrm{Re}_p=\frac{\rho_p\left|U_p\right|D_p}{\mu_p}=\frac{\left|U_p\right|D_p}{\nu_p}$$

while the fluid-fluid shear stresses can be calculated as:

$$T_p^j=f_p^j\frac{\rho_p^jU_p^j\left|U_p^j\right|}{2} \qquad (7)$$

where:

$$U_p^j=U_{p+1}-U_p$$

and:

$$f_p^j=\begin{cases}f_p^i & \left|U_p\right|\geq\left|U_{p+1}\right|\\ f_{p+1}^i & \left|U_p\right|<\left|U_{p+1}\right|\end{cases}\qquad \rho_p^j=\begin{cases}\rho_p & \left|U_p\right|\geq\left|U_{p+1}\right|\\ \rho_{p+1} & \left|U_p\right|<\left|U_{p+1}\right|\end{cases}$$

The interfacial friction factors are therefore evaluated as equal to the wall friction factors pertaining to the faster phase at each interface [2]. Possible definitions of hydraulic diameters for co-current and counter-current flows are reported in Appendix.

A general method for evaluating the wall friction factors in two-phase one-dimensional models is to adopt single-phase flow correlations in terms of the Reynolds number and the pipe roughness. The definition of the interfacial friction factors represents instead a much more controversial question, because of the complex interactions occurring between the two fluids when the interface is not flat [1,2]. Considerable attention has been given to this issue over the past years and many correlation laws have then been elaborated. The same approach has been adopted in three-phase one-dimensional models and friction factors have been obtained partly modifying two-phase flow correlations and partly proposing new ones [4,5].

It is to be stressed that the accuracy of the predictions (i.e. pressure gradient and phase hold-ups) may vary in a significant manner using different correlation laws and even the most appropriate one may still result in considerable discrepancies between predictions and experimental measurements. Clearly, this is true of both two-phase and three-phase flows.

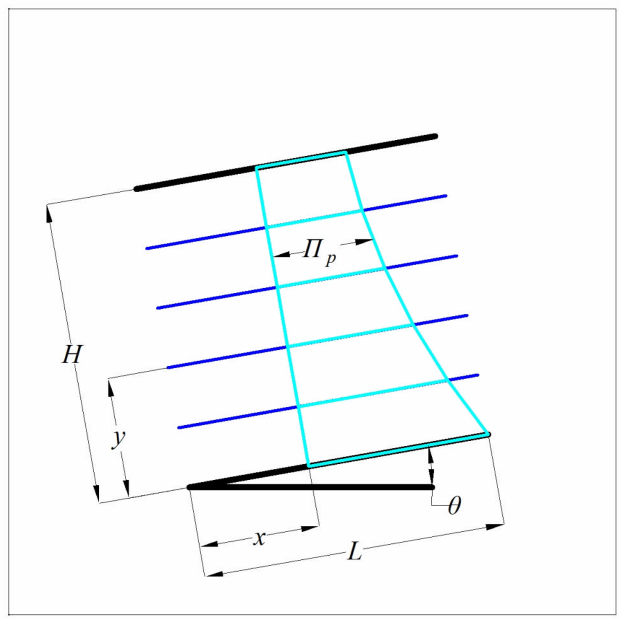

Figure 2. Hydrostatic pressure distribution.

COMPUTATION OF SOLUTIONS

The hypothesis of hydrostatic pressure distribution on the cross-section and previous definitions of shear stresses make it possible to rearrange momentum equations so as so as to yield $n-1$ non-linear algebraic equations:

$$F_p=\frac{dP_p}{dx}-\frac{dP_{p+1}}{dx}=0 \qquad (8)$$

with the functions F_p being defined as:

$$
\begin{aligned}
F_p = & -T_p^{\,i}\frac{P_p^i}{A_p} + T_{p+1}^{\,i}\frac{P_{p+1}^i}{A_{p+1}} \\
& -T_{p-1}^{\,j}\frac{P_{p-1}^j}{A_p} + T_p^{\,j}\left(\frac{P_p^j}{A_p} + \frac{P_p^j}{A_{p+1}}\right) - T_{p+1}^{\,j}\frac{P_{p+1}^j}{A_{p+1}} \\
& -\left(r_p - r_{p+1}\right) g \sin q
\end{aligned} \tag{9}
$$

These equations, when the fluid properties and pipe geometric characteristics are given and the volumetric flow discharges are specified trough continuity equations, depend just on $n-1$ unknowns i.e. the $n-1$ interface levels $\left(H_1,\ldots,H_p,\ldots,H_{n-1}\right)$ and make the problem well posed.

Instead of searching for the zeroes of the non linear system of equations directly, an equivalent and easier to handle problem is here considered. In fact, defining a new function F^* as:

$$F^* = F_p F_p \tag{10}$$

it is easy to show that each solution $\left(H_1^*,\ldots,H_p^*,\ldots,H_{n-1}^*\right)$ which minimises F^* is also a solution of the original problem when the following constraints are satisfied:

$$
\begin{aligned}
& \left(H_1^*,\ldots,H_p^*,\ldots,H_{n-1}^*\right) \in \prod_{p=1}^{n-1} \left]0,H\right[\\
& H_1^* < \ldots < H_p^* < \ldots < H_{n-1}^*
\end{aligned} \tag{11}
$$

$$\det\left(J\left(\boldsymbol{F}\right)\right) \neq 0 \tag{12}$$

with each element of the Jacobian matrix of the original system being defined explicitly as:

$$\frac{dF_p}{dH_q} = \frac{\partial F_p}{\partial H_q} - U_q \frac{P_q^j}{A_q}\frac{\partial F_p}{\partial U_q} + U_{q+1}\frac{P_q^j}{A_{q+1}}\frac{\partial F_p}{\partial U_{q+1}} \tag{13}$$

It is to be stressed that the Jacobian matrix of the shear stress functions is also relevant in performing viscous Kelvin-Helmholtz stability analysis of the flow configurations, as it will be fully shown in Part 3 of this work.

To solve the equivalent optimisation problem an iterative direct method can be adopted. Basically the numerical algorithm generates a sequence of hyper-cubes C_l^{n-1} in which the solution is likely to be found and a sequence of approximated solutions $\left(H_{1l}^*,\ldots,H_{pl}^*,\ldots,H_{n-1l}^*\right)$ which are included in C_l^{n-1}. Mathematically speaking it is:

$$
\begin{aligned}
& \left(H_{1l}^*,\ldots,H_{pl}^*,\ldots,H_{n-1l}^*\right) \in C_l^{n-1} \equiv \prod_{p=1}^{n-1} \left]H_{pl}^a, H_{pl}^b\right[\\
& H_{1l}^* < \ldots < H_{pl}^* < \ldots < H_{n-1l}^*
\end{aligned} \tag{14}
$$

The approximated solutions are obtained examining directly the values assumed by the function F^* on a finite number of points either regularly or irregularly spaced into a grid in C_l^{n-1} and taking the minimum. The region in which to search for solutions is completely defined through the coordinates of the open intervals $\left]H_{pl}^a, H_{pl}^b\right[$. The geometric points H_{pl}^a, H_{pl}^b and the mean points $H_{pl}^c = \left(H_{pl}^a + H_{pl}^b\right)/2$ are then used to orient the search in the next iteration, yielding 2^{n-1} possible alternatives for the choice of C_{l+1}^{n-1}. In particular it is:

$$
\begin{aligned}
& H_{pl+1}^a = H_{pl}^a \\
& H_{pl+1}^b = H_{pl}^c
\end{aligned} \quad \text{if } H_{pl}^* \in \left]H_{pl}^a, H_{pl}^c\right[\tag{15}
$$

while:

$$
\begin{aligned}
& H_{pl+1}^a = H_{pl}^c \\
& H_{pl+1}^b = H_{pl}^b
\end{aligned} \quad \text{if } H_{pl}^* \in \left]H_{pl}^c, H_{pl}^b\right[\tag{16}
$$

Since after each iteration the bounds containing the solutions decrease by a factor of two, the number of iterations required to achieve a given tolerance in the solution can be determined a priori. Once the steady state solutions have been found, all the other geometric quantities follow quite straightforwardly and the actual pressure gradient is calculated as the average of the pressure gradient in each phase.

The procedure herein illustrated describes how the routine proceeds when a unique solution is expected to exist. Whenever non-unique solutions are expected to exist, instead, the routine finds them by applying the single-solution numerical method to a discrete set of disjoint subintervals of the solution space. Quite obviously, in such a case the procedure may either converge or not, thus indicating respectively that a solution exists or not in a particular subinterval. The maximum number of multiple solutions which can be obtained is clearly equal to the number of disjoint subintervals. Given these premises, it follows immediately that, no matter how many solutions may exist, convergence to at least one solution is always assured as long as the grid points are sufficiently numerous. Note that the choice of the number of grid points is determined by the Frobenius norm of the Jacobian matrix $J\left(\boldsymbol{F}\right)$:

$$\left\|\boldsymbol{JF}\right\|_F = \sqrt{\sum_{p=1}^{n-1}\sum_{q=1}^{n-1}\left|J\left(\boldsymbol{F}\right)_{pq}\right|^2} \tag{17}$$

Different numerical methods can be implemented in order to solve the structure of multi-phase flow [3,4,5]. All those procedures and the one herein described, however, reflect and extend the original approach presented for two-phase flow calculations [6].

CONCLUSIONS

In this paper a general mathematical model aiming at computation of uniform flow solutions in horizontal or nearly horizontal pipes has been developed. The model stems from 1-D continuity and momentum equations in integral form treating an arbitrary number of fluids. Closure relationships have been introduced and possible definitions of hydraulic diameters have been described for co-current and counter-current flows. A numerical procedure to compute phase hold-ups and pressure gradients has been outlined. The iterative and direct method has been shown to be robust and not limited by the choice of the friction factor correlations needed to define shear stresses. Linear equations have been also considered to appropriately use Newton's methods. It has been pointed out that the Jacobian matrix of the shear stress functions is strictly needed to perform viscous Kelvin-Helmholtz stability analysis.

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APPENDIX. HYDRAULIC DIAMETERS FOR CO-CURRENT AND COUNTER-CURRENT FLOWS

In the multi-fluid flow model, equivalent hydraulic diameters are defined according to the relative velocity of the phases. Adjustable definitions need to be adopted because the velocities of the fluids may either be of comparable magnitude (i.e. liquid-liquid systems) or differ significantly from one another (i.e. gas-liquid systems). The general case in which each fluid has a contact surface with the wall and with both the lower and upper fluids is described here in details. Specific cases in which the fluid has a contact surface with the wall and with either the lower or upper fluid are not treated since they can be easily derived from the general case.

Flows are recognised as co-current whenever the lowermost, intermediate and uppermost fluids run along either the positive or negative axis direction (i.e. $U_{p-1}U_p \geq 0$ and $U_pU_{p+1} \geq 0$). Definitions of the hydraulic diameters follow quite straightforwardly from considerations on the relative fluid velocities.

If $\left|U_{p-1}\right| > \left|U_p\right|$ and $\left|U_p\right| > \left|U_{p+1}\right|$ then:

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \tag{A.1}$$

if $\left|U_{p-1}\right| > m\left|U_p\right|$ and $\left|U_p\right| > m\left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.2}$$

if $\left|U_{p-1}\right| > m\left|U_p\right|$ and $\left|U_p\right| < m\left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \tag{A.3}$$

if $\left|U_{p-1}\right| < m\left|U_p\right|$ and $\left|U_p\right| > m\left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.4}$$

if $\left|U_{p-1}\right| < m\left|U_p\right|$ and $\left|U_p\right| < m\left|U_{p+1}\right|$

If $\left|U_{p-1}\right| > \left|U_p\right|$ and $\left|U_p\right| < \left|U_{p+1}\right|$ then:

$$D_p = \frac{4A_p}{P_p^i} \tag{A.5}$$

if $\left|U_{p-1}\right| > m\left|U_p\right|$ and $m\left|U_p\right| > \left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.6}$$

if $\left|U_{p-1}\right| > m\left|U_p\right|$ and $m\left|U_p\right| < \left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.7}$$

if $\left|U_{p-1}\right| < m\left|U_p\right|$ and $m\left|U_p\right| > \left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.8}$$

if $\left|U_{p-1}\right| < m\left|U_p\right|$ and $m\left|U_p\right| < \left|U_{p+1}\right|$

If $\left|U_{p-1}\right| < \left|U_p\right|$ and $\left|U_p\right| > \left|U_{p+1}\right|$ then:

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \tag{A.9}$$

if $m\left|U_{p-1}\right| > \left|U_p\right|$ and $\left|U_p\right| > m\left|U_{p+1}\right|$

$$D_p = \frac{4A_p}{P_p^i} \tag{A.10}$$

if $m|U_{p-1}| > |U_p|$ and $|U_p| < m|U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j + P_p^j} \quad (A.11)$$

if $m|U_{p-1}| < |U_p|$ and $|U_p| > m|U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.12)$$

if $m|U_{p-1}| < |U_p|$ and $|U_p| < m|U_{p+1}|$

If $|U_{p-1}| < |U_p|$ and $|U_p| < |U_{p+1}|$ then:

$$D_p = \frac{4A_p}{P_p^i} \quad (A.13)$$

if $m|U_{p-1}| > |U_p|$ and $m|U_p| > |U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i} \quad (A.14)$$

if $m|U_{p-1}| > |U_p|$ and $m|U_p| < |U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.15)$$

if $m|U_{p-1}| < |U_p|$ and $m|U_p| > |U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.16)$$

if $m|U_{p-1}| < |U_p|$ and $m|U_p| < |U_{p+1}|$

A first case of counter-current flow may be identified whenever the lowermost fluid and the intermediate one move co-currently along either the positive or negative axis direction while the uppermost fluid runs in the opposite way (i.e. $U_{p-1}U_p \geq 0$ and $U_pU_{p+1} \leq 0$). Possible and different velocity distributions may then be discerned yielding various definitions for the hydraulic diameters.

If $|U_{p-1}| > |U_p|$ then:

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \quad (A.17)$$

if $|U_{p-1}| > m|U_p|$

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \quad (A.18)$$

if $|U_{p-1}| < m|U_p|$

If $|U_{p-1}| < |U_p|$:

$$D_p = \frac{4A_p}{P_p^i + P_p^j} \quad (A.19)$$

if $m|U_{p-1}| > |U_p|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j + P_p^j} \quad (A.20)$$

if $m|U_{p-1}| < |U_p|$

A second case of counter-current flow may be identified whenever the uppermost fluid and the intermediate one run co-currently along either the positive or negative axis direction while the lowermost fluid moves in the opposite way (i.e. $U_{p-1}U_p \leq 0$ and $U_pU_{p+1} \geq 0$). Possible and different velocity configurations may then be discerned yielding various definitions for the hydraulic diameters.

If $|U_p| > |U_{p+1}|$:

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j + P_p^j} \quad (A.21)$$

if $|U_p| > m|U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.22)$$

if $|U_p| < m|U_{p+1}|$

If $|U_p| < |U_{p+1}|$:

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.23)$$

if $m|U_p| > |U_{p+1}|$

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j} \quad (A.24)$$

if $m|U_p| < |U_{p+1}|$

The last case of counter-current flow which may be observed occurs whenever the intermediate fluid runs along either the positive or negative axis direction while the lowermost and uppermost fluids flow co-currently in the opposite way (i.e. $U_{p-1}U_p \leq 0$ and $U_pU_{p+1} \leq 0$). The hydraulic diameters result so defined:

$$D_p = \frac{4A_p}{P_p^i + P_{p-1}^j + P_p^j} \quad (A.25)$$