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ON A DICHOTOMY IN NEWTONIAN CONTINUUM MECHANICS
(A SURVEY)

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ABSTRACT

It has been observed that *classical* thermomechanics can be founded upon the requirement that the total energy equation is to be consistent with the first axiom of Newton. Then, since the boost velocity in Galilean transformations is arbitrary, a balance of forces must result. Mass invariance is then a consistency requirement.

Newtonian *continuum* mechanics is not totally consistent since *material response* to applied forces is not constrained by the first axiom of Newton: the larger Euclidean group must be adopted for all constitutive aspects of the theory. This dichotomy allows the Cauchy stress tensor symmetry to be explored from the thermal energy equation covariance under $\mathbb{E}_u$. This scalar equation must be covariant under rigid body motion as a reflection of material response being independent of observer motion. The angular momentum equation holds for consistency.

THE STARTING POINT

The development in the paper of *Green and Rivlin [1]* is expanded into a more complete form, including body moments and surface couples and with additional background detail. The early years of the twentieth century saw a change in the understanding of classical mechanics. Thus, *Tait [2]*, in 1899, could declare that mechanics is founded upon the basic principles of conservation of matter and conservation of energy. After *Minkowski* had introduced Lorentz transformations into the geometric structure of the mechanics of *Einstein*, it became clear that transformation groups and symmetry considerations must be a central feature of any theory of mechanics. The early work on quantum mechanics enhanced this conviction. The adoption of such understanding in classical continuum mechanics has been sparse with the notable exception of *Green and Rivlin [1]* and *Marsden and Hughes [3]*. See also *Noll [4]*. Its adoption in constitutive theory is better known: see *Truesdell and Noll [5]*. Recent works include *Muschik and Restuccia [6]*. The important application to Newtonian cosmology is not considered.

Attention is usually directed at the second axiom of Newton since it provides, directly, a formula for computing body motion. However, it is the first axiom of Newton that takes on a deeper meaning since this axiom implies the existence of inertial frames and the action of the Galilean transformation group, $\mathbb{G}_a$, on the event world $\mathbb{W}_e$ (axiom I below). A symmetry group is thus introduced into the theory (of both particle and deformable body dynamics). Herein (as in *Moulden [7]*) observers, $\mathbf{o} \in \mathbb{O}_B$, are identified as being associated with a coordinate frame on $\mathbb{W}_e$ and hence are relative. Each observer has a set of measurable quantities attached: the thermomechanical observables of the theory.

Write the Galilean group $\mathbb{G}_a$ as:

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{V}_T t + \mathbf{x}_0; \qquad t^* = t + t_0 \tag{1}$$

where: $\mathbf{Q} \in \mathbb{SO}_3$ is a *constant* orthogonal transformation. Equation (1) relates equivalent Newtonian observers to a given Newtonian observer. $\mathbf{V}_T \in \mathbb{R}^3$ is a constant boost velocity while vector $\mathbf{x}_0 \in \mathbb{R}^3$ and the scalar $t_0 \in \mathbb{T} \subset \mathbb{R}$ denote constant space and time translations. Group $\mathbb{G}_a$ is a subgroup of the special Euclidean group $\mathbb{SE}_u$:

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t); \qquad t^* = t + t_0 \tag{2a}$$

where $\mathbf{Q}(t) \in \mathbb{SO}_3$ is the time dependent rotation and $\mathbf{d}(t) \in \mathbb{R}^3$ denotes an arbitrary time dependent translation vector. Attention must also be directed to the full Euclidean group $\mathbb{E}_u$ with:

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t); \qquad t^* = t + t_0 \tag{2b}$$

but now $\mathbf{Q}(t) \in \mathbb{O}_3$ and $\mathbb{E}_u$ relates equivalent *Euclidean observers.* The corresponding velocity transformation is:

$$\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{d}}(t) \tag{3}$$

which reduces to $\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \mathbf{V}_T$ for the Galilean group.

NOTATION

$\mathbb{E}_u$, $\mathbb{G}_a \subset \mathbb{E}_u$	Euclidean and Galilean groups
$\mathbb{M}^3_{sy}$, $\mathbb{M}^3_{sk}$	skew and symmetric matrices
$\mathbb{SO}_3 \subset \mathbb{O}_3$	special and general orthogonal groups
$\mathbf{o} \in \mathbb{O}_B$	set of observers $\{\mathbf{o}\}$
$\mathbf{x} \in \mathbb{S}_t \subset \mathbb{R}^3$	coordinate in space of simultaneity
$\mathbb{SE}_u \subset \mathbb{E}_u$	special Euclidean group
$t \in \mathbb{T} \subset \mathbb{R}$	time and time axis and reals $\mathbb{R}$
$\mathbb{W}_e = \{(\mathbf{x}, t)\}$	the event world
$\mathbf{a}$; $\boldsymbol{\zeta} \in \mathbb{R}^3$	the acceleration and vorticity vectors
$\mathbf{d}(t) \in \mathbb{R}^3$	arbitrary translation

$\mathtt{d}(\mathbf{x}, \mathbf{y})$	metric on Euclidean space
$\mathbf{D}$ and $\mathbf{W}$	symmetric and skew parts of $\mathbf{L}$
$\mathbf{f}_B$; $\mathbf{f}_I$; $\mathbf{f}_S \in \mathbb{R}^3$	body, inertial, surface forces
$\mathbf{L} = \boldsymbol{\nabla}_{\mathbf{x}}(\mathbf{v}) \in \mathbb{L}_3$	velocity gradient
$\mathbf{l}_c$; $\mathbf{m}_c \in \mathbb{R}^3$	body and surface couples
$\mathbf{M} \in \mathbb{L}_3$	couple stress tensor
$\mathbf{Q} \in \mathbb{SO}_3$ or $\mathbb{O}_3$	rotation operator
$\mathbf{t}$ and $\mathbf{T}$	stress vector and Cauchy stress tensor
$\mathbf{v} \in \mathbb{R}^3$; P, $\rho \in \mathbb{R}$	velocity, pressure, density
$\mathbf{V}_T \in \mathbb{R}^3$	arbitrary boost velocity
$\mathbf{Z}(t) = \dot{\mathbf{Q}}\mathbf{Q}^T \in \mathbb{M}^3_{sk}$,	Coordinate spin
$\mathbf{Z}\mathbf{r} = \mathbf{0} \in \mathbb{R}^3$	axial vector, $\mathbf{r}(t)$, of $\mathbf{Z}$
$\mathcal{B}$; $\mathcal{B}^e$	material body and its exterior
$\mho = \mathcal{B} \curlyvee \mathcal{B}^e$	universal body
$\partial\mathcal{B}$; $\mathbf{n}$	boundary of $\mathcal{B}$ and its outward unit normal
$\mathcal{D}_t \subset \mathbb{R}^3$	space occupied by $\mathcal{B}$ at time t
$M(\mathcal{B})$; $V(\mathcal{B}) \in \mathbb{R}$	body mass and volume
$\mathcal{A}(T) \in \mathbb{R}$	the working action
$\mathsf{E}_{tot}(\mathcal{B}) \in \mathbb{R}$	the total energy of $\mathcal{B}$
$\mathsf{P}(\mathcal{B})$; $\mathsf{Q}(\mathcal{B}) \in \mathbb{R}$	mechanical and thermal working
$\mathsf{E}(\mathcal{B})$; $\mathsf{K}(\mathcal{B}) \in \mathbb{R}$	internal and kinetic energy
$\mathsf{R} \in \mathbb{R}$	residual
e; $\eta \in \mathbb{R}$	specific internal energy and entropy
d/dt	material derivative
$\langle \mathbf{a}, \mathbf{b} \rangle = a_i b_i \in \mathbb{R}$	inner product
$\lvert\mathbf{v}\rvert^2 \equiv \langle \mathbf{v}, \mathbf{v} \rangle = v_i v_i \in \mathbb{R}$	vector norm
Λ_0, Λ^*	standard and arbitrary inertial frames
$\vert_m$	conditions at center of mass

NEWTONIAN BACKGROUND

The interest is with classical continuum mechanics wherein matter is spread across space rather than being the average of randomly distributed particles.

Let $\mathcal{B}$ be some identified material body moving in the event world $\mathbb{W}_e = \{(\mathbf{x}, t)\}$ due to the action of forces $\mathbf{f} \in \mathbb{R}^3$. Start with the first axiom of Newtonian mechanics (the following is adapted from the translation by *Cohen and Whitman [8]* of Newton's *"Principia"*):

> **Axiom I:** Each body point $\delta\mathcal{B}$ of *body* $\mathcal{B}$ *remains at rest or moves across* $\mathbb{W}_e$, *with the same velocity* $\mathbf{v}$ *relative to the standard frame*, Λ_0, *in such a way that:*
>
> *a). the speed,* $\lvert\mathbf{v}\rvert$, *is constant.*
>
> *b). the motion is rectilinear.*
>
> *unless* $\mathcal{B}$ *is acted upon by a force or torque.* □

as Galilei's *axiom of inertia*. This axiom also serves to define the *Newtonian observer*. Two Euclidean observers find the velocity at body point $\delta\mathcal{B}$ related as:

$$\mathbf{v}^* = \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{Q}}(t)\mathbf{x} + \dot{\mathbf{d}}(t)$$

If both $\mathbf{v}$ and $\mathbf{v}^*$ are to be constant in accord with the axiom there must be $\dot{\mathbf{Q}} = \mathbf{O}$ (as $\mathbf{x}$ is not constant) and $\dot{\mathbf{d}}$ constant (equal to $\mathbf{V}_T$, say). Integrate these two conditions to find $\mathbf{Q}$ is constant and $\mathbf{d} = \mathbf{V}_T t + \mathbf{x}_0$; $\mathbf{x}_0 \in \mathbb{R}^3$ is constant. Now $\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \mathbf{V}_T$ and $\mathbf{v}^*$ is a constant when $\mathbf{v}$ is constant. The Galilean transformations are recovered with the constant *boost velocity* $\mathbf{V}_T \in \mathbb{R}^3$ and constant rotation $\mathbf{Q} \in \mathbb{SO}_3$. Hence (as in equation (1)):

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{V}_T t + \mathbf{x}_0; \qquad t^* = t + t_0$$

defines an equivalence class of inertial frames and norm $\lvert\mathbf{v}^*\rvert \le \lvert\mathbf{v}\rvert + \lvert\mathbf{V}_T\rvert$. The norm $\lvert\mathbf{V}_T\rvert$ is always assumed to be a bounded constant. The above discussion is invariant under the time translation $t \mapsto t + t_0$: the origin of time is arbitrary. The following treatment is founded on axiom I of classical mechanics but leads directly to a dichotomy in the theory. As a corollary of the transformation group $\mathbb{G}_a$ there is an implied structure for $\mathbb{W}_e$:

> **Corollary I:** $\mathbb{W}_e$ *is a fibre bundle* ∎

where $\mathbb{S}_t$ ($t \in \mathbb{T}$ fixed) identifies the space of simultaneous events: the fibres. $\mathbb{T}$ forms the base space. Usually, $\mathbb{S}_t$ is taken as $\mathbb{R}^3$ which is made into a Hilbert space by the definition of an inner product (a norm induced by the inner product then follows). A metric $\mathtt{d}(\mathbf{x}, \mathbf{y})$ can be defined on $\mathbb{S}_t$ in classical mechanics (but not on the whole $\mathbb{W}_e$) and is again induced by the inner product. It is important to note that the causal structure of the theory must be invariant across all inertial frames. The above comments have shown, unexpectedly, that Euclidean observers constitute too large a set and there must be:

> **Result I:** *Not all observers are Newtonian* ∎

which defines the Newtonian observers (ones that identify with axiom I) as a subset of the Euclidean observers. In other words, axion I of Newton only holds if $\mathbb{E}_u$ is reduced to $\mathbb{G}_a$ as the invariance group of the mechanics. Let Λ_0 be the *standard inertial frame* affixed to the distant stars. A comment is necessary here since equation (1) takes $\mathbf{Q} \in \mathbb{SO}_3$ while axiom I led to the condition $\mathbf{Q} \in \mathbb{O}_3$. The difference being due to the present desire to retain all coordinate frames as right handed.

The requirement of axiom I must, in sympathy with the statement above of *Tait [2]*, also apply to the thermodynamic content of the continuum theory. That is (to formalize the development in *Green and Rivlin [1]*):

> **Axiom II:** *The first principle of thermomechanics must be covariant under* $\mathbb{G}_a$ □

to be consistent with Newtonian mechanics. (It hardly needs to be said that axiom II is not related in any way to the similarly numbered axiom of Newton: but does lead, with axiom I, to the same balance of forces). As the first principle of thermomechanics is expressed by a scalar equation, it is straightforward to ensure its covariance — *Marsden and Hughes [3]*. However, the restriction to $\mathbb{G}_a$ symmetry is not consistent with material response to applied forces where the whole group $\mathbb{E}_u$ (or its subgroup $\mathbb{SE}_u$) is demanded for consistency with observation. The *fundamental dichotomy* in Newtonian continuum mechanics thus arises:

> **Dichotomy:** *Either constitutive theory must be reduced to invariance under* $\mathbb{G}_a$ *or Newtonian mechanics must be extended to* $\mathbb{E}_u$ *covariance.* □

It is well known that the former option allows the stress tensor (of a fluid) to depend upon the spin tensor $\mathbf{W}$ (see below): contrary to *Stokes [9]*. The latter option requires the abandonment of axiom I of Newton. The work of *Šilhavý [10]* attempts to give an extension of Newtonian theory to the more general case of $\mathbb{E}_u$ invariance.

There are also constitutive components to the first principle of thermodynamics. These include the internal energy and the heat flux vector. An additional issue is now raised: the type of partial differential equation appropriate. The classical theory of Fourier heat conduction gives a parabolic heat equation with infinite propagation velocity for information — see *Joseph and Preziosi [11]*. This is unphysical and parallels the finding of infinite shear wave propagation velocity for the Navier Stokes equations in fluid mechanics.

Hidden in the formulation is the fundamental assumption that the time axis, $\mathbb{T}$, is the same for both Newtonian mechanics and classical thermodynamics. That is $\mathbb{T}$ is the *thermomechanical time* (and is, essentially, the absolute time of Newton: see *Raine and Heller [12]* for example). The origin of time is arbitrary however. No attempt is made to justify this equality of time scales for diverse processes on widely different length scales.

RBM vis MFI

Let Λ^* be a non–inertial frame attached to an observer $\mathbf{o} \in \mathbb{O}_B$ and be related to the standard frame Λ_0 as in equations (2a) or (2b):

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t); \qquad t^* = t + t_0$$

Then the observer defines:

a). rigid body motion (RBM) if $\mathbf{Q}(t) \in \mathbb{SO}_3$

b). material frame indifferent (MFI) if $\mathbf{Q}(t) \in \mathbb{O}_3$

The group $\mathbb{O}_3$ includes reflections (that is $det(\mathbf{Q}) = \pm 1$) while $\mathbf{Q} \in \mathbb{SO}_3$ has $det(\mathbf{Q}) = +1$ only. The distinction is only of importance when distinguished material axes are present and chirility considerations are necessary (as in polar materials). These are not of concern herein and only rigid body motion is adopted for the study of covariance in constitutive theory. As defined above, rigid body motion is identified with the group $\mathbb{SE}_u$. Material frame indifference identifies with the whole group $\mathbb{E}_u$.

MASS INVARIANCE

Let $\mathcal{B} \prec \mathbb{U}$; $\mathbb{U}$ the universal body for the situation of interest. Then $\mathcal{B} \curlyvee \mathcal{B}^e = \mathbb{U}$ defines the *exterior* of $\mathcal{B}$. $\partial\mathcal{B}$ is the boundary (or surface) of $\mathcal{B}$. A measure dm is defined over $\mathbb{U}$ such that $M(\mathcal{B}) = \int_{\mathcal{B}} dm$ is a scalar quantity associated with $\mathcal{B}$, the *mass* of $\mathcal{B}$, whose dynamic significance has to be determined. If the measure dm over $\mathcal{B}$ is absolutely continuous with respect to volume measure dV over $\mathcal{D}_t$ then $M(\mathcal{B}) = \int_{\mathcal{D}} (\partial m/\partial V)\, dV \equiv \int_{\mathcal{D}} \rho\, dV$, if $V_t(\mathcal{B}) = \int_{\mathcal{D}} dV$ is the body volume. The Radon–Nikodym derivative $\rho(\mathbf{x}, t) = \partial m/\partial V_t$ is found to be thermodynamic in nature. A result (see *Moulden [7]*) can be noted:

Lemma I: *d/dt commutes with $\int_{\mathcal{B}} (\cdot)\, dm$ iff*

$$d\rho/dt + \rho\, div(\mathbf{v}) = 0 \quad with \quad \rho = \partial m/\partial V$$

PROOF: Let the non–zero smooth, scalar valued function, ϕ, be integrable over body $\mathcal{B}$ and consider the transport theorem:

$$\frac{d}{dt}\int_{\mathcal{B}} \phi\, dm \equiv \frac{d}{dt}\int_{\mathcal{D}} \rho\phi\, dV$$

$$= \int_{\mathcal{D}} \left[\left(\frac{d\rho}{dt} + \rho\, div(\mathbf{v})\right)\phi + \rho\frac{d\phi}{dt}\right] dV$$

provided that $\rho = \partial m/\partial V \in \mathbb{R}$ exists and is continuous over arbitrary volume $\mathcal{D}$. This requires:

$$\frac{d}{dt}\int_{\mathcal{B}} \phi\, dm = \int_{\mathcal{B}} \frac{d\phi}{dt}\, dm \;\; iff: \;\; \frac{d\rho}{dt} + \rho\, div(\mathbf{v}) \equiv 0$$

as requested. ∎

The above lemma is equivalent to the condition of mass invariance. Thus, provided that the above constraint holds at all point in $\mathcal{D}_t$:

$$\frac{\partial\rho}{\partial t} + div(\rho\mathbf{v}) = 0 \quad \Longrightarrow \quad \int_{\mathcal{D}} \left[\frac{\partial\rho}{\partial t} + div(\rho\mathbf{v})\right] dV = 0$$

$$\Rightarrow \frac{d}{dt}\int_{\mathcal{D}} \rho\, dV \equiv \frac{d}{dt}\int_{\mathcal{B}} dm = \frac{d}{dt}M(\mathcal{B}) = 0$$

and $M(\mathcal{B})$, an invariant of the body motion, identifies with the body mass, whose invariance has, since at least the ancient Greeks, been recognized empirically. This constraint of mass invariance emerges from the Galilean invariance of the total energy equation and does not require a separate axiom. This finding is little more than a consistency requirement since the energy equation is written for body $\mathcal{B}$ as a *closed* thermomechanical system. See *Green and Rivlin [1]*. While a Lagrangian formulation is not the present interest it can be noted that *Scholle [13]* shows mass invariance to be a consequence of a symmetry analysis of the Lagrangian.

THERMODYNAMIC BACKGROUND

Introduce the total energy of material body $\mathcal{B}$:

$$\mathsf{E}_{tot}(\mathcal{B}) = \int_{\mathcal{B}} dE + \int_{\mathcal{B}} \langle\mathbf{v}, \mathbf{v}\rangle/2\, dm = \int_{\mathcal{B}} [e + \langle\mathbf{v}, \mathbf{v}\rangle/2] dm$$

as the sum of the internal and kinetic energies; a form found by empirical observation. Here $e = \partial E/\partial m$ is the *specific internal energy* as a Radon–Nikodym derivative assuming that the internal energy is absolutely continuous with respect to mass. The internal energy is specified by thermodynamic constitutive statements. It is *assumed* that the body force is frame indifferent under $\mathbb{G}_a$.

Axiom III *First principle of thermomechanics*

The total energy equation:

$$\mathsf{E}_{tot}(\mathcal{B})(T) - \mathsf{E}_{tot}(\mathcal{B})(0) \equiv \mathcal{A}(T)$$

must hold. ∎

The causal structure of the theory is evident in that the action $\mathcal{A}(T) = \int_0^T \left[\mathsf{P}(\mathcal{B}) + \mathsf{Q}(\mathcal{B})\right] dt$ (which vanishes for cyclic processes) is responsible for the change of $\mathsf{E}_{tot}(\mathcal{B})$:

$$d\,\mathsf{E}_{tot}(\mathcal{B})/dt = \mathsf{P}(\mathcal{B}) + \mathsf{Q}(\mathcal{B}) \tag{4}$$

Quantities in the action are measurable by the observer $\mathbf{o} \in \mathbb{O}_B$. The mechanical $\mathsf{P}(\mathcal{B})$ and thermal $\mathsf{Q}(\mathcal{B})$ workings are defined below. This action is potential provided that it is conservative and vanishes on all cyclic processes (see *Coleman and Owen [14])*. *Šilhavý [15]* starts with $\mathcal{A}(T) = \mathcal{O}$ for a cyclic process and shows that an energy exists but cannot determine its structure, as the sum $\mathsf{E}(\mathcal{B}) + \mathsf{K}(\mathcal{B})$, by appeal to Galilean covariance. Determination of the structure requires Euclidean covariance: using either MFI or RBM see *Šilhavý [10]*. The first principle of thermomechanics can be taken in the form of equation (4) without requiring cyclic processes. So find the equation:

$$d[\mathsf{E}_{tot}(\mathcal{B})]/dt \equiv \int_{\mathcal{B}}[de/dt + \langle \mathbf{a}, \mathbf{v}\rangle]\, dm = \mathsf{P}(\mathcal{B}) + \mathsf{Q}(\mathcal{B})$$

when the above lemma is included. These deliberations are in frame Λ_0. Hence (as $\mathbf{a} \equiv d\mathbf{v}/dt$) it is found that:

a). under Galilean boosts ($\mathbf{v} \mapsto \mathbf{v} + \mathbf{V}_T$):

$$\frac{d}{dt^*}[\mathsf{E}^*(\mathcal{B}) + \mathsf{K}^*(\mathcal{B})] = \frac{d}{dt}[\mathsf{E}(\mathcal{B}) + \mathsf{K}(\mathcal{B})] + \int_{\mathcal{B}}\langle \mathbf{a}, \mathbf{V}_T\rangle dm$$

b). under constant rotation ($\mathbf{v} \mapsto \mathbf{Q}\mathbf{v}$):

$$\frac{d}{dt^*}[\mathsf{E}^*(\mathcal{B}) + \mathsf{K}^*(\mathcal{B})] = \frac{d}{dt}[\mathsf{E}(\mathcal{B}) + \mathsf{K}(\mathcal{B})]$$

as the internal energy is frame indifferent. This statement is another aspect of the classical dichotomy: $e(\mathbf{x}, t)$ is a frame indifferent scalar quantity under $\mathbb{E}_u$.

The inertial force has been introduced above by the identity $\mathbf{f}_I = \int_{\mathcal{B}} \mathbf{f}_I^m dm$ with:

$$\mathbf{f}_I = -\int_{\mathcal{B}} \mathbf{a}\, dm = -\int_{\mathcal{B}} d\mathbf{v}/dt\, dm = -d[\int_{\mathcal{B}} \mathbf{v}\, dm]/dt \quad (5)$$

by lemma I. Note that $\int_{\mathcal{B}} \mathbf{a}\, dm = \mathbf{a}_m M(\mathcal{B}) = -\mathbf{f}_I$ to relate $\mathbf{f}_I$ to the quantity $M(\mathcal{B})$: $\mathbf{a}_m$ being the center of mass acceleration. Hence the term *'inertial force'* is just a name for the material rate of change of linear momentum. Then the integral:

$$\int_{\mathcal{B}} \langle \mathbf{a}, \mathbf{V}_T\rangle dm \equiv -\int_{\mathcal{B}} \langle \mathbf{f}_I^m, \mathbf{V}_T\rangle dm = -\int_{\mathcal{D}} \langle \mathbf{f}_I^v, \mathbf{V}_T\rangle dV$$

represents a virtual contribution to $d\mathsf{E}_{tot}(\mathcal{B})/dt$ due to the boost given by the arbitrary translation velocity $\mathbf{V}_T$.

$$\mathsf{P}(\mathcal{B}) = \int_{\mathcal{D}} \rho[\langle \mathbf{f}_B^m, \mathbf{v}\rangle + \langle \boldsymbol{\zeta}, \mathbf{l}_c\rangle]\, dV + \int_{\partial\mathcal{D}} [\langle \mathbf{t}, \mathbf{v}\rangle + \langle \boldsymbol{\zeta}, \mathbf{m}_c\rangle]\, dA$$

is the *mechanical working* due to the body and surface forces and couples (the latter not being considered in *Green and Rivlin [1])*. The body moment $\mathbf{l}_c$ and surface couple $\mathbf{m}_c$ are assumed to be frame indifferent under $\mathbb{E}_u$ (since the metric $\mathsf{d}(\mathbf{x}, \mathbf{y})$ has such invariance).

$$\mathsf{Q}(\mathcal{B}) = -\int_{\mathcal{D}} div(\mathbf{q})\; dV + \int_{\mathcal{D}} q_s\, dV$$

is the *thermal working* due to Fourier heat conduction and a frame indifferent heat source, $q_s(\mathbf{x}, t)$, over $\mathcal{B}$. Note that the Joule constant is not displayed directly in the thermal working so that dimensional consistency is assumed. Other forms of thermal working can be added if needed.

a). Under Galilean boosts:

$$\mathsf{P}^*(\mathcal{B}) = \mathsf{P}(\mathcal{B}) + \int_{\mathcal{D}} \rho\langle \mathbf{f}_B^m, \mathbf{V}_T\rangle dV + \int_{\partial\mathcal{D}} \langle \mathbf{t}, \mathbf{V}_T\rangle\, dA$$

b). Under constant rotations:

$$\mathsf{P}^*(\mathcal{B}) = \mathsf{P}(\mathcal{B})$$

$\mathsf{Q}(\mathcal{B})^* = \mathsf{Q}(\mathcal{B})$ under $\mathbb{G}_a$. For the final energy equation there is, with the above, the *residual* R defined such that:

a). Under Galilean boosts:

$$d\mathsf{E}^*_{tot}(\mathcal{B})/dt^* - \big[\mathsf{P}^* + \mathsf{Q}^*\big] = d\mathsf{E}_{tot}(\mathcal{B})/dt - \big[\mathsf{P} + \mathsf{Q}\big] - \mathsf{R}_B$$

b). Under constant rotations:

$$d\mathsf{E}^*_{tot}(\mathcal{B})/dt^* - \big[\mathsf{P}^* + \mathsf{Q}^*\big] = d\mathsf{E}_{tot}(\mathcal{B})/dt - \big[\mathsf{P} + \mathsf{Q}\big] - \mathsf{R}_R$$

If the energy equation is to retain Galilean covariance in conformity with axiom II above, the residual terms in these equations, R_B and R_R, must vanish identically:

$$\mathsf{R}_B \equiv \int_{\mathcal{D}} \langle \mathbf{V}_T, [\rho\mathbf{f}_I^m + \rho\mathbf{f}_B^m]\,\rangle\, dV + \int_{\partial\mathcal{D}} \langle \mathbf{V}_T, \mathbf{t}\rangle\, dA = 0$$

and:

$$\mathsf{R}_R \equiv 0$$

This expression for R_B would not hold under $\mathbb{SE}_u$. The scalars R_B and R_R have arisen solely from the Galilean transformations with $\mathbf{V}_T$ an arbitrary constant linear motion of some observer and $\mathbf{Q}$ an arbitrary *constant orientation change* of that same observer. For this reason, the transformation is transparent to the couple stresses, and body moments (which are both assumed to be frame indifferent under $\mathbb{E}_u$ and hence under $\mathbb{G}_a$).

Only the action of the boost component of the Galilean group has produced a non–trivial residual R_B. The terms in R_B arose from the kinetic energy and the mechanical working components of the energy equation; the thermal content playing no part in the development. The thermal working does not enter into the form for the residual as the heat flux vector is frame indifferent. The inertial force is only frame indifferent if the mass is invariant:

Proposition I: $\mathbf{f}_I$ *is frame indifferent under* $\mathbb{G}_a$ *iff* $dM(\mathcal{B})/dt = 0$.

PROOF: By equation (5): $\mathbf{f}_I = -d[\int_{\mathcal{B}} \mathbf{v}\, dm]/dt$ which only equals $-\int_{\mathcal{B}} \mathbf{a}\, dm \equiv -\mathbf{a}_m M(\mathcal{B})$ if lemma I holds (since $\mathbf{V}_T$ is constant). ∎

Thus under $\mathbb{E}_u$, equation (2a) gives:

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t) \mapsto \mathbf{a}^* = \mathbf{Q}\mathbf{a} + 2\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x} + \ddot{\mathbf{d}}$$

so that:

$$\mathbf{f}_I = -\int_{\mathcal{B}} \mathbf{a}\, dm \mapsto \mathbf{Q}\underline{\mathbf{f}}_I$$

only if $\mathbb{E}_u$ reduces to $\mathbb{G}_a$ with $\mathbf{Q}$ constant and $\mathbf{d}(t)$ linear in t. Proposition I, does of course, depend upon the result in lemma I.

Invariance is non–trivial, under boosts, because $\mathsf{R}_B \not\equiv 0$. As the boost velocity $\mathbf{V}_T$ is an arbitrary constant vector, it may be extracted from the integrals to give the scalar valued condition:

$$\langle \mathbf{V}_T, \big[\int_{\mathcal{D}}(\rho\mathbf{f}_I^m + \rho\mathbf{f}_B^m)\, dV + \int_{\partial\mathcal{D}} \mathbf{t}\, dA\big]\rangle \equiv \langle \mathbf{V}_T, (\mathbf{f}_I + \mathbf{f}_B + \mathbf{f}_S)\rangle = 0$$

for the constraint $\mathsf{R}_B = 0$. The term $\langle \mathbf{V}_T, \Sigma\mathbf{f}\rangle$ represents the virtual working of the sum of forces, $\Sigma\mathbf{f}$, due to the arbitrary constant relative velocity, $\mathbf{V}_T$, of the observer. This inner product can only vanish if the condition $\Sigma\mathbf{f} = \mathbf{0}$ holds. As a consequence:

Lemma II: *Balance of forces*
Axiom II implies the balance of forces.

PROOF: As $\mathbf{V}_T$ is an arbitrary constant vector, the inner product above only vanishes if:

$$\int_{\mathcal{D}}\rho[\mathbf{f}_I^m + \mathbf{f}_B^m]\,dV + \int_{\partial\mathcal{D}}\mathbf{t}\,dA = \mathbf{0} \\ \equiv \mathbf{f}_I + \mathbf{f}_B + \mathbf{f}_S \tag{6}$$

Which is the basic global *balance equation* of the continuum and identifies the inertial force from equation (4) with the one referred to by Newton in his axiom II. It is covariant under $\mathbb{G}_a$. ∎

Hence the mass $M(\mathcal{B})$ has the meaning defined by Newton and is a property of the body $\mathcal{B}$. The acceleration is defined by the motion of $\mathcal{B}$ while the forces are specified by the interaction of $\mathcal{B}$ with its exterior. The body force is specified outside of continuum mechanics. Body moments and surface couple stresses do not enter into equation (6) but they are part of the angular momentum equation (9). The result in equation (6) is a consequence of vanishing working by the arbitrary boost velocity. By its derivation, it is frame indifferent under $\mathbb{G}_a$.

The Cauchy theory of stress comes directly from equation (6) as shown in texts on continuum mechanics. That ρ is a thermodynamic variable is shown from experiment (usually in the form of a specific volume). The inertial force component of equation (6) arose from the material rate of change of the total energy while the body and surface forces are present in the mechanical working $\mathsf{P}(\mathcal{B})$. Hence the causal structure contained in the total energy equation carries over to the balance of forces in the second axiom of Newton. The ontological content of the force balance theory is changed, however.

The Cauchy theory of stress takes equation (6) and finds that $\mathbf{t} = \mathbf{Tn}$ provided that there is sufficient continuity of $\mathbf{t}$ as a function of the normal, $\mathbf{n}$, over the boundary $\partial\mathcal{B}$. This places restrictions in the boundary geometry. Some of these restrictions on the Cauchy stress theory are relaxed in *Gurtin et al. [16]*.

Subtract equation (6) from equation (4) to find the *thermal energy equation*:

$$\int_{\mathcal{D}}\Big[\rho\frac{de}{dt} + div(\mathbf{q}) - trace(\mathbf{L}\mathbf{T}^T) - q_s - \rho\langle\boldsymbol{\zeta},\mathbf{l}_c\rangle\Big]dV \\ = \int_{\partial\mathcal{D}}\langle\boldsymbol{\zeta},\mathbf{m}_c\rangle\,dA \tag{7}$$

which *Green and Rivlin [1]* require to be covariant under RBM (covariance under $\mathbb{G}_a$ has already been established). Equation (7) is a relation between certain constitutive quantities (e, $\mathbf{q}$, $\mathbf{T}$), kinematic quantities and the applied couples. The present formulation is an extension since surface couples and body moments are included in equation (7). Under $\mathbb{SE}_u$: $\boldsymbol{\zeta} \mapsto \mathbf{Q}(t)\boldsymbol{\zeta} + 2\mathbf{r}$ where $\mathbf{Zr} = \mathbf{0}$ defines the axial vector, $\mathbf{r}$, of the coordinate spin, while $e \mapsto e$, $q_s \mapsto q_s$ and $\mathbf{q} \mapsto \mathbf{Qq}$ (as it does for the Fourier theory) then $div(\mathbf{q}) \mapsto div(\mathbf{q})$ and $\mathbf{L} \mapsto \mathbf{QLQ}^T + \mathbf{Z}$ under $\mathbb{E}_u$. Place $\mathbf{T} = \mathbf{T}^+ + \mathbf{T}^-$ with $\mathbf{T}^+ \in \mathbb{M}^3_{sy}$ and $\mathbf{T}^- \in \mathbb{M}^3_{sk}$. Since $\mathbf{T}$ is frame indifferent under $\mathbb{E}_u$ so are $\mathbf{T}^+$ and $\mathbf{T}^-$. Hence $trace(\mathbf{L}\mathbf{T}^T) \mapsto trace(\mathbf{L}\mathbf{T}^T) - trace(\mathbf{T}^-\mathbf{Z}^*)$ since the transformations:

$$trace(\mathbf{D}\mathbf{T}^+) \mapsto trace(\mathbf{D}\mathbf{T}^+)$$

$$trace(\mathbf{W}\mathbf{T}^-) \mapsto trace(\mathbf{W}\mathbf{T}^-) + trace(\mathbf{T}^-\mathbf{Z}^*)$$

hold under $\mathbb{E}_u$. Now $trace(\mathbf{T}^-\mathbf{Z}^*) = -2\langle\mathbf{s},\mathbf{Q}^T\mathbf{r}\rangle$ where $\mathbf{s}$ is the axial vector of $\mathbf{T}^-$ (that is $\mathbf{T}^-\mathbf{s} = \mathbf{0}$).

$$\rho\langle\boldsymbol{\zeta},\mathbf{l}_c\rangle \mapsto \rho\langle\boldsymbol{\zeta},\mathbf{l}_c\rangle + 2\rho\langle\mathbf{l}_c,\mathbf{Q}^T\mathbf{r}\rangle$$

$$\int_{\partial\mathcal{D}}\langle\boldsymbol{\zeta},\mathbf{m}_c\rangle\,dA \mapsto \int_{\partial\mathcal{D}}\langle\boldsymbol{\zeta},\mathbf{m}_c\rangle\,dA + 2\int_{\partial\mathcal{D}}\langle\mathbf{m}_c,\mathbf{Q}^T\mathbf{r}\rangle\,dA$$

If $\mathbf{m}_c$ is replaced by $\mathbf{Mn}$ as in *Stokes [17]* then:

$$\int_{\partial\mathcal{D}}\langle\mathbf{m}_c,\mathbf{Q}^T\mathbf{r}\rangle dA \equiv \int_{\partial\mathcal{D}}\langle\mathbf{Mn},\mathbf{Q}^T\mathbf{r}\rangle dA = \int_{\mathcal{D}} div(\mathbf{M})\mathbf{Q}^T\mathbf{r}\,dV$$

by the divergence theorem. Covariance of the thermal energy equation (7) under $\mathbb{E}_u$ then requires that:

case a). $\mathbf{m}_c = \mathbf{0}$; $\mathbf{l}_c = \mathbf{0}$ when:

$$trace(\mathbf{Z}^*\mathbf{T}^-) = 0 \Rightarrow \mathbf{T}^- = \mathbf{O} \Rightarrow \mathbf{T} \in \mathbb{M}^3_{sy} \tag{8}$$

as $\mathbf{Z}^* = \mathbf{Q}^T\mathbf{ZQ}$ is arbitrary.

case b). $\mathbf{m}_c \neq \mathbf{0}$; $\mathbf{l}_c \neq \mathbf{0}$:

$$\int_{\mathcal{D}}\langle(div(\mathbf{M}) + \rho\mathbf{l}_c - \mathbf{s}),\mathbf{Q}^T\mathbf{r}\rangle\,dV \equiv 0$$

Since the quantity $\mathbf{Q}^T\mathbf{r}$ is a function of time, t, only, it may be extracted from the integral:

$$\langle\int_{\mathcal{D}}(div(\mathbf{M}) + \rho\mathbf{l}_c - \mathbf{s})\,dV,\mathbf{Q}^T\mathbf{r}\rangle \equiv 0$$

and then, due to the arbitrariness of the rigid body rotation $\mathbf{Q}(t)$ (and the axial vector $\mathbf{r}(t)$ of $\mathbf{Z}(t)$), there is:

$$\int_{\mathcal{D}}[div(\mathbf{M}) + \rho\mathbf{l}_c - \mathbf{s}]\,dV = \mathbf{0} \tag{9}$$

which is the angular momentum equation in the presence of body and surface couples. Equation (9) reduces to a weak version of (8) when $\mathbf{l}_c = \mathbf{0}$ and $\mathbf{m}_c = \mathbf{0}$. *Stokes [17]* explores equation (9) for a viscous fluid with a specific $\mathbf{M}$. The axiom of universal dissipation completes the theory:

Axiom III *Second axiom of thermomechanics*

The entropy of the universal body $\mathbb{U}$ cannot decrease. ∎

and yields the classical evolution equation for entropy. As in the rest of the paper, attention is only directed at a single constituent material with no phase changes. The existence and uniqueness of solutions to the field equations is a separate issue that can only be addressed after a constitute theory is settled upon.

DEVELOPMENT

Green and Rivlin [1] show that invariance of the thermal energy equation under rigid body rotation implies (in the absence of surface couple stresses) that the Cauchy stress tensor be symmetric. However, this is not a requirement that follows from axiom I of Newton since the group ordering $\mathbb{G}_a \subset \mathbb{SE}_u \subset \mathbb{E}_u$ identifies rigid body rotation as intermediate between $\mathbb{G}_a$ and $\mathbb{E}_u$. A new interpretation is needed and this involves the dichotomy mentioned above: material response demands the group $\mathbb{E}_u$. The Cauchy stress tensor, after all, represents part of that response. It is found that axiom I defines the invariance group for the second axiom of Newtonian mechanics (here written as equation (6) and being a corollary of axiom II above) but not for constitutive theory.

While Newton understood heat as an internal vibration within a material, the need for its inclusion into mechanics was not. Thermomechanics was not developed for another century after the third edition of the *'Principia'* was published.

CONSTITUTIVE THEORY

As is well known, continuum mechanics has nothing to say about the structure of the Cauchy stress tensor beyond its definition in terms of the surface stress vector. Basic empirical evidence, known since antiquity, demands that the structure of $\mathbf{T}$ be invariant under the group $\mathbb{E}_u$ (usually referred to as the principle of material frame indifference — *Truesdell and Noll [5])*. This goes far beyond the confines of Newtonian mechanics and presents a fundamental dichotomy in the theory. It has also generated controversy; which still obtains if the group $\mathbb{SE}_u$ is adopted for covariance under RBM. For example of MFI: the *linear viscous fluid* wherein the Cauchy stress tensor is a function of the velocity gradient $\mathbf{L}$: $\mathbf{T} = \mathbf{T}(\mathbf{L}, \rho)$.

Proposition II: *Under* $\mathbb{E}_u$; $\mathbf{T} = \mathbf{T}(\mathbf{D}, \rho)$ *only*

PROOF: If $\mathbf{T} = \mathbf{T}(\mathbf{L}, \rho)$ then $\mathbf{T}$ is not frame indifferent under $\mathbb{E}_u$ since $\mathbf{L} \mapsto \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t) + \mathbf{Z}$ under $\mathbb{E}_u$. However, $\mathbf{D}$, the symmetric part of $\mathbf{L}$ transforms as $\mathbf{D} \mapsto \mathbf{Q}(t)\mathbf{D}\mathbf{Q}^T(t)$ under $\mathbb{E}_u$ and the form $\mathbf{T}(\mathbf{D}, \rho)$ is frame indifferent. ∎

To illustrate the dichotomy note that:

Corollary II: $\mathbf{T} = \mathbf{T}(\mathbf{L}, \rho)$ *under* $\mathbb{G}_a$.
(since $\mathbf{L} \mapsto \mathbf{Q}\mathbf{L}\mathbf{Q}^T$ under $\mathbb{G}_a$). ∎

Stokes [9], on a heuristic argument, required the pressure to be independent of rotation so that the stress tensor must be based upon $\mathbf{D}$. $\mathbb{G}_a$ invariance is not enough. *Svendsen and Bertram [18]* discuss the issue of frame indifference in constitutive theory.

KINETIC THEORY INTERPRETATION

When material frame indifference was first postulated as a principle for placing constraints upon constitutive equations, (see *Truesdell and Noll [5]*, for example) there was considerable controversy concerning its validity. *Cauchy [19]* had already introduced such concepts into the theory of stress but they were long forgotten. The main criticism of MFI came from kinetic theory arguments but, following *Speziale [20]*, note that the molecular velocity $\mathbf{c}$ contains fluctuations $\mathbf{c}'$ which are frame indifferent under $\mathbb{SE}_u$. Hence, for example, the Cauchy stress tensor, which is given by $\mathbf{T} \propto \mathcal{E}(\mathbf{c}' \otimes \mathbf{c}')$, transforms as $\mathbf{T} \mapsto \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ under both $\mathbb{E}_u$ and $\mathbb{SE}_u$ and is quite consistent with the continuum theory argument.

FINAL REMARKS

Classical continuum mechanics is found to rest upon the first axiom of Newton, the first principle of thermomechanics and the restraint that constitutive theory be invariant under $\mathbb{SE}_u$ (that is, under rigid body motion). The balance of forces, the angular momentum principle and the *conservation* of mass (rather than the simple concept of mass) emerge as consistency constraints on the theory and do not require separate specification.

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