HEFAT2010
7th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics
19-21 July 2010
Antalya, Turkey

THE TAYLOR VORTEX AND THE DRIVEN CAVITY PROBLEMS BY THE VELOCITY-VORTICITY FORMULATION

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ABSTRACT
2D isothermal viscous incompressible fluid flows are presented from the Navier-Stokes equations in velocity-vorticity variables. The flows are obtained with a simple numerical method based on a fixed point iterative process to solve the nonlinear elliptic system that results after time discretization; the iterative process leads to the solution of uncoupled, well-conditioned, symmetric linear elliptic problems for which efficient solvers exist regardless of the space discretization. The numerical experiments are given for the Taylor vortex and the driven cavity problems.

Keywords: velocity-vorticity variables, Reynolds number, Taylor vortex problem, driven cavity problem, fixed point iterative process

INTRODUCTION
2D isothermal viscous incompressible flows from the Navier-Stokes equations in velocity-vorticity variables are presented. The flows are obtained with a numerical procedure based on a fixed point iterative process to solve the nonlinear elliptic system that results once a convenient second order time discretization is made. The iterative process leads to the solution of uncoupled, well-conditioned, symmetric linear elliptic problems for which efficient solvers exist either by finite differences or finite elements as far as rectangular domains are considered. On the driven cavity problem isothermal flows up to Reynolds numbers Re = 3200 are presented; on the Taylor vortex problem up to Reynolds numbers Re = 7500.
It appears that with velocity-vorticity variables is more difficult to solve these flows, at least with a numerical procedure similar to the one applied in stream function-vorticity variables to solve an analogous nonlinear elliptic system, [1].

MATHEMATICAL MODEL AND NUMERICAL METHOD
Let $\Omega \subset R^N$ (N = 2, 3) be the region of the flow of an unsteady thermal viscous incompressible fluid, and Γ its boundary. This kind of flows is governed by the non dimensional system, in $\Omega \times (0,T)$, $T > 0$,

$$\begin{cases} \mathbf{u}_t - \frac{1}{Re}\nabla^2\mathbf{u} + \nabla p + \mathbf{u}\nabla\mathbf{u} = \mathbf{f} & (a) \\ \nabla\cdot\mathbf{u} = 0 & (b) \end{cases} \quad (1)$$

known as the Navier-Stokes equations in primitive variables, velocity u and pressure p. The dimensionless parameter Re is the Reynolds number given by $Re = \frac{UL}{v}$, with $v = \frac{UL}{Re}$ =kinematic viscosity, L and U are the length and the velocity of reference. The dimensionless velocity u is given by $\mathbf{u} = \mathbf{u}/U$, an the dimensionless position x by $\mathrm{x} = \mathrm{x}/L$. The system must be supplemented with appropriated initial and boundary conditions.

Taking the curl in (1a), the vorticity ω equation, in $\Omega \times (0,T)$, reads

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = \omega\cdot\nabla\boldsymbol{u} + \mathrm{f}, \quad (2)$$

where the vorticity vector ω is given by

$$\omega = \nabla\times\mathbf{u} \quad (3)$$

Taking the curl in (3), using the identity $\nabla\times\nabla\times\mathbf{a} = -\nabla^2\mathbf{a} + \nabla(\nabla\cdot\mathbf{a})$ and (1b), a velocity Poisson equation results

$$\nabla^2\mathrm{u} = -\nabla\times\omega \quad (4)$$

Hence, equations (2) and (4), give the Navier-Stokes equations in velocity-vorticity variables. It can be verified that the vorticity, scalar, ω transport equation in $\Omega \times (0,T)$, $\Omega \subset R^2$, is given by

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = 0; \tag{5}$$

moreover, from the 2D restriction in (3),

$$\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{6}$$

and, from (4), two Poisson equations for the velocity components are obtained

$$\begin{cases} \nabla^2 u_1 = \frac{\partial\omega}{\partial y} & (a) \\ \nabla^2 u_2 = \frac{\partial\omega}{\partial x} & (b) \end{cases} \tag{7}$$

Then, the system (2) and (4) is reduced to a scalar system of three equations in *2D*: one given by (5) an two by (7); (5) and (7) are related through (6) from which the boundary condition for ω in (5) should be obtained from that of $\mathbf{u} = (u_1, u_2)$.

For the time derivate appearing in the vorticity equation (5) the following well known second-order approximation is used

$$f_t(x,(n+1)\Delta t) = \frac{3f^{n+1} - 4f^n + f^{n-1}}{2\Delta t}, \tag{8}$$

where $\boldsymbol{x} \in \Omega, n \geq 1, \Delta t$ denotes the time step, and $f^r \equiv f(\mathbf{x}, r\Delta t)$, assuming f is smooth enough.

Then, from (5) and (7), the following fully implicit time-discretization system is obtained, in Ω,

$$\begin{cases} \nabla^2 u_1^{n+1} = -\frac{\partial\omega^{n+1}}{\partial y} \\ \nabla^2 u_2^{n+1} = \frac{\partial\omega^{n+1}}{\partial x}, \qquad \mathbf{u}^{n+1}|_\Gamma = \mathbf{u}_{bc} \\ \alpha\omega^{n+1} - \nu\nabla^2\omega^{n+1} + \mathbf{u}^{n+1}\cdot\nabla\omega^{n+1} = f_\omega, \\ \qquad \omega^{n+1}|_\Gamma = \omega_{bc} \end{cases} \tag{9}$$

where $\alpha = \frac{3}{2\Delta t}$, $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$, $f_\theta = \frac{4\theta^n - \theta^{n-1}}{2\Delta t}$, and $\frac{1}{Re}$ has been replaced by the kinematic viscosity coefficient ν, with $U = L = 1$; $\mathbf{u}_{bc}$ and ω_{bc} denote the boundary condition for $\mathbf{u}$ and ω.

Then, at each time level $(n+1)\Delta t$, a nonlinear system of elliptic equations of the following type must be solved, in Ω,

$$\begin{cases} \nabla^2 u_1 = -\frac{\partial\omega}{\partial y} \\ \nabla^2 u_2 = \frac{\partial\omega}{\partial x}, \qquad \mathbf{u}|_\Gamma = \mathrm{u}_{bc} \\ \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = f_\omega, \ \omega|_\Gamma = \omega_{bc} \end{cases} \tag{10}$$

Since system (10) is of non-potential type, a fixed point iterative process may be used.

If we denote

$$R_\omega(\omega, \mathbf{u}) = \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega \ \text{ in } \Omega$$

system (10) is equivalent, in Ω,

$$\begin{cases} \nabla^2 u_1 = -\frac{\partial\omega}{\partial y} \\ \nabla^2 u_2 = \frac{\partial\omega}{\partial x}, \qquad \mathbf{u}|_\Gamma = \mathbf{u}_{bc} \\ R_\omega(\omega, \mathbf{u}) = 0 \qquad \omega|_\Gamma = \omega_{bc} \end{cases} \tag{11}$$

Then,(11) is solved, at time level $(n+1)$, by the fixed point iterative process:

With $\omega^0 = \omega^n$ given, solve until convergence on ω, in Ω,

$$\begin{cases} \nabla^2 u_1^{m+1} = -\frac{\partial\omega^m}{\partial y} \\ \nabla^2 u_2^{m+1} = \frac{\partial\omega^m}{\partial x}, \qquad \mathrm{u}^{m+1}|_\Gamma = \mathrm{u}_{bc} \\ \alpha\omega^{m+1} = \omega^m - \\ \qquad \rho_\omega(\alpha I - \nu\nabla^2)^{-1}R_\omega(\omega^{m+1}, \mathrm{u}^{m+1}), \\ \omega^{m+1}|_\Gamma = \omega_{bc}^m, \ \ \rho_\omega > 0, \end{cases} \tag{12}$$

then, take$(u_1^{n+1}, u_2^{n+1}, \omega^{n+1}) = (u_1^{m+1}, u_2^{m+1}, \omega^{m+1})$.

Finally, system (12) is equivalent, in Ω, to

$$\begin{cases} \nabla^2 u_1^{m+1} = -\frac{\partial\omega^m}{\partial y} \\ \nabla^2 u_2^{m+1} = \frac{\partial\omega^m}{\partial x}, \qquad \mathbf{u}_{bc}^{m+1}|_\Gamma = \mathrm{u}_{bc} \\ (\alpha I - \nu\nabla^2)\alpha\omega^{m+1} = (\alpha I - \nu\nabla^2)\omega^m - \\ \qquad \rho_\omega R_\omega(\omega^m, \mathbf{u}^{m+1}), \\ \rho_\omega > 0, \ \omega^{m+1}|_\Gamma = \omega_{bc}^m. \end{cases} \tag{13}$$

Therefore, at each iteration, of each time level $(n+1)\Delta t$, *three* uncoupled, symmetric, linear elliptic problems associated with the operator $\alpha I - \nu\nabla^2$ and two with ∇^2 have to be solved. It should be noted that the non-symmetric part for ω in (10) has been taken into the right hand side thanks to the iterative process. For the spatial discretization of linear elliptic problems either finite differences or finite elements may be used, as far as rectangular domains are concerned, in either case very efficient solvers exist. In the finite element case, variational formulations have to be chosen and then restrict them to the finite dimensional finite elements spaces, for instance like those in References [3-4].

For the specific results in the following section, the second or fourth order approximation of the Fishpack solver in Reference [5] has been used, where the algebraic linear systems are solved through an efficient cyclic reduction iterative method, [6].

NUMERICAL EXPERIMENTS

The numerical experiments take place in rectangular cavities $\Omega = (0,a) \times (0,b), a, b > 0$. In connection with the driven with the driven cavity problem, the boundary condition is given by $\mathbf{u} = (1,0)$ on the moving wall $y = b$ and $\boldsymbol{u} = (0,0)$ elsewhere. The initial condition is $(\mathrm{u}(\mathrm{x},\mathrm{y},0), \mathrm{v}(\mathrm{x},\mathrm{y},0)) = (0,0)$. Results converging to the asymptotic steady state are reported.

For the Taylor vortex problem in the square cavity $0 \leq x, y \leq \pi$ the exact solution is given by

$$u(x,y,t) = -\cos(x)\sin(y)e^{-\frac{2t}{Re}}$$

and

$$v(x,y,t) = \sin(x)\cos(y)e^{-\frac{2t}{Re}}.$$

From these expressions and using (6), the exact vorticity is given by

$$\omega(x,y,t) = -2\cos(x)\cos(y)e^{-\frac{2t}{Re}}$$

Then the initial conditions for u, v and ω are given by

$$\begin{cases} u(x,y,0) = -\cos(x)\sin(y), \\ v(x,y,0) = \sin(x)\cos(y), \\ \omega(x,y,0) = -2\cos(x)\cos(y); \end{cases}$$

and the periodic boundary condition for ω is given by

$$\begin{cases} \omega(0,y,t) = -2\cos(y)\, e^{-\frac{2t}{Re}} = \omega(2\pi,y,t), \\ \omega(x,0,t) = -2\cos(x)\, e^{-\frac{2t}{Re}} = \omega(x,2\pi,t), \end{cases}$$

see also [7]. The periodic boundary conditions for u, v are given by

$$\begin{cases} u(0,y,t) = -\sin(y)\, e^{-\frac{2t}{Re}} = u(2\pi,y,t), \\ \quad v(x,0,t) = 0 = v(x,2\pi,t). \end{cases}$$

This problem is a time-dependent problem and we are showing results for different values of t.

The Reynolds numbers considered for the driven cavity problem are $400 \leq Re \leq 3200$ an for the Taylor vortex problem are $100 \leq Re \leq 7500$ The mesh sizes are denoted by h, and the time step by Δt; they will be specified for each case under study.

Figure 1 pictures the flow for the driven cavity problem, at steady state, for $Re = 400$, $h = \frac{1}{100}$ and $\Delta t = 0.01$, streamlines and isovorticity contours (left and right) are shown. Figure 2 pictures the flow for the same driven cavity problem, also at steady state, for $Re = 3200$, $h = \frac{1}{256}$ and $\Delta t = 0.01$. As one can see, the number of subvortexes increases as the Reynolds number increases.We must say that these results were compared with those in [1] and [8] and they agree well.

Figure 3 shows the streamlines (left) and isovorticity contours (right) for the Taylor vortex problem, with $Re = 100$ and $h = \frac{1}{128}$ at $t = 10$. Figure 4 shows the stream function and the vorticity for $Re = 100$ and $t = 1000$ (in 3D); the reason for showing the graph in 3D is because one can see the difference in scales at different times, and for different values of Re. Figure 5 and 6 show the graphs for $Re = 7500$ at $t = 10$ and $t = 1000$.

Since for this problem we know the exact solution, in Table 1 we are showing the relative error for the vorticity obtained in the numerical experiments.

Reynolds number	Time	Relative error
100	10	3.207252e-08
100	1000	3.192999e-08
7500	10	3.166967e-08
7500	1000	3.166967e-08

Table 1 Relative error for the Taylor vortex problem

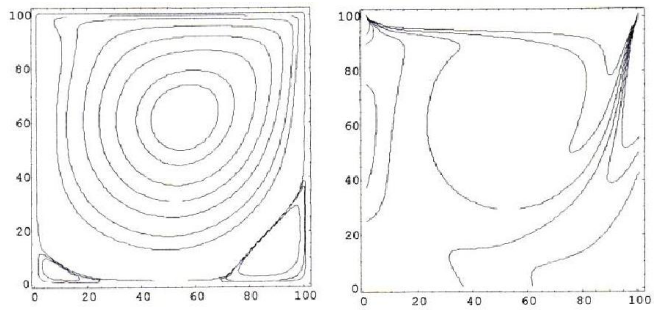

Figure 1. Streamlines (left) and isovorticity contours (right) for $Re = 400, \Delta t = 0.01$ and $h = \frac{1}{100}$

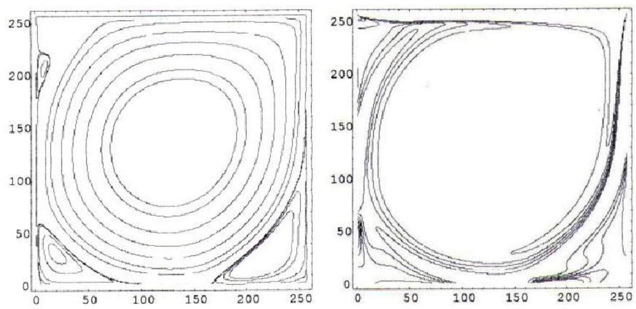

Figure 2. Streamlines (left) and isovorticity contours (right) for $Re = 3200, \Delta t = 0.01$ and $h = \frac{1}{256}$

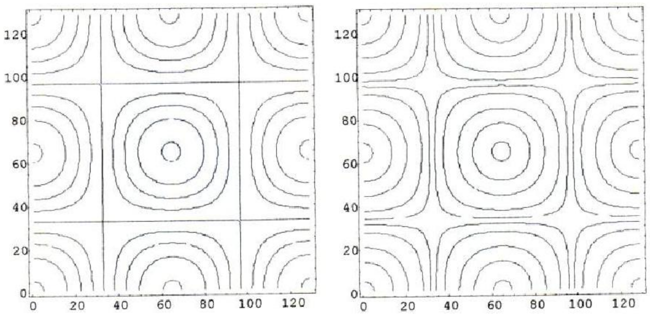

Figure 3. . Streamlines (left) and isovorticity contours (right) for $Re = 100, t = 10$ and $h = \frac{1}{128}$

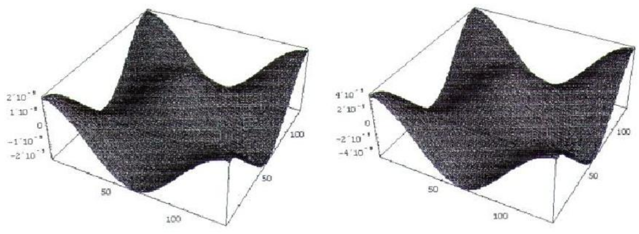

Figure 4. Stream function (left) and vorticity (right) for $Re = 100, t = 1000$ and $h = \frac{1}{128}$

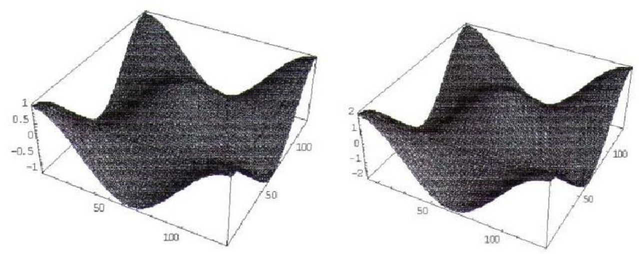

Figure 5. Stream function (left) and vorticity (right) for $Re = 7500, t = 10$ and $h = \frac{1}{128}$

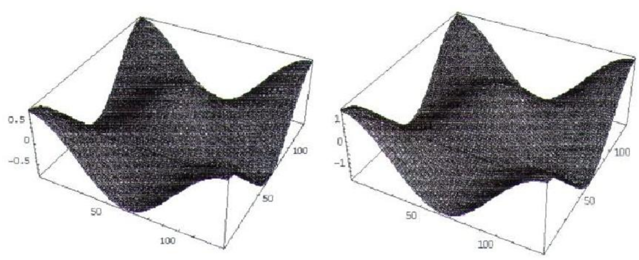

Figure 6. Stream function (left) and vorticity (right) for $Re = 7500, t = 1000$ and $h = \frac{1}{128}$

CONCLUSIONS

In this work we have shown numerical experiments for 2 test problems, the driven cavity problem and the Taylor vortex problem. We have used the velocity-vorticity formulation of the Navier-Stokes equations in order to solve these problems. The results for the driven cavity problem agree very well with those reported in [1] where the stream function-vorticity formulation was used, and with those reported in [8] also.

For the second problem, the Taylor vortex problem, since we know the exact solution we were able to calculate the relative error and the results were very good as it is shown in Table 1.for the vorticity.

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