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FREE OSCILLATIONS OF THREE ARMED LIQUID COLUMNS

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ABSTRACT

A mathematical formulation of the oscillating motion of three interconnected liquid columns open to the atmosphere is presented in this study. It is solved by Runga-Kutta method and compared with the experimental data. The results of the numerical solution are in good agreement with the experimental data. In order to obtain a mathematical statement of the natural frequencies depending on the geometrical parameters of this autonomous dynamic system, the mathematical formulation is obtained by control volume approach. In this way, the variation of the natural frequencies versus the distances between the columns and versus the cross-section areas of the columns is investigated.

INTRODUCTION

The oscillation of liquid columns can be experienced in the pipe networks and interconnected liquid tanks. In this study, the problem of the oscillation of three interconnected liquid columns open to the atmosphere is analyzed in order to investigate the dependency of the oscillation frequencies on the geometrical dimensions of a fluidyne heat machine. In such a machine which pumps water by means of the phase lag between evaporation and condensation processes the oscillation frequencies of the system depend only on the geometrical dimensions, when the amplitude of the vapor pressure variation in the adiabatic section of the fluidyne heat machine is negligible. The columns z_1 and z_2 shown in Figure.1 are connected on the upper sides with an adiabatic section in a Fluidyne heat machine. This adiabatic section has water vapor inside. When the column z_1 is heated and the column z_2 is cooled, a phase lag occurred between the amount of the evaporated and condensed masses. And the columns begin to oscillate by means of this phase lag.

NOMENCLATURE

A_i	[m^2]	Cross sectional area of the vertical columns
A_o,A_d	[m^2]	Cross sectional areas of the horizontal tubes
d_i	[m]	Diameters of the columns
d_o, d_d	[m]	Diameters of the horizontal tubes
F	[N]	External forces in Equation (2)
F	[N]	Friction forces
f		Friction coefficient
g	[m/s^2]	Gravitational acceleration
h_{io}	[m]	Initial heights of the columns
K_{Li}	[-]	Minor loss coefficient (= $\Delta P/(\rho \upsilon^2/2)$)
l_o	[m]	Length of the horizontal tube shown in Fig.1
l_{o1}	[m]	Length between the columns z_1 and z_2 shown in Fig.1
l_d	[m]	Length of the horizontal tube shown in Fig.1
l_{d1}	[m]	Length between the columns z_1 and z_3 shown in Fig.1
n		Normal vector
P	[N/m^2]	Pressure
P_x	[N/m^2]	Pressure at the entrance of the control volume, z_1A_1
P_y	[N/m^2]	Pressure at the entrance of the control volume, $z_2A_2+l_dA_o$
P_z	[N/m^2]	Pressure at the entrance of the control volume, $z_3A_3+l_oA_o$
P_o	[N/m^2]	Atmospheric pressure on the liquid columns
S_A		Mass entrance or exit surfaces
t	[s]	Time
v		Velocity vector
$\mathbf{v_s}$	[m/s]	Velocity at the entrance or exit surface
v_{Li}	[m/s]	Liquid velocities at liquid-air interfaces
v_i	[m/s]	Inlet liquid velocities of the control volumes
V_A		Control volume
V		Volume
W		Work done by the control volume
W_f		Work done by the friction forces
z_i		Column heights
z_o		Oscillations axis of liquid columns
$\dot{z}_i$		Liquid column velocities
Z_i		Dimensionless heights (=z_i/z_o)

Special characters

μ	[Pa.s]	Dynamic viscosity
ρ	[kg/m^3]	density
ω	[rad/s]	frequency

Subscripts

o	Initial value
i	1,2 and 3 show the labels of columns as shown in Fig.1

Elrod [1], Stammers [2] and Geissow [3] developed linear models and analyzed the dependency of the frequencies on the geometrical parameters of the system by assuming small oscillations in the liquid columns. Özdemir and Özgüç [4] proposed a simple mathematical model in order to analyze fluidyne heat machine both thermodynamically and hydrodynamically. The results of their model were in good agreement with the experimental data. When their model is run for the case of the interconnected liquid columns open to the atmosphere, the oscillation behavior of such a system looks like the oscillation of the fluidyne heat machine. Therefore, in this study, the free oscillation of three interconnected liquid columns open to the atmosphere is examined instead of a fluidyne heat machine. Consequently, the relation between frequencies and system geometry is obtained. On the other hand, by constructing two experimental setups, the free oscillation of the three interconnected water columns is observed, and then experimental and theoretical frequencies are compared.

FORMULATIONS

The system of the interconnected three liquid columns is shown in Figure 1. Integral forms of mass, momentum and energy conservation equations for a control volume having deformable boundaries are given as,

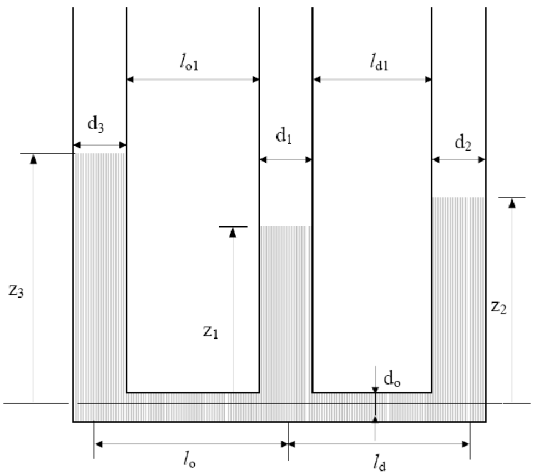

Figure 1 Three armed liquid columns

$$\frac{\partial}{\partial t}\int_{V_A} \rho dV + \int_{S_A} \rho(\mathbf{v}-\mathbf{v}_s)\cdot \mathbf{n} dA = 0 \tag{1}$$

$$\frac{\partial}{\partial t}\int_{V_A} \rho \mathbf{v} dV + \int_{S_A} \rho \mathbf{v}(\mathbf{v}-\mathbf{v}_s)\cdot \mathbf{n} dA = \sum \mathbf{F} \tag{2}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{V_A} \rho\left(\frac{|\mathbf{v}|^2}{2}+gz\right)dV \\ &+\int_{S_A} \rho\left(\frac{P}{\rho}+\frac{|\mathbf{v}|^2}{2}+gz\right)(\mathbf{v}-\mathbf{v}_s)\cdot \mathbf{n} dA = -\dot{W}\end{aligned} \tag{3}$$

respectively [4]. The thermodynamic state of the system does not change so that the energy equation reduces to the conservation of the mechanical energy. The following assumptions can be applied for simplicity.

- There is no mass diffusion at the interfaces between the liquid and air.
- The presence of air on the interfaces does not affect the motion of the liquid columns.
- Surface tension and wall adhesion forces are negligible at the interfaces at three columns.
- Liquid is an incompressible Newtonian fluid.
- Fluid velocities of the liquid columns are uniform.

The system shown in Figure 1 can be divided into three control volumes as $z_1 A_1$, $z_2 A_2 + l_d A_o$ and $z_3 A_3 + l_o A_o$, and their intersection region in order to apply the above conservation equations. After the mass conservation equations are written under the above assumptions and integrated over these control volumes, we obtain

$$v_{L1} = v_1 = \dot{z}_1,\ A_2 v_{L2} = A_2 \dot{z}_2 = A_o v_2,\ A_3 v_{L3} = A_3 \dot{z}_3 = A_o v_3 \tag{4}$$

easily. It is clear that the velocities of material points just on the interfaces are equal to the velocity of the moving boundary.

The conservation of mechanical energy over these control volumes can be written as follows.

$$\begin{aligned}&\rho \frac{d}{dt}\int_0^{z_1}\left(\frac{v^2}{2}+gz\right)A_1 dz \\ &-\rho A_1 v_1\left(\frac{P_x}{\rho}+\frac{v_1^2}{2}\right) = -P_o A_1 \dot{z}_1 - \dot{W}_{f1}\end{aligned} \tag{5}$$

$$\begin{aligned}&\rho \frac{d}{dt}\int_0^{l_d}\frac{v^2}{2}A_o dz + \rho\frac{d}{dt}\int_0^{z_2}\left(\frac{v^2}{2}+gz\right)A_2 dz \\ &-\rho A_o v_2\left(\frac{P_y}{\rho_f}+\frac{v_2^2}{2}\right) = -P_o A_2 \dot{z}_2 - \dot{W}_{f2}\end{aligned} \tag{6}$$

$$\begin{aligned}&\rho \frac{d}{dt}\int_0^{l_o}\frac{v^2}{2}A_o dz + \rho\frac{d}{dt}\int_0^{z_3}\left(\frac{v^2}{2}+gz\right)A_3 dz \\ &-\rho A_0 v_3\left(\frac{P_z}{\rho_f}+\frac{v_3^2}{2}\right) = -P_o A_3 \dot{z}_3 - \dot{W}_{f3}\end{aligned} \tag{7}$$

The first and second terms on the right side of above equations are work done at the moving boundary of the control volumes and work done by the friction forces, respectively. By integrating the energy equations (5 to 7) over those control volumes, it is easily shown that

$$P_x = \rho z_1 \dot{v}_{L1} + \rho g z_1 + P_o + F_1 / A_1 \tag{8}$$

$$P_y = \rho\left(l_d \frac{A_2}{A_o} + z_2 \right)\dot{v}_{L2} + \frac{\rho}{2} v_{L2}^2 \left(1 - \frac{A_2^2}{A_o^2} \right) + P_o + \rho g z_2 + F_2 / A_2 \tag{9}$$

$$P_z = \rho\left(l_o \frac{A_3}{A_o} + z_3 \right)\dot{v}_{L3} + \frac{\rho}{2} v_{L3}^2 \left(1 - \frac{A_3^2}{A_o^2} \right) + P_o + \rho g z_3 + F_3 / A_3 \tag{10}$$

Here, the last terms on the right hand side of these equations represent the friction forces as

$$F_1 = \frac{\dot{W}_{f1}}{v_{L1}} \qquad F_2 = \frac{\dot{W}_{f2}}{v_{L2}} \qquad F_3 = \frac{\dot{W}_{f3}}{v_{L3}}$$

At the intersection region of the columns, we can write mass, momentum and energy conservation equation as :

$$A_1 v_{L1} + A_2 v_{L2} + A_3 v_{L3} = 0 \tag{11}$$

$$\rho\left(v_2^2 - v_3^2\right) = P_z - P_y \tag{12}$$

$$A_1 v_1 \left(\frac{P_x}{\rho} + \frac{v_1^2}{2} \right) + A_o v_2 \left(\frac{P_y}{\rho} + \frac{v_2^2}{2} \right) + A_o v_3 \left(\frac{P_z}{\rho} + \frac{v_3^2}{2} \right) = -\frac{\dot{W}_{fo}}{\rho} \tag{13}$$

Here, $\dot{W}_{fo}$ represents the work done by the friction forces at the intersection region. If P_x, P_y and P_z are eliminated from these equations, we can rearrange them as follows.

$$\begin{aligned} A_k \dot{v}_{L1} = {} & g\left[\frac{A_2}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)(z_2 - z_1) + \left(l_d \frac{A_2}{A_o} + z_2 \right)(z_3 - z_1) \right] + \frac{A_2}{2A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)\left(v_{L2}^2 - v_{L1}^2\right) \\ & + \frac{1}{2}\left(l_d \frac{A_2}{A_o} + z_2 \right)\left(v_{L3}^2 - v_{L1}^2\right) + \frac{A_2^2}{2A_o^2}\left[\left(l_d \frac{A_2}{A_o} + z_2 \right) v_{L2} - \left(l_o \frac{A_3}{A_o} + z_3 \right) v_{L3} \right]\left(\frac{A_3}{A_2} v_{L3} - v_{L2} \right) \\ & + \frac{A_2}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)\left(\frac{F_2}{\rho A_2} - \frac{F_1}{\rho A_1} \right) + \left(l_d \frac{A_2}{A_o} + z_2 \right)\left(\frac{F_3}{\rho A_3} - \frac{F_1}{\rho A_1} \right) - \frac{a}{A_1 v_{L1}} \frac{\dot{W}_{fo}}{\rho} \end{aligned} \tag{14}$$

$$\begin{aligned} A_k \dot{v}_{L2} = {} & g\left[z_1(z_3 - z_2) + \frac{A_1}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)(z_1 - z_2) \right] + \frac{1}{2} z_1\left(v_{L3}^2 - v_{L2}^2\right) \\ & + \frac{A_1}{2A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)\left(v_{L1}^2 - v_{L2}^2\right) - \frac{A_1 A_2}{2A_o^2}\left[z_1 v_{L1} - \left(l_o \frac{A_3}{A_o} + z_3 \right) v_{L3} \right]\left(\frac{A_3}{A_2} v_{L3} - v_{L2} \right) \\ & + z_1\left(\frac{F_3}{\rho A_3} - \frac{F_2}{\rho A_2} \right) + \frac{A_1}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)\left(\frac{F_1}{\rho A_1} - \frac{F_2}{\rho A_2} \right) + \frac{A_1}{A_3}\left(l_o \frac{A_2}{A_o} + z_3 \right) \frac{1}{A_1 v_{L1}} \frac{\dot{W}_{fo}}{\rho} \end{aligned} \tag{15}$$

where

$$A_k = \frac{A_1}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right)\left(l_d \frac{A_2}{A_o} + z_2 \right) + z_1\left[\left(l_d \frac{A_2}{A_o} + z_2 \right) + \frac{A_2}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right) \right] \tag{16}$$

$$a = \frac{A_2}{A_3}\left(l_o \frac{A_3}{A_o} + z_3 \right) + \left(l_d \frac{A_2}{A_o} + z_2 \right) \tag{17}$$

The major and minor loss terms are defined as

$$F_1 = \frac{\dot{W}_{f1}}{v_{L1}} = 8\pi\mu z_1 v_{L1} \tag{18}$$

$$\begin{aligned} F_2 &= \frac{\dot{W}_{f2}}{v_{L2}} \\ &= 8\pi\mu\left(l_d \frac{A_2^2}{A_o^2} + z_2 \right) v_{L2} + A_2 \frac{1}{2}\rho K_{L2} \left|v_{L2}\right| v_{L2} \end{aligned} \tag{19}$$

$$\begin{aligned} F_3 &= \frac{\dot{W}_{f3}}{\dot{z}_3} \\ &= 8\pi\mu\left(l_o \frac{A_3^2}{A_o^2} + z_3 \right) v_{L3} + A_3 \frac{1}{2}\rho K_{L3} \left|v_{L3}\right| v_{L3} \end{aligned} \tag{20}$$

$$\frac{\dot{W}_{fo}}{A_1 v_{L1} \rho} = \frac{1}{2} K_{Lo} \left|v_{L1}\right| v_{L1} \tag{21}$$

by assuming laminar fluid flow with the friction coefficient *f=16/Re*. Here, we must consider the dynamic pressure as $\rho v|v|/2$ for oscillating flow in order to account for the sign of the velocity vector. v_{L3} and z_3 in equations (14) and (15) are not independent variables. v_{L3} can be easily eliminated by using Equation (11). z_3 can also be eliminated by using the conservation of total mass in the system.

$$z_3 = -\frac{A_1}{A_3}(z_1 - h_{10}) - \frac{A_2}{A_3}(z_2 - h_{20}) + h_{30} \tag{22}$$

where h_{10}, h_{20} and h_{30} are the initial liquid column heights.

Two equations are then obtained for $\dot{v}_{L1}$ and $\dot{v}_{L2}$ in the form of

$$\dot{v}_{L1} = f_1(z_1, z_2, v_{L1}, v_{L2})$$
$$\dot{v}_{L2} = f_2(z_1, z_2, v_{L1}, v_{L2})$$

Otherwise, if it is recalled that $\dot{z}_1 = v_{L1}$ and $\dot{z}_2 = v_{L2}$, then the set of four equations can be solved by the fourth-order Runge-Kutta method. The 4th order Runga-Kutta method is widely used since it is fairly accurate, stable and easy to program. The calculations are carried out using a time step of 0.001 s for all cases.

EXPERIMENTAL STUDY

Experimental setup consists of three interconnected glass tubes. Plastic tubes and fittings make the connections between these tubes as shown schematically in Figure 1. Each tube is open to the atmosphere. Two different setups are constructed, according to distance between the tubes. The geometrical dimensions are given in Table 1.

Table 1. Geometrical dimensions of each experimental set-up

	Setup-I	Setup-II
d_1=d_2= d_3	15.3 mm	15.3 mm
d_o	13 mm	13 mm
ℓ_c	1465 mm	1465 mm
ℓ_d	200 mm	200 mm
ℓ_o	200 mm	400 mm

The system is filled with water to a certain level and the water columns in the tubes are at equal level initially. One of the water columns is raised to a certain level by applying vacuum partially on that column. The system of three interconnected water columns, which has gained a potential energy, is motionless at the beginning by means of a valve at the end of that column. The valve is opened suddenly and the damped free oscillation of the system is observed. The system oscillates during a certain time interval due to potential energy difference between the initial and the equilibrium conditions. Then, it is damped by the minor and major friction forces. A digital camera connected to a computer records the variation of the heights of the water columns with time. The recording period is 10 seconds and the time step is 0.04 seconds. The heights of the columns are read on the monitor frame by frame.

An error of approximately ± 2 mm is expected in the readings of the column heights. The summation of the column heights at any time must be constant to satisfy the conservation of mass. But we observed that the error on the total mass was higher than that of the column heights. It was analyzed that the maximum error on the total mass was less than 10 percent. This error varies sinusoidally with time following the motion of the column raised. We estimate that this error is caused by the fact that the water near the walls cannot follow the water in the bulk region exactly.

RESULTS AND DISCUSSION

The experimental results and the solutions of Equations (14) and (15) by Runga-Kutta method are shown in Figures 2, 3 and 4 for the free oscillation of three interconnected water columns. The minor and major loss coefficients that are required for the numerical solution of the theoretical model are modified according to the experimental studies and these coefficients are not changed during all numerical calculations for an experimental setup. As shown in the figures 2, 3 and 4, there is a good agreement between the numerical solutions of the mathematical model and the experimental results.

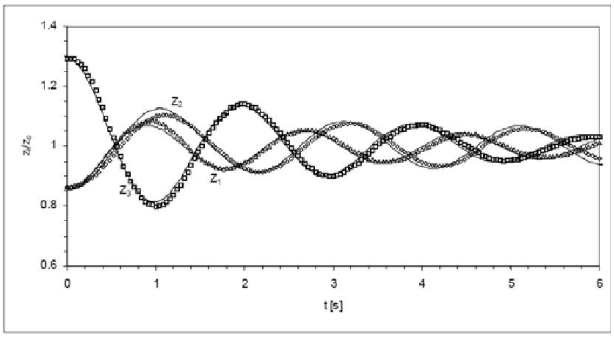

Figure.2 Dimensionless column heights versus time for Setup-I. (z_o=70 cm, h_{30}=90.5 cm, h_{10}=h_{20}=597.5 cm. (---- R-K solution, □ Δ ◊ experiment)

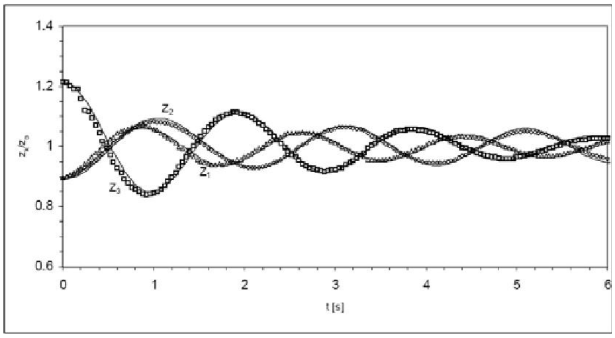

Figure.3 Dimensionless column heights versus time for Setup-I. z_o=70 cm, h_{30}=85 cm, h_{10}=h_{20}=62.5 cm. (----- R-K solution, □ Δ ◊ experiment)

However, at time range where vibration starts and where its dampening speeds up, this agreement fails. The reason of this divergence in the results is thought to be emanating from the fact that the loss expressions are determined according to laminar regime assumption. When loss expressions and loss coefficients are altered, accordance in results is attained in the aforementioned region. But then one starts encountering accordance problems in other regions. The turbulent flow at the beginning gradually becomes laminar first and then; after the 8th second, it starts resembling to creeping flow.

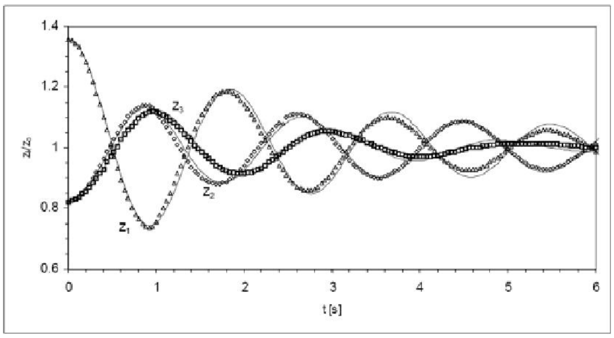

Figure.4 Dimensionless column heights versus time for Setup-II. z_o=70 cm, h_{10}=95 cm, h_{20}=h_{30}=57.5 cm. (---- R-K solution, □ Δ ◊ experiment)

The mathematical solutions may be obtained easily by using the Mathematica software. In this way, the variation of the free oscillation frequencies of three interconnected liquid columns open to atmosphere is obtained mathematically depending on the water mass and the geometry of the system.

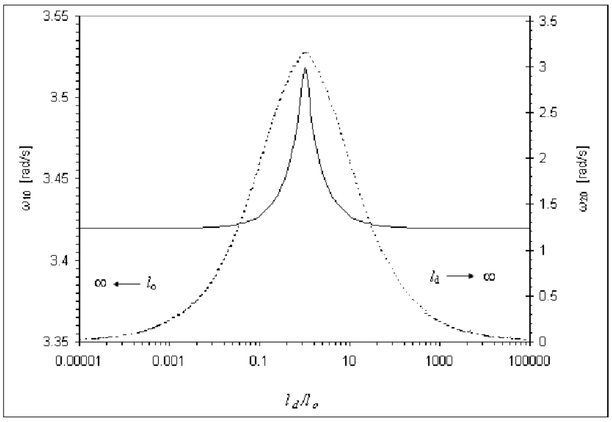

Figure 5 Variation of the natural frequencies versus the lengths between the columns. $A_i / A_0 = 1.385$ (i=1,2 and 3), $z_0 = 0.7$ m. (____ ω_{10} , ……. ω_{20})

In the first order and the un-dampened solution of the mathematical model of this system, two natural frequencies are found as Equations (23) and (25). The evolution of these natural frequencies according to the distance between the columns is shown in Fig. 5. This figure has a symmetry at the point $l_d / l_o = 1$. l_d goes to infinity on the right side of the symmetry line while l_o is constant. On the left side l_o goes to infinity and l_d is constant. As seen in the figure, one of the outer columns is taken away from the other two columns, the frequency ω_{20} approaches to zero. The other frequency ω_{10} approaches to the frequency of a U-tube.

For $l_o \to \infty$ as the frequency given by Eq. (24) approaches to zero, the frequency given by Eq. (26) matches with the frequency of a U-tube having different diameters.

$$\omega_{10} = \lim_{l_o \to \infty} \sqrt{\frac{b+\Delta}{2}} = \sqrt{\frac{g(1+\frac{A_2}{A_1})}{(1+\frac{A_2}{A_1})z_o + \frac{A_2}{A_o} l_d}} \tag{23}$$

and when $l_o \to \infty$ the frequency are obtained:

$$\omega_{10} = \sqrt{\frac{2g}{2z_o + l_d}} \tag{24}$$

Similarly for $l_d \to \infty$:

$$\omega_{20} = \lim_{l_d \to \infty} \sqrt{\frac{b+\Delta}{2}} = \sqrt{\frac{g(1+\frac{A_3}{A_1})}{(1+\frac{A_3}{A_1})z_o + \frac{A_3}{A_o} l_o}} \tag{25}$$

and ω_{20} frequency easily finds that

$$\omega_{20} = \sqrt{\frac{2g}{2z_o + l_o}} \tag{26}$$

In this way, it is shown that the frequency statements given by Equations (23) and (25) match with the natural frequency of a U-tube for limit conditions of l_o and l_d.

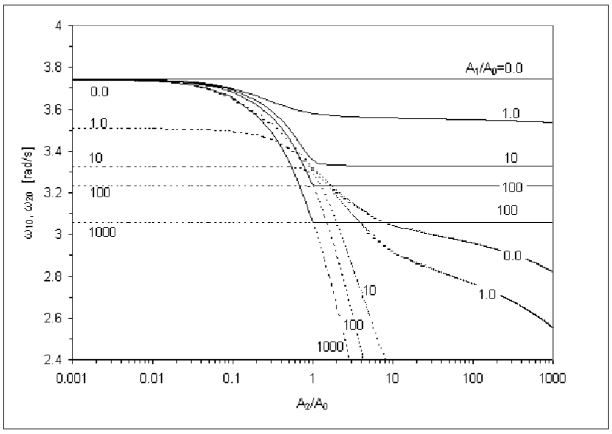

Figure 6 Variation of the natural frequencies versus the cross-section areas of the columns. $l_{d1} / l_{o1} = 1$, $A_3 / A_0 = 1$, $z_0 = 0.7$ m. (—— ω_{10} , ……. ω_{20})

The variation of the natural frequencies versus the column areas A_1 and A_2 is shown in Figure 6. When A_1 / A_o goes to zero, it is seen that ω_{10} approaches to the frequency of a column connected to an infinite pool ($=\sqrt{g/z_o}$), and that ω_{20} approaches to the frequency of a U-tube. The character of ω_{10} and ω_{20} is same as the previous situation for any value of A_1/A_o different from zero, if the value of A_2/A_o goes to zero. ω_{20} approaches to zero value and ω_{10} approaches to the frequency of a U-tube, if the value of A_2/A_o goes to infinity. On the other hand, the frequencies of ω_{10} and ω_{20} approach to the same value, when A_1/A_o increases and A_2/A_o equals to 1.

Such first order solutions approximate the behavior of this non-linear dynamic system having two degrees of freedom. From the fluidyne point of view, in order to increase the work output, the oscillation frequencies must be increased. Consequently, the number of cycles in unit time will increase. The most appropriate fluidyne geometry may be obtained solving the problem of connected case of 1 and 2 columns upper side.

CONCLUSION

In this study, free oscillation experiments are conducted for two different setups of three interconnected liquid columns and the proposed mathematical formulation is solved numerically. The results of the numerical solution are in good agreement with the experimental data. The frequency equations are obtained by solving the mathematical model. These equations give a relation between the frequencies and the geometry of the system. Then, one can optimize the frequency values versus the geometrical parameters. This study contributes to the understanding of the oscillation analysis of three armed liquid columns and Fluidyne systems.

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