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THE FRACTIONAL HEAT EQUATION

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ABSTRACT

This paper extends the method, in which a Volterra-type integral equation that relates the local values of temperature and the corresponding heat flux within a semi-infinite domain, to a transient heat transfer process in a non-isolated system that has a memory about its previous state. To model such memory systems, the apparatus of fractional calculus is used. Based on the generalized constitutive equation with fractional order derivative, the fractional heat equation is obtained and solved. Its analytical solution is given in the form of a Volterra-type integral equation. It follows from the model, developed in this study, that the heat wave, generated in the beginning of ultra-fast energy transport processes, is dissipated by thermal diffusion as the process goes on. The corresponding contributions of the wave and diffusion into the heat transfer process are quantified by a fractional parameter, H, which is a material-dependent constant.

INTRODUCTION

Modeling of heat transfer processes may often become a challenging task due to the fact that different phenomenological concepts (e.g. the classical Fourier model, thermal waves, lagging) may be used for such modeling.

It seems that the method, which leads to a Volterra-type integral equation that relates the local values of temperature and the corresponding heat flux within a semi-infinite domain, is an effective analytical tool for solving many challenging problems in heat transfer. The method was proposed in [1] and applied to solving diffusion problem in [2]. The method was generalized by Frankel for finite domains in [3]. Then, the solution of the heat wave equation was obtained by means of this method accounting for the surface heat flux in work [4] and the both surface heat flux and volumetric source in [5].

This work extends the method to a transient heat transfer process taking place in a non-isolated system that has a memory about its previous state. To model such memory systems, the apparatus of fractional calculus is used. To begin with, the single-phase-lag constitutive equation is considered:

NOMENCLATURE

a	[-]	Positive constant (for Gaussian distribution a = 2.77)
C	[Jm^{-3}K^{-1}]	Volumetric heat capacity
H	[-]	Dimensionless parameter
k	[W/mK]	Thermal conductivity
$\mathbf{q}''$	[W/m^2]	Heat flux vector
q''	[W/m^2]	Axial component of $\mathbf{q}''$
$\mathbf{r}$	[m]	Position vector
T	[K]	Temperature
s	[-]	Laplace variable
T_0	[K]	Initial temperature
t	[s]	Time
x	[m]	Cartesian axis direction

Greek symbols

α	[m^2/s]	Thermal diffusivity
τ	[s]	Relaxation time
ξ	[-]	Dimensionless time
η	[-]	Dimensionless spatial variable

$$\mathbf{q''}(\mathbf{r},t+\tau) = -k\nabla T(\mathbf{r},t), \tag{1}$$

where $\mathbf{q''}(\mathbf{r},t)$ is the heat flux vector, $T(\mathbf{r},t)$ is the temperature, $\mathbf{r}$ is the position vector, t is the time τ is the relaxation time, k is the thermal conductivity. Assuming that τ is small, left part of Eqn. (1) can be extended into the fractional Taylor series [6] as follows

$$\mathbf{q''}(\mathbf{r},t) + \sum_{n=1}^{\infty} \frac{\tau^{nH}}{\Gamma(nH+1)} \, {}_0D_t^{nH} \mathbf{q''}(\mathbf{r},t) = -kT(\mathbf{r},t), \tag{2}$$

where Γ is the Gamma function, H is parameter defined in interval $[0,1]$, D^{nH} is the Riemann-Liouville fractional derivative which is defined as

$$
{}_0D_t^H f(t) = \begin{cases} \dfrac{1}{\Gamma(m-H)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{f(\xi)}{(t-\xi)^{H+1-m}} d\xi, \text{ if } m-1 < H < m; \\ \dfrac{d^m}{dt^m} f(t), \text{ if } H = m \in \mathrm{N}. \end{cases} \quad (3)
$$

Neglecting the second order terms in Eqn. (2) the latter can be simplified as

$$
\mathbf{q''}(\mathbf{r},t) + \tau^H \, {}_0D_t^H \mathbf{q''}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t). \quad (4)
$$

Equation (4) is the generalized constitutive equation with a fractional order derivative. The fractional constitutive equation is the basic assumption in our study, which leads to the fractional differential equation for the temperature. Its analytical solution is obtained in the form of a Volterra-type integral equation that relates the local values of temperature and the corresponding heat flux.

PROBLEM FORMULATION

Let us consider the energy equation which relates the heat flux and temperature by mean of the following relationship

$$
-\nabla \cdot \mathbf{q''}(\mathbf{r},t) = C \frac{\partial}{\partial t} T(\mathbf{r},t), \quad (5)
$$

where C is the volumetric heat capacity. Applying (5) to (4), and accounting for linear properties of (3) [7], the fractional differential equation for the temperature can be written as

$$
{}_0D_t^1 T(\mathbf{r},t) + \tau^H \, {}_0D_t^{1+H} T(\mathbf{r},t) = \alpha \nabla^2 T(\mathbf{r},t), \quad (6)
$$

where α is the thermal diffusivity, which is assumed to be constant, in this study. Finally, let us rewrite Eqn. (6) for one-dimension problem, replacing vector $\mathbf{r}$ by the Cartesian coordinate x, we have

$$
{}_0D_t^1 T(x,t) + \tau^H \, {}_0D_t^{1+H} T(x,t) = \alpha \frac{\partial^2}{\partial x^2} T(x,t). \quad (7)
$$

Equation (7) is the one-dimension partial fractional differential equation that describes the transient temperature behavior a non-isolated system.

System (7) is applied to solve an initial-value problem in a semi-infinite domain under the condition of initial thermal equilibrium of the domain. Therefore, at time $t = 0$, the temperature T is equal to T_0, which is a constant and uniform temperature everywhere within the domain. At $x = 0$, the boundary condition is one of the known heat flux (either adiabatic or uniform nonzero heat flux). For a semi-infinite domain, when $x \to \infty$ the boundary condition is $T \to T_0$.

SOLUTION PROCEDURE

Let us introduce new dimensionless variables, as follows

$$
\xi = t/\tau, \;\; \eta = x/\sqrt{\alpha\tau}. \quad (8)
$$

Using (8), Eqn. (7) can be rewritten

$$
{}_0D_t^1 \theta(\xi,\eta) + {}_0D_t^{1+H} \theta(\xi,\eta) = \frac{\partial^2}{\partial \eta^2} \theta(\xi,\eta), \quad (9)
$$

where $\theta = T - T_0$. Hence, the initial and boundary conditions become as follows

$$
\left.\frac{\partial \theta}{\partial \xi}\right|_{\xi=0} = 0; \;\; \theta\big|_{\xi=0} = 0, \text{ and } \lim_{\eta\to\infty} \theta = 0. \quad (10)
$$

Taking the Laplace transform of (9) (details of the Laplace transform of fractional derivatives can be found in [7]) and taking into account conditions (10), Eqn. (9) becomes

$$
s\Theta(s;\eta) + s^{1+H}\Theta(s;\eta) = \frac{\partial^2}{\partial \eta^2}\Theta(s;\eta), \quad (11)
$$

where $\Theta(s;\eta)$ is the Laplace transform of $\theta(\xi,\eta)$. The general solution of homogeneous equation (11) is

$$
\Theta(s;\eta) = C_1(s)e^{\lambda(s)\eta} + C_2(s)e^{-\lambda(s)\eta}, \quad (12)
$$

where $\lambda(s) = \sqrt{s\left(1+s^H\right)}$. To satisfy the boundary condition, $C_1(s)$ must be identically zero. Denoting $C(s) \equiv C_2(s)$, Eqn. (12) can be rewritten as

$$
\Theta(s;\eta) = C(s)e^{-\lambda(s)\eta}. \quad (13)
$$

To eliminate $C(s)$ from (13) the derivative of $\Theta(\eta,s)$ with respect to η is used:

$$
\frac{\partial\Theta(s;\eta)}{\partial\eta} = -\lambda(s)C(s)e^{-\lambda(s)\eta}. \quad (14)
$$

Combining (14) and (13), we have

$$
\Theta(s;\eta) = -\frac{1}{\lambda(s)}\frac{\partial\Theta(s;\eta)}{\partial\eta}. \quad (15)
$$

The inverse Laplace transform of $1/\lambda(s)$ can be expressed in terms of the generalized function G [8] as follows

$$
L^{-1}\left\{\frac{1}{\sqrt{s\left(1+s^H\right)}}\right\} = G_{H,-\frac{1}{2},\frac{1}{2}}\left(-1,\xi\right). \quad (16)
$$

The description of function G is shown in Appendix. Thus, using (16) and the convolution theorem, the inverse Laplace transform of (15) becomes

$$
\theta(\xi,\eta) = -\int_0^{\xi} \frac{\partial\theta(\xi,\eta)}{\partial\eta} G_{H,-\frac{1}{2},\frac{1}{2}}\left(-1,\xi-\varsigma\right)d\varsigma. \quad (17)
$$

Restoring the original variables, Eqn. (17) can be rewritten as

$$
T(t,x) = T_0 - \sqrt{\frac{\alpha}{\tau}}\int_0^t \frac{\partial T(t,x)}{\partial x} G_{H,-\frac{1}{2},\frac{1}{2}}\left(-1,\frac{t-t^*}{\tau}\right)dt^*. \quad (18)
$$

Equation (18) gives the relationship between the temperature and its spatial derivative at any moment of time and at any location in the non-isolated domain.

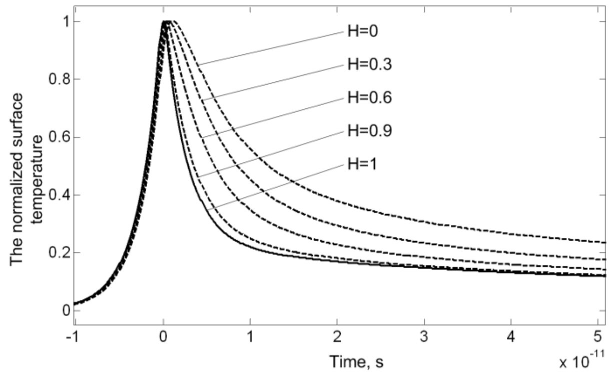

Figure 1. The time evolution of the normalized surface temperature for different values of parameter, *H*.

It is important to note here that if $H=0$, the generalized function $G_{0,-\frac{1}{2},\frac{1}{2}}\left(-1,\frac{t}{\tau}\right)$ becomes simply $\sqrt{\frac{\tau}{\pi t}}$ and the solution of (7) is the classical model considered in [9]. In the case of $H=1$, the generalized function $G_{1,-\frac{1}{2},\frac{1}{2}}\left(-1,\frac{t}{\tau}\right)$ is reduced to $I_0\left(\frac{t}{2\tau}\right)e^{-\frac{t}{2\tau}}$, where I_0 is the modified Bessel function and the solution becomes the same as obtained in [4].

Let us rewrite the generalized constitutive Eqn. (4) for one-dimensional case as

$$q''(x,t)+\tau^H\,{}_0D_t^H q''(x,t)=-k\frac{\partial T(x,t)}{\partial x}. \qquad (19)$$

Substituting (19) into (18), the relationship between the local temperature and the corresponding heat flux for the non-isolated systems is

$$T(t,x)=T_0+\frac{1}{\sqrt{kC\tau}}\int_0^t\left[q''(x,t)+\tau^H\,{}_0D_t^H q''(x,t)\right]\times G_{H,-\frac{1}{2},\frac{1}{2}}\left(-1,\frac{t-t^*}{\tau}\right)dt^*. \qquad (20)$$

RESULTS AND DISCUSSION

Model (20) predicts that in the initial stage, the ultra-fast heat transfer process occurs by means of both thermal waves and diffusion. As time goes on, however, the thermal waves are fully dissipated by thermal diffusion. The corresponding contributions of the wave and diffusion into the heat transfer process are quantified by a fractional parameter, H, which is a material-dependent constant.

Further, Eqn. (20) is solved numerically. In order to compute the surface temperature for a given heat flux at the boundary, let us take the representative physical properties of metals, that are $\alpha=10^{-5}$ $\mathrm{m^2/s}$, $C=10^6$ $\mathrm{J/(m^3K)}$, and $\tau=10$ ps. The heat flux is represented by the Gaussian, namely,

$$q''(t)=\exp\left(-a\left[\frac{t-b}{\sigma}\right]^2\right), \qquad (21)$$

where a is a positive constant (e.g. for the Gaussian distribution a=2.77 [10]), $b=10$ ps and $\sigma=5$ ps. The time evolution of the normalized surface temperature, $\theta=\left(T_s-T_0\right)/\left(T_{\max}-T_0\right)$ for different values of parameter, H is shown in Figure 1.

APPENDIX

Generalized Function G

$$G_{q,\nu,r}(a,t)=\sum_{j=0}^{\infty}\frac{\Gamma(1-r)(-a)^j\,t^{(r+j)q-\nu-1}}{\Gamma(1+j)\Gamma(1-j-r)\Gamma((r+j)q-\nu)}, \qquad (22)$$

and Laplace transform of (22) is

$$L\left\{G_{q,\nu,r}(a,t)\right\}=\frac{s^\nu}{\left(s^q-a\right)^r}, \qquad (23)$$

where Γ is the Gamma function, $\mathrm{Re}\{qr-\nu\}>0$, $\mathrm{Re}\{s\}>0$, $\left|\frac{a}{s^q}\right|>0$.

CONCLUSION

This paper extends the method, in which a Volterra-type integral equation that relates the local values of temperature and the corresponding heat flux within a semi-infinite domain, to a transient heat transfer process in a non-isolated system that has a memory about its previous state. To model such memory systems, the apparatus of fractional calculus is used. Based on the generalized constitutive equation with fractional order derivative, the fractional heat equation is obtained and solved. Its analytical solution is given in the form of a Volterra-type integral equation. It follows from the model, developed in this study, that the heat wave, generated in the beginning of ultra-fast energy transport processes, is dissipated by thermal diffusion as the process goes on. The corresponding contributions of the wave and diffusion into the heat transfer process are quantified by a fractional parameter, H, which is a material-dependent constant.

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