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SOLUTION FOR TRANSIENT HEAT CONDUCTION PROBLEM WITH LOW BIOT NUMBER

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ABSTRACT

Mechanical systems such as brakes and clutches experience short periods of intense heating. The heat lost from the exposed surfaces is small or even negligible compared to the frictional heat generated during a single engagement. The Biot number for such systems is known to be very low. The general transient solution is often obtained by the superposition of a particular solution and the solution of the corresponding homogenous problem. This paper presents the finite element realization of this method when the Biot number is low.

INTRODUCTION

Automotive brakes and clutches involve bodies that are in contact and move relative to one another. The contact pressure *p* and the relative sliding speed *v* yields frictional heating ($q=fvp$), where *f* is the coefficient of friction. The temperature field resulting from the frictional heating causes a thermo-elastic distortion which in turn modifies the contact pressure distribution. This process is found to be unstable when the sliding speed exceeds some critical limit [1] leading to areas of high heat generation or hot spots [2]. To fully understand this process one needs to solve the two systems of equation; the thermo-elastic system for the contact pressure and the heat conduction problem for the temperature evolution. Analytical solution for this type of thermo-mechanical systems is only possible for simplified geometries such as sliding of half-plane against rigid surface [3] or sliding of two-half planes [4]. To account for real geometries, numerical or finite element approaches are often used [5].
An intense amount of frictional heat is generated when the frictional disks in automotive brakes and clutches are engaged [6]. The engagement period in clutches for example is very short and may not exceed half a second. Numerical time integration is one alternative where the transient problem is solved for small time steps [7]. This approach however is found to be computer intensive mainly because of the small time steps that need to be used to capture the quick evolution of the temperature field. Another possible solution is based on superposition of the particular solution (often steady state) and the general solution of the homogenous problem. In this approach the homogenous problem is formulated to yield an eigenvalue problem. The complete solution comprises of a series expansion of the eigenvalues and eigenfunctions [8]. The method of superposition is known to be less computer intensive when compared to the method of time integration.
A large amount of heat is generated during the engagement and the capacity of the outer surface areas to dissipate heat during the engagement period is either small or negligible. Hence the ratio of the heat transfer resistance of the body to the surface of body is small (resulting into a system of low Biot number). The evolution of the temperature field is largely determined by the heat input and the thermal capacity of the system. The order of magnitude of the steady state solution is large when the Biot number is low. This in turn can result into numerical inaccuracy in the transient solution of the heat conduction problem. This paper presents a method that can be used to overcome problem of numerical inaccuracy associated with the method of superposition. The proposed methodology will be tested in the context of a disk sliding against a non-conductive surface.

NOMENCLATURE

T	[^oC]	Temperature
t	[s]	Time
q	[W/m^2]	Heat input
k	[m^2/s]	Thermal diffusivity
K	[$W/m.^oC$]	Thermal Conductivity
h	[$W/m^2.^oC$]	Heat transfer coefficient

TRANSIENT HEAT CONDUCTION PROBLEM

In the general transient heat conduction problem one seek the solution of the heat equation

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t} \tag{1}$$

where T is the temperature in some domain (x, y, z), t is the time and k is the thermal diffusivity of the material. In typical brakes and clutches, the boundary condition will be of the form

$$K\frac{\partial T}{\partial n}(x,y,z,t) = q(x,y,z,t); \qquad (x,y,z)\in\Gamma_q \qquad (2)$$

$$K\frac{\partial T}{\partial n}(x,y,z,t) = -hT(x,y,z,t); \qquad (x,y,z)\in\Gamma_h \qquad (3)$$

where Γ defines the boundary of Ω, n is the outward normal to Γ, q is the frictional heat generation on the boundary Γ_q and h is a heat transfer coefficient. T here is measured with respect to room temperature T_∞. Furthermore, the sought solution must satisfies the initial condition

$$T(x,y,z,0) = T_0(x,y,z)\,. \qquad (4)$$

The general solution to equation (1) can be expressed as the sum of a particular solution T_p and the solution of the homogeneous problem, T_H. The particular solution must satisfies the governing equation and the boundary conditions but not necessarily the initial condition.

$$\nabla^2 T_p = 0 \qquad (5)$$

$$\begin{aligned} K\frac{\partial T_p}{\partial n}(x,y,z,t) &= q(x,y,z,t); \qquad (x,y,z)\in\Gamma_q \\ K\frac{\partial T_p}{\partial n}(x,y,z,t) &= -hT_p(x,y,z,t); \qquad (x,y,z)\in\Gamma_h \end{aligned} \qquad (6)$$

A steady state solution can be regarded for the particular solution. The general solution of the homogeneous problem is defined by

$$\nabla^2 T_H = \frac{1}{k}\frac{\partial T_H}{\partial t} \qquad (7)$$

$$\begin{aligned} K\frac{\partial T_H}{\partial n}(x,y,z,t) &= q(x,y,z,t); \qquad (x,y,z)\in\Gamma_q \\ K\frac{\partial T_H}{\partial n}(x,y,z,t) &= -hT_H(x,y,z,t); \qquad (x,y,z)\in\Gamma_h \end{aligned} \qquad (8)$$

A solution of exponential form can be considered for the homogenous problem [9]

$$T_H = \Theta(x,y,z)\exp(-bt) \qquad (9)$$

Substituting for T_H in equations (7) and (8), gives

$$\nabla^2\Theta + \frac{b}{k}\Theta = 0 \qquad (10)$$

$$\begin{aligned} K\frac{\partial \Theta}{\partial n} &= 0; \qquad (x,y,z)\in\Gamma_q \\ K\frac{\partial \Theta}{\partial n} &= -h\Theta; \qquad (x,y,z)\in\Gamma_h \end{aligned} \qquad (11)$$

There are certain eigenvalues b_i and eigenfunctions Θ_i that satisfies the governing equation (10). The general solution of the homogenous problem can then be written as an eigenfunction series

$$T_H(x,y,z,t) = \sum_{i=1}^{\infty} A_i\Theta_i(x,y,z)\exp(-b_i t) \qquad (12)$$

where A_i are a set of arbitrary constants to be determined from the initial condition.

FINITE ELEMENT SOLUTION

The particular solution can be approximated by

$$T_p(x,y,z) = \sum_{j=1}^{N} T_j^s N_j(x,y,z) \qquad (13)$$

where T_j are a set of nodal temperatures and N_j are independent shape functions. The solution of the heat equation (5) can then be approximated using the Galerkin's method [10].

$$\int_\Omega \left(\nabla^2 T_p\right) N_k(x,y,z)\,d\Omega = 0 \qquad (14)$$

for $k = (1, N)$. The order of the derivative in equation (1) is usually greater than the order of continuity in the shape functions. This difficulty can be overcome by reducing the order of the derivative in equation (5) using the divergence theorem

$$-\sum_{j=1}^{N} T_j^s \int_\Omega \nabla N_j \nabla N_k\,d\Omega + \int_\Gamma N_k \frac{\partial T_p}{\partial n}\,d\Gamma = 0 \qquad (15)$$

Applying the particular solution boundary conditions (6) yields the matrix equation

$$(\mathbf{K}+\mathbf{H})\mathbf{T}^{\mathbf{s}} = \mathbf{Q} \qquad (16)$$

where

$$\begin{aligned} K_{jk} &= \int_\Omega \nabla N_j \nabla N_k\,d\Omega \\ H_{jk} &= \int_{\Gamma_h} \frac{hN_jN_k}{K}\,d\Gamma \\ Q_k &= \int_{\Gamma_q} \frac{qN_k}{K}\,d\Gamma \end{aligned} \qquad (17)$$

A similar procedure can be followed to obtain the finite element solution of the homogenous problem which yields the matrix equation

$$(\mathbf{K}+\mathbf{H})\mathbf{\Theta} = \frac{b}{k}\mathbf{D\Theta} \tag{18}$$

where

$$D_{jk} = \int_\Omega N_j N_k d\Omega \tag{19}$$

Equation (18) defines a general eigenvalue problem that can be solved for the eigenvalues b_i and the eigenvectors $\mathbf{\Theta}_i$. The general solution of the heat conduction problem can be written as

$$\mathbf{T}(t) = \mathbf{T}^s + \sum_{i=1}^{N} A_i \mathbf{\Theta}_i \exp(-b_i t) \tag{20}$$

The particular solution $\mathbf{T}^s$ can be expanded in terms of the eigenvectors

$$\mathbf{T}^s = \sum_{i=1}^{N} B_i \mathbf{\Theta}_i \tag{21}$$

Substituting for $\boldsymbol{T}^s$ in (16) gives

$$\sum_{i=1}^{N} B_i (\mathbf{K}+\mathbf{H})\mathbf{\Theta}_i = \mathbf{Q} \tag{22}$$

Using equation (18) into (22)

$$\sum_{i=1}^{N} \frac{b_i B_i}{k}\mathbf{D\Theta}_i = \mathbf{Q} \tag{23}$$

Since the matrices $\mathbf{K},\mathbf{H}$ and $\mathbf{D}$ are symmetric, one can show that the eigenvectors satisfies the orthogonally condition [11]

$$\mathbf{\Theta}_i^T \mathbf{D\Theta}_j = \delta_{ij} \tag{24}$$

This condition can be used in equation (23) to yield a relation for the particular solution constants

$$B_i = \frac{k}{b_i}\mathbf{\Theta}_j^T \mathbf{Q} \tag{25}$$

The general solution now becomes

$$\mathbf{T}(t) = \sum_{i=1}^{N} B_i \mathbf{\Theta}_i + \sum_{i=1}^{N} A_i \mathbf{\Theta}_i \exp(-b_i t) \tag{26}$$

The initial boundary condition can be used to solve for the constants A_i

$$\mathbf{T}(0) = \sum_{i=1}^{N} (A_i + B_i)\mathbf{\Theta}_i \tag{27}$$

Applying the orthogonally condition (24) gives

$$A_i = -B_i + \mathbf{\Theta}_i^T \mathbf{DT}_0 \tag{28}$$

APPROXIMATION FOR SMALL BIOT NUMBER

In the steady state, the energy balance requires that the total heat input to Ω to be equal to the heat exchanged through the boundary Γ_h.

$$Q = \int_\Omega q(x,y,z) d\Omega = hT_s A_h \tag{30}$$

If the heat transfer coefficient h is small, the steady state temperature must then be large.

$$T_s = \frac{Q}{hA_h} \tag{31}$$

The Biot number can be defined as

$$Bi = \frac{ha}{K} \tag{32}$$

where a is a length representative of the domain Ω and can be defined as the volume of the body divided by the surface area of the body. Applying the divergence theorem to equation (10) shows that

$$\int_\Gamma \frac{\partial \Theta_i}{\partial n} d\Gamma = b_i \int_\Omega \Theta_i d\Omega \tag{33}$$

The zero heat transfer coefficient h corresponds to zero Biot number ($Bi = 0$) and hence $b_i = 0$ satisfies equation (33). In other word as $Bi \to 0$, the first eigenvalue b_1 approaches zero. For small Biot number the constants A_1, B_1 are anticipated to be a very large numbers of opposite sign as seen from equations (25) and (28). Subtracting two large numbers of approximately equal magnitudes will results in numerical inaccuracy. This difficulty can be overcome by separating the terms A_1, B_1 from the series

$$\mathbf{T}(t) = [B_1 + A_1 \exp(-b_i t)]\mathbf{\Theta}_1 + \sum_{i=2}^{N} [B_i + A_i \exp(-b_i t)]\mathbf{\Theta}_i \tag{34}$$

Eliminating A_1 in using equation (28) and defining the exponential function as a power series

$$\begin{aligned}\mathbf{T}(t) = {} & B_1\left(b_1 t - \frac{(b_1 t)^2}{2!} + \frac{(b_1 t)^3}{3!} - \dots\right)\mathbf{\Theta}_1 \\ & + \mathbf{\Theta}_1^T \mathbf{DT}^0 \exp(-b_1 t)\mathbf{\Theta}_1 \\ & + \sum_{i=2}^{N} [B_i + A_i \exp(-b_i t)]\mathbf{\Theta}_i\end{aligned} \tag{35}$$

If $Bi \to 0$, the leading eigenvalue $b_1 \to 0$. The product $B_1 b_1$ is however bounded (equation 25)

$$B_1 b_1 = k\mathbf{\Theta}_1^T \mathbf{Q} \tag{36}$$

Using this result in (35) gives an approximate solution for small Biot number,

$$\mathbf{T}(t) \approx [\mathbf{\Theta}_1^T (\mathbf{Q}t + \mathbf{DT}^0)]\mathbf{\Theta}_1 + \sum_{i=2}^{N} [B_i + A_i \exp(-b_i t)]\mathbf{\Theta}_i \tag{37}$$

TIME DEPENDENT HEAT CONDITION

In brake and clutch systems the sliding speed drops from some initial value to zero which results in time dependent heat condition

$$q(x,y,z,t) = fv(t)\,p(x,y,z) \tag{38}$$

The particular solution here is time dependent and equation (16) changes to

$$(\mathbf{K}+\mathbf{H})\mathbf{T}_p(t) - \frac{1}{k}\mathbf{D}\frac{\partial T_p}{\partial t} = \mathbf{Q}(t) \tag{39}$$

where $\mathbf{T}_p$ is the particular solution. As before, the particular solution can be expanded as eigenfunction series

$$\mathbf{T}_p = \sum_{i=1}^{N} B_i(t)\mathbf{\Theta}_i \tag{40}$$

Substituting (40) into (39), we obtain

$$\sum_{i=1}^{N} B_i(t)(\mathbf{K}+\mathbf{H})\mathbf{\Theta}_i - \frac{1}{k}\sum_{i=1}^{N}\frac{dB_i}{dt}\mathbf{D}\mathbf{\Theta}_i = \mathbf{Q}(t) \tag{41}$$

Multiplying (41) by $\mathbf{\Theta}_j^T$ and using the orthogonality condition (24) yields,

$$\frac{dB_j}{dt} - b_j B_j = -k\mathbf{\Theta}_j^T\mathbf{Q} \tag{42}$$

The solution for B_j can written as

$$B_j(t) = -k\exp(b_j t)\int \exp(-b_j t)\mathbf{\Theta}_j^T\mathbf{Q}(t)dt \tag{43}$$

Alternatively, a step-wise approximation can be regarded for time dependent boundary condition where the heat input is assumed to remain constant over small time steps. The method for constant boundary condition can then be used to obtain the temperature evolution. The temperature at the end of the preceding time step acts as the initial condition. Furthermore, new particular solution must be obtained for each time step.

ONE DISK MODEL

The method presented above was tested in the context of a single steel disk sliding against a non-conductive surface (Figure 1). The disk has an inner radius of 44.5 mm and an outer radius of 57.0 mm. The disk has a thermal conductivity of 54 $W/^oC$, a specific heat capacity of 532 $J/(kg.^oC)$ and density of 7800 kg/m^3. The disk rotates at speed ω of 250 rad/s and has a coefficient of friction f of 0.2. A uniform contact pressure p of 1 MPa is applied along the contact interface. The heat input is proportional to the radial distance r and can be defined as

$$q = f\omega rp \tag{44}$$

The disk was divided into 400 elements of 441 nodes. The temperature evolution was simulated for a single engagement of 0.5 s.

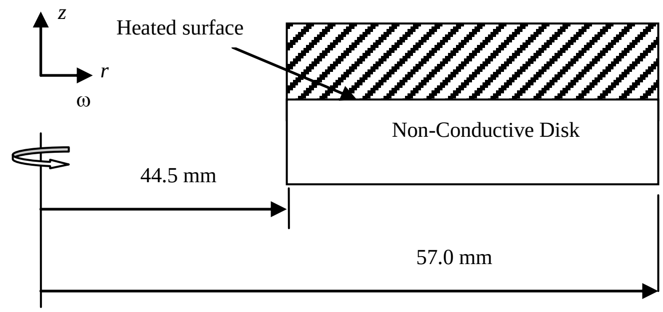

Figure 1: Sliding contact of two disks

The aim here is to investigate the effect of the Biot number (*Bi*) on the temperature evolution and to test the extension of the approximate solution presented in equation (37). The approximate solution is compared with that of equation (34). As explained before, the numerical inaccuracy resulting from the coefficients A_1 and B_1 being very large is eliminated from the solution of equation (34) and therefore this solution is considered to be exact. Figure 2 shows the temperature distribution along the heated surface for different values of *Bi*. The solutions of *Bi* = 0.01 and 0.1 are almost identical. A slight deviation is seen for *Bi* = 1.0. It is worth mentioning that for most brakes and clutches the Biot number is usually less than unity. The heat boundary q dependence on the radial distance (44) resulted in higher heat generation at outer radius of the disk. This explains why the temperature at the outer radius is higher. A significant drop in the temperature is seen at the inner and the outer radius for $Bi = 10.0$. This is mainly because of the cooling effect applied at the inner and outer surfaces. Figures 3, 4, and 5 show a comparison between the exact solution of (34) and the approximate solution of (37) for Biot number of 0.1, 1.0 and 10 respectively. For $Bi = 0.1$ and 1, the two solutions are almost exact. For $Bi = 10$, the approximate solution overestimate the maximum temperature by almost 10%.

A time depended boundary condition was tested in which the sliding speed and therefore the frictional heat input drops linearly from an initial value of 250 rad/s to zero in 0.5 s. A constant-wise approximation is used to represent the heat input time history and is divided into 50 equals time steps. Figures 6, 7, and 8 show the temperature time history at three different locations along the heated surface for Biot numbers of 0.01, 0.1, 1 respectively. The temperature reaches a maximum value and then starts to decrease near the end of the engagement. As seen in figure 8, the temperature time history is the same for the three Biot numbers. A slight discrepancy is seen at the inner and the out radius toward the end of the engagement driven by the cooling effect at these two locations.

What makes this method an attractive alternative to the time integration is the fact that a single time step is needed to obtain the transient solution when the heat condition is constant. The number of time steps for time dependent heat condition was investigated and shown in figure 9. Using 5 time steps is enough to capture the solution with an acceptable accuracy.

System equation (18) resulted into 441 eigenvalues and eigenvectors. All of the eigenvectors are used in the expansion series (34).

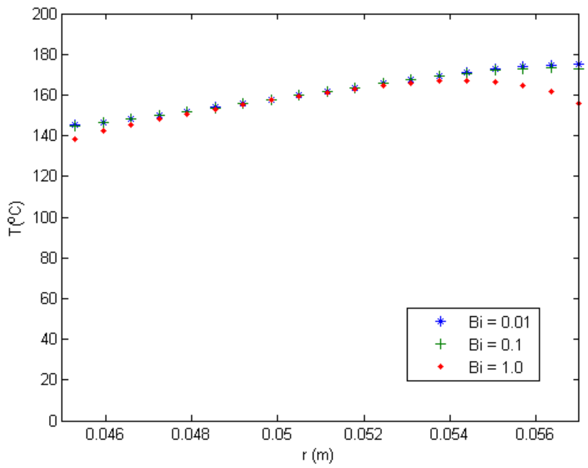

Figure 2: Temperature distribution along the heated surface at the end of the engagement for different Biot numbers.

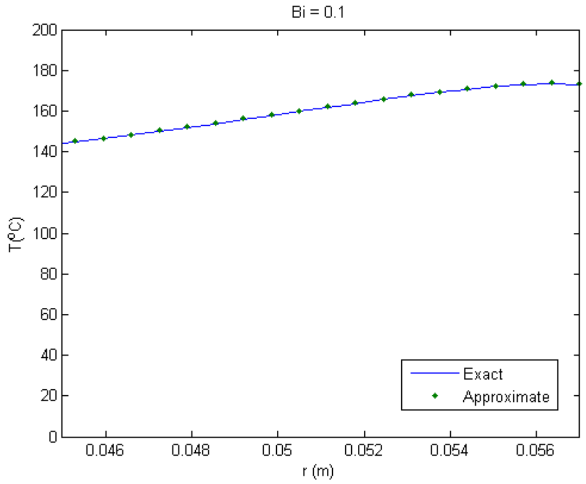

Figure 3: Temperature distribution along the heated surface for $Bi = 0.1$

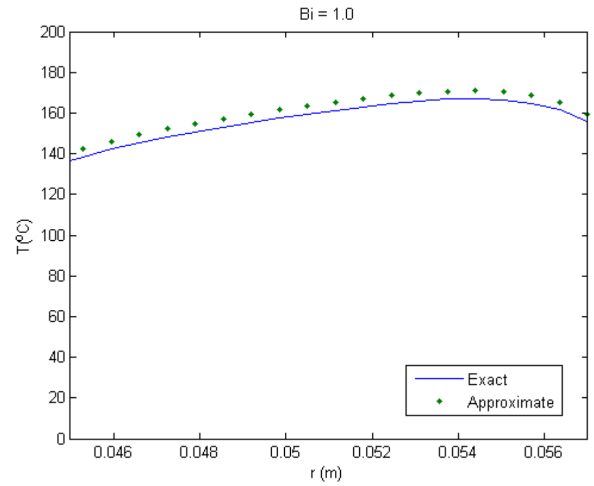

Figure 4: Temperature evolution at three different locations for $Bi = 1.0$

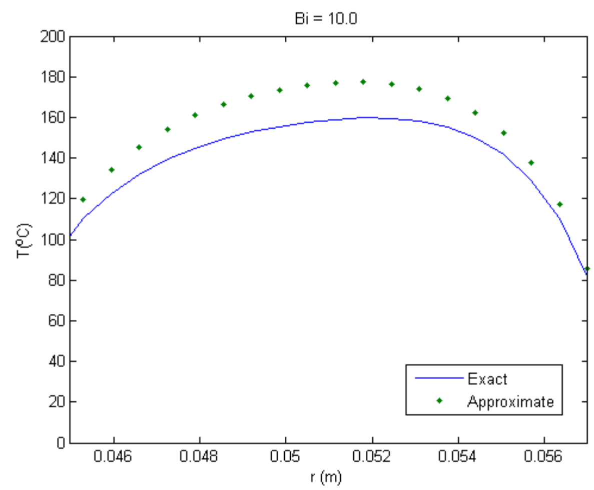

Figure 5: Temperature evolution at three different locations for $Bi = 10.0$

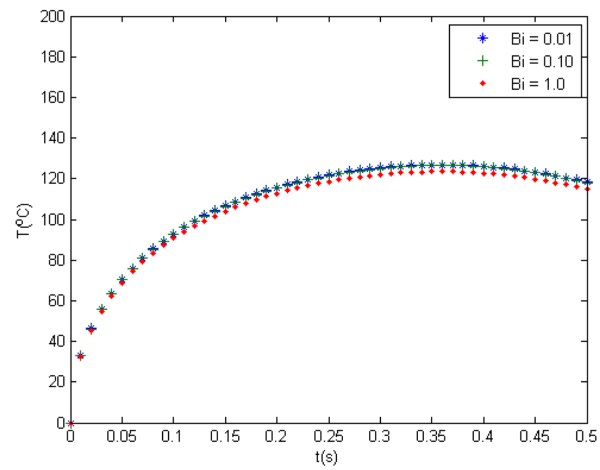

Figure 6: Temperature time history at the inner radius for different Biot numbers

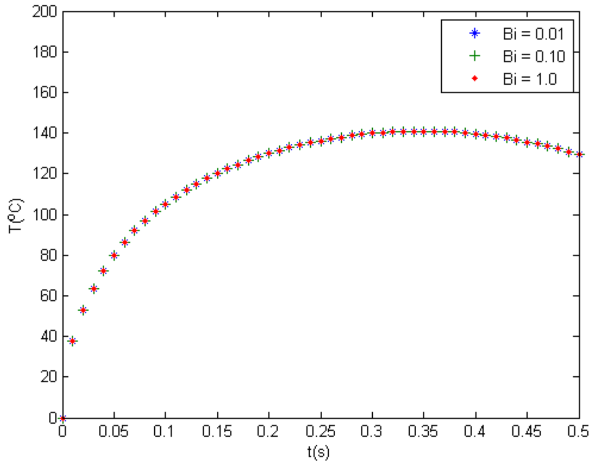

Figure 7: Temperature time history at the mean radius for different Biot numbers

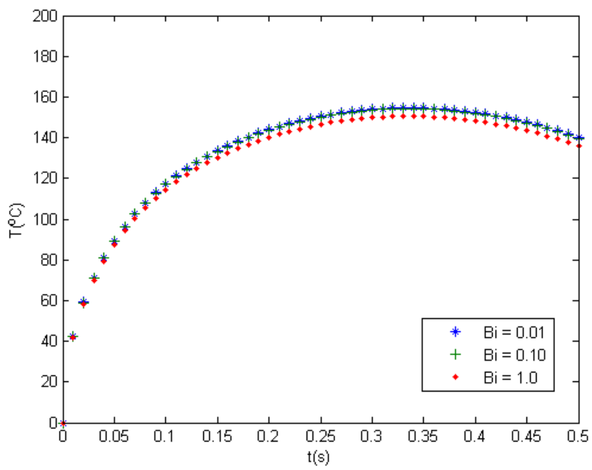

Figure 8: Temperature time history at the outer radius for different Biot numbers

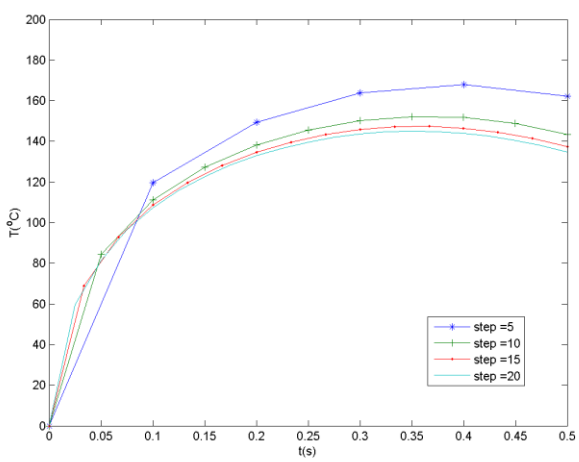

Figure 9: Temperature time history at the mean radius for different number of time steps

CONCLUSION

A methodology for solving the heat conduction problem with low Biot number was investigated. The method is based on a conventional way of solving the transient heat equation where the steady state solution is superimposed to the solution of the homogenous problem. Finite element realization of this method was presented. An approximate solution for heat problem with low Biot number was investigated and tested. The method was proven to work well even when the Biot number is close to unity. Furthermore, this method eliminate the problem of numerical inaccuracy that can result from the direct use of the steady state solution. A solution for a time dependent heat condition was also investigated along with the appropriate number of time steps that need to be used.

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