COMMON FIXED-POINT THEOREMS FOR NONLINEAR WEAKLY CONTRACTIVE MAPPINGS

S. Chandok,[1] M. S. Khan,[2] and M. Abbas[3]

Some common fixed-point results for mappings satisfying a nonlinear weak contraction condition within the framework of ordered metric spaces are obtained. The accumulated results generalize and extend several comparable results well-known from the literature.

Introduction and Preliminaries

The Banach contraction principle is one of the pivotal results in the metric fixed-point theory. It is a popular tool for the solution of existence problems in various fields of mathematics. There are several generalizations of the Banach contraction principle in the related literature on the metric fixed-point theory.

Ran and Reurings [15] extended the Banach contraction principle in partially ordered metric spaces with some applications to linear and nonlinear matrix equations. Nieto and López [14] extended the results of Ran and Reurings and used their main result to obtain a unique solution of the first-order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [3] introduced a concept of mixed monotone mappings and obtained some coupled fixed-point results. Moreover, they applied their results to a first-order differential equation with periodic boundary conditions.

Alber and Guerre-Delabriere [1] introduced a concept of weakly contractive mappings and proved the existence of fixed point for these mappings in Hilbert spaces. In 2001, Rhoades [17] proved the fixed-point theorem which is a generalization of the Banach contraction principle. Weakly contractive mappings are closely related to the mappings of the Boyd–Wong [4] and Reich types [16]. Recently, Doric [9] proved a common fixed-point theorem for generalized (ψ, ϕ)-weakly contractive mappings. Fixed-point problems involving weak contractions and mappings satisfying the inequalities of the weak contractive type were studied by numerous authors (see [1, 5–10, 17] and the references therein). In the present paper, we generalize the Chatterjea-type contraction mappings to (μ, ψ)-generalized Chatterjea-type contraction mappings and deduce some common fixed-point results for single-valued mappings on ordered metric spaces.

First, we recall some basic definitions and notation.

Let (X, d) be a metric space. A mapping $T\colon X \to X$ is said to be:

(a) of the *Kannan type* (see [11]) if there exists a $k \in \left(0, \frac{1}{2}\right]$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$;

(b) of the *Chatterjea type* [7] if there exists a $k \in \left(0, \frac{1}{2}\right]$ such that $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

Khan, et al. [12] initiated the use of a control function that alters the distance between two points in a metric space. Thus, this function was called an altering-distance function.

[1] Khalsa College of Engineering and Technology (Punjab Technical University), Muktsar, Punjab, India.
[2] College of Sciences, Sultan Qaboos University, Al-Khod, Sultanate of Oman.
[3] University of Pretoria, Pretoria, South Africa.

A function $\mu\colon [0,\infty) \to [0,\infty)$ is called an *altering-distance function* if the following properties are satisfied:

(i) μ is monotonically increasing and continuous;

(ii) $\mu(t) = 0$ if and only if $t = 0$.

By using the control function, we generalize the Chatterjea-type contraction mappings as follows:

Suppose that T and f are self-mappings defined on a metric space X. We say that a pair of mappings (T, f) satisfies the (μ,ψ)*-generalized Chatterjea-type contractive condition* if, for all $x, y \in X$,

$$\mu(d(Tx, fy)) \le \mu\left(\frac{1}{2}[d(x, fy) + d(y, Tx)]\right) - \psi(d(x, fy), d(y, Tx)), \tag{1}$$

where $\mu\colon [0,\infty) \to [0,\infty)$ is an altering-distance function and $\psi\colon [0,\infty)^2 \to [0,\infty)$ is a lower semicontinuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Assume that M is a nonempty subset of a metric space X and that a point $x \in M$ is a *common fixed (coincidence) point* of f and T for $x = fx = Tx$ $(fx = Tx)$. The set of fixed (resp., coincidence) points of f and T is denoted by $F(f, T)$ (resp., $C(f, T)$).

Definition 1. *Let $(X, \le)$ be a partially ordered set. Two mappings $f, g\colon X \to X$ are said to be weakly increasing if $fx \le gfx$ and $gx \le fgx$ for all $x \in X$.*

The following example shows that there exist discontinuous not nondecreasing mappings that are weakly increasing.

Example 1. Let $X = (0,\infty)$ be endowed with the ordinary ordering. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} 3x + 2 & \text{if } 0 < x < 1, \\ 2x + 1 & \text{if } 1 \le x < \infty \end{cases}$$

and

$$gx = \begin{cases} 4x + 1 & \text{if } 0 < x < 1, \\ 3x & \text{if } 1 \le x < \infty. \end{cases}$$

For $0 < x < 1$, we have

$$fx = 3x + 2 \le 3(3x + 2) = gfx \qquad \text{and} \qquad gx = 4x + 1 \le 4x + 3 = 2(2x + 1) + 1 = fgx,$$

whereas for $1 \le x < \infty$, we get

$$fx = 2x + 1 \le 3(2x + 1) = gfx \qquad \text{and} \qquad gx = 3x \le 2(3x) + 1 = fgx.$$

Thus, f and g are weakly increasing maps (but not nondecreasing).

Common Fixed-Point Theorem in Ordered Metric Spaces. Suppose that $(X, \preceq)$ is a partially ordered set. A mapping $T\colon X \to X$ is called *monotonically increasing* if, for all $x, y \in X$,

$$x \preceq y \quad \text{if and only if} \quad Tx \preceq Ty. \tag{2}$$

A subset W of a partially ordered set X is called *well-ordered* if every two elements of W are comparable.

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T and f are weakly increasing self-mappings on X satisfying inequality (1) for all comparable elements $x, y \in X$.*

In addition, suppose that either

(i) if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \preceq z$ for every $n \in \mathbb{N}$, or

(ii) T or f is continuous.

Then T and f have a common fixed point. Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point.

Proof. Let $x_0 \in X$. We can choose $x_1, x_2 \in X$ such that $x_1 = Tx_0$ and $x_2 = fx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. As T and f are weakly increasing mappings, we obtain

$$x_1 = Tx_0 \preceq fx_1 = x_2 \preceq Tx_2 = x_3.$$

By induction on n, we conclude that

$$x_1 \preceq x_2 \preceq \ldots \preceq x_{2n+1} \preceq x_{2n+2} \preceq \ldots.$$

Since x_{2n+1} and x_{2n+2} are comparable, by virtue of inequality (1), we get

$$\begin{aligned}
\mu(d(x_{2n+1}, x_{2n+2})) &= \mu(d(Tx_{2n}, fx_{2n+1})) \\
&\leq \mu\left(\frac{1}{2}[d(x_{2n}, fx_{2n+1}) + d(x_{2n+1}, Tx_{2n})]\right) - \psi(d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right) - \psi(d(x_{2n}, x_{2n+2}), 0) \\
&\leq \mu\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right).
\end{aligned}$$

Since μ is a monotone increasing function, for all $n = 1, 2, \ldots,$ we get

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big].$$

This implies that

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}).$$

By using the similar argument, we obtain $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$. Hence,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Thus, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$. Thus,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2} d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big].$$

Passing to the limit as $n \to \infty$, we get

$$r \leq \lim \frac{1}{2} d(x_{2n}, x_{2n+2}) \leq \frac{1}{2} r + \frac{1}{2} r.$$

Therefore, $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 2r$. In view of the continuity of μ and the lower semicontinuity of ψ, we find $\mu(r) \leq \mu(r) - \psi(2r, 0)$. This implies that $\psi(2r, 0) = 0$ and, hence, $r = 0$. Therefore, $d(x_{n+1}, x_n) \to 0$.

We now prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, $d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon$. This means that $d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon$. Hence, we get

$$\begin{aligned}
\epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\
&\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\
&< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).
\end{aligned}$$

Passing to the limit as $k \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{3}$$

Moreover,

$$\begin{aligned}
\epsilon \leq d(x_{2m(k)}, x_{2n(k)}) &\leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}) \\
&\leq 2d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)}, x_{2n(k)}).
\end{aligned}$$

As $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \epsilon. \tag{4}$$

On the other hand, we find

$$\begin{aligned}
d(x_{2m(k)}, x_{2n(k)}) &\leq d(x_{2m(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \\
&\leq d(x_{2m(k)}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).
\end{aligned}$$

In the limit as $k \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)+1}) = \epsilon.$$

In addition,

$$d(x_{2m(k)-1}, x_{2n(k)}) \leq d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)})$$

$$\leq d(x_{2m(k)-1}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).$$

In the limit as $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) = \epsilon.$$

Consider

$$\begin{aligned}
\mu(\epsilon) \leq \mu(d(x_{2m(k)}, x_{2n(k)})) &= \mu(d(Tx_{2m(k)-1}, fx_{2n(k)-1})) \\
&\leq \mu\left(\frac{1}{2}\left[d(x_{2m(k)-1}, fx_{2n(k)-1}) + d(x_{2n(k)-1}, Tx_{2m(k)-1})\right]\right) \\
&\quad - \psi(d(x_{2m(k)-1}, fx_{2n(k)-1}), d(x_{2n(k)-1}, Tx_{2m(k)-1})) \\
&= \mu\left(\frac{1}{2}[d(x_{2m(k)-1}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)})]\right) \\
&\quad - \psi(d(x_{2m(k)-1}, x_{2n(k)}), d(x_{2n(k)-1}, x_{2m(k)})).
\end{aligned}$$

Passing to the limit as $k \to \infty$ and using the continuity of μ and the lower semicontinuity of ψ, we get

$$\mu(\epsilon) \leq \mu\left(\frac{1}{2}[\epsilon + \epsilon]\right) - \psi(\epsilon, \epsilon)$$

and, consequently, $\psi(\epsilon, \epsilon) \leq 0$, which is a contradiction because $\epsilon > 0$. Thus, $\{x_{2n}\}$ is a Cauchy sequence and, hence, $\{x_n\}$ is a Cauchy sequence. As X is a complete metric space, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$. Since $\{x_n\}$ is a nondecreasing sequence, by (i), we have $x_n \preceq t$. Consider

$$\begin{aligned}
\mu(d(x_{2n+1}, ft)) &= \mu(d(Tx_{2n}, ft)) \\
&\leq \mu\left(\frac{1}{2}[d(x_{2n}, ft) + d(t, Tx_{2n})]\right) - \psi(d(x_{2n}, ft), d(t, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}[d(x_{2n}, ft) + d(t, x_{2n+1})]\right) - \psi(d(x_{2n}, ft), d(t, x_{2n+1})).
\end{aligned}$$

In the limit as $n \to \infty$, we obtain

$$\mu(d(t, ft)) \leq \mu\left(\frac{1}{2}d(t, ft)\right) - \psi(d(t, ft), 0)) \leq \mu\left(\frac{1}{2}d(t, ft)\right).$$

This implies that $d(t, ft) = 0$ and, hence, $t = ft$.

Again, consider

$$\mu(d(Tt,t)) = \mu(d(Tt,ft)) \le \mu\left(\frac{1}{2}[d(t,ft)+d(t,Tt)]\right) - \psi(d(t,ft),d(t,Tt))$$

$$= \mu\left(\frac{1}{2}d(t,Tt)\right) - \psi(0,d(t,Tt)) \le \mu\left(\frac{1}{2}d(t,Tt)\right).$$

This implies that $d(Tt,t)=0,\ Tt=t$. Therefore, $t=Tt=ft$, i.e., t is a common fixed point of T and f.

If condition (ii) holds: Assume that T is continuous. Then $t=\lim_{n\to\infty} Tx_n = x_{2n+1} = Tt$. Now

$$\mu(d(t,ft)) = \mu(d(Tt,ft)) \le \mu\left(\frac{1}{2}\left[d(t,ft)+d(t,Tt)\right]\right) - \psi(d(t,ft),d(t,Tt))$$

$$= \mu\left(\frac{1}{2}d(t,ft)\right) - \psi(d(t,ft),0) \le \mu\left(\frac{1}{2}d(t,ft)\right)$$

implies that $d(t,ft)=0,\ ft=t$. Therefore, $t=Tt=ft$, i.e., t is a common fixed point of T and f.

If f is continuous, then following the argument similar to the argument presented above, we get the required result.

We now suppose that the set of common fixed points of T and f is well ordered. We now claim the uniqueness of the common fixed points of T and f. Assume, on the contrary, that $Tu=fu=u$ and $Tv=fv=v$ but $u\ne v$. Consider

$$\begin{aligned}\mu(d(u,v)) &= \mu(d(Tu,fv))\\ &\le \mu\left(\frac{1}{2}[d(u,fv)+d(v,Tu)]\right) - \psi(d(u,fv),d(v,Tu))\\ &= \mu\left(\frac{1}{2}[d(u,v)+d(v,u)]\right) - \psi(d(u,v),d(v,u))\\ &= \mu(d(u,v)) - \psi(d(u,v),d(u,v)).\end{aligned}$$

This implies that $d(u,v)=0$, by the property of ψ. Hence, $u=v$. Conversely, if T and f have only one common fixed point, then the set of common fixed points of f and T (being a singleton) is well ordered.

Theorem 1 is proved.

If $T=f$, then we have the following result:

Corollary 1. *Let $(X,\preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T is a monotone nondecreasing self-mapping on X such that*

$$\mu(d(Tx,Ty)) \le \mu\left(\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right) - \psi(d(x,Ty),d(y,Tx)),$$

is satisfied for all $x,y\in X$ with comparable x and y.

In addition, suppose that either

(i) if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \preceq z$ for every $n \in \mathbb{N}$ or

(ii) T is continuous.

Then T has a fixed point.

If $\mu(t) = t$, then we get the following result:

Corollary 2 (see [5, 10])**.** *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose that T is a monotonically nondecreasing self-mapping on X such that*

$$\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{2}[d(x,Ty) + d(y,Tx)]\right) - \psi(d(x,Ty), d(y,Tx)),$$

is satisfied for all comparable elements $x, y \in X$.

In addition, suppose that either

(i) if $\{x_n\} \subset X$ is a nondecreasing sequence such that $x_n \to z$ in X, then $x_n \preceq z$ for every $n \in \mathbb{N}$ or

(ii) T is continuous.

Then T has a fixed point.

Example 2. Let $M = [0,1]$ be endowed with a partial ordering: $x \preceq y$ if and only if $x \geq y$. Let d be defined as $d(x,y) = |x-y|$. We set $Tx = 0$ and $fx = \dfrac{x^2}{8}$ for all $x \in M$. It is easy to see that f and g are weakly increasing maps. We define $\mu\colon [0,\infty) \to [0,\infty)$ and $\psi\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ by

$$\mu(t) = \frac{t}{2} \qquad \text{and} \qquad \psi(t,s) = \frac{t+s}{16}.$$

Thus, for $x, y \in M$, we get

$$\mu(d(Tx,fy)) = \mu\left(d\left(0, \frac{y^2}{8}\right)\right) = \mu\left(\frac{y^2}{8}\right) = \frac{y^2}{16},$$

and

$$\begin{aligned}
&\mu\left(\frac{1}{2}[d(x,fy) + d(y,Tx)]\right) - \psi(d(x,fy), d(y,Tx)) \\
&= \mu\left(\frac{1}{2}\left[d\left(x, \frac{y^2}{8}\right) + d(y,0)\right]\right) - \psi\left(d\left(x, \frac{y^2}{8}\right), d(y,0)\right) \\
&= \mu\left(\frac{1}{2}\left[\left|x - \frac{y^2}{8}\right| + y\right]\right) - \psi\left(\left|x - \frac{y^2}{8}\right|, y\right) \\
&= \frac{1}{4}\left[\left|x - \frac{y^2}{8}\right| + y\right] - \frac{\left|x - \frac{y^2}{8}\right| + y}{16} \\
&= \frac{3}{16}\left[\left|x - \frac{y^2}{8}\right| + y\right] \geq \frac{3y}{16} \geq \frac{y^2}{16}.
\end{aligned}$$

Hence,

$$\mu(d(Tx, fy)) \leq \mu\left(\frac{1}{2}\left[d(x, fy) + d(y, Tx)\right]\right) - \psi(d(x, fy), d(y, Tx)).$$

Thus, all conditions of Theorem 1 are satisfied. Moreover, T and f have a unique common fixed point 0.

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