An algebraic-analytic framework for the study of intertwined families of evolution operators

by

Wha-Suck Lee

Submitted in partial fulfillment of the requirements for the degree

Philosophiae Doctor

in the Faculty of Natural and Agricultural Sciences
Department of Mathematics and Applied Mathematics

University of Pretoria
Pretoria
January 2015

People To Thank: Part I (Part II on p120)

Oh Most High God, there is none like You. I bow my knee to You and stand in awe at your Goodness and Mercy. Thank you for the name of Jesus and the fire of the Holy Ghost.

...... To the God of Abraham, Isaac and Jacob

Your Blood has set me free

...... To Jesus, the Christ

One went and one rose

...... To My Father

Thank you for loving me

...... To My Mother

Thank you for giving a mute a voice

...... To Professor Sauer

The mustard seed you planted grew into a large tree that gave shelter to this thesis

...... To Professor Diestel

Thank you for teaching me the theory of Banach algebra.

...... To Professor Stroh

Thank you for your efforts in the last stretch.

...... To Professor Anguelov

The PhD seed you planted has finally blossomed. Jacque is another seed that will blossom and speak of the love of Jesus that lives in the marriage union between you and Hannlie.

...... To Professor Ian Le Roux & Hannlie

Oh Captain, My Captain.

...... To Wha-Joon

Goal in! Thats it!

...... To Wha-Choul

Look at what God did through me!

...... To Wha-Yong

We have never met in person, but I believe you have personally trained me in how to walk in the power of great Holy Ghost just as Kenneth E Hagin trained you personally: 1501 - 04. Your free sermon mp3s have imprinted on my spirit and it led to the breakthroughs of this PhD.

...... To Keith Moore

Thank you for the link to the Keith Moore Ministry: it changed my life. You are a good gentle kind Word spreader!

...... To Graham Litchfield

In your friendship I regained my faith in people. I do not think there will be a brighter star out of House 27 for decades to come! Unusual selflessness and compassion flowed in your life, joining into the life of your wife Teresa.

...... To Stable S & Teresa (Stephan Van Staden)

Tea and the English culture were milestones that marked this PhD journey. Integrity was found in abundance in your friendship.

...... To Antoine Smith Dryden Dymond

Acknowledgements

The word *resurrection* describes this PhD in one word. I wrote this PhD during a dark time in my life. The devil attacked me with Ankylosing Spondilitis and blindness, and my father, a father of fathers, with rectal cancer. The great Jesus of Nazareth stepped in when He cured me of both conditions and He took my father into his mansion in heaven. His strength took over when my strength failed. His vision and strength pushed this PhD to its start and completion; He is the alpha and omega of this PhD, the true author of the PhD. The devil lost and the great Jesus won. Life has come.

I acknowledge Jesus for the new paradigm of Section 1.2 and Chapter 2, and the Hille Yosida Kisynski theorem of Theorem 12, Section 4.3. The goodness and mercy of the great Jesus is more astonishing than His breath-taking intelligence and creativity. Every believer must take heart in the goodness of God and His power to deliver them from all forms of bondage. Keep standing and watch the fire of the Holy Ghost make your dreams come true.

To my *Daddy*: This PhD is for you! I hope this work will be a fitting tribute to you.

To *Professor Sauer*: I sat under a prince of mathematics. Your mathematics lives in me now. You defined my mathematical education at the University of Pretoria. An education I take more highly than an education at the University of Princeton. Thanks for arranging National Research Foundation South Africa funding.

To *Jesus*: I bow my knees at your kindness and for taking me out of the dung heap and seating me amongst princes. Mighty Holy Ghost! I end this acknowledgement fittingly with one of the most beautiful verses in the Bible. It captures the beauty of the God of Abraham, Isaac and Jacob, who restores and renews.

Isaiah 61: *3 To appoint unto them that mourn in Zion, to give unto them* **beauty for ashes,** *the oil of joy for mourning, the garment of praise for the spirit of heaviness; that they might be called trees of righteousness, the planting of the LORD, that he might be glorified. 7 For your shame* **ye shall have double;** *and for confusion they shall rejoice in their portion: therefore in their land they shall possess the double: everlasting joy shall be unto them.*

To Jesus, The Christ, the same yesterday, and today, and forever. (Hebrews 13; 8)

An algebraic-analytic framework for the study of intertwined families of evolution operators.
by
Wha-Suck Lee

Supervisor : Prof. Niko Sauer.

Department : Mathematics and Applied Mathematics.

Degree: PhD.

Abstract

Empathy theory replaced semi-group theory for the study of implicit evolution equations $\frac{d}{d\tau}[Bu(\tau)] = Au(\tau)$; A, B not necessarily closed: the evolution process is described with a pair of evolution operators $\langle E(\tau), S(\tau)\rangle_{\tau>0}$ 'intertwined' by the *empathy relation* $S(\tau+\sigma) = S(\tau)E(\sigma)$. The evolution operators associated with the transition probabilities of Markov processes, integrated empathies and semigroups are further cases of intertwined evolution operators.

The framework of vector-valued Schwartz distributions had limited success with certain evolution equations: their convolution theorem rests on a bounded bilinear form; thus it applies to uniformly measurable evolution families. In contrast, the empathy approach to the implicit Cauchy problem leans heavily on the Laplace transform and a convolution theorem which is applicable to strongly measurable evolution families.

This dissertation introduces a new framework which efficiently implements Sauer's approach of giving convolution a central role in the study of intertwined families. A convolution algebra $(\mathscr{A}, *)$ of homomorphisms between a test space of Banach space-valued group-domained functions and the Banach space itself lies at the heart of this new framework: $(\mathscr{A}, *)$ results from the marriage of ideas of vector-valued distributions and abstract harmonic analysis; the product $*$ generalizes the abstract convolution of the well known convolution algebra of abstract harmonic analysis.

The framework replaces families of evolution operators with families of homomorphisms, and their composition with $*$; homomorphisms generalize operators and the product $*$ is more fundamental than operator composition. We call the homomorphisms, *generalized operators*. A new fully developed convolution theorem for families of homomorphisms subsumes the earlier approaches. The framework unifies all the resolvent equations spanning C_0-semigroup, empathy theory and n-times integrated semi-groups.

We next construct a Hille-Yosida generation theorem for the implicit evolution equation set in the more general setting of generalized operators. This

new generation theorem bypasses the assumption of a uniformly bounded empathy by pairing Banach algebra techniques, like the factorization theorem, into operator theoretic problems. Kisynski's equivalent version of the Hille-Yosida Theorem for the abstract Cauchy problem was the forerunner to this approach. Indeed, another convolution algebra $(L^1(0,\infty),*)$ lies at the heart of this approach by Kisynski.

Feller semigroups and processes fit perfectly into the framework of generalized operators. Feller semigroups are intertwined by the Chapman Kolmogorov equation. Our framework handles more complex intertwinements which naturally arise from a dynamic boundary approach to an absorbing barrier of a fly trap model: we construct an entwined pseudo Poisson process which is a pair of stochastic processes entwined by the extended Chapman Kolmogorov equation. Similarly, we introduce the idea of an entwined Brownian motion. We show that the diffusion equation of an entwined Brownian motion involves an implicit evolution equation on a suitable scalar test space. We end off by constructing a new convolution of operator valued measures which generalizes the convolution of Feller convolution semigroups.

Contents

Chapter 1

Introduction

This introduction gives the intuition behind a new algebraic analytic framework for the study of intertwined families of evolution operators such as in empathy theory. Poincaré wrote:

'It is by logic we prove, it is by intuition that we invent. Logic, therefore, remains barren unless fertilised by intuition.'

Empathy theory is involved with a peculiar [1] type of implicit evolution equation which arises naturally in many known situations. Thus, the analysis of empathy theory is very different from that of conventional semigroup theory. These differences motivate the need for a new algebraic-analytical framework to replace that of [5] and [28] used in conventional semigroup theory. This new framework efficiently solves problems in empathy theory.

1.1 History Of Empathy Theory

The study of evolution equations of the form

$$\frac{d}{d\tau}[Bu(\tau)] = Au(\tau), \tag{1.1.1}$$

so-called *implicit evolution equations*, motivated by partial differential equations in continuum mechanics, has been on-going for some time. In (1.1.1) the symbols A and B denote unbounded linear operators with a common domain $\mathscr{D}$ in a Banach space X and range in another Banach space Y. Among the first papers are [30] (in a Hilbert space setting) and [15]. In these papers the operators A and B are closed or closable and the initial condition

[1] For the reader unfamiliar with the terminology and physical intuition behind empathy theory, we refer the reader to Appendix A.3

specifies the initial state $u(0)$ or $Bu(0)$. A detailed account can be found in [16].

There are important situations, pertaining to dynamic boundary conditions, in which the operators A and B are not closable. This was pointed out in the early study [26] of the equation (1.1.1) under the initial condition

$$\lim_{\tau \to 0^+} [Bu(\tau)] = y \in Y. \tag{1.1.2}$$

This study also revealed that closedness [2] of the operators A and B was not crucial. It was replaced by the notion of *joint closedness* which states that the operator pair $\langle A, B \rangle : \mathscr{D} \to Y \times Y$ is closed. Somewhat later it was realized that there were two families $\langle S(\tau), E(\tau) \rangle$ of bounded evolution operators involved [27], *intertwined* by the *empathy relation*

$$S(\tau + \sigma) = S(\tau)E(\sigma), \tag{1.1.3}$$

where $S(\tau) : Y \to X$ and $E(\tau) : Y \to Y$; $\sigma, \tau > 0$. We call the pair $\langle S(\tau), E(\tau) \rangle$ an empathy. It was assumed that $E(\tau)$ was a semigroup:

$$E(\tau + \sigma) = E(\tau)E(\sigma). \tag{1.1.4}$$

This idea was further extended in [28] by neither assuming that the family $E(\tau)$ was a semigroup, nor that an operator such as B existed. Instead, the analysis was very much based on the Laplace transform for vector-valued functions as it was done, in part, in [26, 27] and virtually at the same time for semigroups, [1], where the notion of *integrated semigroup* was introduced. It should also be mentioned that earlier this notion was implicitly used in [15].

The points of departure in [28] were two-fold: In the first place, since the Cauchy-problem (1.1.1), (1.1.2) makes perfect sense for $\tau > 0$, the relation (1.1.3) only needs to hold for $\sigma, \tau > 0$. Secondly, it was assumed that the Bochner integrals

$$\left.\begin{aligned} R(\lambda)y &= \int_{(0,\infty)} e^{-\lambda\tau} E(\tau)y\, d\tau; \\ P(\lambda)y &= \int_{(0,\infty)} e^{-\lambda\tau} S(\tau)y\, d\tau, \end{aligned}\right\} \tag{1.1.5}$$

existed in Y and X respectively, for every $y \in Y$ and $\lambda > 0$. In this setting the *empathy pseudo resolvent equations*

$$\left.\begin{aligned} R(\lambda) - R(\mu) &= (\mu - \lambda)R(\lambda)R(\mu); \\ P(\lambda) - P(\mu) &= (\mu - \lambda)P(\lambda)R(\mu); \end{aligned}\right\} \tag{1.1.6}$$

[2] The initial condition u_0 and $Bu(0)$ requires $u(0) \in \mathscr{D}$. This initial condition is not feasable in many physical systems. The initial condition (1.1.2) is more feasable in applications. If B is closed and u_0 exists, then the initial condition (1.1.2) also implies that $u(0) \in \mathscr{D}$.

were obtained from (1.1.4) -(1.1.3). We call this pair $\langle R, P\rangle$ an *empathy pseudo-resolvent*. Under the *invertibility assumption* (Section 4.2), the linear operators A and B defined on a domain $\mathscr{D} \subset X$ could be constructed such that $P(\lambda) = (\lambda B - A)^{-1}$. It is shown in [28] that for $y \in B[\mathscr{D}]$ the function $u(\tau) = S(\tau)y$ is the solution of the Cauchy problem (1.1.1) - (1.1.2). The constructed operator pair $\langle A, B\rangle$ is called the *generator* of the empathy $\langle S(\tau), E(\tau)\rangle$. Following [15], the operators $P(\lambda) = (\lambda B - A)^{-1}$ are called *generalized resolvents*.

A characterization of the generator of uniformly bounded empathies in terms of properties of the generalized resolvent was given under the restrictive assumption that the space Y had the Radon-Nikodým property. This restriction was first by-passed in [5], by introducing the notion of 'integrated empathy' similar to the notion of integrated semigroup introduced in [1] (see also [2]).

Integrated semigroups and empathies are instances of evolution operators that are highly intertwined. Convolution semigroups used by William Feller [17, p. 284] for the study of Markov processes are derived from the convolution of probability measures. The evolution operators associated with the transition probabilities of Markov processes in [17, p. 284] are special cases of intertwined evolution operators.

The development of a dynamic systems approach to the problems described above was based on rather pedestrian and tedious approaches as in [28] and [5]. In [28] and [5], the algebraic-analytic setting was time-domained Z-valued functions which were Bochner integrable over open intervals. We shall denote this setting as $\mathcal{L}^1_{loc}((0,\infty), Z); Z$ is a Banach space. In this vectorization of $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$, where a Banach space Z replaces $\mathbb{R}$, the Laplace Transform and a general form of the convolution theorem reported in [5] was central.

Also important to note is that certain evolution equations have been treated within the framework of vector-valued Schwartz distributions with limited success (see *e.g.* [14]). We highlight one of the reasons for the limited success in Section 2.9, remark 11.

1.2 The New Framework

This dissertation develops an algebraic-analytic setting in which the earlier approaches may by included. For this we adapt some of the ideas of vector-valued distributions and abstract harmonic analysis to define a product of homomorphisms between a 'test space' of Banach space-valued functions and the Banach space itself. The homomorphisms are, of course, linear operators. We shall, however, persist in calling them homomorphisms to emphasize their particular role.

As in the Laurent Schwartz theory of vector-valued distributions (see *e.g.* [14]) , we introduce in Section 2.1, the notion of *test spaces*. They consist of Banach space -valued functions defined on an Abelian group G. In contrast to the theory of vector-valued distributions where differentiability is prominent, the requirement rather is invariance under translation. Similar to distribution theory the homomorphisms from the space of 'test functions' to the underlying Banach space are important. The notion of *admissible homomorphism*, when the homomorphism of a translated test function is once again a test function, is central. We introduce a product of admissible homomorphisms, borrowed from abstract harmonic analysis (*e.g.* [18, Chap. V, §19], [23, 1.9.7]) that satisfies a *power law*, crucial in the work. Under this product the homomorphisms form an associative algebra with a unit. Bounded linear operators in the Banach space give rise to admissible homomorphisms. Conversely, for suitable test spaces, bounded homomorphisms are admissible and they correspond to bounded linear operators in the test space, called dualisms. The product of such homomorphisms corresponds to composition of dualisms so that traditional operations can be carried out in the test space. It should be noted that in abstract harmonic analysis, the homomorphisms in question are linear functionals and the product is convolution.

In Section 2.2, we introduce analytic structures to the theory by considering continuous test functions when the group G is topological. The pervading emphasis is on a Fréchet test space of vector valued continuous test functions. This analytic structure ensures the study of time-dependent homomorphisms, as a replacement for families of evolution operators, from the point of view of their integrals (Section 2.3.1), their Laplace transforms (Section 2.4.1), their convolutions (Section 2.5) and the convolution theorem (Section 2.6). Indeed, we construct a convolution theorem with fully developed Laplace transform theorems (Section 2.6.3) under a crucial closedness assumption (Section 2.4.1). This fuller resemblance to the classical case subsumes the brute force approach based on the convolution theorem of [5].

Causal relations such as the semigroup, empathy (Section 2.7) and integrated empathy (Section 2.8.2) are considered with the standard composition of operators replaced by the product of homomorphisms introduced earlier. The corresponding resolvent equations under the 'new' product are derived *efficiently* with a *unified approach* as the convolution of the Laplace transforms of two unlike parameters (Theorem 12, Section 2.7). In Section 2.8, the resolvent equations of the classical evolution operators, as discussed above, are placed in the present framework. Thus, we efficiently implement Sauer's approach of giving the convolution a central role [5].

We introduce a new analytic structure by merely changing the test space.

Another pervading emphasis is on a Banach test space of bounded uniformly continuous test functions (Section 3.1). In Section 5.2, Feller's observation that probability measures give rise to bounded linear operators on spaces of continuous functions, is brought into the framework of the constructed algebra of admissible homomorphisms. It turns out that the product introduced here corresponds to the convolution of probability measures.

With this new analytic structure, we re-study families of evolution operators in Section 3.2 by re-considering time-dependent *normed* homomorphisms, their integrals and their Laplace transforms. Since the integrals involved are Bochner integrals over (mostly) open time intervals, the relation to the integrals and the behaviour of the integrands at time zero is investigated in Section 3.2.2. To some extent the results here correspond to results in [19]. Resolvent equations under the 'new' product are derived and some properties of so-called star-semigroups, analogous to more properties of traditional semigroups are shown to hold. In Section 3.2.3, the similarity goes deeper: the canonical family associated with the double family $\langle \mathcal{S}, \mathcal{E} \rangle$ of empathy theory [28] is an *isometric* representation; the corresponding dualisms are the known semigroup and empathy relations as far as strongly continuous semi-groups are concerned. Indeed, in Section 3.5, we show that the classical result that every measurable semigroup is strongly continuous is also true for normed $*$-semigroups.

The before-mentioned derived resolvent equations in the setting of our convolution algebra of admissible homomorphisms or generalized operators, form the basis for the definition and characterization of the 'generators' of causal relations under the product of homomorphisms. In Section 4.2.1, we perform a preliminary analysis of 'star-empathies' along lines similar to [28] by constructing integral representations similar to those for classical semigroups (Lemma 2.7, [28]); just as in the case of the empathy(Lemma 2.7, [28]), these integral representations show what the pair of generators of the implicit Cauchy problem should look like. Here the roles of the test space and the dualisms in an algebraic sense become clear. Then, in Section 4.3, we apply the Kisynski Generation Theorem for C_0-semigroups to a generation theorem for the implicit Cauchy problem in this new algebraic-analytic setting of admissible homomorphisms: the empathy theory approach to the implicit Cauchy problem has solutions on a non-closed dense subspace of the regularity space generated for C_0-semigroups which can be identified with T; this dense subspace, Δ^2_K, can be identified with T^2 . It then follows immediately that the implicit evolution equation for Banach spaces (1.1.1) has solutions also on Δ^2_K. Indeed, we give an exact way to measure how far empathy theory differs from semi-group theory (Corollary 5, Section 4.3.2) and Remark 12).

The notion of a convolution semigroup is natural in our framework of admissible homomorphisms: in Section 5.2, we identify random variables as admissible linear functionals on a suitable scalar test space, and the Feller convolution with the product of admissible homomorphisms. In Section 5.3 we approach an absorbing barrier of a Markov process with the philosophy of dynamic boundary condition where the boundary is a body in its own right. The absorbing boundary is seen as a distinct collection of states with zero intensity: this is a more realistic way to model fly trap models. Therefore, this approach gives rise to two distinct state spaces and stochastic processes which fits in perfectly with the two state theory of empathy. This pair of stochastic processes gives rise to a pair of transition probability density functions which satisfy the *extended Chapman Kolmogorov equation* (5.3.3). We call this process an *entwined pseudo Poisson process.* Indeed, in Section 5.4.1, we introduce the idea of an entwined Brownian motion in the form of a pair of random variables entwined by the extended Chapman Kolmogorov equation. We show that the diffusion equation of such a pair of entwined random variables involves an implicit evolution equation on a suitable scalar test space (Theorem 3).

For a certain test space, we vectorize the Feller convolution semigroups into a semigroup of operator valued measures in Section 5.5: random variables are a special case of *dominated* admissible homomorphisms on a vectorized test space; such homomorphism are uniquely identified with an operator valued measure; in particular, we construct a new convolution of operator valued measures so that the measure associated with the product of such homomorphisms is the convolution of their respective operator valued measures (Section 5.5.3). An immediate consequence is a vector valued version of the extended Chapman-Kolmogorov relation (Section 5.6).

Chapter 2

The New Framework: Fréchet Test Space

The convolution algebra of abstract harmonic analysis (Appendix B.2) consists of the class of admissible linear functionals, $x' : \Phi \to \mathbb{C}$; Φ is a *translation invariant* function space of group G-domained scalar functions. We vectorize the convolution algebra of abstract harmonic analysis in two steps.

Firstly, we vectorize the function space Φ by choosing a suitable translation-invariant subspace of group G-domained Z-valued functions; Z is a Banach space (Section 2.1). We call this vectorized Φ a *test space*. Secondly, we vectorize x' by replacing $\mathbb{C}$ with Z itself (Section 2.1.1). The group G allows certain homomorphisms, $x' : \Phi \to Z$ to play a second role as endomorphisms $X' : \Phi \to \Phi$, just as in abstract harmonic analysis. We call such homomorphisms *admissible homomorphisms*; the endomorphisms shall be called *dualisms*; $\mathscr{A}$ is the class of all admissible homomorphisms. We then construct an algebra product $*$ on $\mathscr{A}$ which generalizes the abstract convolution of the convolution algebra of abstract harmonic analysis. Therefore, we call $*$ a *convolution*.

Operators on Banach spaces are a special case of admissible homomorphisms for a suitable test space Φ. In this chapter, we set Φ to be $C(G, Z)$, the space of continuous functions with the topology of uniform convergence of compacts; the topological group G is locally compact and second countable. Φ is then Frechét (Section 2.2). The analytic structure induced by the Frechet test space Φ allows us to consider time-dependent *bounded* homomorphisms (Section 2.3), their integrals (Section 2.3.1) and their Laplace transforms (Section 2.4.1). The integrals involved are Bochner integrals over (mostly) open time intervals. Indeed, the theory of time-dependent *bounded* homomorphisms have a general version of the integrability theorem of Hille ([19], Theorem 3.8.2, p.85) (Theorem 6, Section 2.3.1).

Thus we study families of bounded homomorphisms, as a replacement for families of bounded evolution operators. Specifically, we consider time-dependent bounded homomorphisms integrable over every finite open interval. We denote this class by $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$; $\mathscr{A}_B$ is the algebra of bounded homomorphisms.

The framework $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$ will be the new algebraic-analytic setting for the study of intertwined families of evolution operators (Section 2.3.2). Tradition vectorized $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$ by replacing the algebra $\mathbb{R}$ with a Banach space Z ([1],[2]) to arrive at $\mathcal{L}^1_{loc}((0,\infty),Z)$, the traditional algebraic-analytic setting for empathy theory. Our vectorization of $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$ replaces the algebra $\mathbb{R}$ naturally with another (convolution) algebra $\mathscr{A}_B$, to arrive at the *new* algebraic-analytic setting $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$. Therefore, the vectorization relies on the algebra product $*$ of the convolution algebra to transfer the scalar convolution theorem as opposed to a *bounded* bilinear form as in well known vector valued convolution theorems (Appendix 2.9, Proposition 8). Bounded bilinear forms require the family of evolution operators to be uniformly measurable (Section 2.9, Remark 11). This assumption is too strong for empathy theory but not for vector valued distributions for which these vector valued convolution theorems were made for. We show that the convolution theorem $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$ like its forerunner, the convolution theorem of [5], requires no additional assumptions by virtue of defining the Laplace Transform and convolution pointwise. Indeed, the versatility of the framework rests on the freedom to change the test space Φ according to the applications.

Under the crucial closedness assumption (Section 2.4.1), we transfer the full computational power of the classical convolution theorem and the Laplace Transform Theorems (Section 2.6.2 and 2.6.3): the convolution theorem of $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$ gives rise to the Laplace transform theorems which are identical in nature to the classical ones (Section 2.6.3). We use this full computational power to *efficiently* show that the causal relations such as semigroups, empathy (Section 2.7) and integrated empathy (Section 2.8.2), phrased in terms of the introduced product, yield resolvent-like equations analogous to the ones that are crucial to the classical studies; the diverse resolvent equations under the 'new' product or convolution are all uniformly computed as the convolution of the Laplace transforms of two unlike parameters (Theorem 12, Section 2.7) of their respective causal relations. The resolvent equations derived here will form the basis for the definition and characterization of the 'generators' of causal relations (Section 4.3) under the product of homomorphisms. Thus, the convolution operation plays a central role in the analysis of intertwined families of evolution operators, by virtue of a fully developed convolution theorem and Laplace transform theorems.

2.1 Vectorized Test Space

In our considerations, the translation operators play a critical role with an Abelian group playing the supporting role. If f is a function defined on an Abelian group $\langle G, +\rangle$, then the Abelian [1] group G gives rise to translation operators R^s defined by

$$R^s f : t \in G \mapsto f(t-s) \in Z, \tag{2.1.1}$$

for each fixed $s \in G$; Z is a vector space. We introduce new notation for the translation operator R^s of equation (2.1.1) to distinguish the parameter s from the variable t of the function f in the form of R^s_t:

$$R^s_t f : t \in G \mapsto f(t-s) \in Z. \tag{2.1.2}$$

The symbol R^s_t needs to be read as '*t is replaced by $t-s$*'. When s is intended as a 'parameter', the notation $f_s(t) = f(t-s)$ will be used. The translation operators follow certain rules which can be verified directly. These are:

$$R^s_t R^r_t = R^{(s+r)}_t = R^r_t R^s_t; \tag{2.1.3a}$$

$$R^s_r R^r_t = R^r_t R^{-s}_t; \tag{2.1.3b}$$

$$R^s_r R^{-r}_t = R^s_t R^{-r}_t; \tag{2.1.3c}$$

$$R^s_t [g \circ f] = g \circ R^s_t f = g \circ f_s. \tag{2.1.3d}$$

We shall use the notation R^s when replacement of t is understood.

A class Φ of functions defined on G and the same range is called *translation invariant* if $f \in \Phi$ implies that for every $s \in G$, $f_s := R^s f \in \Phi$; that is, Φ is an $\mathscr{R}$-group; $\mathscr{R}$ is the set of all the translation operators R^s. Translation invariant classes will be crucial in this work. From this point onward, it will be required throughout that Φ be translation invariant. We shall call $\Phi = \Phi(G, Z)$ a *test space* and refer to members of Φ as *test functions*. Examples of test spaces are:

1. The space of all functions from G to Z.
2. The space of constant functions. This is the classical case where $\Phi = Z$.
3. If G is a finite group then Φ is equivalent to a finite Cartesian power of Z. When G is the trivial group, Φ is the same as the test space of constant functions.

[1] If the group G is not Abelian then another translation operator is possible: $L^s f : t \in G \mapsto f(-s+t) \in Z$ since the group operation $+$ is not commutative. For our purposes, there is no need to consider both types of translation operators. Therefore we take G as Abelian so that $R^s = L^s$.

4. If G is a topological group, the following possibilities are available: $\Phi = C(G, Z)$, the space of continuous functions from G to Z; $\Phi = BC(G, Z)$, the space of bounded continuous functions; $\Phi = BUC(G, Z)$, the space of bounded $(G, \mathcal{U}) \to (Z, \mathcal{V})$ uniformly continuous functions; $\mathcal{V}$ is the uniformity induced by the norm on Z; $\mathcal{U}$ is the uniformity induced by the topological group G. The latter choice will be prominent in our considerations. If G is a compact topological group, $\Phi = C(G, Z) = BUC(G, Z)$.

2.1.1 Admissible Homomorphisms

We shall be interested in (vector space) homomorphisms $x' : \Phi \to Z$; Φ, Z are vector spaces. The notation $\langle f, x' \rangle = x'(f)$ reminiscent of usage in functional analysis, underlines our intention to treat these operators as if they were linear functionals. We shall interchangeably call them *vector-valued* linear functionals or homomorphisms. Such a homomorphism is called *admissible* if the mapping

$$^{x'}\!f : p \mapsto \langle f_{-p}, x' \rangle, \tag{2.1.4}$$

is in Φ. When clarity demands, we use the notation $x'_{\cdot r}$ to indicate that the homomorphism removes dependence on $r \in G$; $f : r \in G \mapsto f(r) \in Z$; $f \in \Phi$. The class of admissible homomorphisms is denoted by $\mathscr{A}_\Phi$ or $\mathscr{A}$ when Φ is understood. Admissible homomorphisms take center stage: we mostly work with admissible homomorphisms.

For a fixed $f \in \Phi$, the translation mappings R^{-q} generate a bundle of "curves" $\{f_{-q} : q \in G\}$ in Z which is then mapped by x' to the curve $^{x'}\!f$. Admissibility requires that this curve should be in Φ. Hence the translation mappings allow each admissible homomorphism x' to play two roles. The default role is simply as a homomorphism of $\Phi \to Z$. In the second role they induce the endomorphism X' on Φ defined by

$$X'f = {}^{x'}\!f; \quad f \in \Phi. \tag{2.1.5}$$

The endomorphisms so induced will be called *dualisms*. Examples of admissible homomorphisms are:

6. The point evaluation maps $\theta'_q : f \in \Phi \mapsto f(q) \in Z$ are admissible homomorphisms. In fact, $^{\theta'_q}\!f(p) = f(p + q) = R^{-q} f(p) = [\Theta'_q f](p)$. Thus, Θ'_0 is the identity on Φ.

7. If $A : Z \to Z$ is a bounded linear operator, the mapping $x'_A : f \mapsto A\langle f, \theta'_0 \rangle$ is an admissible homomorphism when $\Phi = \mathrm{C}(G, Z)$. Indeed, $[A'f](q) = {}^{x'_A}\!f(q) = Af(q)$. We shall refer to the relation $A \to x'_A$ as

the *canonical mapping* and x'_A as the *associated canonical homomorphism.*

8. When the group G is trivial, admissible homomorphisms are classical operators.

Thus, the class $\mathscr{A}_\Phi$ can be quite large depending on the choice of Φ.

Remark 1. *For a non trivial G, admissible homomorphisms generalize operators. If $A, B \in \mathcal{L}(Z)$, the space of all bounded linear operators on Z, then*

$$\langle K_z, a' \rangle = Az; \langle K_z, a' * b' \rangle = (A \circ B)z,$$

where $K_z \in \Phi : t \in G \to z \in Z$ is the constant function and a', b' is the associated canonical homomorphism. Therefore, if the test space Φ strictly contains the set $\{K_z | z \in Z\}$ of constant functions, then a' and b' is a strict generalization of A and B, respectively.

The test space Φ is an $\mathscr{A}_\Phi$-group. For our purposes, we need Φ to be both an $\mathscr{A}_\Phi$-group and $\mathscr{R}$-group. We give the class $\mathscr{A}$ of admissible homomorphisms an algebra structure by introducing an associative product. Let x' be a homomorphism (not necessarily in $\mathscr{A}$) and suppose that $y' \in \mathscr{A}$. The product homomorphism $x' * y'$ is defined as:

$$\langle f, x' * y' \rangle = \langle\, {}^{y'}f, x' \rangle; f \in \Phi. \tag{2.1.6}$$

The above product $*$ on $\mathscr{A}$ is based on the fact that Φ is simultaneously an $\mathscr{A}$-group and an $\mathscr{R}$-group; the double role each $x' \in \mathscr{A}$ plays allows (2.1.6) to have a meaning.

In the case the classical convolution algebra, the associativity of the product $*$ of the linear functionals follows from the translation invariance of the linear functionals (Appendix B.2, equation (B.2.2)). We now show that this is also the case for our vectorized convolution algebra.

Lemma 1 (Commutation rule)**.** *For arbitrary $x' \in \mathscr{A}$, $s \in G$ and $f \in \Phi$, ${}^{x'}[R^s f] = R^s[{}^{x'}f]$. That is, $X'[R^s f] = R^s[X' f]$.*

Proof. Direct computation shows that $X'[R^s(f)](t) = x'(R^{-t}(R^s(f))) = x'(R^{s-t}(f))$ and $R^s(X'f)(t) = (X'f)(t-s) = x'(R^{-(t-s)}f) = x'(R^{s-t}(f))$. The above can be rewritten as:

$$R^s_t[\,{}^{x'}f(t)] = R^s_t\langle R^t_r f, x'_{\cdot r}\rangle = \langle R^s_t R^t_r f, x'_{\cdot r}\rangle = \langle R^{-t}_r R^s_r f, x'_{\cdot r}\rangle = {}^{x'}[R^s_r f(t)],$$

having made use of the translation rules (2.1.3). $\square$

The commutation rule plays a decisive role in the following calculations:

$$\begin{aligned}
[\,^{(x'*y')}f](t) &= [x'*y'](R^{-t}(f)) = x'(Y'(R^{-t}(f))) \\
&= x'(R^{-t}(Y'f)) \\
&= X'(Y'f)(t) \\
&= {}^{x'}(\,^{y'}f)(t),
\end{aligned}$$

for all $t \in G$. Equivalently $\langle R_r^{-t}f, x'*y'\rangle = \langle\,^{y'}[R_r^{-t}f](r), x'_{\cdot r}\rangle = \langle R_r^{-t}[\,^{y'}f](r), x'_{\cdot r}\rangle$. Thus, we have derived the power rule which is in fact valid for $y' \in \mathscr{A}$ and arbitrary x' [2]:

Corollary 1 (Power Rule)**.** *For $x', y' \in \mathscr{A}$ and $f \in \Phi$, $x' * y' \in \mathscr{A}$ and*

$$^{x'}[^{y'}f] = ^{x'*y'}[f]. \tag{2.1.7}$$

This leads to the *associativity* of the product $*$ on $\mathscr{A}$ [3].

Theorem 1. *The admissible homomorphisms $\mathscr{A}_\Phi$ equipped with the product $*$ is an associative set/class with unit θ_0'.*

The product $*$ introduced here is related to composition of linear transformations in the following way: If x' and y' are in $\mathscr{A}_\Phi$, then by (2.1.7) and (2.1.5)

$$^{(x'*y')}f = X' \circ Y'f. \tag{2.1.8}$$

Thus, the product of homomorphisms corresponds to composition of their dualisms. Therefore, the mapping

$$\Gamma : x' \in \mathscr{A}_\Phi \mapsto X' \in L(\Phi), \tag{2.1.9}$$

is an isomorphism in the algebra $L(\Phi)$ of linear transformations on Φ.

Remark 2. *Recall in Remark 1, that when the group G is non trivial, admissible homomorphisms are generalized operators. Now, the product $*$ is a generalized composition of operators: if $A_1, A_2 \in \mathcal{L}(Z)$ and $a_1', a_2' \in \mathscr{A}_B$ are the associated canonical homomorphisms, then $A_1 \circ A_2$ has the associated canonical homomorphism $a_1' * a_2'$.*

---

[2] Just as long as $^{x'*y'}f$ is a function on G : it need not be in Φ. Therefore, the relation $^{x'}[^{y'}f] = ^{x'*y'}[f]$ applies without the assumption that $^{x'*y'}f \in \Phi$. Indeed it follows from $^{x'}[^{y'}f] = ^{x'*y'}[f]$ that if $x', y' \in \mathscr{A}$ then $x' * y' \in \mathscr{A}$: $^{y'}f \in \Phi$ implies $^{x'}(^{y'}f) \in \Phi$.

[3] The preceding constructions are natural extensions of the well known construction of a convolution product on the dual space X' of classical $\mathbb{C}$-valued functionals of a translation-invariant function space X [§ 19 [18]] or [Section 1.9.7 [23]]. This vindicates our use of the the notation x' for members in $\mathscr{A}$, which is normally reserved for linear functionals of the dual space X'.

A question to ask at this point is which linear operators on the test space are dualisms? That is, which of the linear operators on the test space give rise to admissible homomorphisms from Φ to Z? Let Δ be a (vector) subspace of Φ. A linear mapping $M : \Delta \to \Phi$ is called *translatable* if Δ is translation invariant and for every $p \in G$ and $f \in \Delta$,

$$R^p M f = M R^p f.$$

The following result can be verified directly once it is realized that M induces a homomorphism m' defined by $\langle f, m' \rangle := \langle Mf, \theta_0' \rangle$. It should be noted that the induced homomorphism m' is restricted to the 'domain' Δ. We call such homomorphisms, *restricted homomorphisms*.

Theorem 2. *A linear mapping $M : \Phi \to \Phi$ is translatable if and only if it is a dualism of a restricted homomorphism. If M is a one-to-one dualism and $\Delta := M[\Phi]$, then $M^{-1} : \Delta \to \Phi$ is translatable. Sums and compositions of translatable mappings are translatable.*

Proof. For the sufficient condition, the induced homomorphism m' is admissible since ${}^{m'}f = M(f)$ for all $f \in \Phi$:

$${}^{m'}f : s \mapsto (MR^{-s}f)(0) = (R^{-s}Mf)(0) = Mf(s).$$

The necessary condition is immediate from Lemma 1.

The translation invariance of Δ follows from the translation invariance of Φ: consider $f := Mf' \in \Delta$; $f_p = M(R^p f') \in \Delta$; $f' \in \Phi$. Similarly, direct computation shows M^{-1} commutes with R^p. □

We impose a linear structure on the algebra $\mathscr{A}$ by requiring both Φ and Z to be vector spaces over the scalar field $\mathbb{C}$ (or $\mathbb{R}$); vector addition coincides with the Abelian group operation in Φ. The linear structure is defined as follows:

$$\langle f, x' + y' \rangle : = \langle f, x' \rangle + \langle f, y' \rangle \quad \textit{and} \quad \langle f, \lambda x' \rangle : = \lambda \langle f, x' \rangle.$$

The linear structure on Φ ensures that if x', y' are admissible, then so are $x' + y'$ and $\lambda x'$: ${}^{(x'+y')}(f) = {}^{x'}(f) + {}^{y'}(f)$ and ${}^{\lambda x'}f = {}^{x'}(\lambda f)$. Furthermore, the homogenity of x', y' ensures that the relation

$$(\lambda x') * y' = \lambda(x' * y') = x' * (\lambda y').$$

Theorem 3. *If the admissible homomorphisms in $\mathscr{A}$ are linear transformations (the Abelian groups Φ and Z are vector spaces; vector addition coincides with the Abelian group operation), then the ring $\langle \mathscr{A}, +, * \rangle$ is a $\mathbb{C}$-algebra with unit θ_0'.*

We shall take the symbol $\mathscr{A}$ to mean the $\mathbb{C}$-algebra $\langle \mathscr{A}, +, * \rangle$ and members of $\mathscr{A}$ be called Φ-admissible vector valued functionals.

2.2 Test Space With Fréchet Topology

When the group G is topological, continuous test functions may be considered, as remarked. It also opens up the possibility of introducing vector topologies for $\Phi = C(G, Z)$ such that the translation map

$$p \in G \mapsto R^p f \in \Phi, \tag{2.2.1}$$

is continuous for each fixed $f \in \Phi$. This ensures that the admissibility of each bounded $x' \in \mathrm{Hom}(\Phi, Z)$: ${}^{x'}f : p \in G \mapsto x' \circ f_{-p}$. If G is a topological group and $\Phi := C(G, Z)$ one could define a locally convex topology for Φ based on the family of seminorms $|f|_K = \sup_{t \in K} \|f(\tau)\|_Z$; $K \subset G; K$ compact. If G is locally compact and has a countable neigbourhood basis, Φ can be made into a Fréchet space (Theorem 7.19, [31]). The Fréchet property will turn out to be critical in the study of strong integrability. Since Z is a complete metric space, under this topology of uniform convergence on compact sets, the translations (equation (2.2.1)) are continuous for each fixed $f \in \Phi$. This follows from the fact that a continuous metric space valued function f on a compact set $I \subset G$ is uniformly continuous.

Theorem 4. *The mapping $p \mapsto f_p \in \Phi$ is continuous.*

Proof. The exponential property (2.1.3a) makes it sufficient to prove continuity at the origin for the non-linear map $p \to f_p$ which is immediate from the uniform continuity of f [4] when restricted to a compact domain. □

In this chapter, we shall take this Fréchet space as our test space Φ. Let $\mathscr{A}_B$ denote the class of bounded homomorphisms. From Theorem 4, $(\mathscr{A}_B, *)$ is an algebra:

Theorem 5. *If x' is a bounded homomorphisms, it is admissible.*

Remark 3. *The translation operators impose the following minimum requirements on the test space: the test space is to be (i) translation invariant and (ii) to have a topology which ensures that the map (equation (2.2.1)) is continuous for each fixed $f \in \Phi$.*

[4]We illustrate for $G = \mathbb{R}$. We now show that $\frac{1}{n} \to 0$ implies $f_{-\frac{1}{n}} \rightrightarrows f$ on compacts. The fixed f, when restricted to compact domain K, is uniformly continuous: for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow \|f(x) - f(x')\| < \epsilon.$$

Let H_n denote the function $f_{-\frac{1}{n}} : x : f(x + \frac{1}{n})$. Then choose n large enough such that $\frac{1}{n} < \delta$. Then $\|H_n(x) - f(x)\| < \epsilon$ for all $x \in K$. That is, $H_n \rightrightarrows f$ on compact K since $\sup_{x \in K} \|H_n(x) - f(x)\| < \epsilon$.

2.3 Time-dependent Homomorphisms

With Remark 1-2, the canonical mapping between bounded linear operators and bounded admissible homomorphisms motivates the study of *time-dependent* members of bounded admissible homomorphisms, instead of traditional families of bounded evolution operators with composition of operators replaced by the product $*$. Thus we consider a family $\mathfrak{X}' = \{x'(\tau) \in \mathscr{A}_B | \tau \in \mathbb{T} = \mathbb{R}^+ := (0, \infty)\}$. The family $\mathfrak{X}' := \{x'(\tau)|\tau > 0\}$ can be identified with a time-dependent $\mathscr{A}_B$-valued function $\tau \mapsto x'(\tau) \in \mathscr{A}_B$.

2.3.1 Strong Integrability

In classical semi-group theory, the strong integrability of the semi-group $\{E(\tau)|\tau \geq 0\}$ (time-dependent family of bounded evolution operators) ensured the existence of the smoothing operators $V(\tau) := \int_{(0,\tau)} E(\sigma)d\sigma$. These operators established fundamental properties of the semigroup such as strong continuity implies differentiability ([12], Lemma 1.3, p. 50).

Likewise, we shall be most interested in the 'strong integrability' of the family $\mathfrak{X}'$; $\mathfrak{X}'$ is a replacement of the family of bounded evolution operators. We say that $\mathfrak{X}'$ is *(strongly) integrable over an interval* $I \subseteq (0, \infty)$ if for every $f \in \Phi$, the Bochner integral $\int_I \langle f, x'(\tau)\rangle d\tau$ exists in Z. If this is the case, we define the smoothing homomorphism x'_I by the relation

$$\langle f, x'_I\rangle = \int_I \langle f, x'(\tau)\rangle d\tau; f \in \Phi. \tag{2.3.1}$$

Therefore, Z needs to be a real or complex Banach space.

Central to the study of strong integrability of $\mathfrak{X}'$ is the following version of a well-known theorem ([19], Theorem 3.8.2, p.85). The test space Φ being Fréchet plays a critical role in our version of Theorem 3.8.2.

Theorem 6. *Suppose the functions $\tau \to \langle f, x'(\tau)\rangle$; $\tau \in (0, \infty)$; $f \in \Phi$ are (strongly) measurable and the homomorphisms $x'(\tau)$ are bounded. If for an interval $I \subseteq (0, \infty)$ the integral*

$$\langle f, x'_I\rangle = \int_I \langle f, x'(\tau)\rangle d\tau, \tag{2.3.2}$$

exists for every $f \in \Phi$, the homomorphisms $x'_I : \Phi \to Z$ are bounded and hence, admissible.

Proof. Let $W : f \in \Phi \to Wf \in L^1(\mathbb{R}, Z)$ be defined by $[Wf](\tau) = \langle f, x'(\tau)\rangle$. To see that W is continuous we show that it is closed and use the closed graph theorem (Φ is Fréchet). Suppose that $f_n \to f$ in Φ and that $Wf_n \to g$

in $L^1(I, Z)$. Then at least a subsequence of $[Wf_n](\tau)$ converges to $g(\tau)$ in Z for almost every $\tau \in I$. By the continuity of $x'(\tau)$, $[Wf_n](\tau) \to [Wf](\tau)$ in Z for every $\tau \in I$. Hence $g = Wf$. Finally we note that $\|\langle f, x'_I \rangle\| \leq \|Wf\|_{L^1(I,Z)}$ □

2.3.2 New Framework

From now on we shall assume that the functions $\tau \to \langle f, x'(\tau) \rangle$ are strongly Lebesgue measurable. Additionally, integrals will be in the sense of Bochner. If for all $I = (0, \tau)$; $\tau > 0$, the smoothing homomorphism $x'_I \in \mathscr{A}_B$, we say that $\mathfrak{X}'$ is *integrable near the origin*. We let $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ denote the class of all such time-dependent $\mathscr{A}_B$-valued functions. This will be the new algebraic-analytic framework for the study of intertwined families of evolution operators.

The framework $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ vectorizes $\mathcal{L}^1_{loc}((0, \infty), \mathbb{R})$ by replacing the algebra $\mathbb{R}$ with another algebra $\mathscr{A}_B$, rather than with a plain Banach space Z. In fact, $\mathcal{L}^1_{loc}((0, \infty), \mathbb{R})$ embeds in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$: we associate with each time-dependent scalar valued function $\phi \in \mathcal{L}^1_{loc}((0, \infty), \mathbb{R})$, a time dependent $\mathscr{A}_B$-valued function x'_ϕ by the relation, $x'_\phi(\tau) := \phi(\tau)\theta'_0$. Clearly $x'_\phi \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$. Now, one of the motivations for this more natural vectorization, is to construct a convolution theorem in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ that resembles its counterpart in $\mathcal{L}^1_{loc}((0, \infty), \mathbb{R})$. A dominant theme of this new framework is to see *how far these $\mathscr{A}_B$ -valued functions behave like scalar valued functions in terms of convolution.*

Therefore, we need the notion of Laplace closed families for our convolution theorem and Laplace transform theorems in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ to resemble the scalar convolution theorem of $\mathcal{L}^1_{loc}((0, \infty), \mathbb{R})$ (Theorem 11.9B [13]). We will show that the algebra product of the convolution algebra transfers the scalar convolution theorem, as opposed to a bounded bilinear form used in other well known vectorizations (Section 2.9).

2.4 Closedness

Fundamental to our investigation is the product $y' * x'_I$ for a fixed $y' \in \mathscr{A}_B$. We thus consider the family $[y' * x'](\tau) := y' * x'(\tau)$. We say that the homomorphism $y' \in \mathscr{A}_B$ is *closed* over the interval I with respect to the family $\mathfrak{X}'$ if for every $f \in \Phi$ the mapping $\tau \mapsto \langle f, [y' * x'](\tau) \rangle$ is strongly measurable in Z, the integral $[y' * x']_I$ exists and $\langle f, y' * x'_I \rangle = \langle f, [y' * x']_I \rangle$:

$$y' * \int_I x'(\tau)\, d\tau = \int_I (y' * x'(\tau))\, d\tau. \tag{2.4.1}$$

From (2.4.1) it is seen that closedness in the present sense resembles a well-known theorem concerning the commutation of closed linear operators and Bochner integrals. Indeed, the canonical homomorphisms [5] are closed over any I with respect to any family $\mathfrak{X}'$. From direct computation it follows that

Proposition 1. *Let y' be a canonical homomorphism. Then it is closed over I with respect to $\mathfrak{X}'$.*

Remark 4. *I wish to point out the important role Professor Diestel played in showing us the validity of Theorem 3.8.2 [19]. It is his proof that we extended for the proof of Theorem 6.*

2.4.1 Laplace Transform and Closedness

At this stage we introduce the *Laplace transform* of the family $\mathfrak{X}'$ formally, for $f \in \Phi$ at $\lambda > 0$, by the expression

$$\langle f, \widehat{x'}(\lambda) \rangle = \int_0^\infty e^{-\lambda\tau} \langle f, x'(\tau) \rangle d\tau = \int_0^\infty \langle f, e^{-\lambda\tau} x'(\tau) \rangle d\tau. \qquad (2.4.2)$$

The family $\mathfrak{X}'$ is $\mathscr{A}_B$-Laplace transformable if $\widehat{x'}(\lambda) \in \mathscr{A}_B$. We denote the class of all such functions by $Lap(\lambda, \mathscr{A}_B)$. In that case, also the integrals x'_I exist for all finite intervals I. Clearly if the Laplace transform exists for some λ_0 is exists for all $\lambda > \lambda_0$. An immediate consequence of Theorem 6 is

Theorem 7. *If for some $\lambda > 0$ the Laplace transform $\widehat{x'}(\lambda)$ exists, it is bounded and therefore, in $\mathscr{A}_B$. Additionally, the integrals x'_I are bounded for all finite intervals $I \subset (0, \infty)$: if $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$, then $\mathfrak{X}' \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$.*

In developing a convolution theorem in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$, we need to investigate the distributive nature of the Laplace transform over products. If $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$, then the homomorphism y' is said to be *Laplace-closed* with respect to $\mathfrak{X}'$ if it is closed over $\mathbb{R}^+$ with respect to the family $\{e^{-\lambda\tau} x'(\tau)\}$, that is,

$$y' * \widehat{x'}(\lambda) = \int_{(0,\infty)} e^{-\lambda\tau} (y' * x'(\tau))\, d\tau. \qquad (2.4.3)$$

Theorem 8. *Suppose that $y' \in \mathscr{A}_B$ and that the Laplace transform $\widehat{x'}(\lambda)$ exists. Then*

$$\int_0^\infty e^{-\lambda\tau} \langle f, x'(\tau) * y' \rangle\, d\tau = \langle f, \widehat{x'}(\lambda) * y' \rangle. \qquad (2.4.4)$$

[5] Given our canonical identification of a bounded operator A with a canonical map y'_A this should not be surprising.

If y' is Laplace-closed over $\mathbb{R}^+$ with respect to $\mathfrak{X}'$, then the Laplace transform $\widehat{[y' * x']}(\lambda)$ exists and

$$\langle f, \widehat{[y' * x']}(\lambda) \rangle = \langle f, y' * \widehat{x'}(\lambda) \rangle. \tag{2.4.5}$$

Proof. The proof of (2.4.4) is fairly straightforward and a direct calculation shows that the integral exists. When $x'(\tau)$ and y' are interchanged in the product the matter is more complicated. To begin with consider

$$\int_0^\infty e^{-\lambda\tau} \langle f_{-s}, x'(\tau) \rangle \, d\tau = \langle f_{-s}, \widehat{x'}(\lambda) \rangle = [\widehat{X'(\lambda)}](\sigma) \in \Phi. \tag{2.4.6}$$

If we apply y' to (2.4.6) the expresion (2.4.5) follows from (2.4.1). □

A family $\mathfrak{Y}'$ is *Laplace-closed* with respect to $\mathfrak{X}'$ if for each fixed $\sigma > 0$, $y'(\sigma)$ is Laplace-closed with respect to $\mathfrak{X}'$. We say that $\mathfrak{Y}'$ is a *canonical family* if $y'(\tau) = A(\tau)\theta_0'$; $A(\tau) \in \mathcal{L}(Z)$. The canonical families provides a wide class of Laplace closed families.

Proposition 2. *Let $\mathfrak{Y}'$ be a canonical family and $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$. Then $\mathfrak{Y}'$ is Laplace-closed with respect to $\mathfrak{X}'$.*

Proof. This is immediate from Proposition 1, Section 2.3.1. □

2.4.2 Empathy Laplace Transformability

In empathy theory, the family $\mathcal{S} := \{S(\tau) \in \mathcal{L}(Z) : \tau \in \mathbb{T}\}$ of bounded operators is Laplace transformable at $\lambda > 0$ if the Bochner integral $P(\lambda)z := \int_{(0,\infty)} e^{-\lambda\tau} S(\tau)z \, d\tau$ exists in Z for every $z \in Z$ and $\lambda > 0$; we call $P(\lambda)$ the resolvent operator. The canonical mapping between a bounded operator and a homomorphism associates a canonical family $\mathfrak{X}'$ defined by the homomorphisms $x'(\tau) = S(\tau)\theta_0'$. We call $\mathfrak{X}'$ the *associated canonical family* or the *$\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$-analogue* of $\mathcal{S}$. These two families are related by Theorem 7, Section 2.4.1:

Corollary 2. *Let the family $\mathcal{S} := \{S(\tau) \in \mathcal{L}(Z) : \tau \in \mathbb{T}\}$ of bounded operators be Laplace transformable. Then the associated canonical family $\mathfrak{X}'$ is $\mathscr{A}_B$-Laplace transformable.*

It also is immediate from Proposition 2 that,

Corollary 3. *Let the families $\mathcal{S} := \{S(\tau) \in \mathcal{L}(Z) : \tau \in \mathbb{T}\}$, $\mathcal{E} := \{E(\tau) \in \mathcal{L}(Z) : \tau \in \mathbb{T}\}$ be Laplace transformable and $\mathfrak{X}', \mathfrak{Y}'$ be their associated canonical families. Then $\mathfrak{X}'$ is closed with respect to $\mathfrak{Y}'$.*

2.5 Convolution

Let $\mathfrak{X}', \mathfrak{Y}'$ be integrable near the origin. Then $\mathfrak{X}', \mathfrak{Y}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$. We define the *convolution* $\mathfrak{X}' \circledast \mathfrak{Y}'$ like its scalar counterpart, by the Bochner integrals

$$\langle f, (x' \circledast y')(\tau)\rangle = \int_{(0,\tau)} \langle f, x'(\tau - \sigma) * y'(\sigma)\rangle \, d\sigma; \quad f \in \Phi, \tag{2.5.1}$$

by virtue of the natural product $*$ in $\mathscr{A}_B$.

The definition is formal since the existence is not known to be guaranteed unlike in the case of $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$. Theorem 6 gives an easy to check criteria to ensure that $x' \circledast y' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$.

Proposition 3 (Existence of Convolution)**.** *Let $\mathfrak{X}', \mathfrak{Y}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$. For fixed $\tau > 0$, if the map*

$$\sigma \mapsto \langle f, x'(\tau - \sigma) * y'(\sigma)\rangle,$$

is Bochner integrable over $(0, \tau)$ for every fixed $f \in \Phi$, then $(x' \circledast y')(\tau)$ is bounded and hence admissible.

Corollary 4. *Let $X = \{S(\tau) : \tau \in \mathbb{T}\}$ and $Y = \{E(\tau) : \tau \in \mathbb{T}\}$ be Laplace transformable families for each $\lambda > 0$ interrelated by the empathy causal relation $S(\tau + \sigma) = S(\tau)E(\sigma)$ for all $\tau, \sigma \in \mathbb{T}$. Then their associated canonical families $\mathfrak{X}', \mathfrak{Y}'$ have the property that $\mathfrak{X}' \circledast \mathfrak{Y}'$ is integrable near the origin.*

2.6 Laplace Transform Theorems

We now construct a convolution theorem and Laplace transform theorems in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ for Laplace closed families that resembles the scalar convolution theorem and their Laplace transform theorems (Theorem 11.9B [13]).

2.6.1 Transfer Lemma

The following lemma transfers the proof of the classical scalar convolution theorem, word for word, into the proof of the convolution theorem in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$.

Lemma 2. *Let $\mathfrak{X}'$ be Laplace-closed with respect to $\mathfrak{Y}'$ and $\mathfrak{X}', \mathfrak{Y}' \in Lap(\lambda, \mathscr{A}_B)$ (equation (2.4.3), Section 2.4.1). Then for every $f \in \Phi$, we can construct a measurable function H on $D\colon = (0,\infty) \times (0,\infty)$ as follows :*

$$H : D \ni (\sigma, \tau) \mapsto e^{-\lambda(\sigma+\tau)}(x'(\sigma) * y'(\tau))(f);$$

which has the following properties:

$$\lim_{B\to\infty}\int_0^B\int_0^B H(\sigma,\tau)d\sigma d\tau = \int_0^\infty\int_0^\infty H(\sigma,\tau)d\sigma d\tau; \quad (2.6.1)$$

$$\int_{\sigma=0}^B\int_{\tau=0}^B \|H(\sigma,\tau)\|_Z d\tau d\sigma < \infty \text{ for all } B>0; \quad (2.6.2)$$

$$\|\int_{R_B} H(\sigma,\tau)dA_{\sigma\tau} - \int_{W_B} H(\sigma,\tau)dA_{\sigma\tau}\|_Z \to 0 \quad \text{as} \quad B\to\infty, \quad (2.6.3)$$

where $R_B\colon = (0,B)\times(0,B)$, W_B *is the open wedge defined by the lines* $\sigma+\tau = 2B, \tau = 0$ *and* $\sigma = 0$.

Proof. By assumption, $\widehat{x'}(\lambda), \widehat{y'}(\lambda) \in \mathscr{A}_B$. Then $\widehat{x'}(\lambda) * \widehat{y'}(\lambda) \in \mathscr{A}_B$ since $\mathscr{A}_B$ is closed with respect to $*$. Furthermore by the Laplace-closedness assumption,

$$\langle f, \widehat{x'}(\lambda) * \widehat{y'}(\lambda)\rangle = \int_{\sigma=0}^\infty\int_{\tau=0}^\infty H(\sigma,\tau)d\tau d\sigma;$$

hence establishing the measurability of H. The absolute summability of the Bochner integral,

$$\int_{\sigma=0}^\infty\int_{\tau=0}^\infty \|H(\sigma,\tau)\|_Z d\tau d\sigma < \infty,$$

follows and we have the existence of $\int_D H(\sigma,\tau)dA_{\sigma\tau}$. Therefore, $\int_{R_B} H(\sigma,\tau)dA_{\sigma\tau}$ exists and is equivalent to $\int_0^B\int_0^B H(\sigma,\tau)d\sigma d\tau$, again by Fubini's theorem (Theorem 1.1.9 [2]). Thus, condition (2.6.2) follows by the absolute summability of the Bochner integral.

Now it follows by the dominated convergence theorem (Theorem 1.1.8 [2]) that

$$\lim_{n\to\infty}\int_{R_n} H(\sigma,\tau)dA_{\sigma\tau} = \int_D H(\sigma,\tau)dA_{\sigma\tau}; \quad (2.6.4)$$

$$\lim_{n\to\infty}\int_{W_n} H(\sigma,\tau)dA_{\sigma\tau} = \int_D H(\sigma,\tau)dA_{\sigma\tau}. \quad (2.6.5)$$

Set $f_n : (\sigma,\tau)\in D : \mathcal{X}_{R_n}H$ so that $H(\sigma,\tau) = \lim_{n\to\infty} f_n(\sigma,\tau)$ for every $(\sigma,\tau)\in D$; $\mathcal{X}_{R_n}$ is the indicator function of the set $R_n = (0,n)\times(0,n)$. Trivially $\|f_n\| \le \|H\|$ where $\|H\|$ is integrable over D. Therefore, by the dominated convergence theorem, $\lim_{n\to\infty}\int_D f_n(\tau)d\tau = \int_D HdA_{\sigma\tau}$. Equation (2.6.5) follows similarly.

Therefore, equation (2.6.1) is now immediate and (2.6.3) follows by additionally noting that

$$\|\int_{R_B} H(\sigma,\tau)dA_{\sigma\tau} - \int_{W_B} H(\sigma,\tau)dA_{\sigma\tau}\|_Z$$

is dominated by

$$\begin{aligned}\| \int_{R_B} H(\sigma,\tau)dA_{\sigma\tau} &- \int_D H(\sigma,\tau)dA_{\sigma\tau}\|_Z \\ &+ \| \int_D H(\sigma,\tau)dA_{\sigma\tau} - \int_{W_B} H(\sigma,\tau)dA_{\sigma\tau}\|_Z.\end{aligned}$$

□

2.6.2 Convolution Theorem

The following convolution theorem, like the lemma, is a property purely of the vector space $Lap(\lambda, \mathscr{A}_B)$. It is independent of the choice of Fréchet test space Φ.

Theorem 9. *Let $\mathfrak{X}', \mathfrak{Y}' \in Lap(\lambda, \mathscr{A}_B)$. If $(\mathfrak{X}' \circledast \mathfrak{Y}')$ is integrable near the origin and $\mathfrak{X}'$ is Laplace-closed respect to $\mathfrak{Y}'$, then $\widehat{x' \circledast y'}(\lambda)$ exists and*

$$\widehat{x' \circledast y'}(\lambda) = \widehat{x'}(\lambda) * \widehat{y'}(\lambda). \tag{2.6.6}$$

Proof. We start at the right hand side of equation (2.6.6) just as in the proof of the convolution theorem in $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$. Therefore, consider the product $p(B) := (\int_{\sigma=0}^{B} e^{-\lambda\sigma}x'(\sigma)d\sigma * \int_{\tau=0}^{B} e^{-\lambda\tau}y'(\tau)d\tau)$. It turns out that, $p(B)$ is an approximation of $\widehat{x'}(\lambda) * \widehat{y'}(\lambda)$. Then

$$p(B)(f) = \int_{\sigma=0}^{B}\int_{\tau=0}^{B} H(\sigma,\tau)d\tau d\sigma,$$

by the Laplace-closedness assumption.

Now, invoke Lemma 2. Then $\lim_{B\to\infty} p(B)(f) = \langle f, \widehat{x'}(\lambda) * \widehat{y'}(\lambda)\rangle$ by (2.6.1); $p(B)(f)$ is a double integral over R_B by (2.6.2); it suffices to take $p(B)(f)$ as a double integral over the wedge W_B in the $S-T$ plane to evaluate $p(B)(f)$ as $B \to \infty$ by (2.6.3). In order to evaluate $p(B)(f)$ as a double integral over the wedge W_B in the $S-T$ plane:

$$\int_{W_B} H(\sigma,\tau)dA_{\sigma\tau} = \int_{W_B} e^{-\lambda(\sigma+\tau)}(x'(\sigma) * y'(\tau))(f)dA_{\sigma\tau},$$

first consider the plane-to-plane transformation $J : (\sigma,\tau) \in S \times T \mapsto (\sigma = f(\sigma,\tau), \sigma' = g(\sigma,\tau)) \in S\times\Sigma$ where $f(\sigma,\tau) = \sigma$ and $g(\sigma,\tau) = \sigma+\tau$. Then its inverse $K : (\sigma,\sigma') \in S \times \Sigma \mapsto (\sigma = h(\sigma,\sigma'), \tau = i(\sigma,\sigma'))$ where $h(\sigma,\sigma') = \sigma$ and $i(\sigma,\sigma') = \sigma' - \sigma$ has 1 as its Jacobian. Furthermore K maps the wedge, W_B', in the $S\times\Sigma$ plane defined by the Y-axis, line $\sigma' = \sigma$ and $\sigma' = 2B$ into the wedge W_B of the $S \times \Sigma$. Therefore, integrating $H(\sigma,\tau)$ over W_B' as a type 2 region in the $S \times \Sigma$ -plane, we have

$$\begin{aligned}\int_{W_B} H(\sigma,\tau)dA_{\sigma\tau} &= \int_{W_B'} H(\sigma,\tau)dA_{\sigma\sigma'} \\ &= \int_{\sigma'=0}^{\sigma'=2B} e^{-\lambda\sigma'} \int_{\sigma=0}^{\sigma=\sigma'} (x'(\sigma) * y'(\sigma'-\sigma))(f)d\sigma d\sigma',\end{aligned}$$

by virtue of the *integrability near the origin of* $(\mathfrak{X}' \circledast \mathfrak{Y}')$. We are done on passing the limit $B \to \infty$. □

Remark 5. *For scalar valued functions, the convolution product* $\circledast$ *smooths in general: the convolution product of two scalar functions is at least as nice a function as either of its factors. The* $\mathscr{A}_B$*-Laplace transformability of the convolution* $\mathfrak{X}' \circledast \mathfrak{Y}'$ *confirms the property that the convolution is at least as nice a function as either of its factors.*

2.6.3 Laplace Transform Theorems

Laplace-closed $\mathscr{A}_B$ -valued functions behave in the same way as scalar valued functions of $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$ as far as the convolution theorem is concerned; the proof of the classical convolution theorem transfers word for word[6]. Furthermore, if $\phi \in Lap(\lambda)$, then $\widehat{x'_\phi}(\lambda) = \widehat{\phi}(\lambda)\theta'_0$.

Corollary 5. *Let* $\phi \in Lap(\lambda)$ *and* $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$. *Then* $\widehat{x'_\phi \circledast x'}(\lambda)$ *exists and*

$$\widehat{x'_\phi \circledast x'}(\lambda) = \widehat{\phi}(\lambda)\widehat{x'}(\lambda). \tag{2.6.7}$$

Proof. By virtue of the fact that θ'_0 is the identity of $\mathscr{A}_B$, x'_ϕ is Laplace-closed with respect to $\mathfrak{X}'$ and

$$\langle f, (x'_\phi \circledast x')(\tau)\rangle = \phi * h(\tau);$$

where $h : \sigma \mapsto \langle f, x'(\sigma)\rangle \in Z$; $\phi * h(\tau) := \int_{(0,\tau)} \phi(\tau-\sigma)h(\sigma)d\sigma$. It is well known that if both ϕ and h are integrable near the origin, then so is their convolution $\phi * h$ ([2] p23). Now $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$ implies that h is integrable near the origin so the hypothesis of the convolution theorem (Theorem 9) is met and the conclusion follows from it. □

[6] Recall that this transfer is by virtue of the absolute summability of the Bochner integral, the integrability near the origin of the convolution $x' \circledast y'$ and the Bochner integral $\widehat{x'}(\lambda) * \widehat{y'}(\lambda)(f) = \int_{\sigma=0}^{\infty}\int_{\tau=0}^{\infty} H(\sigma,\tau)d\tau d\sigma$ as well as the Laplace-closedness assumption. See the proof of Theorem 9.

The following Laplace transform theorems provide a computational power of this new algebraic-analytic framework $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$. They hold in the vector space $Lap(\lambda,\mathscr{A}_B)$ and hence are independent of the choice of Fréchet test space Φ. They further confirm the intuition that Laplace transformable $\mathscr{A}_B$ -valued functions behave in the same way as scalar valued functions of $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$.

The classical translation [7] theorems of scalar valued functions derived new Laplace transforms from the known ones. Their proofs carry over for $\mathscr{A}_B$-valued functions once we define the product $e^{\lambda\tau}\mathfrak{X}'(\tau)$ as the product $(\mathfrak{X}'_{e^{\lambda\cdot}}*\mathfrak{X}')(\tau) := x'_{e^{\lambda\cdot}}(\tau)*x'(\tau)$, the translated function ${}^{\sigma}x'(\tau) := x'(\tau-\sigma)$ if $\tau-\sigma>0$; 0 otherwise, the dilated function $x'(\sigma\cdot) := x'(\sigma\tau)$; $x'_{e^{\lambda\cdot}}$ is the $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$-analogue of the scalar exponential function $e^{\lambda\cdot}:\tau\mapsto e^{\lambda\tau}$.

Theorem 10 (Translation on the λ-axis)**.** *If $\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$ and $a\in\mathbb{R}$, then $(\mathfrak{X}'_{e^{a\cdot}}*\mathfrak{X}')\in Lap(\lambda,\mathscr{A}_B)$ and*

$$\widehat{x'_{e^{a\cdot}}*x'}(\lambda)=\widehat{x'}(\lambda-a). \tag{2.6.8}$$

Proof. It suffices to first prove equation (2.6.8) as functionals since $\widehat{x'}(\lambda-a)\in\mathscr{A}_B$ will imply that $\mathfrak{X}'_{e^{a\cdot}}*\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$. The equality as functionals follows directly from the definition of $\widehat{x'}(\lambda-a)$ just as in the scalar case. □

Theorem 11 (Translation on the τ-axis)**.** *Let $\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$ and fix $\sigma>0$. Then ${}^{-\sigma}\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$ and*

$$\widehat{{}^{-\sigma}x'}(\lambda)=e^{\lambda\sigma}\widehat{x'}(\lambda)-(x'_{e^{\lambda\cdot}}\circledast x')(\sigma), \tag{2.6.9}$$

and ${}^{\sigma}\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$, where

$$\widehat{{}^{\sigma}x'}(\lambda)=e^{-\lambda\sigma}\widehat{x'}(\lambda)+(x'_{e^{-\lambda\cdot}}\circledast Mx')(\sigma), \tag{2.6.10}$$

and the mirrored function $Mx'(\tau):=x'(-\tau)$.

Proof. Once again the assumption $\widehat{x'}(\lambda)\in\mathscr{A}_B$ makes it sufficient to prove equations (2.6.9) - (2.6.10) as functionals by virtue of Corollary 5. The proof of equality as functionals follows word for word as for the scalar case by the basic property of the Bochner integral. □

[7] The dilation theorems of scalar valued functions of $\mathcal{L}^1_{loc}((0,\infty),\mathbb{R})$ also carry over but are not needed in our current work.**(Positive Dilation on the t-axis)***Let $\mathfrak{X}'\in Lap(\lambda,\mathscr{A}_B)$ and fix $\sigma>0$. Then $\mathfrak{X}'(\sigma\cdot)\in Lap(\lambda,\mathscr{A}_B)$ and $\widehat{x'(\sigma\cdot)}(\lambda)=\frac{1}{\sigma}\widehat{x'}(\frac{\lambda}{\sigma})$.* The proof follows directly from the definition of $\widehat{x'}(\frac{\lambda}{\sigma})$ just as in the scalar case.

2.7 Star Causal Relations

The only non-constant continuous functions f which satisfy the functional equation

$$f(\tau)f(\sigma) \;=\; f(\sigma+\tau), \tag{2.7.1}$$

are the exponential functions of the form $f(\tau) = e^{a\tau}$. Thus, functional equations axiomatically define certain time-dependent scalar function. Semi-group and empathy relations, as discussed in Section 1.1, are examples of functional equations for time-dependent operator valued functions. Such *causal relations* of [29] are built upon composition of evolution operators.

We introduce similar relations for $\mathscr{A}_B$-valued functions based on the product $*$ in $\mathscr{A}_B$. Consider the following functional equations involving two time-dependent $\mathscr{A}_B$-valued functions $\mathfrak{X}', \mathfrak{Y}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$:

$$y'(\tau) * y'(\sigma) \;=\; y'(\tau+\sigma) = \,^{-\sigma}y'(\tau); \tag{2.7.2}$$
$$x'(\tau) * y'(\sigma) \;=\; x'(\tau+\sigma) = \,^{-\sigma}x'(\tau); \tag{2.7.3}$$
$$x'(\tau) * y'(\sigma) \;=\; (x'_{-\sigma_\phi} \circledast x')(\tau); \tag{2.7.4}$$
$$x'(\tau) * y'(\sigma) \;=\; (x'_\phi \circledast {}^{-\sigma}x')(\tau); \tag{2.7.5}$$
$$x'(\tau) * y'(\sigma) \;=\; (x'_\phi \circledast {}^{-\sigma}x')(\tau) - (x'_{-\sigma_\phi} \circledast x')(\tau), \tag{2.7.6}$$

where $\phi \in Lap(\lambda)$(that is, ϕ is a scalar function Laplace transformable at λ); $\lambda > 0$ and $\sigma, \tau > 0$; $^{-\sigma}\phi : \tau \mapsto \phi(\tau+\sigma)$ if $\tau+\sigma > 0$ is the shifted scalar function; $x'_{-\sigma_\phi}$ is defined as in Section 2.6.3 ;$^{-\sigma}x' : \tau \mapsto x'(\tau+\sigma)$ if $\tau+\sigma > 0$ is the translated function. We call $\mathfrak{Y}'$ a *star-semigroup* should (2.7.2) hold. Likewise, we call the double family $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ a *star-empathy* and *integrated star empathy* should (2.7.3) and (2.7.6) hold respectively.

It is worth noting that in the case where G is the trivial group, the *star-semigroup relation* (2.7.2) is the traditional semigroup. Similarly, the *star-empathy relation* (2.7.3) is similar to the empathy relation discussed in Section 1.1; the product $*$ playing the role of composition of operators in [28]. Before long we shall show that the similarity goes deeper. The relation (2.7.6) will be called the *integrated star-empathy.* Its equivalent form is:

$$\langle f, x'(\tau) * y'(\sigma) \rangle = \textstyle\int_0^\tau \langle f, x'(\sigma+\eta) - x'(\eta) \rangle \, d\eta, \tag{2.7.7}$$

where ϕ is the scalar constant $K_1 : t \mapsto 1$ function. The previous ones, (2.7.2) and (2.7.3), should of course also be interpreted in terms of evaluations at $f \in \Phi$. It should also be noted that the relation (2.7.7) presupposes integrability of $\tau \mapsto \langle f, x'(\tau) \rangle$ over finite subintervals of $\mathbb{R}^+$.

The Laplace transform derived the resolvent equation of empathy theory from the empathy causal relation (equation (1.1.1), Section 1.1). Suppose $\mu \neq \lambda$. First take the Laplace transform, $\mathcal{L}_\mu d\tau$, at parameter μ of the causal relation and then, $\mathcal{L}_\lambda d\sigma$, at parameter λ, [28]. The framework $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}_B)$ captures this technique as the product of Laplace transforms of unlike parameters.

Proposition 4. *Suppose $\mu \neq \lambda$. Let $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$ and $\mathfrak{Y}' \in Lap(\mu, \mathscr{A}_B)$, where $\mathfrak{X}'$ is Laplace-closed with respect to $\mathfrak{Y}'$. Then*

$$\widehat{x'}(\lambda) * \widehat{y'}(\mu) = \mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau (x'(\tau) * y'(\sigma)), \tag{2.7.8}$$

where $\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau (x'(\tau) * y'(\sigma)) := \int_0^\infty e^{-\lambda\sigma} d\sigma \int_0^\infty e^{-\mu\tau}(x'(\tau) * y'(\sigma)) d\tau$

Proof. This is immediate from the definition of Laplace-closedness. □

Theorem 12 (General Resolvent Equation)**.** *Suppose $\mu \neq \lambda$. Let $\mathfrak{X}', \mathfrak{Y}' \in Lap(\lambda, \mathscr{A}_B)$ for all $\lambda > 0$ and $\mathfrak{X}'$ be Laplace-closed with respect to $\mathfrak{Y}'$.*

If $x'(\tau) * y'(\sigma)$ is a linear combination of ${}^{-\sigma}x'(\tau)$ and $(x'_\phi \circledast \ {}^{-\sigma}x') - (x'_{-\sigma_\phi} \circledast x')(\tau)$, then $\widehat{x'}(\lambda) * \widehat{y'}(\mu)$ is a linear combination of $\widehat{x'}(\lambda)$ and $\widehat{x'}(\mu)$; $\phi \in Lap(\lambda) \bigcap Lap(\mu)$.

Proof. We assume initially that $\lambda > \mu$. We compute $\widehat{x'}(\lambda) * \widehat{y'}(\mu)$ directly as $\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau$ of the right hand side terms, which only involve the family x'; invoking the appropriate Laplace transform theorems of Section 2.6.3 dictated by the right hand side terms of (2.7.3) - (2.7.5):

$$\begin{aligned}
\mathcal{L}_\mu d\tau \ {}^{-\sigma}x'(\tau) &= e^{\mu\sigma}\widehat{x'}(\mu) - (x'_{e^{\mu\cdot}} \circledast x')(\sigma); & (2.7.9)\\
\mathcal{L}_\mu d\tau (x'_\phi \circledast {}^{-\sigma} x')(\tau) &= \widehat{\phi}(\mu)[e^{\mu\sigma}\widehat{x'}(\mu) - (x'_{e^{\mu\cdot}} \circledast x')(\sigma)]; & (2.7.10)\\
\mathcal{L}_\mu d\tau (x'_{-\sigma_\phi} \circledast x')(\tau) &= [e^{\mu\sigma}\widehat{\phi}(\mu) - (\phi_{e^{\mu\cdot}} \circledast \phi)(\sigma)]\widehat{x'}(\mu), & (2.7.11)
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau \ {}^{-\sigma}x'(\tau) &= -\frac{1}{\lambda - \mu}(\widehat{x'}(\lambda) - \widehat{x'}(\mu)); & (2.7.12)\\
\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau (x'_\phi \circledast^{-\sigma} x')(\tau) &= -\frac{1}{\lambda - \mu}\widehat{\phi}(\mu)(\widehat{x'}(\lambda) - \widehat{x'}(\mu)); & (2.7.13)\\
\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau (x'_{-\sigma_\phi} \circledast x')(\tau) &= -\frac{1}{\lambda - \mu}(\widehat{\phi}(\lambda) - \widehat{\phi}(\mu))\widehat{x'}(\mu). & (2.7.14)
\end{aligned}$$

The right hand side of equation (2.7.6) is a *linear combination* of the right hand side terms of (2.7.4) - (2.7.5). For this case, by the linearity of $\mathcal{L}_\lambda d\sigma \mathcal{L}_\mu d\tau$,

$$\widehat{x'}(\lambda) * \widehat{y'}(\mu) = -\frac{1}{\lambda - \mu}(\widehat{\phi}(\mu)\widehat{x'}(\lambda) - \widehat{\phi}(\lambda)\widehat{x'}(\mu)). \tag{2.7.15}$$

The case of $\mu > \lambda$ is treated by reversing the roles of μ and λ. This role reversal does not change the right hand side of (2.7.12) and (2.7.15). Hence, (2.7.12) and (2.7.15) is valid for $\mu \neq \lambda$. □

Therefore, the Laplace transform theorems and the functional equations form a potent computational tool in evaluating $\widehat{x'}(\mu) * \widehat{y'}(\lambda)$.

Corollary 6. *Assume the hypothesis of Theorem 12. In the special cases of (i) $x'(\tau) * y'(\sigma) = {}^{-\sigma}x'(\tau)$ and (ii) $x'(\tau) * y'(\sigma) = (x'_\phi \circledast {}^{-\sigma}x')(\tau) - (x'_{-\sigma\phi} \circledast x')(\tau)$*

$$\widehat{x'}(\lambda) * \widehat{y'}(\mu) = -\frac{1}{\lambda - \mu}(\widehat{x'}(\lambda) - \widehat{x'}(\mu)); \tag{2.7.16}$$

$$\frac{\widehat{x'}(\lambda)}{\widehat{\phi}(\lambda)} * \frac{\widehat{y'}(\mu)}{\widehat{\phi}(\mu)} = -\frac{1}{\lambda - \mu}(\frac{\widehat{x'}(\lambda)}{\widehat{\phi}(\lambda)} - \frac{\widehat{x'}(\mu)}{\widehat{\phi}(\mu)}), \tag{2.7.17}$$

respectively.

Corollary 7. *Consider the the canonical families $\mathfrak{X}', \mathfrak{Y}'$ of Corollary 4. Then*

$$\widehat{x'}(\lambda) * \widehat{y'}(\mu) = -\frac{1}{\lambda - \mu}(\widehat{x'}(\lambda) - \widehat{x'}(\mu)). \tag{2.7.18}$$

Remark 6. *Resolvent equations are a result of the theory of Fréchet spaces. Indeed, the Fréchet test space $\Phi = C(G, Z)$ is a very rough test space by the standards of test spaces in distribution theory.*

Remark 7. *The assumptions of the general resolvent equations are weaker than that of the general convolution theorem (Section 2.6, Theorem 9) since one of the factors of the convolution only involves a 'scalar' function (see Corollary 5).*

2.8 Applications to Resolvent Equations of Classical theory

We now capture the resolvent equations of classical theory (C_0-semigroups, intertwined empathy of the implicit evolution equation and n-times integrated semi-groups) into our new framework $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$. We have already indicated that for the case where the group G is trivial, the framework developed here is precisely the framework for one-parameter semigroups of bounded linear operators in the Banach space Z.

The fairly recent theory of empathy [28], as described in the introduction involving two Banach spaces X and Y and two families $S(\tau) : Y \to X$ and $E(\tau) : Y \to Y$, seems to be quite different. This can be forged into the present consideration by letting $Z = X \times Y$ and defining the operators $s'(\tau)$ and $e'(\tau)$ as follows: We represent the elements of Z as $z = (x, y)$

(with obvious notation). Then, we isometrically embed evolution operators $S(\tau) : Y \to X$ and $E(\tau) : Y \to Y$ as $Z \to Z$ operators $s'(\tau) : z \mapsto (S(\tau)(y), 0)$ and $e'(\tau) : z \mapsto (0, E(\tau)(y))$, respectively. The empathy relation $S(\tau + \sigma) = S(\tau)E(\sigma)$ then transfers as $s'(\tau + \sigma) = s'(\tau)e'(\sigma)$. We take $S(\tau), E(\tau)$ to always mean $s'(\tau), e'(\tau)$. Therefore, *we work in a single state space Z as opposed to two state spaces X, Y.* Formally,

Convention 1 (Operator Identification)**.** *Let A be any operator from $U \to V$ where U, V are subspaces of X, Y, respectively. We identify A with the operator $a' : Z \to Z$ where $a'(x, y) := (0, Ax)$ if $x \in U$ and $(0, 0)$ if $x \notin U$. The other combinations of the subspaces being in Y, X, or Y, Y or X, X respectively are done similarly.*

Evidently, for more general groups G used in the test space Φ, these causal relations go far beyond the classical. In particular, the new framework $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ gives the convolution a central role by virtue of a fully developed convolution theorem (Section 2.6, Theorem 9). Thus we efficiently implement Sauer's approach of giving convolution a central role in the analysis of implicit evolution equations: a single general resolvent equation in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ captures all the resolvent equations spanning C_0-semigroups, intertwined empathy of the implicit evolution equation and n-times integrated semi-groups.

2.8.1 Resolvent Equations of Empathy Theory

We capture the causal relations of empathy theory into our new framework $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ as follows: (i) We associate homomorphisms $x'(\tau), y'(\tau) \in \mathscr{A}_B$ to each bounded evolution operators $S(\tau), E(\tau)$ by the canonical mapping between operators on Z and homomorphisms (Section 2.1.1); (ii) We show that the resulting pair of time dependent algebra valued families $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is integrable near the origin: we call $\mathfrak{X}', \mathfrak{Y}'$ the $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$-analogue of the evolution family $\mathcal{S}, \mathcal{E}$ respectively. (iii) We show that $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ replaces $\langle \mathcal{S}, \mathcal{E} \rangle$ of [28] as far as causal relations are concerned. (iv) We invoke the machinery of $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ on $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ to derive the fundamental properties of $\langle S, E \rangle$.

We now consider *causal relations* such as the semigroup, empathy (Section 2.7) for the pair of canonical families $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ with the standard composition of operators replaced by the product of homomorphisms. The resulting relations are logically equivalent to the standard causal relations by virtue of $*$ being a generalized composition (Section 2.1):

Proposition 5 (Causal relations in $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$)**.** *Let $\mathfrak{X}'$ and $\mathfrak{Y}'$ be the* *associated canonical families of $\mathcal{S}$ and $\mathcal{E}$ respectively. Then,*

$$x'(\tau) * y'(\sigma) = x'(\sigma) * y'(\tau) \Leftrightarrow S(\tau)E(\sigma) = S(\sigma)E(\tau); \tag{2.8.1}$$

$$y'(\tau) * y'(\sigma) = y'(\sigma) * y'(\tau) \Leftrightarrow E(\tau)E(\sigma) = E(\sigma)E(\tau), \tag{2.8.2}$$

and for causal relations,

$$x'(\tau) * y'(\sigma) = x'(\tau+\sigma) \Leftrightarrow S(\tau)E(\sigma) = S(\tau+\sigma); \tag{2.8.3}$$

$$y'(\tau) * y'(\sigma) = y'(\tau+\sigma) \Leftrightarrow E(\tau)E(\sigma) = E(\tau+\sigma). \tag{2.8.4}$$

Proof. Recall that $x'(\tau) : f \mapsto S(\tau)[\theta_0(f)]$ and $y'(\tau) : f \mapsto E(\tau)[\theta_0(f)]$. Therefore, ${}^{y'(\tau)}f : \sigma \mapsto E(\tau)f(\sigma)$ and $x'(\tau) * y'(\sigma) : f \mapsto S(\tau)E(\sigma)f(0)$. Consequently, the necessary conditions of equations (2.8.1) - (2.8.4) is immediate. The sufficient condition of equations (2.8.1)-(2.8.4) follows from noting that the constant functions $K_{(0,y)} : t \mapsto (0, y)$ belongs to the test space Φ; $y \in Y$. □

Remark 8. *Homomorphisms act on the function space Φ as opposed to evolution operators acting on the Banach space Z. The constant function $K_z \in \Phi$ plays the role of the state $z = (0, y) \in Z$; $K_z : t \mapsto z$ [Proposition 5].*

The Laplace transformability of the pair of canonical families $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is *logically equivalent* conditions to that of empathy theory (Corollary 2, Section 2.4.1).

Proposition 6 (Equivalent Notions of Laplace Transformability)**.** *Let $\mathfrak{X}'$ and $\mathfrak{Y}'$ be the $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$-analogue of the evolution family $\mathcal{S}$ and $\mathcal{E}$ respectively of [28]. Then,*

$\mathcal{S}$ is Laplace transformable if and only if $\mathfrak{X}'$ is $\mathscr{A}_B$-Laplace transformable

Proof. By direct computation, the operator $\widehat{x'}(\lambda) : f \mapsto P(\lambda)[\theta_0(f)]$, in analogy to $\widehat{x'_\phi}(\lambda) = \widehat{\phi}(\lambda)\theta_0$ if $\phi \in Lap(\lambda)$. Therefore, ${}^{\widehat{x'}(\lambda)}f : \tau \in \mathbb{R} \mapsto P(\lambda)f(\tau)$. Consequently, the sufficient condition follows from noting that $P(\lambda)$ is bounded [Theorem 3.8.2 [19]]. The converse follows from $\Phi \supset \{K_{(0,y)} | y \in Y\}$. □

Remark 9. *Proposition 6 applies equally well to the other evolution family $\mathcal{E}$. Therefore, we will not make explicit references to $\mathcal{E}$ unless need arises.*

We invoke the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ (Corollary 7, Section 2.7) for the pair $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ to end up with the resolvent equations (6), (7) of [28] (recall our identification of states with constant functions). Formally,

Corollary 8 (The Resolvent Equations for $\langle \mathcal{S}, \mathcal{E} \rangle$)**.** *Consider $\langle \mathcal{S}, \mathcal{E} \rangle$ of [28]. The resolvent equations (6), (7) of [28] are special cases of the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$* [8].

Proof. Fix $f := K_{(0,y)}$. Then $\widehat{x'}(\lambda) * \widehat{y'}(\mu)(f) = P(\lambda)R(\mu)y$; $\widehat{y'}(\lambda) * \widehat{y'}(\mu)(f) = R(\lambda)R(\mu)y$. □

Therefore, the pair $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ replaces the pair $\langle \mathcal{S}, \mathcal{E} \rangle$ for all purposes (Propositions 5 - 6, Corollary 8). The pair $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ has $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ as its underlying structure; the general convolution theorem and the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ is thus available.

2.8.2 Resolvent Equation of n-times Integrated Empathy

The machinery of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ generates the functional equations and the fundamental resolvent equations of the integrated semi-groups. Integrated semi-groups are semigroups that result from applying a convolution transform on a given semigroup. Hence they embody the concept of a moving average of the entire history of the given semi-group. In general, the convolution product smoothes [9] and therefore, the resulting integrated semi-group is smoother than the given semigroup. We define the *convolution transform* $\mathfrak{X}''$, of $\mathfrak{X}'$ with K' as the kernel of the transform as:

$$\mathfrak{X}''(\tau) = \int_{(0,\infty)} K'(\tau - \sigma) * \mathfrak{X}'(\sigma) d\sigma = (K' \circledast \mathfrak{X}')(\tau). \tag{2.8.5}$$

Thus we replace the product $\cdot$ of the convolution transform in $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$ [10] with $*$ to arrive at our definition of a convolution transform in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$.

Consider $\mathfrak{X}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$. Then fixing a $\mathfrak{K}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ we

[8] The resolvent equations for C_0-semigroups are proved similarly since the Lebesgue integral generalizes the Riemann integral.

[9] Loosely speaking, the convolution product of two functions is at least as nice a function as either of its factors, that is, the better of the factors. For example, the convolution of two discontinuous indicator functions is a continuous function. In a sense, a convolution product with a nice enough function can be a well-defined function even when the other factor is not a function in the classical sense. For example, this happens when that "other factor" is Dirac's δ distribution which is, almost by definition, the neutral element for the convolution operation:

$$\delta \circledast f = f \circledast \delta = f.$$

[10] The Laplace Transform $x''(\tau) := \int_{(0,\infty)} e^{-\tau\sigma} x'(\sigma) d\sigma$ of a function x in $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$ is a convolution transform after a suitable change of variables; $x''(\tau) = e^{\tau} x(e^{-\tau})$; $k'(\tau) = e^{\tau} exp(-e^{\tau})$; $x'(\sigma) = x(e^{-\sigma})$; $\tau > 0$ (Section 7, Example A [32])

transform $\mathfrak{X}'$ into another $\mathscr{A}_B$-valued function $\mathfrak{X}'' := \mathfrak{K}' \circledast \mathfrak{X}'$, where

$$\begin{aligned} x''(\tau) \quad &:= \int_{(0,\infty)} k'(\tau - \sigma) * x'(\sigma) d\sigma \\ &= \int_0^\tau k'(\tau - \sigma) * x'(\sigma) d\sigma = (k' \circledast x')(\tau), \end{aligned} \tag{2.8.6}$$

where $\tau > 0$. The map $\mathfrak{X}' \mapsto \mathfrak{X}''$ is called the convolution transform with kernel $\mathfrak{K}'$ of $\mathfrak{X}'$ into $\mathfrak{X}''$. Should $\mathfrak{X}''$ be in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ one can then view the transform Φ as transforming one evolution family $\mathfrak{X}'$ into another evolution family $\mathfrak{X}''$.

Consider the simplest case of a scalar valued kernel $\mathfrak{K}' := x'_\phi$ where $\phi \in Lap(\lambda)$ for $\lambda > 0$. Let $\mathfrak{X}', \mathfrak{Y}'$ denote the associated canonical families of the double family $\langle \mathcal{S}, \mathcal{E} \rangle$ respectively of empathy theory. Then the convolution transform is well defined by Section 2.6.3, Corollary 5 and it takes the form

$$x''(\tau) : f \mapsto \int_0^\tau \phi(\tau - \sigma) S(\sigma) f(0) d\sigma.$$

In particular, set ϕ_n to be the power function $\phi_n(\tau) := \frac{\tau^{n-1}}{(n-1)!}$ where $n \in \mathbb{N}^+$. We then call the convolution transform $\mathfrak{X}''_n := x'_{\phi_n} \circledast \mathfrak{X}'$ the n-times integrated family [11] of $\mathfrak{X}'$; the same applies equally well to the other family $\mathfrak{Y}'$ so we do not mention $\mathfrak{Y}''$.

The machinery of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ generates the causal relation of the n-times integrated empathy $\langle \mathfrak{X}''_n, \mathfrak{Y}''_n \rangle$ (equation (2.7.6), Section2.7) from the causal relation $x'(\tau) * y'(\sigma) = x'(\tau + \sigma)$ of the empathy family $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$.

Proposition 7. *Let $\mathfrak{X}', \mathfrak{Y}'$ be the $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$-analogue of the evolution family $\mathcal{S}, \mathcal{E}$ respectively of empathy theory. Then the functional equation*

$$x''_n(\tau) * y''_n(\sigma) = (x'_{\phi_n} \circledast^{-\sigma} x''_n)(\tau) - (x'_{-\sigma\phi_n} \circledast x''_n)(\tau). \tag{2.8.7}$$

of the n-times integrated empathy $\langle \mathfrak{X}''_n, \mathfrak{Y}''_n \rangle$ follows from the functional equation $x'(\tau) * y'(\sigma) = x'(\tau + \sigma)$ of the empathy family $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$.

Proof. The proof follows by direct computation. Let $f \in \Phi$. Then,

$$\begin{aligned} \langle f, x''_n(\tau) * y''_n(\sigma) \rangle \quad &= \int_0^\tau dt \int_0^\sigma \phi_n(\tau - t)\phi_n(\sigma - s) S(t) E(s) f(0) ds \quad (2.8.8) \\ &= \int_0^\tau \phi_n(\tau - t) d\tau \int_0^\sigma \phi_n(\sigma - s) S(s + t) f(0) ds \quad (2.8.9) \end{aligned}$$

[11] If we set $n = 1$, then ϕ_1 is the constant $\mathbf{1}$ function so the convolution transform $\mathfrak{X}''(\tau)$ is the *integrated empathy semigroup* $\int_0^\tau x'(\sigma) d\sigma : f \mapsto \int_0^\tau S(\sigma) f(0) d\sigma$.

By the change of variables $s' = s+t$, the integral in equation (2.8.9) becomes

$$\int_0^\tau \phi_n(\tau - t)d\tau \int_t^{\sigma+t} \phi_n(\sigma + t - s')S(s')f(0)ds'. \tag{2.8.10}$$

But $\int_t^{\sigma+t} = \int_0^{\sigma+t} - \int_0^t$ so that $\int_t^{\sigma+t} \phi_n(\sigma+t-s')S(s')f(0)ds'$ is

$$x''_n(\sigma + t)f - (x'_{-\sigma_{\phi_n}} \circledast x')(t)f. \tag{2.8.11}$$

Denoting $(x'_{-\sigma_{\phi_n}} \circledast x')(t)f$ by $F(t)$, the computation ends on noting that

$$\int_0^\tau \phi_n(\tau - t)x''_n(\sigma + t)fdt = (x'_{\phi_n} \circledast^{-\sigma} x''_n)(\tau). \tag{2.8.12}$$

and

$$\begin{aligned}
\int_0^\tau \phi_n(\tau - t)F(t)dt &= (x'_{\phi_n} \circledast F)(\tau) = (x'_{\phi_n} \circledast x'_{-\sigma_{\phi_n}} \circledast x')(\tau) \\
&= (x'_{-\sigma_{\phi_n}} \circledast x'_{\phi_n} \circledast x')(\tau) \\
&= (x'_{-\sigma\,\phi_n} \circledast x''_n)(\tau).
\end{aligned}$$

□

A *single* application of Section 2.6.3, Corollary 5 to $x''_n := x'_{\phi_n} \circledast x'$, as opposed to integrating by parts n-times on p. 336 [1], yields the relation

$$\widehat{x''_n}(\lambda) = \frac{\widehat{x'}(\lambda)}{\lambda^n},$$

since x'_ϕ is Laplace-closed with respect to x'. The convolution of unlike parameters yields

$$\begin{aligned}
\widehat{x''_n}(\lambda) * \widehat{y''_n}(\mu) &= \frac{1}{\lambda^n \mu^n}\widehat{x'}(\lambda) * \widehat{y'}(\mu) &(2.8.13)\\
&= -\frac{1}{\lambda - \mu}\frac{\widehat{x'}(\lambda) - \widehat{x'}(\mu)}{\lambda^n \mu^n}, &(2.8.14)
\end{aligned}$$

by virtue of the resolvent equation for the empathy family $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$. Thus,

$$\begin{aligned}
\frac{\widehat{x''_n}(\lambda)}{\widehat{\phi}_n(\lambda)} * \frac{\widehat{y''_n}(\mu)}{\widehat{\phi}_n(\mu)} &= -\frac{1}{\lambda - \mu}(\widehat{x'}(\lambda) - \widehat{x'}(\mu)) &(2.8.15)\\
&= -\frac{1}{\lambda - \mu}(\frac{\widehat{x''_n(\lambda)}}{\widehat{\phi}_n(\lambda)} - \frac{\widehat{x''_n}(\mu)}{\widehat{\phi}_n(\mu)}). &(2.8.16)
\end{aligned}$$

Writing $P_n(\lambda) := \lambda^n \widehat{x'}(\lambda)$ and $R_n(\lambda) := \lambda^n \widehat{y'}(\lambda)$, we have the resolvent equation of n-times integrated empathy. We now formally define an n-times integrated empathy as follows: a double family $\langle \mathfrak{X}'_n, \mathfrak{Y}'_n \rangle$ is an n-times integrated empathy should the functional equation

$$x'_n(\tau) * y'_n(\sigma) = (x'_{\phi_n} \circledast^{-\sigma} x'_n)(\tau) - (x'_{-\sigma\,\phi_n} \circledast x'_n)(\tau),$$

hold. The general resolvent equation (Theorem 12 (2.7.17)) immediately yields

$$\widehat{x''_n}(\lambda) * \widehat{y''_n}(\mu) = \widehat{x''_n}(\mu) * \widehat{y''_n}(\lambda) = -\frac{1}{\lambda - \mu}(\widehat{x''_n}(\lambda) - \widehat{x''_n}(\mu)),$$

where $\widehat{x''_n}(\lambda) = \lambda^n \widehat{x'_n}(\lambda), \widehat{y''_n}(\lambda) = \lambda^n \widehat{y'_n}(\lambda)$. We end up with the pseudo resolvent equations on n-times integrated semigroups of [5] once we identify the state $y \in Y$ with the constant function $K_{(0,y)} \in \Phi$, and restrict the above operators to these functions.

Theorem 13 (Pseudo-Resolvent Equations for the integrated empathy)**.** *Let $\langle \mathcal{S}, \mathcal{E} \rangle$ be an n-times integrated empathy : definition (11.1) [5]. Then the pseudo-resolvent equations (3.5) and (3.8) of [5] are special cases of the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$. The pseudo-resolvent equations (3.5) and (3.8) follow from the functional equation (11.1), [5].*

Remark 10 (Invertible resolvent operators)**.** *Note that the resolvent equations (6), (7) of [28] follow from the functional equations (2.8.3) - (2.8.4) of empathy theory. The functional equation (2.8.4) follows from the strong requirement that the resolvent operator $P(\lambda)$ be invertible for some fixed λ* [12]

2.9 A Comparison With Bilinear Formed Vector Convolution Theorems

Let E, F and G be three Banach spaces with the bounded bilinear map $\cdot : E \times F \to G | (u,v) \mapsto u \cdot v$. Let f, g denote time dependent E and F - valued functions so that the ordinary convolution $f * g$ is analogously defined as $(f * g)(\tau) := \int_0^t f(t-s) \cdot g(\sigma) ds$, where the integral is the Bochner integral over $(0, \tau)$. We write $f \in Lap(\lambda, E)$ whenever $\hat{f}(\lambda) := \int_{(0,\infty)} e^{-\lambda\tau} f(\tau) d\tau \in E$, the integral being defined in the sense of an E-valued Bochner integral.

Proposition 8. *Suppose that $f \in Lap(\lambda, E)$ and $g \in Lap(\lambda, F)$. If in addition $|f|, |g| \in Lap(\lambda)$, then the $G-$ valued Laplace transform of $f * g \in Lap(\lambda, G)$ where*

$$\widehat{f * g}^{\lambda} = \widehat{f}^{\lambda} \cdot \widehat{g}^{\lambda}.$$

Remark 11. *For the bounded bilinear form $\cdot : (S(\tau), y) \in \mathcal{L}(Y, X) \times Y \mapsto S(\tau) y \in X$, Proposition 8 requires $\mathcal{S}$ to be locally measurable in the uniform*

[12] The double family $\langle \mathcal{S}, \mathcal{E} \rangle$ is called an *empathy* when this condition is combined with the empathy relation $S(\tau+\sigma) = S(\tau)E(\sigma)$ or its logically equivalent condition $x'(\tau+\sigma) = x'(\tau) * y'(\sigma)$. This combination is potent: $E(\tau+\sigma) = E(\tau)E(\sigma)$ and $P(\lambda)$ is invertible for all $\lambda > 0$ then follows.

norm in order for $\mathcal{S} \in Lap(\lambda, \mathcal{L}(Y, X))$. *Indeed in these frameworks, Laplace transformability will require the over-restrictive requirement of uniform exponential boundedness. This is is too strong an assumption for empathy theory [28].*

Theorem 13 of Section 2.8.2, demonstrates that our full vectorization of the scalar convolution theorem requires no such additional assumptions.

Chapter 3

Normed star-semigroups: Banach Test Space

In Chapter 2, time-dependent families of homomorphisms replaced time-dependent families of bounded evolution operators as far as resolvent equations are concerned. For the Fréchet test space $\Phi := C(G, Z)$, bounded homomorphisms of $\mathscr{A}_B$ do not have a norm like the bounded evolution operators they replace. For this reason, in this chapter, we take our test space Φ to be $BUC(G, Z)$, the Banach subspace of bounded uniformly continuous test functions with the supremum norm; G is a locally compact topological group. Then like the previous test space, Φ is (i) Fréchet, (ii) every bounded $\Phi \to Z$ homomorphism is Φ-admissible and (iii) the map $p \in G \mapsto f_{-p} \in \Phi$ is continuous for each fixed $f \in \Phi$ (Theorem 1, Section 3.1).

We re-study time-dependent members of $\mathscr{A}_B$, that is, time-dependent *normed* homomorphisms, as a replacement of traditional families of evolution operators to show the advantage of working with normed admissible homomorphisms. Specifically, we study the behaviour of time-dependent normed homomorphisms near $\tau = 0$ along the lines set out in [19] (Section 3.2.2). Under special cases, normed star semigroups are isometric representations of the double family of evolution operators (Section 3.2.3).

We get sharper results by studying $\mathfrak{X}' = \{x'(\tau) \in \mathscr{A}_B | \tau > 0\}$ in conjunction with its isometric counterpart $X' := \{\Gamma x'(\tau) \in \mathscr{A}'_B | \tau > 0\}$; $\mathscr{A}'_B := \Gamma[\mathscr{A}_B]$; Γ is defined as in equation (2.1.9), Section 2.1. In Section 3.4.1, we study the measurability of $\mathfrak{X}'$ (that is, the measurability of the map $\tau \mapsto x'(\tau)f$) in conjunction with the measurability of its isometric function space valued counterpart X' (that is, the measurability of the map $\tau \mapsto X'(\tau)f)$. It is evident that the measurability of X' implies the measurability of $\mathfrak{X}'$. We show that the converse is true under suitable conditions on G, provided $\mathfrak{X}'$ is a family of bounded homomorphisms uniformly bounded in norm on compact intervals. Such $\mathfrak{X}'$ are called d-measurable. Indeed, if G is Lindelöff, the interplay between dualism and uniform boundedness theorem establishes the topological measurability of X' (Lemma 2, Section 3.4.1). The resulting equicontinuity is then expressed as separability by strengthening G to be compact and metrizable (Theorem 10, Section 3.4.1). Thus, the metric-topological structure of the group G sidesteps the need to study the complex nature of the duals of spaces of vector valued continuous functions in the study of the measurability of the function space valued map $\tau \mapsto X'(\tau)f$. If we additionally assume that $\tau \mapsto \|X'(\tau)f\|_\infty$ is integrable for each $f \in \Phi$, then the induced integral is a Φ-valued Bochner integral (Proposition 2, Section 3.4.2). Then, as in the classical case, every bounded homomorphism is closed over I with respect to $\mathfrak{X}'$ (Corollary 1, Section 3.4.2). This sharper closedness condition for $\mathfrak{X}'$ shows that closedness is related to the problem of the transfer of measurability of $\mathfrak{X}'$ to its isometric counterpart X'.

Indeed, the strong integrability of $\mathfrak{X}'$ induces an 'integral' on the operator valued family X'. We call this the induced integral or p-integral (Section 3.4.3). In fact, the strong integrability of $\mathfrak{X}'$ over an interval I exactly happens when the dual family X' is p-integrable over I (Section 3.4.3, Proposition 3). We let $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ denote the class of X' p-integrable over any $I \subset \mathbb{R}^+$. Therefore, the dualism induces a new equivalent framework $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ of time-dependent operator valued families. We then construct an appropriate Laplace transform of these families which will allow the dualism to also transfer the general convolution theorem and the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ word for word into $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ (Theorem 11, Section 3.4.3).

In classical semigroup theory, every measurable semigroup is strongly continuous. This is also the case for every measurable d-normed star semigroup (Theorem 12, Section 3.5). For strongly continuous star semigroups $\mathfrak{X}'$, the group G need only be locally compact for the dual X' to be strongly continuous. Therefore, if $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is the canonical family of homomorphisms associated with the strongly continuous pair $\langle \mathcal{S}, \mathcal{E} \rangle$, then the dual (operator) pair $\langle X', Y' \rangle$ is an isometric identification with $\langle \mathcal{S}, \mathcal{E} \rangle$ (Proposition 5, Section 3.6).

3.1 Banach Test Space

In our initial translation invariant test space of $\Phi = C(G, Z)$, Φ was (i) Fréchet, (ii) every bounded $\Phi \to Z$ homomorphism is Φ-admissible and, (iii) the map $p \in G \mapsto f_{-p} \in \Phi$ was continuous for each fixed $f \in \Phi$. The subspace $BC(G, Z)$ of bounded continuous functions is a Banach test space under the supremum norm. Unfortunately, condition (iii) is not met:

consider the case of $G = \mathbb{R}$ and $f(t) = [\sin t^2]z$, $z \in Z$. The even smaller supremum-normed test space $C_0(\mathbb{R}, Z)$ meets all these requirements except possibly (ii); for the Banach space $Z := \mathbb{C}$, (ii) is met ([18] Lemma 19.5); it is unknown to the author for which Banach spaces, condition (ii) is met. [1]

Consider the subspace $(\mathrm{BUC}(G, Z), \|\cdot\|_\infty)$ of bounded $(G, \mathcal{U}) \to (Z, \mathcal{V})$ uniformly continuous functions; $\mathcal{V}$ is the canonical uniformity induced by the norm on Z; $\mathcal{U}$ is the canonical uniformity induced by the topological group G. If the topological group G is locally compact, then $\mathrm{BUC}(G, Z)$ is Banach. Indeed, $(\mathrm{BUC}(G, Z), \|\cdot\|_\infty)$ meets *all* the requirements (i) - (iii). From this point onward, for the test space $\mathrm{BUC}(G, Z)$, we shall take G as a locally compact topological group. Note that $C_0(G, Z) \subset \mathrm{BUC}(G, Z)$. In what follows we shall take our test space to be $\Phi = \mathrm{BUC}(G, Z)$.

Theorem 1. *Let Φ be the Banach space $(BUC(G, Z), \|\cdot\|_\infty)$. Then the mappings $s \in G \mapsto f_s \in \Phi$; $f \in \Phi$ are uniformly continuous.*

3.2 Time-dependent Normed Homomorphisms

3.2.1 Strong Integrability

Every Banach space is a Fréchet space. Therefore, our version of a well-known theorem (Theorem 3.8.2, [19], p.85) (Theorem 6, Section 2.3.1) which is central to the study of strong integrability of $\mathfrak{X}'$ is also valid.

Theorem 2. *Suppose the functions $\tau \to \langle f, x'(\tau)\rangle$; $\tau \in (0, \infty)$; $f \in \Phi$ are (strongly) Lebesgue measurable in Z. If, for an interval $I \subset (0, \infty)$ the Bochner integral*

$$\langle f, x'_I \rangle = \int_I \langle f, x'(\tau)\rangle \, d\tau, \tag{3.2.1}$$

exists for every $f \in \Phi$, the homomorphism $x'_I : \Phi \to Z$ is bounded.

Likewise, we transfer its immediate consequence:

Theorem 3. *If for some $\lambda > 0$ the Laplace transform $\widehat{x'}(\lambda)$ exists, it is bounded and therefore, in $\mathscr{A}_B$. Additionally, the integrals x'_I are bounded for all finite intervals $I \subset (0, \infty)$: if $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$ then $\mathfrak{X}' \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$.*

3.2.2 Behaviour near the initial moment

Since the families of homomorphisms we consider are only defined for $\tau > 0$, we need to obtain information about their behaviour near $\tau = 0$. Our

[1] It would be interesting exercise to find conditions on the Banach space Z for which each bounded $x' \in \mathrm{Hom}(\Phi, Z)$ is admissible; $\Phi = (C_0(\mathbb{R}, Z), \|\cdot\|_\infty)$.

investigation will follow the lines set out in [19]. Central to the investigation is the following 'net' version of the uniform boundedness theorem:

Theorem 4. *(Corollary 7.1.6 [3]) Suppose that $A_\tau, \tau \in (0,1]$, is a family of bounded linear operators with common Banach space domain X such that for every $x \in X$, the limit $\lim_{\tau\to 0} A_\tau x$ exists. Then there exists a $\delta > 0$ such that $\sup_{0<\tau<\delta} \|A_\tau\|$ is finite.*

To begin with, if the limit $\langle f, i'\rangle := \lim_{\eta\to 0^+}\langle f, x'(\eta)\rangle$exists for every $f \in \Phi$, it follows from the uniform boundedness theorem (Theorem 4) that the homomorphism $i' : \Phi \to Z$ is bounded [2] and therefore, in $\mathscr{A}_B$. If this is the case, the family $\mathfrak{X}'$ is called C_0-*summable* (Class C_0).

In the second place, if it is assumed that for the interval $I = (0,1)$ the integral x'_I exists, we define the homomorphisms $j'(\eta)$ by

$$\langle f, j'(\eta)\rangle = \eta^{-1}\int_0^\eta \langle f, x'(\tau)\rangle\, d\tau = \int_0^1 \langle f, x'(\eta\xi)\rangle\, d\xi. \tag{3.2.2}$$

For $0 < \eta < 1$ this is well-defined and by Theorem 2, $j'(\eta) \in \mathscr{A}_B$ is bounded. If the limit $\langle f, j'\rangle := \lim_{\eta\to 0^+}\langle f, j'(\eta)\rangle$ exists for every $f \in \Phi$, it follows as before that the homomorphism $j' \in \mathscr{A}_B$ because it is bounded. In this case we say that $\mathfrak{X}'$ is C_1-*summable* (Class C_1).

Thirdly, if the Laplace transform $\widehat{x'}(\lambda)$ exists for some $\lambda > 0$, let $k'(\lambda)$ be defined by

$$\langle f, k'(\lambda)\rangle = \langle f, \lambda\widehat{x'}(\lambda)\rangle = \int_0^\infty \exp\{-\xi\}\langle f, x'(\xi/\lambda)\rangle\, d\xi; \quad f \in \Phi. \tag{3.2.3}$$

Theorem 3, ensures that these are in $\mathscr{A}_B$. If the limit $\langle f, k'\rangle = \lim_{\lambda\to\infty}\langle f, \lambda\widehat{x'}(\lambda)\rangle$ exists for every $f \in \Phi$, the homomorphism $k' : \Phi \to Z$ is once again admissible. If this is so, we call $\mathfrak{X}'$ *Abel summable* (Class A).

Under certain conditions the homomorphisms i', j' and k' are related.

Theorem 5. *If $\mathfrak{X}'$ is of Class C_0, it is of Class C_1 and Class A. Additionally, $i' = j' = k'$.*

Proof. We observe first that

$$\langle f, j'(\eta)\rangle = \int_0^1 \langle f, x'(\eta\sigma)\rangle\, d\sigma. \tag{3.2.4}$$

The uniform boundedness theorem applied to the integrand on the right ensures that the limit and the integrand can be switched. The other statement follows from (3.2.3) in a similar way. □

[2] $\|\langle f, i'\rangle\| = \lim_{\eta\to 0^+} \|\langle f, x'(\eta)\rangle\|$

The next is more technical. The family $\mathfrak{X}'$ is *exponentially bounded* if for every $\delta > 0$ there exist constants $M = M_\delta$ and $\omega = \omega_\delta \geq 0$ such that for all $\tau \geq \delta$, $\|x'(\tau)\| \leq M \exp\{\omega\tau\}$.

Theorem 6. *If $\mathfrak{X}'$ is exponentially bounded and of class C_1, then the Laplace transform $\widehat{x'}(\lambda)$ exists for sufficient large λ, $\mathfrak{X}'$ is Abel summable and $k' = j'$.*

Proof. First we notice that, because $\mathfrak{X}'$ is of class C_1 and is exponentially bounded, the estimate

$$\|\langle f, j'(\eta)\rangle\|_Z \leq \frac{1}{\delta}\int_0^\delta \|\langle f, x'(\tau)\rangle\| \, d\tau + \frac{M}{\omega\eta}\exp\{\omega\eta\}\|f\|_\Phi \quad \text{for } \eta \geq \delta, \tag{3.2.5}$$

holds true. The term $\frac{M}{\omega\eta}$ results from a direct integration of $\frac{1}{\eta}\int_\delta^\eta Me^{\omega t}\|f\|_\Phi \, d\tau$. In addition,

$$\frac{d}{d\eta}[\eta\langle f, j'(\eta)\rangle] = \langle f, x'(\eta)\rangle \quad \text{for almost all } \eta. \tag{3.2.6}$$

If we (formally) take the Laplace transform of (3.2.6), the result is

$$\langle f, \widehat{x'}(\lambda)\rangle = \lambda\int_0^\infty \eta \exp\{-\lambda\eta\}\langle f, j'(\eta)\rangle \, d\eta. \tag{3.2.7}$$

The boundary term at $\tau = 0$ vanishes because of the assumption that $\mathfrak{X}'$ is of Class C_1. The boundary term for large η vanishes because of (3.2.5) if $\lambda > \omega$. Similarly, the integral on the right of (3.2.7) exists. Hence the Laplace transform in question exists for $\lambda > \omega$.

From (3.2.7) we also see that

$$\langle f, k'(\lambda)\rangle = \lambda^2\int_0^\delta \eta \exp\{-\lambda\eta\}\langle f, j'(\eta)\rangle \, d\eta + \lambda^2\int_\delta^\infty \eta\exp\{-\lambda\eta\}\langle f, j'(\eta)\rangle \, d\eta. \tag{3.2.8}$$

Use of (3.2.5) shows that the second term on the right of (3.2.8) converges to zero when $\lambda \to \infty$. We also note that

$$\langle f, j'\rangle = \lambda^2\int_0^\delta \eta\exp\{-\lambda\eta\}\langle f, j'\rangle \, d\eta + \lambda^2\int_\delta^\infty \eta\exp\{-\lambda\eta\}\langle f, j'\rangle \, d\eta. \tag{3.2.9}$$

Once again the second term converges to 0 when $\lambda \to \infty$. Hence, a combination of (3.2.5) and (3.2.9) shows that

$$\langle f, k'(\lambda) - j'\rangle \sim \lambda^2\int_0^\delta \eta\exp\{-\lambda\eta\}\langle f, j'(\eta) - j'\rangle d\eta \quad \text{as } \lambda \to \infty. \tag{3.2.10}$$

By (3.2.10), $\langle f, k'(\lambda) - j'\rangle \to 0$ as $\lambda \to \infty$. To see this, first note that since $\mathfrak{X}'$ is of class C_1, for $\eta \in (0, \delta)$, there exists a constant K_δ such that

$$\|\langle f, j'(\eta) - j'\rangle\| < K_\delta. \tag{3.2.11}$$

Then note that for arbitrary $\epsilon > 0$, L'Hospital's rule ensures that for λ sufficiently large,

$$|\lambda^2 \eta \exp\{-\lambda\eta\}| < \frac{\epsilon}{K_\delta}. \tag{3.2.12}$$

Combining (3.2.11) and (3.2.12), for $\epsilon > 0$, there exists $\lambda_0(\epsilon)$, such that $\lambda > \lambda_0(\epsilon)$

$$\|\lambda^2 \eta \exp\{-\lambda\eta\}\langle f, j'(\eta) - j'\rangle\| < \epsilon, \tag{3.2.13}$$

for $\eta \in (0, \delta)$. Thus, the right hand side integral of (3.2.10) has norm bounded by $\epsilon\delta$; δ fixed. □

3.2.3 Normed star semigroups

We transfer the star causal relations of Section 2.7. The relations of particular interest are the *star-semigroup relation* (3.2.14), the *star-empathy relation* (3.2.15) and the *integrated star-empathy relation* (3.2.16).

$$y'(\tau) * y'(\sigma) = y'(\tau + \sigma); \quad \tau, \sigma \in \mathbb{R}^+; \; y' \in \mathscr{A}_B; \tag{3.2.14}$$

$$x'(\tau) * y'(\sigma) = x'(\tau + \sigma); \quad \tau, \sigma \in \mathbb{R}^+; \; x', y' \in \mathscr{A}_B. \tag{3.2.15}$$

$$\langle f, x'(\tau) * y'(\sigma)\rangle = \int_0^\tau \langle f, x'(\sigma + \eta) - x'(\eta)\rangle d\eta; \quad \tau, \sigma \in \mathbb{R}^+; \; x', y' \in \mathscr{A}_B. \tag{3.2.16}$$

These relations (3.2.14) - (3.2.16) generate the "resolvent equations" via the machinery of $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$ (Section 2.7). In these situations we shall use special notation for the Laplace transform such as $\mathfrak{r}(\lambda) = \widehat{y'}(\lambda)$, $\mathfrak{p}(\lambda) = \widehat{x'}(\lambda)$. It should be noted that $\mathfrak{r}$ and $\mathfrak{p}$, if they exist, represent bounded homomorphisms from Φ to Z. Formally,

Theorem 7. *Suppose that $y'(\tau)$ satisfies* (3.2.14) *and the Laplace transforms $\mathfrak{r}(\lambda)$ exist, for $\lambda \geq \lambda_0$. If $\mathfrak{Y}'$ is Laplace-closed with respect to itself, then*

$$\mathfrak{r}(\lambda) - \mathfrak{r}(\mu) = (\lambda - \mu)\mathfrak{r}(\lambda) * \mathfrak{r}(\mu), \tag{3.2.17}$$

for $\lambda, \mu \geq \lambda_0$.

Suppose that $x'(\tau), y'(\tau)$ satisfy (3.2.15) *and the Laplace transforms $\mathfrak{r}, \mathfrak{p}$ exist. If $\mathfrak{X}'$ is Laplace-closed with respect to $\mathfrak{Y}'$ then*

$$\mathfrak{p}(\lambda) - \mathfrak{p}(\mu) = (\lambda - \mu)\mathfrak{p}(\lambda) * \mathfrak{r}(\mu), \tag{3.2.18}$$

for λ, μ sufficiently large. For $x'(\tau), y'(\tau)$ that satisfy (3.2.16) *the corresponding expression is*

$$\mathfrak{P}(\lambda) - \mathfrak{P}(\mu) = (\lambda - \mu)\mathfrak{P}(\lambda) * \mathfrak{R}(\mu), \tag{3.2.19}$$

with $\mathfrak{P}(\lambda) = \lambda\mathfrak{p}(\lambda)$ and $\mathfrak{R}(\lambda) = \lambda\mathfrak{r}(\lambda)$.

Proof. Equation (3.2.19) follows from equation (2.7.7) of Section 2.7, and equation (2.7.17) of Corollary 6. □

Consider the special case of normed star-semigroups defined by the canonical mapping. Such normed star-semigroups are *isometric* representations of the double family of evolution operators: consider $x'(\tau) : f : S(\tau)\theta'_0 \in \mathscr{A}_B$; $\|x'(\tau)\| \leq \|S(\tau)\|$ follows from $\|S(\tau)f(0)\| \leq \|S(\tau)\|\|f(0)\| \leq \|S(\tau)\|\|f\|_\infty$; the reverse inequality is immediate on noting that for each z in the unit ball of Z the function zf is in the unit ball of $\mathrm{BUC}(G, Z)$, where f is in the unit ball of $\mathrm{BUC}(G, Z)$ such that $f(0) = 1$. This is the advantage of working with the sup-normed test space $\Phi = \mathrm{BUC}(G, Z)$ as opposed to the Fréchet test space $\Phi := C(G, Z)$. Before long we shall show that the similarity goes deeper: we will show that the corresponding dualisms $X'(\tau)$ and $Y'(\tau)$ are the known semigroup and empathy relations in the test space $\Phi = \mathrm{BUC}(G, Z)$.

3.3 Operator Norm

For our purposes, the choice of norm on the test space Φ is important. In order to meet two conditions: $(\mathscr{A}_B, *)$ is (i) a unital normed algebra which (ii) embeds isometrically as a *unital* normed subalgebra of the operator algebra $\mathrm{Hom}_B(\Phi, \Phi)$. As a non-example, $\Phi := L^p(\mathbb{R}, Z)$ with the L^p-norm fails to meet these requirements: the identity $\theta'_0 \in \mathscr{A}$ is non-closeable unbounded operator for $\Phi = L^p(\mathbb{R}, Z); 1 \leq p < \infty$. Conditions (i)-(ii) ensure that each homomorphism $x' \in \mathscr{A}_B$ embeds isometrically as a bounded operator on Φ. In such situations, the norm on $\mathscr{A}_B$ is called an *operator norm.* We show that the supremum norm, such as in $\Phi = (\mathrm{BUC}(G, Z), \|\cdot\|_\infty)$, induces operator norms on $\mathscr{A}_B$.

3.3.1 Dualism Norm: Supremum Normed Test Space

Consider the unital algebra $(\mathscr{A}_\Phi, *)$, where $\Phi := (L^p(\mathbb{R}, Z), \|\cdot\|_p)$. The identity functional $\theta'_0 \in \mathscr{A}_\Phi$ is badly behaved. However, its dualism $\mathbf{1}_\Phi$ is the bounded identity operator on Φ. The admissibility of θ'_0 allows us to re-norm θ'_0 with the dualism norm or d-norm $\|\cdot\|_\Gamma$: $\|\theta'_0\|_\Gamma := \|\mathbf{1}_\Phi\| = 1$. The d-norm is well defined since Γ is 1-1 [3] . Indeed, Γ is an algebra isomorphism by the Power Rule (Lemma 2, Section 1.3.1).

The admissible homomorphism $x' \in \mathscr{A}$ is *dualism-normed* (d-normed) if and only if $X' = \Gamma(x') \in \mathcal{L}(\Phi)$. We let $\mathscr{A}_C$ denote the set of all d-normed

[3]Injectivity follows from the relation$({}^{x'}f)(0) = \langle f, x'\rangle$; $x'_1 \neq x'_2$ if and only if there exists $f \in \Phi$ such that $\langle f, x'_1\rangle \neq \langle f, x'_2\rangle \Leftrightarrow ({}^{x'_1}f)(0) \neq ({}^{x'_2}f)(0) \Leftrightarrow ({}^{x'_1}f) \neq ({}^{x'_2}f) \Leftrightarrow x'_1 \neq x'_2$

admissible homomorphisms in $\mathscr{A}$. Then the dualism norm is an operator norm on $\mathscr{A}_C$.

Proposition 1. *The dualism norm is an operator norm on $\mathscr{A}_C$.*

Proof. It is easy to verify that $\mathscr{A}_C$ is a normed vector space under $\|\cdot\|_\Gamma$: $\|x'+y'\|_\Gamma := \|X'+Y'\| \leq \|X'\|+\|Y'\| = \|x'\|_\Gamma+\|y'\|_\Gamma$. Also note that $\Gamma(\theta_0') = 1_\Phi$. Finally, submultiplicativity of $\|\cdot\|_\Gamma$ follows from the submultiplicative nature of the operator norm $\|\cdot\|$: $\|x' * y'\|_\Gamma = \|\Gamma(x' * y')\| = \|X' \circ Y'\| \leq \|X'\|\|Y'\| = \|x'\|_\Gamma\|y'\|_\Gamma$ □

The norm of $\mathscr{A}_B$ is an operator norm once we show $\mathscr{A}_B = \mathscr{A}_C$ up to norms. Growth conditions on the translation operators [4] ensure that the subalgebra $\mathscr{A}_B \subset \mathscr{A}_C$.

Lemma 1. *Let $M := \sup_{s\in G}\{\|R^s\| | s \in G\} < \infty$. Then $\mathscr{A}_B \subset \mathscr{A}_C$. If additionally $M \leq 1$, then $\|x'\|_\Gamma \leq \|x'\|$.*

Proof. For each fixed $s \in G$, $\|\langle R^{-s}f, x'\rangle\| \leq \|x'\|\|R^{-s}f\|$. Hence, $\|X'\| \leq M\|x'\|$. The condition $\|x'\|_\Gamma \leq \|x'\| < \infty$ for all $x' \in \mathscr{A}_B$ ensures that $\mathscr{A}_B \subset \mathscr{A}_C$. □

The inclusion $\mathscr{A}_B \subset \mathscr{A}_C$ is strict for the test space $\Phi := L^1(\mathbb{R}, Z)$ [5]: although $M = 1$ (each translation operator R^τ is an isometry), the identity $\theta_0' \notin \mathscr{A}_B$.

A supremum normed test space ensures the reverse inclusion $\mathscr{A}_B \supset \mathscr{A}_C$ by the reverse inequality $\|x'\|_\Gamma \geq \|x'\| : |\langle f, x'\rangle| = |(X'f)(0)| \leq \|X'f\|_\infty$. Indeed, $\mathscr{A}_B = \mathscr{A}_C$ by Lemma 1: each translation operator R^{-s} is an isometry; $\|R^{-s}f\|_\infty = \|f\|_\infty$. Thus, the norm of $\mathscr{A}_B$ coincides with its d-norm.

Theorem 8. *Let the test space Φ be supremum-normed. Then the norm of $\mathscr{A}_B$ is an operator norm: the norm coincides with the dualism norm. Furthermore, the map $\Gamma : x' \mapsto X'$ is an identity preserving isometric embedding of the unital algebra $\mathscr{A}_B$ into the Banach algebra $\mathcal{L}(\Phi)$.*

Theorem 9. *Let Φ be the Banach space $(BUC(G, Z), \|\cdot\|_\infty)$. The mapping $\Gamma : x' \to X'$ for bounded x' is norm-preserving and the algebra $\mathscr{A}_B$ of bounded homomorphisms is a Banach algebra: the norm on $\mathscr{A}_B$ is an operator norm.*

[4] This is not surprising given that the dualism norm is constructed from Γ.

[5] The elements of Φ are equivalence classes of functions which differ from one another only on a null set. Therefore, one can define the action of the translation operator R^τ and $x' \in \mathscr{A}_B$ as if these equivalence classes are functions: $f = g$ *a.e* implies $R^\tau f = R^\tau g$ *a.e* and $x'(f) = x'(g)$ *a.e.*

Remark 1. *The results in this section have shown the importance of the norm of the test space Φ and a growth condition on the translation operators. It would be interesting exercise to find other norms on function spaces for which the dualism norm and the usual norm coincide.*

3.4 Operator Normed Homomorphisms

We re-study an *operator normed* family $\mathfrak{X}' = \{x'(\tau)|\tau \in \mathbb{T}\}$, in conjunction with its isometric counterpart $\mathbb{X}' = \{X'(\tau) = \Gamma(x'(\tau)) \in \mathscr{A}'_B|\tau \in \mathbb{T}\}$; $\mathscr{A}'_B := \Gamma[\mathscr{A}_B]$. This will extend the corresponding results for $\mathfrak{X}'$ as a (plain) normed homomorphism family of Section 2.3. In particular, the closedness condition of Proposition 1, Section 2.4. The correspondence $\mathfrak{X}' \leftrightarrow \mathbb{X}'$ is unique: Γ is an algebra isomorphism; $\mathscr{A}_B$ is isometrically isomorphic to the unital algebra $\mathscr{A}'_B \subset \mathcal{L}(\Phi)$ (Theorem 9). We call $\mathbb{X}'$ the dual of $\mathfrak{X}'$ and denote it as $\Gamma(\mathfrak{X}')$.

3.4.1 d-measurability

Closedness may under circumstances be viewed in terms of the problem of the transfer of measurability of the mapping $\tau \to \langle \varphi, x'(\tau) \rangle$ in Z to the measurability of the mapping $\tau \to X'(\tau)\varphi$ in Φ. In short, how does 'dualism' transfer the property of measurability: if $\tau \mapsto x'(\tau)f$ is measurable, that is, $\mathfrak{X}'$ is strongly measurable in Z, then is it true that $X' : \tau \mapsto X'(\tau)f$ is measurable,that is, its dual $\mathbb{X}'$ is strongly measurable in Φ?

We say that $\mathfrak{X}'$ is *dualism-measurable* (d-measurable) if for every $f \in \Phi$ the mapping $\tau \to X'(\tau)f$ is measurable in Φ (we always assume that $\tau \mapsto x'(\tau)f$ is measurable). Now, the measurability of $\tau \mapsto X'(\tau)f$ implies the measurability of $\tau \mapsto x'(\tau)f$: $\langle f, x'(\tau) \rangle = \langle X'(\tau)f, \theta'_0 \rangle$ and θ'_0 is bounded. Thus, d-measurability of $\mathfrak{X}'$ implies strong measurability of $\mathfrak{X}'$ in Z. The difficulty with the converse statement lies in the complex nature of the duals of spaces of continuous functions. The essence of the problem seems to be in the structure of the topological group G. If, for example, G is at most countable, there is no problem. For a more general G we need the following lemma:

Lemma 2 (Topological Measurability)**.** *Let $\Phi = BUC(G, Z)$, where G is Lindelöff. If $\tau \mapsto X'(\tau)f$ is separably valued and $\mathfrak{X}' = \{x'(\tau)|\tau > 0\}$ is a family of bounded homomorphisms uniformly bounded in norm on compact intervals, then $\tau \mapsto X'(\tau)f$ is topologically measurable (that is, the inverse image of every open subset of Φ is a measurable subset of $(0, \infty)$); $f \in \Phi$ is fixed.*

Proof. Since the Φ-valued map $\tau \mapsto X'(\tau)f$ is separably valued, we assume without loss of generality that the the metric space Φ is separable or equivalently Lindelöff. Therefore, to prove the topological measurability of $\tau \mapsto X'(\tau)f$, it suffices to show that for every $\epsilon > 0$, there exists a non-trivial measurable set M containing τ_0 such that

$$\tau \in M \Rightarrow \sup_{q \in G}\{\|[X'(\tau)f](q) - [X'(\tau_0)f](q)\|\} < \epsilon,$$

for every fixed $\tau_0 > 0$.

This follows essentially from a diagram chase of neighbourhoods. Now, for any fixed $q_1 \in G$

$$\begin{aligned}\|[X'(\tau) \quad f \quad](q) - [X'(\tau_0)f](q)\| &< \|[X'(\tau)f](q) - [X'(\tau)f](q_1)\| \\ &+ \quad \|[X'(\tau)f](q_1) - [X'(\tau_0)f](q_1)\| + \|[X'(\tau_0)f](q_1) - [X'(\tau_0)f](q)\|.\end{aligned} \tag{3.4.1}$$

The measurability of the map $W_{q_1} : \tau \mapsto \langle f_{-q_1}, x'(\tau)\rangle$ and the uniform continuity of the function $X'(\tau_0)f \in \Phi$ ensures that there is a measurable set $M(\frac{\epsilon}{3}; \tau_0, q_1) \subset \mathbb{R}^+$ containing τ_0 and a neighbourhood $q_1 + U(\frac{\epsilon}{3}; \tau_0) \subset G$, such that for $\tau \in M(\frac{\epsilon}{3}; \tau_0, q_1)$ and $q \in q_1 + U(\frac{\epsilon}{3}; \tau_0)$, the latter two proximities of equation (3.4.1) are both less than $\frac{\epsilon}{3}$.

For the first proximity, the equicontinuity of $\{H(\tau) := X'(\tau)f | \tau \in K\}$; K is a compact set containing τ_0 (Appendix C.2.2,Theorem 1 & Proposition 4), ensures that there exists a neighbourhood $q_1 + V(\frac{\epsilon}{3}, K)$ such that for $q \in q_1 + V(\frac{\epsilon}{3}, K)$ the former inequality is less that $\frac{\epsilon}{3}$ for each $\tau \in K$: the neighbourhood $q_1 + V(\frac{\epsilon}{3}, K)$ is independent of $\tau \in K$.

Therefore, for $\tau \in N(\epsilon; \tau_0, q_1) := M(\frac{\epsilon}{3}; \tau_0, q_1) \bigcap K$,

$$\sup_{q \in q_1 + W(\epsilon)} \|[X'(\tau)f](q) - [X'(\tau_0)f](q)\| < \epsilon,$$

where $W(\epsilon; q_1) = V(\frac{\epsilon}{3}, K) \bigcap U(\frac{\epsilon}{3}; \tau_0)$. Now G has the Lindelöff property so any $q \in G$ will belong to one of the countably many $q_i + W(\epsilon; q_i)$; $i \in \mathbb{N}$. Thus,

$$\sup_{q \in G} \|[X'(\tau)f](q) - [X'(\tau_0)(f)](q)\| < \epsilon,$$

for $\tau \in M := \bigcap_1^\infty N(\epsilon; \tau_0, q_i)$ (measurable sets are closed under countable intersection by the σ-algebra property of measurable sets). □

Remark 2. *This proof was an interplay between the uniform boundedness theorem and dualism: the uniform boundedness theorem ensured the equicontinuity of the family $\mathfrak{X}'$ over compacts and then the dualism, by virtue of the continuity of the map $p \to f_{-p}$, generates an equicontinuous subset of continuous functions $\{X'(\tau)f | \tau \in [\alpha, \beta]$ in the nice space $\Phi = BUC(G, Z)$ which then ensures that the desirable properties of $\tau \to X'_\tau(f)$.*

We now have,

Theorem 10. *Suppose that G is a compact metric abelian group and that $\mathfrak{X}' = \{x'(\tau)|\tau > 0\}$ is a family of bounded homomorphisms uniformly bounded in norm on compact intervals. If $\mathfrak{X}'$ is strongly measurable in Z, it is d-measurable.*

Proof. The compactness of G immediately ensures that G is Lindelöff. Topological measurability and measurability coincides for essentially separably valued functions. Therefore, by Lemma 2, we need only show that $\tau \to X'(\tau)f$ is essentially separably valued in Φ. For the metric space Φ, separability and total boundedness are equivalent. We now prove separability in Φ as total boundedness.

Let I be a compact nontrivial subinterval of $\mathbb{R}^+$. For fixed $f \in \Phi$ the mapping $\tau \to \langle f, x'(\tau)\rangle$ is measurable in Z. There is, therefore, a set $N_0 \subset I$ of measure zero such that the set $\{\langle f, x'(\tau)\rangle : \tau \in I_0 := I \setminus N_0\} \subset Z$ is separable.

Since G is compact and metrizable, it is separable. Suppose the set $\{s_n : n = 1, 2, \dots\}$ is dense in G. By the argument above, there are "null-sets" $N_n \subset I$ such that $\{X'(\tau)f(s_n) = \langle f_{-s_n}, x'(\tau)\rangle : \tau \in I_n := I \setminus N_n\}$ is separable. Let $N_\infty := \cup_{n=1}^{\infty} N_n$, and let $I_\infty = I \setminus N_\infty$. Then for each fixed $s \in G$, the set

$$\{X'(\tau)f(s) : \tau \in I_\infty\} \subset Z,$$

is separable by the uniform continuity of the the map $s \mapsto f_{-s}$.

Let us consider the set $\Upsilon_I := \{X'(\tau)f : \tau \in I_\infty\} \subset \Phi$. From the local uniform boundedness of $X'(\tau)$, it follows that Υ_I is equicontinuous. We come to the conclusion, by the Ascoli-Arzelà theorem (*e.g.* [4, p. 210]), that it is totally bounded and therefore, separable in Φ. Since $\mathbb{R}^+$ can be covered by a countable sequence of compact subintervals, it now follows that the function $\tau \to X'(\tau)f$ is essentially separably valued in Φ. □

3.4.2 d-integrability

The transfer of the strong measurability of $\mathfrak{X}'$ by dualism (Theorem 10) allows us to consider the integrability of the dual $\mathbb{X}'$: we say that a d-measurable $\mathfrak{X}'$ is *dualism-integrable* (d-integrable) [6] if $\tau \mapsto \|X'(\tau)f\|_\infty$ is

[6]For the function space valued map $\tau \mapsto X'(\tau)f$ to be Bochner integrable it must be measurable and $\tau \mapsto \|X'(\tau)f\|_\infty$ must be integrable. The former condition eliminates well known cases of the latter case being true and the function not being Bochner integrable. Indeed, the former case only ensures that a sequence of Φ-valued simple functions converges uniformly to $\tau \mapsto X'(\tau)f$: uniform convergence does not ensure convergence in L^1 as in the case of $s_n := \frac{1}{n}\mathcal{X}_{[0,n]}$; uniform convergence with L^1-convergence ensures that Bochner integrability in Φ.

integrable over I for each fixed $f \in \Phi$. The d-integrability of $\mathfrak{X}'$ sets the stage for the test space Φ to do work just like its counterpart in the Laurent Schwartz theory of vector valued distributions:

Proposition 2. *If $\mathfrak{X}'$ is d-integrable, then $X'_I f := (\Gamma x'_I)f = (\Gamma \int_I x'(\tau)d\tau)f$ is a Bochner integral, $\int_I X'(\tau)f d\tau$, for each fixed $f \in \Phi$.*

Proof. We bootstrap the case of $\tau \mapsto X'(\tau)f$ being a step function in Φ which is elementary. The bootstrap rests upon the well known fact that uniform convergence in Φ implies pointwise convergence. □

Closedness may under some circumstances be viewed as the problem of the transfer of measurability of the mapping $\tau \to \langle f, x'(\tau)\rangle$ in Z to the d-integrability of the mapping $\tau \to X'(\tau)f$ in Φ:

$$\langle f, y' * \int_I x'(\tau)d\tau\rangle = \langle(\Gamma \int_I x'(\tau)d\tau)f, y'\rangle.$$

Once the dualism $\Gamma \int_I x'(\tau)d\tau$ is a Bochner integral $\int_I X'(\tau)f d\tau$ in Φ, we sharpen the closedness condition of Proposition 1, Section 2.4, with the well known theorem concerning the commutation of closed linear operators and Bochner integrals:

Corollary 1 (Closedness)**.** *Assume the hypothesis of Theorem 10. If $\mathfrak{X}'$ is d-integrable over I, then every $y' \in \mathscr{A}_B$ is closed over I with respect to $\mathfrak{X}'$.*

Proof. $\langle \int_I X'(\tau)f d\tau, y'\rangle = \int_I \langle X'(\tau)f, y'\rangle d\tau = \int_I \langle f, y' * x'(\tau)\rangle d\tau$. □

Remark 3. *The closedness condition of Proposition 1, Section 2.4, involved 'trivial' bounded homomorphisms in the form of canonical homomorphisms. The compact metrizability of the group G allows our test space Φ to do work: the Φ-valued Bochner integral is equivalent to the induced integral. This is the key for a sharper closedness condition for $\mathfrak{X}'$ by studying $\mathfrak{X}'$ in conjunction with its isometric counterpart $\mathbb{X}'$. This sharper closedness condition for $\mathfrak{X}'$ is similar to the classical case of bounded evolution operators.*

3.4.3 Equivalent Framework: induced integral

The dualism $X'_I := \Gamma \int_I x'(\tau)d\tau$ may be seen as an 'induced' integral. The family $\mathfrak{X}'$ is homomorphism valued so its integral $x'_I = \int_I x'(\tau)d\tau$ is also a homomorphism; the dual family $\mathbb{X}'$ is operator valued so its 'integral' X'_I should be an $\Phi \to \Phi$ operator. We construct the 'integral' X'_I (operator valued) with exactly the same philosophy behind constructing x'_I: *we drop $X' : \tau \mapsto X'(\tau)$ to a time dependant Z-valued family by evaluating X' at a fixed $f \in \Phi$, denoted $X'(f)$, and then at a fixed s in the common domain*

G of the functions in Φ; the second evaluation is needed since $X'(f)$ is a function space Φ-valued time dependant function.

The action of X'_I on $f \in \Phi$ is another function in Φ. We denote this action as ${}^{X'_I}f$ in order to make it clear we are dealing with a function. We now define X'_I as follows:

$$^{X'_I}f : s \mapsto \int_I X'_s(f)d\tau \in Z; f \in \Phi. \tag{3.4.2}$$

Therefore, the integral X'_I may be seen as some induced integral which we shall denote as $p - \int_I X'(\tau)d\tau$. We call this the *induced integral*: For each fixed $f \in \Phi$, the group elements $s \in G$ generate a bundle of time domained "curves" $\{X'_s(f) : \tau \to [X'(\tau)f](s)\}$ in Z. The Bochner integral $\int_I X'_s(f)d\tau \in Z$ exists should the functions $\tau \mapsto \langle f, x'(\tau)\rangle$; $f \in \Phi$, be strongly Lebesgue measurable; $[X'(\tau)(f)](s) = x'(\tau)(R^{-s}f)$. Then the (operator-valued) integral X'_I

$$X'_I f : s \mapsto \int_I X'_s(f)d\tau \in Z; f \in \Phi, \tag{3.4.3}$$

over the interval $I \subset (0, \infty)$ is well defined.

We say that the family X' is *locally p-integrable*, if $X'_I \in \mathscr{A}'_B$ for every $I = (0, \tau) \subset (0, \infty); \tau > 0$. We let $\mathcal{L}^1_{loc}((0, \infty), \mathscr{A}'_B)$ denote the class of all such families. Like for their $\mathscr{A}_B$-valued counterpart an immediate question is: when will the the integral X'_I be in $\mathscr{A}'_B$? The dualism $\Gamma(\mathfrak{X}'_I)$ is precisely X'_I by direct computation: Γ and $p - \int$ are interchangeable. This vindicates our notation (3.4.3) and choice of p-integral.

$$^{\mathfrak{X}'_I}f = ^{X'_I}f. \tag{3.4.4}$$

That is, the strong integrability of $\mathfrak{X}'$ over I ensures that the dual family X' is p-integrable over I: the strong integrability of $\mathfrak{X}'$ induces an 'integral' on X'. Indeed these two integrals 'coincide':

Proposition 3 (Equivalent Framework)**.** *Let $X' \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}'_B)$. Then $X' \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}'_B)$ if and only if $\mathfrak{X}' \in \mathcal{L}^1_{loc}((0, \infty), \mathscr{A}_B)$.*

Proof. Write $X' = \Gamma(\mathfrak{X}')$. The crux of the proof is the important observation $X'(\tau)(f)(s) = \langle R^{-s}f, x'(\tau)\rangle$ since $\Gamma(x'(\tau)) = X'(\tau)$. Therefore,

$$X'_s(f) = \mathfrak{X}'(R^{-s}f).$$

For the sufficient condition, the admissibility of $\mathfrak{X}'_I \in \mathscr{A}_B$ ensures that $X'_I \in \mathscr{A}'_B$ (Theorem 5, Section 2.2). For the necessary condition, firstly each $x'(\tau)$ is bounded by the continuity of both $X'(\tau)$ and θ_0. The strong integrability of $\mathfrak{X}'$ is immediate from $\langle f, \mathfrak{X}'_I\rangle = [\ ^{X'_I}f](0)$. The continuity and thus, admissibility of the homomorphism $\mathfrak{X}'_I$ is then immediate. $\square$

At this stage we similarly introduce the induced Laplace transform of the family X' formally. If for each $s \in G$, the Laplace transforms $\int_{(0,\infty)} e^{-\lambda\tau}[X'(\tau)f](s)d\tau$ exists, then we naturally define the operator valued induced Laplace transform as

$$[\widehat{X'(\lambda)}^p f] : s : \int_{(0,\infty)} e^{-\lambda\tau}[X'(\tau)f](s)d\tau \in Z; f \in \Phi. \tag{3.4.5}$$

We say that the family X' is *p-Laplace transformable* if and only if $\widehat{X'}^p(\lambda) \in \mathscr{A}'_B$. We let $Lap(\lambda, \mathscr{A}'_B)$ denote the class of all such families. It is immediate from the proof of Proposition 3 that $X' \in Lap(\lambda, \mathscr{A}'_B)$ if and only if $\mathfrak{X}' \in Lap(\lambda, \mathscr{A}_B)$. Formally,

Proposition 4. *The map* Γ *commutes with the p-Laplace Integral Transform:*

$$\widehat{X'}^p(\lambda) = \Gamma\widehat{\mathfrak{X}}'(\lambda). \tag{3.4.6}$$

We can now transfer the general convolution theorem and the general resolvent equation of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ by formally defining their convolution $(X' \circledast Y')$ for $X', Y' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ by the Bochner integrals

$$X' \circledast Y'(\tau)f : s \in G : \int_{(0,\tau)} \langle R^{-s}f, x'(\tau-\sigma) * y'(\sigma)\rangle\, d\sigma, \tag{3.4.7}$$

where $f \in \Phi; s \in G$. Therefore, $X' \circledast Y' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ if and only if $\mathfrak{X}' \circledast \mathfrak{Y}' \in \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$.

Theorem 11 (Transfer from $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ into $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$)**.** *The general convolution theorem and the general resolvent equation of* $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$ *transfer word for word into* $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ *by merely replacing* $*$ *of* $(\mathscr{A}_B, *)$ *with* $\circ$ *of* $(\mathscr{A}'_B, \circ)$.

3.5 d-Normed Star Semigroups

For the time-dependent d-normed family $\mathfrak{X}'$, some of the results of classical semigroup theory carry over to star-semigroups. For example we have a Miyadera type result ([19] Lemma 10.2.1) in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$, where strong measurability combined with the semigroup property implies uniform boundedness on compacts:

Lemma 3 (Miyadera in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_B)$)**.** *Let* $\Phi = BUC(G, Z)$*. If* $\mathfrak{X}'$ *is a star-semigroup and for every* $f \in \Phi$ *the function* $\tau > 0 \mapsto \langle f, x'(\tau)\rangle$ *is measurable, then* $\|x'(\tau)\|$ *is uniformly bounded on compact subintervals of* $\mathbb{R}^+$.

Proof. The mapping $\Gamma : x' \mapsto X'$ is norm preserving so that $\|\langle f, x' * y' \rangle\| \leq \|x'\|\|y'\|\|f\|_\infty$. It is then immediate that the proof by contradiction of Lemma 10.2.1 [19] carries over word for word. □

The interplay between the uniform boundedness in norm on compacts and the compact metrizability of the abelian group G of the test space $\Phi = \mathrm{BUC}(G, Z)$ established the d-measurability of $\mathfrak{X}'$ (Theorem 10). Therefore, it is immediate from Lemma 3 that every strongly measurable d-normed star-semigroup is d-measurable:

Corollary 2 (Dualism Preserves Measurability)**.** *Suppose that G is a compact metric abelian group. If $\mathfrak{X}'$ is a star-semigroup and for every $f \in \Phi$ the function $\tau > 0 \mapsto \langle f, x'(\tau) \rangle$ is measurable, then $\tau > 0 \mapsto X'(\tau)f$ is (uniformly) measurable; $f \in \Phi$ is fixed.*

Now recall the well known result that a strongly measurable semigroup is strongly continuous (Lemma 3, Chap. VIII [11]). The strong continuity property of measurable classical semigroups, however, is not analogously true for star-semigroups. To investigate this, we consider the dualisms $X'(\tau)$ induced by the homomorphisms $x'(\tau)$. From Theorem 1, Section 3.1, it is seen that the family $\mathfrak{X}' = \{X'(\tau)|\tau > 0\}$ is a semigroup of bounded linear operators in the space Φ. Hence if $\mathfrak{X}'$ is d-measurable, then the semigroup $\mathfrak{X}$ is strongly continuous.

Therefore, under suitable restrictions on G, from Corollary 2, the d-measurability of $\mathfrak{X}'$ transfers the proof of Lemma 3, Chap. VIII [11] word for word. Indeed, the submultiplicativity of the norm: $\|x' * y'\| \leq \|x'\|\|y'\|$ ensures that the proof of exponential boundedness (Corollary 5, Chap. VIII [11]) to also carry over word for word. Therefore,

Theorem 12. *Let $\Phi = BUC(G, Z)$; G is a compact metric abelian topological group. If $\mathfrak{X}'$ is a star-semigroup and for every $f \in \Phi$ the function $\tau > 0 \mapsto \langle f, x'(\tau) \rangle$ is measurable, then $\|x'(\tau)\|$ is uniformly bounded on compact subintervals of $\mathbb{R}^+$, $x'(\tau)$ is strongly continuous and exponentially bounded.*

Thus, for star-semigroups, Theorems 5 and 6 of Section 3.2.2, apply.

3.6 Isometric Embedding Of Empathy Theory

In the study of d-normed star semi-groups (Section 3.5), the strong measurability of $\mathfrak{X}'$ is the starting assumption in the investigation of $\mathfrak{X}'$. This is in line with [19], [11] for classical semigroups. We likewise demonstrated that strong measurability is a more fundamental concept than strong continuity (Theorem 12).

The strong continuity of the evolution family is the starting assumption for some authors. Indeed, the assumptions of empathy theory ensure strong continuity of the double family of evolution operators $\langle \mathcal{S}, \mathcal{E} \rangle$ (Theorem 2.2, [28]). Should this also be our starting assumption for $\mathfrak{X}'$, that is, $\tau > 0 \mapsto \langle f, x'(\tau) \rangle$ is continuous for each $f \in \Phi$, then under much weaker restrictions on the group G, the isometric counterpart X' is strongly continuous. More precisely, if G is locally compact, then the dualism transfers the strong continuity of the family $\mathfrak{X}'$ (uniformly bounded over compacts) into the the strong continuity of X' (Appendix C.3.1, Theorem 2).

Suppose $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is the canonical family of homomorphisms associated with the strongly continuous pair $\langle \mathcal{S}, \mathcal{E} \rangle$; $x'(\tau) := S(\tau)\theta_0'$ and $y'(\tau) := E(\tau)\theta_0'$ (Section 2.8). Then $\langle \mathbb{X}', \mathbb{Y}' \rangle$ is an isometric identification with $\langle \mathcal{S}, \mathcal{E} \rangle$ as far as strongly continuous semi-groups are concerned; $X'(\tau) = \Gamma(x'(\tau))$ and $Y'(\tau) = \Gamma(y'(\tau))$.

Proposition 5 (Operator Identification)**.** *Let* $\Phi = BUC(G, Z), \| \cdot \|_\infty)$*;* G *is locally compact. Then the pairs* $\langle S(\tau), E(\tau) \rangle$*,* $\langle P(\lambda), R(\lambda) \rangle$ *embeds isometrically as pairs* $\langle X'(\tau), Y'(\tau) \rangle$*,* $\langle \widehat{X'}^p(\lambda), \widehat{Y'}^p(\lambda) \rangle$*, respectively. Furthermore, the functional equations (resolvent equations) are preserved. Indeed, if* $\langle S(\tau), E(\tau) \rangle$ *is strongly continuous, then so is* $\langle X'(\tau), Y'(\tau) \rangle$ *.*

Proof. It suffices to prove it for the family $\mathbb{X}'$ (the proof for $\mathbb{Y}'$ is similar). Since $\Gamma : x' \mapsto X'$ is norm preserving, $\|X'(\tau)\| = \|S(\tau)\|$. The proof of the induced Laplace integrals $\langle \widehat{X'}^p(\lambda), \widehat{Y'}^p(\lambda) \rangle$ is done similarly.

The strong continuity and the local boundedness of $\mathcal{S}$ (Theorem 2.2, [28]) ensures that the family $\mathfrak{X}'$ of bounded homomorphisms is strongly continuous and simply bounded over compacts. Therefore, the family $\mathbb{X}'$ is strongly continuous (Theorem 2, Appendix C.3.1). Finally, the algebra isomorphism Γ transfers the causal relations in $\mathscr{A}_B$ of Proposition 5, Section 2.8.1, into $\mathscr{A}_B'$. □

Chapter 4

Generation Theorem: Star Implicit Cauchy Problem

The well known convolution algebra $(L^1(0,\infty),*)$ lies at the heart of Kisynski's equivalent formulation of the Hille-Yosida Theorem: a Banach algebra representation T on $(L^1(0,\infty),*)$ generates the C_0-semigroup $(E(\tau))_{\tau\geq 0}$ satisfying the abstract Cauchy problem. The solution or regularity space for the abstract Cauchy problem can be constructed purely from T alone. Thus, we can identify this regularity space with T. For brevity, we shall use the notation $L^1(0,\infty)$ to denote the convolution algebra $(L^1(0,\infty),*)$.

In this chapter, we adapt Kisynski's formulation to construct a Hille-Yosida-Kisynski generation theorem for the implicit Cauchy problem (Equations (1.1.1)-(1.1.2), Section 1.1). We show that the solution or regularity space for the implicit Cauchy problem is a non-closed dense subspace of the solution space for the abstract Cauchy problem; indeed, the regularity space for the implicit Cauchy problem can be identified with T^2 . Furthermore, this empathy theory adaptation of the Kisynski approach shows how much empathy theory differs from semi-group theory.

We then demonstrate the versatility of Kisynski's formulation by adapting it also to a Hille-Yosida-Kisynski generation theorem for the implicit Cauchy problem cast in our more general framework of admissible homomorphisms or generalized operators. This should not be surprising since the framework of generalized operators is based on another convolution algebra, $(\mathscr{A}_B,*)$.

4.1 Generation Theorem: Classical Implicit Cauchy Problem

Recall that for the implicit Cauchy problem (equations (1.1.1)-(1.1.2), Section 1.1), the operators A and B are not closable, thus, precluding the commuting of B with the time derivative or the limit at $\tau = 0^+$ in (1.1.1) and (1.1.2). The classical *generation problem (for implicit evolution equations)* is to assume an empathy pseudo-resolvent $\langle R, P \rangle$ defined for a scalar $\lambda \in U$, where U is the half-ray $(0, \infty)$ (Section 1.1); that is, $R(\lambda) \in \mathcal{L}(Y)$ and $P(\lambda) \in \mathcal{L}(Y, X)$ satisfies (1.1.6). Then, find a 'generator' of an empathy in the form of an operator pair $\langle A, B \rangle$ and empathy $\langle S(\tau), E(\tau) \rangle$ (Section 1.1) related to the implicit Cauchy problem (1.1.1)-(1.1.2).

We adapt Kisynski's approach [22] to the Hille-Yosida theorem (for the abstract Cauchy problem) of constructing a representation T of a Banach-subalgebra of pseudo-resolvents $R(\lambda)$ by the convolution algebra $L^1(0, \infty)$. We show that for the empathy approach, a different 'representation' T' by $L^1(0, \infty)$ is needed to represent $P(\lambda)$ and generate $S(\tau)$ of an empathy $\langle S(\tau), E(\tau) \rangle$, which is *not* defined at $\tau = 0$, where $u(\tau) = S(\tau)y$ solves (1.1.1) for y in a non-closed isomorphic dense subspace of the solution space of the abstract Cauchy problem.

Suppose $\langle S(\tau), E(\tau) \rangle_{\tau > 0}$ is an empathy. Then the integral representations of the action of the assumed empathy on the subspace $\Delta_Y := R(\lambda)[Y]$ gives a clue of how $\langle A, B \rangle$ should look like (Lemma 2.7 [28]). Indeed, by these integral representations, the behaviour of $S(\tau)$ and $E(\tau)$, which is not defined at $\tau = 0$ on Δ_Y, is expressed as an asymptotic property of the Laplace transforms of the empathy resolvents which is well defined on Δ_Y.

Theorem 1. *[28] Assume the existence of the empathy $\langle S(\tau), E(\tau) \rangle_{\tau > 0}$, the empathy pseudo-resolvent $\langle R(\lambda), P(\lambda) \rangle$ and the invertibility assumption. Then the implicit Cauchy problem* (1.1.1)-(1.1.2) *has solution $u(\tau) = S(\tau)y$ for almost all $\tau > 0$ on $y \in \Delta_Y$. For any $y \in \Delta_Y$,*

$$\lim_{\lambda \to \infty} \lambda R(\lambda)y = y, \lim_{\lambda \to \infty} \lambda P(\lambda)y = Jy, \tag{4.1.1}$$

where $J := P(\lambda)R(\lambda)^{-1}$ need not be closed.

Remark 1. *The empathy approach of introducing a pair of evolution families and a pair of invertible resolvents showed that closedness of the operators A and B was not crucial: the 'impossible' commutation was bypassed to a specially constructed space $B[\mathscr{D}] = \Delta_Y$ of Y.*

Therefore, an empathy $\langle S(\tau), E(\tau) \rangle$ deviates much from a C_0-semigroup $E(\tau)$: T' is a linear map on $L^1(0, \infty)$ and is not an algebra representation and need not be closed; furthermore, $S(0) = J$ on $\overline{\Delta_Y}$ deviates from the

C_0-semigroup behaviour $E(0) = \mathbf{1}$ in the sense that J need not be closed; indeed, $S(0) = J$ extends the behaviour in [28], of the assumed empathy $\langle S(\tau), E(\tau)\rangle$ which is not defined at $\tau = 0$ on Δ_Y but expressed as an asymptotic property of the strong Laplace transforms (Theorem 1 (4.1.1)). The notation $S(0) = J$ on $\overline{\Delta_Y}$ means $\lim_{\tau\to 0+} S(\tau)y = Jy$ for $y \in \overline{\Delta_Y}$.

4.1.1 Kisynski Regularity Space

Consider a pseudo-resolvent R, where U is the half-ray $(0,\infty)$. Then the Kisynski regularity space $\Delta_K \subset Y$ defined as

$$\Delta_K := \{y \in Y | \lim_{\lambda\to\infty} \|\lambda R(\lambda)y - y\| = 0\}, \tag{4.1.2}$$

and (ii) an operator A_E defined by its graph

$$\{(y, y') \in Y \times Y | \lim_{\lambda\to\infty} \|\lambda(\lambda R(\lambda)y - y) - y')\| = 0\}. \tag{4.1.3}$$

The Kisynski regularity space Δ_K is a space where the resolvent $(R(\lambda))_{\lambda>0}$ behaves asymptotically as the identity operator [1]; the domain of the operator A_E is the space where the Yosida approximants converge [2].

Under the growth condition $\|\lambda R(\lambda)\| = O(1)$, namely, $\limsup_{\lambda\to\infty}\lambda\|R(\lambda)\| < \infty$, the space Δ_K is the closed subspace $\overline{\Delta_Y}$ of Y and as such it is a Banach space.

Proposition 1. *(Proposition 5.2 [22]) If* $\lim sup_{\lambda\to\infty}\lambda\|R(\lambda)\| < \infty$*, then* Δ_K *is closed.*

Proof. Let $y \in \overline{\Delta_K}$. Then writing $y = (y - y_n) + y_n$,

$$\lambda R(\lambda)y - y = [1 + \lambda R(\lambda)](y - y_n) + [\lambda R(\lambda)y_n - y_n],$$

where (y_n) is a sequence of elements of Δ_K that converges to y. With $\limsup_{\lambda\to\infty}\|\lambda R(\lambda)y_n - y_n\| = 0$,

$$\limsup_{\lambda\to\infty}\|\lambda R(\lambda)y - y\| \leq [1 + \limsup_{\lambda\to\infty}\lambda\|R(\lambda)\|]\|y_n - y\|,$$

for every $n \in \mathbb{N}$. □

[1]This is a translation (to resolvents) of the strong continuity assumption of a semigroup $E(\tau)$ on Φ_Y behaving like the identity operator near $\tau = 0$: $\lim_{\tau\to 0^+} E(\tau)f = f$ for all$f \in \Phi_Y$. The numerical heuristic of the construction (4.1.2) lies in the relation $\frac{\lambda \mathbf{1}}{\lambda - A} = \lambda R(\lambda)$; that is, the Hille pseudo resolvent coincides with the Banach algebraist's resolvent. Taking the limit $\lambda \to \infty$ of the left hand side term leads to $\mathbf{1}$.

[2]This is a translation (to resolvents) of the classical result that the infinitesimal generator of a semigroup coincides with the generator of the resolvent. The motivation for the construction (4.1.3) lies in the well known Yosida approximants of the infinitesimal generator of a C_0-semi-group (Appendix D, Definition 4).

Furthermore, the pseudo-resolvent R becomes a resolvent $R(\lambda, A_E) = (\lambda - A_E)^{-1}$; with generator A_E. We consider the restriction, $R(\lambda)|_{\Delta_K}$, of $R(\lambda)$ to the subspace Δ_K in order to construct a C_0-semigroup on the Banach space Δ_K with generator A_E so that $(\lambda - A_E)^{-1} = R(\lambda)$ on Δ_K. Indeed, $R(\lambda)|_{\Delta_K}$ is a 1-1 continuous extension of $R(\lambda)$ restricted to Δ_Y, $R(\lambda)|_{\Delta_Y}$. We let $[R(\lambda)|_{\Delta_K}]^{-1}$ denote the inverse of $R(\lambda)|_{\Delta_K}$.

Proposition 2. *[22] Let $\|\lambda R(\lambda)\| = O(1)$. Then*
(i) The Kisynski regularity space Δ_K is a closed subspace of Y.
(ii) The generator A_E is closed, where $Dom(A_E)$ is dense in Δ_K.
(iii) For each $\lambda \in U$, (a) $R(\lambda)[\Delta_K] \subset \Delta_K$ and (b) $R(\lambda)|_{\Delta_K}$ is the resolvent $R(\lambda, A_E)$. That is, $R(\lambda)|_{\Delta_K}$ is a 1-1 continuous extension of $R(\lambda)|_{\Delta_Y}$.
(iv) $R : \lambda \to R(\lambda)|_{\Delta_K}$ is locally analytical.

Proof. Now (ii) is immediate from Corollary 5.3 [22]. For (iii), Proposition 5.4, [22], ensures that $R(\lambda)|_{\Delta_K}$ is the resolvent $R(\lambda, A_E) = (\lambda - A_E)^{-1}$, where the resolvent set of A_E contains $U := (0, \infty)$. Thus, for each $\lambda \in U$, $R(\lambda)[\Delta_K] \subset \Delta_K$ and $R(\lambda)|_{\Delta_K}$ is 1-1.

Finally (iv) follows from (iii) by Proposition 1.3 (ii), Section IV.1, [12]. □

Now, if we assume the more stringent Widder Growth Condition,

$$\sup_{\lambda>0;k\in\mathbb{N}} \{\|[\lambda R(\lambda)]^k\|\} < \infty,$$

then every pseudo-resolvent R is the image of a bounded *algebra representation* [3] T of the canonical pseudo-resolvent $(r(\lambda) := e_{-\lambda})_{\lambda\in(0,\infty)}$ of the convolution algebra $L^1(0, \infty)$,

$$r(\lambda) - r(\mu) = (\mu - \lambda) r(\lambda) * r(\mu), \tag{4.1.4}$$

where $e_{-\lambda} : \tau \in (0, \infty) \mapsto e^{-\lambda\tau}$. Indeed, the identification $T : r(\lambda) \mapsto R(\lambda)$ is unique in the following sense:

Theorem 2. *[22] Let $R(\lambda); \lambda > 0$, satisfy the Widder Growth Condition. Then there exists a unique (bounded) Banach algebra representation $T : L^1(0, \infty) \to \mathcal{L}(Y)$ such that (i) $T(e_{-\lambda}) = R(\lambda)$ and (ii) T algebraically reconstructs the Kisynski regularity space Δ_K as $\Delta_{K'}$:*

$$\Delta_K = \overline{\Delta}_Y = \bigcup_{\phi\in L^1(0,\infty)} T(\phi)[Y] =: \Delta_{K'}. \tag{4.1.5}$$

[3] We say that a representation T of an algebra L on a linear space Φ is an algebra homomorphism T from L into the algebra $\mathcal{L}(\Phi)$.

Proof. Firstly T is an algebra homomorphism on the total subset $S := \{e_{-\lambda}|\lambda > 0\}$ of $L^1(0,\infty)$: $T(e_{-\lambda} * e_{-\mu}) = T(e_{-\lambda})T(e_{-\mu})$ by virtue of the resolvent equation (4.2.6). Now T extends uniquely to an *algebra representation* on the entire $L^1(0,\infty)$ by the Widder Growth Condition .

The algebraic reconstruction (4.1.5) of the Banach space $\Delta_{_K}$ follows from the factorization theorem for representations of the Banach algebra $\mathcal{L}(Y)$ (Theorem 5.2.2 [23]). □

With the algebraic $\Delta_{_{K'}}$, as opposed to the equivalent $\Delta_{_K}$, a C_0-semigroup $\mathcal{E} = (E(\tau) : \Delta_{_{K'}} \to \Delta_{_{K'}})_{\tau\geq 0}$ on $\Delta_{_{K'}}$ is constructed by right shift maps, R^τ, on the function space $L^1(0,\infty)$. For every $\tau \geq 0$ and $\phi \in L^1(0,\infty)$,

$$E(\tau)[T(\phi)] := [T(R^\tau\phi)], \tag{4.1.6}$$

where the right translate of ϕ by τ is the element $R^\tau\phi \in L^1(0,\infty)$; $R^\tau\phi(\xi) = \phi(\xi-\tau)$ if $\xi \in (\tau,\infty)$ and $R^\tau\phi(\xi) = 0$ if $\xi \in (0,\tau]$. We call the space $\Delta_{_{K'}}$, the *T-regularity space* since T uniquely reconstructs the space $\Delta_{_K}$.

Theorem 3. *[22] The construction* (4.1.6) *uniquely defines a C_0-semigroup $(E(\tau) : \Delta_{_{K'}} \to \Delta_{_{K'}})_{\tau\geq 0}$ on the T-Kisynski regularity space $\Delta_{_{K'}}$. That is, for $y = T(\phi)y' \in \Delta_{_{K'}}$,*

$$E(\tau)y = E(\tau)[T(\phi)y']$$

is independent of representation of y. $E(0)$ is the identity operator on $\Delta_{_{K'}}$ and not the whole space Y.

Proof. The independence of representation follows from $E(\tau)y = \frac{d}{d\tau}G(\tau,y)$, where $G(\tau,y) : \tau \mapsto [T(\mathcal{X}_{(0,\tau)})](y)$ (Theorem 4.2, (4.9), [22]) since $R^\tau\phi = \frac{d}{d\tau}[\mathcal{X}_{(0,\tau)} * \phi]$. □

Remark 2. *In Kisynski's approach [22] to the Hille Yosida theorem, the reconstructed C_0-semi-group $(E(\tau))_{\tau\geq 0}$ is defined on $\Delta_{_{K'}}$ as opposed to the reconstructed C_0-semi-group of the Hille Yosida theorem which is defined on the whole space Y. Kisynski's approach is an equivalent formulation of the Hille Yosida theorem [7]. Therefore, there is no loss of generality in $E(0)$ being the identity operator on $\Delta_{_{K'}}$ and not the whole space Y.*

Therefore, every semigroup constructed from a pseudo-resolvent which satisfies the Widder Growth Condition is essentially an image of a translation semigroup. From this point onward, we shall always assume the Widder Growth Condition for any pseudo-resolvent R. The Widder Growth Condition was critical in constructing the T-regularity space, $\Delta_{_{K'}}$, the solution space for the abstract Cauchy problem. We now introduce *T^2-regularity* *space*, $\Delta^2_{K'}$, an isomorphic dense subspace of the T-regularity space. We will show $\Delta^2_{K'}$ is a solution space for the more general star implicit Cauchy problem. Thus, the T^2-regularity space is a large solution space.

Remark 3. *We can represent the construction* (4.1.6) *as the commuting diagram:*

$$\begin{array}{ccccc}
\phi \in L^1(0,\infty) & \xrightarrow{\quad T \quad} & T(\phi) \in \mathcal{L}(Y) & \xrightarrow{\quad \Theta_{f'} \quad} & T(\phi)f \in \Delta_{K'} \\
\Big\downarrow{\scriptstyle R^\tau} & & \Big\downarrow{\scriptstyle Y'(\tau)} & & \Big\downarrow{\scriptstyle Y'(\tau)} \\
R^\tau\phi \in L^1(0,\infty) & \xrightarrow{\quad T \quad} & Y'(\tau)(T(\phi)) \in \mathcal{L}(Y) & \xrightarrow{\quad \Theta_{f'} \quad} & Y'(\tau)(T(\phi)f) \in \Delta_{K'}
\end{array}$$

Diagram 1. C_0 *– semigroup* $(E(\tau))_{\tau\geq 0}$ *on the T-Kisynski regularity space* $\Delta_{K'}$*.*

4.1.2 Empathy Regularity Space

Consider the subspace, $\Delta^2_{K'}$, of the T-regularity space:

$$\Delta^2_{K'} := R(\lambda)[\Delta_{K'}] \tag{4.1.7}$$

Thus, $\Delta^2_{K'}$ is the domain of the inverse $[R(\lambda)|_{\Delta_K}]^{-1} = [R(\lambda)|_{\Delta_{K'}}]^{-1}$.

Proposition 3. *The* T^2-*regularity space,* $\Delta^2_{K'}$ *is (i) an isomorphic dense subspace of* $\Delta_{K'}$ *and (ii) the representation* (4.1.7) *is independent of the choice of* λ*.*

Proof. (i) is immediate from firstly noting that $R(\lambda)|_{\Delta_{K'}}$ is 1-1 (Proposition 2 (ii)) and secondly that $\Delta^2_{K'} = \text{Ran}(R(\lambda)|_{\Delta_{K'}}) = \text{Dom}([R(\lambda)|_{\Delta_{K'}}]^{-1}) = \text{Dom}(\lambda - A_E) = \text{Dom}(A_E)$ which is a dense subspace of $\Delta_{K'}$ (Proposition 5.4 [22]). For (ii), we show that $R(\lambda)[\Delta_{K'}] \subset R(\mu)[\Delta_{K'}]; \lambda, \mu \in U$. Consider the dense subspace $R(\lambda)[\Delta_Y]$ of $R(\lambda)[\Delta_{K'}]$ [4] . Then $y \in R(\lambda)[\Delta_Y]$ implies $y = R(\lambda)R(\mu)y'$ for some $y' \in Y$ since Δ_Y is independent of representation. Therefore, $y = R(\mu)R(\lambda)y' \in R(\mu)[\Delta_Y]$ by the commutativity of the resolvents. Thus, $R(\lambda)[\Delta_Y] \subset R(\mu)[\Delta_Y]$. The reverse inclusion is proved similarly. □

[4] Δ_Y is a dense subspace of $\Delta_{K'}$ (Theorem 2, (4.1.5)). The boundedness of $R(\lambda)$ ensures that $R(\lambda)[\Delta_Y]$ is a dense subspace of $R(\lambda)[\Delta_{K'}]$.

With the T^2-regularity space, we construct an empathy $\langle E(\tau), S(\tau)\rangle_{\tau>0}$ from the pseudo-resolvent pair $\langle R, P\rangle$. The construction of $S(\tau)$ on the T^2 regularity space is inspired by the critical identity (5), Lemma 2.3 [28]. We construct $(S(\tau) : \Delta^2_{K'} \to \Delta_X)_{\tau>0}$ as follows: For every $\tau > 0$ and $\phi \in L^1(0, \infty)$,

$$S(\tau)[R(\lambda)T(\phi)] := P(\lambda)[E(\tau)T(\phi)], \tag{4.1.8}$$

where $\Delta_X = P(\lambda)[Y]$.

For construction (4.1.8) to be well defined, we show that the T^2-regularity space is invariant under $E(t)$.

Lemma 1. *For each $\tau \geq 0$, $E(\tau)[\Delta^2_{K'}] \subset \Delta^2_{K'}$. Indeed, for every $\phi \in L^1(0, \infty)$, $T(\phi)[\Delta^2_{K'}] \subset \Delta^2_{K'}$:*

Proof. This follows from the fact that $E(\tau)$ and $R(\lambda)$ commute on Δ_K and $\Delta_{K'}$ is invariant under $E(\tau)$ [5] . The latter statement follows from the fact that convolution is commutative in $L^1(0, \infty)$:

$$\begin{aligned} T(\phi)(R_\lambda T(\varphi)y) &= T(\phi)T(e_{-\lambda})T(\varphi)y = T(\phi * e_{-\lambda} * \varphi)y = T(e_{-\lambda} * \phi * \varphi)y \\ &= R_\lambda T(\phi * \varphi)y, \end{aligned}$$

where $\phi * \varphi \in L^1(0, \infty)$ □

Theorem 4. *Construction* (4.1.8) *uniquely defines an empathy $\langle S(\tau), E(\tau)\rangle_{t>0}$ and is independent of representation of $y \in \Delta^2_{K'}$. The empathy relation* (1.1.3) *holds on $\Delta^2_{K'}$. On the T^2-regularity space $\Delta^2_{K'}$,*

$$S(0) = J, \textit{that is}, \lim_{\tau\to 0+} S(\tau)y = Jy, \tag{4.1.9}$$

$y \in \Delta^2_{K'}$.

Proof. The following proof is purely algebraic. We show that if there is $y', y'' \in Y$ such that

$$R(\lambda)T(\phi)y' = R(\mu)T(\phi')y''$$

then

$$P(\lambda)[E(\tau)T(\phi)y'] = P(\mu)[E(\tau)T(\phi')y'']$$

[5] For $y \in \Delta_{K'}$, $R_\lambda y$ has Bochner integral representation $\int_{(0,\infty)} e^{-\lambda\tau}E(\tau)y$. Commutativity follows from the semi-group relation $E(\tau)E(\sigma) = E(\tau + \sigma)$, where $E(\sigma) : \Delta_{K'} \to \Delta_{K'}$ since $(E(\tau))_{\tau\geq 0}$ is a semi-group on $\Delta_{K'}$ (cf. Lemma 2.3 (4) [28]) .

The crux is the invertibility of $R(\lambda)|_{\Delta_{K'}}$ (Proposition 2 (ii)) and independence of representation of $J(\lambda)|_{\Delta^2_{K'}} := P(\lambda)[R(\lambda)|_{\Delta_K}]^{-1}$ which we shall denote as R_λ and J_λ, respectively; similarly denoting the restriction $P(\lambda)|_{\Delta_{K'}}$ as P_λ:

$$\begin{aligned}
P(\lambda)[E(\tau)T(\phi)y'] &= P_\lambda E(\tau)[R_\lambda^{-1}R_\mu T(\phi')y'']; && (4.1.10)\\
&= J_\lambda R_\lambda[E(\tau)R_\lambda^{-1}R_\mu T(\phi')y'']; && (4.1.11)\\
&= J_\lambda E(\tau)[R_\lambda R_\lambda^{-1}R_\mu T(\phi')y'']; && (4.1.12)\\
&= J_\lambda E(\tau)R_\mu T(\phi')y''; && (4.1.13)\\
&= J_\mu R_\mu E(\tau)T(\phi')y''; && (4.1.14)\\
&= P_\mu E(\tau)T(\phi')y'', && (4.1.15)
\end{aligned}$$

where (4.1.11) and (4.1.15) follows from equation $J|_{\Delta^2_{K'}} R_\lambda = J_\lambda R_\lambda = P_\lambda$ since both $R_\lambda[E(\tau)R_\lambda^{-1}R_\mu T(\phi')y'']$ and $R_\mu[E(\tau)T(\phi')y'']$ belong to $\Delta^2_{K'}$ (both $E(\tau)R_\lambda^{-1}R_\mu T(\phi')y''$ and $E(\tau)T(\phi')y''$ belong to $\Delta_{K'}$); (4.1.12) and (4.1.14) follows from R_λ commuting with $E(\tau)$ on $\Delta_{K'}$; (4.1.13) follows from $R_\lambda R_\lambda^{-1}R_\mu T(\phi) = R_\mu T(\phi)$.

The empathy relation which is well defined by Lemma 1 follows from the semi-group property of $(E(\sigma))_{\sigma\geq 0}$ and the construction (4.1.6) of Theorem 3. Suppose $y = R(\lambda)T(\phi)y' \in \Delta^2_{K'}$. Then,

$$S(\tau+\sigma)y = S(\tau+\sigma)R(\lambda)T(\phi)y' = P(\lambda)E(\tau)T(R^\sigma\phi)y'.$$

By the commutativity of $E(\sigma)$ and $R(\lambda)$ on $\Delta_{K'}$,

$$S(\tau)E(\sigma)y = S(\tau)E(\sigma)R(\lambda)T(\phi)y' = S(\tau)R(\lambda)E(\sigma)T(\phi)y'.$$

Now invoking the construction (4.1.6) of Theorem 3 and then (4.1.8),

$$S(\tau)E(\sigma)y = S(\tau)R(\lambda)T(R^\sigma\phi)y' = P(\lambda)E(\tau)T(R^\sigma\phi)y'.$$

The initial condition(4.1.9) follows from the commutativity of the $\lim_{\tau\to 0+}$ operator with bounded $P(\lambda)$ and $E(0)$ being the identity operator on $\Delta_{K'}$, that is, $\lim_{\tau\to 0+} E(\tau)y = y$ for all $y \in \Delta_{K'}$. □

Remark 4. *The constructed empathy $\langle S(\tau), E(\tau)\rangle$ can be well defined at $\tau = 0$ on $\Delta_{K'} = \overline{\Delta_Y}$ by (4.1.8) since $E(0)$ is well defined on $\Delta_{K'}$. However, since the empathy relation $S(\tau+\sigma) = S(\tau)E(\sigma)$ only makes sense for $\sigma, \tau > 0$, we do not define $\langle S(\tau), E(\tau)\rangle$ at $\tau = 0$. Therefore, the notation $S(0) = J$ on $\Delta^2_{K'}$ means $\lim_{\tau\to 0+} S(\tau)y = Jy$ for $y \in \Delta^2_{K'}$. Note however a direct substitution $\tau = 0$ into (4.1.8) also yields $S(0) = J$.*

Therefore, the empathy $\langle S(\tau), E(\tau)\rangle$ deviates much from a C_0-semigroup $E(\tau)$: $S(0) = J$ on $\Delta^2_{K'}$ deviates from C_0-semigroup behaviour $E(0) = \mathbf{1}$ in the sense that J need not be closed; furthermore, the linear map T' which represents $P(\lambda)$ on $L^1(0,\infty)$ is not an algebra representation and need not be closed; in contrast, a bounded algebra representation T on $L^1(0,\infty)$ represents a resolvent $R(\lambda)$; $S(\tau)$ on $\Delta^2_{K'}$ is essentially the image under T' of a exponentially pre-convolved translation semigroup; in contrast, the semigroup $E(\tau)$ on $\Delta_{K'}$ is essentially the image under T of a translation semigroup.

Corollary 1. *The homomorphism $T' = JT$ on the convolution algebra $L^1(0,\infty)$ represents $P(\lambda)$ and generates $S(\tau)$ on the domain $\Delta^2_{K'}$ as follows:*

$$T'(e_{-\lambda}) = P(\lambda); T'(e_{-\lambda} * R^\tau\phi) = S(\tau)[R(\lambda)T(\phi)], \tag{4.1.16}$$

where $\phi \in L^1(0,\infty)$. T' is not an algebra representation and need not be closed.

Proof. $T'(e_{-\lambda}) = P(\lambda)$ is immediate from $T(e_{-\lambda}) = R(\lambda)$ and $J = P(\lambda)R^{-1}(\lambda)$. By the canonical resolvent equation (4.1.4), T' cannot be an algebra representation; $T' = P(\lambda)R^{-1}(\lambda)T$, where $R^{-1}(\lambda)T$ is closed since it is the product of T bounded and $R^{-1}(\lambda)$ closed; thus, T' is the product of $P(\lambda)$ bounded and $[R^{-1}(\lambda)T]$ closed and so need not be closed. Finally, from (4.1.8), $X'(\tau)[R(\lambda)T(\phi)] = JR(\lambda)[E(\tau)T(\phi)] = JE(\tau)[R(\lambda)T(\phi)] = JT(e_{-\lambda})T(R^\tau\phi)$ $= JT(e_{-\lambda} * R^\tau\phi) = T'(e_{-\lambda} * R^\tau\phi)$. □

We represent the above construction diagrammatically as follows, where we use the following notation $\mathbb{A}_Y := \mathcal{L}(Y), \mathbb{A}_X := \mathcal{L}(Y,X)$. The diagram starts with a $\phi \in L^1(0,\infty)$:

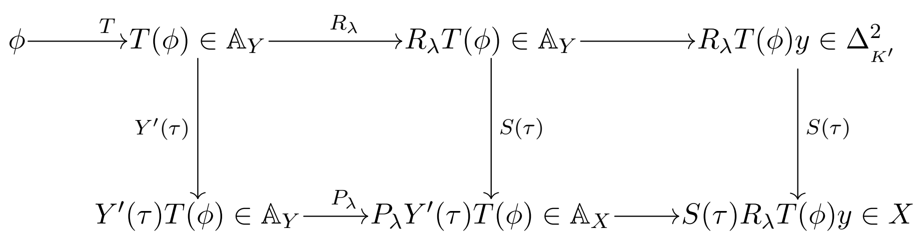

Diagram 2 : Construction of transition map $(S(\tau))_{\tau>0}$ on $\Delta^2_{K'}$ by the representation T';

Remark 5. *By* (4.1.16)*, one can identify $\langle P(\lambda), S(\tau)\rangle$ with T'. Similarly, one can identify $\langle R(\lambda), E(\tau)\rangle$ with T. Therefore, a pair $\langle T, T'\rangle$ is used to generate an empathy $\langle E(\tau), S(\tau)\rangle$. One can identify the domains of the empathy $\langle \Delta_{K'}, \Delta^2_{K'}\rangle$ with the pair $\langle T, T^2\rangle$.*

4.1.3 Invertibility assumption

Up to this point we have not used the invertibility assumption. The invertibility assumption constructs the operator $B := R(\lambda)P(\lambda)^{-1}$ which ensures that the critical identity on $\Delta^2_{K'}$

$$BS(\tau) = E(\tau), \tag{4.1.17}$$

when one notes that $\Delta^2_{K'} \subset \Delta_Y$. With the empathy $\langle S(\tau), E(\tau)\rangle$, for each $y \in \Delta^2_{K'}$,

$$u : t \mapsto u(\tau) = S(\tau)y, \tag{4.1.18}$$

is a solution of the implicit evolution equation (1.1.1), Section1.1: apply B to (4.1.18) and then invoke (4.1.17):

$$v : t \mapsto Bu(\tau) = E(\tau)y. \tag{4.1.19}$$

Hence

$$\frac{d}{dt}v(\tau)y = A_E E(\tau)y = A_E BS(\tau)y =: A_S S(\tau)y, \tag{4.1.20}$$

where $A_S := A_E B$, for every $t > 0$. Furthermore, $P(\lambda) = (\lambda B - A_S)^{-1}$ since $B = R(\lambda)P^{-1}(\lambda)$ and $A_S = \lambda B - R^{-1}(\lambda)B$ by $A_E = \lambda - R^{-1}(\lambda)$. Thus, $\langle A_S, B\rangle$ is the generator of the empathy $\langle S(\tau), E(\tau)\rangle$.

With (4.1.20), we are left with the initial condition of the implicit Cauchy problem. Once again from (4.1.17), for each $y \in \Delta^2_{K'}$, $\lim_{t\to 0^+} BS(\tau)y = y$ since $(E(\tau))_{t>0}$ is a C_0-semigroup on $\Delta_{K'} \supset \Delta^2_{K'}$. Therefore, we arrive at a Hille-Yosida-Kisynski theorem for the implicit evolution equation on a non-closed dense isomorphic subspace of the solution space for the Hille-Yosida-Kisynski theorem for C_0-semigroups (Theorem 5.5 [22]) .

Theorem 5 (Hille-Yosida-Kisynski Generation-A)**.** *Consider the empathy pseudo-resolvent $\langle R(\lambda), P(\lambda)\rangle$. Under the assumption that R satisfies the Widder Growth Condition and the invertibility assumption, we can construct an empathy $\langle S(\tau), E(\tau)\rangle_{t>0}$ and a pair of generators $\langle A_S, B\rangle$ such that for $y \in \Delta^2_{K'}$, the implicit Cauchy problem* (1.1.1)-(1.1.2)*, Section 1.1, is satisfied when one sets $u(\tau) = S(\tau)y$ and A to be the generator A_S. Indeed, $P(\lambda) = (\lambda B - A)^{-1}$.*

Remark 6. *Technically, the operator $S(\tau)$ of an empathy $\langle S(\tau), E(\tau)\rangle_{\tau>0}$ is defined on a Banach space. It suffices to define $S(\tau)$ on the dense subspace $\Delta^2_{K'}$ of the Banach space $\Delta_{K'}$ since there is a unique extension by the boundedness of $S(\tau)$.*

However, it is important to note that just as in the case of the abstract Cauchy problem, $\Delta^2_{K'}$ represents the solution space for the implicit Cauchy problem (1.1.1)*-*(1.1.2)*, and not the domain of the evolution operator. The solution space for the abstract Cauchy problem is a non-closed dense subspace of the domain of the evolution operator, (Proposition 3.1.9 (h) [2] or Theorem 6.7 [12]) . Likewise, $\Delta^2_{K'}$ is a non-closed dense subspace of the domain $\Delta_{K'}$, of the evolution operator $E(\tau)$.*

Remark 7. *Our algebraic version of the Hille -Yosida-Kisynski theorem for the implicit Cauchy problem improves Theorem 8.2 [28], where Y was assumed to have the Radon-Nikodym property and $\|\lambda P(\lambda)\| = O(1)$. Theorem 8.2 [28] was based on the standard Hille-Yosida Theorem. In contrast, the proof of theorem 5 was based on Kisynski's equivalent algebraic formulation of the standard Hille-Yosida Theorem. It is thus, not surprising the proof of theorem 5 is algebraic in nature.*

4.1.4 Bochner Integral Representation

Empathy theory ([28]) assumes that the $R(\lambda)y$ and $P(\lambda)y$ are (Bochner) Laplace integrals. Indeed such representations force the boundedness of $R(\lambda)$ and $P(\lambda)$ (Theorem 3.8.2 [19]). The extra assumption that B is closed, ensures that this is the case for $\Delta^2_{K'}$ for the constructed empathy $\langle S(\tau), E(\tau)\rangle$ even though in [28], B need not be closed:

Proposition 4. *Assume the invertibility assumption. Let $y \in \Delta^2_{K'}$. If B is closed, then $R(\lambda)y = \int_0^\infty e^{-\lambda t}E(\tau)ydt$, and $P(\lambda)y = \int_0^\infty e^{-\lambda t}S(\tau)ydt$ are Bochner Laplace integrals in Y and X respectively; $\lambda > 0$.*

4.2 Star Implicit Cauchy Problem

We now lift the results of Section 4.1 into the more general setting of admissible homomorphisms. In this setting, the implicit Cauchy problem takes the more general form

$$\frac{d}{d\tau}\langle f, b' * x'(\tau)\rangle = \langle f, a' * x'(\tau)\rangle, \lim_{\tau\to 0^+}\langle f, b' * x'(\tau)\rangle = \langle f, \theta'_0\rangle, \qquad (4.2.1)$$

where $\langle\theta'_0, f\rangle = f(0)$ is the unit of $(\mathscr{A}_B, *)$; 0 is the identity of the group G. We call (4.2.1) the *star-implicit Cauchy equation*. It is important to note that for the special case of G being the trivial group, $a' = A\theta'_0$ and $b' = B\theta'_0$, the star-implicit Cauchy equation (4.2.1) becomes the implicit Cauchy equation of (1.1.1), Section 1.1, with initial condition (1.1.2). In this section, we take $\Phi := BUC(G, Z)$ and the topological group G to be locally compact.

Suppose Φ_X, Φ_Y are two closed translation invariant subspaces of Φ; Φ_X, Φ_Y are then Banach spaces. Heuristically, Φ_X, Φ_Y play the roles of the Banach spaces X, Y of Section 4.1, respectively. We say that the half-ray U-domained pair of functions $\langle \mathfrak{r}', \mathfrak{p}' \rangle$ is a *star empathy pseudo-resolvent pair on* $\langle \Phi_Y, \Phi_X \rangle$ if

$$\mathfrak{r}'(\lambda) - \mathfrak{r}'(\mu) = (\lambda - \mu)\mathfrak{r}'(\lambda) * \mathfrak{r}'(\mu); \tag{4.2.2}$$
$$\mathfrak{p}'(\lambda) - \mathfrak{p}'(\mu) = (\lambda - \mu)\mathfrak{p}'(\lambda) * \mathfrak{r}'(\mu), \tag{4.2.3}$$

for $\lambda, \mu \in U$; $\mathfrak{r}'(\lambda), \mathfrak{p}'(\lambda) : \Phi_Y \to Z$ are admissible homomorphisms with dualisms $\mathfrak{R}'(\lambda) : \Phi_Y \to \Phi_Y$, $\mathfrak{P}'(\lambda) : \Phi_Y \to \Phi_X$. Likewise, the pair $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is a *star-empathy* if $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ satisfies (i) the star-empathy relation $y'(\tau+\sigma) = y'(\tau) * x'(\sigma)$ and (ii) the star-semigroup relation $x'(\tau + \sigma) = x'(\tau) * x'(\sigma)$ for $\sigma, \tau > 0$.

The problem at hand is, given a star empathy pseudo-resolvent $\langle \mathfrak{r}', \mathfrak{p}' \rangle$, construct a 'generator' of a star-empathy $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ in the form of a homomorphism pair $\langle a', b' \rangle$ related to the star implicit Cauchy equation (4.2.1), where $\mathfrak{p}'(\lambda) = (\lambda b' - a')^{-1}$. We call this the *star generation problem (for implicit evolution equation).*

4.2.1 A First analysis of Star Empathy

The star-empathy $\mathfrak{X}', \mathfrak{Y}'$ play the roles of the empathy $\mathcal{S} := \{S(\tau)|\tau > 0\}, \mathcal{E} := \{E(\tau)|\tau > 0\}$ of [28], respectively. We analyze the star-empathy along the lines similar to [28], by constructing integral representations similar to those for classical semi-groups (Lemma 2.7, [28] and Appendix D.2, Lemma 1). Just as in the case of empathy theory (Lemma 2.7, [28]), these integral representations show what the generators $\langle a, b \rangle$ should look like. There are differences forced by the generality of the framework, though. A point that will become clear is that the analysis needs to be performed in the space Z because measurability is not transfered to the dualisms. On the other hand the dualisms lend themselves perfectly to pure algebra.

Suppose the families $\mathfrak{X}'$ and $\mathfrak{Y}'$ satisfy the star-empathy relation (3.2.15) (Section 3.2.3). The hypotheses of Theorem 7, Section 3.2.3, will be continued here. In particular, that the family $\mathfrak{Y}'$ is Laplace-closed with respect to itself and $\mathfrak{X}'$ is closed with respect to $\mathfrak{Y}'$.

Theorem 6. *The family $\mathfrak{Y}'$ is a star-semigroup. In addition, the following identities hold:*

$$y'(\tau) * \mathfrak{r}'(\lambda) = \mathfrak{r}'(\lambda) * y'(\tau); \tag{4.2.4}$$
$$x'(\tau) * \mathfrak{r}'(\lambda) = \mathfrak{p}'(\lambda) * y'(\tau). \tag{4.2.5}$$

Proof. The first assertion is a consequence of the identity $\langle f, x'(\rho+\sigma+\tau)\rangle = \langle f, x'(\rho)*y'(\sigma+\tau)\rangle = \langle f, x'(\rho+\sigma)*y'(\tau)\rangle = \langle f, x'(\rho)*[y'(\sigma)*y'(\tau)]\rangle$ derived from(3.2.15) (Section 3.2.3). By taking the Laplace transform with respect to ρ at λ one obtains $\langle f, \widehat{y'}(\lambda) * [x'(\sigma+\tau) - x'(\sigma) * x'(\tau)]\rangle = 0$ for all λ and all f. From the uniqueness of the Laplace transform [2, Theorem 6.2.3] the conclusion is evident.

To derive the identities (4.2.4) and (4.2.5) it is sufficient to observe that $y'(\sigma) * y'(\tau) = y'(\tau) * y'(\sigma)$ and $x'(\sigma) * y'(\tau) = x'(\tau) * y'(\sigma)$. □

As a consequence of Theorem 12, Section 3.5, applied to $\mathfrak{X}'$ we have

Corollary 2. *The family $\mathfrak{X}'$ is locally uniformly bounded in norm and exponentially bounded.*

4.2.2 Integral Representations

We proceed to study the Laplace transforms $\mathfrak{r}'(\lambda), \mathfrak{p}'(\lambda)$ and in the process use the associated dualisms $\mathfrak{R}'(\lambda), \mathfrak{P}'(\lambda)$ without assuming d-measurability. The dualisms, therefore, cannot be considered as Laplace transforms; they are induced integrals. It is expedient, though, to note that in terms of dualisms, the identities (3.2.17)-(3.2.18), (4.2.4) -(4.2.5) for the pair $\langle \mathfrak{r}'(\lambda), \mathfrak{p}'(\lambda)\rangle$, transcribe into the following identities for the pair $\langle \mathfrak{R}'(\lambda), \mathfrak{P}'(\lambda)\rangle$:

Theorem 7. *Consider the pair of operator valued dualisms $\mathfrak{R}'(\lambda), \mathfrak{P}'(\lambda)$. Then,*

$$\mathfrak{R}'(\lambda) - \mathfrak{R}'(\mu) = (\mu - \lambda)\mathfrak{R}'(\lambda)\mathfrak{R}'(\mu) = (\mu - \lambda)\mathfrak{R}'(\mu)\mathfrak{R}'(\lambda); \quad (4.2.6)$$
$$\mathfrak{P}'(\lambda) - \mathfrak{P}'(\mu) = (\mu - \lambda)\mathfrak{P}'(\lambda)\mathfrak{R}'(\mu) = (\mu - \lambda)\mathfrak{P}'(\mu)\mathfrak{R}'(\lambda). \quad (4.2.7)$$

and

$$Y'(\tau)\mathfrak{R}'(\lambda) = \mathfrak{R}'(\lambda)Y'(\tau); \quad (4.2.8)$$
$$\mathfrak{P}'(\lambda)Y'(\tau) = X'(\tau)\mathfrak{R}'(\lambda). \quad (4.2.9)$$

Proof. Recall that the map Γ is an algebra homomorphism (equation(2.1.8), Section 2.1.1): the algebraic relations in $*$ are then transcribed as relations in $\circ$. □

Next we define the domains $\Delta_{Y'} = \mathfrak{R}'(\lambda)[\Phi_Y]$ and $\Delta_{X'} = \mathfrak{P}'(\lambda)[\Phi_Y]$, both being vector subspaces of Φ. From (4.2.6)-(4.2.7) it is clear that these domains do not depend on the choice of λ. From the identities (4.2.8)-(4.2.9) it is seen that

$$Y'(\tau)[\Delta_{Y'}] \subset \Delta_{Y'} \text{ and that } X'(\tau)[\Delta_{Y'}] \subset \Delta_{X'}.$$

Now we are set up for the following integral representation of the action of the homomorphisms $x'(\tau), y'(\tau)$ on the subspace $\Delta_{Y'}$:

Lemma 2. *Suppose that* $f = \mathfrak{R}'(\lambda)f_\lambda \in \Delta_{Y'}$. *Then*

$$\langle f, y'(\tau)\rangle = \exp\{\lambda\tau\}\left[\langle f, \theta_0'\rangle - \int_0^\tau \exp\{-\lambda\sigma\}\langle f_\lambda, y'(\sigma)\rangle\, d\sigma\right], \qquad (4.2.10)$$

$$\langle f, x'(\tau)\rangle = \exp\{\lambda\tau\}\left[\langle f_\lambda, \mathfrak{p}'(\lambda)\rangle - \int_0^\tau \exp\{-\lambda\sigma\}\langle f_\lambda, x'(\sigma)\rangle\, d\sigma\right]. \quad (4.2.11)$$

Proof. To derive (4.2.10) we note that

$$\langle f, y'(\tau)\rangle = \langle \mathfrak{R}'(\lambda)f_\lambda, y'(\tau)\rangle = \langle f_\lambda, y'(\tau) * \mathfrak{r}'(\lambda)\rangle = \langle f_\lambda, \mathfrak{r}'(\lambda) * y'(\tau)\rangle,$$

by (4.2.4) in the final step. Then

$$\langle f, y'(\tau)\rangle = \int_0^\infty e^{-\lambda\sigma}\langle f_\lambda, y'(\sigma) * y'(\tau)\rangle d\sigma = \int_0^\infty e^{-\lambda\sigma}\langle f_\lambda, y'(\sigma+\tau)\rangle d\sigma,$$

since $y'(\tau)$ is a $*$-semigroup. By the change of variable $\sigma' := \sigma + \tau$ and the relation $\int_\tau^\infty = \int_0^\infty - \int_0^\tau$,

$$\langle f, y'(\tau)\rangle = e^{\lambda\tau}[\langle f_\lambda, \mathfrak{r}_\lambda\rangle - \int_0^\tau e^{-\lambda\sigma}\langle f_\lambda, y'(\sigma')\rangle d\sigma'].$$

We are done on noting that $\langle f_\lambda, \mathfrak{r}'(\lambda)\rangle = \langle \mathfrak{R}'(\lambda)f_\lambda, \theta_0'\rangle = \langle f, \theta_0'\rangle$. The representation (4.2.11) is obtained in a likewise manner. □

The 'domain' $\langle \Delta_{Y'}, \Delta_{X'}\rangle$ play an important role in the behaviour of $x'(\tau)$ and $y'(\tau)$ at $\tau = 0$. This follows from Lemma 2:

Corollary 3 (Behaviour at the Origin)**.** *For* $f \in \Delta_{Y'}$, $\lim_{\tau\to 0^+}\langle f, y'(\tau)\rangle = \langle f, \theta_0'\rangle$. *There exists a linear map* $j_0' : \Delta_{Y'} \to Z$ *such that* $\lim_{\tau\to 0^+}\langle f, x'(\tau)\rangle = \langle f, j_0'\rangle$. *If every dualism* $\mathfrak{R}'(\lambda); \lambda > 0$ *is one-to-one, then* j_0' *has the representation*

$$\langle f, j_0'\rangle = \langle f_\lambda, \mathfrak{p}'(\lambda)\rangle. \qquad (4.2.12)$$

Proof. The first assertion follows directly from (4.2.10). The second assertion is more difficult. From (4.2.11) it is seen that the limit exists and equals $\langle f_\lambda, \mathfrak{p}'(\lambda)\rangle$.

Suppose $f = \mathfrak{R}'(\lambda)f_\lambda = \mathfrak{R}'(\mu)f_\mu$. To show that the representation (4.2.12) is independent of λ, it suffices to show $\mathfrak{P}'(\lambda)f_\lambda = \mathfrak{P}'(\mu)f_\mu$: $\mathfrak{P}'(\lambda)f_\lambda = \mathfrak{P}'(\mu)[f_\lambda + (\mu-\lambda)\mathfrak{R}'(\mu)f_\mu]$ by (4.2.7); $f_\mu = f_\lambda + (\mu-\lambda)\mathfrak{R}'(\mu)f_\mu$ by the invertibility of $\mathfrak{R}'(\mu)$ and (4.2.6). □

If it is assumed that *for some* $\xi > 0$ *the dualism* $\mathfrak{P}'(\xi)$ *is invertible*, then there is a purely algebraic proof, based on (4.2.6) and (4.2.7), that every $\mathfrak{P}'(\lambda)$ and every $\mathfrak{R}'(\lambda)$ is invertible (see [28]). Indeed, the linear map $\mathfrak{J}' = \mathfrak{P}'(\lambda)[\mathfrak{R}'(\lambda)]^{-1} : \Delta_{Y'} \to \Delta_{X'}$ does not depend on λ and

$$\langle f, j'_0 \rangle = \langle f_\lambda, \mathfrak{p}'(\lambda) \rangle = \langle \mathfrak{J}' f, \theta'_0 \rangle.$$

Proposition 5. *The operator* $\mathfrak{J}' := \mathfrak{P}'(\lambda)[\mathfrak{R}'(\lambda)]^{-1}$ *is independent of the choice of* λ*. If it is assumed that for some* $\xi > 0$ *the dualism* $\mathfrak{P}'(\xi)$ *is invertible, then* $\mathfrak{J}'$ *maps* $\Delta_{Y'}$ *onto* $\Delta_{X'}$ *in a one-to-one way.*

Proof. The crux of the proof is that $R : \lambda \mapsto \mathfrak{R}'(\lambda)$ is a pseudo-resolvent on Φ; that is, a $\mathrm{Hom}_B(\Phi, \Phi)$-valued function defined on a scalar $\lambda \in U \subset \mathbb{C}$ satisfying (4.2.6); Φ is a Banach space. The resolvent equation (4.2.6) ensures that every $\mathfrak{R}'(\lambda)$ has common null space (kernal) N_E, and common range $\Delta_{Y'}$. Now, R becomes the resolvent exactly when $N_E = \{0\}$: $N_E = \{0\}$ if and only if $\mathfrak{R}'(\lambda)$ is the resolvent $R(\lambda, A)$ of a closed operator $A := \lambda 1_X - \frac{1}{\mathfrak{R}'(\lambda)}$, this representation being independent of the choice of λ (Theorem 1, Chapter VIII, p. 216 [33]).

Let $f \in \mathrm{Dom}(\mathfrak{R}'(\lambda)^{-1}) = \Delta_{Y'}$; $f := \mathfrak{R}'(\lambda) f_\lambda$; $f_\lambda \in Y$. Then $\mathfrak{P}'(\lambda)\mathfrak{R}'(\lambda)^{-1} f$ $= \mathfrak{P}'(\lambda) f_\lambda = \mathfrak{P}'(\mu)\mathfrak{R}'(\lambda)^{-1} f + (\mu - \lambda)\mathfrak{P}'(\mu) f = \mathfrak{P}'(\mu)[\mu - (\lambda - \mathfrak{R}'(\lambda)^{-1})] f$. Now invoke the relation that $\mu - \mathfrak{R}'(\mu)^{-1} = \lambda - \mathfrak{R}'(\lambda)^{-1}$ ([33] p. 216). □

Remark 8. *The independence of representation of* $\mathfrak{J}'$ *in Proposition 5, does not require the invertibility the dualism* $\mathfrak{P}'(\xi)$*.*

The inverse of $\mathfrak{J}'$ has the representation $\mathfrak{J}'^{-1} = \mathfrak{R}'(\lambda)[\mathfrak{P}'(\lambda)]^{-1} =: B'$, and B' is independent of the choice of λ [6]. B' is a 'backward' one-to-one map from $\Delta_{X'}$ onto $\Delta_{Y'}$. Now applying (4.2.11) to $f := B'g$; $g = \mathfrak{P}'(\lambda) g_\lambda \in \Delta_{X'}$, the result is

$$\langle B'g, x'(\tau) \rangle = \exp\{\lambda\tau\} \left[\langle g, \theta'_0 \rangle - \int_0^\tau \exp\{-\lambda\sigma\} \langle g_\lambda, x'(\sigma) \rangle \, d\sigma \right]. \quad (4.2.13)$$

The above integral representation (4.2.13) allows us to differentiate $\langle B'g, x'(\tau) \rangle$ as the product $f(\tau)g(\tau)$, where $g(\tau) = \left[\langle g, \theta'_0 \rangle - \int_0^\tau \exp\{-\lambda\sigma\} \langle g_\lambda, x'(\sigma) \rangle \, d\sigma \right]$

[6] Originally ([26]), the map B' conceptualized a backward map from effect X to cause space Y in the sense of $BS(\tau)y = E(\tau)y$. However, this holds only for certain $y \in \mathfrak{Y}$. empathy theory got around this problem by adopting a resolvent based approach: let the resolvents P, R be the analogue to $\mathcal{S}, \mathcal{E}$ respectively; then we require analogously that B' transform P to R. In empathy theory, we achieve this by defining $B := R_\lambda P_\lambda^{-1}$ (independent of the choice of λ) by the invertibility assumption. This then does indeed hold for all $y \in Y$.

and $f(\tau) = e^{\lambda\tau}$. We differentiate by the product rule to get for almost all $\tau > 0$,

$$\frac{d}{d\tau}\langle B'g, x'(\tau)\rangle = \lambda\langle B'g, x'(\tau)\rangle - \langle[\mathfrak{P}'(\lambda)]^{-1}g, x'(\tau)\rangle. \qquad (4.2.14)$$

Inspired by (4.2.14), we define another 'backward' linear mapping $A' : \Delta_{X'} \to \Phi$ as $A' := \lambda B' - [\mathfrak{P}'(\lambda)]^{-1}$. Once again it can be proved (algebraically) from (4.2.7) that A' does not depend on the choice of λ.

The following identities on the action of the operators $B'X'(\tau), B'Y'(\tau)$ on the subspace $\Delta_{Y'}$ will be of importance to us:

Lemma 3. *Let $f = \mathfrak{R}'(\lambda)f_\lambda \in \Delta_{Y'}$. Then*

$$B'X'(\tau)f = \mathfrak{R}'(\lambda)Y'(\tau)f_\lambda = Y'(\tau)f; \qquad (4.2.15)$$

$$A'X'(\tau)f = Y'(\tau)[\lambda f - f_\lambda]. \qquad (4.2.16)$$

Proof. These identities are derived from Theorem 7, (4.2.8) and (4.2.9). □

Remark 9. *It is appropriate to remark that* (4.2.15) *is in some accordance with the theory of B-evolutions [26].*

The operators A' and B' induce admissible homomorphisms a' and b':

$$\langle f, a'\rangle := \langle A'f, \theta'_0\rangle; \qquad (4.2.17)$$

$$\langle f, b'\rangle := \langle B'f, \theta'_0\rangle. \qquad (4.2.18)$$

This follows from $\mathfrak{P}'(\lambda)^{-1} : \Delta_{X'} \to \Phi$ being translatable; $\mathfrak{P}'(\lambda)$ is a one-to-one dualism (Theorem 2, Section 2.1.1). Therefore, we can translate the operator identities (4.2.15) - (4.2.16) to (admissible) homomorphisms:

$$\langle f, b' * x'(\tau)\rangle = \langle f_\lambda, r'(\lambda) * y'(\tau)\rangle; \qquad (4.2.19)$$

$$\langle f, a' * x'(\tau)\rangle = \langle \lambda f - f_\lambda, y'(\tau)\rangle. \qquad (4.2.20)$$

We are now set up for integral representations of the action of the homomorphisms $b' * x'(\tau)$ on the subspace $\Delta_{Y'}$.

Theorem 8. *For $f = \mathfrak{R}'(\lambda)f_\lambda \in \Delta_{Y'}$,*

$$\langle f, b' * x'(\tau)\rangle = \exp\{\lambda\tau\}\left[\langle f, \theta'_0\rangle - \int_0^\tau \exp\{-\lambda\sigma\}\langle f_\lambda, y'(\sigma)\rangle\, d\sigma\right].$$

Proof. The equation (4.2.15) can be re-phrased in the form $\langle f, b' * x'(\tau)\rangle = \int_0^\infty \exp\{-\lambda\rho\}\langle f_\lambda, y'(\rho) * y'(\tau)\rangle\, d\rho$ by virtue of (4.2.19). Use of Theorem 6 and an obvious change of variable as in the proof of Appendix D.2, Lemma1, leads to the required result. □

Our final result is

Theorem 9. *Let* $\langle x'(\tau), y'(\tau)\rangle$ *be a star empathy. If* $\mathfrak{P}'(\xi)$ *is invertible for some* $\xi > 0$ *and* $f \in \Delta_{Y'}$ *then the implicit Cauchy problem*

$$\begin{aligned} \frac{d}{d\tau}\langle f, b' * x'(\tau)\rangle &= \langle f, a' * x'(\tau)\rangle \text{ for almost all } \tau; \\ \lim_{\tau\to 0^+}\langle f, b' * x'(\tau)\rangle &= \langle f, \theta'_0\rangle, \end{aligned}$$

is satisfied by $\langle a', b'\rangle$ *defined by* (4.2.17)-(4.2.18).

Proof. From Theorem 8, (4.2.19) and (4.2.20), $\frac{d}{d\tau}\langle f, b' * x'(\tau)\rangle = \lambda\langle f, b' * x'(\tau)\rangle - \langle f_\lambda, y'(\tau)\rangle = \langle \lambda f - f_\lambda, y'(\tau)\rangle = \langle f, a' * x'(\tau)\rangle$. The limit part is evident from Corollary 3. □

The pair $\langle a', b'\rangle$ will be called the *generator* of the empathy $\langle \mathfrak{X}', \mathfrak{Y}'\rangle$.

Remark 10. *The case of star-semigroups is also covered in the analysis above. Evidently this happens when* $\mathfrak{Y}' = \mathfrak{X}'$ *and then, of course,* $\mathfrak{p}' = \mathfrak{r}'$*. The homomorphism* b' *in this case reduces to* θ'_0*, the unit in the algebra of homomorphisms.*

4.3 Star Generation problem

We do not directly solve the star generation problem from the given homomorphism valued star empathy pseudo-resolvents $\langle \mathfrak{r}', \mathfrak{p}'\rangle$ on $\langle \Phi_Y, \Phi_X\rangle$. Instead, we first solve the generation problem with the operator valued dualisms $\langle \mathfrak{R}', \mathfrak{P}'\rangle; U = (0,\infty)$; $\mathfrak{R}'(\lambda) = \Gamma(\mathfrak{r}'(\lambda)), \mathfrak{P}'(\lambda) = \Gamma(\mathfrak{p}'(\lambda))$, without assuming d-measurability; $\mathfrak{R}'(\lambda), \mathfrak{P}'(\lambda)$ are induced integrals only.

Our first important observation is:

Proposition 6. $\mathfrak{R}'$ *is a pseudo-resolvent on* Φ_Y*;* $U = (0, \infty)$.

Proof. The map Γ translates (4.2.2) into the resolvent equation (4.2.6). □

4.3.1 T-Regularity Space

We now lift the results of Section 4.1.1 for the pseudoresolvent R on a Banach space Y into the (operator) dualism $\mathfrak{R}'$ on a function test space $\Phi = BUC(G, Z)$. Therefore, we will borrow the notations of Section 4.1.1.

Consider the pseudo-resolvent $\mathfrak{R}'$, where U is the half-ray $(0, \infty)$. From $\mathfrak{R}'$, we analogously construct (i) a *Kisynski regularity space* $\Delta_K \subset \Phi_Y$ defined as

$$\Delta_K := \{f \in \Phi_Y \,|\, \lim_{\lambda\to\infty} \|\lambda\mathfrak{R}'(\lambda)f - f\| = 0\}, \tag{4.3.1}$$

and (ii) an operator $A'_{Y'}$ defined by its graph

$$\{(f, f') \in \Phi_Y \times \Phi_Y \,|\, \lim_{\lambda\to\infty} \|\lambda(\lambda\mathfrak{R}'(\lambda)f - f) - f')\| = 0\}. \tag{4.3.2}$$

Proposition 2, Section 4.1.1, carries over word for word for the pair $\langle \Delta_K, A'_{Y'} \rangle$. We extend Proposition 2, Section 4.1.1, by showing that Δ_K is translation invariant.

Proposition 7. *Let* $\|\mathfrak{R}'(\lambda)\| = O(1)$. *Then*
(i) The Kisynski regularity space Δ_K *is a translation invariant closed subspace and as such it is a Banach space closed under translations.*
(ii) The generator $A'_{Y'}$ *taken as an operator is closed, where the inclusion* $Dom(A'_{Y'}) \subset \Delta_K$ *is dense.*
(iii) For each $\lambda \in U$, *(a)* $\mathfrak{R}'(\lambda)[\Delta_K] \subset \Delta_K$ *and (b)* $\mathfrak{R}'(\lambda)|_{\Delta_K}$ *is the resolvent* $\mathfrak{R}'(\lambda, A'_{Y'}) = (\lambda - A'_{Y'})^{-1}$. *That is,* $\mathfrak{R}'(\lambda)|_{\Delta_K}$ *is a 1-1 continuous extension of* $\mathfrak{R}'(\lambda)|_{\Delta_{Y'}}$.
(iv) $\mathfrak{R}' : \lambda \to \mathfrak{R}'(\lambda)|_{\Delta_K}$ *is locally analytical.*

Proof. For (i), it suffices to show that $\Delta_{Y'} = \mathfrak{R}'(\lambda)[\Phi_Y]$ is translation invariant [7]. This is immediate from the translation invariance of Φ_Y when one notes that $\mathfrak{R}'(\lambda)$ is a dualism and hence trivially commutes with the the shift map.

(ii)-(iv) follows similarly as in Proposition 2, Section 4.1.1. □

Our approach is to solve the generation problem initially with the (operator) dualisms and then backtrack these dualisms as homomorphisms. Therefore, we show that the dualism $\mathfrak{R}'(\lambda)$ is translatable.

Corollary 4. *Let* $\|\mathfrak{R}'(\lambda)\| = O(1)$. *Then,* $\mathfrak{R}'(\lambda)|_{\Delta_K}$ *is translatable for each* $\lambda \in U$.

Proof. The boundedness of $\mathfrak{R}'(\lambda)|_{\Delta_K}$ and the shift operators, transfers the translatability of $\mathfrak{R}'(\lambda)|_{\Delta_{Y'}}$ from the translation invariant subspace $\Delta_{Y'}$ to its closure Δ_K (also translation invariant) by writing each $f \in \Delta_K$ as a limit of a sequence in $\Delta_{Y'}$. □

[7]By the dense inclusion, for any $f \in \Delta_{K'}$, there exists a sequence $(f_n) \subset \Delta_{Y'}$ which converges to f. Then $R^s f_n \to R^s f$ by the boundedness of the shift map R^s. The Cauchy sequence $(R^s f_n)$ is a sequence in $\Delta_{Y'}$ by the translation invariance of $\Delta_{Y'}$. Thus, $R^s f \in \Delta_{K'}$ by the dense inclusion.

Under the Widder Growth Condition $\sup_{\lambda>0;k\in\mathbb{N}}\{\|[\lambda\mathfrak{R}'(\lambda)]^k\|\} < \infty$, Section 4.1.1 and Theorem 2 - 3 carries over word for word for the dualism $\mathfrak{R}'(\lambda)$.

Theorem 10 (Reconstructed Kisynski Regularity Space)**.** *Let $\mathfrak{R}'(\lambda); \lambda > 0$, satisfy the Widder Growth Condition. Then, there exists a unique (bounded) Banach algebra representation $T : L^1(0,\infty) \to \mathcal{L}(\Phi_Y)$ such that (i) $T(e_{-\lambda}) = \mathfrak{R}'(\lambda)$ and (ii) T algebraically reconstructs the Kisynski regularity space Δ_K as $\Delta_{K'}$*

$$\Delta_K = \overline{\Delta}_Y = \bigcup_{\phi\in L^1(0,\infty)} T(\phi)[\Phi_Y] =: \Delta_{K'}. \tag{4.3.3}$$

With the algebraic $\Delta_{K'}$, as opposed to the equivalent Δ_K, we construct a strongly continuous semigroup $Y'(\tau) : \Delta_{K'} \to \Delta_{K'}$ by right shift maps, R^τ, on $L^1(0,\infty)$ alone; for every $\tau \geq 0$ and $\phi \in L^1(0,\infty)$,

$$Y'(\tau)[T(\phi)] = [T(R^\tau\phi)], \tag{4.3.4}$$

where the right translate of ϕ by τ is the element $R^\tau\phi \in L^1(0,\infty)$; $R^\tau\phi(\xi) = \phi(\xi - \tau)$ if $\xi \in (\tau,\infty)$ and $R^\tau\phi(\xi) = 0$ if $\xi \in (0,\tau]$. We shall call $\Delta_{K'}$ the *T-Kisynski regularity space* since T uniquely reconstructs the space Δ_K.

Theorem 11 (C_0-semigroup on T-Kisynski Regularity space)**.** *The construction* (4.3.4) *uniquely defines a C_0-semigroup $Y' = (Y'(\tau) : \Delta_{K'} \to \Delta_{K'})_{\tau\geq 0}$ on the T-Kisynski regularity space $\Delta_{K'}$. That is, for $f = T(\phi)f' \in \Delta_{K'}$,*

$$Y'(\tau)f = Y'(\tau)[T(\phi)f'],$$

is independent of representation of f. $Y'(0)$ is the identity operator on $\Delta_{K'}$ and not the whole space Y.

From this point onward we shall always assume the Widder Growth Condition for any pseudo-resolvent $\mathfrak{R}'$.

4.3.2 T^2-Regularity Space

We now lift the results of Section 4.1.2, for the generalized resolvent P on a Banach space Y into the (operator) dualism $\mathfrak{P}'$ on a function test space $\Phi = BUC(G,Z)$. Therefore, we will borrow the notations of Section 4.1.2.

Consider the subspace, $\Delta^2_{K'}$, of the T-regularity space:

$$\Delta^2_{K'} := \mathfrak{R}'(\lambda)[\Delta_{K'}]. \tag{4.3.5}$$

We extend Proposition 3, Section 4.1.2 by showing that $\Delta^2_{K'}$ is translation invariant

Proposition 8. *The T^2-Kisynski regularity space, $\Delta^2_{K'} := \mathfrak{R}'(\lambda)[\Delta_{K'}]$ is (i) an isomorphic dense subspace of $\Delta_{K'}$, (ii) independent of representation $\lambda \in U$ and (iii) is translation invariant. Thus, the inclusion*

$$\Delta^2_{K'} \subset \Delta_{K'},$$

is a dense inclusion.

Proof. (i)-(ii) is proved as in Proposition 3, Section 4.1.2. For (iii), the translation invariance of $\Delta^2_{K'} := \mathfrak{R}'(\lambda)[\Delta_{K'}]$ follows from the translation invariance of $\Delta_{K'}$ by the commutativity of the shift operators R^s and the dualisms $\mathfrak{R}'(\lambda)$. □

We may now define a translatable operator $\mathfrak{P}'(\lambda)[\mathfrak{R}'(\lambda)|_{\Delta_K}]^{-1}$ on Δ^2_K. We shall denote this operator as $\mathfrak{J}'(\lambda)|_{\Delta^2_K}$. Formally,

Lemma 4. *The translatable operator $\mathfrak{J}'(\lambda)|_{\Delta^2_K} : \Delta^2_K \to \Delta_{X'}$ is independent of representation $\lambda \in U$; $\Delta_{X'} := \mathfrak{P}'(\lambda)[\Phi_{Y'}]$. We thus, denote $\mathfrak{J}'(\lambda)|_{\Delta^2_K}$ as $\mathfrak{J}'|_{\Delta^2_K}$.*

Proof. The independence of representation of $\mathfrak{J}'(\lambda)|_{\Delta^2_K}$ follows word for word from the proof of Proposition 5, Section 4.2.2. The translatability follows from composition preserving dualisms. □

Then as in Lemma 1, Section 4.1.2, $\Delta^2_{K'}$ is an invariant subspace of each $Y'(\tau)$, $\tau \geq 0$ and $T(\phi)$, $\phi \in L^1(0, \infty)$:

Lemma 5. *For each $\tau \geq 0$, $Y'(\tau)[\Delta^2_{K'}] \subset \Delta^2_{K'}$. Indeed, for every $\phi \in L^1(0, \infty)$, $T(\phi)[\Delta^2_{K'}] \subset \Delta^2_{K'}$:*

With the T^2-regularity space, we construct an empathy (dualism) $\langle X'(\tau), Y'(\tau)\rangle_{\tau>0}$ from the pseudo-resolvent pair $\langle \mathfrak{R}', \mathfrak{P}'\rangle$, just as in equation(4.1.8), Section 4.1.2. The construction of the second transition map $(X'(\tau) : \Delta^2_{K'} \to \Delta_{Y'})_{\tau\geq 0}$ on the T^2 regularity space is also inspired by the critical identity (5), Lemma 2.3 [28]. For every $\tau \geq 0$ and $\phi \in L^1(0, \infty)$

$$X'(\tau)[\mathfrak{R}'(\lambda)T(\phi)] := \mathfrak{P}'(\lambda)[Y'(\tau)T(\phi)], \tag{4.3.6}$$

where $\Delta_{X'} = \mathfrak{P}'(\lambda)[\Phi_Y]$.

Theorem 12 (Empathy on T^2-Kisynski Regularity space). *The construction (4.3.6) uniquely defines a transition map $X' := (X'(\tau) : \Delta^2_{K'} \to$*

$\Delta_{X'})_{\tau>0}$, *on the* T^2*-Kisynski regularity space* $\Delta^2_{K'}$. *That is, for* $f = \mathfrak{R}'(\lambda)T(\phi)f' \in \Delta^2_{K'}$,

$$X'(\tau)f = X'(\tau)[\mathfrak{R}'(\lambda)T(\phi)f'] = \mathfrak{P}'(\lambda)[Y'(\tau)T(\phi)f'],$$

is independent of representation of f. *Indeed, for every* $f = \mathfrak{R}'(\lambda)T(\phi)f' \in \Delta^2_{K'}$,

$$X'(\tau+\sigma)f = X'(\tau)Y'(\sigma)f. \tag{4.3.7}$$

On the T^2*-regularity space* $\Delta^2_{K'}$,

$$X'(0) = \mathfrak{J}', \text{that is, } \lim_{\tau\to 0+} X'(\tau)f = \mathfrak{J}'f, \tag{4.3.8}$$

$f \in \Delta^2_{K'}$.

Remark 11. *The constructed empathy* $\langle X'(\tau), Y'(\tau)\rangle$ *can be well defined at* $\tau = 0$ *on* $\Delta_{K'} = \overline{\Delta_{Y'}}$ *by* (4.3.6) *since* $Y'(0)$ *is well defined on* $\Delta_{K'}$. *However, since the empathy relation* $X'(\tau+\sigma) = X'(\tau)Y'(\sigma)$ *only makes sense for* $\sigma, \tau > 0$, *we do not define* $\langle X'(\tau), Y'(\tau)\rangle$ *at* $\tau = 0$. *Therefore, the notation* $X'(0) = \mathfrak{J}'$ *on* $\Delta^2_{K'}$ *means* $\lim_{\tau\to 0+} X'(\tau)f = \mathfrak{J}'f$ *for* $f \in \Delta^2_{K'}$. *Note however a direct substitution* $\tau = 0$ *into* (4.3.6) *also yields* $X'(0) = \mathfrak{J}'$.

Corollary 1, Section 4.1.2, carries over word for word for the dualism $\mathfrak{P}'(\lambda)$:

Corollary 5. *The homomorphism* $T' = \mathfrak{J}'T$ *on the convolution algebra* $L^1(0,\infty)$ *represents* $\mathfrak{P}'(\lambda)$ *and generates* $X'(\tau)$ *on the domain* $\Delta^2_{K'}$ *as follows:*

$$T'(e_{-\lambda}) = \mathfrak{P}'(\lambda); T'(e_{-\lambda} * R^{\tau}\phi) = X'(\tau)[\mathfrak{R}'(\lambda)T(\phi)], \tag{4.3.9}$$

where $\phi \in L^1(0,\infty)$. T' *is not an algebra representation and need not be closed.*

Remark 12. *By* (4.3.9), *one can identify* $\langle \mathfrak{P}'(\lambda), X'(\tau)\rangle$ *with* T'. *Similarly, one can identify* $\langle \mathfrak{R}'(\lambda), Y'(\tau)\rangle$ *with* T. *Therefore, a pair* $\langle T, T'\rangle$ *is used to generate an empathy* $\langle Y'(\tau), X'(\tau)\rangle$. *One can identify the domains of the empathy* $\langle \Delta_{K'}, \Delta^2_{K'}\rangle$ *with the pair* $\langle T, T^2\rangle$.

4.3.3 Invertibility assumption

The invertibility assumption takes the form *for some* $\xi > 0$, *the dualism* $\mathfrak{P}'(\xi)$ *is invertible.* Then as in [28], there is a purely algebraic proof, based on (4.2.6) and (4.2.7), that every $\mathfrak{P}'(\lambda)$ and every $\mathfrak{R}'(\lambda)$ is invertible. This assumption with the relation $\Delta^2_{K'} \subset \Delta_{Y'}$ ensures that the critical identity (4.2.15) for $f \in \Delta^2_{K'}$. Indeed,

Proposition 9. *Let $\mathfrak{R}'$ satisfy the Widder Growth Condition; $U := (0, \infty)$. If we assume the invertibility assumption, then $\mathfrak{J}'|_{\Delta_{Y'}}$ of Proposition 5, Section 4.2.2, is a 1-1 closed extension of $\mathfrak{J}'|_{\Delta^2_{K'}}$.*

Proof. This is immediate from the relation $\Delta^2_{K'} \subset \Delta_{Y'}$. □

By Lemma 3, equation (4.2.15), Section 4.2.2, it is then immediate from the relation $\Delta^2_{K'} \subset \Delta_{Y'}$ that

$$B'X'(\tau)f = Y'(\tau)f \text{ for all } f \in \Delta^2_{K'}. \tag{4.3.10}$$

Indeed, the pair $\langle Y', X' \rangle$ satisfies the generation problem for the implicit Cauchy problem in the operator algebra $\mathrm{Hom}_B(\Phi, \Phi)$: set

$$u : \tau \mapsto u(\tau) = X'(\tau)f; f \in \Delta^2_{K'}. \tag{4.3.11}$$

Then apply B' to (4.3.11) and then invoke (4.3.10):

$$v : \tau \mapsto B'u(\tau) = Y'(\tau)f \text{ for each } f \in \Delta^2_{K'}. \tag{4.3.12}$$

Hence

$$\frac{d}{d\tau}v(\tau)f = A'_{Y'}Y'(\tau)f = A'_{Y'}B'X'(\tau)f; \tag{4.3.13}$$

$$= A'_S X'(\tau)f, \tag{4.3.14}$$

where $A'_S := A'_{Y'}B'$, for every $\tau > 0$. Note that (4.3.14) is true for all $\tau > 0$; this is an improvement of Theorem 9, Section 4.2.2 which is only true for almost all $\tau > 0$. Furthermore, $\mathfrak{P}'(\lambda) = (\lambda B' - A'_S)^{-1}$ since $B' = \mathfrak{R}'(\lambda)\mathfrak{P}'^{-1}(\lambda)$ and $A'_S = \lambda B' - \mathfrak{R}'^{-1}(\lambda)B'$ by $A'_{Y'} = \lambda - \mathfrak{R}'^{-1}(\lambda)$. Thus, $\langle A'_S, B' \rangle$ is the generator of the empathy $\langle Y', X' \rangle$.

4.3.4 Backtrack

We backtrack the operator valued pair $\langle Y', X' \rangle$ into the (admissible) homomorphism valued pair $\langle \mathfrak{Y}', \mathfrak{X}' \rangle$ by the θ'_0 map: $y'(\tau) := \theta'_0 Y'(\tau), x'(\tau) := \theta'_0 X'(\tau)$. Therefore, we need to show that the operators $Y'(\tau)$ and $X'(\tau)$ are translatable. We need the following lemma:

Lemma 6. *Consider the map $T : L^1(0, \infty) \to \mathcal{L}(\Phi_Y)$ of Section 4.3.1, Theorem 10. If every operator $T(e_{-\lambda})$ is translatable, then $T(\phi)$ is translatable for every $\phi \in L^1(0, \infty)$*

Proof. First consider $\psi \in S := \text{span}\{e_{-\lambda} | \lambda > 0\}$. Then the linearity of the shift map R^s ensures that $T(\psi)$ is translatable. Now consider an arbitrary $\phi \in L^1(0, \infty)$. Then there is a sequence $(\psi_n)_{n\geq 0}$ in S that converges in L^1 to ϕ (S is a total set of $L^1(0,\infty)$). Then for each fixed $f \in \Phi$,

$$R^s T(\psi_n) f \to R^s T(\phi) f$$

in the norm of Φ_Y since $T(\psi_n) \to T(\phi)$ in operator norm of $\mathcal{L}(\Phi_{_Y})$ (T is bounded) and $R^s : \Phi_{_Y} \to \Phi_{_Y}$ is bounded. Noting that $T(\psi_n) R^s f \to T(\phi) R^s f$ completes the proof. □

Proposition 10. *Let $\mathfrak{R}'$ satisfy the Widder Growth Condition; $U := (0,\infty)$ and $(Y'(\tau))_{\tau \geq 0}$ be the associated C_0-semi-group defined on the Banach space $\Delta_{_{K'}}$ of Theorem 11, Section 4.3.1. Let $(X'(\tau) : \Delta^2_{_{K'}} \to \Delta_{_{X'}})_{\tau\geq 0}$, be the transition map defined on the T^2-Kisynski regularity space $\Delta^2_{_{K'}}$ by equation (4.3.6), Theorem 12, Section 4.3.1. Then each $Y'(\tau) : \Delta_{_{K'}} \to \Delta_{_{K'}}$ and $X'(\tau) : \Delta^2_{_{K'}} \to \Delta_{_{X'}}$ is translatable.*

Proof. $Y'(\tau)$ is translatable on $\Delta_{_{K'}}$ [8] follows directly from Lemma 6 ($T(r(\lambda)) = \mathfrak{R}'(\lambda)$ is a dualism and so is translatable). From equation (4.3.6), Theorem 12, Section 4.3.1, the transition map $X'(\tau)$ is also translatable on $\Delta^2_{_{K'}}$ since composition of translatable mappings are translatable (Theorem 2, Section 2.1.1). □

Likewise, we backtrack the translatable operator valued pair of generators $\langle A'_S, B' \rangle$ into the pair $\langle a'_s, b' \rangle$ by the θ'_0 map: $a'_s := \theta'_0 A'_S, b := \theta'_0 B'$ are admissible (not necessarily bounded) homomorphisms $a'_s := \theta'_0 A'_S, b' := \theta'_0 B'$ both defined on $\Delta_{X'}$. By the invertibility assumption, $\mathfrak{P}'(\lambda)$ and $\mathfrak{R}'(\lambda)$ are one-to-one dualisms on the translation invariant subspace $\Phi_{Y'}$. Therefore, $B' := \mathfrak{R}'(\lambda)[\mathfrak{P}'(\lambda)]^{-1}$ is translatable (Theorem 2, Section 2.1.1). Now $A'_{Y'} = \lambda - \mathfrak{R}'(\lambda)^{-1}$. Thus, $A'_{Y'}$ and consequently $A'_S := A'_{Y'} B'$ are translatable by Theorem 2, Section 2.1.1.

Indeed, the relations (4.3.11) - (4.3.14) for the operator valued pair $\langle Y', X' \rangle$ carry over to the (admissible) homomorphism valued pair $\langle \mathfrak{Y}', \mathfrak{X}' \rangle$:

Lemma 7. *Let $f \in \Delta^2_{_{K'}}$. Then the pair $\langle \mathfrak{X}', \mathfrak{Y}' \rangle$ is a star empathy and*
(i) $\langle f, b' * x'(\tau) \rangle = \langle f, y'(\tau) \rangle$
(ii) $\frac{d}{d\tau} \langle f, b' * x'(\tau) \rangle = \langle f, a'_S * x'(\tau) \rangle$ for all $\tau > 0$

Proof. By Theorem 12, (4.3.7),

$$\langle f, x'(\tau) * y'(\sigma) \rangle = \langle X'(\tau) Y'(\sigma) f, \theta'_0 \rangle = \langle X'(\tau + \sigma) f, \theta'_0 \rangle = \langle f, x'(\tau + \sigma) \rangle.$$

[8] $\Delta_{_{K'}}$ is translation invariant.

For (i), by (4.3.10),

$$\langle f, b' * x'(\tau)\rangle = \langle B'X'(\tau)f, \theta_0'\rangle = \langle Y'(\tau)f, \theta_0'\rangle = \langle f, y'(\tau)\rangle.$$

For (ii), first note that the time derivative $\frac{d}{d\tau}X'(\tau)f$ commutes with the bounded operator θ_0' since the definition of the time derivative involves only linear operations and the limit operation. Therefore, by (4.3.14)

$$\begin{aligned} \frac{d}{d\tau}\langle f, b' * x'(\tau)\rangle &= \frac{d}{d\tau}\theta_0' B'X'(\tau)f = \theta_0'\frac{d}{d\tau}B'X'(\tau)f; \\ &= \theta_0' A_S' X'(\tau)f = \langle f, a_S' * x'(\tau)\rangle. \end{aligned}$$

□

We now address the initial condition of the implicit Cauchy problem. From (4.3.10), for each $f \in \Delta^2_{K'}$, $\lim_{\tau\to 0^+} B'X'(\tau)f = f$ since $(Y'(\tau))_{\tau>0}$ is a C_0-semigroup on $\Delta_{K'} \supset \Delta^2_{K'}$. Then, by the boundedness of θ_0',

$$\lim_{\tau\to 0^+}\langle f, b' * x'(\tau)\rangle = \langle f, \theta_0'\rangle,$$

for each $f \in \Delta^2_{K'}$. Therefore, we solve the generation problem for the star implicit evolution equation on a non-closed dense subspace of the regularity space for the Hille-Yosida-Kisynski theorem (Theorem 5.5 [22]) for C_0-semigroups.

Theorem 13 (Hille-Yosida-Kisynski Generation-B)**.** *Consider the star pseudo-resolvent pair $\langle \mathfrak{r}', \mathfrak{p}'\rangle$. Let $\langle \mathfrak{R}', \mathfrak{P}'\rangle$ denote the operator valued dualisms. Under the assumption that $\mathfrak{R}'$ satisfies the Widder Growth Condition and the invertibility assumption, we can construct an empathy $\langle \mathfrak{X}' = (x'(\tau)), \mathfrak{Y}' = (y'(\tau))\rangle_{\tau\geq 0}$ and a pair of generators $\langle a', b'\rangle$ such that for $f \in \Delta^2_{K'}$, the star implicit Cauchy problem* (4.2.1) *is satisfied when one sets the generators $b' := \theta_0' B'$ and $a' = a_s' := \theta_0' A_S'$. Indeed, $\mathfrak{p}'(\lambda) = (\lambda b' - a')^{-1}$.*

Proof. From $\mathfrak{P}'(\lambda) = (\lambda B' - A')^{-1}$, that is, $\mathfrak{P}'(\lambda)(\lambda B' - A') = (\lambda B' - A')\mathfrak{P}'(\lambda) = \mathbf{1}$, it is immediate that $\mathfrak{p}'(\lambda) = (\lambda b' - a')^{-1}$ when one notes that $\theta_0'(X'Y')(f) = \langle f, x' * y'\rangle$. □

4.3.5 Bochner Integral Representation

Empathy theory ([28]) assumes that $\mathfrak{r}'(\lambda)f, \mathfrak{p}'(\lambda)f$ are (Bochner) Laplace integrals. Indeed, such representation force the boundedness of $\mathfrak{r}'(\lambda)$ and $\mathfrak{p}'(\lambda)$ (Theorem 3.8.2 [19]). The extra assumption that the dualism $B' = \mathfrak{R}'(\lambda)\mathfrak{P}'^{-1}(\lambda)$ is closed, ensures that this is the case for $\Delta^2_{K'}$ for the constructed empathy $\langle x'(\tau), y'(\tau)\rangle$ even though in [28], B' need not be closed: we first show that the dualisms $\mathfrak{R}'(\lambda), \mathfrak{P}'(\lambda)$ are d-integrable:

Lemma 8. *Assume the invertibility assumption and that the dualism B' is closed. Let $f \in \Delta^2_{K'}$. Then the dualisms*

$$\mathfrak{R}'(\lambda)f = \int_0^\infty e^{-\lambda\tau} Y'(\tau) f d\tau; \tag{4.3.15}$$

$$\mathfrak{P}'(\lambda)f = \int_0^\infty e^{-\lambda\tau} X'(\tau) f d\tau, \tag{4.3.16}$$

are Bochner Laplace integrals in Φ; $\lambda > 0$.

Proof. The representation (4.3.15) follows from $\Delta^2_{K'} \subset \Delta_{K'}$ and the assumption that $\mathfrak{R}'$ satisfies the Widder Growth Condition; $U := (0, \infty)$ (Theorem 5.5 [22]). From (4.3.10) and the commutativity of the Bochner integral with the closed operator B', we have: $\mathfrak{R}'(\lambda)f = B' \int_0^\infty e^{-\lambda\tau} X'(\tau) f d\tau$. We are done on noting that $\mathfrak{J}'_\lambda = (B')^{-1}$. □

The boundedness of θ'_0 immediately ensures that representations of $\mathfrak{r}'(\lambda)f$ and $\mathfrak{p}'(\lambda)f$ as Bochner Laplace integrals.

Corollary 6. *Assume that $\mathfrak{R}'$ satisfy the Widder Growth Condition and the invertibility assumption. Let $f \in \Delta^2_{K'}$. If the dualism B' is closed, then,*

$$\mathfrak{r}'(\lambda)f = \int_0^\infty e^{-\lambda\tau} y'(\tau) f d\tau; \mathfrak{p}'(\lambda)f = \int_0^\infty e^{-\lambda\tau} x'(\tau) f d\tau, \tag{4.3.17}$$

are Bochner Laplace integrals in Z; $\lambda > 0$.

Remark 13. *Under the additional assumptions of Lemma 8, we have a quick proof of Lemma 2, equations* (4.2.10)-(4.2.11)*, Section 4.2.2. The Laplace representations* (4.3.15)- (4.3.16) *for $X'(\tau)f$ and $Y'(\tau)f$ of Lemma 8, ensures that the integral representations of Lemma 2.7, [28], carry over word for word to $X'(\tau)f$ and $Y'(\tau)f$ since the dualisms $X'(\tau)$ and $Y'(\tau)$ are operator valued. Then we are done on applying the bounded operator θ'_0. Indeed, noting that $\Delta^2_{K'} \subset \Delta_{Y'}$, Theorem 8, Section 4.2.2 carries over word for word.*

Therefore, under this stronger condition of d-integrability, we do not obtain integral representations of the homomorphism actions for $\langle f, x'(\tau)\rangle$ and $\langle f, y'(\tau)\rangle$ directly, as we did in Section 4.2.2. Instead, we first obtain integral representations of the *operator valued* actions for the dualisms $X'(\tau)f$ and $Y'(\tau)f$ and then 'backtrack' these operator valued representations into homomorphism valued representations by the relations $\langle f, x'(\tau)\rangle = \langle X'(\tau)f, \theta'_0\rangle$ and $\langle f, y'(\tau)\rangle = \langle Y'(\tau)f, \theta'_0\rangle$. This approach is reminiscent of the philosophy of the classical Laplace transform approach to solving differential equations which converts the original calculus problem into a 'dual' algebra world: the dualisms that live here are algebraic induced Laplace transforms. Thus, the map Γ maps analytic homomorphisms into algebraic dualisms.

Chapter 5

Feller Semigroups and Processes

So far in applications, admissible homomorphisms were of an elementary nature: canonical homomorphisms of the form $A\theta_0'$ where A was a bounded operator; if G is the trivial group, admissible homomorphisms are classical operators. In this chapter, we show Feller semigroups and processes are admissible homomorphisms and the Feller convolution is the product $*$ of $(\mathscr{A}_\Phi, *)$ by another judicious choice of test space Φ.

5.1 Scalar Test Spaces

A real valued random variable $\mathbf{Y}$ uniquely induces a finite regular Borel measure $\mu_{\mathbf{Y}}$ (called the *distribution or probability law of* $\mathbf{Y}$) or a unique distribution function, $F_{\mathbf{Y}}$, of $\mu_{\mathbf{Y}}$. The Riesz Representation Theorem identifies $\mu_{\mathbf{Y}}$ *isometrically* with a bounded admissible homomorphism, $y'_{\mu_{\mathbf{Y}}}$, on the scalar test space, $\Phi_S := C_0(G, Z); Z := \mathbb{R}$ [1] .

$$\langle f, y'_{\mu_{\mathbf{Y}}} \rangle \quad = \quad \int_{\mathbb{R}} f d\mu_{\mathbf{Y}} =: E_{\mu_{\mathbf{Y}}} f. \tag{5.1.1}$$

The value of $y'_{\mu_{\mathbf{Y}}}$ at f is nothing but the "expectation" of $f(\mathbf{Y})$ in terms of $\mu_{\mathbf{Y}}$ [31, Theorem 5.6 p. 291]. The integral in (5.1.1) is defined as in [11, IV.8.10]. Since $\mu_{\mathbf{Y}}$ is a probability measure, $y'_{\mu_{\mathbf{Y}}}$ is bounded. In fact $\|y'_{\mu_{\mathbf{Y}}}\| = 1$. Thus $y'_{\mu_{\mathbf{Y}}} \in \mathscr{A}_B$. We shall always take the topological group G as the locally compact abelian group $(\mathbb{R}, +)$ in this chapter.

Remark 1. *Two finite Borel measures μ and ν on $\mathbb{R}$ are identical if $\int f d\mu = \int f d\nu$ for all $f \in \Phi := BC(G, Z); G, Z = \mathbb{R}$ ([3], Proposition 1.2.20). Thus*

[1] Every bounded homomorphism on Φ_S is admissible ([18] Lemma 19.5).

the test space Φ is a domain of definition for finite Borel measures. Indeed, in probability theory, the weak convergence of measures is defined on Φ.

Remark 2. *We work with the distribution μ instead of the random variable itself $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ in order to shift focus from the complicated sample space $(\Omega, \mathcal{F}, \mathbb{P})$ to the easier space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$; that is, $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ effectively becomes the new sample space. This change of sample space is fine when we are concerned with properties like the expectations of random variables: $E(Y) = \int_\Omega Y(\omega) d\mathbb{P}(\omega) = \int_\mathbb{R} y d\mu(y); E(f(Y)) = \int_\Omega f(Y)(\omega) d\mathbb{P}(\omega) = \int_\mathbb{R} f(y) d\mu(y)$.*

William Feller's treatment of limit theorems in probability [17, Chap. VIII], on the other hand, leads to the larger test space $C[\mathbb{R}]$ of continuous real-valued functions f on $\mathbb{R}$ for which the limits at $\pm\infty$ exist. With obvious adaptation of notation, let $\Phi = C[\mathbb{R}; Z]$. Since members of Φ are bounded and uniformly continuous, it falls well within the framework of a Banach test space $\Phi = \mathrm{BUC}(\mathbb{R}, Z)$ of Chapter 3. Indeed, $C[\mathbb{R}]$ is isomorphic to $C[0,1]$ and for any linear functional x' on $C[\mathbb{R}]$ there exists a unique Borel measure μ on $\mathbb{R}$ and two unique numbers a and b such that $\langle f, x' \rangle = \int f d\mu + af(+\infty) + bf(-\infty)$ ([3], Corollary 5.2.10).

Remark 3. *The so-called 'weak convergence' of probability theory describes a mode of convergence of the distributions of the random variables, that is, measures on $\mathbb{R}$, and not the random variables themselves; it is not the values of the random variables converging but the measures, taken as linear functionals, in the weak*-topology, on the function space, $BC(\mathbb{R})$, of real-valued, bounded and continuous functions on $\mathbb{R}$ (Definition18.1, [24]). We shall nevertheless, refer to this type of convergence as weak convergence from this point onwards. The Central Limit Theorem is an example of weak convergence.*

Sometimes, $C_0(\mathbb{R})$ is not the best choice of test space to study the weak convergence of measures on $\mathbb{R}$ which escape to infinity. The sequence of probability distributions $\mu_n := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n$ converges to an improper distribution $\frac{1}{2}\delta_0$ (mass is a half) in $C_0(\mathbb{R})$. In $C[\mathbb{R}]$, μ_n converges to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_\infty$ (mass is 1); $\langle f, \delta_\infty \rangle = f(+\infty)$.

Neither is the test space $BC(\mathbb{R})$ for reasons explained in Section 3.1.

5.2 The Feller convolution

A time homogenous Markov transition probability function Q plays a dual role as a Borel measurable function and a probability measure ([17], VI. 11, Definition 1). To capture this dual role, Feller introduces the Feller star or convolution $A \star B$ (Definition 1, [17], p. 141), to denote integration with respect to a measure A [2] as defined in [11, IV.8.10] or [31, Definition 4.21 p. 227]. The idea is based on the well-known result that the distribution function of the sum of two independent random variables with distribution functions F and G is given by the convolution function $F \star G$ (notation of [17, V, equation (1.10) p. 131])

$$F \star G(x) = \int_{-\infty}^{\infty} G(x-y)F\{dy\}, \tag{5.2.1}$$

and the observation that for functions in the space $C[\mathbb{R}]$, the convolution function $F \star f$ belongs to $C[\mathbb{R}]$

$$x \to (F \star f)(x) = \int_{-\infty}^{\infty} f(x-y)F\{dy\}, = (\mathfrak{F}f)(x), \tag{5.2.2}$$

and therefore, the distribution function F defines a bounded linear operator $\mathfrak{F} : f \in C[\mathbb{R}] : F \star f \in C[\mathbb{R}]$ in $C[\mathbb{R}]$.

Indeed, the Feller convolution captures the operator representation of a *space-homogenous* time homogenous Markov transition probability function Q:

Definition 1. *([6] Definition 2.9.7) The unique operator $T_\tau : BC(\mathbb{R}) \to BC(\mathbb{R})$ associated with the Markov transition function Q is called the operator representation of Q_τ:*

$$(T_\tau f)(x) = \int_{\mathbb{R}} f(y)Q_\tau(x|\mathbb{R}) = E_{Q_\tau(x|\mathbb{R})}[f(y)].$$

We say Q has a space-homogenous density if $Q_\tau(x|\Gamma) = \int_\Gamma k_\tau(x,y)dy = \int_\Gamma \phi_\tau(x-y)dy$. Brownian motion is a typical example.

Proposition 1. *Consider a time homogenous Markov transition probability function $Q_\tau(x|\Gamma) := P\{Y(\sigma+\tau) \in \Gamma | Y(\sigma) = x\}$ which is independent of σ ([17] X.1 (1.1)). Then if Q has a space-homogenous density, the operator representation T_τ is a Feller convolution:*

$$(T_\tau f)(x) = (Q_\tau(x|\mathbb{R}) \star f)(x)$$

5.2.1 Capturing the Feller Convolution in the framework

We now show that the Feller convolution $\star$ is a special case of the product $*$ of admissible homomorphisms. In the light of Proposition 1, we formally define the Feller operator representation of the random variable $\mathbf{Y}$:

[2] The second factor B is free to be a function or a measure and in that case $A \star B$ is then accordingly a function or measure, respectively. For instance, $\mu \star f$ denotes the function $(\mu \star f)(x) := \int_{\mathbb{R}} f(x-y)d\mu(y)$; f is a function locally integrable near the origin and μ the σ-finite Borel measure and $\mu \star \nu$ denotes the convolution measure $\mu \star \nu(E) := \int_G \nu(E-y)d\mu(y)$; ν is a measure.

Definition 2. *Let* $\mathbf{Y}$ *be a random variable defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with probability distribution* μ. *Then we define the (Feller) operator representation of* $\mathbf{Y}$ *to be the dualism* $Y'_\mu := \Gamma(y'_\mu)$ *where,*

$$Y'_\mu f : x \mapsto \int_G f(x+y) d\mu(y), \tag{5.2.3}$$

where $G := \mathbb{R}$ *and* $f \in \Phi_S$ *or* $C[\mathbb{R}]$.

Remark 4. *Equation* (5.2.3) *coincides with the definition of Section 2.3.21 [3].*

Theorem 1. *Let* y'_F *and* y'_G *be the homomorphisms induced by the probability distributions* F *and* G, *then* $y'_F * y'_G$ *is induced by the convolution* $F \star G$ *defined in* (5.2.1). *Moreover,*

$$^{y'_F * y'_G} f = Y'_F Y'_G f = ^{[F \star G]'} f. \tag{5.2.4}$$

Additionally, the dualism Y'_F *to* y'_F *has the form* $Y'_F f = \mathfrak{F} f$.

Proof. It is well known that for every Borel measurable function f integrable over G,

$$\int_G f(t) d(\mu * \nu)(t) = \int_{G^2} f(x+y) d(\mu \times \nu)$$

μ, ν are σ-finite Borel measures. Thus, the convolution measure of the measures identifiable with y'_F and y'_G in $\mathscr{A}_B$ coincides with the measure identifiable with their product linear functional $y'_F * y'_G \in \mathscr{A}_B$. □

Remark 5. *For our purposes, we are only concerned about the weak convergence of a sequence of probability distributions* μ_n *to another probability distribution, like* μ. *In such cases, this implies that the corresponding sequence of (operator) dualisms* $Y'_{\mu_n} := \Gamma(y'_{\mu_n})$ *in* $C[\mathbb{R}]$, *converges strongly to* Y'_μ *since* $BC(\mathbb{R}) \supset C[\mathbb{R}]$ *([3], Lemma 5.4.18 p. 173 or [17], VIII.3 Theorem 1 p. 255 / Theorem 1a. p. 257). Since* $\Phi_S \subset C[\mathbb{R}]$, *we obviously have strong convergence as well on this smaller test space.*

Let $\mathscr{M}(Z)$ denote the Banach space of distributions, that is, Z-valued regular measures of bounded variation on the σ-ring, $\mathscr{B}$, of Borel sets of G. Let Z be the Banach space $\mathbb{R}$. For the smaller test space $\Phi_S \subset C[\mathbb{R}]$, we have a sharper form of Theorem 1: the identification $J : \mu_{\mathbf{Y}} \mapsto y'_{\mu_{\mathbf{Y}}}$ is an isometric Banach space isomorphism between $\mathscr{M}(Z)$ and $\mathrm{Hom}_B(\Phi, Z)$ by the Riesz Representation. Then we rephrase Theorem 1 as follows:

Theorem 2 (Feller Star is the algebra product)**.** *Let* $Z := \mathbb{R}$ *and the test space* $\Phi := \Phi_S$. *Then the Feller star* $\star$ *can be identified with the product* $*$ *of* $\mathscr{A}_B$. *That is, the identification* $J : \mu_{\mathbf{Y}} \mapsto y'_{\mu_{\mathbf{Y}}}$ *lifts into an isometric* *algebra isomorphism between the unital Banach algebras* $(\mathscr{M}(Z), \star, \delta_0)$ *and* $(\mathscr{A}_B, *, \theta_0')$ *which preserves the identities*[3]. *The products* $\star$ *and* $*$ *will be used interchangeably.*

Remark 6. *In Feller's approach the space* Z *is simply the real line. As remarked in [17, X.6] the case of multivariate distribution functions is treated by letting* $G = \mathbb{R}^n$.

5.3 Convolution semigroups and beyond

Consider a family $\mathbf{Y} := \{\mathbf{Y}(\tau) | \tau \in \mathbb{T}\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call $\mathbf{Y}$ a *time continuous stochastic process* if the indexing set $\mathbb{T}$ is $(0, \infty)$ or $[0, \infty)$. Now $\mathbf{Y}$ has *stationary* increments if the *distribution* μ_τ of the increments $(\mathbf{Y}_{\sigma+\tau} - \mathbf{Y}_\sigma)$ depends only on the length τ of the interval and not on σ; $\mathbf{Y}$ has *independent* increments if the increments $(\mathbf{Y}(\tau_{j+1}) - \mathbf{Y}(\tau_j))$ are independent whenever $0 \le \tau_1 < \tau_2 < ... < \tau_n$; $1 \le j < n$; $n \ge 1$. We say $\mathbf{Y}$ is *autonomous* when it has stationary independent increments. We identify an autonomous processes $\mathbf{Y} := \{\mathbf{Y}(\tau) | \tau \in \mathbb{T}\}$ with its family of distributions $\{\mu_\tau | \tau \in \mathbb{T}\}$.

It is well known that the one-parametric family of probability distributions $\{\mu_\tau | \tau \in \mathbb{T}\}$ of an autonomous stochastic process satisfies the *Chapman-Kolmogorov relation* ([17] VI. 4, (4.1))

$$\mu_{\tau+\sigma} = \mu_\tau \star \mu_\sigma; \sigma, \tau > 0. \tag{5.3.1}$$

Indeed, Adam Bobrowski [3, Definition 7.6.1] introduced the more restrictive notion of a *convolution semigroup of measures* (defines μ_τ at $\tau = 0$) which is a star-semigroup of bounded homomorphisms:

Definition 3 (Convolution Semigroup of Measures)**.** *A family* $\{\mu_\tau | \tau \ge 0\}$ *of Borel measures on* $\mathbb{R}$ *is said to be a convolution semigroup of measures iff (a)* $\mu_0 = \delta_0$ *(b)* μ_τ *converges weakly to* δ_0*, as* $\tau \to 0+$*, and (c)*

$$\mu_{\tau+\sigma} = \mu_\tau \star \mu_\sigma; \sigma, \tau \ge 0. \tag{5.3.2}$$

It is well known ([17], VI 4, p. 177) that the only family of distributions $\{\mu_\tau | \tau \ge 0\}$ that play the role of the distributions of $(\mathbf{Y}_{\sigma+\tau} - \mathbf{Y}_\sigma)$ [4] of a time continuous *autonomous* process $\mathbf{Y}$ are those that satisfy (5.3.1).

Within the confines of Section 5.2, if we let $y_F'(\tau)$ be the (admissible) homomorphism induced by $F(\cdot, \tau)$, this leads to $y_F'(\tau)$ being a star-semigroup

[3] $J(\delta_0) = \theta_0'$; δ_0 is the Dirac point mass measure which assigns measure 1 to any set A containing 0 and measure 0 otherwise.

[4] The random variable $(\mathbf{Y}_{\sigma+\tau} - \mathbf{Y}_\sigma)$ has an infinitely divisible distribution. Hence the associated distribution satisfies equation (5.3.1).

of bounded homomorphisms. If we (partially) follow the Feller text, $y'_F(\tau)$ is strongly continuous and all the results of Section 3.2.2 apply. Moreover, the results of Section 3.2.3 concerning star-semigroups apply.

Proposition 2. *Let $y'(\tau)$ be the admissible homomorphism induced by μ_τ of a convolution semi-group of measures. Then, $y'(\tau)$ is a strongly continuous $*$-semigroup of bounded homomorphisms satisfying equation* (3.2.14)*, Section 3.2.3 and the dualisms $Y'(\tau)$ a strongly continuous Feller convolution semigroup [17, IX. 2, Definition 1].*

Proposition 3. *Every convolution semigroup $\{\mu_\tau|\tau \geq 0\}$ of Definition 3, Section 5.3, has a uniquely associated C_0-semi-group on $\Phi_S := C_0(G)$.*

Proof. Each distribution μ_τ or y'_{μ_τ} is isometrically identified with the *bounded* operator (dualism) $E(\tau) := Y'_{\mu_\tau} \in \mathrm{Hom}_B(\Phi_S)$. Now, Definition 3(a), Section 5.3, is equivalent to $y'_{\mu_0} = \theta'_0$ which is equivalent to $E(0) = \mathbf{1}_{\mathbf{\Phi_S}}$; Definition 3 (b), Section 5.3, implies $\langle f, y'_{\mu_\tau}\rangle \to \langle f, y'_{\mu_0}\rangle$ as $\tau \to 0^+$ for each $f \in \Phi_S$ which implies $E(\tau)f = f$ as $\tau \to 0^+$ for each $f \in \Phi_S$ (Remark 5, Section 5.2); Definition 3 (c), Section 5.3, is equivalent to $y'_{\mu_{\tau+\sigma}} = y'_{\mu_\tau} * y'_{\mu_\sigma}$ which is equivalent to $E(\tau+\sigma) = E(\tau) \circ E(\sigma)$. In short, $\mathcal{E}$ is a C_0-semigroup. □

5.3.1 Extended Chapman Kolmogorov Equation

Now, consider another time continuous stochastic process $\mathbf{X} := \{\mathbf{X}(\tau)|\tau > 0\}$. We say that $\mathbf{X}$ has *empathetic stationary independent increments* with $\mathbf{Y}$ if (i) the *distribution* ν_τ of the increments $(\mathbf{X}_{\sigma+\tau} - \mathbf{Y}_\sigma)$ depends only on the length τ of the interval and not on σ, for any σ, τ and (ii) the increments $(\mathbf{X}(\tau_{j+1}) - \mathbf{Y}(\tau_j))$ are independent of the increments $(\mathbf{Y}(\tau_j) - \mathbf{Y}(\tau_{j-1}))$ whenever $0 \leq \tau_1 < \tau_2 < ... < \tau_n$; $1 \leq j < n$; $n \geq 1$. Then, ν_τ satisfies the the *extended Chapman-Kolmogorov relation*

$$\nu_{\tau+\sigma} = \nu_\tau \star \mu_\sigma; \sigma, \tau > 0. \tag{5.3.3}$$

Remark 7. *It is important to note that $\{\nu_\tau|\tau > 0\}$ is not be defined at $\tau = 0$, although $\{\mu_\tau|\tau \in \mathbb{T}\}$ can be defined at $\tau = 0$. Furthermore, $\mathbf{Y}$ is* **not** *independent of $\mathbf{X}$. We call $\langle \mathbf{Y}, \mathbf{X}\rangle$ a convolution empathy.*

The notion of convolution empathy extends the notion of a convolution-semigroup. We call the pair of distributions $\langle \mu_\tau, \nu_\tau\rangle_{\tau>0}$ for which (5.3.1) and (5.3.3) holds, a *convolution empathy.*

Proposition 4. *Let $x'(\tau), y'(\tau)$ be the admissible homomorphisms induced by the distributions μ_τ, ν_τ of a convolution empathy $\langle \mu_\tau, \nu_\tau\rangle$, respectively; $\tau > 0$. Then the pair $\langle x'(\tau), y'(\tau)\rangle$ is a star-empathy and the dualisms $\langle X'(\tau), Y'(\tau)\rangle$ satisfies the empathy relation $Y'(\tau+\sigma) = Y'(\tau)X'(\sigma); \sigma, \tau > 0$.*

Proof. From Theorem 1, it is immediate the pair $\langle x'(\tau), y'(\tau)\rangle$ is a star-empathy. Then the empathy relation is immediate from Γ preserving $*$ as $\circ$. □

Convolution semigroups are complexly intertwined. Much more so are convolution empathies. It should also be noted that Feller's approach amounts to the assumption that the dualisms are continuous in τ and therefore, measurable in Φ and d-measurability is implied.

5.3.2 Entwined Pseudo Poisson Process

We approach an absorbing barrier of a Markov process with the philosophy of dynamic boundary condition where the boundary is taken a body in its own right. The absorbing boundary is seen as a distinct collection of states with zero intensity: this is a more realistic way to model fly trap models. Therefore, this approach gives rise to two distinct state spaces and stochastic processes which fits in perfectly with the two state theory of empathy. Indeed, we construct a new stochastic process with state space $\mathbb{N}$ which is intertwined with the classical pseudo Poisson process where the intensity (> 0) of the states are all equal. This pair of stochastic processes gives rise to a pair of transition probability density functions which satisfy the extended Chapman Kolmogorov equation. We call this process an entwined pseudo Poisson process.

The conditional version of the classical Chapman Kolmogorov equation (5.3.1) is

$$Q_{\tau+\sigma}(x,\Gamma) = \int_{y\in\Gamma} Q_\tau(x,\Gamma)Q_\sigma(y,\Gamma); \sigma,\tau > 0. \tag{5.3.4}$$

where $Q_\tau(x,\Gamma)$ is the measure and $Q_\sigma(y,\Gamma)$ the function; $\Gamma = \mathbb{R}$. Typical examples of time continuous processes which satisfy (5.3.4) arise even in processes with state space $\mathbb{N}$:

Let $X := \{\bar{1}, ..., \bar{n}\}$ denote the $\bar{n}$ fly traps; $Y := \{1, ..., m\}$ are the safe spots. In a 1-D setting, X will be a singleton (an absorbing barrier).

These two distinct state spaces X, Y give rise to two distinct transitions: the n-step transition $r^n_{ij} = P(Y_n = j|Y_0 = i)$ within the safe states of Y and the n-step transition $t^n_{i\bar{j}} = P(X_n = \bar{j}|Y_0 = i)$ into the fly traps of X.

$$r^n_{ij} = \sum_{k=1}^{|Y|} r^{n-1}_{ik} p_{kj}; r^1_{ij} = p_{ij}; \tag{5.3.5}$$

$$t^n_{i\bar{j}} = \sum_{k=1}^{|Y|} r^{n-1}_{ik} s_{kj}; t^1_{i\bar{j}} = s_{i\bar{j}}; \tag{5.3.6}$$

where p_{ij} denotes the one step transition $P(Y_{n+1} = j|Y_n = i)$; $s_{i\bar{j}}$ denotes the one step transition $P(X_{n+1} = \bar{j}|Y_n = i)$. Indeed, direct computation shows

$$t_{i\bar{j}}^{p+q} = \sum_{k=1}^{|Y|} r_{ik}^{q} t_{k\bar{j}}^{p} = \sum_{k=1}^{|Y|} r_{ik}^{p} t_{k\bar{j}}^{q}; p, q \in \mathbb{N}; \tag{5.3.7}$$

Now we arrive at the continuous time version of the discrete Markov chains (5.3.5) - (5.3.6) by conditioning on the number of transitions occurring in time interval $(0, t]$:

$$Q_\tau(i, \Gamma = \{j\}) \;=\; P(Y_\tau = j|Y_0 = i) = e^{-\lambda\tau} \sum_{n=0}^{\infty} r_{ij}^{n} \frac{(\lambda\tau)^n}{n!}; \tag{5.3.8}$$

$$R_\tau(i, \Gamma = \{\bar{j}\}) \;=\; P(X_\tau = \bar{j}|Y_0 = i) = e^{-\lambda\tau} \sum_{n=0}^{\infty} t_{i\bar{j}}^{n} \frac{(\lambda\tau)^n}{n!}; \tag{5.3.9}$$

That is, all the states in Y have the same intensity $\lambda > 0$ and all the states in X have the same intensity of 0.

The pseudo Poisson process (5.3.8) satisfies the requirements of the Chapman Kolmogorov equation (5.3.4) which takes the form

$$P(Y_{\tau+\sigma} = j|Y_0 = i) = \sum_{k=1}^{|Y|} P(Y_{\tau+\sigma} = j|Y_\tau = k)P(X_\tau = k|Y_0 = i).$$

The measure $Q_\sigma(x, \Gamma)$ of (5.3.4) is defined on measure space $(\Gamma = \mathbb{N}, \mathcal{P}(\mathbb{N}))$ as the function $Q_\sigma(y, \Gamma)$ is a countably valued $\mathbb{N}$-domained function. Similarly the entwined pseudo Poisson process (5.3.9) satisfies the (countable) conditional version of the extended Chapman Kolmogorov equation:

$$R_{\tau+\sigma} = R_\tau \star Q_\sigma; \sigma, \tau > 0. \tag{5.3.10}$$

The extended Chapman Kolmogorov equation (5.3.10) has the conditional form:

$$R_{\tau+\sigma}(x, \Gamma) = \int_{y\in\Gamma} R_\tau(x, \Gamma) Q_\sigma(y, \Gamma); \sigma, \tau > 0. \tag{5.3.11}$$

where $R_\tau(x, \Gamma)$ is the measure and $Q_\sigma(y, \Gamma)$ the function; $\Gamma = \mathbb{R}$. For $\Gamma = \mathbb{N}$, we have the countable conditional version of the extended Chapman Kolmogorov equation

$$P(X_{\tau+\sigma} = \bar{j}|Y_0 = i) = \sum_{k=1}^{|Y|} P(X_{\tau+\sigma} = \bar{j}|Y_\tau = k)P(Y_\tau = k|Y_0 = i).$$

The notion of convolution empathy extends the notion of a convolution-semigroup. We call the pair of distributions $\langle \mu_\tau = Q_\tau, \nu_\tau = R_\tau \rangle_{\tau>0}$ for which (5.3.10) and (5.3.1) holds, a *convolution empathy.*

Proposition 5. *Let $y'(\tau), x'(\tau)$ be the admissible homomorphisms induced by the distributions μ_τ, ν_τ of a convolution empathy $\langle \mu_\tau, \nu_\tau \rangle$, respectively; $\tau > 0$. Then the pair $\langle x'(\tau), y'(\tau) \rangle$ is a star-empathy and the dualisms $\langle X'(\tau), Y'(\tau) \rangle$ satisfies the empathy relation $X'(\tau + \sigma) = X'(\tau)Y'(\sigma); \sigma, \tau > 0$.*

Proof. From Theorem 1, it is immediate the pair $\langle x'(\tau), y'(\tau) \rangle$ is a star-empathy. Then the empathy relation is immediate from Γ preserving $*$ as $\circ$. □

5.4 Fokker Planck Equations

Having established a precise mathematical meaning of the relationship between probability distributions and operators, let us follow William Feller a little further by considering a family $\{\mu_\tau | \tau > 0\}$ of distributions which satisfies the *convolution equation* (5.3.1) (compare Defintion 3). The *convolution equation* (5.3.1) plays a central role in the development of limit theorems and Markov processes. In the latter (5.3.1) expresses the Chapman-Kolmogorov relation. If we let $x'(\tau)$ be the (admissible) homomorphism induced by μ_τ. By Theorem1 $x'(\tau)$ is a star-semigroup of bounded homomorphisms and the dualism $X'(\tau)$ (in Feller's words) a *convolution semigroup.* For a Feller convolution semigroup, if it assumed that the distributions μ_τ have variance $\sigma^2 = c\tau$, it follows ([17], IX.4) that the convolution semigroup has a generator A' whose domain includes the class C^∞ of functions whose derivatives are also in $C[\mathbb{R}]$, and hence it is strongly continuous.

In the special case where μ_τ is the normal (Gaussian) distribution with zero mean and variance τ, the generator can be calculated:

$$A'f = \frac{1}{2}\frac{d^2 f}{dx^2}$$

for $f \in C^\infty$. We may now apply the results of Section 4.2.1 for semigroups, taking B' as the identity. For $f \in C^\infty$, $v(\tau, x) = X'(\tau)f(x)$ satisfies the *Fokker-Planck equation*

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}. \tag{5.4.1}$$

If we translate back to the original homomorphisms the equation (5.4.1) becomes

$$\frac{\partial}{\partial \tau}\int_{\mathbb{R}} f(x)\mu_\tau\{dx\} = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} f(x)\mu_\tau\{dx\}. \tag{5.4.2}$$

If (5.4.2) can be phrased is terms of probability densities, it is the proper formulation of the diffusion equation so often used in physics against which Feller issues a stern warning.

5.4.1 Entwined Brownian Motion

Let $\mathbf{X} := \{W(\tau) : \Omega \to \mathbb{R} | \tau \geq 0\} \equiv \{\mu_\tau := \mu_{W(\tau)} | \tau > 0\}$ be a Brownian motion in one dimension; Ω is the space of right continuous functions $\omega : [0, \infty) \to \mathbb{R}$ with finitely many jumps in any finite time interval. Then it is well known that $\{\mu_\tau := \mu_{W(\tau)} | \tau \geq 0\}$ is a convolution semigroup. For the test space $\Phi := \Phi_S$, the induced family of homomorphisms $\mathfrak{x}'$ is strongly continuous. Therefore, $X' := \{X'(\tau) := \Gamma(x'(\tau)) | \tau \geq 0\}$ is a strongly continuous contraction semi-group on Φ_S (Theorem 2, Appendix C.3).

Remark 8. *By Proposition 1, Section 5.2, the dualism Y'_τ is precisely the standard transition operators $T(\tau)f : s \in G \mapsto \int_{y\in G} p_\tau(s,y)f(y)dy$ associated with the transition kernels or density $p_\tau(x,y) = \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\Delta^2}{2\tau}}$; $\Delta = x - y$; $Y'_\tau f : s \in G \mapsto \int_{x\in G} f(x+s)p_\tau(0,x)dx$, where $p_\tau(0,x) = p_\tau(s, x+s)$. Indeed, the Brownian motion $\{Y'_\tau | \tau > 0\}$ is a Feller process on G.*

The diffusion equation associated with the Brownian motion is immediate from the Hille-Yosida Theorem for contraction semi-groups. We call $\langle \mathbf{Y}, \mathbf{X}\rangle$ an entwined Brownian motion should $\langle \mathbf{Y}, \mathbf{X}\rangle$ be a convolution empathy, where $\mathbf{X}$ is a Brownian motion in one dimension. The diffusion equation associated with entwined Brownian motion is also immediate from the Hille Yosida Generation theorem for the implicit evolution equation (Theorem 13, Section 4.3.3). In the notation of Theorem 13, Section 4.3.3.

Theorem 3. *Consider an entwined Brownian motion $\langle \mathbf{Y}, \mathbf{X}\rangle$ as a double family of homomorphisms (or finite regular Borel measures) on Φ_S. If we assume the existence of the star pseudo-resolvent pair $\langle \mathfrak{r}', \mathfrak{p}'\rangle$ and the invertibility assumption, then for $f \in \Delta^2_{K'}$*

$$\frac{d}{d\tau}\langle f, b' * y'(\tau)\rangle = \langle f, a' * y'(\tau)\rangle; \lim_{\tau\to 0^+}\langle f, b' * y'(\tau)\rangle = \langle f, \theta'_0\rangle,$$

where $b' := \theta'_0 B$ and $a' = a'_s := \theta'_0 A_S$. Indeed, $\mathfrak{p}'(\lambda) = (\lambda b' - a')^{-1}$.

Remark 9. *We shall construct a new convolution of vector valued measures to give a vector valued version of the extended Chapman-Kolmogorov relation* (5.3.3) *for a suitable vector valued test space defined on the locally compact Abelian group $G = (\mathbb{R}, +)$.*

5.5 Vector Valued Feller construction

Another rendition of Feller's construction would be to let $\Phi = C[\mathbb{R}]$ and to consider vector-valued measures (*e.g.* [8]). The difficulty would then be to define convolution of measures. For the smaller test space $\Phi_S \subset \Phi = C[\mathbb{R}]$

our vectorization of the Feller construction proceeds in the same way we vectorized the admissible homomorphisms of classical convolution algebra of abstract harmonic analysis; after all, distributions are isometrically identifiable with (scalar) admissible homomorphisms on the (scalar) test space Φ_S.

5.5.1 Vectorized Feller Test Space

We set the Banach test space $\Phi_V := \overline{C_{00}(G,Z)}$; $C_{00}(G,Z)$ denotes the space of continuous Z-valued functions with compact support[5]; Z is a Banach space. We need only define *continuous admissible* homomorphisms on the dense subspace $C_{00}(G,Z)$ since there is a unique extension to Φ_V. Then Φ_V is (i) translation invariant, (ii) Frechet and (iii) the map $p \in G \mapsto f_{-p} \in \Phi_V$ is continuous for each fixed $f \in \Phi_V$.

Remark 10. *We shall mainly consider the strong convergence of sequences of equibounded homomorphisms, taken as operators, on Φ_V. Homomorphisms induced by probability measures, are special cases of equibounded homomorphisms. For sequences of equibounded homomorphisms, it suffices to establish strong convergence on the dense subspace $C_{00}(G,Z)$ ([3], Section 5.4.17). In the setting of probability theory, strong convergence is referred to as weak convergence .*

Proposition 6. *For the vector valued, translation invariant Banach test space Φ_V, the mappings $s \in G \mapsto f_s \in \Phi$; $f \in \Phi$ are uniformly continuous. For the special case of $Z = \mathbb{R}$, every bounded homomorphism is admissible; that is, $\mathscr{A}_B$ and $\mathscr{M}(Z)$ are isomorphic Banach spaces.*

Proof. For $Z = \mathbb{R}$, the admissibility of each bounded homomorphism follows from Lemma 19.6 [18]. Therefore, every distribution can be taken as an element of $\mathscr{A}_B$. $\square$

Remark 11. *It would be an interesting exercise to find conditions on the Banach space Z for which each bounded $x' \in Hom(\Phi, Z)$ is admissible.*

5.5.2 Vectorized Admissible Homomorphisms

Distributions (random variables) were identified with bounded admissible homomorphisms on a scalar valued test space Φ_S (Theorem 2, Section 5.2). In this section, we 'vectorize' these homomorphisms as *dominated* bounded admissible homomorphisms on the vectorized test space Φ_V. An alternative approach is to vectorize the range by considering operators on the test space

[5]Note the inclusion $C_{00}(G,Z) \subset C_0(G,Z) \subset BUC(G,Z)$ under the sup norm. For the case of $Z = \mathbb{R}$, $\overline{C_{00}(G,Z)} = C_0(G,Z)$ by the Urysohn Lemma.

Φ_S into $\mathcal{L}(Z)$. These two approaches are equivalent (Theorem 1, Section III, §19, p. 377 [10]).

We say that a bounded homomorphism on Φ_V is *dominated* if there exists a regular positive measure λ dominating the homomorphism x' in the sense:

$$\|\langle f, x'\rangle\| \leq \int |f| d\lambda, \tag{5.5.1}$$

for every $f \in C_{00}(G, Z)$. Then such homomorphisms have a partial Riesz representation theorem:

Theorem 4 (Vector Riesz Representation). *(Theorem 2, III, §19, [10]) Each dominated bounded admissible homomorphism x' can be identified uniquely with an operator-valued measure $\bar{\mu} \in \mathscr{M}(\mathcal{L}(Z))$ by the following identification:*

$$\langle f, x'\rangle := \int f d\bar{\mu} \tag{5.5.2}$$

for every $f \in C_{00}(G, Z)$.

Theorem 5. *Let $x' \in Hom_B(\Phi_V, Z)$ be a dominated bounded admissible homomorphism. Then x' has a unique extension to a bounded functional $\widetilde{x}'$ on $\mathcal{L}^1_\lambda(G, Z)$ such that the unique operator $\mathcal{L}(Z)$-valued measure $\bar{\mu}$ associated with x' is of the form:*

$$\bar{\mu}(A) : z \in Z \mapsto \langle z\mathcal{X}_A, \widetilde{x}'\rangle, \tag{5.5.3}$$

for every Borel set $A \in \mathscr{B}$; $\mathcal{X}_A$ is the indicator function on A.

Proof. Intuitively, the dominance condition (5.5.1) ensures that if x' is continuous on $C_{00}(G, Z)$ under the supremum norm, then x' is also continuous on $C_{00}(G, Z)$ under the $\mathcal{L}^1_\lambda(G, Z)$-topology. Since $C_{00}(G, Z)$ is a dense subspace of $\mathcal{L}^1_\lambda(G, Z)$, there is a unique continuous extension denoted by $\widetilde{x}'$, to $\mathcal{L}^1_\lambda(G, Z)$. □

Remark 12. *Consider an autonomous time continuous stochastic process* **X** *with its family of (time-independent transition) distributions $\{\mu_\tau | \tau \in \mathbb{T}\}$ (Section 5.3). For the test space $C[\mathbb{R}], C_0(\mathbb{R})$ or $C(K)$; K compact, the family $\{x'_{\mu_\tau} | \tau > 0\}$ is an equibounded family of dominated bounded admissible homomorphisms.*

Thus, a dominated bounded admissible homomorphism $x' \in Hom_B(\Phi_V, Z)$ can be regarded as a vectorized probability transition distribution.

It is important to note that the subclass, $\mathscr{A}_D$, of dominated admissible homomorphisms, form a subalgebra of $\mathscr{A}_B$.

Proposition 7. *Let $x', y' \in \mathscr{A}_D$. Then $x' * y', x' + y', \lambda x' \in \mathscr{A}_D$.*

Proof. Without loss of generality, we can assume that $x', y' \in \mathscr{A}_B$ is dominated by the same regular positive measure λ: suppose x' and y' are dominated by the regular positive Borel measures γ and μ respectively; then $\lambda := \gamma + \mu$ [6] is a regular positive Borel measure (the set of regular positive Borel measures is a linear space) which dominates both x' and y'.

By the dominance of first x' and then y', $\|\langle Y'f, x'\rangle\| \leq \int \|Y'f\| d\lambda \leq \int \|f_{-s}\| d\lambda \leq \int \|f\| d\lambda$. □

5.5.3 Vectorized Feller Star

Let $\bar{\mu}$ and $\bar{\nu}$ be the unique operator-valued measures in $\mathscr{M}(\mathcal{L}(Z))$ uniquely associated with the *dominated* admissible homomorphisms x' and y' on Φ_V, respectively (Theorem 4, Section 5.5.2). Then, just as in the scalar case, *we would like the convolution measure $\bar{\mu} * \bar{\nu}$ to coincide with the unique measure associated with the product $x' * y' \in \mathscr{A}_B$* (the product $x' * y'$ is also dominated [Proposition 7]), in order for the convolution semigroup equation (5.3.2), Section 5.3, to be a special case of the convolution of operator-measures.

Now, given two measures $\bar{\mu}, \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$, their convolution $\bar{\mu} * \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$ since operator composition is a $\mathcal{L}(Z) \times \mathcal{L}(Z) \to \mathcal{L}(Z)$ bilinear map, where $\bar{\mu} * \bar{\nu}(A) := \bar{\mu} \times \bar{\nu}(A_2)$; for each Borel set $A \in \mathscr{B}$, the pullback set $A_2 := \{(s,t) : s + t \in A\}$ belongs to $\mathscr{B}(\mathbb{R} \times \mathbb{R})$ [Theorem IV.2, [20]]. Unfortunately, the convolution $*$ of measures in $\mathscr{M}(\mathcal{L}(Z))$ is based on the bilinear form of operator composition and hence has a highly restrictive Fubini Theorem (Theorem III.1 [20]) requiring separability of one of the measures: *Let $A \in \mathscr{B}(G)$ and $A_2 := \{(s,t) \in G \times G | s + t \in A\}$. For every $A_2 \in \mathcal{B}(G \times G)$ and each fixed s, the section $A_2^s := \{t \in G | (s,t) \in A_2\} = A - s \in \mathscr{B}(G)$; $A - s$ is the set A shifted s to the left. Then, the separability of $\bar{\nu}(\mathscr{B}(G))$ is needed to ensure that the operator-valued function $\varphi_{A_2} : s \mapsto \bar{\nu}(A_2^s)$ is uniformly measurable, for the Bochner integral in $\mathcal{L}(Z)$, of following equation to make sense: $\bar{\mu} * \bar{\nu}(A) = \bar{\mu} \times \bar{\nu}(A_2) = \int \varphi_{A_2}(s) d\bar{\mu}(s)$.*

Therefore, we construct a new convolution of measures in $\mathscr{M}(\mathcal{L}(Z))$ which we shall denote as $\bigstar$, such that (i) $\bar{\mu} \bigstar \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$ and (ii) $\bar{\mu} \bigstar \bar{\nu} = \bar{\gamma}$ for the purpose of getting $\bar{\mu} * \bar{\nu}$ to coincide with λ *without any further assumptions*; $\bar{\gamma}$ is the unique measure in $\mathscr{M}(\mathcal{L}(Z))$ associated with $x' * y'$. Just as in the Feller star $\star$, the symbol $A \bigstar B$ stands for integration with respect to the measure $A \in \mathscr{M}(\mathcal{L}(Z))$. We formally define this new convolution, $\bigstar$, of measures in $\mathscr{M}(\mathcal{L}(Z))$, since existence has not yet been proven:

[6] $\lambda(A) := \gamma(A) + \mu(A)$ for each $A \in \mathscr{B}$

Definition 4 (Convolution in $\mathscr{M}(\mathcal{L}(Z))$)**.** *Let $\bar{\mu}, \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$. Let $A \in \mathscr{B}(G)$. Then the weak-convolution $\bar{\mu}\bigstar\bar{\nu}$ is an operator-valued set domained function on the Borel σ-field $\mathscr{B}$, where the operator $\bar{\mu}\bigstar\bar{\nu}(A) \in \mathcal{L}(Z)$, is defined pointwise at each $z \in Z$ as follows:*

$$\bar{\mu}\bigstar\bar{\nu}(A) : z \in Z \mapsto \int \varphi_{A_2,z}(s)d\bar{\mu}(s) \in Z, \tag{5.5.4}$$

where $\varphi_{A_2,z} : s \mapsto \bar{\nu}(A-s)z \in Z$.

Remark 13. *The Z-valued function $\varphi_{A_2,z} : s \mapsto \bar{\nu}(A-s)z \in Z$ can be regarded as the weak*-version of the operator-valued function φ_{A_2}: the weak-convolution, $\bigstar$, is based on the bilinear form B on $\Phi \times Hom(\Phi, Z) \to Z$, $B(f, x') = \langle f, x' \rangle$ as opposed to the standard bilinear form of operator composition on $\mathcal{L}(Z) \times \mathcal{L}(Z) \to \mathcal{L}(Z)$.*

We begin by showing that given two measures $\bar{\mu}, \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$, their weak convolution $\bar{\mu} * \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$. Pleasantly, the admissibility of the uniquely associated homomorphisms x', y' is a sufficient condition for the existence of $\bar{\mu}\bigstar\bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$. We need the following lemma which follow from the inclusions $C_{00}(G, Z) \subset \Phi_V \subset \mathcal{L}^1_\lambda(G, Z)$ being $\|\cdot\|_\infty$ and $\|\cdot\|_1$ dense, respectively:

Lemma 1. *Let x' be a bounded Φ_V-admissible homomorphism dominated by a regular positive measure λ. Then x' is $\mathcal{L}^1_\lambda(G, Z)$-admissible.*

Proof. The crux of the lemma is that the dualism X' preserves continuous extensions. The isometric shift operators on $(\Phi, \|\cdot\|_\infty)$ ensures that the operator X' is bounded on $(\Phi, \|\cdot\|_\infty)$ into itself. Furthermore, X' is bounded on $(\Phi, \|\cdot\|_1)$ into itself by the dominance of x' by a regular positive measure λ together with the isometric shift operators on $(\Phi, \|\cdot\|_1)$: $\|\langle R^{-s}f, x'\rangle\| \leq \int \|R^{-s}f\|d\lambda = \int \|f\|d\lambda$.

Let $\overline{X'}$ denote the unique continuous extension of X' to $\mathcal{L}^1_\lambda(G, Z)$. Consider the operator $\tilde{X}' : f \in \mathcal{L}^1_\lambda(G, Z) : \langle R^{-s}f, \tilde{x}'\rangle$. $\overline{X'} = \tilde{X}'$ since they agree on the dense subspace $(C_{00}(G, Z), \|\cdot\|_1)$. □

We now show that $\mathcal{L}^1_\lambda(G, Z)$-admissibility is the key to the weak convolution $\bar{\mu} * \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$.

Theorem 6. *Let $\bar{\mu}, \bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$ be the unique operator-valued measures in $\mathscr{M}(\mathcal{L}(Z))$ associated with the dominated admissible homomorphisms x', y' on Φ_V, respectively. Then*

(i) $\bar{\mu}\bigstar\bar{\nu} \in \mathscr{M}(\mathcal{L}(Z))$;
(ii) $\bar{\mu}\bigstar\bar{\nu} = \bar{\gamma}$, where $\bar{\gamma}$ is the unique measure in $\mathscr{M}(\mathcal{L}(Z))$ associated with

the dominated admissible homomorphism $x' * y'$.
(iii) $\bigstar$ *is associative.*

For the case of $Z = \mathbb{C}$, *the Feller star* $\star$ *and* $\bigstar$, *and the convolution* $*$ *of measures coincide.*

Proof. Firstly, the function $\varphi_{A_2,z}$ of Definition 4, (5.5.4) is the function $\widetilde{Y}'(z\mathcal{X}_A) \in \mathcal{L}^1_\lambda(G,Z)$ by the $\mathcal{L}^1_\lambda(G,Z)$-admissibility of y' [Lemma 1]; $\widetilde{Y}'$ is the dualism of y'. By direct computation,

$$\bar{\nu}(A_2^s)(z) := \langle z\mathcal{X}_{A-s}, \tilde{y}'\rangle = \langle R^{-s}z\mathcal{X}_A, \tilde{y}'\rangle = \widetilde{Y}'(z\mathcal{X}_A)(s) \qquad (5.5.5)$$

Now, $\bar{\mu}\bigstar\bar{\nu}(A)(z) := \int \varphi_{A_2,z}(s)d\bar{\mu}(s) = \langle \widetilde{Y}'(z\mathcal{X}_A), \tilde{x}'\rangle$ by the Vector Riesz RepresentationTheorem (Theorem 4, (5.5.2)) [7]. Thus,

$$\bar{\mu}\bigstar\bar{\nu}(A)(z) = \langle z\mathcal{X}_A, \tilde{x}' * \tilde{y}'\rangle = \langle z\mathcal{X}_A, \widetilde{x' * y'}\rangle,$$

On the other hand, the *unique* operator-valued measure $\bar{\gamma} \in \mathscr{M}(\mathcal{L}(Z))$ associated with the *dominated* admissible homomorphisms $x' * y'$ is given by, $\bar{\gamma}(A) : z \in Z \mapsto \langle z\mathcal{X}_A, \widetilde{x' * y'}\rangle$ (Theorem 5, (5.5.3)) Therefore, $\bar{\gamma} = \bar{\mu}\bigstar\bar{\nu}$ proving both (i) and (ii). Property (iii) is immediate from the associativity of the product of homomorphisms in $\mathscr{A}_B$.

Finally if one notes that for $\mathbb{C}$ the ordered set $\{1\}$ is a basis, setting $z = 1$ in the construction (5.5.4) reduces $\bar{\mu}\bigstar\bar{\nu}$ to the traditional convolution measure $\bar{\mu} * \bar{\nu}$. Hence, $\star$, $\bigstar$, and $*$ coincide for $Z = \mathbb{C}$. □

5.5.4 A More General Dominating Condition

In Section 5.5.3, we constructed the weak convolution of operator-valued measures that vectorized the classical Feller star or convolution (Definition 1, [17], p. 141) . This construction rested on the concept of a *dominated admissible* homomorphism (equation (5.5.1), Section 5.5.2). We introduce a new 'norm' on (vectorized) homomorphisms $x' \in \mathrm{Hom}(\Phi_V, Z)$ which attempts to easily characterize whether x' is dominated.

Definition 5 (Operator norm relative to Borel set A)**.** *Let* $x' \in Hom(\Phi_V, Z)$. *Fix a Borel set* $A \in \mathscr{B}$. *Let* $C_{00}(G, Z|A)$ *be the subspace of* $C_{00}(G,Z)$ *consisting of functions with compact support* $\subset A$. *Then*

$$\|x'\|_A := \sup|\langle f, x'\rangle|, \qquad (5.5.6)$$

where f *belongs to the unit ball of* $C_{00}(G, Z|A)$. *We call* $\|\cdot\|_A$ *the operator norm relative to* A.

[7] For each $f \in L^1_\lambda(G,Z)$, $\int f d\bar{\mu} = \langle f, \tilde{x}'\rangle$ by the density of $C_{00}(G,Z)$ in $\mathcal{L}^1_\lambda(G,Z)$, the boundedness of x' and the completeness of the Banach space Z

Now, dominated (scalar) homomorphisms on Φ_S have an easy characterization in terms of this norm:

Theorem 7 (Relative Operator Norm)**.** *(Section §19, Corollary 1, [10]) A homomorphism x' on Φ_S is dominated if and only if $\|x'\|_A < \infty$ for every Borel set $A \in \mathscr{B}$.*

For the vector valued case Φ_V, each fixed $z^* \in Z^*$ induces a dominated $x'_{z^*} \in \mathrm{Hom}_B(\Phi, \mathbb{C})$ as follows:

$$x'_{z^*} : f \mapsto \langle x'(f), z^* \rangle,$$

where $x' \in \mathrm{Hom}(\Phi_V, Z)$. Hence, there is a regular Z^*-valued measure $\bar{\mu}_{z^*}$ with bounded variation such that

$$x'_{z^*} : f \mapsto \int f d\bar{\mu}_{z^*}. \tag{5.5.7}$$

Therefore, we naturally construct a $\mathcal{L}(Z, Z^{**})$-valued additive set function, $\bar{\mu}$, on $\mathscr{B}$ as follows:

$$\bar{\mu}(A) : x \in Z \mapsto (z^* \in Z^* \mapsto \bar{\mu}_{z^*}(A)x \in \mathbb{C}), \tag{5.5.8}$$

where $(z^* \in Z^* \mapsto \bar{\mu}_{z^*}(A)x \in \mathbb{C})$ is an element of Z^{**}. Therefore,

Proposition 8. *Consider a homomorphism $x' \in Hom_B(\Phi_V, Z)$. Let $\|x'\|_A < \infty$ for every Borel set $A \in \mathscr{B}$. Then we can construct a $\mathcal{L}(Z, Z^{**})$ additive set function $\bar{\mu}$ of bounded semi variation such that*

$$\langle f, x' \rangle = \int f d\bar{\mu} \tag{5.5.9}$$

Remark 14. *The left hand term of equation (5.5.9) is Z-valued, but the right hand term is Z^{**}-valued. Therefore, we take Z as a subspace of Z^{**} for equation (5.5.9) to make sense.*

Proof. The construction of the $\mathcal{L}(Z, N'^*)$-valued Borel measure $\bar{\mu}$, rests on the existence of a norming space N' for Z. We say that $N' \subset Z^*$ is a norming space for Z provided if for every $z \in Z$, we have $|z| = \sup\{|\langle z, n' \rangle| n' \in N'_1\}$ where N'_1 is the unit ball of the subspace N'. □

5.6 Vector Valued Chapman-Kolmogorov

The notion of a convolution semi-group of probability measures on $\mathbb{R}$ began with an autonomous stochastic process $\mathbf{X} := \{X(\tau) | \tau > 0\}$ with its unique family of transition distributions $\{\mu_\tau | \tau > 0\}$, which satisfy the Chapman

Kolmogorov relation (5.3.1). For the (scalar) test space Φ_S, the framework of dualisms $\mathcal{L}^1_{loc}((0,\infty),\mathscr{A}'_B)$ identified the convolution semi-group with a C_0-semigroup on Φ_S (Proposition 3, Section 5.3). In this section, we vectorize Proposition 3, Section 5.3, to operator-valued probability transition distributions.

5.6.1 Dominated Operator-valued Distributions

Each transition distribution μ_τ is uniquely identified with a bounded *dominated* admissible homomorphism, x'_{μ_τ}, on the scalar valued test space Φ_S. A dominated admissible homomorphism on the vector valued test space Φ_V, that is, changing the domain of x'_{μ_τ} to Φ_V, is equivalent to vectorizing the range of x'_{μ_τ}, that is, considering x'_{μ_τ} as an operator from Φ_S to $\mathcal{L}(Z)$ (Theorem 1, Section III, §19, p. 377 [10]). Therefore, we shall consider operator $\mathcal{L}(Z)$-valued transition distributions $\{\bar{\mu}_\tau|\tau>0\}$; $\bar{\mu}_\tau$ is a $\mathcal{L}(Z)$-valued Borel measure of bounded variation.

Since weak convolution $\bigstar$ is a vectorized Feller star $\star$ (Theorem 6, Section 5.5.3), we introduce the notion of an *extended Chapman-Kolmogorov relation* for $\{\bar{\mu}_\tau|\tau>0\}$ as:

$$\bar{\mu}_{\tau+\sigma} = \bar{\mu}_\tau \bigstar \bar{\mu}_\sigma; \tau,\sigma>0, \tag{5.6.1}$$

Now, for (5.6.1) to make sense, we need to additionally assume that each $\bar{\mu}_\tau$ is an operator-valued measure in $\mathscr{M}(\mathcal{L}(Z))$ uniquely associated with a *dominated* admissible homomorphism on Φ_V (Theorem 6, Section 5.5.3). We shall call such measures *dominated $\mathcal{L}(Z)$-valued Borel measures.*

Remark 15. *For the case $Z=\mathbb{R}$,* (5.6.1) *becomes the classical Chapman-Kolmogorov relation,* (5.3.1)*.*

Indeed, we formally define a dominated operator-valued convolution semigroup as follows:

Definition 6 (Dominated Convolution Semigroup)**.** *A family $\{\bar{\mu}_\tau|\tau\geq 0\}$ of dominated $\mathcal{L}(Z)$-valued Borel measures on $\mathbb{R}$ is said to be a dominated operator-valued convolution semigroup of measures iff (a) $\bar{\mu}_0=\bar{\delta}_0$ (b) $\bar{\mu}_\tau$ converges v-weakly to $\bar{\delta}_0$, as $\tau\to 0+$, and (c)*

$$\bar{\mu}_{\tau+\sigma} = \bar{\mu}_\tau \star \bar{\mu}_\sigma; \sigma,\tau\geq 0, \tag{5.6.2}$$

where we define the v-weak convergence of a sequence $\bar{\mu}_n$ of dominated $\mathcal{L}(Z)$-valued Borel measures on $\mathbb{R}$ to another dominated $\mathcal{L}(Z)$-valued Borel measure $\bar{\mu}$ as follows:

Definition 7 (V-weak Convergence)**.** *A sequence $\bar{\mu}_n$ of dominated $\mathcal{L}(Z)$-valued Borel measures on $\mathbb{R}$ v-weak converges to another dominated $\mathcal{L}(Z)$-valued Borel measure $\bar{\mu}$ provided the corresponding sequence of (operator) dualisms $X'_{\bar{\mu}_n} := \Gamma(x'_{\bar{\mu}_n})$ on Φ_V converges strongly to $X'_{\bar{\mu}} := \Gamma(x'_{\bar{\mu}})$.*

Remark 16. *Definition 7 is not vacuous by virtue of Appendix C.3.1, Theorem 2.*

Just as in the classical case, by choosing Φ_V as the test space, the framework $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}'_B)$ turns out to be the framework to identify the dominated operator-valued convolution semigroup $\{\bar{\mu}_\tau)|\tau \geq 0\}$ with a C_0-semigroup, $\{E(\tau)|\tau > 0\}$, on the Banach space Φ_V.

5.6.2 Generalized Random Variables

Let $x'_{\bar{\mu}_\tau} \in \mathscr{A}_B$ denote the dominated bounded admissible homomorphism uniquely associated with the dominated $\mathcal{L}(Z)$-valued Borel measure $\bar{\mu}_\tau$ (Theorem 4, Section 5.5.2). *Dualism* isometrically identifies each $x'_{\bar{\mu}_\tau}$, with the *bounded* (dualism) operator $X'(\tau)$ on Φ_V; $X'(\tau) := \Gamma(x'_{\bar{\mu}_\tau}) \in \mathscr{A}'_B$ [Theorem 8, Section 3.3.1]; we call $X'(\tau)$ an operator representation of a generalized random variable. We will show that $E(\tau) := X'(\tau)$ gives the desired C_0-semigroup.

Remark 17. *The dominance condition ensures that $\|X'(f)\|_\infty \leq \|f\|_1$; $X'(f) \in \Phi_V$.*

Definition 6(c) of Section 5.6.1, is equivalent to $x'_{\mu_{\tau+\sigma}} = x'_{\mu_\tau} * x'_{\mu_\sigma}$, which is equivalent to $E(\tau + \sigma) = E(\tau) \circ E(\sigma)$ in similar manner to (c) of Proposition 3, Section 5.3. Indeed, $\bar{\mu}_{\tau+\sigma} = \bar{\mu}_\tau \bigstar \bar{\nu}_\sigma$ iff $y'_{\bar{\mu}_{\tau+\sigma}} = y'_{\bar{\mu}_\tau} * x'_{\bar{\nu}_\sigma}$ iff $Y'_{\tau+\sigma} = Y'_\tau \circ X'_\sigma$ for appropriately defined families. Finally, Definition 6(b) of Section 5.6.1, implies $E(\tau)f = f$ as $\tau \to 0^+$ for each $f \in \Phi_V$.

Proposition 9. *Every dominated operator-valued convolution semigroup $\{\bar{\mu}_\tau|\tau \geq 0\}$ of Definition 6, Section 5.6.1, has an associated C_0-semi-group on Φ_V.*

Appendix A

Intuition of Empathy Theory

Causality is the study of causes, effects and *causation*; causation explains why the *system* changes from cause to effect; the term 'system' stands for the well identifiable object under study.

An analytical description of causality needs the notion of (i) a *state*, which roughly speaking is a good description of the system, (ii) *transition maps* which are associations between states at different times capturing the observer analytically, (iii) *causal relations* which are the relationships between the transition maps capturing the notion of *how* the system changes from one state to another as opposed to a mere description and (iv) *causation* which captures the model analytically.

This appendix shows how the intertwined families of empathy theory arise naturally as *causal relations* and *causation*.

A.1 Foundations

Causality has as its basis the notions of *time* and *state*. The essence of time is 'before' and 'after'. *Time*, denoted by T, is simply a linear ordered sequence of events on a chronometer (a universal stopwatch which does not interfere with the system) and is thus mapped in a one-one manner onto a a continuum of real numbers. The elements of T will be denoted by the real number τ. *State* pertains to a mathematical description of whatever is considered to be adequate to represent the system. Indeed, in the classical mechanics of a particle, the state of a particle is the ordered tuple of data such that knowing this data at instant τ and the laws of the model, determines future values of the ordered tuple of data for time $\rho > \tau$. States in general will be denoted by the symbol x, y. For example, let the system be a single Newtonian particle and the property of study its position as it moves through space. Then since the future position of a particle is determined by the particle's current position q and it's velocity v, the ordered pair of functions (q, v) will denote the *state* of the particle. As another example, let the system be an open solid which we represent as a subset Ω in three dimensional Euclidean space E^3 and the property of study be heat transfer. Then the states will be the ordered pair $(u(p), q(p))$ where p is a point on Ω and $u(p), q(p)$ represents the temperature and thermal energy density functions at point p.

The *state space* is a set of all the possible states of the system and we denote it by $\mathfrak{Y}$. For example, the state space of the before mentioned Newtonian particle will be the set of all pairs of position and velocity it passes through its journey. Likewise the state space of the solid Ω will be the set of all pairs of functions over the entire body of the object. The requirement of a set is not trivial: every two considered state should be distinct. The state space is therefore, the pure data set for the observer: it is important to note that there is no notion of time at all in the state space. In order to capture the notion of when a state a is close to another state b, we endow the state space $\mathfrak{Y}$ with a topology. From this point onward, we take $\mathfrak{Y}$ as a topological space.

A.1.1 Phase Space

The concept of the *phase space* marries the notion of time and state. Indeed, a fundamental assumption of causality is that the state changes as time goes by: the position of the particle changes with time, tracing out a *directed* path in the state space $\mathfrak{Y}$; the temperature distribution at each point of the body changes with time. Therefore, we consider states that depend on time. Therefore, the notion of phase space captures the observer watching the system evolve in time. We capture this notion with mappings $u : \tau \mapsto u(\tau) \in \mathfrak{Y}$ where τ denotes time on the time continuum T. As it is sensible to associate a unique state to a given time τ, the mapping u has to be a function. The set of such functions is denoted by $\Phi(T; \mathfrak{Y})$ and is called the *phase space over the continuum* T. The phase space can be considered a visualization of change. It mathematically encodes *a transition map*: a description of the state $u(\tau)$ of a dynamic process evolving to a state $u(\rho)$ at some later time ρ. It is important to note that the concept of a state is indeed powerful : no matter *how* the particle (the system) arrived at state $u(\tau)$, once in state $u(\tau)$, its future state $u(\rho)$ is determined for $\rho > \tau$. The state $u(\tau)$ is considered the 'event' that gives rise to - *causes* - the event (*effect*) state $u(\rho)$.

A.1.2 Causal Relations

The notion of *causal relations* explains *how* the state $u(\tau)$ evolves to a state $u(\rho)$ at some later time ρ as opposed to a mere description of the phase curve. The manner in which the states evolve is represented by *causal relations.* We formulate these causal relations as functions C defined on the state space into itself: they map the point $u(\tau)$ on a phase curve to the point $u(\rho)$ on the same curve for a time $\rho > \tau$. That is,

$$u(\rho) = C(\rho, \tau)u(\tau). \tag{A.1.1}$$

The expression (A.1.1) says the event $u(\rho)$ evolved from the event $u(\tau)$.

A very fundamental assumption is that (A.1.1) holds for all τ and ρ in the time continuum T for which $\tau < \rho$: all points on a given phase space curve evolved from earlier points on the curve. Then, if there is some time σ between τ and ρ $(\tau < \sigma < \rho)$, the state of $u(\sigma)$ is result of $u(\tau)$: $u(\sigma) = C(\sigma, \tau)u(\tau)$ and, in turn is the cause of $x(\rho)$: $u(\rho) = C(\rho, \sigma)u(\sigma)$. Therefore,

$$u(\rho) = C(\rho, \sigma)[C(\sigma, \tau)u(\tau)]. \tag{A.1.2}$$

Since (A.1.2) has to hold for all $t \in T$, it is reasonable to postulate the *relation*

$$C(\rho, \tau) = C(\rho, \sigma) \circ C(\sigma, \tau), \tag{A.1.3}$$

for all $\tau < \sigma < \rho$. Relation (A.1.3) has so far yielded very few results.

A special case of (A.1.3) which did yield a mathematically rich and deep theory was based on the assumption that the *effect of the state $u(\tau)$ on any later state ρ is determined solely by the the difference $\eta = \rho - \tau$.* We call such systems *autonomous.* Under this assumption the function C is of the form:

$$C(\rho, \tau) = E(\rho - \tau). \tag{A.1.4}$$

Therefore, the causal relation (A.1.3) becomes

$$E(\rho - \tau) = E(\rho - \sigma) \circ E(\sigma - \tau), \tag{A.1.5}$$

where the composition of function E amounts to the addition of the arguments. With E as a function of one positive real variable, the increase in time, by (A.1.5) we have

$$E(\xi + \gamma) = E(\xi) \circ E(\gamma) = E(\gamma) \circ E(\xi). \tag{A.1.6}$$

One can interpret the time increases ξ and γ as time if time $\tau = 0$ is part of the continuum T since $\xi = \xi - 0$ will be the increase since the initial instant $\tau = 0$. We use the term *semigroup* to denote the evolution according to (A.1.6).

A.1.3 Causation

Causation addresses the question why states change. In mathematical modelling, the mechanism of an evolution process, the *causation* is often captured in an equation of evolution in the following differential form

$$u'(\tau) = Au(\tau) + f(\tau, u(\tau)), \tag{A.1.7}$$

with $u'(\tau)$ being the time rate of change (a total derivative) along the trajectory of the process in phase space, and A some 'operator' which captures the mechanism of the causation. The operator A represents the influence of internal agents and f the influence of external agents. Equation (A.1.7) should be interpreted as: *the rate at which the states change is caused by the work of the internal agents to which it is added the influence of external agents.*

If the evolution equation (A.1.7) under the initial condition

$$\lim_{\tau \to 0^+} u(\tau) = a, \tag{A.1.8}$$

has a unique solution, the considerations of the previous sections are applicable. For then the mapping $t \mapsto u(\tau)$ defines a curve in the state space $\mathfrak{Y}$ which represents some causal relation.

A.2 Intuition of Semigroups

Semigroups are the simplest examples of states which evolve in time from a given *initial state* a in the state space $\mathfrak{Y}$. Indeed, if we let $u(\tau) = E(\tau)a$ for $\tau > 0$, the curve $u(\tau)$ represents a continuum of states which originate from a; $u(\tau)$ is a curve in the state space $\mathfrak{Y}$. In order to bring meaning to the notion of 'tangent vector' to each of the points of the curve $u(\tau)$, 'the states a and b are close to each other', and the curve '$u(\tau)$ is continuous', additional structure is required of $\mathfrak{Y}$. Therefore, we enrich the time continuum $T = \{0 < \tau < \infty\}$ and the phase space $\Phi(T; \mathfrak{Y})$ with the following additional topological structures: T is endowed with with the usual topology of the real line; the state space $\mathfrak{Y}$ is a Banach space. Then, mathematical properties of semi-flows such as their long-term behaviour, which lead to the concept of *strange attractors* and *attractors*, can be derived. If in addition the mappings $E(\tau)$ are bounded linear operators, the theory yields very deep and spectacular results. The assumption of boundedness means that $u(\tau) := E(\tau)a$ (τ fixed) varies continuously with the initial data a; the motion $\{u(\tau)|\tau > 0\}$ in the phase space $\Phi(T; \mathfrak{Y})$ is assumed to satisfy a linear homogenous dynamical system; therefore, the Superposition Principle imposes linearity of each $E(\tau)$. Therefore, it makes sense to define the state space $\mathfrak{Y}$ as a vector space. Indeed, state spaces are taken as Banach spaces to facilitate analysis.

In particular, if the causal relation is a semigroup (A.1.6), it is known there is an operator A for which the problem (A.1.7), (A.1.8) with $f = 0$ has a unique solution, for a well-defined subset of the state space $\mathfrak{Y}$. In fact, the causation mechanism A is already captured in the transition map : the state y is in the domain $D(A)$ of the linear operator A if the limit

$$Ay := \lim_{h \to 0^+} \frac{E(h)y - y}{h}, \tag{A.2.1}$$

exists. If for all $a \in D(A)$ the curve $u(\tau) = E(\tau)a$ in $\Phi(T; \mathfrak{Y})$ is continuous, the derivative $u'(\tau)$ exists in $\mathfrak{Y}$ and

$$u'(\tau) = Au(\tau). \tag{A.2.2}$$

For transition maps defined under less restrictive condition such a theorem is not in general known.

A.3 Intuition of Empathy Theory

There are situations where a single state space is inadequate for a good description of transition map. Consider for example, the thermal interaction between a potato and its skin. For this case, two state spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are considered; space $\mathfrak{Y}$ represents the whole system (the potato and its skin) and $\mathfrak{X}$ a significant part of it (what is inside the skin). Then there would be two phase spaces $\Phi(T; \mathfrak{X})$ and $\Phi(T; \mathfrak{Y})$. The interactive transition map is described by two families of *causal mappings* . The one, $E(\tau)$ acts totally within the space $\mathfrak{Y}$ while the other, $S(\tau)$, maps from $\mathfrak{Y}$ to $\mathfrak{X}$ to describe change in the 'world' $\mathfrak{X}$ in 'empathy' with the changes in $\mathfrak{Y}$. The causal relation postulated is

$$S(\tau + \sigma) = S(\tau) \circ E(\sigma) = S(\sigma) \circ E(\tau), \tag{A.3.1}$$

and is called the *empathy relation*; we say that the transition map $E(\sigma)$ intervenes empathetically with the transition map $S(\tau)$; indeed, the transition map $S(\tau)$ makes the transition map $E(\sigma)$ behave in a conventional way as a semigroup: $E(\tau + \sigma) = E(\tau)E(\sigma)$ under additional assumptions. The empathy relation describes a way in which curves in the state spaces $\mathfrak{X}$ and $\mathfrak{Y}$ evolve in interaction with each other. Analysis of empathy relations differs from that of semi-flows or semigroups since it involves mappings between different spaces. Nonetheless, under some additional assumptions, it turns out that the causation in this case is reflected by *implicit* evolution equations of the form

$$[Bu(\tau)]' = Au(\tau), \tag{A.3.2}$$

with A and B being mappings from a subset of $\mathfrak{X}$ to the space $\mathfrak{Y}$: the additional assumptions ensure the family $E(\tau)$ is a semigroup and that the two families $E(\tau)$ and $S(\tau)$, acting in unison by virtue of (A.3.1), generate a pair of linear operators A and B and a domain $\mathscr{D}_Y \subset \mathfrak{Y}$ such that for $y_0 \in \mathscr{D}_Y, u(\tau) = S(\tau)y_0$ satisfies the evolution equation (A.3.2). It is important to note that A and B map *'backwards'* from effect $\mathfrak{X}$ to cause $\mathfrak{Y}$. The initial condition for (A.3.2) (analogous to (A.1.8)) is :

$$\lim_{\tau \to 0^+} Bu(\tau) = y_0 \in \mathfrak{Y}. \tag{A.3.3}$$

In this case, we require the solution $u(\tau) = S(\tau)y_0$ of (A.3.2) to additionally satisfy the initial condition (A.3.3). Thus it can be said that the state $y_0 \in \mathfrak{Y}$ 'causes' the 'effect' $u(\tau) \in \mathfrak{X}$ (in general, $\mathfrak{Y}$ represents the state of causes and $\mathfrak{X}$ a space of effects).

A.3.1 Departures From Semigroup Theory

Empathy theory is a strict generalization of the abstract Cauchy problem : set $\mathfrak{X} = \mathfrak{Y}$ and $B = 1_{\mathfrak{X}}$. Indeed, we note four major differences. Firstly, *a pair of state spaces* $\mathfrak{Y}, \mathfrak{X}$. The analysis of empathy theory involves operators between Banach spaces $\mathfrak{Y}$ and $\mathfrak{X}$, the solution (effect) and the causal state space respectively. It is impossible to define the notion of powers of operators and the identity operator for such operators. In semigroup theory these notions gives rise to the analyticity of the resolvent $R(\lambda)$ and the concept of the infinitesimal generator. Secondly, *empathy theory works with a pair of evolution families:* $\mathcal{E} : \tau > 0 : E(\tau) : \mathfrak{Y} \to \mathfrak{Y}$; $\mathcal{S} : \tau > 0 : S(\tau) : \mathfrak{Y} \to \mathfrak{X}$. The analysis of empathy theory insists that the pair of evolution families are only defined on the open interval $(0, \infty)$. Differentiation is, after all, properly defined on open sets, and initial conditions are in the form of limits as $\tau \to 0^+$. Indeed, *no assumption whatsoever is made of the evolution operators at the origin* $\tau = 0$. This is in contrast to the well-known theory of semigroups where the point $\tau = 0$ is included (C_0-semigroups). Thirdly, *a pair of non-closeable generators* A, B *which are* **backward** *maps from the solution or effect space* $\mathfrak{X}$ *into the space of causes* $\mathfrak{Y}$. The non-closeability of B precludes interchange of B with $\frac{d}{d\tau}$ or $\int d\tau$ in equation (1.1.1); it is impossible to reduce it to the well known abstract Cauchy problem. Fourthly, empathy theory involves a *a pair of resolvents* $\langle R, P \rangle$ *satisfying a pair of resolvent equations* $R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$ *and* $P(\lambda) - P(\mu) = -(\lambda - \mu)P(\lambda)R(\mu)$. In short, the objects of study in empathy theory, comes in pairs.

Appendix B

Convolution Algebra of Abstract Harmonic Analysis

Convolution is the basic operation of harmonic analysis. In a very general sense, convolutions correspond to *weighted averages*. For example, given two 2π-periodic functions f, g, their convolution $f * g$ is defined as $f * g(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)ds$. If we set $g = 1$ then $f * g(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)ds$ which is the average of f over the 'circle' $[-\pi, \pi]$.

The objects of harmonic analysis are diverse: functions and measures defined on *any* topological group like $(-\infty, -\infty)$ or the circle group $S^1 := \{z \in \mathbb{C} | |z| = 1\}$. A very general definition of convolution is therefore required to capture the different definitions of the convolutions of the diverse objects of study.

B.1 Translation invariant linear functionals

For this purpose, we view the objects of study uniformly as *translation invariant linear functionals*. This we do by virtue of distributions on the topological group in question. The following example is instructive.

Let f be a function defined on the topological group S^1; $f \in L^1(S^1)$. First set the test space, $D(S^1)$, to be the space of all the scalar valued S^1-domained smooth test functions. Then the space, $D'(S^1)$, of all distributions on the circle group S^1 consists of all the linear functionals x^* on $D(S^1)$ that are continuous in the usual distributional sense [1]. We identify $f \in L^1(S^1)$

[1]Firstly, we say that a sequence $(\tau(n))$ of test functions in $D(S^1)$ is null iff $\lim_{n\to\infty} \|\tau(n)\|_{C^p} = 0$ where $\|\tau(n)\|_{C^p} = \sum_{0\le j\le p} \|\tau^{(j)}(n)\|_\infty$ for every $p \in \mathbb{N}$. Then the linear functional x^* on $D(S^1)$ is continuous in the sense that $\lim_{n\to\infty} x^*(\tau(n)) = 0$ for all null sequences $(\tau(n))$ in $D(S^1)$.

with the distribution $x_f^* : \tau \in D(S^1) \mapsto \int_{S^1} \tau f$; $x_f^* \in D'(S^1)$.

Now x_f^* plays a second role as an operator, X_f^*, on $D(S^1)$ by the group structure of S^1:

$$X_f^*(\tau) : a \in S^1 \mapsto \langle \tau, R^a x_f^* \rangle,$$

where we define the right translated distribution $R^a x_f^*$ as $\langle \tau, R^a x_f^* \rangle := \langle R^{\overline{a}} \tau, x_f^* \rangle$; R^a is the right shift map which maps each test function τ of $D(S^1)$ into the shifted function $\tau(\cdot - a)$; $a \in S^1$. Indeed, both $D(S^1)$ and $D'(S^1)$ are translation invariant spaces.

We say that x_f^* is a translation invariant functional provided its *dualism* X_f^* commutes with every shift operator R^a:

$$X_f^*[R^a \phi] = [R^a] X_f^* \phi. \tag{B.1.1}$$

Theorem 1. [Functions as translation invariant linear functionals] *Every $f \in L^1(S^1)$ is a translation invariant linear functional on $D(S^1)$.*

Remark 1. *Traditionally, objects of harmonic analysis are studied uniformly as translation invariant operators on the distribution space $D'(S^1)$: each $R^a : x^* : R^a x^*$ is viewed as an operator on $D'(S^1)$ and each $x^* \in D(S^1)$ is viewed as the operator on $D'(S^1) : L_{x^*} : y^* : x^* * y^*$ where*

$$x^* * y^* : \tau : x^*[Y^* \tau]$$

for every $\tau \in D(S^1)$ and x^* is translation invariant in the sense that

$$R^a L_{x^*} = L_{x^*} R^a. \tag{B.1.2}$$

Proposition 1. *If every distribution in $D'(S^1)$ is translation invariant in the sense of (B.1.1), our definition of x_f^* as a translation invariant functional (B.1.1) is a sufficient condition for (B.1.2)*

Proof.

$$\langle \tau, R^a(x_f^* * x^*) \rangle = \langle R^{\overline{a}} \tau, x_f^* * x^* \rangle = \langle \tau', x_f^* \rangle;$$

where $\tau' : a : \langle R^{\overline{a}} \tau, R^a x^* \rangle$ while

$$\langle \tau, x_f^* * R^a x^* \rangle = \langle \tau', x_f^* \rangle;$$

since $\langle \tau, R^a(R^a x^*) \rangle = \langle R^{\overline{a}} \tau, R^a x^* \rangle$. □

B.2 Convolution Algebra

Hewitt and Zukermann (§19 [18]) developed a very general notion of *convolution* called the *convolution on dual spaces* (Chapter 1.9.7 [23]) to capture the notion of the convolution of translation invariant linear functionals. It has the following three constructs motivated by the previous example:

Construct 1. Test space. Let G be an abelian group (not necessarily topological). Then the linear space D of all scalar valued functions on G plays the role of the test function space.

We insist that the domain of the test functions be a group G so that for any element $a \in G$ one can use the group structure to define the right translation map $R^a : D \to D | f \mapsto R^a f$ where

$$R^a f : t \in G \mapsto f(t - a). \tag{B.2.1}$$

It is critical to note that the new function $R^a f$ is in D again for every $a \in G$. Since this is the case for every $f \in D$ we say that D is *translation invariant*. We shall always take D as translation invariant.

Construct 2. Translation invariant linear functional. We insist that every linear functional x' (not necessarily bounded) [2] defined on D is *translation invariant*, that is,

$$x'[R^a f] = R^a[x' f]; \tag{B.2.2}$$

for every $a \in G$. Now, for (B.2.2) to make sense, we need to impose an additional requirement on each x' that it plays an extra role as an endomorphism on D. Therefore, each x' plays a double role as a linear functional and as an endomorphism X' on D defined by

$$X' f = {}^{x'} f; \qquad f \in D; \tag{B.2.3}$$

where

$${}^{x'} f : a \in G \mapsto x'[R^{-a} f]. \tag{B.2.4}$$

The endomorphisms so induced will be called *dualisms*: $X' f(0) = \langle f, x' \rangle$.

Note that the group structure once again is crucial in definition (B.2.4). Therefore, we rewrite equation (B.2.2) as

$${}^{x'}[R^a f] = R^a[\ {}^{x'} f]. \tag{B.2.5}$$

Construct 3. Convolution Algebra. By the double role of each translation invariant functional, we can now meaningfully define the product functional $x' * y'$ as follows:

[2] The linear functional x' is not necessarily bounded to make the framework as general as possible; it plays the role of a distribution in Example 1.

$$x' * y' : f \in D \mapsto x'(\,^{y'}f). \tag{B.2.6}$$

Let $\mathscr{A}$ be a space of translation invariant linear functional on D which is closed with respect to $*$. Then we call $\mathscr{A}$ a convolution algebra.

Hewitt (§19 [18]) calls the product $*$ a *convolution* : let D be the space $C_0(G)$ of all continuous scalar valued functions defined on a locally compact abelian group G which vanish at infinity [3]. Then the dual space $C_0(G)^*$ of all *bounded* linear functionals on $C_0(G)$ is $M(G)$ the space of all complex regular Borel measures on G. Taking $x', y' \in C_0(G)^*$, their product $x' * y'$ according to equation (B.2.6) coincides with the linear functional identifiable with the convolution measure of the measures identifiable with x' and y'.

[3] A function f on G is said to vanish at infinity if for every $\epsilon > 0$, there exists a compact subset K of G such that $|f(t)| < \epsilon$ for all $t \in G \backslash K$

Appendix C

Dualism Transfers Strong Continuity

Consider the function space Φ of continuous functions. Compact sets of Φ are necessarily equicontinuous. Equicontinuous subsets are also (topologically) finite. We study equicontinuous sets of admissible homomorphisms acting on subsets of Φ. Indeed, such a set behaves like a singleton set as far as the image of a compact set is concerned (Appendix C.1, Proposition 1).

Given an equicontinuous set of admissible homomorphisms $\{x'(\tau)|\tau \in I\}$, when is the 'dual' operator valued set $\{X'(\tau) = \Gamma(x')|\tau \in I\}$ equicontinuous? For the test spaces $\Phi = C(G, Z), C_0(G, Z)$, and $BUC(G, Z)$, the subset $\{X'(\tau)f|\tau \in I\}$ is an equicontinuous subset of Φ for each fixed $f \in \Phi$ (Appendix C.2.1, Proposition 2). Indeed, simple boundedness ensures that if the set of admissible homomorphisms $\{x'(\tau)|\tau \in I\}$ is equicontinuous then the dual set of operators $\{X'(\tau) = \Gamma(x')|\tau \in I\}$ is equicontinuous (Appendix C , Proposition 3 & Theorem 3).

Given a strongly continuous family of admissible homomorphisms $\mathfrak{X}' := \{x'(\tau)|\tau \in \mathbb{R}^+\}$, the 'dual' family of operators $\mathbb{X}' := \{X'(\tau)|\tau \in \mathbb{R}^+\}$ is strongly continuous if $\mathfrak{X}'$ is simply bounded over compacts (AppendixC.3, Theorem 2). The crux is that the simple boundedness of $\mathfrak{X}'$ over compacts ensures the equicontinuity of $\mathbb{X}'$ over compacts.

C.1 Equicontinuous Sets of Linear Mappings

A compact subset of a general topological space is *finite* (small) in a topological sense: every open cover has a finite subcover. A compact subset of a *function space* of *continuous* functions is necessarily *equicontinuous* [1]. An equicontinuous set, H, is also *finite* (small) topologically: every $f \in H$ behaves like a single fixed function $f_0 \in H$ as far as the topological concept of continuity (and hence uniform continuity) [2] and vanishing at infinity [3]. An equicontinuous subset of a function space of continuous functions behaves like a singleton set.

Consider an equicontinuous homomorphism valued subset $H = \{x'(\tau)|\tau \in I\} \subset \mathscr{A}_B$ [4]. Then H behaves like a single fixed bounded homomorphism $x'(\tau_0) \in H$ as far as the image of a compact set is concerned:

Proposition 1 (Equicontinuous image of a compact set is bounded)**.** *Let $H := \{x'(\tau)|\tau \in I\} \subset Hom_B(\Phi, Z)$ be an equicontinuous subset of bounded homomorphisms; $\Phi := C(G, Z); Z$ is a metric space; G is a topological group. If $K \subset \Phi$ is compact, then the image set $\{x'(\tau)[K]|\ \tau \in I\} := \bigcup_{\tau \in I}\{x'(\tau)[K]\}$ is a bounded subset of Z.*

Proof. Let V denote an ϵ-open ball about $x'(\tau_0)f \in Z$ for a fixed $\tau_0 \in I$. By the equicontinuity of the set $\{x'(\tau)|\tau \in I\}$, there is a neighbourhood U about $f \in \Phi$ such that $x'(\tau)[U] \subset V$ for every $\tau \in I$. Now $\{g + U|g \in K\}$ is an open covering of K and by the compactness of K, K is a subset of a finite subcover of this family:

$$K = \bigcup_{1 \leq i \leq k} g_i + U.$$

Therefore, the image set

$$\{x'(\tau)[D]|\ \tau \in I\} \subset \bigcup_{1 \leq i \leq k} \{x'(\tau)(g_i) + V|\tau \in I\},$$

is a bounded subset of Z since V is bounded, $\{x'(\tau)(g_i)|\tau \in I\}$ is a bounded subset and a finite union of bounded subsets is bounded. □

[1]Equicontinuity in a function space of continuous functions is the analogue of precompactness in a general topological space.

[2]Saying $f \in \Phi := C(\mathbb{R}, Z)$; Z a metric space, is continuous is equivalent to: *for each neighbourhood $V \subset Z$, there is a neighbourhood $U \subset \mathbb{R}$ with $f[U] \subset V$*. If $H \subset C(\mathbb{R}, Z)$ is equicontinuous, then the same neighbourhood $U \subset \mathbb{R}$ works for all the $f' \in H$ for every challenge V.

[3]Saying $f \in \Phi := C_0(\mathbb{R}, Z)$; Z a metric space, vanishes at infinity is to say that for every $\epsilon > 0$ there exists a compact set $K(\epsilon) \subset \mathbb{R}$ such that $|f(x)| < \epsilon$ for every $x \in \mathbb{R} \backslash K(\epsilon)$: *there exists a $U := \mathbb{R} \backslash K(\epsilon)$ such that $f[U] \subset B_\epsilon(0)$; U is open so U and $V := B_\epsilon(0)$ are both neighbourhoods* . If $H \subset \Phi$ is equicontinuous then the same neighbourhood U works for all the functions in H for every challenge V. Therefore, for the same compact set $K(\epsilon)$, $|f(x)| < \epsilon$ for every $x \in \mathbb{R} \backslash K(\epsilon)$ for every $f \in H$.

[4]To say that each linear mapping $x' \in \text{Hom}(\Phi, Z)$ is continuous is equivalent to

For each neighbourhood $V \subset Z$, there is a neighbourhood $U \subset \Phi$ with $x'[U] \subset V$.

If the same neighbourhood U works for each $x'(\tau) \in H$ then H is called equicontinuous.

C.2 Dualism Transfers Equicontinuity

Each $x'(\tau) \in \mathscr{A}_B$ plays a dual role as an operator $X'(\tau) \in \mathrm{Hom}(\Phi, \Phi)$. Under what condition does dualism transfer equicontinuity: given an equicontinuous set of admissible homomorphisms $H := \{x'(\tau)|\tau \in I\}$, when is the 'dual' operator valued set $H' = \{X'(\tau) = \Gamma(x')|\tau \in I\} \subset \mathscr{A}'_B$ equicontinuous? We give answers for the following test spaces: $\Phi = C(G, Z)$ (Chapter 2.2) and the smaller test spaces $\Phi_0 = C_0(G, Z)$ and $\Phi_1 = BUC(G, Z)$.

C.2.1 Pointwise Equicontinuity

Let $\Phi := C(G, Z); Z$ is a metric space. In this section we take G to be a topological group. We say that the 'dual' set $H' = \{X'(\tau)|\tau \in I\}$ is *pointwise equicontinuous* if $\{X'(\tau)f|\tau \in I\} \subset \Phi$ is equicontinuous for each $f \in \Phi$. Dualism transfers equicontinuity in the following sense: the equicontinuity of the family $\{x'(\tau)|\tau \in I\}$ implies the pointwise equicontinuity of the dual family $\{X'(\tau)|\tau \in I\}$.

Proposition 2 (Dualism respects equicontinuity)**.** *Let $\Phi := C(G, Z)$. Let $\{x'(\tau)|\tau \in I\} \subset \mathscr{A}_B$ be an equicontinuous family of continuous admissible homomorphisms. Then $\{X'(\tau)|\tau \in I\}$ is pointwise equicontinuous.*

Proof. For each fixed $f \in \Phi$, $X'(\tau)f : q : \langle f_{-q}, x'(\tau)\rangle$ is the composition of $x'(\tau) \in \mathrm{Hom}(\Phi, Z)$ with the continuous map $T : \omega \in G \mapsto f_{-\omega} \in \Phi$. It is straightforword to show that the continuity of T transfers the equicontinuity of the set $\{x'(\tau)|\tau \in I\} \subset \mathrm{Hom}(\Phi, Z)$ into the equicontinuity of the set $\{X'(\tau)f|\tau \in I\} \subset \Phi$. □

Remark 1. *Propositions 1 - 2 are equally valid for the smaller test spaces Φ_0 and Φ_1: continuous homomorphisms are admissible for these test spaces.*

Remark 2. *In Proposition 2, the dualism (by virtue of the continuity of the map $p \mapsto f_{-p}$) generates an equicontinuous subset $\{X'(\tau)f|\tau \in I\}$ of continuous functions in the nice function space Φ.*

C.2.2 Simple Boundedness

A simply bounded family [5] of continuous homomorphisms on a Frechet space is equicontinuous (uniform boundedness theorem [Theorem 3, [25]]). Thus if dualism transfers simple boundedness, then both the linear mapping valued families $\{x'(\tau)|\tau \in I\} \subset \mathscr{A}_B$ and $\{X'(\tau)|\tau \in I\} \subset \mathscr{A}'_B$ are equicontinuous

[5]Let F be a Frechet space and G a convex space. Then we say that the family $T := \{t_\alpha : F \to G|\alpha \in I\}$ of continuous operators is simply bounded if for each fixed point $f \in \Phi$, the set $\{\langle f, t_\alpha\rangle|\alpha \in I\} \subset G$ is bounded.

should the former family be simply bounded. This is the case for $\Phi = C(G, Z)$ should we assume that G is a second countable topological group.

Proposition 3 (Dualism transfers equicontinuity)**.** *Let $\Phi = C(G, Z)$; G is a second countable topological group. Let the family $\mathfrak{X}' := \{x'(\tau)|\tau \in I\} \subset Hom_B(\Phi, Z)$ of continuous homomorphisms be simply bounded. Then the families $\{x'(\tau)|\tau \in I\}$and $\{X'(\tau)|\tau \in I\}$ are equicontinuous.*

Proof. Since G is second countable, $\Phi = C(G, Z)$ is Frechet. Therefore, by the uniform boundedness theorem [Theorem 3, [25]], it suffices to show that the set $\{X'(\tau)|\tau \in I\} \subset \Phi$ is simply bounded since each $X'(\tau)$ is continuous. That is,

$$\sup_{\tau \in I}\{\|X'(\tau)f|_K\|\} < \infty,$$

for any compact set $K \subset \mathbb{R}^+$; $f \in \Phi$ fixed [6]. This is immediate from noting that the set $\{X'(\tau)f|_K;\ \tau \in [\alpha, \beta]\} = \{x'(\tau)[D];\ \tau \in I\}$; $D := \{R^p f|p \in K\} \subset \Phi$ is an equicontinuous image of a compact set (Appendix C.2.1, Proposition 1). □

For the Banach test spaces Φ_0, Φ_1, the set of bounded homomorphisms $\{x'(\tau)|\tau \in I\})$ is equicontinuous if and only if $\|x'(\tau)\| < M$ for all $\tau \in I$. Thus, the operator valued family $\{X'(\tau)|\tau \in I\}$ is simply bounded [7] since

$$\|\langle f_{-q}, x'(\tau)\rangle\| \leq M\|f\|_\infty,$$

for all $q \in G$; $\|f_{-q}\|_\infty = \|f\|_\infty$. By the uniform boundedness theorem [Theorem 3, [25]] [8], for any topological group G, dualism transfers equicontinuity for these test spaces:

Theorem 1 (Dualism transfers equicontinuity)**.** *Let $\Phi = BUC(G, Z)$ or $C_0(G, Z)$. Let the family $\mathfrak{X}' := \{x'(\tau)|\tau \in I\} \subset \mathscr{A}_B$ be simply bounded. Then the families $\{x'(\tau)|\tau \in I\}$ and $\{X'(\tau)|\tau \in I\}$ are equicontinuous.*

Remark 3. *In view of the fact that the product $*$ is a generalized composition it is not surprising that $*$ preserves equicontinuity: if $\{x'(\tau)\}$ and $\{Y'(\tau')\}$ are equicontinuous on the compacts $[\alpha, \beta]$ and $[\alpha', \beta']$, then $x'(\tau) * y'(\tau')$ is jointly equicontinuous [9].*

[6] The seminorms $\rho_K(f, g) := \|(f - g)|_K\|_\infty$; $K \subset G$ is compact, generate a convex topology on Φ.

[7] $\sup\{\|X'(\tau)f\|_\infty|\tau \in I\} < \infty$; $f \in \Phi$ is fixed.

[8] Note that each $X'(\tau)$ is bounded (Chapter 3.3.1, Theorem 8) and that Φ_0, Φ_1 are Banach so trivially Frechet

[9] That is, for each ϵ-ball $B_{x'(\tau)*y'(\tau')f}(\epsilon)$ about $x'(\tau) * y'(\tau')f \in Z$ there exists a neighbourhood $U_f \subset \Phi$ about f such that $x'(\tau) * y'(\tau')[U_f] \subset B_{x'(\tau)*y'(\tau')f}(\epsilon)$ for all $\tau \in [\alpha, \beta], \tau' \in [\alpha', \beta']$. This follows immediately from the definitions.

We end off by noting that $\{X'(\tau)|\tau \in I\}$ being equicontinuous is a sufficient condition for $\{X'(\tau)|\tau \in I\}$ being pointwise equicontinuous.

Proposition 4 (Equicontinuity implies Pointwise Equicontinuity)**.** *Let* $\Phi = C(G,Z), BUC(G,Z)$ *or* $C_0(G,Z)$*. If* $\{X'(\tau)|\tau \in I\} \subset \mathscr{A}_B'$ *is an equicontinuous family of bounded operators, then* $\{X'(\tau)|\tau \in I\}$ *is pointwise equicontinuous.*

Proof. The proof of Appendix C.2.1, Proposition 2 follows through by noting that $X'(\tau)f : q : \langle f_{-q}, x'(\tau)\rangle$ is the composition $\theta_0 \circ X'(\tau) \circ T$ of three continuous maps. □

C.3 Dualism Transfers Strong Continuity

The family of admissible homomorphisms $\mathfrak{X}' := \{x'(\tau)|\tau \in \mathbb{R}^+\}$ is strongly continuous if the mapping $\tau \in \mathbb{R}^+ \mapsto \langle f, x'(\tau)\rangle \in Z$ is continuous; $f \in \Phi$ is fixed. Likewise, the 'dual' family of operators $\mathbb{X}' := \{X'(\tau)|\tau \in \mathbb{R}^+\}$ is strongly continuous if the mapping $\tau \in \mathbb{R}^+ \mapsto X'(\tau)f \in \Phi$ is (uniformly) continuous for each fixed $f \in \Phi$. Under what condition does dualism transfer strong continuity: given a strongly continuous family $\mathfrak{X}'$ when is the 'dual' operator valued family $\mathbb{X}'$ strongly continuous? We give answers for the following test spaces: $\Phi = C(G,Z)$, $C_0(G,Z)$ and $BUC(G,Z)$.

C.3.1 Simple boundedness over compacts

We now show that if the strongly continuous family $\mathfrak{X}'$ is simply boundedness over compacts then the dual family $\mathbb{X}'$ is also strongly continuous. We first prove the result for the test space $\Phi = C(G,Z)$; G is metrizable second countable topological group.

Proposition 5 (Dualism transfers Strong Continuity)**.** *Let* $\Phi = C(G,Z)$*;* G *is metrizable second countable topological group. Let the strongly continuous family* $\mathfrak{X}' := \{x'(\tau)|\tau \in \mathbb{R}^+\} \subset \mathscr{A}_B$ *be simply bounded over compacts* [10]*. If* $\mathfrak{X}'$ *is strongly continuous, then* $\{X'(\tau)|\tau \in \mathbb{R}^+\}$ *is strongly continuous.*

Proof. For a first countable space $\mathbb{R}^+$, continuity and sequential continuity are equivalent. We prove the strong continuity of $\mathbb{X}'$ by showing that for each fixed $\tau_0 \in \mathbb{R}^+$, if $\tau_n \to \tau_0$ then

$$H_n := X'(\tau_n)f \rightrightarrows_K H_0 := X'(\tau_0)(f), \tag{C.3.1}$$

[10]We say that $\mathfrak{X}' := \{x'(\tau)|\tau \in \mathbb{R}^+\}$ is simply bounded over compacts if for every compact interval $[\alpha, \beta]$, $\sup\{x'(\tau)f|\tau \in [\alpha,\beta]\} < \infty$ for each fixed $f \in \Phi$.

where $f \in \Phi$ is fixed; $H_n \rightrightarrows_K H_0$ iff the convergence is uniform on any compact set K of the common domain G of the functions H_n, H by virtue of the fact that G is metrizable.

The crux of the proof is that a sequence of continuous functions that converges pointwise also converges uniformly on compacts *provided that the sequence of functions is equicontinuous*: the strong continuity of $\mathfrak{X}'$ immediately ensures the pointwise convergence of the sequence of functions H_n:

$$[X'(\sigma_n)f](\omega) \to [X'(\tau_0)(f)](\omega). \tag{C.3.2}$$

Therefore, once we show $\{X'(\tau_n)\}$ is pointwise equicontinuous, the proof is complete. The local compactness of $\mathbb{R}^+$ allows us to assume without loss of generality that $\{\tau_n | n \in \mathbb{N}\}$ belongs to a compact neighbourhood of τ_0 thus ensuring $\{X'(\tau_n)\}$ is equicontinuous (Appendix C.2.2, Proposition 3) and hence pointwise equicontinuous (Appendix C.2.2, Proposition 4). □

If the topological group G is locally compact, then the test space $\Phi = BUC(G, Z)$ is Banach and hence Frechet. Therefore, we can drop the requirement that G is second countable should G be locally compact. Formally:

Corollary 1 (Dualism Respects Continuity)**.** *Let $\Phi = BUC(G, Z)$; G is locally compact and metrizable. Let the strongly continuous family $\mathfrak{X}' := \{x'(\tau) | \tau \in \mathbb{R}^+\} \subset \mathscr{A}_B$ of bounded homomorphisms be simply bounded over compacts. Then $\mathbb{X}' = \{X'(\tau) | \tau \in \mathbb{R}^+\}$ is strongly continuous.*

Proof. By virtue of $BUC(G, Z) \subset C(G, Z)$, all the statements in the proof of Proposition 5 follow through with the help of Appendix C.2.2, Theorem 1 [11]. Therefore, to complete the proof we now show $H_n \rightrightarrows_K H_0$ implies $H_n \rightrightarrows H_0$. This is immediate on noting that G locally compact and Hausdorff implies each of its points has a compact neighbourhood; hence the uniform continuity ensures uniform convergence. □

For the test space $\Phi = C_0(G, Z)$, one can even drop the local compactness of G:

Corollary 2 (Dualism Respects Continuity)**.** *Let $\Phi = C_0(G, Z)$; G is metrizable. Let the strongly continuous family $\mathfrak{X}' := \{x'(\tau) | \tau \in \mathbb{R}^+\} \subset \mathscr{A}_B$ of bounded admissible homomorphisms be simply bounded over compacts. If $\mathfrak{X}'$ is strongly continuous then $\mathbb{X}' = \{X'(\tau) | \tau \in \mathbb{R}^+\}$ is strongly continuous.*

[11] Appendix C.2.2, Theorem 1 establishes that $\{H_n\} \subset \Phi_1$, is an equicontinuous set of uniformly continuous functions

Proof. By virtue of $C_0(G,Z) \subset C(G,Z)$, all the statements in the proof of Proposition 5 follow through with the help of Appendix C.2.2, Theorem 1 [12]. Therefore, to complete the proof we now show $H_n \rightrightarrows_K H_0$ implies $H_n \rightrightarrows H_0$. The crux of the matter is that $\{H_n\} \subset \Phi_0$, is an equicontinuous set of continuous functions which vanish at infinity. This ensures that for every $\epsilon > 0$, there exists a compact $K_\epsilon \subset G$ such that $\|H_n|_{G\backslash K_\epsilon}\|_\infty < \epsilon$ for every n. Now, H_0 also vanishes at infinity. Hence there exists a compact $K'_\epsilon \subset G$ such that $\|H_0|_{G\backslash K'_\epsilon}\|_\infty < \epsilon$. Now consider the compact set

$$K''_\epsilon := K_\epsilon \bigcup K'_\epsilon.$$

Since $G\backslash K''_\epsilon$ is a subset of both $G\backslash K_\epsilon$ and $G\backslash K'_\epsilon$,

$$\|(H_n - H_0)|_{G\backslash K''_\epsilon}\|_\infty < 2\epsilon,$$

for every n. Now by virtue of $H_n \rightrightarrows_K H_0$, for n large enough

$$\|(H_n - H_0)|_{K''_\epsilon}\|_\infty < 2\epsilon.$$

Hence $H_n \rightrightarrows H_0$. □

A topological space that is locally compact, Hausdorff and second countable is metrizable (Theorem 7.16 [31]). Therefore,

Theorem 2 (Dualism Transfers Strong Continuity)**.** *Let $\Phi = C(G,Z)$, $BUC(G,Z)$ or $C_0(G,Z)$; G is locally compact, Hausdorff and second countable. If the strongly continuous family $\mathfrak{X}' := \{x'(\tau)|\tau \in \mathbb{R}^+\} \subset \mathscr{A}_B$ of bounded admissible homomorphisms is simply bounded over compacts, then $\mathbb{X}' = \{X'(\tau)|\tau \in \mathbb{R}^+\}$ is strongly continuous.*

[12] Appendix C.2.2, Theorem 1 establishes that $\{H_n\} \subset \Phi_0$, is an equicontinuous set of continuous functions

Appendix D

C_0-semigroup Generation Theorems

D.1 Infinitesimal Generator

Let $\{E(\tau)|\tau \geq 0\}$ denote a uniformly continuous semigroup on the Banach space Y. Then

$$E(\tau) = e^{\tau A}; \quad \text{(D.1.1)}$$
$$E'(\tau) = AE(\tau), \quad \text{(D.1.2)}$$

for some bounded operator A. We say that A is the *infinitesimal generator* of the semigroup E. The infinitesimal generator A is independent of time or *time invariant*, unlike the family $\{E(\tau)|\tau \geq 0\}$, implying a sense of permanence. Now A contains all the information on $\{E(\tau)|\tau \geq 0\}$: A generates or germinates into $\{E(\tau)|\tau \geq 0\}$ under the exponential function [1] ; indeed, the association $A \leftrightarrow \{E(\tau)|\tau \geq 0\}$ is unique. The causal relation $E(\tau+\sigma) = E(\tau)E(\sigma)$ ensures the differentiability of $\{E(\tau)|\tau \geq 0\}$ so the causation mechanism (D.1.2) makes sense. Furthermore A is evaluated as the right hand side time derivative at $\tau = 0$: $A = \frac{d}{d\tau}E(0^+)$.

For many natural semigroups defined on concrete function spaces, the requirement for uniform continuity is too strong. Consequently uniform continuity is replaced by strong continuity and we arrive at the theory of C_0-semigroups and an analogue of the notion of a generator of a C_0-semigroup.

The analogue of the generator A of a uniformly continuous semigroup is a *closed* unbounded operator $(A, D(A))$ defined on a *dense* subspace $D(A)$ of Y: each $y \in Y$ induces a phase curve $E_y : \tau \mapsto E(\tau)y$ in the phase space;

[1]The exponential function is constructed by virtue of the functional calculus on the Banach algebra of bounded operators on a Banach space.

we define Ay as the right hand side derivative $\frac{d}{d\tau}E_y(0^+)$ analogous to the uniform case and $D(A) := \{y \in Y | \frac{d}{d\tau}E_y(0^+)$ *exists*$\}$. Then $\frac{d}{d\tau}E_y = AE_y$ for each $y \in D(A)$. Hence $D(A)$ plays a pivotal role. Formally,

Definition 1 (Infinitesimal Generator of a C_0-semigroup)**.** *Let $\{E(\tau)|\tau \geq 0\}$ denote a strongly continuous semigroup on the Banach space Y. Then we define the generator of the semigroup $\{E(\tau)|\tau \geq 0\}$ as the closed unbounded operator operator $(A, D(A))$ defined on a dense subspace $D(A)$ of Y:*

$$Ay \text{ is the right hand derivative } \tfrac{d}{d\tau}E_y(0^+),$$

and

$$D(A) := \{y \in Y | \frac{d}{d\tau}E_y(0^+) \text{ exists}\}.$$

Therefore, the strong operator topology enables us to construct the analogue of the generator A of a uniformly continuous semigroup in the theory of strongly continuous semigroups. The price to pay is that $(A, D(A))$ *has to be closed, have a dense domain and have a spectrum in a proper left half plane.*

The generation problem for C_0-semigroups is to characterize those linear operators A generate a strongly continuous semigroup $\mathcal{E} := \{E(\tau)|\tau \geq 0\}$ satisfying the ACP,

$$\frac{d}{d\tau}E(\tau)y = AE(\tau)y; \tag{D.1.3}$$

$$\lim_{\tau \to 0^+} E(\tau)y = y, \tag{D.1.4}$$

for y in a special subspace of Y

D.2 Integral Representations

Consider a semigroup $\{E(\tau) : Y \to Y | \tau \geq 0\}$ on a Banach space Y. We say that $\{E(\tau) : Y \to Y | \tau \geq 0\}$ is a E-semigroup if for every $y \in Y$ and $\lambda > 0$ the Laplace transforms

$$R(\lambda)y := \int_{(0,\infty)} e^{-\lambda\tau}E(\tau)y dt, \tag{D.2.1}$$

exists as a Bochner integral.

Theorem 1. *Let the family $\mathcal{E}$ be an E-semigroup. Then the following identities hold:*

$$R(\lambda)E(\tau) = E(\tau)R(\lambda); \tag{D.2.2}$$

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) = -(\lambda - \mu)R(\mu)R(\lambda), \tag{D.2.3}$$

for arbitrary positive λ, μ, τ

Proof. The identity (D.2.2) follows from $E(\tau)E(\sigma) = E(\sigma)E(\tau)$ after taking Laplace transforms with respect to σ. The identity (D.2.3) follows from the semigroup property. □

Next we define the domain $\Delta_Y := R(\lambda)[Y]$ which is a vector subspace of Y. From (D.2.3), it is clear that this domain does not depend on the choice of λ. From (D.2.2), $E(\tau) : \Delta_Y \to \Delta_Y$ for each t. We study the action of $E(\tau)$ on this subspace to see what the infinitesimal generator A should look like. After all, A is not defined everywhere. We call Δ_Y, the Sauer regularity domain.

We obtain an important integral representation of $E(\tau)y$ for $y \in \Delta_Y$.

Lemma 1. *Let $y = R(\lambda)y_\lambda \in \Delta_Y$ where $y_\lambda \in Y$. Then*

$$E(\tau)y = e^{\lambda\tau}[y - \int_0^\tau e^{-\lambda\sigma} S(\sigma)y_\lambda ds]. \tag{D.2.4}$$

Proof. Since $y = R(\lambda)y_\lambda$, from direct calculations (τ is fixed)

$$E(\tau)y = e^{\lambda\tau} \int_0^\infty e^{-\lambda(\sigma+\tau)} E(\sigma + \tau)y_\lambda ds.$$

The change of variable $\sigma' = \sigma + \tau$ gives

$$E(\tau)y = e^{\lambda\tau} \int_\tau^\infty e^{-\lambda\sigma'} E(\sigma')y_\lambda d\sigma'.$$

Writing $\int_\tau^\infty$ as $\int_0^\infty - \int_0^\tau$,

$$E(\tau)y = e^{\lambda\tau}[R(\lambda)y_\lambda - \int_0^\tau e^{-\lambda\sigma} S(\sigma)y_\lambda ds].$$

□

The above integral representation allows us to calculate $\frac{d}{d\tau}E(\tau)y$ as the product $f(\tau)g(\tau)$ where $f(\tau) = e^{\lambda\tau}$ and $g(\tau) = y - \int_0^\tau e^{-\lambda\sigma} S(\sigma)y_\lambda ds$. We differentiate by the product rule to get

$$\frac{d}{d\tau}E(\tau)y = [\lambda - R^{-1}(\lambda)]E(\tau)y \text{ for almost all } \tau,$$

provided $R(\lambda)$ is invertible on Δ_Y for all $\lambda > 0$ [2]. Furthermore, by the integral representation,

$$\lim_{\tau \to 0^+} E(\tau)y = y,$$

[2]The commutativity of $E(\tau)$ and $R^{-1}(\lambda)$ follows from pre and post multiplying the identity (D.2.2)

for $y \in \Delta_Y$.

Therefore, we define the operator $A : \Delta_Y \to Y$ as $A := \lambda - R^{-1}(\lambda)$. Once again it can be proved (algebraically) that A does not depend no the choice of λ. Finally, we have

Theorem 2. *If $\mathcal{E}$ is an E-semigroup and $R(\lambda)$ is invertible on Δ_Y for all $\lambda > 0$ and $y \in \Delta_Y$, then the abstract Cauchy problem*

$$\frac{d}{d\tau}E(\tau)y = AE(\tau)y \quad \text{for almost all } \tau; \tag{D.2.5}$$

$$\lim_{\tau \to 0^+} E(\tau)y = y, \tag{D.2.6}$$

is satisfied for almost all τ.

D.3 Approximations of Infinitesimal Generator

The pseudo-resolvent operators $R(\lambda)$ (D.2) constructed the infinitesimal generator A of an E-semigroup $\mathcal{E}$. For a plain semigroup one cannot initially assume the existence of such resolvents. Hille extended the Banach algebraist's formulation of a resolvent of a bounded operator A to the resolvent of a closed operator A in order to construct bounded operators A_n which approximate A.

Definition 2 (Unbounded Generator of a Resolvent)**.** *Fix an unbounded operator $A : D(A) \subset Y \to Y$. Then A is called a generator of the resolvent $R_{\lambda,A} : \lambda \in \rho(A) \mapsto R(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{L}(Y)$ where $(\lambda - A)$ is 1-1.*

It turns out that such resolvents are equivalent to the resolvents $R(\lambda)$ (D.1, equation (D.2.1)) of an E-semigroup (D.3.2, Theorem 4). This observation is the crux of Yosida's approximation of the infinitesimal generator A by bounded operators A_n (D.3.2, Definition 4).

D.3.1 Hille approximations

Hille treated the semigroup $\mathcal{E}$ generated by the operator A as an exponential function $t \mapsto e^{\tau A}$. Thus, the generation problem for the abstract Cauchy problem can be posed as follows: can a closed densely defined operator $(A, D(A))$ be constructed in such a way that the family $E(\tau)y = [e^{\tau A}]y$ is a strongly continuous semigroup satisfying the ACP. Diagrammatically,

$$A \xrightarrow{\text{what sense } [e^{\tau A}]y} E(\tau)y$$

Diagram 3 : Hille approach to the Generation Problem for ACP

The well known one-dimensional case formula $e^{\tau A} = \lim_{n\to\infty}(1+\frac{\tau}{n}A)^n$ is problematic for unbounded operators since powers of unbounded operators are involved: convergence is unlikely. Therefore, we resort to an equivalent form involving negative powers

$$e^{\tau A} = \lim_{n\to\infty}(1 - \frac{\tau}{n}A)^{-n},$$

which can be rewritten using *bounded* resolvent operators $R(\lambda, A) := (\lambda 1 - A)^{-1} \in \mathcal{L}(Y)$ in the form of

$$E(\tau)y := [e^{\tau A}]y = \lim_{n\to\infty}[\frac{n}{\tau}R(\frac{n}{\tau}, A)]^n y. \qquad \text{(D.3.1)}$$

Hille then showed that the above limit exists when a growth condition is imposed on the powers of the resolvent $R(\lambda, A)$:

Theorem 3 (Generation theorem)**.** *Let $(A, D(A))$ be a densely defined closed operator on a Banach space Y. Then the above limit (D.3.1) exists and defines a strongly continuous semigroup $\{E(\tau)|\tau \geq 0\}$ satisfying:*

(i) $E(\tau+\sigma) = E(\tau)E(\sigma)$*;*
(ii) $E(0) = \mathbf{1}$*;*
(iii) $\lim_{t\to 0^+} E(\tau)y = y, y \in Y$*,*

if and only if there exists constants $\omega \in \mathbb{R}, M > 0$ such that the resolvent set of A contains the half-line (ω, ∞) and the growth condition

$$[(\lambda - \omega)R(\lambda, A)]^n \leq M,$$

is imposed.

D.3.2 Yosida Approximants

Yosida's approach approximated A by a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded operators and hope that

$$E(\tau)y := [e^{\tau A}]y = \lim_{n\to\infty}[e^{\tau A_n}]y, \qquad \text{(D.3.2)}$$

since the exponential function for bounded operators is well defined. The motivation behind the construction of the approximants $A_n \in \mathcal{L}(Y)$ lies in the fact that the pseudo-resolvent $R(\lambda)$ coincides with the resolvent $R(\lambda, A)$ of the infinitesimal generator A of the semigroup

Definition 3 (Pseudo Resolvent of C_0-semigroup)**.** *Let $\{E(\tau)|\tau \geq 0\}$ be a C_0-semigroup. Then we define the pseudo resolvent R as the function of the scalar λ into $\mathcal{L}(Y)$; $R : \lambda \mapsto R(\lambda) \in \mathcal{L}(Y)$ where $R(\lambda) : y \in Y \mapsto \int_{[0,\infty)} e^{-\lambda\tau} E_y(\tau)dt \in Y$.*

Theorem 4. *Fix the generator A of the resolvent $R(\lambda, A)$ to be the generator of the C_0-semigroup $\{E(\tau)|\tau \geq 0\}$. Then the (bounded) resolvent $R(\lambda, A)$ coincides with the pseudo-resolvent $R(\lambda) : y \in Y \mapsto \int_{[0,\infty)} e^{-\lambda\tau} E_y(\tau)dt$ on $\lambda \in \rho(A)$.*

Therefore, $R(\lambda)(\lambda - A)$ is the identity operator on $D(A)$. For all $y \in D(A)$,

$$AR(\lambda)y = \lambda R(\lambda)y - y$$

Set $\lambda := n$. Now the numerical analogue of $nAR(n, A)$ is the number $A_n := \frac{nA}{n-A} = \frac{A}{1-n^{-1}A}$ where $A_n \overset{n\to\infty}{\longrightarrow}$ A. Therefore, we define the Yosida approximations of A by

$$A_n = n^2R(n, A) - n = nAR(n, A)$$

and indeed $A_ny \to Ay$ for all $y \in D(A)$. Formally,

Definition 4 (Yosida Approximants)**.** *Let $\{E(\tau)|\tau \geq 0\}$ be a C_0-semigroup and A the infinitesimal generator of the semigroup. Then we define the Yosida approximations of A by the bounded operators*

$$A_n = n^2R(n, A) - n = nAR(n, A).$$

Diagrammatically,

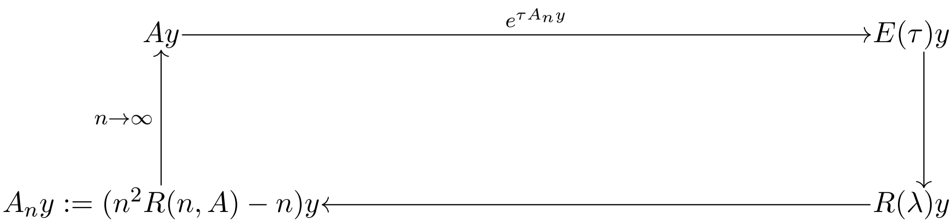

Diagram 4 : Yosida Approximants

Theorem 5. *Let $\{E(\tau)|\tau \geq 0\}$ be a C_0-semigroup. Then the limit (D.3.2) exists for all $y \in Y$; $A_n \in \mathcal{L}(Y)$ and $A_n = n^2R(n, A) - nI$: A_n is constructed purely from the resolvent. Furthermore, the association $A \leftrightarrow \{E(\tau)|\tau \geq 0\}$ is unique.*

Remark 1. *The generalization of the concept of a resolvent of an unbounded operator (Definition 2) is adequate for C_0 semigroup theory on Banach spaces by virtue of Theorem 6: Definition 2 forces A to be closed.*

Constructing the family $\{E(\tau)|\tau \geq 0\}$ using equation (D.3.2) satisfies the Generation Theorem 3. Formally,

Theorem 6. *The family* $\{E(\tau)|\tau \geq 0\}$ *constructed using equation (D.3.2) satisfies the Generation Theorem 3;* $A_n = n^2R(n,A) - n = nAR(n,A)$. *Diagrammatically,*

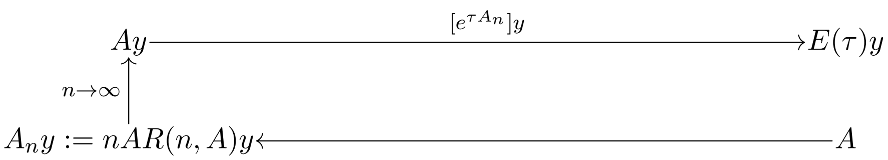

Diagram 5 : Yosida solution to the Generation Problem for the ACP

Bibliography

[1] W. Arendt, Vector-valued Laplace transforms and Cauchy problems. *Israel J. Math.*, **59** (1987), 327–352.

[2] W. Arendt, C.J. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems : The General Theory*, Birkhäuser Verlag, Basel-Boston-Berlin, Monographs in Mathematics, Vol 96, 2001.

[3] A. Bobrowski, *Functional Analysis for Probability and Stochastic Processes*, Cambridge University Press, New York, 2005.

[4] D. S. Bridges, *Foundations of Real and Abstract Analysis,* Graduate Texts in Mathematics, Springer, New York, 1997.

[5] T.J. Brown and N. Sauer, Double families of integrated evolution operators, *J. Evol. Equ.*, **4** (2004), 567–590.

[6] V Capasso and D. Bakstein, *An Introduction to Continuous-Time Stochastic Processes, Theory, Models, and Applications to Finance, Biology, and Medicine*, Birkhauser, Boston 2005.

[7] W. Chojnacki, On the equivalence of a Theorem of Kisynski and the Hille-Yosida Generation Theorem, *Proc. Amer. Math. Soc.*, Volume 126, Number 2, 491-497.

[8] J. Diestel and J.J. Uhl, Vector Measures, *American Mathematical Society, Providence, RI. Mathematical Surveys and Monographs, Vol 15*, 1977.

[9] J. A Dieudonne, *History of Functional Analysis*, North-Holland Mathematics Studies 49, North-Holland, Amsterdam, 1983.

[10] N. Dinculeanu, *Vector Measures*, Pergamon Press, Veb Deutscher Verlag Der Wissenschaften, Berlin, 1967.

[11] N. Dunford and J. T. Schwartz, *Linear Operators Part I: General Theory*, Interscience Publishers, Inc., New York, 1958.

[12] K. J Engel, R. Nagel , *One-Parameter Semigroups for Linear Evolution Equations* Springer Verlag, New York, 2000.

[13] A. G. Fadell, *Vector Calculus and Differential Equations*, D. Van Nostrand, United States, 1968.

[14] H. O. Fattorini, The Cauchy Problem, *Encyclopedia of Mathematics and its Applications vol 18*, Addison-Wesley Publishing Company, London (1983).

[15] A. Favini, Laplace transform method for a class of degenerate evolution problems, *Rend. Mat.*, **3 - 4** (1979), 511–536.

[16] ——— and A. Yagi, *Degenerate Differential Equations in Banach Spaces*, Marcel Dekker, Inc., New York-Basel-Hong Kong, 1999.

[17] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol II*, Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons Inc, New York, 1966.

[18] E. Hewitt and K. A. Ross, *Abstract harmonic Analysis Volume I : Structure of Topological Groups, Integration Theory, Group Representations*, Springer Verlag, New York,1963.

[19] E. Hille and R.S. Phillips, *Functional Analysis and Semi groups*, American Mathematical Society Colloquium Publications, Rhode Island, Volume 31, 2000.

[20] J.E Huneycutt, *Products and convolutions of vector valued set functions, Studia Math. 41, 1972, 119 -129*

[21] Y. Katznelson, *An introduction to Harmonic Analysis*, Third Edition, Cambridge University Press, Cambridge, 2004.

[22] J. Kisynski, *The Widder spaces, representations of the convolution algebra* $L^1(\mathbb{R})^+$ *and one parameter semigroups of operators*, Preprint no. 588, Institute of Mathematics, Polish Academy of Sciences, Warsaw, June 1998.

[23] T. W. Palmer, *Banach Algebras and The General Theory of *-Algebras; Volume I : Algebras and Banach Algebras*, Encyclopaedia of Mathematics and its Applications, Cambridge University Press, New York, 1994.

[24] J. Jacod and P. Protter , *Probability Essentials*, Springer-Verlag Berlin Heidelberg, 2000.

[25] A. P. Robertson, *Topological Vector Spaces*, Cambridge at the University Press, Cambridge, 1964.

[26] N. Sauer, Linear evolution equations in two Banach spaces. *Proc. Royal Soc. Edinburgh*, **91A** (1982), 287–303.

[27] ——— and J.E. Singleton, Evolution operators in empathy with a semigroup, *Semigroup Forum*, **39** (1989), 85–94.

[28] ——— Empathy Theory and the Laplace transform. In: *Linear Operators*, Banach Center Publ., Institute of Mathematics, Polish Academy of Sciences, Warszawa **38** (1997), 325 – 338.

[29] ——— Causality and Causation: What we learn from mathematical dynamic systems theory, Trans Roy Soc S Afr. Volume **65** (2010), 65–68 .

[30] R.E. Showalter and T.W. Ting, Partial differential equations of Sobolev-Galpern type. *Pacific J. Math.*, **31** (1969), 787–794.

[31] J. N McDonald, N. A. Weiss, *A Course in Real Analysis*, Academic Press, San Diego, 1999.

[32] D. V. Widder, *An Introduction to Transform Theory*, Academic Press, New York, 1971.

[33] K. Yosida, *Functional Analysis*, Springer Verlag, Berlin-Göttingen-Heidelberg, 1965.

People To Thank : Part II

The loyalty and faithfulness of Jesus lives in you.

...... To Mrs McDermot

You are a fine man of God. Thank you for your friendship.

...... To Rudolf Venter

Thank you for accepting me into the Africa Missions Church. It is an outpost for the Kingdom of God. Moksaanim allow the Blood of Jesus to speak for you; Saamonim continue being a walking example of Christian integrity. Your living sacrifice now flows through into the Blood (of Jesus) washed lives of a compassionate son, Chan Hee, and a gentle daughter Moon Hee.

...... To Pastor and Mrs Lee.

Expect your kindness and care to come back to you one hundred fold in the name of Jesus.

...... To Mrs Cronje

My idol! Like my idol I am flying the House 27 flag proud! Your marriage to Louisa, a woman who has a relationship with Jesus, further cements the title 'idol'!

...... To Minora

Keep your vision large: a doctor ten times better than the other doctors!

...... To Louisa

Without your fine library service, this work would not be possible

...... To Mrs Boschoff

Your footprints are on Section 5.3.2, page 81. Jesus steps with your every footsteps. The born again experience now awaits!

...... To Claire Dufourd

The high point of your friendship was your integrity! Tell your wife I am honoured to know you.

...... To Rabah Sehar

Shepherd James, Lee Jae Eel, your passion for spreading the Gospel is exemplary. Saamonim Josephine, Kim Won Cha, you are what a real Christian should be like. Thank you for your care and love over this PhD.

...... To James and Josephine Lee

Today I have shown what God can do through a mongrel!

...... To Jonathan Jude Ramnanun

This PhD is for the Kingdom Of God just like your marriage.

...... To Conrad and Letitia Pienaar

Your Majesty, King Khaled, I have delivered by the power of the Holy Ghost.

...... To Khaled Ahmad Ibrahim

King Jibsaanim, thank you for your humour and believing in me. You are an example of being young at heart.

...... To Kim Sang Hoon

You have a special Friendship with the living Christ Jesus, Yeshua. You manifested this Friendship with your Holy Spirit inspired friendship towards me. In your Friendship, I saw the Blood of Jesus silence all your enemies and speak victory and restoration into your life.Continue to manifest the unstoppable power of Jesus, Friend of Jesus.

...... To Mulatu Fekadu Zerihun

The gentle touch of Jesus shines through your gentle heart. Always remember Jesus values you to the point He died for you. His Blood speaks victory for you and your Blood washed family.

...... To Myung Jibsaanim and family

Thanks for your special and unique friendship.As unique as Pooh the Bear!

...... To Lee Hyo Joon and family

You have a wisdom and peace beyond your years. What Jesus did for me in this thesis, He will do for you since He is no respector of persons (Acts 10:34). Continue being a powerful witness of the love of Jesus.

...... To Charlie (Jong-il)

The creative power of Jesus is working right now in your life, taking all the broken pieces of shattered dreams and resurrecting them into a new powerful unstoppable dream. Unstoppable resurrection power of Christ Jesus already declared for you in the throne room of Elohim! (Isaiah 55: 11)

...... To Shin Gi-Yoon

Remember your value: Jesus saw you as worth dieing for. Life is good: Jesus is alive.

...... To Michael (Fengnian Zhang)

Your life is a gentle witness to the beauty of Jesus.

...... To Gwendolyn Elizabeth Barnes

Index

DECLARATION

I, Mr Wha-Suck Lee, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Name: Wha-Suck Lee.

Date: 22 January, 2015.

Signature: