Quasi-orthogonality and real zeros of some ${}_2F_2$ and ${}_3F_2$ polynomials

SJ Johnston

Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, South Africa

K Jordaan[1]

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

Abstract

In this paper, we prove the quasi-orthogonality of a family of ${}_2F_2$ polynomials and several classes of ${}_3F_2$ polynomials that do not appear in the Askey scheme for hypergeometric orthogonal polynomials. Our results include, as a special case, two ${}_3F_2$ polynomials considered by Dickinson in 1961. We also discuss the location and interlacing of the real zeros of our polynomials.

Keywords: Hypergeometric polynomials; Quasi-orthogonal polynomials; zeros; ${}_3F_2$ polynomials; ${}_2F_2$ polynomials.
2000 MSC: 33C05, 33C45, 33D15, 33D45

1. Introduction

A sequence $\{P_n\}$ of real polynomials of exact degree $n \in \mathbb{N}$ is orthogonal with respect to a positive-definite moment functional $\mathcal{L}$ if (cf. [3])

$$\mathcal{L}[R_m(x)R_n(x)] = 0 \text{ for } m \in \{0, 1, \ldots, n-1\}.$$

A well-known consequence of orthogonality is that the n zeros of $P_n(x)$ are real and simple and lie in the supporting set of $\mathcal{L}$ (cf. [3]). The zeros of P_n depart

Email addresses: johnssj@unisa.ac.za (SJ Johnston), kjordaan@up.ac.za (K Jordaan)
[1]Research by the second author is partially supported by the National Research Foundation of South Africa

from the supporting set of $\mathcal{L}$ in a specific way when the parameters are changed to values where the polynomials are no longer orthogonal and this phenomenon can be explained in terms of the concept of quasi-orthogonality.

We say that a polynomial sequence $\{R_n\}$ is quasi-orthogonal of order $r \geq 1$, $r \in \mathbb{N}$ with respect to a moment functional $\mathcal{L}$ if

$$\mathcal{L}[R_m(x)R_n(x)] = 0, \ |n-m| \geq r+1$$

$$\exists \ s \geq r \text{ such that } \mathcal{L}[R_{s-r}(x)R_s(x)] \neq 0.$$

It is equivalent to say that

$$\mathcal{L}[x^m R_n(x)] = 0, \ m \in \{0, 1, \dots, n-(r+1)\}, \quad n \geq r+1$$

$$\exists \ s \geq r \text{ such that } \mathcal{L}[x^{s-r} R_s(x)] \neq 0.$$

Furthermore, R_n has at least $n-r$ distinct, real zeros in the supporting set of $\mathcal{L}$ (cf. [3]).

Quasi-orthogonal polynomials of order 1 were first introduced by Riesz [22] in 1923 in his solution of the Hamburger moment problem and Fejér [13] considered quasi-orthogonality of order 2 in 1933. In 1937, Shohat [23] generalised the concept of quasi-orthogonality to any order and showed that whenever there exists an orthogonal polynomial sequence $\{P_n\}$ for $\mathcal{L}$, then $\{R_n\}$ being a quasi-orthogonal polynomial sequence of order $r \geq 1$ with respect to $\mathcal{L}$, is equivalent to

$$R_n(x) = \sum_{\nu=n-r}^{n} c_{n,n-\nu} P_\nu(x), \qquad n \in \{r, r+1, \dots\}, \tag{1.1}$$

whilst

$$R_n(x) = \sum_{\nu=0}^{n} c_{n,n-\nu} P_\nu(x), \qquad n \in \{0, \dots, r-1\},$$

and $\exists \ s \geq r$ such that $c_{s,s-r} \neq 0$.

A more general definition of quasi-orthogonality was given in 1957 by Chihara (cf. [2]), who discussed quasi-orthogonality in the context of three-term recurrence relations, proving that a quasi-orthogonal polynomial of any order r satisfies a three-term recurrence relation whose coefficients are polynomials of appropriate degrees. Draux [5] proved the converse of one of Chihara's results and Dickinson [4] improved Chihara's result by deriving a system of recurrence relations that is both necessary and sufficient for quasi-orthogonality. Dickinson applied this method to some special cases of Sister Celine's polynomials

$$f_n(a,x) = {}_3F_2\left(\begin{array}{c} -n, n+1, a \\ \frac{1}{2}, 1 \end{array}; x\right) = \sum_{m=0}^{n} \frac{(-n)_m (n+1)_m (a)_m}{\left(\frac{1}{2}\right)_m (1)_m} \frac{x^m}{m!}$$

and proved that $f_n\left(\frac{3}{2}, x\right)$ and $f_n(2,x)$ are quasi-orthogonal of order 1 on the interval $(0,1)$ with respect to the weight functions $(1-x)$ and $x^{-1/2}(1-x)^{3/2}$ respectively. Algebraic properties of the linear functional associated to quasi-orthogonality are given in [5, 18, 19, 20]. More recent results, particularly on the zeros of order 1 and 2 quasi-orthogonal polynomials, are due to Brezinski, Driver and Redivo-Zaglia [1] and Joulak [16]. For the convenience of the reader, we summarise some of these results.

Lemma 1.1. *Let $\{P_n\}$ be real, monic polynomials of exact degree n that are orthogonal with respect to a positive-definite moment functional $\mathcal{L}$ with supporting set (a,b) and let $x_{i,n}, i = 1,2,\ldots,n$, be the zeros of $P_n(x)$ and $y_i, i = 1,2,\ldots,n$, the zeros of $R_n(x)$, where*

$$R_n(x) = P_n(x) + a_n P_{n-1}(x)$$

with $a_n \neq 0$. Let $f_n(x) = P_n(x)/P_{n-1}(x)$. Then

- *(a) $y_1 < a$ if and only if $-a_n < f_n(a) < 0$;*
- *(b) $b < y_n$ if and only if $-a_n > f_n(b) > 0$;*
- *(c) R_n has all its zeros in (a,b) if and only if $f_n(a) < -a_n < f_n(b)$;*
- *(d) $x_{i,n} < y_i < x_{i,n-1}$ for $i = 1,\ldots,n-1$, and $x_{n,n} < y_n$ if and only if $a_n < 0$;*
- *(e) $x_{i-1,n-1} < y_i < x_{i,n}$ for $i = 2,\ldots,n$ and $y_1 < x_{1,n}$ if and only if $a_n > 0$;*
- *(f) $y_{1,n+1} < y_{1,n} < y_{2,n+1} < \cdots < y_{n,n+1} < y_{n,n} < y_{n+1,n+1}$ if and only if $f_{n+1}(y_{n,n}) + a_{n+1} < 0$ when $a_n < 0$ or $f_{n+1}(y_{1,n}) + a_{n+1} > 0$ when $a_n > 0$.*

Proof. Parts (a), (b) and (c) are proved in [16, Theorem 4], parts (d) and (e) in [16, Theorem 5] and (f) in [16, Theorem 6]. □

Lemma 1.2. *Let $\{P_n\}$ be real polynomials of exact degree n that are orthogonal with respect to a positive-definite moment functional with supporting set (a, b), and let $x_{i,n}, i = 1, 2, \dots, n$, be the zeros of $P_n(x)$ and $y_i, i = 1, 2, \dots, n$, the zeros of $R_n(x)$, where*

$$R_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x)$$

with $b_n \neq 0$. Let $f_n(x) = P_n(x)/P_{n-1}(x)$. Then

(a) If $b_n < 0$ then all of the zeros of R_n are real and distinct and at most two of them lie outside the interval (a, b).

In particular,

(b) if $b_n < 0$ then the zeros of R_n are such that $y_1 < x_{1,n-1}, x_{i-1,n-1} < y_i < x_{i,n-1}$ for $i = 2, ..., n1$, and $x_{n-1,n-1} < y_n$. Additionally,
   - *(i) $y_n < x_{n,n}$ if $-a_n - b_n/f_{n-1}(x_{n,n}) < 0$ and $y_n > x_{n,n}$ if $-a_n - b_n/f_{n-1}(x_{n,n}) > 0$;*
   - *(ii) $y_n < b$ if $-an-b_n/f_{n-1}(b) < f_n(b)$ and $y_n > b$ if $-a_n-b_n/f_{n-1}(b) > f_n(b)$;*
   - *(iii) $y_1 < x_{1,n}$ if $-a_n - b_n/f_{n-1}(x_{1,n}) < 0$ and $y_1 > x_{1,n}$ if $-a_n - b_n/f_{n-1}(x_{1,n}) > 0$;*
   - *(iv) $y_1 < a$ if $-a_n - b_n/f_{n1}(a) < f_n(a)$ and $y_1 > a$ if $-an-b_n/f_{n-1}(a) > f_n(a)$.*

Proof.

(a) This is proved in [23, Theorem VII].

(b) This is proved in [1, Theorem 5]. □

In this paper, we prove the quasi-orthogonality of some general classes of hypergeometric polynomials of the form

$${}_pF_q\left(\begin{array}{c}-n, \beta_1 + k, \alpha_3 \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{array}; x\right) = \sum_{m=0}^{n} \frac{(-n)_m(\beta_1 + k)_m(\alpha_3)_m \dots (\alpha_p)_m}{(\beta_1)_m \dots (\beta_q)_m} \frac{x^m}{m!},$$

$k \in \{1, 2, \dots, n-1\}$, which do not appear in the Askey scheme for hypergeometric orthogonal polynomials (cf. [17, p. 183]). One of our results includes the two polynomials considered by Dickinson [4] as special cases. Results on the location and interlacing of the real zeros related to the quasi-orthogonality of the polynomials under consideration are also discussed. The location of real zeros of hypergeometric functions plays an important role in, for example, problems arising in weighted Bergman spaces (cf. [25]) and convergence of rational approximants of hypergeometric functions (cf. [7]). The zeros of classes of hypergeometric polynomials that are not orthogonal are of interest and the need to find new results specifically for more general hypergeometric polynomials was highlighted at the NATO conference on Special Functions: Current Perspectives and Future Directions held in Tempe, Arizona in 2000.

Some progress has been made and interesting results concerning the zeros of several classes of ${}_3F_2$ and ${}_4F_3$ polynomials that can be written as products of ${}_2F_1$ polynomials (cf. [6], [8], [11] and [12]) have been obtained. The asymptotic behaviour of the zeros of certain families of ${}_3F_2$ functions were studied by Driver and Jordaan in [9] using Hurwitz theorem with [24] by Srivastava, Zhou and Wang completing this work. In [10], Driver, Jordaan and Martínez-Finkelstein prove that the zeros of some families of ${}_3F_2$ hypergeometric polynomials are all real and negative using the theory of Pólya frequency sequences and functions. As a consequence, they establish the asymptotic distribution of these zeros when the degree of the polynomials gets large.

Section 2 of this paper contains our results on quasi-orthogonality and resulting new information for the zeros of a class of ${}_2F_2$ polynomials while Section 3 focuses on several classes of ${}_3F_2$ polynomials.

2. Quasi-orthogonality and real zeros of a class of ${}_2F_2$ polynomials.

We begin by proving the following lemma which is used extensively in our proofs.

Lemma 2.1. *Let $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n-1\}$ and $\alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R}$ with $\alpha_2, \ldots, \alpha_p, \beta_1, \ldots \beta_q \notin \{0, -1, -2, \ldots, -n\}$ and $\alpha_2 \notin \{0, 1, \ldots, k-1\}$. Then*

$$ {}_pF_q \left(\begin{array}{c} -n, \alpha_2 + 1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) = a_k \ {}_pF_q \left(\begin{array}{c} -n + k, \alpha_2 - k + 1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) + \ldots $$

$$+ a_1 \; {}_pF_q \left(\begin{array}{c} -n+1, \alpha_2-k+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right)$$

$$+ a_0 \; {}_pF_q \left(\begin{array}{c} -n, \alpha_2-k+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) \tag{2.2}$$

for some constants $a_i, \; i \in \{0, 1, \ldots, k\}$ *depending on the parameters* n, α_2 *and* k.

Proof. A contiguous relation for the ${}_pF_q$ function (cf. [21, p. 82, eqn. (14)]) with $\alpha_1 = -n$ and $\alpha_k = \alpha_2$ is given by

$$\alpha_2 \; {}_pF_q \left(\begin{array}{c} -n, \alpha_2+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) \tag{2.3}$$

$$= (n+\alpha_2) \; {}_pF_q \left(\begin{array}{c} -n, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) - n \; {}_pF_q \left(\begin{array}{c} -n+1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right).$$

Applying (2.3) iteratively to both polynomials on the right-hand side of (2.3) $k-1$ times, we find that ${}_pF_q \left(\begin{array}{c} -n, \alpha_2+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right)$ may be written as the linear combination (2.2). Since the use of (2.3) introduces a denominator $\alpha_2(\alpha_2-1)\ldots(\alpha_2-k+1)$ for each a_i, we have to assume that $\alpha_2 \notin \{0, 1, \ldots, k-1\}$. □

Theorem 2.2. *Let* $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $\beta, \beta+k \notin \{0, -1, \ldots, -n\}$ *with* $\alpha > -1$, $k \in \{1, \ldots, n-1\}$ *fixed. Then the polynomial* ${}_2F_2 \left(\begin{array}{c} -n, \beta+k \\ \alpha+1, \beta \end{array} ; x \right)$ *is quasi-orthogonal of order* k *on* $(0, \infty)$ *with respect to the Laguerre weight* $e^{-x}x^{\alpha}$ *and has at least* $n-k$ *distinct real positive zeros.*

Proof. Let $p = q = 2$, $\alpha_2 = \beta+k-1$, $\beta_1 = \alpha+1$ and $\beta_2 = \beta$ in (2.2). Then there exist constants $a_i, i \in \{0, \ldots, k\}$ such that

$${}_2F_2 \left(\begin{array}{c} -n, \beta+k \\ \alpha+1, \beta \end{array} ; x \right)$$

$$= a_0 \; {}_2F_2 \left(\begin{array}{c} -n, \beta \\ \alpha+1, \beta \end{array} ; x \right) + \ldots + a_k \; {}_2F_2 \left(\begin{array}{c} -n+k, \beta \\ \alpha+1, \beta \end{array} ; x \right)$$

$$= a_0\ {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array};x\right) + \ldots + a_k\ {}_1F_1\left(\begin{array}{c} -n+k \\ \alpha+1 \end{array};x\right)$$

$$= a_0\ \frac{n!}{(\alpha+1)_n} L_n^{\alpha}(x) + \ldots + a_k\ \frac{(n-k)!}{(\alpha+1)_{(n-k)}} L_{n-k}^{\alpha}(x)$$

where $L_n^{\alpha}(x) = \dfrac{(\alpha+1)_n}{n!}\ {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array};x\right)$ denotes Laguerre polynomials (cf. [17, (9.12.1)]). The result then follows from equation (1.1) together with the orthogonality of Laguerre polynomials (cf. [17, (9.12.2)]). □

It follows from Theorem 2.2 that the hypergeometric polynomial

$${}_2F_2\left(\begin{array}{c} -n, \beta+1 \\ \alpha+1, \beta \end{array};x\right)$$

is quasi-orthogonal of order 1 on the positive real line and consequently has at least $(n-1)$ real and distinct positive zeros for any $\beta \in \mathbb{R}$.

Theorem 2.3. *Let $\alpha > -1$ and $n \in \mathbb{N}$, $\beta \notin \{0, -1, \ldots - n - 1\}$. Denote the zeros of $F_n = {}_2F_2\left(\begin{array}{c} -n, \beta+1 \\ \alpha+1, \beta \end{array};x\right)$ by y_i, $i \in \{1, 2, \ldots, n\}$ and those of the monic Laguerre polynomials $\tilde{L}_n^{\alpha} = (-1)^n n!\ L_n^{\alpha}$ by $x_{i,n}$. Then*

(i) $y_i > 0$ for $i \in \{1, 2, \ldots, n\}$ if and only if either $\beta < -n$ or $\beta > 0$;

(ii) $y_1 < 0$ and $y_i > 0$ for $i = 2, 3, \ldots, n$ if and only $-n < \beta < 0$;

(iii) The zeros of F_n interlace with those of $\tilde{L}_n$ and $\tilde{L}_{n-1}$ as follows:

(a) $y_1 < x_{1,n}$ and $x_{i-1,n-1} < y_i < x_{i,n}$ $\forall\ i = 2, \ldots, n$ if and only if $\beta > -n$;

(b) $x_{i-1,n-1} < y_i < x_{i,n}$ $\forall\ i = 2, \ldots, n$ and $x_{n,n} < y_n$ if and only if $\beta < -n$.

Proof. Letting $p = q = 2, \alpha_2 = \beta + k - 1, \beta_1 = \alpha + 1, \beta_2 = \beta$ and $k = 1$ in (2.3), we obtain

$${}_2F_2\left(\begin{array}{c} -n, \beta+1 \\ \alpha+1, \beta \end{array};x\right) = \frac{n+\beta}{\beta}\ {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array};x\right) - \frac{n}{\beta}{}_1F_1\left(\begin{array}{c} -n+1 \\ \alpha+1 \end{array};x\right)$$

and hence

$$\frac{(\alpha+1)_n}{(-1)^n}\frac{\beta}{(n+\beta)}\ {}_2F_2\left(\begin{array}{c}-n,\beta+1\\ \alpha+1,\beta\end{array};x\right) \quad = \quad \tilde{L}_n^{\alpha}(x)+a_1\ \tilde{L}_{n-1}^{\alpha}(x)$$

where $a_1=\dfrac{n(n+\alpha)}{n+\beta}$. Let $f_n(x)=\dfrac{\tilde{L}_n^{\alpha}(x)}{\tilde{L}_{n-1}^{\alpha}(x)}$, then we see that $\lim\limits_{b\to\infty} f_n(b)=\infty$.

(i) Since $f_n(0)=-(n+\alpha)<-a_1$ if and only if either $\beta<-n$ or $\beta>0$, the result follows from Lemma 1.1(c).

(ii) $-a_n<f_n(0)<0$ if and only if $-n<\beta<0$ and hence the result follows from Lemma 1.1(a).

(iii) This is a direct consequence of Lemma 1.1 (d) and (e) with the constant a_1 either positive or negative. □

For $n\in\mathbb{N}$ fixed, it follows from Theorem 2.2 that ${}_2F_2\left(\begin{array}{c}-n,\beta+2\\ \alpha+1,\beta\end{array};x\right)$ is quasi-orthogonal of order 2 on the positive real line and consequently has at least $(n-2)$ real and distinct positive zeros for any β real. The remaining two zeros can be two real zeros, one double zero, or two complex conjugate zeros.

Theorem 2.4. *Let $\alpha>-1$ and $n\in\{2,3,\dots\}$, $\beta\notin\{0,-1,\dots,-n-2\}$. Denote the zeros of the polynomial ${}_2F_2\left(\begin{array}{c}-n,\beta+2\\ \alpha+1,\beta\end{array};x\right)$ by z_i, $i\in\{1,2,\dots,n\}$ and those of the monic Laguerre polynomials $\tilde{L}_n^{\alpha}$ by $x_{i,n}$. If $-n-1<\beta<-n$, the zeros of ${}_2F_2\left(\begin{array}{c}-n,\beta+2\\ \alpha+1,\beta\end{array};x\right)$*

(i) are all real and distinct and at most two of them are negative;

(ii) interlace with the zeros of $\tilde{L}_n^{\alpha}$ and $\tilde{L}_{n-1}^{\alpha}$ such that $z_1<x_{1,n-1}$, $x_{i-1,n-1}<z_i<x_{i,n-1}$ for $i=2,\dots,n-1$ and $x_{n-1,n-1}<z_n$.

Proof. Iterating (2.3) twice, we obtain

$$\frac{\alpha_2(\alpha_2+1)}{(n+\alpha_2)(n+\alpha_2+1)}\,{}_pF_q\left(\begin{array}{c}-n,\alpha_2+2,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{array};x\right) \tag{2.4}$$

$$= {}_pF_q\left(\begin{matrix} -n, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix}; x\right)$$
$$- \frac{2n}{n+\alpha_2+1}\ {}_pF_q\left(\begin{matrix} -n+1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix}; x\right)$$
$$+ \frac{n(n-1)}{(n+\alpha_2)(n+\alpha_2+1)}\ {}_pF_q\left(\begin{matrix} -n+2, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix}; x\right).$$

In (2.4), let $p = q = 2, \alpha_2 = \beta, \beta_1 = \alpha + 1, \beta_2 = \beta$. Then

$$\frac{\beta(\beta+1)}{(n+\beta)(n+\beta+1)}{}_2F_2\left(\begin{matrix} -n, \beta+2 \\ \alpha+1, \beta \end{matrix}; x\right)$$
$$= {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right) - \frac{2n}{n+\beta+1}\ {}_1F_1\left(\begin{matrix} -n+1 \\ \alpha+1 \end{matrix}; x\right) +$$
$$\frac{n(n-1)}{(n+\beta)(n+\beta+1)}\ {}_1F_1\left(\begin{matrix} -n+2 \\ \alpha+1 \end{matrix}; x\right)$$

and hence

$$\frac{\beta(\beta+1)(-1)^n(\alpha+1)_n}{(\beta+n)(\beta+n+1)}{}_2F_2\left(\begin{matrix} -n, \beta+2 \\ \alpha+1, \beta \end{matrix}; x\right) \tag{2.5}$$
$$= \tilde{L}_n^{\alpha} + \frac{2n(\alpha+n)}{\beta+n+1}\tilde{L}_{n-1}^{\alpha} + \frac{n(n-1)(\alpha+n)(\alpha+n-1)}{(\beta+n)(\beta+n+1)}\tilde{L}_{n-2}^{\alpha}$$

and the result follows from Lemma 1.2 when the coefficient of $\tilde{L}_{n-2}^{\alpha}$ in (2.5) is negative. □

3. Quasi-orthogonality and real zeros of some ${}_3F_2$ polynomials

Our first result for ${}_3F_2$ polynomials generalises work done by Dickinson [4].

Theorem 3.1. *Let $n \in \mathbb{N}$, $b,\ c,\ d \in \mathbb{R}$, $c,\ d \notin \{0, -1, \ldots\}$, $k \in \{1, 2, \ldots, n-1\}$ and $b \notin \{1-2n, 3-2n, \ldots, 2k-1-2n\}$. Then the hypergeometric polynomial ${}_3F_2\left(\begin{matrix} -n, b+n, c+k \\ c, d \end{matrix}; x\right)$ is quasi-orthogonal of order k with respect to the weight function $|x^{d-1}(1-x)^{b-d+k}|$ on* *(i)* $(0,1)$ *for* $d>0$ *and* $b>d-2$;

(ii) $(-\infty,0)$ *for* $d>0$ *and* $b<1-2n$;

(iii) $(1,\infty)$ *for* $d-2<b<1-2n$.

Remark. The quasi-orthogonality of order 1 of the polynomial $f_n(\frac{3}{2},x)$ with respect to $(1-x)$ on $(0,1)$ considered by Dickinson (cf. [4]) is a special case of Theorem 3.1(i) when $b=1,\ k=1\ ,c=\frac{1}{2}$ and $d=1$ while the quasi-orthogonality of order 1 of $f_n(2,x)$ with respect to $x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}$ on $(0,1)$ also follows from Theorem 3.1(i) by taking $b=1,\ k=1\ ,c=1$ and $d=\frac{1}{2}$.

Corollary 3.2. *Let* $n\in\mathbb{N}$, $b,\ c,\ d\in\mathbb{R}$, $c,\ d\notin\{0,-1,\ldots\}$ *and* $k\in\{1,2,\ldots,n-1\}$ *fixed. Then the polynomial* ${}_3F_2\left(\begin{array}{c}-n,n+b,c+k\\ c,d\end{array};x\right)$ *has at least* $n-k$ *real distinct zeros in*

(i) $(0,1)$ *for* $d>0$ *and* $b>d-2$;

(ii) $(-\infty,0)$ *for* $d>0$ *and* $b<1-2n$;

(iii) $(1,\infty)$ *for* $d-2<b<1-2n$.

The proof of Theorem 3.1 makes use of the orthogonality of the ${}_2F_1$ hypergeometric polynomials which follows from the connection with Jacobi polynomials (cf. [21, p. 254])

$$ {}_2F_1\left(\begin{array}{c}-n,n+\alpha+\beta+1\\ \alpha+1\end{array};x\right)=\frac{n!}{(1+\alpha)_n}P_n^{(\alpha,\beta)}(1-2x). $$

Lemma 3.3. *Let* $F_n(x)={}_2F_1\left(\begin{array}{c}-n,b+n\\ d\end{array};x\right)$ *where* $n\in\mathbb{N},\ b,\ d\in\mathbb{R}$ *and* $d\notin\mathbb{Z}^-$. *The sequence of polynomials* $\{F_n(x)\}_{n=0}^{\infty}$ *is orthogonal with respect to the positive weight function* $|x^{d-1}(1-x)^{b-d}|$ *on the intervals*

(i) $(0,1)$ *for* $d>0$ *and* $b>d-1$;

(ii) $(-\infty,0)$ *for* $d>0$ *and* $b<1-2n$;

(iii) $(1,\infty)$ *for* $d-1<b<1-2n$.

Proof. See Driver and Johnston [6], replacing b with $b-n$. $\square$

Remark. Note that the orthogonality weight and the restrictions on the parameters in Lemma 3.3(i) are independent of the degree of the polynomial concerned whereas the same is not true in cases (ii) and (iii).

Proof of Theorem 3.1. Letting $p=3, q=2$ and using the substitutions $\alpha_2 = c+k-1$, $\alpha_3 = b+n$, $\beta_1 = c$, and $\beta_2 = d$ in (2.2) we obtain

$$
\begin{aligned}
&{}_3F_2\left(\begin{array}{c}-n, b+n, c+k\\ c, d\end{array}; x\right)\\
&\quad = a_0\ {}_2F_1\left(\begin{array}{c}-n, b+n\\ d\end{array}; x\right) + \ldots + a_k\ {}_2F_1\left(\begin{array}{c}-n+k, b+n\\ d\end{array}; x\right)
\end{aligned}
\tag{3.6}
$$

for some constants $a_i,\ i \in \{0, 1, \ldots, k\}$ depending on the parameters n, c and k. Next, we recall the contiguous relation for ${}_2F_1$ polynomials given in [21, p. 71, eqn. (9)]

$$
\begin{aligned}
&(b-a)(1-x)\ {}_2F_1\left(\begin{array}{c}a, b\\ d\end{array}; x\right)\\
&\qquad = (d-a)\ {}_2F_1\left(\begin{array}{c}a-1, b\\ d\end{array}; x\right) - (d-b)\ {}_2F_1\left(\begin{array}{c}a, b-1\\ d\end{array}; x\right)
\end{aligned}
\tag{3.7}
$$

and use it to write

$$
\begin{aligned}
&(b+2n-1)(1-x)\ {}_2F_1\left(\begin{array}{c}-n+1, b+n\\ d\end{array}; x\right)\\
&= (d+n-1)\ {}_2F_1\left(\begin{array}{c}-n, b+n\\ d\end{array}; x\right) - (d-b-n)\ {}_2F_1\left(\begin{array}{c}-n+1, b+n-1\\ d\end{array}; x\right)
\end{aligned}
\tag{3.8}
$$

by letting $a = -n+1$ and replacing b with $b+n$. Applying (3.8) iteratively until all the ${}_2F_1$ terms in (3.6) are of the form ${}_2F_1\left(\begin{array}{c}-n+l, b+n-l\\ d\end{array}; x\right)$, $l \in \{0, 1, \ldots, k-1\}$, we obtain

$$
(1-x)^k {}_3F_2\left(\begin{array}{c}-n, b+n, c+k\\ c, d\end{array}; x\right) = \sum_{i=0}^{k} g_{k-i}(x)\ {}_2F_1\left(\begin{array}{c}-n+i, b+n-i\\ d\end{array}; x\right)
$$

where g_i is a polynomial of degree i.

Multiplying both sides by the weight function $w_1(x) = \left|x^{d-1}(1-x)^{b-d}\right|$ and by a factor of x^j, and integrating with respect to x on the interval $(0,1)$ we have

$$\int_0^1 x^j [(1-x)^k {}_3F_2 \left(\begin{array}{c} -n, b+n, c+k \\ c, d \end{array} ; x \right)] w_1(x) dx$$
$$= \sum_{i=0}^{k} \int_0^1 x^j g_{k-i}(x) \ {}_2F_1 \left(\begin{array}{c} -n+i, b+n-i \\ d \end{array} ; x \right) w_1(x) dx. \tag{3.9}$$

Each term $x^j g_{k-i}(x)$ will be of degree $j+k-i$ and, applying Lemma 3.3(i), we see that the integrals on the right-hand side of (3.9) will be equal to zero when $d > 0$ and $b > d-2$ for $j+k-i \in \{0, 1, \ldots, n-1\}$, i.e for $j \in \{0, 1, \ldots, n-k-1\}$. Thus

$$\int_0^1 x^j {}_3F_2 \left(\begin{array}{c} -n, b+n, c+k \\ c, d \end{array} ; x \right) x^{d-1}(1-x)^{b-d+k} dx = 0$$

for $j \in \{0, 1, \ldots, n-k-1\}$ which proves Theorem 3.1(i). The proofs of (ii) and (iii) follow using an analogous argument, applying Lemma 3.3 (ii) and (iii) respectively. □

Theorem 3.4. *Let $n \in \mathbb{N}$, $b, c \in \mathbb{R}$, $c, c+k \notin 0, -1, \ldots, -n$, $a < -n$, $k \in \{1, 2, \ldots, n-1\}$ fixed and $a \notin \{-n, -n+1, \ldots, -n+k-1\}$. Then the hypergeometric polynomial ${}_3F_2 \left(\begin{array}{c} -n, n+2a+1, c+k \\ c, a+1+ib \end{array} ; \frac{1-ix}{2} \right)$ is quasi-orthogonal of order k with respect to the Pseudo-Jacobi weight function $(1+x^2)^a (1+ix)^k e^{2b \arctan x}$ supported on $\mathbb{R}$ and has at least $n-k$ distinct real zeros.*

Proof. Letting $p = 3, q = 2$ and $\alpha_2 = c+k-1$, $\alpha_3 = n+2a+1$, $\beta_1 = c$ and $\beta_2 = a+1+ib$ in (2.2), we can write

$$\begin{aligned} & {}_3F_2 \left(\begin{array}{c} -n, n+2a+1, c+k \\ c, a+1+ib \end{array} ; \frac{1-ix}{2} \right) \\ = \quad & a_0 \ {}_2F_1 \left(\begin{array}{c} -n, n+2a+1 \\ a+1+ib \end{array} ; \frac{1-ix}{2} \right) \\ & + a_1 \ {}_2F_1 \left(\begin{array}{c} -n+1, n+2a+1 \\ a+1+ib \end{array} ; \frac{1-ix}{2} \right) + \end{aligned}$$

$$\ldots + a_k \; {}_2F_1\left(\begin{matrix} -n+k, n+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \tag{3.10}$$

for some constants a_i, $i \in \{0, 1, \ldots, k\}$ which depend on the parameters n, c and k. Letting $a = -n+1$, $b = n+2a+1$, $d = a+1+ib$ and replacing x with $\frac{1-ix}{2}$ in (3.7), we obtain

$$\begin{aligned}
&(2n+2a)\left(\frac{1+ix}{2}\right) \; {}_2F_1\left(\begin{matrix} -n+1, n+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \\
&= \quad (a+n+ib) \; {}_2F_1\left(\begin{matrix} -n, n+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \\
&\qquad + (a+n-ib) \; {}_2F_1\left(\begin{matrix} -n+1, n+2a \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right)
\end{aligned} \tag{3.11}$$

We now apply (3.11) iteratively until all the ${}_2F_1$ terms in (3.10) are of the form

$${}_2F_1\left(\begin{matrix} -n+l, n-l+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right),$$

$l \in \{0, 1, \ldots, k-1\}$ to obtain

$$\begin{aligned}
&(1+ix)^k {}_3F_2\left(\begin{matrix} -n, n+2a+1, c+k \\ c, a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \\
&= \sum_{m=0}^{k} g_{k-m}(x) \; {}_2F_1\left(\begin{matrix} -n+m, n-m+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right)
\end{aligned} \tag{3.12}$$

where g_i is a polynomial of degree i with complex coefficients. The polynomials

$$\left\{ {}_2F_1\left(\begin{matrix} -n, n+2a+1 \\ a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \right\}_{n=0}^{\infty}$$

are known as Pseudo-Jacobi polynomials (cf. [17, eqn. (9.9.1)]). Note that the normalisation used here is different from that in [17].

Multiplying both sides of (3.12) by $x^j \; (1+x^2)^a e^{2b \arctan x}$ and integrating with respect to x yields

$$\int_{-\infty}^{\infty} x^j \left[(1+ix)^k {}_3F_2\left(\begin{matrix} -n, n+2a+1, c+k \\ c, a+1+ib \end{matrix}; \frac{1-ix}{2}\right) \right] (1+x^2)^a e^{2b \arctan x} dx$$

$$= \sum_{m=0}^{k} \int_{-\infty}^{\infty} x^j g_{k-m}(x) \; {}_2F_1\left(\begin{array}{c} -n+m, n-m+2a+1 \\ a+1+ib \end{array} ; \frac{1-ix}{2}\right)(1+x^2)^a e^{2b\arctan x} dx \tag{3.13}$$

where each term $x^j g_{k-m}(x)$ is of degree $j+k-m$. The result now follows from equation (1.1) together with the orthogonality of Pseudo-Jacobi polynomials with respect to the weight function $(1+x^2)^a e^{2b\arctan x}$ on $\mathbb{R}$ given in [17, eqn. (9.9.2)] (see also [15]) by using an argument analogous to that used in the proof of Theorem 3.1. $\square$

Theorem 3.5. *Let $n \in \mathbb{N}$, b, $c \in \mathbb{R}$, $c, c+k, b \notin \{0,-1,\ldots\}$ and $k \in \{1,2,\ldots,n-1\}$ fixed with $x \in \mathbb{C}$ the independent variable. Then the polynomial* ${}_3F_2\left(\begin{array}{c} -n,-x,c+k \\ c,b \end{array};t\right)$ *is discrete quasi-orthogonal of order k with respect to*

(i) $\dfrac{(1-t)^{-x}(b)_x}{x!}$ *on $(0,\infty)$ for $b>0$, $t<0$ and $n \in \{0,1,2\ldots\}$;*

(ii) $\dfrac{(b)_{-x-b}(1-t)^{x+b}}{(-x-b)!}$ *on $(-\infty,-b)$ for $b>0$, $0<t<1$ and $n \in \{0,1,2\ldots\}$;*

(iii) $\dbinom{N}{x} t^{-x}\left(\dfrac{t-1}{t}\right)^{N-x}$ *on $(0,N)$ for $b=-N$, $N \in \mathbb{N}$, $t>1$ and $n \in \{0,1,2\ldots,N\}$,*

and has, in each case, at least $n-k$ real distinct zeros in the respective given interval.

Proof. Letting $p=3, q=2$ and $\alpha_2 = c+k-1$, $\alpha_3=-x$, $\beta_1 = c$, $\beta_2 = b$ and $x=t$ in (2.2), we see that

$${}_3F_2\left(\begin{array}{c} -n,-x,c+k \\ c,b \end{array};t\right)$$

may be written as a linear combination of ${}_2F_1\left(\begin{array}{c} -n,-x \\ b \end{array};t\right)$, ${}_2F_1\left(\begin{array}{c} -n+1,-x \\ b \end{array};t\right)$, $\ldots$, ${}_2F_1\left(\begin{array}{c} -n+k,-x \\ b \end{array};t\right)$. Meixner polynomials (cf. [17, (9.10.1)]), defined by

$$p_n(x;b,t) = (b)_n \; {}_2F_1\left(\begin{array}{c} -n, -x \\ b \end{array} ; t\right),$$

are orthogonal with respect to the discrete weight function $\dfrac{(b)_x}{x!(1-t)^x}$ on the interval $(0,\infty)$ when $b>0$ and $t<0$ (cf. [17, (9.10.2)]) while they are orthogonal on $(-\infty,-b)$ with respect to the discrete weight $\dfrac{(b)_{-x-b}(1-t)^{x+b}}{(-x-b)!}$ for $b>0$, $0<t<1$ (cf. [3, p. 177, eqn. (3.7)] and [17, (9.10.2)]). Equation (1.1) now yields (i) and (ii) respectively. The orthogonality for a finite number of Meixner polynomials

$$p_n(x;-N,t) = (N)_n \; {}_2F_1\left(\begin{array}{c} -n, -x \\ -N \end{array} ; t\right), \qquad n \in \{0,1,\dots,N\}$$

with respect to the discrete weight $\binom{N}{x} p^x \,(1-p)^{N-x}$ when $t>1$ and $b=-N$, $N \in \mathbb{N}$ is precisely that of the Krawtchouk polynomials (cf. [17, (9.11.2)] and [14]) and this, together with equation (1.1), yields (iii). $\square$

Theorem 3.6. *Let $n \in \mathbb{N}$, $\lambda > 0$, $0<\phi<\pi$, $c, c+k \notin \{0,-1,\dots -n\}$ and $k \in \{1,2,\dots,n-1\}$ fixed. Then the polynomial ${}_3F_2\left(\begin{array}{c} -n, \lambda+ix, c+k \\ c, 2\lambda \end{array} ; 1-e^{-2i\phi}\right)$ is quasi-orthogonal with respect to the weight function $e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2$ on $\mathbb{R}$ and has at least $n-k$ real distinct zeros.*

Proof. The proof uses the orthogonality of Meixner-Pollaczek polynomials defined by (cf. [17, eqn. (9.7.2)])

$$P_n^{\lambda}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{array}{c} -n, \lambda+ix \\ 2\lambda \end{array} ; 1-e^{-2i\phi}\right)$$

with respect to the weight $e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2$ on $\mathbb{R}$ for $\lambda>0$ and $\phi \in (0,\pi)$ and is analogous to that of Theorem 3.5. $\square$

4. Concluding remarks

More detailed information on the location and interlacing of the zeros of the four classes of ${}_3F_2$ polynomials considered in Section 3 can be obtained using analogous arguments to those given in Section 2 for the cases for where the polynomials are quasi-orthogonal of order 1 and 2.

The orthogonality of Hahn, Continuous Hahn, Dual Hahn, Continuous Dual Hahn, Wilson and Racah polynomials (cf. [17]) in conjunction with equation (1.1) and 2.1 can be used to prove similar results for the quasi-orthogonality of certain classes of ${}_4F_3$ and ${}_5F_4$ polynomials.

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