Outlining a practical approach to price and hedge minimum rate of return guarantees embedded in recurring-contribution life insurance contracts

by

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Declaration

I, the undersigned, hereby declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract

This dissertation tackles the current life insurance industry challenge to price and hedge minimum rate of return guarantees (MRRG) embedded in recurring-contribution life insurance contracts in a practical manner. The key contribution to the literature is to outline a practical approach to quantify and project the impact of dynamic hedging strategies for such options. MRRGs are typically very long-dated and as a result the validity of using typical financial economics options pricing models under incomplete market conditions remains a debate. However, life insurers need robust, practical solutions to assist them to manage market risk exposures for day-to-day solvency and income statement management. Literature specific to the topic of MRRG pricing and hedging over recurring-contribution life insurance products is sparse but Schrager and Pelssers' significant contribution (Schrager and Pelsser 2004) provided a basis on which this dissertation was built. Schrager and Pelsser show these options to be analogous to Asian options written over a stochastically-weighted average of the underlying unit fund price. This dissertation demonstrates the effects of stochastic interest rates on MRRGs increase with maturity, as shown by Schrager and Pelsser. Consequentially, users should be aware of the effect and limitations of their choice of interest rate model when pricing MRRGs. Sensitivities for the various maturity terms of MRRG benefits are shown and provide readers with insight into the factors driving the dynamics of such options. A simple dynamic hedging program is outlined and projected under real-world evolutions on a daily basis, thus allowing the effectiveness of the hedging program to be tested.

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1 The challenge to price and hedge minimum rate of return guarantees (MRRG) embedded in life insurance contracts

1.1 Embedded guarantees and options are common features in life insurers' products

"Traditionally regarded by the Street as behind the times in terms of product innovation, US life companies have emerged as major equity derivatives shops. They are now writers of equity options in all but name, at a scale and sophistication that would leave many dealers in their wake." Patel 2006

Life insurance companies have, for many decades, sold investment products which contain some form of minimum investment return guarantee. These investment guarantees provide for clients needs both in terms of accumulation phase build-up guarantees as well as decumulation phase retirement income guarantees.

These guarantees are typically termed variable annuities (or equity-index annuities) in the United States. In the United Kingdom and Europe they typically go by the name of guaranteed unit-linked contracts, and in Canada they are often called segregated fund guarantees.

1.2 Embedded investment return guarantees take a range of forms

Typical life insurance minimum investment return guarantees consist of a basket of underlying unit-linked investment funds where the client has the choice of a variety of guarantees (often called riders) to attach to the contract.

These guarantee benefits can generally be broken down into four broad groups. The general terminology when describing variable annuity products follows these classifications (Table 1).

Table 1: Classification of the four broad types of variable annuities sold

A Guaranteed Minimum Accumulation Benefit offers clients the certainty that they will achieve some minimum guaranteed investment return despite the actual performance of their chosen investment fund/s.

The Guaranteed Minimum Income Benefit offers a guarantee which entitles the policyholder to convert a lump sum into a retirement income through an annuity at a pre-specified amount.
A Guaranteed Minimum Death Benefit guarantees to typically pay the greater of a pre-specified guaranteed amount and the client's fund value on the policyholder's death.
A Guaranteed Minimum Withdrawal Benefit is a complex form of guarantee whereby the policyholder is entitled to continue withdrawing a specified percentage of notional from a fund account or in some cases until death irrespective of investment market performance on their underlying fund choice.

Source: Milliman

1.3 Fair value accounting and risk-based solvency measures have highlighted risks

"Historically, many of these options were included in the contract without explicitly being priced. Many of the options were thought to be conservatively designed and would rarely, if ever, come into play. However, the recent low interest rate period has certainly proven that theory wrong." Hill, Visser et al. 2008

For insurers and regulators the challenge is that embedded options guarantees are sufficiently understood, priced correctly and the risks are managed so that the solvency of the insurer is maintained. While these requirements go without saying for any insurance or investment product, the drive towards fair value accounting, market-consistent liabilities and economic capital calculations has brought the management of embedded options and guarantees to the fore.

To ensure ongoing solvency and profitability insurers will need to charge sufficiently for these guarantees and options so that the actuarial insurance risks, market risks as well as the interaction between the two are allowed for.

This dissertation focuses exclusively on the case of the Minimum Rate of Return Guarantee (MRRG). This guarantee is also commonly referred to as the Guarantee Minimum Accumulation Benefit (as referred to in Table 1). More specifically, the case of the MRRG written over recurring contribution (or premium) savings contracts are analysed for pricing and hedging purposes.

1.4 Single contribution MRRG cases have a closed form solution

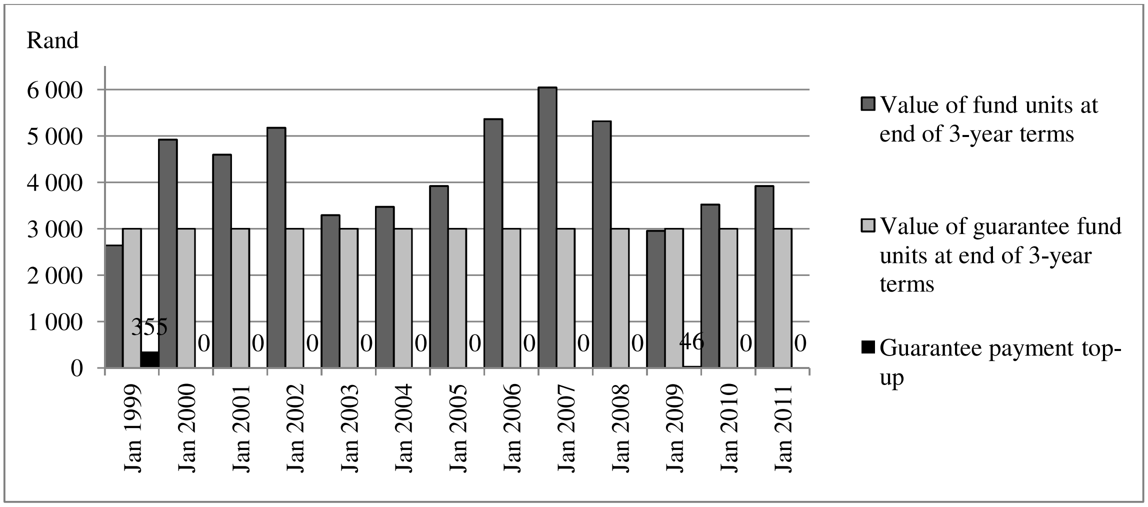
For the single-premium case of a unit-linked investment guarantee closed-form solutions can often be found. The basis of these solutions is similar, if not identical, to that of the Black Scholes European put option pricing solution (Black and Scholes 1973). Further detail on the application of these methods to life insurance can be found in the following papers: Brennan and Schwartz 1976, Bacinello and Ortu 1993, Nielsen and Sandmann 1995, Nielsen and Sandmann 1996 and Nielsen and Sandmann 2002. Finding an exact closed form solution for the value of a guarantee or embedded option is an ideal scenario for an insurer but unfortunately the guarantees and embedded options sold by insurers today are often far too complex for analytical solutions to be found (Finkelstein, McWilliam et al. 2003). This is because the bulk of savings products sold by the insurance industry are regular contribution, rather than single contribution, committed savings contracts.

1.5 Investment guarantees on regular contribution savings contracts exhibit path-dependent payoffs

As an introduction to the complexity of a recurring contribution MRRG We introduce an example. We take the case of a simple 3-year regular annual premium savings product with a minimum return of contribution guarantee (0% minimum rate of return guarantee). Figure 1 shows the value of the client's investment fund units compared to his or her R1000 per annum of notional guarantee build-up at each of the three year periods ending 1 January 1999 to 1 January 2012.

If a life insurance company were to have sold a minimum of return of contribution guarantee on an investment fund earning the JSE All Share Total Return Index over each of these 3-year periods then it would have had to top-up the clients fund value at the guarantee maturity date by R355 on 1 January 1999 and by R46 on 1 January 2009.

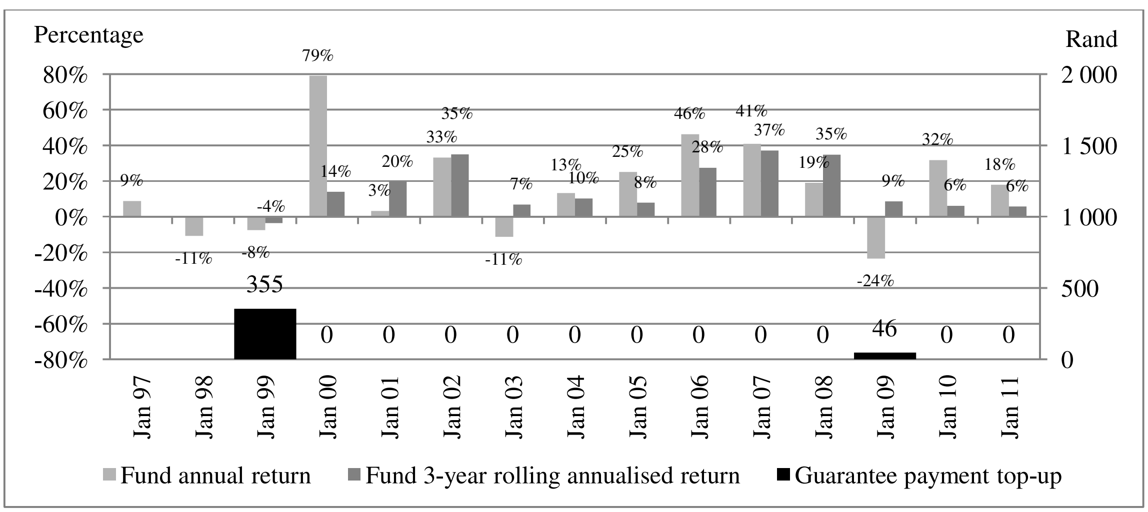
Figure 1: Guarantee top-up requirements for cohorts of 3-year policies of R1000 p.a. paid annually in advance accruing the total return of the JSE All Share Total Return Index for periods ending 1 January 1999 to 1 January 2012 with 0% rate of return guarantees



Source: Inet data, own calculations

Figure 2 shows detail of the annual rates of return achieved on the investment fund in the 3-year periods prior to each contract maturity. In the case of 1 January 2009 top-up the return is on the three prior calendar years were 41%, 19% and -24% in 2006, 2007 and 2008 respectively. This equates to a 3-year annualised return of positive 9% per annum. Despite this, the 0% per annum MRRG bites and a top-up is required on 1 January 2009.

Figure 2: Illustration of guarantee top-up requirement on 1 January 2009 despite a positive 3-year time-weighted annualised return of 9% per annum having been achieved on the JSE All Share Total Return Index



Source: Inet data, own calculations

A guarantee top-up is required because the MRRG, which is written over a recurring contribution product, is the return achieved on the money-weighted average of the returns over the period and not on the time-weighted cumulative return achieved over the period. Table 2 shows that the average return on the JSE All Share Total Return Index performance weighted on the three contributions was c.-1.5% despite the JSE All Share Total Return Index having increased over the period. If the index returns are now modified, such that the index level achieved at 1 January 2008 is achieved at 1 January 2007, as in Table 3, then the contribution weighted average return drops even further to c.-6.3%. This is because the return on the second contribution to maturity now drops, as can be seen in Table 3.

Table 2: Actual realised return on contributions to 1 January 2009 maturity

Date	Fund index (J200T)	Return on contribution to maturity	Contribution weighted
02/01/06	1673.83	28.1%	33.3%
01/01/07	2358.35	-9.1%	33.3%
01/01/08	2805.72	-23.6%	33.3%
01/01/09	2144.23	n/a	n/a
Average return		-1.5%	100.0%

Source: Inet data, own calculations

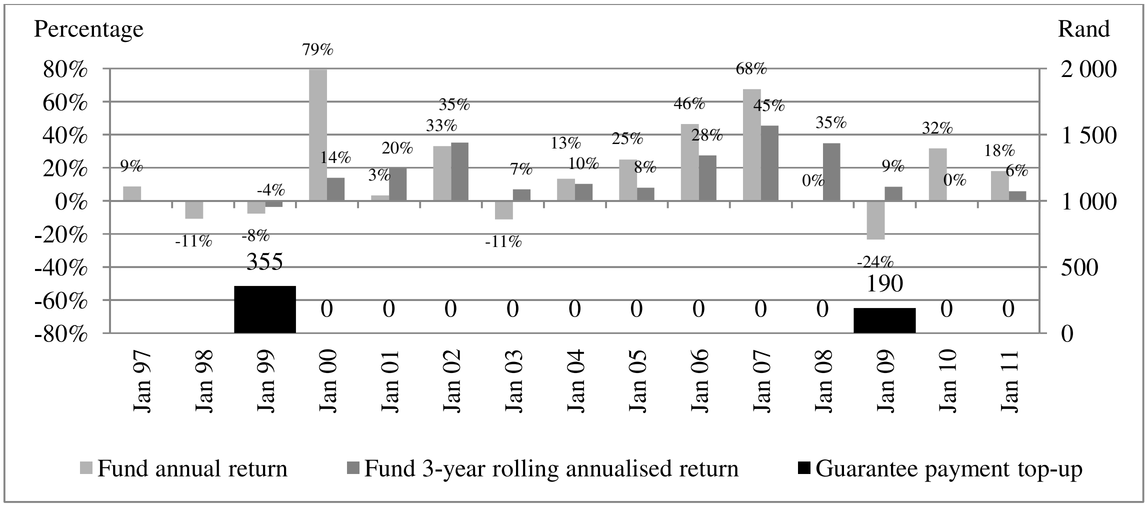
Table 3: Demonstration of path-dependency via accelerating the 2007 index performance to 2006

Date	Fund index (J200T)	Return on contribution to maturity	Contribution weighted
02/01/06	1673.83	28.1%	33.3%
01/01/07	2805.72	-23.6%	33.3%
01/01/08	2805.72	-23.6%	33.3%
01/01/09	2144.23	n/a	n/a
Average return		-6.3%	100.0%

Source: Inet data, own calculations

Figure 3 illustrates the effect of the of the index level being achieved one year earlier, as was the case in Table 3. The top-up at 1 January 2009 now needs to be R190 (rather than R46 previously). MRRG benefits written over recurring contribution contracts are therefore affected by the index level (or fund unit price) at each contribution date and are therefore path-dependent.

Figure 3: Illustration of the increase in guarantee payoff resulting from bringing forward the actual returns achieved in 2008 by one year



Source: Inet data, own calculations

1.6 Regular contribution MRRG benefits can be interpreted as put options based on a stochastically-weighted average of the underlying

This dissertation will draw extensively from the detailed work of Schrager and Pelsser (Schrager and Pelsser 2004) to outline a formulaic expression for the case of a recurring contribution (or premium) investment contract with a MRRG. In doing so, I adopt Schrager and Pelssers' notation (Schrager and Pelsser 2004).

Schrager and Pelsser let S_t be the underlying risky asset price at time t. An example of this underlying risky asset may be the underlying unit-linked fund price (or an index level). Schrager and Pelsser setup a contract where the first investment premium (contribution) is paid at time 0 and subsequent investment premium payments be made at time i, where $i = 0, 1, \ldots n - 1$ and denote these premium payments as P_i. Then assume that at t_0 the balance of the client's fund is zero in our initial workings. i.e. construct a new recurring-premium contract. By not applying any premium deductions or charges and adopt the simplification that the full investment premium is allocated to the client's chosen investment fund account when payment is made. The number of units purchased with each premium P_i is thus equal to P_i / S_i. Similar to Schrager and Pelsser, let $T = t_n$ be the time at which the policy expires and the guarantee payment is made. The unit price at time T is denoted as S_T.

Then, as shown by Schrager and Pelsser (Schrager and Pelsser 2004), the client's fund value at the policy expiry, at time *T,* is given by FV_n where FV denotes the fund value, as in Equation 1.

Equation 1

$$\mathrm{FV}_n = \sum_{i=0}^{n-1} P_i\left(S_T/S_{t_i}\right)$$

Using Schrager and Pelssers' notation, *K*, to denote the minimum maturity payment offered by the guarantee. The resulting contract maturity payoff is given by Equation 2 (Schrager and Pelsser 2004).

Equation 2

$$\max(\mathrm{FV}_n, K) = \max\left(\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}, K\right) = \sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}} + \left(K - \sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}\right)^+$$

The insurer's payoff is therefore a function of the client's accumulated fund value, at contract expiry, plus a put option on $\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}$. This can be interpreted as a put option based on a stochastically weighted average of the underlying fund unit prices over the duration of the contact at expiry (Schrager and Pelsser 2004).

For the purposes of this dissertation I setup the guarantee, *K*, to be the guaranteed fund value given by accumulating the client's invested contributions at a rate of return guarantee. *K* is therefore given by Equation 3 where R denotes the continuous form of the annual guaranteed rate of return guarantee.

Equation 3

$$K = \sum_{i=0}^{n-1} P_i e^{R(T-t_i)}$$

Schrager and Pelsser extend the above notation to allow for the likely characteristics of typical regular premium investment contacts. For example, they extend to include an early death guarantee payment and premium fee deductions (deterministic deduction for expenses and/or mortality changes). They also demonstrate proofs of the independence of mortality on early payment as well as on premium fee deductions. The consequence of their proofs being that I can ignore such independent items as these demographic policy factors can be “pulled out” of the payoff formula outlined in Equation 2 (Schrager and Pelsser 2004).

1.7 Recurring contribution MRRG benefits are analogous with Asian options

The result of the structure of Equation 2 is that of the guarantee is dependent on the underlying unit fund (or index level) prices at different time points. This, as discussed by Schrager and Pelsser leads to the analogy with Asian options (Schrager and Pelsser 2004). Under a few simplifying assumptions for the return volatility and the short rate processes, Schrager and Pelsser’s analogy with Asian options leads to an insightful introduction to the mathematical characteristics of the recurring contribution MRRG payoff.

In order to prove this analogy, Schrager and Pelsser setup a case of market completeness and no arbitrage. Further to this they setup a simple Black-Scholes process for the underlying equity index/unit fund price, denoted S_{t_i}. In addition, they assume the stock price return volatility, σ_s, and the short rate, r, are both constant over time (Schrager and Pelsser 2004).

Under these assumptions Schrager and Pelsser setup a recurring contribution savings policy where the premium invested are a level amount, P_i, where P_i equals $1/n^{th}$ of the initial equity index / unit fund price level, S_o (Schrager and Pelsser 2004). Under these assumptions they prove the equality between the price of an average price Asian Put and a MRRG (both with the same guarantee level, K). i.e. They prove Equation 4 (Schrager and Pelsser 2004).

Equation 4

$$\mathrm{e}^{-rT}E^{Q}\left[\left(K-\frac{1}{n}\sum_{i=1}^{n}S_{t_i}\right)^{+}\right]=\mathrm{e}^{-rT}E^{Q}\left[\left(K-\sum_{i=0}^{n-1}P_i\frac{S_T}{S_{t_i}}\right)^{+}\right]$$

Their approach is to prove this equality by showing that the first two moments of $\frac{1}{n}\sum_{i=1}^{n}S_{t_i}$ and $\sum_{i=0}^{n-1}P_i\frac{S_T}{S_{t_i}}$ are equal under the risk-neutral measure, Q. Demonstrating equality of the first two moments alone is sufficient to conclude the proof as, by assumption, the processes are lognormal random variables and therefore specified fully by the first two moments (Schrager and Pelsser 2004).

Schrager and Pelsser's work shows us that under Black-Scholes assumptions a similar level of randomness exists in each premium payment period (Schrager and Pelsser 2004). Schrager and Pelsser go on to conclude that this equality will still hold under a generalised case which allow for other forms of stationary stochastic volatility (Schrager and Pelsser 2004).

1.8 MRRG prices are sensitive to the stochastic interest rates

However, the same is not true with regards to interest rate risk (Schrager and Pelsser 2004). The Asian option (given by the term denoted on the left in Equation 4) is sensitive to stochastic interest rates from time zero until each of the Asian average calculation dates i.e. $[0,t_i]$ for $i = 1,2, \ldots,n$ (Schrager and Pelsser 2004). However, the unit-linked guarantee (given by the term denoted on the right in Equation 4) is sensitive to interest rates from the date of payment until maturity i.e. over the interval $[t_i,T]$ for $i = 0,1, \ldots,n - 1$ (Schrager and Pelsser 2004). The implications of this are that one cannot generalise the results calculated under Schrager and Pelsser's assumptions to allow for stochastic interest rate properties or for any form of time period dependence with non-stationary features in the equity index / unit fund volatility (Schrager and Pelsser 2004).

Schrager and Pelsser interpret this as the risk in, S_T / S_{t_i}, being split into interest rate (or forward bond price) risk, from time zero to time i, t_i, and equity (or forward stock price) risk from time t_i to maturity time, T (Schrager and Pelsser 2004). They concluded that this outline of the unit-linked guarantees mathematical characteristics provides us with insight as to potentially appropriate hedging approaches. What this means is that the risk changes over the time the contract is in force. For example, early in the contracts term the risk is mainly related to interest rate sensitivity but as the contract approaches maturity the underlying equity index / unit fund price risks dominate (Schrager and Pelsser 2004). Therefore, early in a contracts life interest rate hedges such as caps and floors may be appropriate while stock options (forward starting put options, for example) may have the characteristics of being good hedges later as the contract approaches maturity (Schrager and Pelsser 2004).

1.9 Quantification and projection of dynamic hedging strategies is a key challenge

The Society of Actuaries 2007 Survey on variable Annuity Hedging Programs for Life Insurance Companies (Gilbert, Ravindran et al. 2007) outlines relative levels of difficulty of the potential implementation challenges (Table 4). The quantification and projection of the impact of dynamic hedging strategies under various assumptions bases remains a key industry challenge. With this as a backdrop, this dissertation aims to outline a practical approach to price and hedge MRRG products in the more complex path-dependent case of the recurring contribution life insurance contract.

Table 4: Rankings of the difficulty of the potential variable annuity hedging program implementation challenges

Rank	Description of Implementation Challenge	Extremely Difficult	Somewhat Difficult	Relatively Easy
1	Attribution analysis	3	12	1
2	Quantification and projection of impact of specific dynamic hedging strategies on an economic basis	6	8	4
3	Quantification and projection of impact of specific dynamic hedging strategies under FAS 133	5	9	4
4	Personal acquisition and retention	4	11	3
5	Calibrating models	4	12	3
6	Analysis of various risk management strategies	1	16	1
7	Development and/or acquisition of requisite software and technology	2	14	2
8	Formulating specific hedging strategies	0	17	1

Source: Gilbert, Ravindran et al. 2007

1.10 Research objective: To outline a practical approach to price and hedge MRRG embedded in recurring contribution life insurance contracts

"Due to the rapid growth of equity-linked business, it is important to address the question of 'correct' pricing of equity-linked products in general. From the perspective of the insurance industry, the effects of failing to adopt adequate pricing and risk management models can be devastating. Clearly, some of the events causing large losses to insurance companies cannot be predicted (natural disasters, terrorist activities, etc.). However, fluctuations in the prices

of risky assets and mortality patterns can be analyzed quantitatively and qualitatively to help build proper pricing tools for insurance firms. Thus the question of finding hedging methodologies that can assess and value financial and insurance risks, and provide appropriate risk management strategies, is of great interest and significance from both theoretical and practical perspectives." Melnikov and Romanyuk 2006

Chapter 1 introduces the practical and mathematical challenges faced when quantifying a fair price for MRRGs embedded in recurring contribution life insurance contracts. Chapter 2 provides a literature review of the various approaches currently adopted when pricing embedded rate of return guarantees. Chapter 3 continues the literature review with a particular emphasis on the mathematical relationships between the recurring contribution rate of return guarantees price and the stochastic processes for the underlying market variables being modelled. I demonstrate the pricing of a typical life insurance policy minimum rate of return guarantee in Chapter 4. This pricing is performed under the Black-Scholes Hull-White model. Sensitivities are shown as to how the calculated price of the MRRG changes under different market input variables as well under different policy characteristics. For example, these include different guarantee levels and contract maturity terms. Chapter 5 draws from the literature in discussing current market practice for the hedging of such guarantees. A simple hedging program is outlined. In Chapter 6 I draw from the literature on real-world economic variable simulation. A method is calibrated to the South African market and applied to the quantification and projection of a dynamic hedging program under a simple daily hedging program. In Chapter 7 I summarise the outline of a practical pricing and hedging approach and show its effectiveness. This concludes this dissertation by meeting the research objective. Chapter 8 outlines the limitations of this analysis and discusses areas of potential future research. Chapter 9 contains various technical appendices and Chapter 10 provides a list of references used.

2 Literature Review: Various approaches adopted to price embedded rate of return guarantees

2.1 Deterministic pricing of embedded guarantees is inappropriate

The vast majority of products sold with embedded options and guarantees of some form represent an aggregate risk to the insurer in that a significant portion of their exposure is likely to experience payout at a similar point in time.

This is different to the case of mortality (with the exception of catastrophic mortality risk), where a higher number of policies would diversify the risk to the insurer across its pool of lives (Boyle, Hardy et al. 2007). This principle is due to the law of large numbers where the experience will tend to the mean and from the Central Limit Theorem the distribution of claims will tend to the normal distribution in the case where the variances are finite (Boyle, Hardy et al. 2007).

The major shortcoming of the deterministic approach is that it fails to account for variability of the potential future outcome of risky asset returns. Therefore by pricing on expectation alone, via the use of deterministic assumptions, the probability and size of potential fund shortfall payment on the guarantee contracts which expire in the money is not captured. The use of deterministic pricing for diversifiable risks is discussed in Boyle and Schwartz 1977, Lin and Tan 2003 and Boyle, Hardy et al. 2007

To compensate for these shortcomings a margin is often added. For example, Dahl 2004, states: “insurance firms have traditionally calculated premiums and reserves based on deterministic mortality and interest rates, and to compensate for this, have overpriced financial and insurance risks”. The product being that this approach results in higher premiums than necessary (on average) and thus leave room for error between charged and expected mortality experience and interest rates.

Deterministic pricing therefore lacks a robust approach to incorporate the variability (or volatility) of potential parameter outcomes.

2.2 The probability of ruin concept gives an indication of the potential real-world payoff

An understanding of this fundamental difference in treatment of diversifiable and non-diversifiable risk gave rise to considerable concern for the actuarial community in the 1970's. In response to this the Institute and Faculty of Actuaries (UK) commissioned a report by the Maturity Guarantees Working Party over three decades ago. The Working Party report (Benjamin, Ford et al. 1980) found that the traditional valuation methods in life insurance were deterministic (or expected value) pricing in nature with prudence margins incorporated implicitly in the basis. The Working Party report also found that these methods were appropriate for the case of independent risks, such as mortality, where large numbers of independent lives were exposed to risks resulting in low variability of claims. However, investment contracts with guarantees were not independent of one another as the external market events which drive payoffs would affect all policies at the same time. As a result of this effect the variability in the insurers' claims is in fact far greater (Benjamin, Ford et al. 1980).

The Working Party (Benjamin, Ford et al. 1980) aimed to find a distribution for unit prices over time and suggested that simulating returns under appropriate model assumptions should be the basis for reserving for maturity guarantees. This method of stochastic simulation would recognize the variability of potential investment returns and associated guarantee claim costs (Benjamin, Ford et al. 1980).

This approach requires a distribution assumption for the range of potential outcomes for the random variable in question (in our case the fund value at maturity). The insurer could then setup a reserve such that they would hold sufficient reserves for 99% of the time, say. This concept of the "probability of ruin" would give the insurer comfort that they were holding sufficient reserves to cover almost all future events that could lead to both significant over estimation of guarantee costs as well as prove to be insufficient in extreme events (Benjamin, Ford et al. 1980). The method does not, however, give any indication as the manner in which assets backing reserves should be invested. i.e. the probability of ruin approach merely gives an indication of how much a insurers liability may be within a given degree of confidence (Benjamin, Ford et al. 1980). This method does not provide an outline of the behaviour of the price for such a guarantee under different forecasts and thus falls short of providing the user with insights into potential immunisation or hedging strategies (Benjamin, Ford et al. 1980).

The approach therefore only lends itself to a passive strategy of simply holding what is believed to be a sufficient amount of assets to meet its obligations with a chosen high level of probability. It does not, however, remove the risk the insurer will have to pay out more than the reserve held or find itself holding too much reserve for the expected payment x% of the time. This "over reserving" is inefficient in that it would lead to a cost to the insurer on all but the few occasions that the guarantee payment would need to be met.

2.3 Financial economics approaches are increasingly being applied

2.3.1 Introduction to typical financial economics options pricing approaches

The advent of Black and Scholes research papers on the pricing of options and corporate liabilities (Black and Scholes 1973) laid the foundations for much of the last four decades of work on the topic of option pricing. For insurers, the theory was considered soon thereafter by Brennan and Schwartz (Brennan and Schwartz 1976; Brennan and Schwartz 1979). These papers, along with the various literature items, which followed, further developed the thinking that maturity guarantees could be viewed as similar, or in some cases identical, to put options. In the two decades that followed a number of authors investigated how financial economics approaches could be used to price single premium life insurance contracts. Some of these authors are: Boyle and Schwartz 1977; Bacinello and Ortu 1993; Bacinello and Ortu 1993; Nielsen and Sandmann 1996; Boyle and Hardy 1997.

The publications by Brennan and Schwartz and by Boyle and Schwartz, amongst others, suggested that an answer to the question of how to price the single premium case of an embedded option was found. The Maturity Guarantees Working Party (Benjamin, Ford et al. 1980) considered this point when it investigated Fagen's publication (Fagen 1977) which suggested that in the case of savings contracts with maturity guarantees, the risks could be reduced or even mitigated entirely via an appropriate immunization strategy. This type of strategy would typically suggest that a lower number of underlying fund units should be purchased to maintain an immunized position (Fagen 1977). Under this approach the pricing (and potential hedging) of the embedded options and guarantees were performed under the equivalent martingale measure (Fagen 1977).

The Working Party, at the time, investigated the Black and Scholes paper (Black and Scholes 1973) but concluded that the theory was based on complex mathematics and there were still varying degrees of confidence that the mathematics were indeed sound (Benjamin, Ford et al. 1980). They also noted that the theory was reliant on several underlying assumptions which would not easily hold in practice, and in particular in the context of insurer's exposures. The Working Party (Benjamin, Ford et al. 1980) suggested that at that point in time it was not clear that following immunization strategies could conclusively be shown to reduce maturity guarantee requirements but it did conclude that the subject warranted further investigation.

Despite financial economics principles having resided in the financial engineering literature for some time, the application of these principles to insurance has largely evolved over the past ten to fifteen years (Boyle, Hardy et al. 2007). This is in part due to the fact that the majority of the financial economics literature has focussed on short-rate models for relative pricing of short-term traded derivatives (Boyle, Hardy et al. 2007). Boyle, Hardy, et al. state that the direct application of this research to the case of the non-relative pricing (or absolute pricing) of long-term guarantees and options is not necessarily appropriate (Boyle, Hardy et al. 2007).

Boyle and Hardy, in their paper titled Reserving for maturity guarantees: Two approaches (Boyle and Hardy 1997), describe the stochastic simulation approach suggested by the Working Party (Benjamin, Ford et al. 1980) and option pricing methods discussed above then go on to compare the two approaches. They point out that the potential financial economics approach provides the insurer with insight into actively managing the risk through dynamically adjusting its investment strategy on an ongoing basis (Boyle and Hardy 1997).

They state that the Black and Scholes option price model is underpinned by a simple geometric Brownian motion model for the equity returns and comment that while this may be appropriate for its original intention it may not be the case for the long-dated options written by insurers. Under the Black Scholes assumptions equity returns are assumed to be independent and identically distributed with a constant mean and variance and this is seen to be unrealistic in the context of longer-term options (Boyle and Hardy 1997).

The arbitrage theory developed by Black and Scholes (Black and Scholes 1973) takes the price dynamics of certain economic variables (equity, rates, etc) and tries to calculate the prices of other derivatives of those underlying (contingent claims) through arbitrage considerations alone. What this means is that by adopting this approach for pricing a derivative implies that no opportunities for arbitrage should exist (Harrison and Kreps 1979).

This arbitrage-free pricing of contingent claims has been extensively researched in: Merton 1973; Ross 1976; Merton 1977. This modelling approach is called modelling in the risk-neutral measure. The risk-neutral measure is built on the concept of a market price of risk (Hull 2003). The traditional risk-neutral world assumes that all market prices of risk are zero. In other words, investors do not require risk premiums to invest (Hull 2003). Using the risk neutral model does not necessarily require that investors are indeed risk neutral in behaviour but it is merely a convenient method for calculating option prices in a market-consistent manner (Hull 2003). Under the risk-neutral probability measure, the expected return on all assets is equal to the risk free rate (Hull 2003).

Under the practical implementation of this approach the yield curve (swap or bond) is used alongside assumptions for market-implied volatility for the interest rate process and underlying risky assets to calibrate an economic scenario generator under so as to generate a series of market-consistent risk-neutral simulations. This requires an economic scenario generator, which in turn requires some form of parametric term structure model to specify its dynamics.

2.3.2 Incomplete market dynamics bring the validity of the option pricing theory approaches into question

"The main assumptions of the option pricing theory, i.e., no-arbitrage, dynamic hedging, and market completeness. Of these three hypotheses, the least realistic one is that of market completeness, namely, it is possible to replicate the payoff of any claim in the market by means of a self-financing strategy". Consiglio and Giovanni 2007

Consiglio and Giovanni give a number of motivations as to why investment return options typically sold by insurance companies seldom satisfy the requirement for market completeness (Consiglio and Giovanni 2007). For example, they state that the underlying stochastic processes often exhibit jumps in the real world, underlying assets sometimes exhibit heteroscedasticity, discrete hedging is expensive and market frictions such as trading costs and limited short sales ability exist (Consiglio and Giovanni 2007).

Other authors have investigated the pricing of options under incomplete market assumptions further, for example: Follmer and Sondermann 1986; Follmer and Schweizer 1991; Schweizer 1996; Moeller 1998; Moeller 2001 all discuss various aspects of financial economics approaches in the context of life insurance guarantees.

Nyholm and Rebonato (Nyholm and Rebonato 2007) describe the two sub-classes of such risk-neutral model approaches. They state that a number of relative-pricing models are available for the yield curve evolution over the user's choice of time horizon but split these into two sub-classes with distinguishing characteristics (Nyholm and Rebonato 2007).

Nyholm and Rebonato firstly describe fundamental models which use a joint specification of the risk premia and the real-world dynamics. These models describe a process for the driving factors of the yield curve and hence the risk aversion of the market participants. Models such as Vasicěk (Vasicek 1977), Cox, Ingersoll, Ross (Cox, Ingersoll et al. 1985) and Longstaff and Schwartz (Longstaff and Schwartz 1992) fall into this category (Nyholm and Rebonato 2007). They too state that the translation from the risk-neutral to real-world drift can be accommodated by adjusting for the markets risk aversion characteristics via the drift factors (Nyholm and Rebonato 2007). However, while all of these models do offer users modelling ability none of them can reproduce exactly today's yield curve nor its evolution over time (Rebonato, Mahal et al. 2005).

Nyholm and Rebonato state that the fundamental problem with these fundamental model approaches is that their parametric form has limited factors and thus yield curve movements are limited. This is particularly a problem in the context of MRRG pricing where is has been shown the term structure of the yield curve is a significant determinant of the price as will be shown later (Nyholm and Rebonato 2007).

Secondly they describe, reduced-form approaches, such as Rebonato, Mahal et al. (Rebonato, Mahal et al. 2005) such as Ho and Lee (Ho and Lee 1986), Hull and White (Hull and White 1990) and Heath, Jarrow and Morton (Heath, Jarrow et al. 1989) and note that these are also being constructed for the purpose of relative pricing and therefore only contain drift terms governed by no arbitrage conditions. While this is also appropriate for relative pricing this is inappropriate for modelling long-term real-world yield curve evolutions in that the evolutions arising are dictated by the no-arbitrage drift term and therefore do not resemble reality (Nyholm and Rebonato 2007).

Rebonato et al. states that these short-rate models prescribe curve evolution to replicate the interest-rate derivatives with no arbitrage so while these models are based on the Girsanov transformation (Girsanov 1960) between real-world and pricing measure the restrictions on this transformation are absolute continuity and equivalence between the real-world and pricing measure (Rebonato, Mahal et al. 2005). They note that if perfect replication is not possible, through incompleteness, there exists a variety of pricing measures consistent with the absence of arbitrage and therefore there exists a variety of drifts to avoid arbitrage (Rebonato, Mahal et al. 2005).

As a result of this it has been suggested by Rebonato et al. (Rebonato, Mahal et al. 2005) that semi-parametric interest-rate model approaches are appropriate for assessing hedging program performance, asset/liability investment mismatch strategies and for economic-capital calculations. All of these applications, are of interest to life insurers writing long-term products, as they require the evolutions of the yield curve to forecast for lengthy time periods (often many decades) as well as that they must accurately represent the potential population of future yield curves (Rebonato, Mahal et al. 2005).

2.4 The concept of the "fair price" of a MRRG remains a debate

2.4.1 Market consistency can't be achieved for such long-dated guarantees

At the core of the challenge of finding a "fair price" for a MRRG lies the question: How will risky underlying assets, such as equity, perform over the next few decades? It is fair to acknowledge that major economic trends, the nature of corporate earnings and governments interaction, demographic shifts, politics and technological change will play their part in future risky asset returns. To quantify how these, and other factors, will affect expected returns (firstly), and the variability of these returns (secondly) seems extremely ambitious, at best.

The market does not provide us with a price for this long-term risky asset return risk. Traded equity options in South Africa seldom have maturities beyond three years, let alone as far out as thirty years. And in addition, the range of moneyness levels traded decreases rapidly as tenors stretch out into the future.

Further, the interrelated nature of long-term risky asset returns and risk-free asset returns becomes more central to the problem as the effects of discounting start to play a greater role in the determination of present value pricing. Against this backdrop, true market consistent pricing is a challenge and simplification is almost always required.

2.4.2 Acknowledging this, and moving to find a pragmatic model and associated hedge recipe is of value for life insurers

This difficulty should not stop insurers from taking pragmatic decisions to manage key risks embedded in such long-dated MRRGs. The argument behind this statement is that it would be better to have a robust, uncomplicated, model for the reasonable cost of such a guarantee than none at all. Putting such a model in place would facilitate the testing of guarantee price changes under stressed market conditions and allow management to act to protect the insurance company's balance sheet against key market risks.

The remainder of this dissertation acknowledges this practical reality and aims to provide a robust, reasonable, basis for the day-to-day management of the key risks emanating from such MRRG products in the case of the recurring contribution life insurance contracts.

2.4.3 Insurers day-to-day income statement and solvency management objectives are likely to be driven by local accounting measures and/or regulatory solvency

"There is still divergence in approaches to the pricing and hedging of life insurance guarantees. Proponents of the risk-neutral approach will advocate for the application of those methodologies on the basis that it provides a no arbitrage price for the option but critics comment that the onerous assumptions underlying no arbitrage models and argue that these assumptions as inappropriate for use in the modelling of the behaviour of the option, and consequentially the hedge behaviour over the life of the option in the real world." Haastrecht, Lord et al. 2008

The practical reality is that the pricing approach adopted will likely be driven by the desire to meet some commercial goal. This is likely to be either income statement volatility management or economic capital management (or both).

2.4.4 The Black Scholes Hull White model provides a robust, mathematically tractable basis

I choose to use the Black Scholes Hull White (BSHW) basis for pricing. Some of the positive features of this model are that the model has high mathematical tractability, secondly it can be calibrated directly to the market yield curve (which has benefits for market consistent yield curve pricing) and it allows for the correlations between short rates and the risk underlying equity process.

Thus the model provides flexibility, but is not overly complex and requires multiple instances of manual calibration. Given the limited availability of traded market instruments to infer the market price of risk for variables such as correlations between equity and rates, or across the rates curve, extensions beyond this model would start to become spurious in the South African context.

2.5 Introducing the hybrid approach and explaining why real-world and risk-neutral simulations will both be required for assessing a hedging program

Boyle and Hardy also outline the issues life insurers have faced in introducing dynamic hedging approaches. They describe the long-term nature of the benefits, the fact that options are struck far out of the money, the practical issues of discreet hedging and transaction costs as well as the fact that mortality means that the time to pay out is often random to a degree (Boyle, Hardy et al. 2007). They suggest the hybrid approach of combining both real-world stochastic simulations and the risk-neutral approach in which options are priced under the risk neutral measure but that the cost quantification of a hedging strategy is performed under real world projections.

This approach allows users to estimate the hedged liability arising from discreet hedging, the associated transaction costs and the model error. Practically this approach is performed by using the real-world measure to project the behaviour of various asset classes over time so that the value of the hedging portfolio can be compared to the new risk-neutral price of the MRRG under the projection of the real-world value of each of the economic inputs. Users can thus infer the unhedged liability and then quantify the cost of re-hedging the position at that time. This process can then be repeated over the course of a number of time-steps and by discounting these costs the cost of the hedging program (by way of the gaps arising from noncontinuous hedging as well as transaction cost) can be found (Boyle, Hardy et al. 2007). Boyle and Hardy point out that the hybrid approach to risk management is permitted for Canadian insurers writing equity-linked contracts with guarantees (Boyle, Hardy et al. 2007). Accordingly, I adopt this approach when outlining a simple approach to project and quantify hedge program effectiveness in the real-world.

3 Literature Review: Outlining the relationships between a recurring contribution MRRG price and the underlying stochastic processes

3.1 Recapping the basics – the essence of the pricing problem is to find the fair price of a MRRG written over an stochastic equity path

In section 1.5 I showed that the case of a MRRG written over a recurring contribution life insurance contract is path dependent. The consequence of this path-dependency on pricing is that Monte Carlo simulation methods need to be employed.

3.2 Demonstration of the effects of stochastic interest rates

Schrager and Pelsser (Schrager and Pelsser 2004) use the Levy approximation (Levy 1992) to analyse the effects of stochastic interest rates on the MRRG price. They show that this amounts to approximating the distribution of the fund value (the weighted average of sum of the fund unit prices) at maturity, $\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}$, with a lognormal distribution with the same mean and variance (Schrager and Pelsser 2004). This is because under the assumption of log-normality the first two moments fully specify the distribution of the fund value outcomes, and in turn the associated guarantee (Schrager and Pelsser 2004). Schrager and Pelsser compute the first two moments of the fund value process under the T-forward measure to show this.

They demonstrate that the benefit of making the lognormal approximation assumption is that we can analyse the effects of the stochastic interest rates via analyzing only the first two moments (Schrager and Pelsser 2004). What Schrager and Pelsser find is that the first moment doesn't show the effects of stochastic interest rates and therefore, by implication, the full effect of stochastic interest rates is captured in the second moment alone (under the lognormal assumption for the fund value at maturity process). This results from the assumption that the expectation, under the T-forward measure, for the return on the underlying unit funds' from t_i to T is given by the continuously compound forward rate for the period (Schrager and Pelsser 2004).

The formulaic breakdown of the second moment, is therefore key to understanding the impact of stochastic interest rates on the guarantee payoff. To calculate the second moment, however, assumptions about the dynamics of the price of the underlying risky asset (unit fund price or stock price index) are needed. Schrager and Pelsser parameterize their model on the assumption of lognormal stock prices and a Gaussian interest rate model (Schrager and Pelsser 2004). This assumption implies that the volatility of forward unit fund prices and bond prices are deterministic functions of time (Schrager and Pelsser 2004). This setup, using Schrager and Pelssers' notation (Schrager and Pelsser 2004), amounts to the *T*-forward unit fund price, F_T, and the *T*-forward bond price, $D^T(t,S)$, following the price process dynamics of Equation 5 and Equation 6 respectively.

Equation 5

$$dF_t^T = \sigma_F(t)\, F_t^T\, dW_t^T$$

Equation 6

$$dD^T(t,S) = \sigma_{D_S^T}(t)\, D^T(t,S)\, dW_t^{ST}$$

Here, W_t^T and W_t^{ST} are standard Brownian motions under the *T*-forward measure and σ_F and $\sigma_{D_S^T}$ are deterministic functions of time (Schrager and Pelsser 2004). Schrager and Pelsser assume that the correlation between the two Brownian motions are given by Equation 7 and Equation 8.

Equation 7

$$dW_t^T dW_t^{ST} = \rho_{F^T D_S^T}$$

Equation 8

$$dW_t^{ST} dW_t^{UT} = \rho_{D_S^T D_U^T}$$

Schrager and Pelsser compute the formulaic breakdown of the second moment in their paper (Schrager and Pelsser 2004) and arrive at the expression given by Equation 9. In this expression time t_i represents the time at which the i^{th} contribution is made and t_j is some later date. Note that t_j can be set to the maturity time.

Equation 9

For $t \leq t_i \leq t_j$, by the Change of Numeraire Theorem, which allows the measure of worth which is used to price an asset to be changed, the second moment is given by (Schrager and Pelsser 2004):

$$E_t^{Q^t}\left[\left(\sum_{i=0}^{n} P_i \frac{S_T}{S_{t_i}}\right)^2\right] = \sum_{i=0}^{n-1}(P_i)^2 E_t^{Q^t}\left[\left(\frac{S_T}{S_{t_i}}\right)^2\right] + 2\sum_{i=0}^{n-2}\sum_{j>i} P_i P_j E_t^{Q^t}\left[\left(\frac{S_T}{S_{t_i}S_{t_j}}\right)^2\right]$$

Equation 9 allows Schrager and Pelsser to provide an insightful explanation as to the effects of stochastic interest rates and underlying stock price volatility affect the MRRG price. The first term of Equation 9 is independent of both the assumed stock price and bond price volatility and therefore does not give rise to any volatility (Schrager and Pelsser 2004).

The three integrals in the second term of Equation 9 quantify the instantaneous covariance's of $\ln(S_T / S_{t_i})$ and $\ln(S_T / S_{t_j})$ for the relevant time intervals (Schrager and Pelsser 2004). Schrager and Pelsser describe these three integrals as:

- The first integral (before t_i, i.e. from t to t_i) describes the correlation between normalised bonds with maturity t_i and t_j (Schrager and Pelsser 2004). This is because the uncertainty relates to the T-forward bond price processes. This integral can therefore be interpreted as the quadratic covariance between the forward bond processes. Practically, this means that the risk from outset until a contribution is invested is only related to interest rates (Schrager and Pelsser 2004).
- The second integral (interval between t_i and t_j) describes the covariance between the forward equity index / unit fund and $D(t,t_j) / D(t,T)$ (Schrager and Pelsser 2004). Thus the uncertainty relates to the quadratic covariance of the forward equity / unit fund and forward bond price processes. This is because after t_i , S_{t_i} is now known and the risk remaining in S_T / S_{t_i} is accounted for by the T-forward asset price (Schrager and Pelsser 2004).
- The third integral (after t_j, i.e. ranging from t_j to T) relates only to forward stock price risk (Schrager and Pelsser 2004). This risk is described by the implied equity index / unit fund volatility of a forward start stock option (Schrager and Pelsser 2004). The

lack of interest rate risk results from, after t_j the unit fund prices, S_{t_i} and S_{t_j} would be known constants (Schrager and Pelsser 2004).

Schrager and Pelsser's interpretation of the second moment demonstrates the effect of stochastic interest rates on the price of our recurring premium MRRG. Their findings show that, when interest rates are set at expected value (or deterministic), there is no interest rate risk from time zero to t_i and thus the first and second integral equal zero (Schrager and Pelsser 2004). This is because, when interest rates are deterministic then the forward bond price volatility will be zero.

What this means is that under the assumption of log-normality, the effect of stochastic interest rates on the volatility of the fund value at maturity, and hence the guarantee value, is given by the quadratic covariance between the two forward bond price processes and between the forward bond price process and the forward unit fund price process (Schrager and Pelsser 2004).

3.3 A more sophisticated model to better reflect the interrelationship between the movement of yields at different maturities is needed

Schrager and Pelssers' analysis of recurring contribution MRRG's shows us that we require a model that is sophisticated enough to capture the interrelationships between different points in the interest rate term structure.

Fortunately, models for the term structure of interest rates can be quite complex and many models would be able to satisfy this requirement. This is because the behaviour of the entire (zero-coupon bond) yield curve is required, volatilities along the yield curve may be different and interest rates are used for discounting as well as for defining the payoff (Hull 2003).

3.4 Equilibrium term structure models don't always allow for market-consistent pricing... a key requirement for insurance liability valuation

The Vasicek Model (Vasicek 1977) is one model of a family of one-factor equilibrium models. By virtue of its parametric form the Vasicek Model, along with other one-factor equilibrium models have only one source of uncertainty – that being the short-rate process. The specification of the Vasicek Model is that the instantaneous drift and the instantaneous standard deviation are functions of the short-rate but are both independent of time (Vasicek

1977). The Vasicek Model, like other one-factor models, has the restriction that all rates move in the same direction over a short time interval (Hull 2003). The Vasicek Model takes the form: $dr = a(b-r)dt + \sigma.dz$ where a, b and σ are all constants. In essence, the Vasicek Model is mean-reverting in that the short-rate (r) reverts to level (b) at a reversion rate given by parameter (a) (Hull 2003). The key drawback of equilibrium models, such as the Vasicek Model, is that they do not automatically fit the current term structure (Hull 2003). The result of this is that insurance companies would not be able to calculate their liability valuation under the current market yield curve as required by fair value accounting and actuarial guidance. This is seen as a major shortcoming.

3.5 No arbitrage term structure models can calibrate to the current yield curve

No arbitrage models are designed to calibrate exactly to the current yield curve (Hull 2003). Hull attributes the essential difference between equilibrium models and no arbitrage models to the fact that the term structure of interest rates is an output in the case of the former, while it is an input in the case of the later model type (Hull 2003). One of the extensions to the Vasicek Model by Hull and White (Hull and White 1990) can be used to provide an exact fit to the initial yield curve. The model can be characterized as the Vasicek model with a time-dependent reversion level (Hull 2003). The extension is now more commonly known as the Hull-White model takes the following parametric form: $dr = [\theta(t) - ar]dt + \sigma.dz$ where a and σ are constants and $\theta(t)$ is time-dependent (Hull and White 1990).

The Hull-White model of the term structure of interest rates is popular as it is analytically tractable while being non-trivial. I therefore choose to model under the Hull-White approach to outline a practical approach for pricing and hedging MRRG benefits.

In the Black-Scholes Hull-White framework stock price indices (or in our case unit fund prices) have constant volatility and the short-rate follows the Hull-White process in the risk-neutral measure. The Hull-White model is based on the stochastic process of the instantaneous short rate. The model assumes that the short rate process is normal and mean-reverting. i.e. Equation 10 holds.

Equation 10

$$dr_t = (\theta_t - ar_t)\,dt + \sigma_r dW_{2,t}$$

where r_t is the short rate, σ_r is the short rate volatility, *a* is the mean-reversion strength and θ_t is the deterministic drift function (Hull 2003).

The instantaneous forward rate volatility, at time *T*, is $v(t,t+T) = \sigma.\exp[-a.T]$ (Hull 2003). What this means, is as *T* decreases to zero the volatility of the short rate reduces to σ. A drawback of the Hull-White Model is its Gaussian property in that since its short rate is normally distributed there is a chance that the short rate is negative at all times *t* (Hull 2003).

3.5.1 Modelling the Black Scholes process for the risky equity underlying component

Schrager and Pelsser (Schrager and Pelsser 2004) make the assumption that the (instantaneous) correlation between the stock price and the short rate is constant, given by $Correl^Q[d(\ln S_t);\ dr_t] = \rho\, dt$. Thus, in the Black-Scholes Hull-White model all equity volatilities are deterministic functions (of time). This assumption is then represented by the processes, dS_t, for the stock (unit price) as in Equation 11 (Schrager and Pelsser 2004).

Equation 11

$$dS_t = r_t\, S_t\, \mathrm{dt} + \sqrt{1-\rho^2}\,\sigma_S\, S_t\, \mathrm{d}W_{1,t} + \rho\sigma_S S_t \mathrm{d}W_{2,t}$$

3.5.2 Demonstrating that stochastic interest rates under Hull-White lead to complex guarantee pricing formulae

Under the Hull-White dynamics it is typical to define the term, $B(t,T) = \left(1/_a\right)\left[1 - e^{-a(T-t)}\right]$, so as to simplify notation (Hull 2003). In the case of the BSHW model the volatilities are deterministic functions of time i.e. constant stock volatility and a fixed correlation, ρ, between the changes in the stock and the short-rate. The implication of this is that under these price dynamics , Schrager and Pelsser derive the price dynamics of the *T*-forward stock (unit price), F_t and *T*-forward bond price as in Equation 12 and Equation 13 (Schrager and Pelsser 2004).

Equation 12

$$dD^T(t,S) = -\sigma_r[B(t,S) - B(t,T)]D^T(t,S)\mathrm{d}W_{2,t}^T$$

Equation 13

$$dF_t^T = \sqrt{1-\rho^2}\,\sigma_S\, F_t^T\, \mathrm{d}W_{1,t}^T + \rho\sigma_S F_t^T \mathrm{d}W_{2,t}^T + \sigma_r B(t,T) F_t^T \mathrm{d}W_{2,t}^T$$

or equivalently

$$dF_t^T = \sqrt{\sigma_S^2 + 2\rho\sigma_S\sigma_r B(t,T) + \sigma_r^2\left(B(t,T)\right)^2}\; F_t^T \mathrm{d}Z_t^T$$

$W_{1,t}^T, W_{2,t}^T$ and Z_t^T are standard Brownian motions in the *T*-forward measure. What this means is that users no longer need to infer interest rate volatility from market data, but rather can calibrate by parameterisation to the formulae above. This form makes the volatility of the forward asset price explicit. i.e. the component of confined to the square root. Schrager and Pelsser derive the instantaneous covariance of F_t^T and $D^T(t,S)$ to be Equation 14.

Equation 14

$$\rho_{F,S}(t)\sigma_F(t)\sigma_S(t) = -\rho\sigma_S\sigma_r[B(t,S) - B(t,T)] + \sigma_r^2\left(B(t,T)\right)^2 - \sigma_r^2 B(t,S)B(t,T)$$

Schrager and Pelsser parameterise Equation 12 and Equation 13 can be parameterised to resemble Equation 5 and Equation 6. In this case Schrager and Pelsser (Schrager and Pelsser 2004) show that Equation 15 holds. This shows the volatility of interest rates, or the forward bond volatility.

Equation 15

$$\begin{aligned} E_t^{Q^t}\left[\int_t^T \sigma_i(s)\mathrm{d}W_S^i \int_t^T \sigma_{ij}(s)\mathrm{d}W_S^j\right] &= \left(+\sigma_r^2 \int_t^{t_i}[B(s,T) - B(s,t_i)]\left[B(s,T) - B\left(t,t_j\right)\right]\mathrm{d}s\right) \\ &+ \left(-\rho\,\sigma_S\;\; \sigma_r \int_{t_i}^{t_j} B\left(s,t_j\right) - B(s,T)\,\mathrm{d}s + \sigma_r^2 \int_t^{t_i}\left(B(s,T)\right)^2 - B(s,T)B\left(s,t_j\right)\mathrm{d}s\right) \\ &+ \left(\sigma_r^2(T - t_j) + 2\rho\,\sigma_S\;\; \sigma_r \int_{t_j}^{T} B(s,T)\mathrm{d}s + \sigma_r^2 \int_{t_j}^{T}\left(B(s,T)\right)^2\mathrm{d}s\right) \end{aligned}$$

Schrager and Pelsser (Schrager and Pelsser 2004) breakdown Equation 15 expression into three parts for explanation purposes. These parts correspond to the procedure followed in breaking down Equation 9:

- The first term, $\sigma_r^2 \int_t^{t_i}[B(s,T) - B(s,t_i)]\left[B(s,T) - B\left(t,t_j\right)\right]\mathrm{d}s$ corresponds to $\int_t^{t_i} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\mathrm{d}s$ (in Equation 9). This term represents the correlation between the bonds with maturity t_i and t_j. This follows from, for a one factor model, the correlation between forward bond prices equalling one (Schrager and Pelsser 2004).

- The second term, $-\rho\, \sigma_S\ \ \sigma_r\ \int_{t_i}^{t_j} B(s, t_j) - B(s, T)\, \mathrm{d}s + \ \sigma_r^2 \int_t^{t_i} \big(B(s, T)\big)^2 - B(s, T) B(s, t_j)\, \mathrm{d}s$, corresponds to $\int_{t_i}^{t_j} \rho_{F,j}(s) \sigma_F(s) \sigma_j(s) \mathrm{d}s$ (in Equation 9). This term represents the covariance between the forward asset price and $D(t,\ t_j)/\ D(t,\ t_T)$ (Schrager and Pelsser 2004).
- The final term, $\sigma_r^2(T - t_j) + 2\rho\, \sigma_S\ \ \sigma_r\ \int_{t_j}^{T} B(s, T) \mathrm{d}s + \ \sigma_r^2 \int_{t_j}^{T} \big(B(s, T)\big)^2 \mathrm{d}s$ corresponds to $\int_{t_j}^{T}\ \sigma_F^2(u) \mathrm{d}u$ (in Equation 9). This term represents the implied volatility of a forward start option on a risky asset (Schrager and Pelsser 2004).

Schrager and Pelsser's breakdown gives an idea as to the drivers of volatility of the MRRG and therefore what potential hedging instruments may be used to capture some of these characteristics. Aspects of the MRRG formulae suggest forward starting equity options could act as potential hedges (Schrager and Pelsser 2004).

4 Demonstration of the pricing of a recurring premium rate of return guarantee under Black Scholes Hull White assumptions

4.1 Calibrating the BSHW model

4.1.1 The Hull-White term structure model has high analytical tractability

In the Hull-White model, at time t the short rate reverts to $\theta(t)/a$ at a rate a (Hull 2003). $\theta(t)$ can be calculated off the initial term structure of interest rates, using Equation 16 (Hull 2003). Often, however the last term is simply ignored and the resulting equation implies that, on average, the short rate follows the initial slope of the instantaneous forward rate curve. When the curve deviates from the initial forward curve it reverts back to it at rate a (Hull 2003).

Equation 16

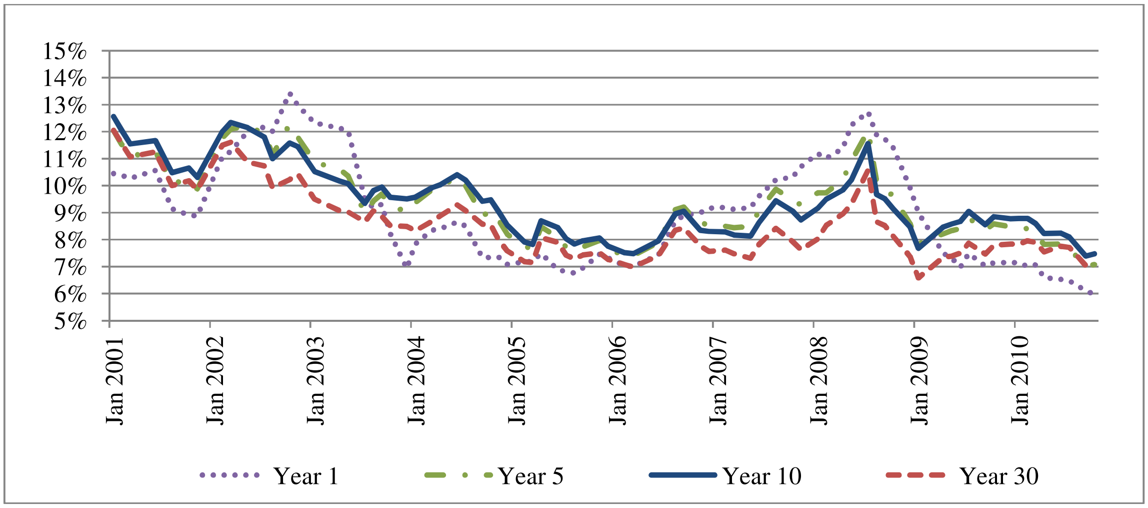
$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

$F(0,t)$ is the instantaneous forward rate and $F_t(0,t)$ is the change in $F(0,t)$. What this means is that $P(0,t)$, the price of a zero-coupon bond maturating at time t, valued at time 0, must be a twice differentiable function. As the Nelson-Siegel (Nelson and Siegel 1987) yield curve fitting approach satisfies this requirement it is a logical choice.

4.1.1.1 Using observable swap rates as input traded yields

Observed traded swap rates are used as market-consistent inputs for the pricing of the MRRG benefits in this dissertation. The time period October 2001 to September 2010 was used in the historic data analysis, as shown in Figure 4.

Figure 4: Historic data of the South African swap rates from October 2001 to September 2010



Source: Bloomberg, own calculations

4.1.1.2 Application of the Nelson-Siegel approach to fit traded market swap rates

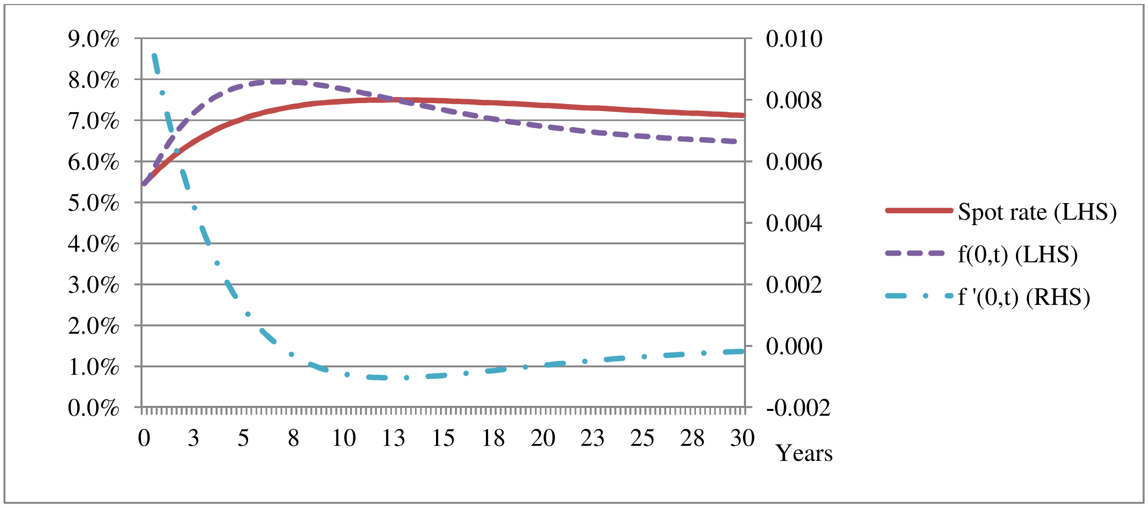
The Nelson-Siegel approach for fitting a yield curve ensures the interpolation approach is twice differentiable. This is due to the Nelson-Siegel (Nelson and Siegel 1987) formulae taking the form shown in Equation 17 where $\beta_0, \beta_1, \beta_2$ and τ represent the yield curve level, short-term components, medium-term components and the decay speed.

Equation 17

$$y(t) = \beta_0 + \beta_1 \left[\frac{1 - e^{-t/\tau}}{t/\tau}\right] + \beta_2 \left[\frac{1 - e^{-t/\tau}}{t/\tau} - e^{-t/\tau}\right]$$

Calibration of a term-structure model requires choosing the model parameters, $\beta_0, \beta_1, \beta_2$ and τ, so as to fit the traded prices as closely as possible. This would involve some form of goodness-of-fit measure such as $\sum_{i=0}^{n}(M_i - N_i)^2$ where M_i is the market price for the calibrating instrument and N_i is the price calculated for the calibrating instrument from the model (Hull 2003). Naturally, the number of parameters to be found must be less than the number of calibrating instruments. The resulting parameters are shown in Figure 62 and Figure 63 in the appendix. The twice differentiable nature of the fitted Nelson-Siegel yield curve is shown in Figure 5.

Figure 5: Demonstration of the twice differentiable nature of the fitted Nelson-Siegel yield curve



Source: Bloomberg , Own calculations

4.1.1.3 Simple formulae describe the volatility of the term structure of interest rates under the Hull-White model

The volatility structure of the Hull-White model is determined by both σ and a (Hull 2003). Firstly, the volatility at time t of the price of a zero-coupon bond maturing at time T is given by Equation 18 (Hull 2003).

Equation 18

$$\frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right)$$

Secondly, the instantaneous standard deviation at time t of the zero-coupon interest rate maturing at time T is given by Equation 19 (Hull 2003):

Equation 19

$$\frac{\sigma}{a(T-t)}\left(1 - e^{-a(T-t)}\right)$$

Thirdly, the instantaneous standard deviation of the T-maturity instantaneous forward rate is given by Equation 20 (Hull 2003):

Equation 20

$$\sigma e^{-a(T-t)}$$

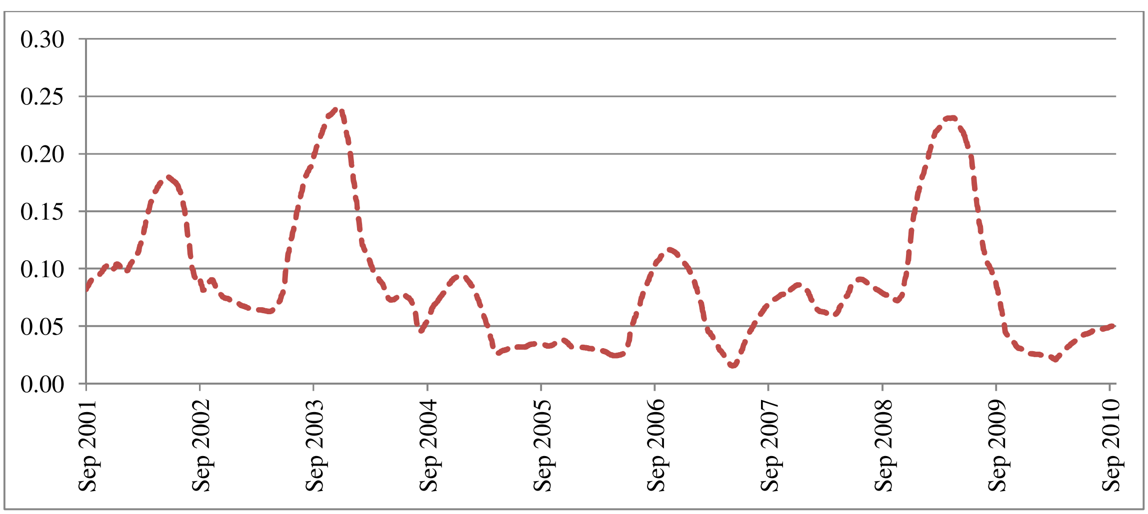
Hull's conclusions can be seen directly from these functions. The parameter σ determines the short rate's instantaneous standard deviation (Hull 2003). The reversion rate parameter, a, determines the rate at which bond price volatilities increase with maturity and the rate at which interest rate (forward rate) standard deviations decrease with maturity (Hull 2003).

4.1.1.4 Setting reasonable interest volatility parameters to calibrate the Hull-White model

In calibrating the Hull-White model the following steps were taken. Firstly, as suggested by Hull and White (Hull and White 1990) the σ parameter should be calibrated to the modellers view of the future volatility of the short rate. Typically, to achieve market consistency the implied volatility of swaptions or caps are used. My approach has been to choose a sigma that represents the historic realised standard deviations of continuously compound forward rates. The average of the standard deviations of the forward rates at different maturities on the yield curve is taken as the choice of sigma, σ, parameter.

I set our choice of sigma to: $\sigma = \sqrt{252}\, times\ the\ standard\ deviation\ of\ (-ln(1 + r(0,1))$ *over the preceding 181 trading days up to 30 September 2010.* This results in a setting $\sigma = 0.05$. Thus underlying historic data for the continuous compound short rate standard deviation is shown in Figure 6.

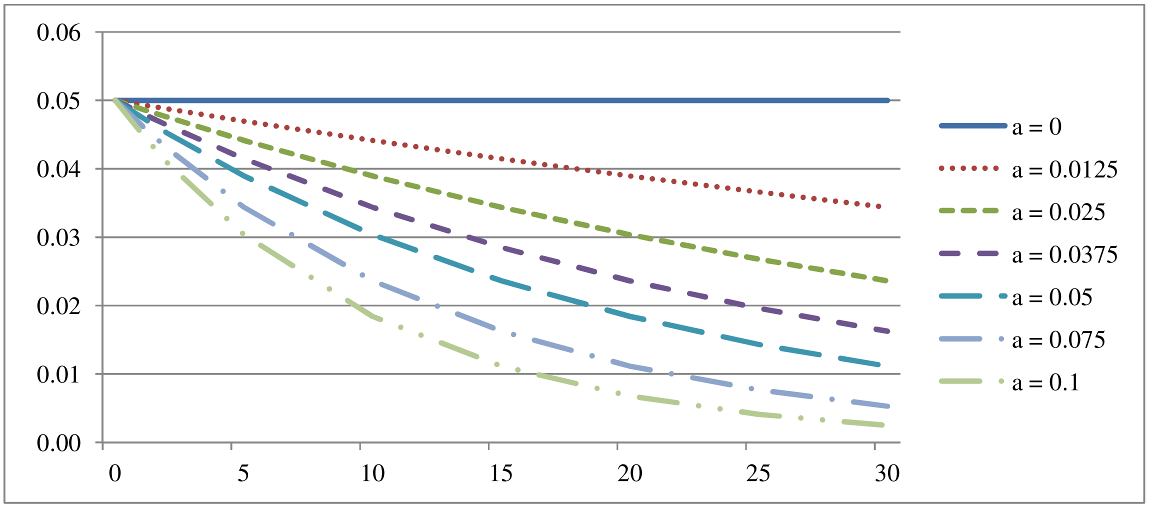
Figure 6: Continuously compounded short rate standard deviation (annual basis) from September 2001 to September 2010



Source: Bloomberg , Own calculations

It can also be seen that fixing $a = 0$ sets the zero-coupon bond price volatilities as linear functions of maturity and the instantaneous standard deviations of the forward rates being constant (Hull 2003). The effect of various choices of a on the instantaneous standard deviations for the T-maturity instantaneous forward rate can be seen inFigure 7. For the Hull-White model, a choice of reversion rate "a" is required, such that the future instantaneous standard deviations for the T-maturity forward rates are reasonable.

Figure 7: Instantaneous standard deviations for the T-maturity instantaneous forward for a range of *a* parameter choices under the Hull-White model



Source: Bloomberg , Own calculations, assumptions: σ = 0.05

4.1.1.5 Calibrating the Hull-White model to traded swap rates

In the Hull-White Model zero coupon bond prices are given by Equation 21 (Hull 2003):

Equation 21

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \text{ where}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)} e^{\left\{B(t,T)f(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right\}}$$

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

What this allows us to do is solve for a based on our calibration of σ. In doing so, least squares minimisation of the square of the difference between the market and model bond price is applied (Equation 8).

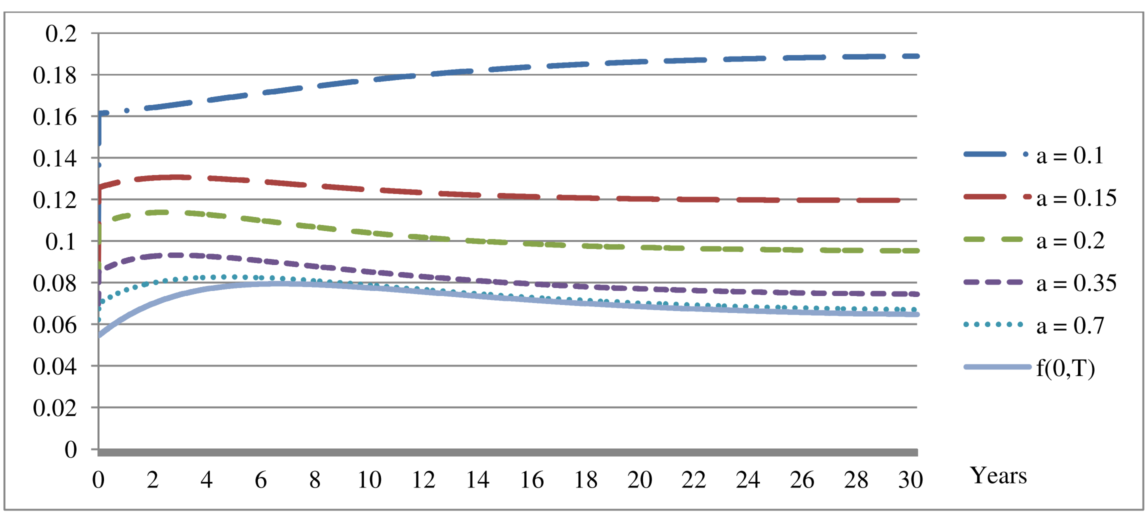
Table 5: Demonstration of how the Hull-White model fits to a choice of alpha "*a*"

Description	P(0,1)	P(0,2)	P(0,5)	P(0,10)	P(0,15)	P(0,20)	P(0,25)	P(0,30)	Sum
Market	0.944	0.885	0.711	0.487	0.339	0.241	0.174	0.127	
Model	0.944	0.885	0.711	0.487	0.339	0.241	0.174	0.127	
B(0,t)	0.929	1.728	3.518	5.179	5.964	6.335	6.510	6.593	
A(0,t)	0.993	0.972	0.862	0.646	0.469	0.341	0.249	0.182	
r(0)	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	
Difference model to market	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
Difference model to market ^2	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**

Source: Own calculations

$\theta(t)$ follows directly from Equation 16. The choice of a, the reversion rate, determines how closely $\theta(t)/a$ tracks the forward rates given by the initial calibrated yield curve. Thus if the choice of a is large, then we are effectively setting the strength of the force to revert to the forward rate, while a small a would not pull back to the initial forward rates as quickly. This can be seen in Figure 8.

Figure 8: Reversion of theta(*t*)/*a* to the forward rates implied by the initial curve at time *t*



Source: Own calculations

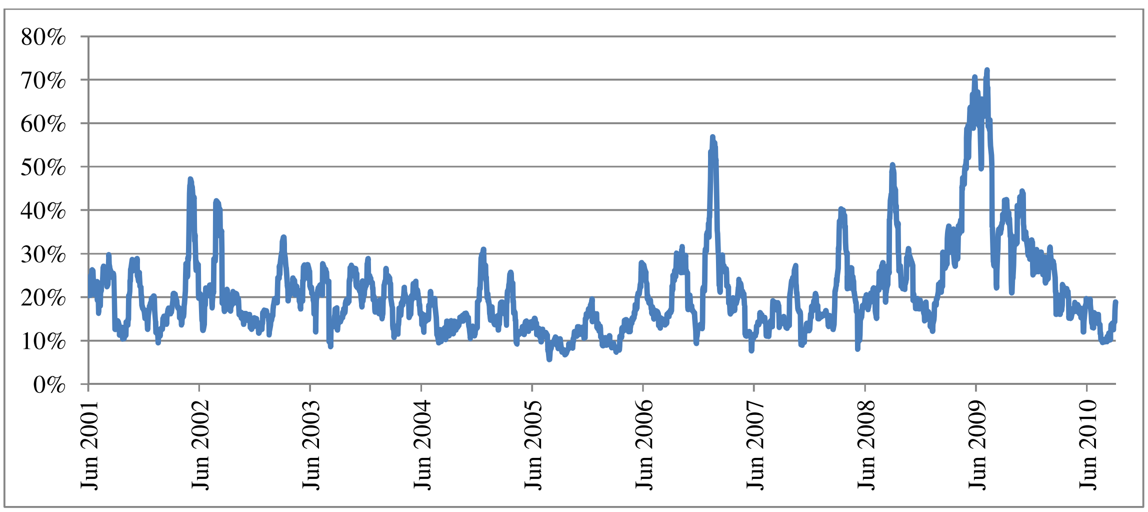
4.2 Calibrating the BSHW model to equity market inputs

Next, calibration of the Black Scholes process to describe the risky equity assets behavior is required. A calibration process is followed to reflect the equity parameters in a market consistent manner, where possible. When pricing in the risk neutral measure we do not make any allowance for equity risk premiums.

4.2.1 Analyzing equity market volatility

The historic 180-day annualized standard deviation of JSE Shareholder-weighted Top40 Index returns for the period September 2001 to September 2010 averaged c.23%. I calculate this as daily JSE Shareholder-weighted Top40 Index returns and annualize by multiplying by $\sqrt{252}$. As shown in Figure 9, realized standard deviations have, at times, been high. These market volatility spikes tend to pass and standard deviations tend to drift lower towards the c.20% mark.

Figure 9: 180-day annualised standard deviation of JSE Top40 index daily returns - June 2001 to September 2010



Source: Bloomberg data, Own calculations

4.2.1.1 Setting the equity volatility "σ_E" parameter in the Black Scholes setting

In the JSE Shareholder-weighted Top40 Index equity options seldom have expiries beyond three years. In addition, it is not likely that the limited number of traded options are sufficient in number to fully calibrate the total portfolio of the unit-linked investment guarantees at various moneyness levels. For example, a specific policy may have been in force in excess of ten years and, under above-average investment returns now exceed the current minimum rate of return guarantee. In this instance this policy's guarantee is far out of the money. It is therefore appropriate to consider this on calibration to equity market volatility when calculating the MRRG price for each specific policy.

The topic of the estimation of the appropriate mathematical form and parameterization of the term structure of equity volatility is one in itself. In general, South African insurance companies choose to calibrate, as far as appropriate and possible, to the near-term traded equity options and taper the estimated volatility parameters out to some long-term value. The implicit assumption in these approaches being that equity market volatility is mean-reverting, in the long-term. There exists a significant volume of literature on various volatility term structure fitting. As these topics are significantly beyond the scope of this dissertation, I have chosen to set the equity standard deviation parameter at 25% p.a. at all future time points, i.e. $\sigma_E = 25\%$

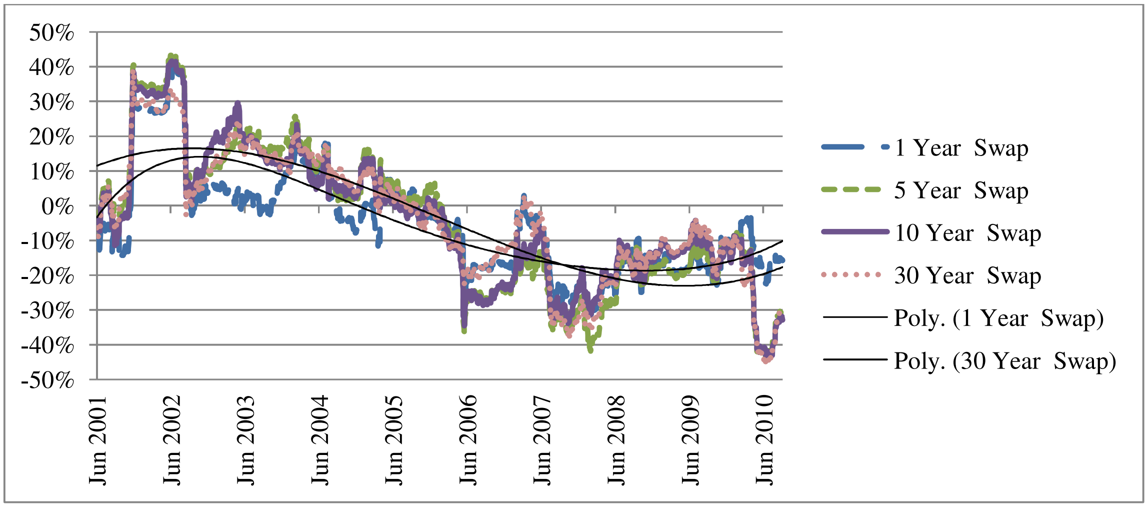
4.2.2 Analyzing the correlation structure between equity returns and interest rates

The Black Scholes Hull White model requires an estimate of the correlation between movements in the short rate process and the risky asset (equity) return process. This single parameter is then used in each time period to capture the effects of the relationship between the two independent input processes. These being the Hull-White short rate process and the Black Scholes process.

4.2.2.1 Correlations between equity returns and interest rates tend to be cyclical

Historically, the correlation between equity returns and various swap rate movement changes have followed a cyclical behavior. This can be observed in Figure 10, where changes in swap rates tended to be positively correlated with equity returns between 2001 and 2005, while the reverse is true from 2006 to 2010.

Figure 10: Historic annualised correlation between JSE Shareholder-weighted Top40 Index and changes in various swap rates over the 180-days prior to the yield curve calibration date



Source: Bloomberg, own calculations

As a result of this cyclical behavior, the overall correlation between interest rate changes and equity index returns in the sample is close to zero. A full set of the correlations between equity index returns and different swap rates is shown in Figure 11.

Figure 11: Correlation between daily JSE Top40 moves and changes in various swap rates

Correlation between daily moves in the South African JSE Top40 Index and the changes in various swap rates (between October 2000 and September 2010)							
1 Year Swap	2 Year Swap	5 Year Swap	10 Year Swap	15 Year Swap	20 Year Swap	25 Year Swap	30 Year Swap
-2.33%	-1.20%	-2.30%	-1.84%	-1.30%	-2.09%	-2.63%	-2.23%

Source: Bloomberg, own calculations

4.2.2.2 Setting the "$\rho_{E,IR}$" correlation parameter

As a result of the cyclical nature of the correlation between changes in interest rates and equity index returns I have chosen to set the base case correlation at 0%, i.e. $\boldsymbol{\rho_{E,IR}}$ = 0%. The effect is that our model of future equity returns, under the Black Scholes process, will be independent from the Hull White short rate process in our base case model. Sensitivities around this base case assumption are shown later.

4.3 Outlining practical BSHW simulation generation in a spreadsheet

4.3.1 BSHW simulations in a spreadsheet requires discretization

In practice, Monte Carlo implementations of short rate models, such as the Hull-White model, requires discretization of the stochastic process. Hill, Visser et al. state the one common approach is to use the BGM formulation. Under this method discrete rates are used and the volatility term structure is replaced by a step function (Hill, Visser et al. 2008). I write Equation 10 in its Euler form in Equation 22.

Equation 22

$$r(t_{i+1}) = r(t_i) + [\theta(t_i) - ar(t_i)].\Delta T + \sigma W_{2,t} \sqrt[2]{\Delta T}$$

Discretizing Equation 11 leads to the derivation given by Equation 23

Equation 23

$$S(t_{i+1}) = S(t_i) \exp\left\{ \left(f(t_i) - {}^{1}\!/_{2}\, \sigma_S^2\right) \Big/ {}_{\Delta T} + \sqrt[2]{{}^{1}\!/_{\Delta T}} \,.\, \sigma_S . \left[\sqrt{1-\rho^2}\ W_{1,t} + \rho W_{2,t}\right] \right\}$$

Where $W_{1,t}$ and $W_{2,t}$ are defined as before. i.e. the standard normal random variables relating to the underlying equity process, S_t and the short rate respectively.

4.3.2 Outline of a spreadsheet-based model structure

The spreadsheets built to facilitate the calibration; economic scenario generation and MRRG price calculation are described in the Appendix. In addition, an outline of how the various spreadsheet models relate to one another is also shown in the Appendix for reference.

4.4 Demonstration of the simulations generated by the BSHW model

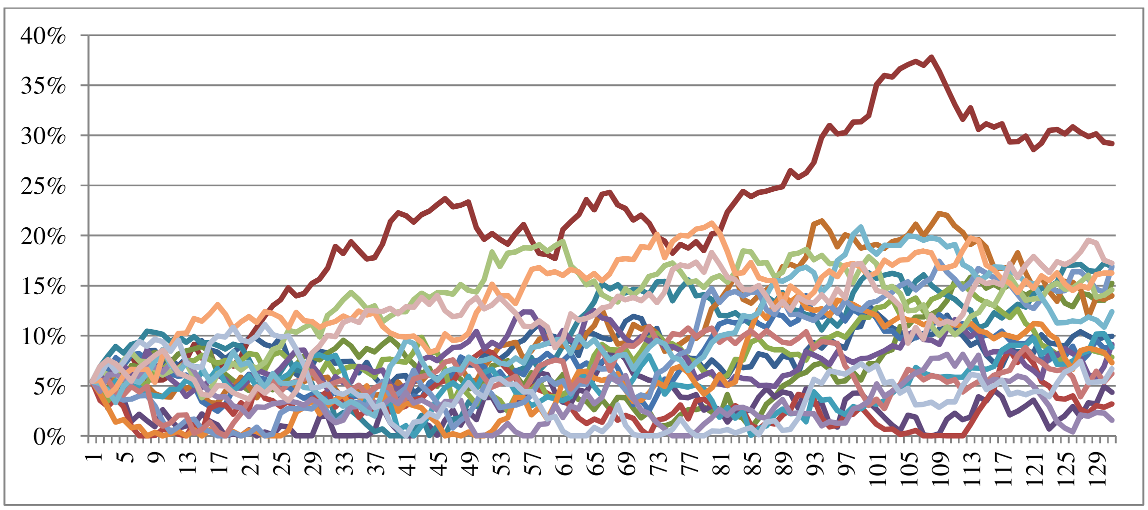
4.4.1 Demonstration of the Hull-White simulations

The date used for the initial base case pricing of a minimum rate of return guarantee was 30 September 2010. As outlined in section 4.1.1.2 the swap rates at 30 September 2010 were used to calibrate a Nelson-Siegel curve off which forward rates could be found. In section 4.1.1.5 I showed the choice of short-term interest rate volatility set at 0.05. The choice of *a* determines the strength with which the future deviations away from the initial calibrated market rate revert in the Hull-White model.

In section 4.2.1.1 I concluded that an equity market standard deviation of 25% is an appropriate simplifying choice for future equity volatility. I used a 0% correlation between the underlying risky asset (or equity) process and the short-rate in the pricing of the base case guarantee. This assumption was set in section 4.2.2.2.

1000 simulations were generated and an illustration of the first 20 of these simulations for the short rate and the equity index modeled for a time period of 30 quarters (or 7.5 years). The short rate simulations are shown in Figure 12 while these simulations of the underlying equity index are shown in Figure 13.

Figure 12: Illustration of first 20 Hull-White simulations over a 5-year period shown in 130 fortnightly time steps



Source: Own calculations

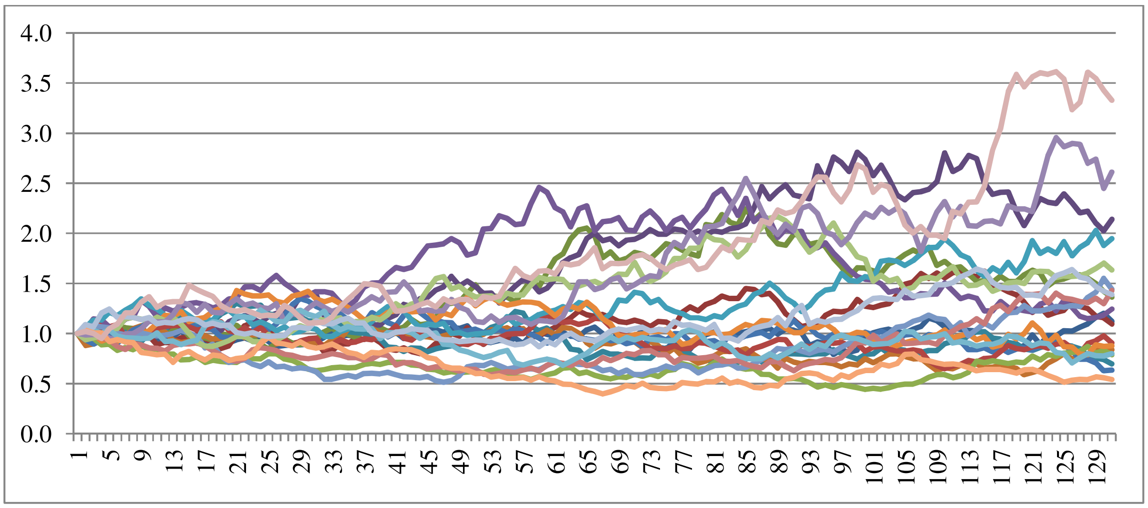
4.4.2 Testing the reasonability of the Hull-White short rate simulations by pricing a bond

Reasonability check is: $P(0,30) = \frac{1}{2000}\sum_{i=1}^{2000} exp\{-\sum_{j=0}^{120} 0.25 f(t_{i,j})\}$ where j represents the 120 quarterly time intervals in a 30 year period, and i represents the 1000 simulations. In our model for 30 September 2010, $P(0,30) = 0.13$ which implied that $r(0,30) = 7.03\%$. This was comparable to the 30-year swap rate source data input of 7.13%.

4.4.3 Demonstrating the BSHW simulations for the underlying risky asset

I set out base case market input assumptions at 30 September 2010 for the correlation of the short rate process and the equity index return process at 0%, i.e. $\boldsymbol{\rho_{E,IR}}$ = 0%, and the volatility of the risky underlying asset process by σ_E = 25%.

Figure 13: Illustration of first 20 BSHW simulations over a 5-year period shown in 130 fortnightly time steps



Source: Own calculations

4.4.4 Reasonability checking the Black Scholes equity simulations via a Martingale test

A check that the equity simulations generated under the BSHW process satisfy the condition that the average of the discounted value of our simulated equity index process across the simulations approximately equals 1. This is a necessary requirement for our simulations to be a martingale. That is $\frac{1}{2000}\sum_{i=1}^{2000} S_i(0,30).P_i(0,30) = 0.996 \ldots \cong 1$

4.5 Pricing a MRRG guarantee under BSHW

4.5.1 Introduction to the pricing of a typical MRRG

Commence the pricing of a MRRG under BSHW by setting some basic policy assumptions. I assume that the premium contributions to the underlying savings product are paid quarterly, at a rate of R1000 per quarter. At first I assume the series of contributions to be made over the products life are constant over time (i.e. 0% p.a. contribution escalation). A 5% p.a.

MRRG has been assumed to be the typical product offered. These typical (or base case) product demographic characteristics are shown in Table 6 below.

Table 6: Basic demographic assumptions for typical (base case) MRRG products

Premium, product and guarantee inputs	
Quarterly contribution	R 1000
Annual Contribution increase	0.00%
Guarantee percentage (annually)	5.00%

Source: Own calculations

In our initial pricing, the product being priced and analysed is assumed to be a new product. Therefore there is no initial fund value built up at the start of the policy and as there are no past paid premiums there is also no guarantee at the start of the policy. Later in our analysis an initial fund value balance and initial guarantee is assumed. This will allow the effect of the moneyness level of the fund value relative to the guarantee to be analysed.

The initial market value inputs used in pricing the base case MRRG are shown in Table 7. These market values and assumptions represent the traded South African swap rates on 30 September 2010 and the associated BSHW parameters derived in Chapter 4.1.

Table 7: Basic economic assumptions for typical (base case) MRRG products

5 Year swap	10 Year swap	15 Year swap	20 Year swap	25 Year swap	30 Year swap	Standard Deviation	Hull-White - Alpha	Hull-White - Sigma	Underlying process volatility	Correlation of underlying and short-rate
7.06	7.49	7.46	7.36	7.24	7.13	0.00316	0.15	0.05	0.25	0.00

Source: Bloomberg, own calculations

Contract terms of 5, 10, 15, 20, 25 and 30 years have been considered. Table 8 shows the outcome of the MRRG price calculations. For the case of the 5-year MRRG product the nominal quarterly contributions total R20000. These contributions have a discounted value of R16595 under the base yield curve generated. The calculated MRRG cost for a 5-year term product is R1465. This expected guarantee cost equates to 8.83% of the discounted value of the contributions expected on the product. As can be seen in Table 8 the cost of the MRRG (expressed as percentage of discounted contributions) decreases as the term of the guaranteed products increases.

Table 8: MRRG absolute price and price relative to contributions for various MRRG terms

Type of output	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Total nominal contributions	20 000	40 000	60 000	80 000	100 000	120 000
Discounted value of the contributions	16 595	27 108	33 415	37 177	39 432	40 776
Minimum Rate of Return Guarantee price	1 465	2 330	2 367	1 999	1 545	1 312
Guarantee price as a percentage of the PV of total contributions	8.83%	8.60%	7.08%	5.38%	3.92%	3.22%

Source: Own calculations

4.5.1.1 Variation in the minimum rate of return guaranteed, the annual contribution increases and the term of the MRRG contract

Table 9 demonstrates the extent to which changes in the demographic assumptions of the contract affect the MRRG price.

Firstly, comparison is made between a 0% annual contribution increase (ACI) and a 10% ACI. It can be seen that the MRRG price of the 10 % ACI (expressed as a percentage of discounted contributions) is slightly lower for short terms but higher than that of the 0% ACI as the term extends. This is because in the case of the 10% ACI the weighted average of the contributions is longer, and therefore means that on average the cash flows are invested for a shorter period of time until the maturity date. The guarantee cost therefore rises as the effect of the MRRG term seen in Table 8 takes effect.

Secondly, the calculations show that the cost of a 0% guarantee tapers quickly to well within 1% of the present value of contributions as terms stretch beyond 20 years. For this reason I have elected to demonstrate the costs of a 5% p.a. MRRG option in the sensitivity analysis that follows. This will allow analysis of various sensitivities under a more financially significant guarantee cost.

Table 9: Demonstration of the effects of different demographic assumptions on the price of a range of MRRG terms

ACI %	Guar.. % p.a.	Type of output	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG

0%	0%	Nominal contributions	20000	40000	60000	80000	100000	120000
		PV of the contributions	16595	27107	33414	37176	39432	40776
		MRRG price	775	868	621	395	192	135
		Price as % of PV contributions	4.7%	3.2%	1.9%	1.1%	0.5%	0.3%
0%	5%	Nominal contributions	20000	40000	60000	80000	100000	120000
		PV of the contributions	16595	27107	33414	37176	39432	40776
		MRRG price	1466	2330	2367	1999	1545	1312
		Price as % of PV contributions	8.8%	8.6%	7.1%	5.4%	3.9%	3.2%
10%	0%	Nominal contributions	24420	63750	127090	229100	393388	657976
		PV of the contributions	19949	40237	59820	78640	96809	114239
		MRRG price	902	1261	1195	989	669	774
		Price as % of PV contributions	4.5%	3.1%	2.0%	1.3%	0.7%	0.7%
10%	5%	Nominal contributions	24420	63750	127090	229100	393388	657976
		PV of the contributions	19949	40237	59820	78640	96809	114239
		MRRG price	1678	3210	3898	3976	3843	4272
		Price as % of PV contributions	8.4%	8.0%	6.5%	5.1%	4.0%	3.7%

Source: Own calculations

The same random seed has been used when comparing the MRRG price under different scenarios. This stops further random fluctuations being introduced and allows for easier comparison in the sections which follow. This decision was taken in the context of the pricing only being performed on a limited number of simulations due to run-time constraints. Ideally a sufficient number of simulations would have been run to ensure convergence of the pricing problem. If this were the case then the choice of seed for the simulation generation would no longer influence results.

4.5.1.2 Parallel yield curve shifts

The first economic parameter sensitivities analysed are those to changes in swap rate inputs. This tells us how the market consistent cost of the MRRG changes under different parallel swap rate shifts. Table 10 shows how the discounted value of the future contributions and the MRRG price changes under a 10bps upward and downward shift in all swap rate inputs.

The cost of the MRRG increases as swap rates drop, all else equal. That change in the price increases as the term of the option increases.

Table 10: Sensitivity to parallel up and down shifts in the base swap rates on the price of a 5% p.a. MRRG

Parallel shifts in the swap inputs	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
All durations - 10bps	16624	27183	33532	37329	39610	40974
Base swap rates	16595	27109	33414	37176	39431	40776
All durations + 10bps	16567	27034	33296	37023	39252	40577

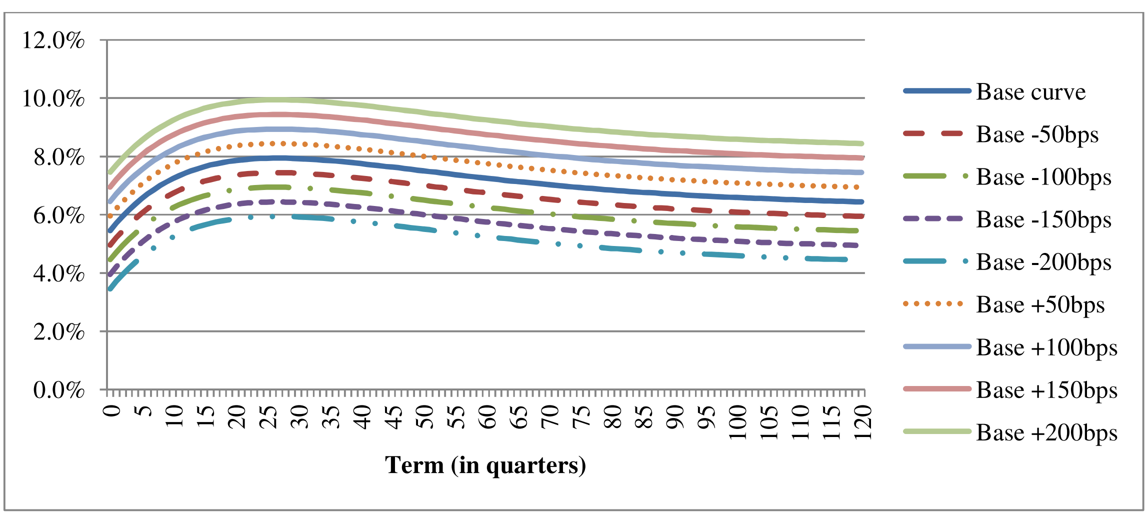
Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
All durations - 10bps	1435	2154	2151	1821	1345	1154
Base swap rates	1421	2125	2113	1780	1306	1119
All durations + 10bps	1408	2096	2075	1739	1269	1086

Change in MRRG price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
All durations - 10bps	0.94%	1.35%	1.82%	2.32%	2.92%	3.06%
Base swap rates	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
All durations + 10bps	-0.94%	-1.35%	-1.81%	-2.30%	-2.87%	-2.97%

Source: Own calculations

The analysis has been furthered to a wider range of parallel yield curve stress tests. To do so, a range of 50bps increment stresses were made, both upward and downward. These are shown in Figure 14.

Figure 14: Illustration of the yield curves fitted under a range of parallel swap rate stresses



Source: Own calculations

In that case of a 5% p.a. MRRG guarantee (with 0% p.a. ACI), there is a more pronounced relative increase in the price of the MRRG cost as swap rates fall compared to when they increase. This is observed in Table 11 where the nominal change in the MRRG price is larger for decreasing parallel movements in the yield curve as opposed to increases in the yield curve. For example, the 30-year MRRG price will decrease by R538 if swap rates rise by 200bps but increase by R896 if swap rates fall by 200bps.

Table 11: Outline of the relative increases (decreases) in the MRRG price under the stressed parallel curve inputs

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base +200bps	7.20%	6.24%	4.62%	3.23%	2.01%	1.58%
Base +150bps	7.55%	6.62%	5.02%	3.58%	2.28%	1.83%
Base +100bps	7.90%	7.02%	5.43%	3.95%	2.58%	2.11%
Base +50bps	8.23%	7.42%	5.87%	4.35%	2.92%	2.42%
Base curve	8.56%	7.84%	6.32%	4.79%	3.31%	2.75%
Base -50bps	8.89%	8.26%	6.79%	5.26%	3.73%	3.12%
Base -100bps	9.22%	8.69%	7.25%	5.77%	4.16%	3.54%
Base -150bps	9.53%	9.13%	7.72%	6.30%	4.62%	3.99%
Base -200bps	9.82%	9.57%	8.20%	6.85%	5.10%	4.51%

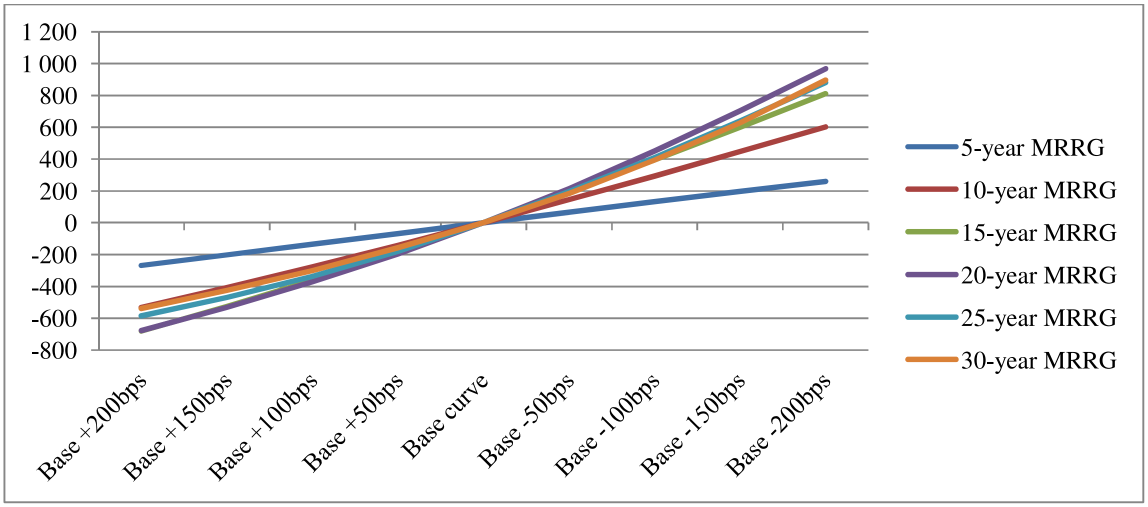
Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base +200bps	1153	1594	1431	1101	722	581
Base +150bps	1220	1720	1587	1248	838	693
Base +100bps	1287	1849	1750	1409	970	820
Base +50bps	1355	1985	1926	1583	1126	962
Base curve	1421	2125	2113	1780	1306	1119
Base -50bps	1489	2271	2308	1995	1503	1305
Base -100bps	1555	2420	2508	2231	1716	1513
Base -150bps	1619	2572	2713	2482	1945	1747
Base -200bps	1681	2728	2924	2747	2188	2015

Change in Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base +200bps	-268	-531	-681	-678	-585	-538
Base +150bps	-201	-406	-526	-532	-468	-426
Base +100bps	-134	-276	-362	-370	-336	-300
Base +50bps	-67	-141	-187	-196	-180	-157
Base curve	0	0	0	0	0	0
Base -50bps	67	146	195	215	196	185
Base -100bps	134	295	395	451	410	394
Base -150bps	198	447	600	703	638	628
Base -200bps	260	603	811	967	882	896

Source: Own calculations

A graphic illustration of this interest rate sensitivity is shown in Figure 15. Therefore the MRRG price has positive interest rate convexity.

Figure 15: Illustration of positive convexity of the MRRG to parallel yield curve changes



Source: Own calculations

To accurately quantify this positive interest rate convexity one would need to calculate the second partial derivatives, or "gammas", for the eight swap rate input instruments in our model. While this would be of interest it goes beyond the scope of current industry intest rate hedging practice as discussed in section 5.1.

4.5.1.3 Different yield curve shapes

Five different sets of swap rate inputs have been chosen to test the sensitivity of the MRRG price to changes in the shape of the yield curve. These swap rates were chosen manually to capture the effect of flat, downward sloping, upward sloping, positively humped and negatively humped yield curves. Table 12 outlines these manually setup swap rates.

Table 12: Swap rate scenario input assumptions for different yield curve shapes

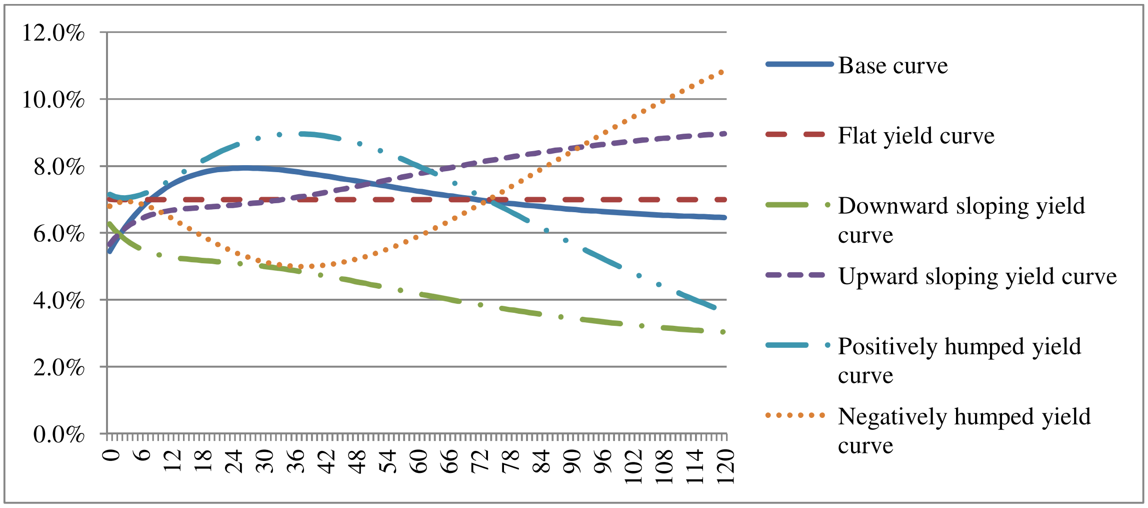
Swap rate inputs	1 Year Swap	2 Year Swap	5 Year Swap	10 Year Swap	15 Year Swap	20 Year Swap	25 Year Swap	30 Year Swap
Base yield curve	5.97	6.27	7.06	7.49	7.46	7.36	7.24	7.13
Flat yield curve	7.00	7.00	7.00	7.00	7.00	7.00	7.00	7.00
Downward sloping yield curve	5.97	5.72	5.47	5.22	4.97	4.72	4.47	4.22
Upward sloping yield curve	5.97	6.22	6.47	6.72	6.97	7.22	7.47	7.72

Positively humped yield curve	7.00	7.25	7.50	8.00	8.50	8.00	7.50	7.00
Negatively humped yield curve	7.00	6.75	6.50	6.00	5.50	6.00	6.50	7.00

Source: Own calculations

A graphic illustration of the Nelson-Siegel yield curve fit calculated for each of the sets of swap rate inputs is shown in Figure 16.

Figure 16: Graphic illustration of resulting yield curve scenarios



Source: Own calculations

The BSHW model assumptions, other than the swap rate inputs shown above, were kept constant at the base run parameters outlined in Table 7 above. These parameters are shown again in Table 13.

Table 13: Base run BSHW parameters used under interest rate sensitivity tests

Standard Deviation Cont. Comp.	Hull-White Parameters		Underlying "equity" process parameters		Correlation between the risky underlying and the short-rate
Short rate	a	sigma	Drift	Volatility	Correlation
0.0032	0.15	0.0503	0.00%	25.00%	0.00%

Source: Own calculations

The market consistent price of the MRRG's under different yield curve shapes is shown in Table 14. These results are shown for the case of the 5% p.a. MRRG guarantee and 0% p.a. ACI. This shows that the downward sloping yield curve gives rise to far higher MRRG costs (expressed as a % of the PV of contributions). This is seen in that the 30-year MRRG cost is c.6.0% under a downward sloping yield curve whereas the cost of a 30-year MRRG costs c.1.9% under an upward sloping yield curve. This is consistent with our findings above that lower yield curves increase the market consistent cost the MRRG, all else equal.

Similar effects can be seen when the humped curves are analyzed. If the base yield curve is adjusted to reflect a positively-humped yield curve shape then it can be seen that MRRG's with terms shorter than the point at which the positive hump peaks drop in price. However, MRRG's with longer terms experience increases in price. This can be seen in Table 14 where the 25-year and 30-year MRRG cost increase to c.3.4% and c.3.3% of discounted contributions compared to c.3.3% and c.2.7% of discounted contributions as under the base yield curve. The opposite effect is true for the case of the negatively humped swap rate inputs.

Table 14 Result of various swap rate scenarios on the MRRG price for various terms

Discounted value of the contributions still to be paid	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	16595	27108	33415	37177	39432	40776
Flat yield curve	16553	27299	33851	37762	40084	41449
Downward sloping yield curve	16970	28533	35984	40760	43847	45839
Upward sloping yield curve	16721	27613	34191	38011	40188	41403
Positively humped yield curve	16447	26686	32634	36139	38289	39646
Negatively humped yield curve	16656	27898	35077	39416	41899	43239

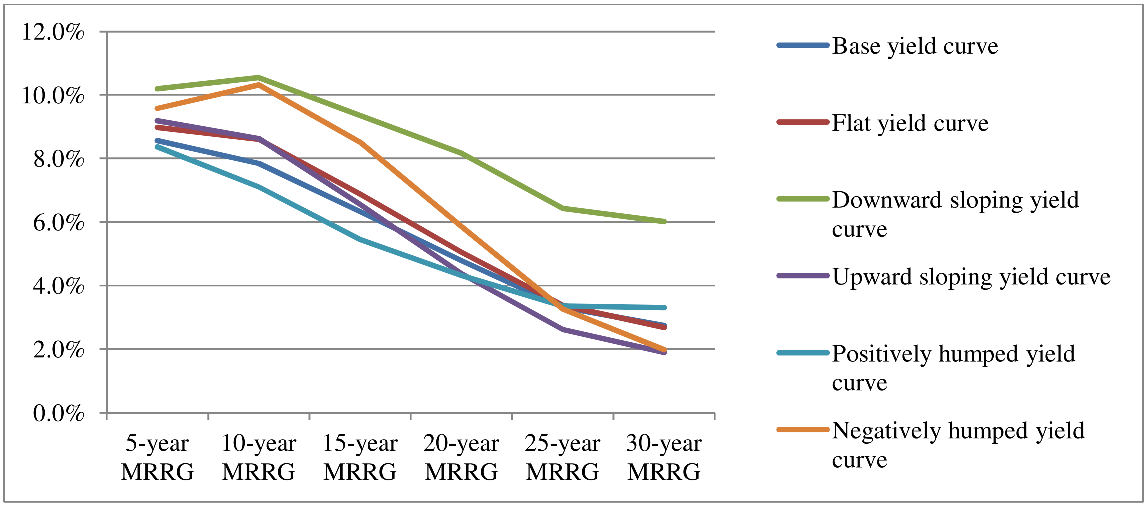
Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	1421	2126	2114	1780	1306	1118
Flat yield curve	1486	2349	2331	1907	1352	1112
Downward sloping yield curve	1729	3009	3364	3327	2820	2755
Upward sloping yield curve	1536	2383	2239	1664	1052	783
Positively humped yield curve	1376	1895	1778	1559	1283	1310
Negatively humped yield curve	1596	2879	2986	2310	1366	855

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	8.6%	7.8%	6.3%	4.8%	3.3%	2.7%
Flat yield curve	9.0%	8.6%	6.9%	5.1%	3.4%	2.7%
Downward sloping yield curve	10.2%	10.5%	9.3%	8.2%	6.4%	6.0%
Upward sloping yield curve	9.2%	8.6%	6.5%	4.4%	2.6%	1.9%
Positively humped yield curve	8.4%	7.1%	5.4%	4.3%	3.4%	3.3%
Negatively humped yield curve	9.6%	10.3%	8.5%	5.9%	3.3%	2.0%

Source: Own calculations

A graphic illustration of the calculated cost of the MRRG price of different terms under the various yield curve shapes is shown in Figure 17 below. It can be seen that a negatively sloping yield curve environment has the greatest increase in the price of the MRRG.

Figure 17 Illustration of a MRRG price as a percentage of discounted contributions under different yield curves



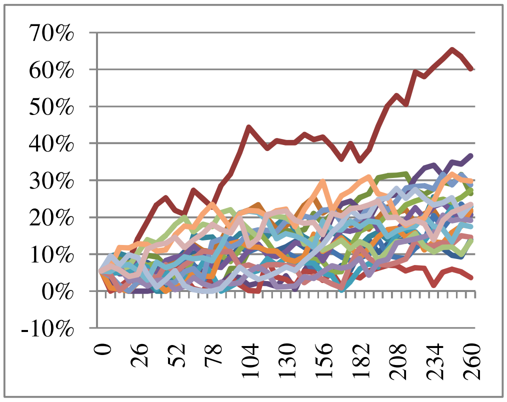
Source: Own calculations

4.5.1.4 Variation in the modeled short rate volatility

4.5.1.4.1 Changes in the "*a*" parameter in the Hull-White model

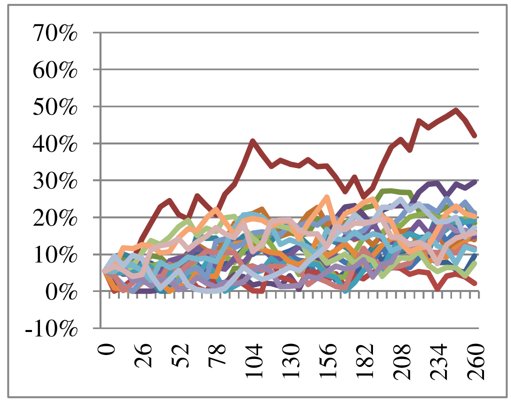
The purpose of the analysis is to better understand how changes in the variability of the yield curve simulations affect the MRRG price. To do this I analyse how changes in the Hull White "*a*" reversion parameter affect our pricing. Figure 18 to Figure 21 show the effect of the choice of "*a*" reversion parameter in the Hull-White model has on the first 20 simulation outcomes. The same random seed was used in all cases. Under a choice of "*a*" = 0.01 the reversion is very weak and simulations spread widely. This can be compared to the case where "*a*" is set to 0.25 and the resulting simulations are far more clustered.

Figure 18: Short rate simulations under Hull-White "a" = 0.01



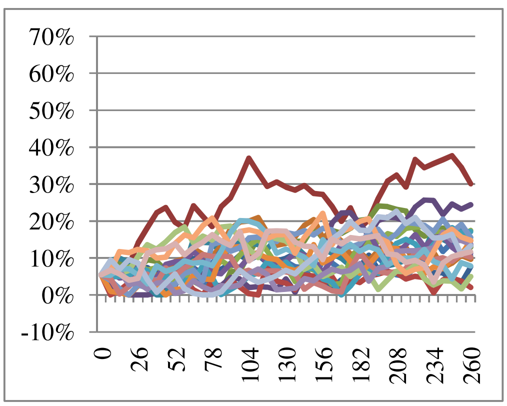
Source: Own calculations

Figure 19: Short rate simulations under Hull-White "a" = 0.075



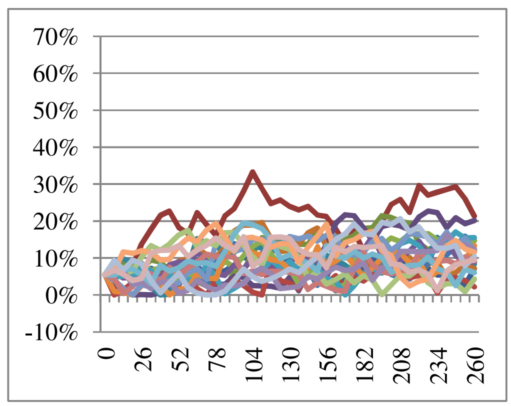
Source: Own calculations

Figure 20 Short rate simulations under Hull-White "a" = 0.15



Source: Own calculations

Figure 21: Short rate simulations under Hull-White "a" = 0.25



Source: Own calculations

The impact of changes in the "*a*" Hull-White parameter are shown for the case of the 5% p.a. MRRG in Table 15. It can be seen is that as the "*a*" parameter drops so too does the price of the MRRG. This is explained by a lower "*a*" parameter indicating a lower reversion strength and as a result forward rate simulations which tend to drift higher.

Table 15: Effect of the choice of "a" reversion parameter in the Hull-White model on MRRG pricing

Change in Hull White "a" parameter	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
a = 0.01	16461	26011	30370	31868	32230	32289
a = 0.075	16530	26611	32087	34863	36208	36831
a = 0.15	16595	27108	33415	37177	39432	40776

a = 0.25	16663	27562	34529	39042	42003	43948

Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
a = 0.01	1286	1209	420	59	7	0
a = 0.075	1356	1689	1214	667	287	139
a = 0.15	1421	2126	2114	1780	1306	1118
a = 0.25	1492	2569	3024	3133	2855	2955

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
a = 0.01	7.8%	4.6%	1.4%	0.2%	0.0%	0.0%
a = 0.075	8.2%	6.3%	3.8%	1.9%	0.8%	0.4%
a = 0.15	8.6%	7.8%	6.3%	4.8%	3.3%	2.7%
a = 0.25	9.0%	9.3%	8.8%	8.0%	6.8%	6.7%

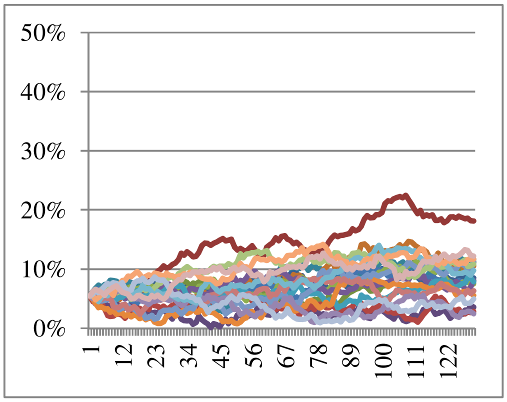
Source: Own calculations

What is evident is that the choice of “a” has a significant effect on the ultimate price of the MRRG thus small arising from manual calibration processes are likely to give rise to spurious accuracy. Demonstration of the process to project and test a hedging program relies on our ability to accurately measure differences in MRRG prices under changing input assumptions and therefore I have chosen to keep the stochastic volatility parameters constant in forecasting in the later sections.

4.5.1.4.2 Changes in “sigma” parameter in the Hull-White model

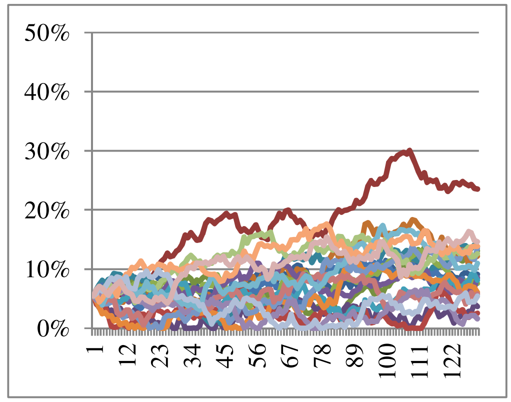
The “sigma” parameter in the Hull-White model drives the volatility of the deviations in the short rate process. The lower the “sigma” term the smaller these deviations are. This effect can be seen for different choices of “sigma” in Figure 18 to Figure 21. What this shows us is that for higher “sigma” inputs the higher the short rate process simulations are on average.

Figure 22: Short rate simulations under Hull-White "sigma" = 0.0251



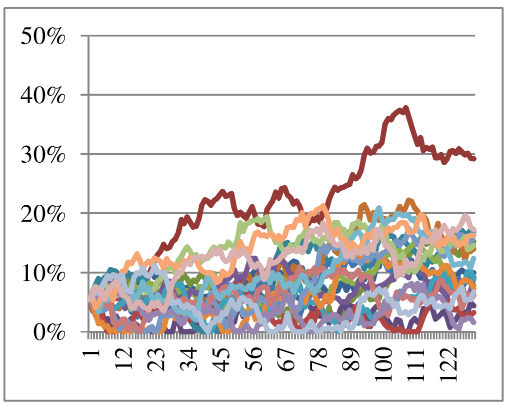
Source: Own calculations

Figure 23: Short rate simulations under Hull-White "sigma" = 0.0377



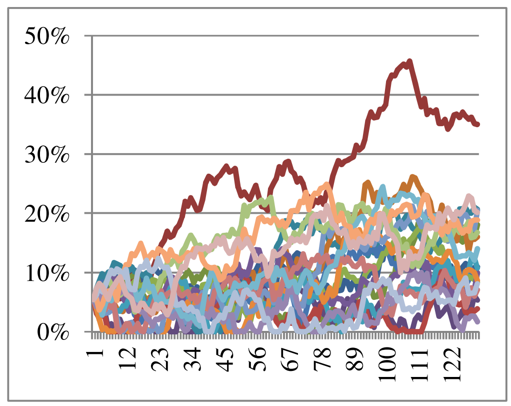
Source: Own calculations

Figure 24 Short rate simulations under Hull-White "sigma" = 0.0503



Source: Own calculations

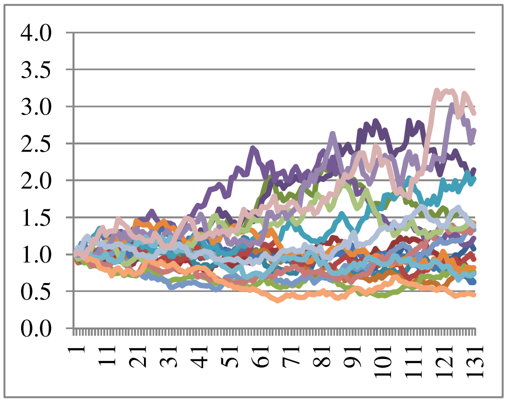
Figure 25: Short rate simulations under Hull-White "sigma" = 0.0629



Source: Own calculations

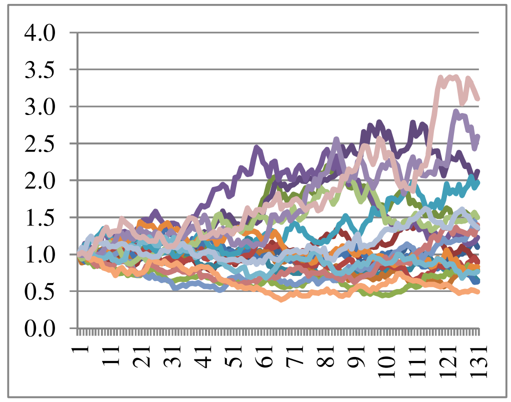
What is also worth noting is that choices of "sigma" also directly affect the equity process simulations in the BSHW model. This effect can be seen in Figure 26 to Figure 29 where it is evident that the higher the choice of "sigma" the greater spread of simulations in the equity index price process. On closer inspection of these figures it is also apparent that a higher choice of "sigma" leads to a slightly higher equity index level all else equal.

Figure 26: Equity simulations under Hull-White "sigma" = 0.0251



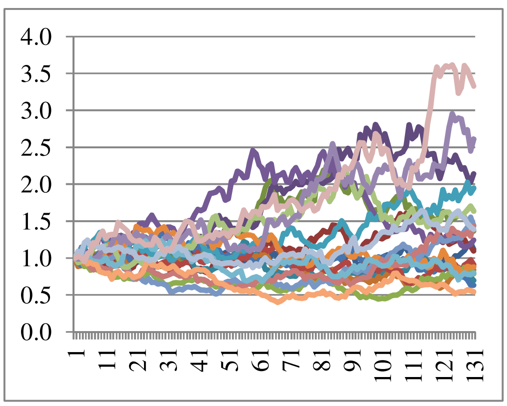
Source: Own calculations

Figure 27: Equity simulations under Hull-White "sigma" = 0.0377



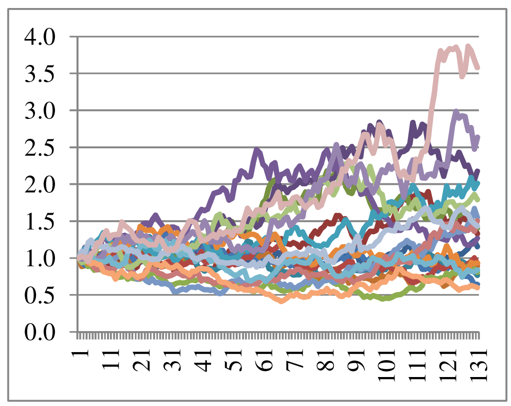
Source: Own calculations

Figure 28 Equity simulations under Hull-White "sigma" = 0.0503



Source: Own calculations

Figure 29: Equity simulations under Hull-White "sigma" = 0.0629



Source: Own calculations

Under higher "sigma" inputs the MRRG discounting would be under a higher yield curve, on average, which would reduce the discounted price as well as make the guarantee level bite on fewer occasions due to the generally higher equity index level relative to the fixed rand guarantee. These two effects would both work to reduce the MRRG price.

This effect is quantified in Table 16. The effect of increased Hull-White process volatility, as represented by higher "sigma" inputs has a significant effect on the MRRG price.

Table 16: Effect of the choice of "sigma" parameter in the Hull-White model on MRRG pricing

Change in Hull White "sigma" parameter	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
sigma = 0.0251	16883	28508	36463	41988	45880	48630
sigma = 0.0377	16792	28001	35266	39989	43088	45118
sigma = 0.0503	16595	27109	33414	37176	39431	40776
sigma = 0.0629	16335	26051	31388	34278	35844	36685

Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
sigma = 0.0251	1693	3399	4438	5328	5645	6181
sigma = 0.0377	1590	2826	3303	3402	3025	3016
sigma = 0.0503	1421	2125	2113	1780	1306	1119
sigma = 0.0629	1234	1552	1255	859	513	346

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
sigma = 0.0251	10.0%	11.9%	12.2%	12.7%	12.3%	12.7%
sigma = 0.0377	9.5%	10.1%	9.4%	8.5%	7.0%	6.7%
sigma = 0.0503	8.6%	7.8%	6.3%	4.8%	3.3%	2.7%
sigma = 0.0629	7.6%	6.0%	4.0%	2.5%	1.4%	0.9%

Source: Own calculations

4.5.1.5 Equity market input parameters

4.5.1.5.1 A range of equity market volatility inputs

The assessment of the relationship between equity volatility assumptions and the MRRG price is shown in Table 17. For the case of the 5% p.a. MRRG the price calculations suggests that increased equity volatility leads to sharp increases in the MRRG price. Considering the case of 35%p.a. equity volatility the price of a 5-year MRRG is c.2.5x greater than under a 25%p.a. equity volatility assumption. As the duration of the MRRG increases the sensitivity to equity volatility assumptions also increases. By comparison the price of a 30-year MRRG increases c.5x when equity volatilities are assumed to be 35% p.a. rather than 25% p.a. Note that changes in equity volatility has no effect on the discounted value of the MRRG contributions, as expected.

Table 17: Demonstration of the effect of increases in the equity volatility on the price of the MRRG

Change in Equity	5-year	10-year	15-year	20-year	25-year	30-year

Volatility parameter	MRRG	MRRG	MRRG	MRRG	MRRG	MRRG
Equity volatility = 15%	16595	27109	33414	37176	39431	40776
Equity volatility = 25%	16595	27109	33414	37176	39431	40776
Equity volatility = 35%	16595	27109	33414	37176	39431	40776

Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Equity volatility = 15%	106	121	64	27	2	1
Equity volatility = 25%	847	859	483	328	136	93
Equity volatility = 35%	2190	2299	1649	1104	644	451

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Equity volatility = 15%	0.6%	0.4%	0.2%	0.1%	0.0%	0.0%
Equity volatility = 25%	5.1%	3.2%	1.4%	0.9%	0.3%	0.2%
Equity volatility = 35%	13.2%	8.5%	4.9%	3.0%	1.6%	1.1%

Source: Own calculations

This suggests that MRRG's are highly sensitive to equity volatility assumptions and poses an interesting predicament for insurers as to how to price the long-dated MRRGs being sold. This results from term limitations in the traded equity market in South Africa (as well as to a lesser extent in other international markets) and therefore medium- and long-term equity option implied volatilities are not observable. In practice, South African life insurers follow the approach to mark to model.

4.5.1.5.2 Variation in the correlation between equities and interest rates

The effect of the correlation between equity returns and the short rate is shown in Table 18. What this shows is that the negative correlation between equity returns and the short rate process lowers the cost of the MRRG.

This suggests that if the underlying assets, over which the MRRG is written, tend to rise in value as interest rates fall then this effect will dampen, or even offset, the effect of falling interest rates. Therefore, all else being equal, writing MRRGs over funds with negative correlations with interest rate movements will be cheaper.

Table 18: Calculations of the effect of correlations between short rates and equity processes on the MRRG price

Correlation between equity returns and the short rate	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Correlation of -20%	16595	27109	33414	37176	39431	40776
Correlation of 0%	16595	27108	33415	37177	39432	40776

Correlation of +20%	16597	27116	33425	37189	39444	40788

Minimum Rate of Return Guarantee price	**5-year MRRG**	**10-year MRRG**	**15-year MRRG**	**20-year MRRG**	**25-year MRRG**	**30-year MRRG**
Correlation of -20%	1365	1940	1840	1544	1080	901
Correlation of 0%	1421	2126	2114	1780	1306	1118
Correlation of +20%	1468	2333	2367	1998	1544	1313

Guarantee price as a percentage of the PV of total contributions	**5-year MRRG**	**10-year MRRG**	**15-year MRRG**	**20-year MRRG**	**25-year MRRG**	**30-year MRRG**
Correlation of -20%	8.2%	7.2%	5.5%	4.2%	2.7%	2.2%
Correlation of 0%	8.6%	7.8%	6.3%	4.8%	3.3%	2.7%
Correlation of +20%	8.8%	8.6%	7.1%	5.4%	3.9%	3.2%

Source: Own calculations

4.5.1.5.3 Variation in moneyness due to instantaneous changes in the underlying equity index

Up until this point the MRRG pricing and sensitivity analysis shown has only considered new products. Therefore there has been no fund value at the start of the contract and all MRRG pricing related to future expected cash flows under the product. This section introduces an initial fund balance and considers the pricing of the MRRG part way through the products life on the insurers' books. This is of interest to life insurance companies as many of them have large in-force books of business on which MRRG's have existed for some years.

This analysis also serves as an indication of how the market-consistent MRRG price changes over the time in-force as equity markets and yield curves fluctuate. In this section the moneyness of the MRRG is used as an additional initial input. The moneyness represents the percentage by which the current actual fund balance exceeds the guaranteed fund balance at a point in time.

As in Table 19, I have set the initial fund value to 20 times the initial quarterly contribution. This equates to five years worth of contributions having been invested at the date the MRRG price was calculated.

Table 19: BSHW and initial moneyness level assumptions

Standard Deviation Cont. Comp.	Hull-White Parameters - alpha	Hull-White Parameters - sigma	Underlying "equity" process parameters - Drift	Underlying "equity" process parameters - Volatility	Correlation between the risky underlying and the short-rate	Initial fund value as multiple of initial quarterly contribution
0.003168	0.15	0.05	0.00	0.25	0.00	2000%

Source: Own calculations

The results of this analysis are shown in Table 20. As expected, moneyness does not have an effect on the discounted value of the future expected contributions of the MRRG product. However, there is a strong inverse relationship between the MRRG price and initial moneyness. For example, in the case of a 5-year MRRG, if the initial moneyness is 200% then the price is c.0.9% of discounted contributions but if the moneyness is 50% then the price rises to c.37% of discounted contributions. This makes sense as if the fund value is half the guarantee value then the charge required over the next five years needs to represent the high chance of the fund value still being in shortfall of its guarantee at maturity. As a result the price reflects this higher required charge. The opposite is true in that if the fund value far exceeds the guarantee level to date then the fair price for the MRRG in the coming five years should reflect the reduced likelihood of the guarantee paying out at maturity.

The sensitivity to moneyness decreases as the term of the MRRG increases. This is explained by the fact that, the initial moneyness level becomes less relevant as the proportion of future expected cash flows increases. For example, even if the fund value is only half that of the guarantee value (i.e. moneyness is 50%) then the price of a 30-year MRRG only increases to c.1.2% of discounted contributions as opposed to c.1.0% under a 100% moneyness.

Table 20: Demonstration of the sensitivity of the MRRG price to changes in the initial moneyness

Change in initial moneyness	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Moneyness = 200%	16595	27109	33414	37176	39431	40776
Moneyness = 150%	16595	27109	33414	37176	39431	40776
Moneyness = 100%	16595	27109	33414	37176	39431	40776
Moneyness = 75%	16595	27109	33414	37176	39431	40776
Moneyness = 50%	16595	27109	33414	37176	39431	40776

Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Moneyness = 200%	154	519	523	532	286	254
Moneyness = 150%	556	959	812	711	386	314
Moneyness = 100%	1953	1832	1320	950	539	391
Moneyness = 75%	3539	2541	1703	1126	639	441
Moneyness = 50%	6161	3570	2214	1366	769	506

Guarantee price as a percentage of the PV of total contributions	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Moneyness = 200%	0.9%	1.9%	1.6%	1.4%	0.7%	0.6%
Moneyness = 150%	3.3%	3.5%	2.4%	1.9%	1.0%	0.8%
Moneyness = 100%	11.8%	6.8%	3.9%	2.6%	1.4%	1.0%
Moneyness = 75%	21.3%	9.4%	5.1%	3.0%	1.6%	1.1%
Moneyness = 50%	37.1%	13.2%	6.6%	3.7%	1.9%	1.2%

Source: Own calculations

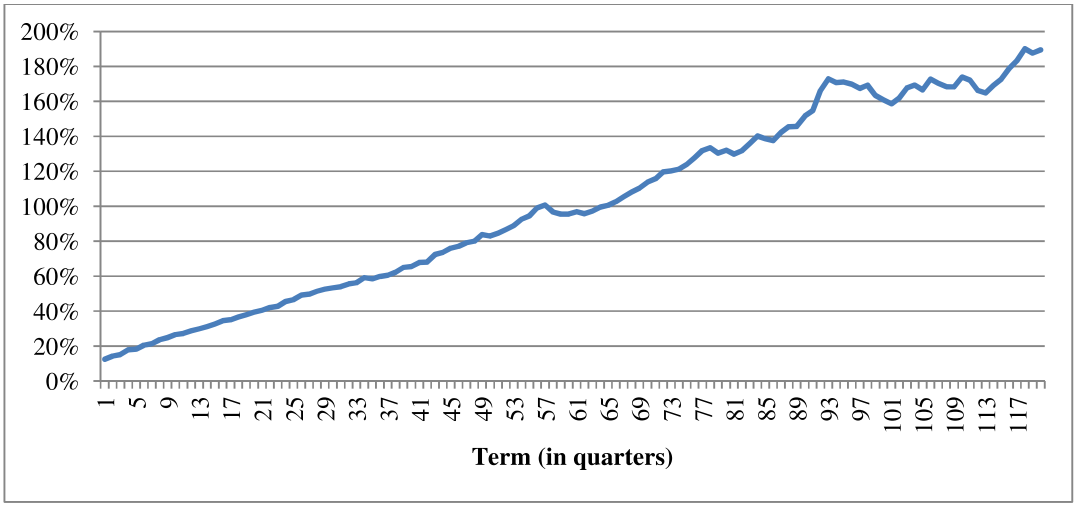
What this analysis suggests is that moneyness levels have greater effect on the MRRG price the closer one gets to expiry. This suggests that hedging of the underlying fund units becomes more important the closer one gets to the maturity date of the MRRG.

4.5.2 Showing the effect of stochastic interest rates on the volatility of the equity fund value at maturity

I model the volatility of the simulated fund value at maturity under the BSHW model to show the effect of changes in the input assumptions. Pelsser and Schrager state that since the mean of the fund value is independent of the parameters of the model that all volatility in the fund value must be attributable to the volatility of the Black Scholes lognormal approximation for the equity process (Schrager and Pelsser 2004). Pelsser and Schrager also state that the effect of stochastic interest rates on the fund value volatility increases with maturity and they call this effect the convexity correction effect (Schrager and Pelsser 2004).

The analysis indicates that there is an increased volatility of fund value at maturity as maturity term increases. I illustrate this in Figure 30 where the standard deviation of the equity fund value simulations divided by the mean of the equity fund value simulations increases at an increasing rate with maturity.

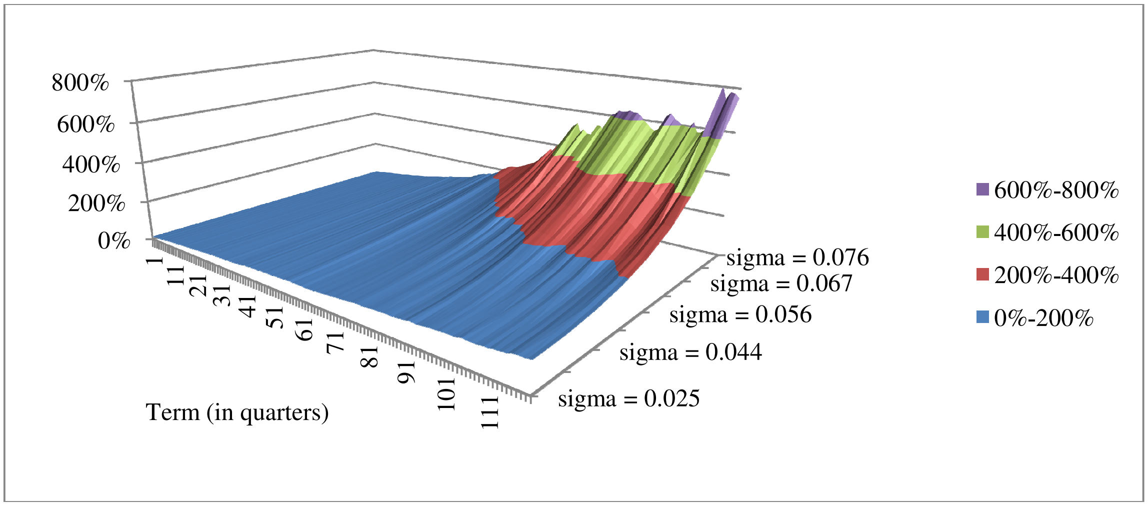
Figure 30: Standard deviation of the equity fund value simulations divided by the mean of the equity fund value simulations in the case of Hull-White sigma = 0.0503



Source: Own calculations

The same effect can be seen in Figure 31, but for changes in the Hull-White interest rate volatility parameter "sigma". Here, changes in "sigma", while holding the reversion parameter "*a*" constant, show that there is a positive relationship between stochastic short rates and the volatility of the equity fund value at maturity. i.e. the higher the short rate volatility, the higher the volatility of the equity fund value at maturity. The impact of this is that the choice of the stochastic short rate model and the required parameter choices impacts the equity fund value simulation output and therefore has a direct impact on the ultimate price of the MRRG.

Figure 31: Standard deviation of the equity fund value simulations divided by the mean of the equity fund value simulations for various choices of Hull-White sigma parameters



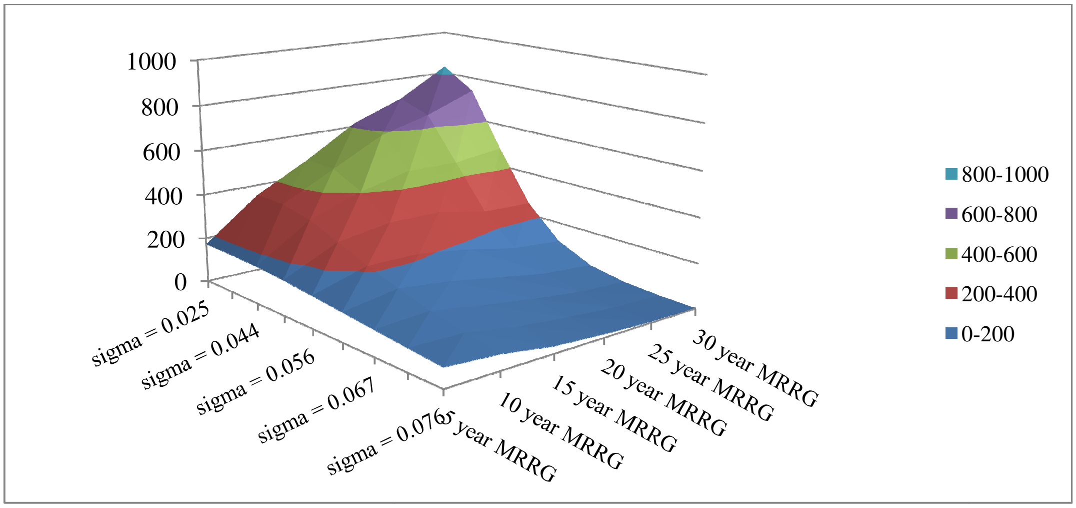
Source: Own calculations

Pelsser and Schrager attribute this maturity “time effect” to the increasing number of cross correlations (at a rate of T^2) between the contribution dates at longer maturity terms (Schrager and Pelsser 2004).

This finding leads Pelsser and Schrager to further investigate comparisons of alternative short rate models, for example, the Libor Market Model (LMM). They reach the conclusion that the effect of stochastic interest rates, in the case of long-term recurring-contribution MRRGs, is in fact non-negligible (Schrager and Pelsser 2004). The implication is that the Hull-White model representation of stochastic volatility differs from other short rate models and this would suggest that the choice of short rate model has a material effect on the price of the MRRG.

The effect on the MRRG prices can be seen in Figure 32 where the lower the sigma input the higher the MRRG price, all else equal.

Figure 32: MRRG prices under various sigma parameter choices and maturity terms



Source: Own calculations

5 Practical hedging of a MRRG

5.1 Current industry practice for hedging MRRG

In 2007 the Society of Actuaries' Committee on Finance Research, along with the ALM Institute, conducted a survey on the various approaches and practical implementation techniques to manage variable annuities. This survey considered the hedging practice of 50 of the largest North American life insurance companies (Gilbert, Ravindran et al. 2007).

Responses were received from 20 variable annuity guarantee writers in North America. 17 out of the 20 respondents were issuing Guaranteed Maturity Accumulation Benefits (comparable to MRRG benefits). While six of the 17 were not hedging, nine were using equity futures, seven were using equity options, seven were using interest-rate futures and swap but zero were using interest-rate options (Gilbert, Ravindran et al. 2007).

This shows us that a number of major life insurance companies are managing variable annuity guarantee risk via some form of hedging program. It appears that the most common hedging techniques deal with equity market movements. These changes being in the underlying equity market price as well as in the equity volatility. Interest rate hedging appears to only be used to a lesser extent – with interest rate volatility risk not seeming to be managed in many instances.

In this Chapter, I introduce typical hedging approaches for the four key market input sensitivities. Thereafter, practical examples of each of these hedging approaches is shown and the limitations of such approaches discussed in the context of our BSHW model.

5.2 Introduction to typical equity and interest rate risk hedging techniques

5.2.1 Typical approaches to manage changes in swap rates

The hedging of interest rate derivatives begins with understanding the sensitivity to movements in the yield curve. These sensitivity effects are split into first partial derivative effects, called deltas, second partial derivative effects, called gammas, etc. Hull describes the simplest delta sensitivity as the impact of a small (typically one-basis-point) parallel shift in the entire yield curve (Hull 2003). The preferred delta sensitivity approach, according to Hull (Hull 2003), is to calculate the impacts of small changes in each of the underlying instruments used to contrast the yield curve. Hull argues that in practice the only way the yield curve can change is via changes in one or more of the quoted instruments. Therefore focussing sensitivity analysis to exposures arising from changes in the underlying instruments is preferred. In our case this means hedging changes in the eight swap rate inputs.

Hull also state that the number of Gamma sensitivities increases rapidly with the increasing number of input instruments. This is because Gamma sensitivities pick up the cross-gamma sensitivities between changes at two different points along the yield curve, for example Hull states that in the case of ten input instruments there would be 55 different gamma measures (Hull 2003). These cross-gamma sensitivities would typically be ignored and only the second partial derivative of each input instruments would be considered.

5.2.2 Outlining a program to manage changes in swap rates

5.2.2.1 Common hedging approaches use PV01 stresses of the underlying swap rates

The common approach to test interest rate sensitivity is to stress the input swap rates independently and then to assess the change in the price of the option post the stress. Table 21 illustrates the 10bps stresses made to each of the input swap rates independently and simultaneously.

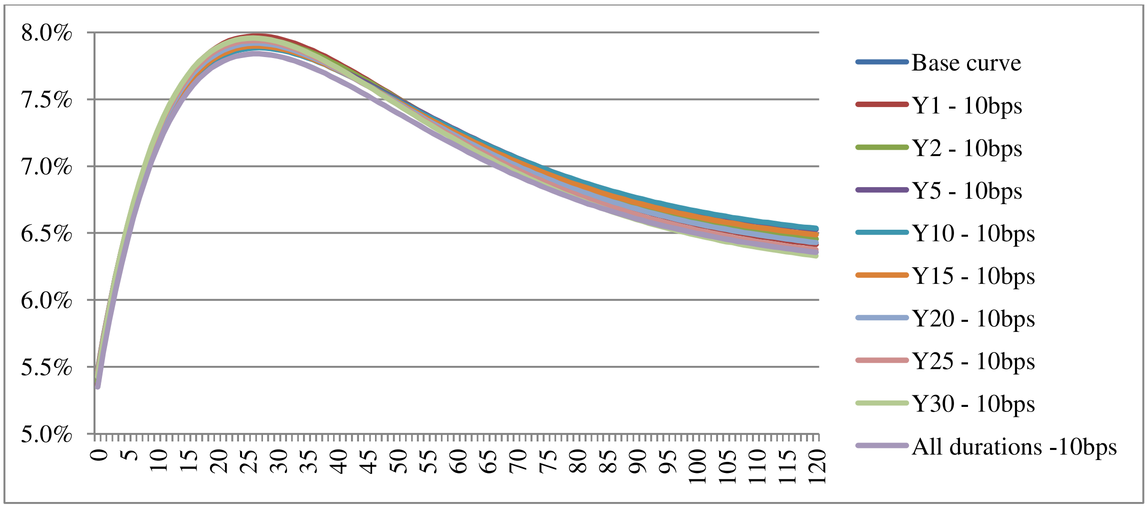
Table 21: 10bps stresses to each input swap rate comprising our base yield curve

Swap rate inputs	1 Year Swap	2 Year Swap	5 Year Swap	10 Year Swap	15 Year Swap	20 Year Swap	25 Year Swap	30 Year Swap
Base yield curve	5.97	6.27	7.06	7.49	7.46	7.36	7.24	7.13
1 Year Swap -10bps	5.87	6.27	7.06	7.49	7.46	7.36	7.24	7.13
2 Year Swap -10bps	5.97	6.17	7.06	7.49	7.46	7.36	7.24	7.13
5 Year Swap -10bps	5.97	6.27	6.96	7.49	7.46	7.36	7.24	7.13
10 Year Swap -10bps	5.97	6.27	7.06	7.39	7.46	7.36	7.24	7.13
15 Year Swap -10bps	5.97	6.27	7.06	7.49	7.36	7.36	7.24	7.13
20 Year Swap -10bps	5.97	6.27	7.06	7.49	7.46	7.26	7.24	7.13
25 Year Swap -10bps	5.97	6.27	7.06	7.49	7.46	7.36	7.14	7.13
30 Year Swap -10bps	5.97	6.27	7.06	7.49	7.46	7.36	7.24	7.03
All durations -10bps	5.87	6.17	6.96	7.39	7.36	7.26	7.14	7.03

Source: Own calculations

Figure 33 shows the resulting Nelson-Siegel fitted curves which are generated under each of these independent and simultaneous swap rates stresses. As can be seen the fitted curves remain smooth.

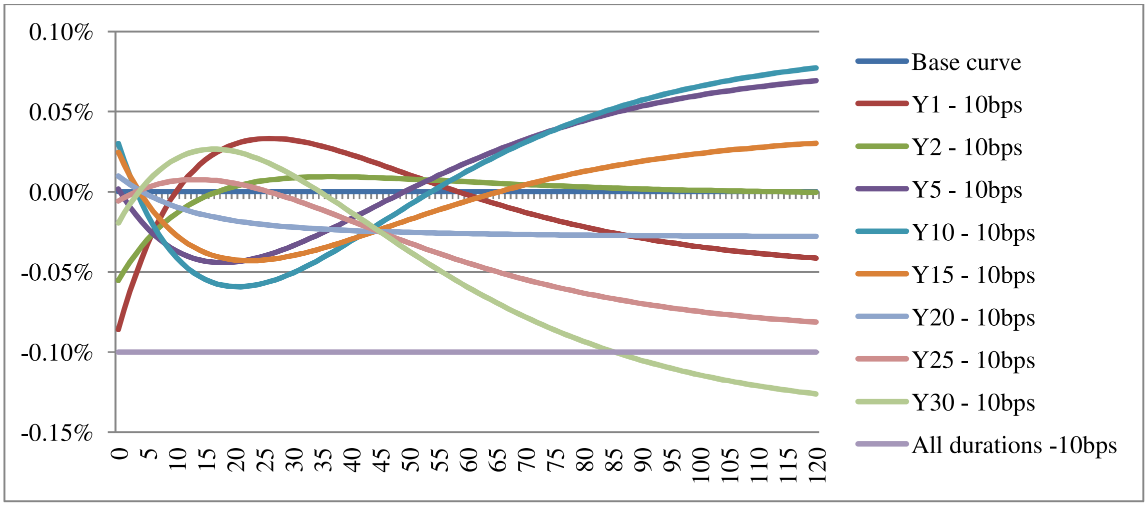
Figure 33: Illustration of resulting Nelson-Siegel yield curve fits under each 10bps yield curve stress



Source: Own calculations

On closer inspection, Figure 34 shows that the changes in the fitted Nelson-Sigel yield curve do not actually capture the independent stresses of each swap rate that well. This can be seen in that a reduction in one area of the yield curve is offset somewhat by an increase in another region of the curve. This effect can be explained by the limited number of Nelson-Siegel fitting parameters. The only fitted yield curve which closely represents the intended stresses is that of the simultaneous stress of all swap rate inputs.

Figure 34: Illustration of the resulting Nelson-Siegel fitted yield curve changes under each swap rate input stress



Source: Own calculations

The consequence of the limitations of the Nelson-Sigel fitting approach can be seen in the "noisy" results of the changes in the MRRG price. Table 22 shows the calculated price of the MRRG under each swap rate input stress as well as the change in the MRRG relative to the base yield curve price under each stress. What can be seen is that due to the smooth yield curve fit brought about by the Nelson-Siegel yield curve fitting the changes in the MRRG can be counter intuitive. For example, in the case of a 10bps reduction in the 5-year swap rate input the cost of MRRG's of all durations (5-years and above) should increase. However, as seen in Table 22 the cost of the 5-year MRRG does increase but as the duration of the MRRG's priced increases the change in the price starts in increase by a lesser amount and post the 20-year duration the cost starts to decrease.

Table 22: Output of the calculation of the MRRG price under each swap rate input stress

Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	1421	2126	2114	1780	1306	1118
1 Year Swap -10bps	1419	2119	2109	1780	1309	1123
2 Year Swap -10bps	1422	2125	2112	1778	1305	1117
5 Year Swap -10bps	1426	2135	2118	1777	1298	1108
10 Year Swap -10bps	1427	2139	2123	1780	1299	1108
15 Year Swap -10bps	1425	2136	2123	1785	1307	1117

20 Year Swap -10bps	1423	2132	2123	1790	1316	1127
25 Year Swap -10bps	1420	2127	2122	1795	1323	1136
30 Year Swap -10bps	1418	2124	2121	1799	1330	1144
All durations -10bps	1434	2155	2153	1822	1344	1152

Change in Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	0.00	0.00	0.00	0.00	0.00	0.00
1 Year Swap -10bps	-1.78	-7.21	-5.59	-0.77	2.64	4.77
2 Year Swap -10bps	0.79	-1.40	-2.18	-2.00	-1.43	-0.95
5 Year Swap -10bps	5.32	9.08	4.35	-3.49	-7.91	-10.33
10 Year Swap -10bps	6.52	13.11	8.44	-0.84	-6.61	-9.96
15 Year Swap -10bps	4.50	10.31	9.12	4.46	0.90	-1.41
20 Year Swap -10bps	1.71	5.70	8.69	10.06	9.62	8.80
25 Year Swap -10bps	-0.83	1.37	8.02	14.78	17.19	17.80
30 Year Swap -10bps	-3.02	-2.41	7.34	18.71	23.63	25.50
All durations -10bps	13.37	28.70	38.45	41.24	38.19	34.23
Sum of stresses to each swap rate	13.21	28.55	38.19	40.90	38.02	34.23

Source: Own calculations

The same pattern can be seen for the 5-year swap rate decrease stress in Table 23. Here, the percentage changes in the MRRG price are shown. I attribute this dynamic to the limited manner in which the Nelson-Sigel curve fitting approach captures small swap rates stresses. Therefore, caution should be applied when trying to test the MRRG price's sensitivity to small swap rate input changes as the limitations of yield curve fitting approaches has an influence of the validity of results.

Table 23 shows us that under a 10bps reduction in all swap rate inputs the price of the MRRG increases by c.1% in the case of a 5-year MRRG but as much as c.3% in the case of a 30-year MRRG. Thus it is apparent that MRRG benefits with longer maturity terms are more sensitive to changes in the input swap rates. This suggests that one should place greater emphasis on managing interest rate risks in the longer term MRRG maturities than under short-term contracts.

Table 23: Percentage changes in the MRRG price under swap rate input stresses

Change in MRRG price under each swap rate input stress	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Base yield curve	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

1 Year Swap -10bps	-0.13%	-0.34%	-0.26%	-0.04%	0.20%	0.43%
2 Year Swap -10bps	0.06%	-0.07%	-0.10%	-0.11%	-0.11%	-0.08%
5 Year Swap -10bps	0.37%	0.43%	0.21%	-0.20%	-0.61%	-0.92%
10 Year Swap -10bps	0.46%	0.62%	0.40%	-0.05%	-0.51%	-0.89%
15 Year Swap -10bps	0.32%	0.48%	0.43%	0.25%	0.07%	-0.13%
20 Year Swap -10bps	0.12%	0.27%	0.41%	0.56%	0.74%	0.79%
25 Year Swap -10bps	-0.06%	0.06%	0.38%	0.83%	1.32%	1.59%
30 Year Swap -10bps	-0.21%	-0.11%	0.35%	1.05%	1.81%	2.28%
All durations -10bps	0.94%	1.35%	1.82%	2.32%	2.92%	3.06%

Source: Own calculations

5.2.2.2 Limitations of stressing swap rate inputs to generate PV01's under Hull-White

Two limitations with regards to swap rate hedging in the BSHW setting are apparent. Firstly, due to the limited number of parameters used in fitting smooth Nelson-Siegel yield curve the stress tests on the individual swap rates cannot be isolated from broader moves in the yield curve.

Therefore, practically, the swap rate sensitivities for individual swap rates should be used with caution. What this means is that under BSHW the inability to accurately unpack the sensitivity of the MRRG price to individual swap rates means that hedging would need to make some form of assumption as to the term structure sensitivity of the MRRG.

Secondly, the effect of positive convexity, seen in Figure 15, suggests that either hedging instruments with similar positive interest rate convexity should be purchased or continual delta hedging of the interest rate exposures would be required.

5.2.2.3 Demonstration of a simple interest rate hedging program

A practical hedging approach to manage swap rate sensitivity would be to buy or sell interest rate sensitive instruments of appropriate duration. Table 22 shows that a 10bps drop in interest rates at all durations would add between R13.37 and R41.24 to the MRRG price depending on the MRRG maturity term. A simple approach would be to purchase zero coupon bonds which offset the increase in the MRRG price under a 10bps parallel decrease in all swap rates.

Table 24 shows the calculated rand per point sensitivity of a basic R1million nominal value zero coupon bond contract of maturity 5, 10, 15, 20, 25 and 30 years.

Table 24: Rand per point sensitivity of a zero coupon bond contract of various maturities.

PV01 of Zero Coupon Bond Contract (R 1m notional)					
5-year Zero Coupon Bond	**10-year Zero Coupon Bond**	**15-year Zero Coupon Bond**	**20-year Zero Coupon Bond**	**25-year Zero Coupon Bond**	**30-year Zero Coupon Bond**
-332	-452	-474	-451	-407	-355

Source: Own calculations

Therefore, selling 0.004, 0.0063, 0.008, 0.0091, 0.0093 and 0.0096 Zero Coupon Bond contracts of each of the 5, 10, 15, 20, 25 and 30 year terms respectively will provide some protection to offset the change in the MRRG price as small changes in swap rates occur (Table 25).

Table 25: Calculation of the number of Zero Coupon Bond contracts required to hedge parallel moves in swap rates

	5-year Zero Coupon Bond	**10-year Zero Coupon Bond**	**15-year Zero Coupon Bond**	**20-year Zero Coupon Bond**	**25-year Zero Coupon Bond**	**30-year Zero Coupon Bond**
PV01 of Zero Coupon Bond Contract	-332	-452	-474	-451	-407	-355
PV10 of MRRG	13.21	28.55	38.19	40.90	38.02	34.23
PV01 of MRRG	1.32	2.85	3.82	4.09	3.80	3.42
Number of Zero Coupon Bond Contracts to hedge	-0.0040	-0.0063	-0.0080	-0.0091	-0.0093	-0.0096

Source: Own calculations

Due to the limitations of the BSHW model being used as well as the fact that I are only constructing a hedge against the first derivative with respect to changes in the swap rates the hedge will not be perfect.

5.2.3 Outlining a practical approach to manage changes in interest rate volatility inputs

5.2.3.1 Estimating the sensitivity of the MRRG to changes in interest rate volatility

In Section 4.5.1.4 the analysis of the effect of changes in the "*a*" and "sigma" parameters in the Hull-White model on the MRRG's prices was performed. This analysis showed that the higher the volatility of interest rate inputs the lower the resulting price of the MRRG.

5.2.3.2 Limitations of hedging interest rate volatility under Hull White

The BSHW model used has limitations in terms of the numbers of parameters with which users can set the volatility of all future interest rates. This has the drawback that the full term structure of market implied interest rate volatility cannot be captured in the model. A simplified approximation is therefore required, and in doing so the amount of information we can draw from the MRRG interest rate sensitivity stress is reduced.

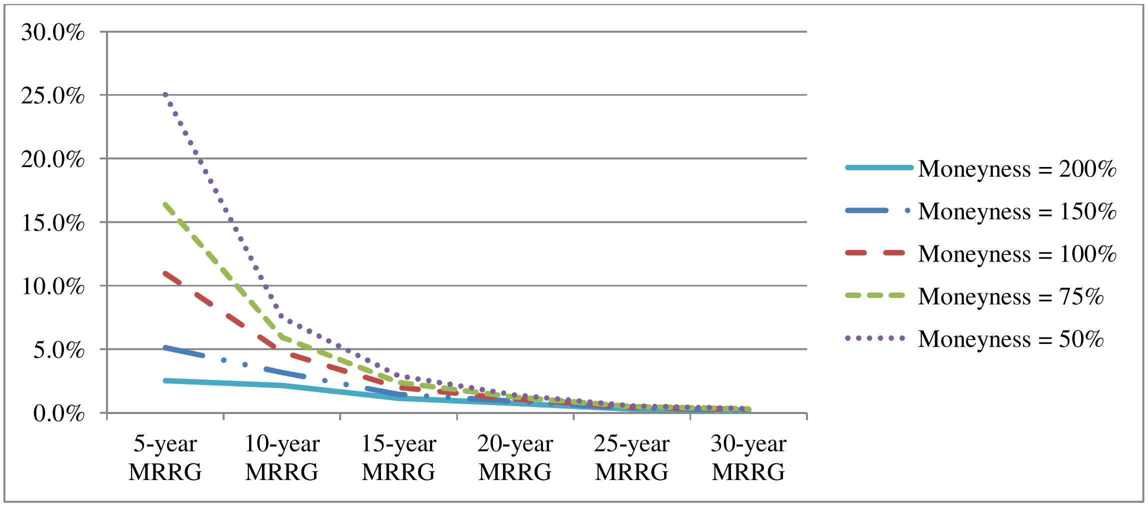
However, there is a more fundamental point with regards to Hull-White interest rate sensitivity. Figure 7 showed that the choice of the reversion parameter "*a*" has significant effect on the T-maturity instantaneous forward in the Hull-White model.

It can also be seen that fixing $a = 0$ sets the zero-coupon bond price volatilities as linear functions of maturity and the instantaneous standard deviations of the forward rates being constant (Hull 2003). The effect of various choices of a on the instantaneous standard deviations for the T-maturity instantaneous forward rate were seen above. The Hull-White model, requires the choice of reversion rate "*a*" such that the future instantaneous standard deviations for the *T*-maturity forward rates are reasonable.

5.2.4 Outlining a practical approach to manage changes in the underlying equity market price

As shown in Section 4.5.1.5.3 there is an inverse relationship between the moneyness of a MRRG and its price. Figure 35 provides an illustration of the prices of the MRRGs of various terms under different moneyness levels. What this shows us is that the shorter term MRRG prices are more sensitive to moneyness level than the longer termed MRRG. This is consistent with the analysis shown in section 4.5.1.5.3 and suggests that management of the underlying equity market price risk becomes more important the closer the MRRG benefit gets to the maturity date.

Figure 35: Inverse relationship between the prices of the MRRG and the moneyness levels



Source: Own calculations

5.2.4.1 Practical approaches to hedging changes in the underlying equity price

Managing the sensitivity to changes in the price of the underlying risky asset over which the MRRG is written can be done by shorting the underlying asset (Fagen 1977). The quantum to which the underlying asset should be shorted is calculated to offset the changes in the price of the MRRG when underlying asset prices change.

Table 26 shows that changes in the price of a MRRG for each 2.5% change in the underlying risky equity index over which the MRRG was written. What this table shows us is, like seen in Table 20, short-term MRRGs are far more sensitive to changes in the underlying asset price. This is evident from the price of the 5-year MRRG increasing sharply from R3934 to R5166 while the 30-year MRRG price only increased from R1588 to R1676 when the underlying equity index price decreased by c.20.4% in both cases.

Table 26: Changes in the price of the MRRG for changes in the underlying equity (shown in 2.5% increments compounded)

Initial moneyness level	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Equity index level stress of 0%	3934	4147	3535	2784	1933	1588

Equity index level stress of -2.5%	4069	4218	3580	2812	1949	1599
Equity index level stress of -4.9%	4205	4289	3624	2840	1964	1609
Equity index level stress of -7.3%	4341	4361	3667	2867	1979	1619
Equity index level stress of -9.6%	4479	4435	3710	2893	1994	1629
Equity index level stress of -11.9%	4615	4507	3753	2918	2008	1639
Equity index level stress of -14.1%	4752	4580	3796	2944	2022	1648
Equity index level stress of -16.2%	4890	4655	3838	2970	2036	1658
Equity index level stress of -18.3%	5027	4729	3880	2995	2050	1667
Equity index level stress of -20.4%	5166	4802	3921	3020	2064	1676

Change in Minimum Rate of Return Guarantee price	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Equity index level stress of 0%	0	0	0	0	0	0
Equity index level stress of -2.5%	135	71	44	28	16	10
Equity index level stress of -4.9%	271	143	88	55	31	21
Equity index level stress of -7.3%	407	215	132	82	46	30
Equity index level stress of -9.6%	545	288	175	108	61	40
Equity index level stress of -11.9%	681	361	218	134	75	50
Equity index level stress of -14.1%	818	434	260	160	89	60
Equity index level stress of -16.2%	956	508	302	186	103	69
Equity index level stress of -18.3%	1093	582	344	211	117	79
Equity index level stress of -20.4%	1232	656	386	236	131	88

Source: Own calculations

To hedge against such changes life insurance comanies should sell/short the appropriate number of equities (or the underlying fund units) to offset this sensitivity. Table 27 shows the short equity (or unit fund) exposure required to hedge against a small decrease in the underlying equity index (or unit fund) price in the case of each MRRG. This short exposure was calculated by taking the increase in the MRRG price under a 2.5% decrease in the underlying equity index and dividing this by -2.5%. An alternative approximation would be to take the exposure implied by the average of a small increase and a small decrease in the underlying equity index (or unit fund) price.

Table 27: Calculations of the short exposure required to hedge against a small decrease in the underlying risky asset

Description of item	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
Equity index level stress of -2.5%	135	71	44	28	16	10
Offsetting change in equity index required	-135	-71	-44	-28	-16	-10
Equity exposure required to achieve offsetting change	-5411	-2851	-1776	-1118	-623	-415

Source: Own calculations

5.2.4.2 Limitations to hedging the sensitivity to underlying equity deltas under BSHW

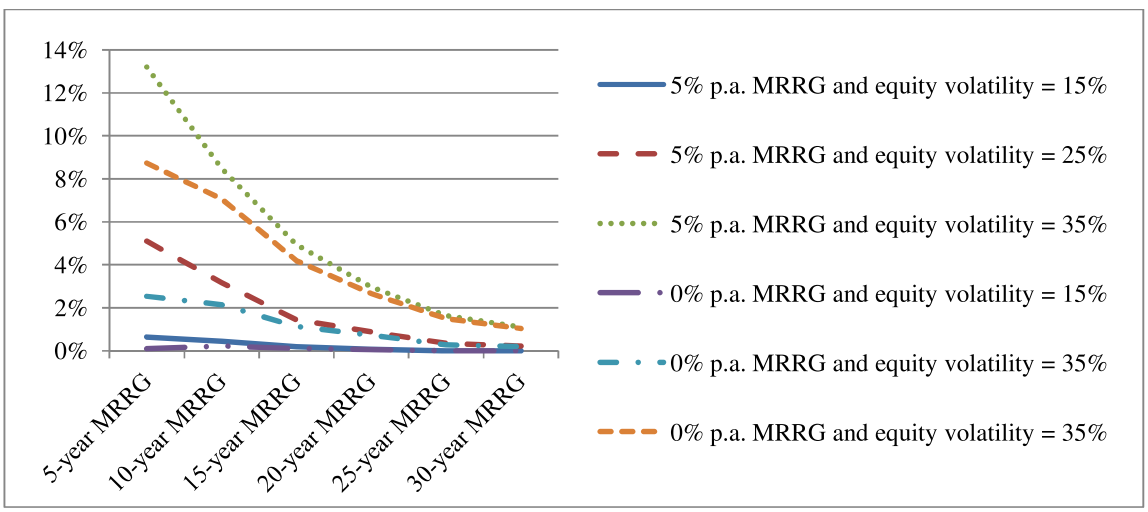
Table 20 showedthe increase in the MRRG price is greater for a 50% reduction in the underlying equity index price than as opposed to the decrease in the underlying equity index increased by 50%. This was seen, for example, in the case of the 5-year MRRG, where a 50% increase in the equity index would reduce the price by R1397 (from R1953 to R556) but a 50% reduction in the equity index sees the price increase by R4208 (from R1953 to R6161). Thus, due to the non-linear sensitivity of changes in the underlying equity fund process there is a need to monitor equity delta exposures and rebalance hedges on an ongoing basis.

5.2.5 Outlining a practical approach to manage changes in the market's price of equity volatility in the future

5.2.5.1 Showing equity volatility sensitivity

Figure 36 shows the effect of different equity volatility inputs on the price of the MRRG. As was seen in Section 4.5.1.5.1 the MRRGs with shorter maturity terms are more sensitive to changes in the equity volatility input parameters.

Figure 36: Illustration of the effect of equity volatility inputs on MRRG prices



Source: Own calculations

5.2.5.2 Limitations to hedging the equity market volatility under BSHW

Table 17 showed the changes in the MRRG price as equity volatility reduced from 25% to 15% as well as increased from 25% to 35%. It is worth noting that it is unlikely that traded equity options of sufficient term can be purchased to protect against these equity volatility changes. It is therefore likely that a practical blend of short-term equity options will be purchased to "generally" hedge against changes in the short-term MRRGs prices but the long term MRRG's will generally be left unhedged. As the MRRG's age and the term to maturity decreases the sensitivity to the underlying equity volatility will increase. Thus as the MRRG's approach maturity the importance of equity volatility sensitivity monitoring will increase. Thus a practical compromise may be to consider hedging the equity volatility once the option has less than 5 years to run until maturity, say.

6 Quantifying and projecting the impact of a specific dynamic hedging program

6.1 Real-world simulations are required to forecast the evolution of the MRRG and the hedging instruments

In Chapter 5 the sensitivities of a range of MRRGs to changes in economic assumption inputs was shown. However, these sensitivities were all at an instantaneous point in time. Projecting potential sensitivities over time requires projections of how the various economic input variables evolve over time. To do this a credible real-world evolution model is required. This chapter outlines an approach to implement a real-world evolution model that I can quantify and project the impact of a dynamic hedging program.

6.2 Introducing semi-parametric yield curve evolution approaches

In this chapter the semi-parametric approaches to evolving the yield curve in the real-world measure are applied. The methods of Rebonato, Mahal, Joshi, Bucholz and Nyholm (Rebonato, Mahal et al. 2005) are outlined and later applied in the context of the South African market in recent years. According to Nyholm and Rebonato, the application of the methods suggested allows for the coherent modelling of the cash flows from assets and liabilities over long time periods in a consistent manner (Nyholm and Rebonato 2007). They also state that few models produce multi-factor evolutions of the entire yield curve under the real-world measure and over forecasting horizons relevant for long-term financial decision making as typically seen in life insurance (Nyholm and Rebonato 2007). Throughout this dissertation the Rebonato, Mahal, Joshi, Bucholz and Nyholm method will be abbreviated as RMJBN.

The RMJBN is intuitive in that the model is based on semi-parametric approach which re-samples to capture historic changes in yields to generate a yield curve evolution. The only 'structural' feature (as is termed in Nyholm and Rebonato 2007) is the spring functions which are introduced to model the behaviours of arbitragers in the market.

Rebonato, Mahal, Joshi, Bucholz and Nyholm (Rebonato, Mahal et al. 2005) state that their proposed model exhibits a number of positive features while still remaining simple and intuitive. Some of these positive features are that it reproduces the unconditional variance asymptotically, the skewness and the kurtosis of the one-day changes in all the reference rates. They note that it also recovers approximately the distribution of yield curve curvatures, the unconditional variance and the serial autocorrelation for many-day changes in the traded swap yield inputs. Finally, and most importantly, they state that the most significant model parameters introduced (the “spring constants”) are financially motivated as they represent the activities of arbitragers (Rebonato, Mahal et al. 2005).

Importantly the RMJBN model does not need to make any assumptions about a driving stochastic process nor does it assume that the innovations are independent or identically distributed (Rebonato, Mahal et al. 2005).

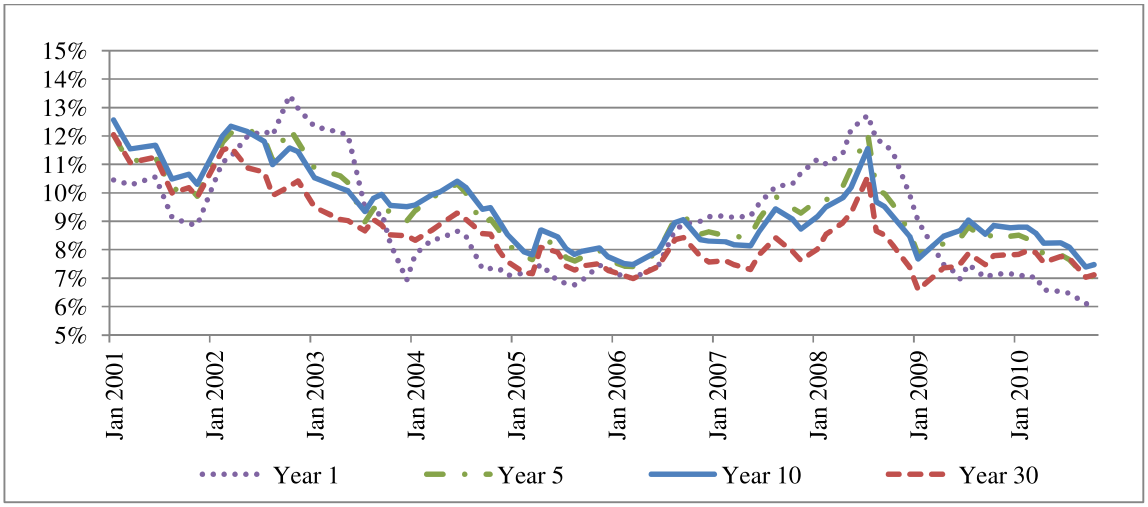
6.3 Implementing a real-world scenario generator for South African yield curve data

In this chapter I apply the same data analysis methods to the South African market as those applied by Rebonato, Mahal et al. (Rebonato, Mahal et al. 2005) in their work. The purpose of this is to describe the features of the South African interest rate environment and to motivate for the use of the RJMBN model as a method of capturing the features exhibited.

6.3.1 Description of past data for the South Africa yield curve

It is advisable to work with yields rather than forward rates (or other) curve information as the traded yields can be directly observed in a model-free way (Rebonato, Mahal et al. 2005). The empirical data used in this historical collection of South African swap rates (1-, 2-, 5-, 10-, 15-, 20-, 25- and 30-year) has been sourced from Bloomberg. The dates included in this analysis were October 2000 to September 2010. What this data set shows is that the shorter-term interest rates are more volatile than the long-term interest rates (Figure 37). RMBJN’s choice to work with swap rates was due to the ability to observe these inputs directly in the market (in a model-independent way).

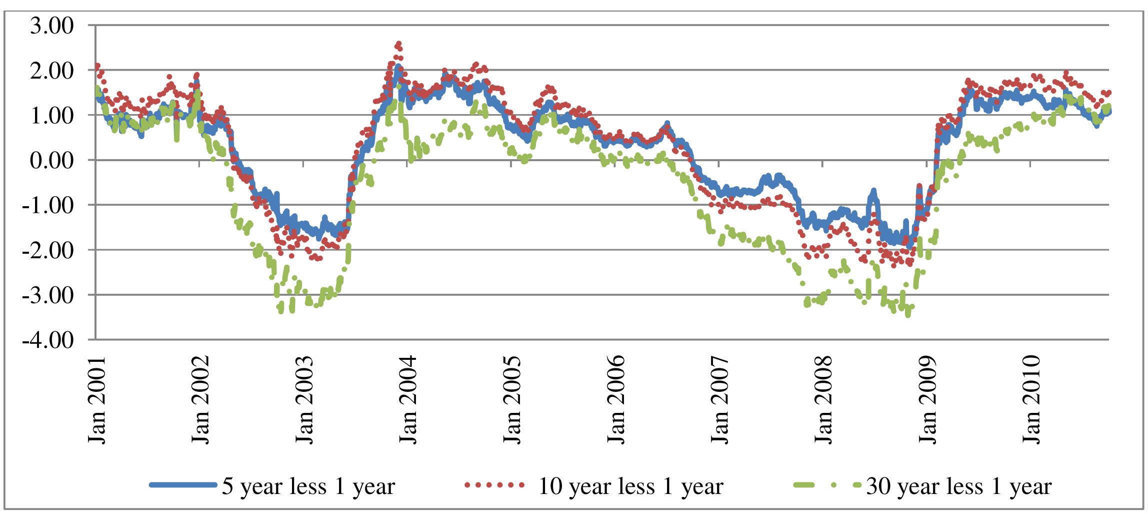
Figure 37: Historic data from October 2001 to September 2010 for the South African swap rates



Source: Bloomberg, own calculations

It is clear that yields have not only moved in parallel over the sample period. In fact, there have been instances where the difference between the 30 year and 1 year swap rate has changed by more than 5% within the space of one year. These extreme changes in differences can be seen in Figure 38 between January 2003 and January 2004, and to a lesser extent between January 2009 and January 2010.

Figure 38: Empirical data of the percentage difference between longer-term swap rates and short-term (1-year) rates over the period of October 2000 to September 2010



Source: Bloomberg, own calculations

Rebonato, Mahal et al. suggest that the changes in the swap rates are more informative than the level of the swap rates themselves and thus derive descriptive statistics for these changes (Rebonato, Mahal et al. 2005). I have calculated the same descriptive statistics for the South African yield curve data as seen in Figure 37. The mean of the changes is -0.02% (for all swap rates included in the dataset). This is reasonable given the general reduction in swap rates over the period. Standard deviations of the changes in daily swap rates increase slightly with the tenor of the swap.

Figure 39: Descriptive statistics for the changes in each swap rate duration over the period of October 2000 to September 2010

Changes in swap rate	1 Year Swap	2 Year Swap	5 Year Swap	10 Year Swap	15 Year Swap	20 Year Swap	25 Year Swap	30 Year Swap
Minimum	-7.43%	-13.13%	-8.57%	-7.97%	-11.45%	-11.57%	-11.53%	-11.60%
Maximum	12.16%	13.10%	12.76%	10.94%	10.93%	11.24%	11.99%	12.39%
Average	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%
Standard Deviation	0.84%	0.96%	0.91%	0.90%	0.99%	1.06%	1.12%	1.08%
Variance	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
Skewness	1.11	0.52	0.82	0.67	0.04	0.07	0.27	0.33
Kurtosis	35.10	39.55	24.84	18.23	18.89	17.38	20.57	15.36

Source: Bloomberg, own calculations

6.3.2 Unconditional variances

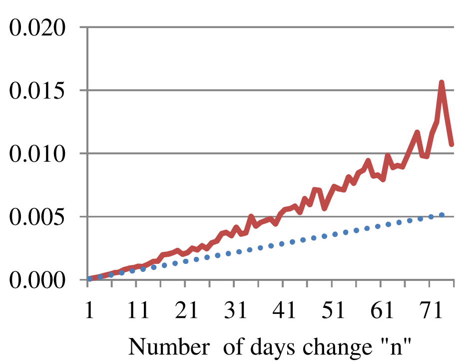
RMBJM uses a concept they call unconditional variances to test independence of increments across time. They do this by calculating the variance of non-overlapping changes in the period s of length m-days (Rebonato, Mahal et al. 2005). For our South African dataset for the period October 2000 to September 2010 the total number of observations was 2000 days. Thus, where m= 50, there would be 40 non-overlapping 50-day periods.

This methodology allows RMBJM to conclusion that the series (one for each swap rate) are not independent and that the variances also lack independence across time horizons (Rebonato, Mahal et al. 2005). A comparable investigation is completed for the South African market over the period October 2000 to September 2010. The straight line in Figure 40 to Figure 45 represents the extension of the one-day change variances. i.e. $var\,_m^i = m.\,var_1^i$ where i denotes the number of the swap-rate series in question. As RMBJM explain, if the serial variances of each of the swap-rate series were independent and identically distributed (i.i.d.) then the calculated serial variances would fall on the straight lines shown in the respective figures.

This is, however, not the case in RMBJN's finding and similarly in the South African data over the time period October 2000 to September 2010. Like RMBJN's study, it is observed that there are definite deviations away from the extensions of the one-day variances (Rebonato, Mahal et al. 2005). In particular, as the length of the non-overlapping periods increases, the short maturity rates (1-year, 2-year and 5-year) exhibit higher variances than would otherwise be expected if the rates were i.i.d. (Rebonato, Mahal et al. 2005). For long maturities (20-year and 30-year), the variances increase less than linearly, as the non-overlapping period lengths increase (Rebonato, Mahal et al. 2005).

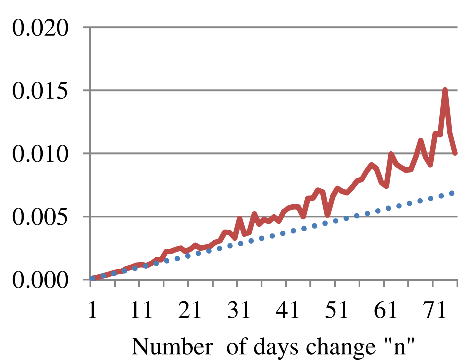
RMBJN note that a "less-than-linear" increase in the *m*-day variances could be compatible with positive autocorrelation and/or mean reversion. The key observation from our analysis is that there is a definite lack of independence of the variances (Rebonato, Mahal et al. 2005).

Figure 40: Historic serial variance: 1-year rate



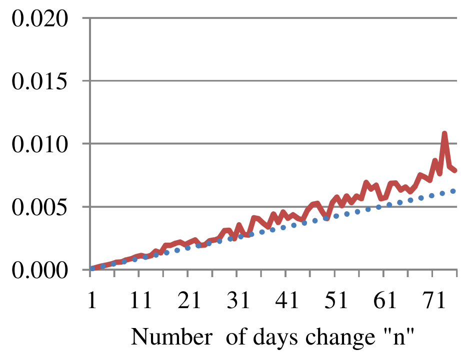
Source: Bloomberg, own calculations

Figure 41: Historic serial variance: 2-year rate



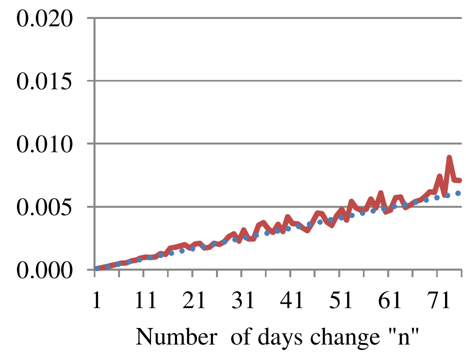
Source: Bloomberg, own calculations

Figure 42: Historic serial variance: 5-year rate



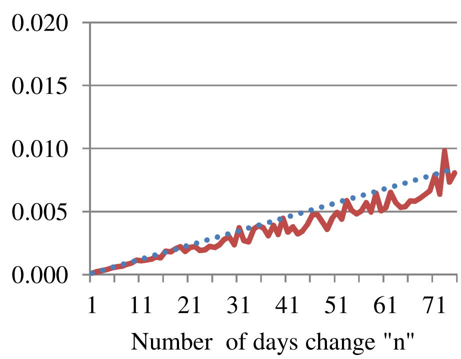
Source: Bloomberg, own calculations

Figure 43: Historic serial variance: 10-year rate



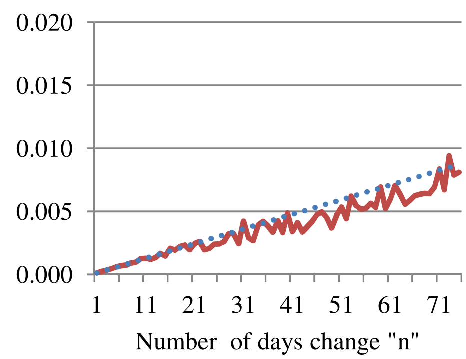
Source: Bloomberg, own calculations

Figure 44: Historic serial variance: 20-year rate



Source: Bloomberg, own calculations

Figure 45: Historic serial variance: 30-year rate



Source: Bloomberg, own calculations

6.3.3 Curvatures

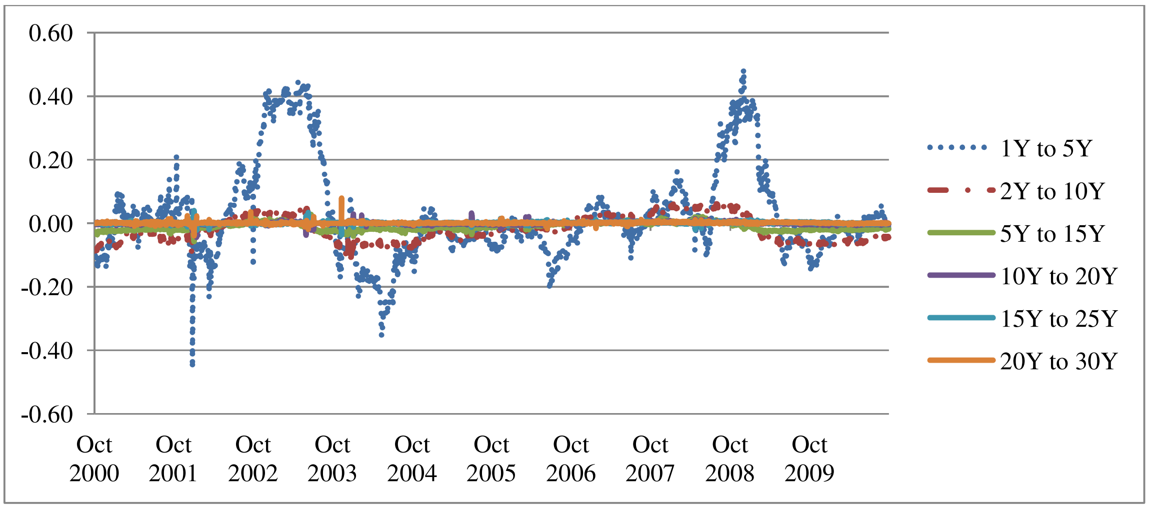
RMBJM analyse the yield curve curvatures, ξ_i , at a time point τ_i where $\tau_i = \frac{T_{i+1} - T_{i-1}}{2}$, $i = 2,3,\dots,7$. By defining the curvatures between three successive points on the yield curve as Equation 24 an approximation of a second derivative can be found (Rebonato, Mahal et al. 2005). The advantage of this approach is that RMBJN are able to observe the curvatures in the actual market data and not have to make any interpolation assumptions (Rebonato, Mahal et al. 2005).

Equation 24

$$\xi_i = \frac{\frac{y_{i+1} - y_i}{T_{i+1} - T_i} - \frac{y_i - y_{i-1}}{T_i - T_{i-1}}}{\frac{T_{i+1} + T_i}{2} - \frac{T_i + T_{i+1}}{2}}$$

Figure 46 demonstrations my calculations of the curvatures of the South African swap rates from October 2000 to September 2010, when calculated in the same manner as by RMBJN (Rebonato, Mahal et al. 2005). What this shows is that the curvatures of the near-term swap rates (such as 1Y to 5Y, say) are significantly more volatile than those between longer-term rates (such as 20Y to 30Y, say).

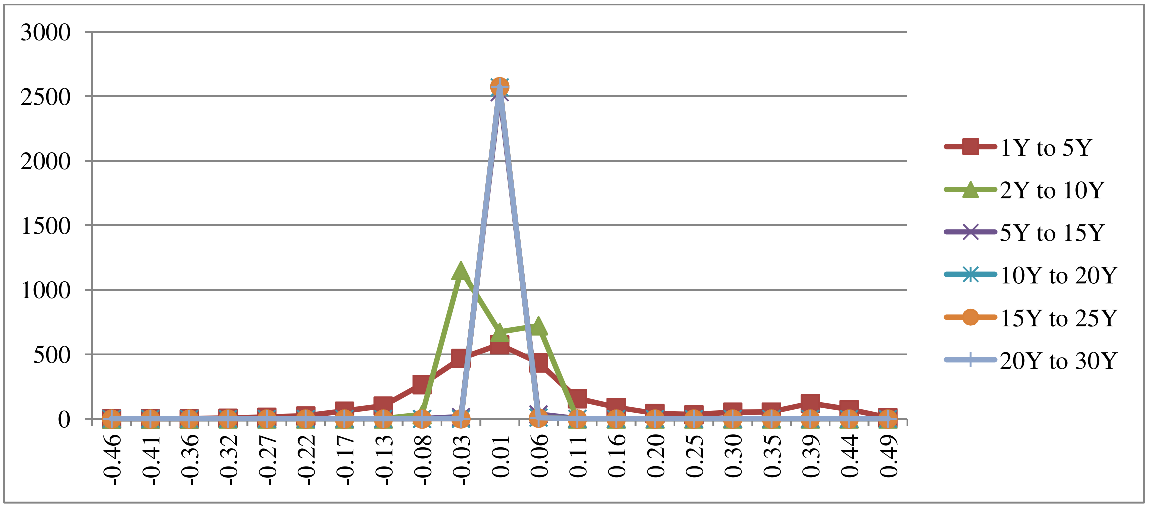
Figure 46: Curvatures from October 2000 to September 2010 under various approximate maturities



Source: Bloomberg, own calculations

This effect can also be seen in the density of the curvatures between successive swap rates in Figure 47. Here, similar to the findings by RMBJN, the density of the curvatures between the shorter-term swap yields is far more spread than the density of the curvatures of the longer-term swap rates (Rebonato, Mahal et al. 2005).

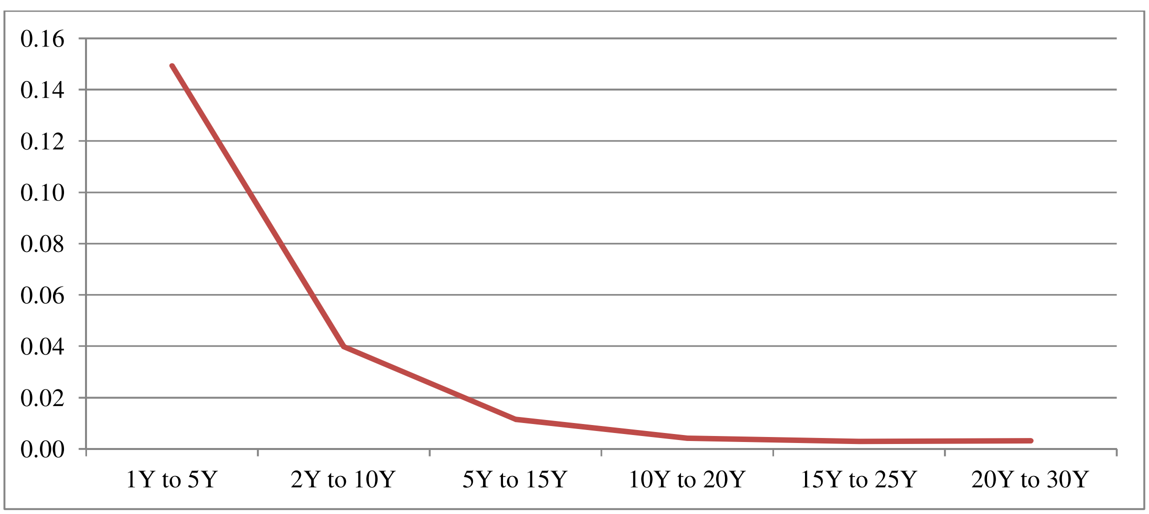
Figure 47: Frequency of the distribution of curvatures between different swap rate points



Source: Bloomberg, own calculations

The result of this feature of observed swap rate behaviour is summarised in Figure 48 where the standard deviation of the curvatures (as functions of approximate maturity) are shown (Rebonato, Mahal et al. 2005). This feature, which both RMBJM and I find, shows that the swap rate data suggests the long end of the yield curve should be less twisted (or curved) at longer maturities. Rebonato, Mahal et al. therefore motivate that real-world simulations should reflect this, and thereby introduce spring mechanisms into their swap rate forecast formulae (Rebonato, Mahal et al. 2005).

Figure 48: Standard Deviation of curvatures for each duration bucket

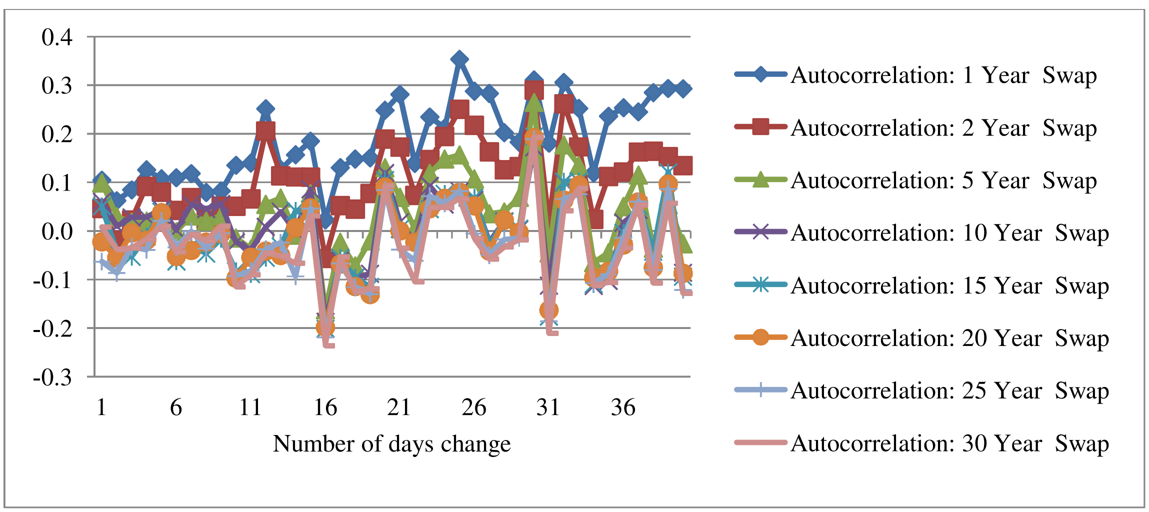


Source: Bloomberg, own calculations

6.3.4 Serial Autocorrelations

RMBJN also calculate the serial autocorrelations of non-overlapping *n*-day changes in the various swap rates (Rebonato, Mahal et al. 2005). Figure 49 shows these autocorrelations between the non-overlapping *n*-day changes for the various swap curve maturities. (Rebonato, Mahal et al. 2005) state that for all maturities they observe a positive serial autocorrelation that increase as a function of *n* (Rebonato, Mahal et al. 2005). In my analysis of the autocorrelations of the *n*-day changes in the swap rates the effects of autocorrelation are less pronounced for longer-term swaps but still evident. What this analysis means is that, as was the case in RMBJN's findings, there exists a strong positive correlation between the *n*-day changes in swap rates, in particular in the short-term rates. Thus from a modelling perspective it is inadequate to assume independence of changes in swap rates from one period to the next (Rebonato, Mahal et al. 2005). Rather, as described by RMBJN this analysis demonstrates the fact that monetary authorities tend to act in a loosening or tightening cycle (Rebonato, Mahal et al. 2005).

Figure 49: Lag 1 autocorrelations



Source: Bloomberg, own calculations

6.4 RMBJM's proposed model for real-world swap rate forecasting

The method proposed by RMBJN also includes the effects of less kinked/straighter yield curves at longer maturities but not at short-term rates. I have adopted this same approach in my forecasting of the South African market yield curve evolution. This effect is captured in the third term of RMBJN's proposed model in Equation 25. The value of the *i*th swap rate *M* days after today is given by Equation 25 (Rebonato, Mahal et al. 2005):

Equation 25

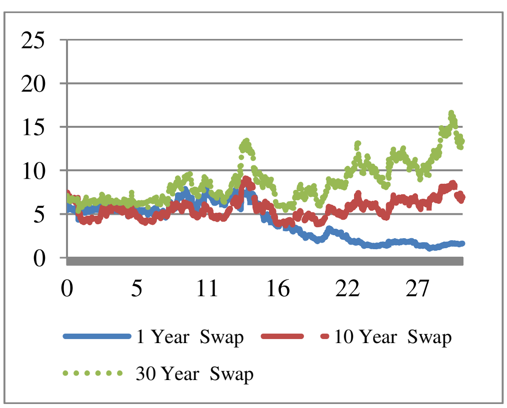
$$y_i^{N+M} = y_i^N + \sum_{r=1,M} \Delta y_i^{U_r} + \sum_{r=1,M} k_i \xi_i^{N+r} \quad \text{for } i = 2,3, \dots 7, \text{and}$$

$$y_i^{N+M} = y_i^N + \sum\nolimits_{r=1,M} \Delta y_i^{U_r} + \sum\nolimits_{r=1,M} h_i(y_{\ i} - y_i^{N+r}) \quad \text{for } i = 1 \text{ and } 8$$

6.5 Demonstration of forecast future yield curve evolutions

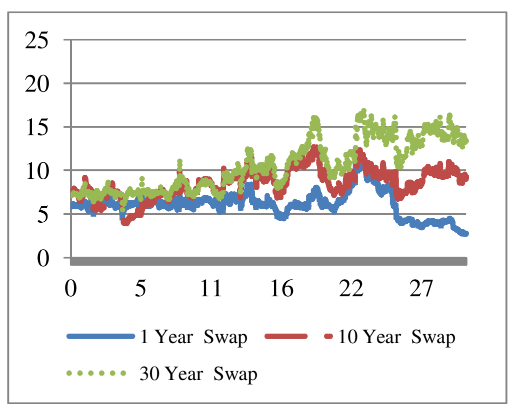
Figure 50 to Figure 55 show six of the potential swap rate evolutions generated by the RMBJN model when calibrated to the South African swap rates at September 2010.

Figure 50: Real-world scenario example 1



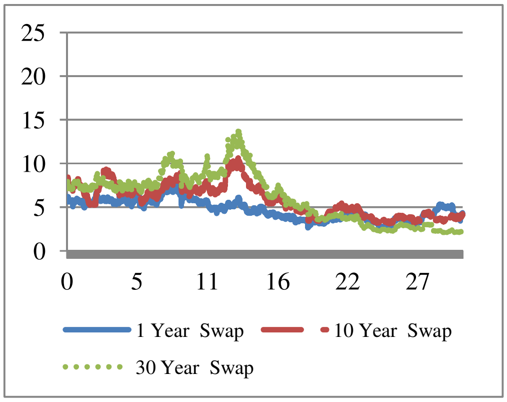
Source: Own calculations

Figure 51: Real-world scenario example 2



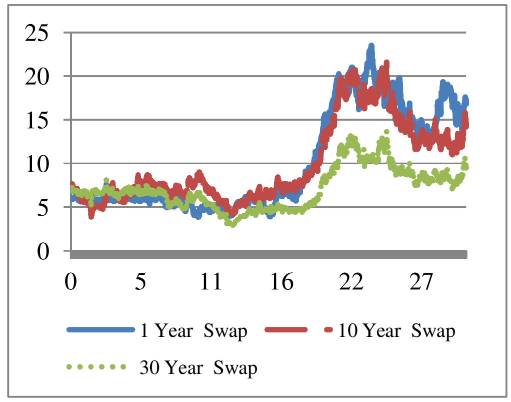
Source: Own calculations

Figure 52: Real-world scenario example 3



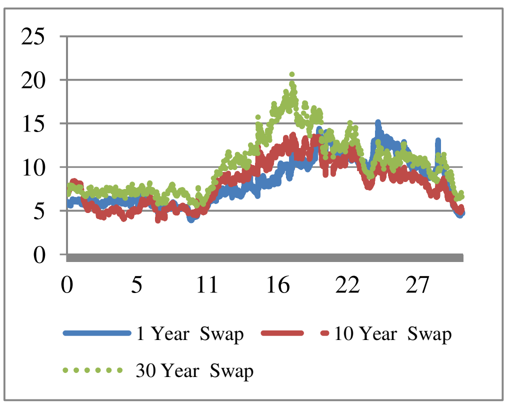
Source: Own calculations

Figure 53: Real-world scenario example 4



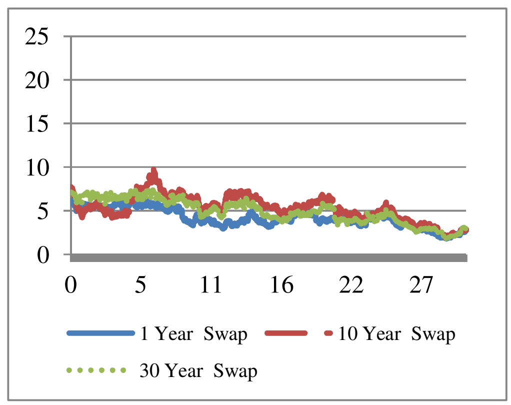
Source: Own calculations

Figure 54: Real-world scenario example 5



Source: Own calculations

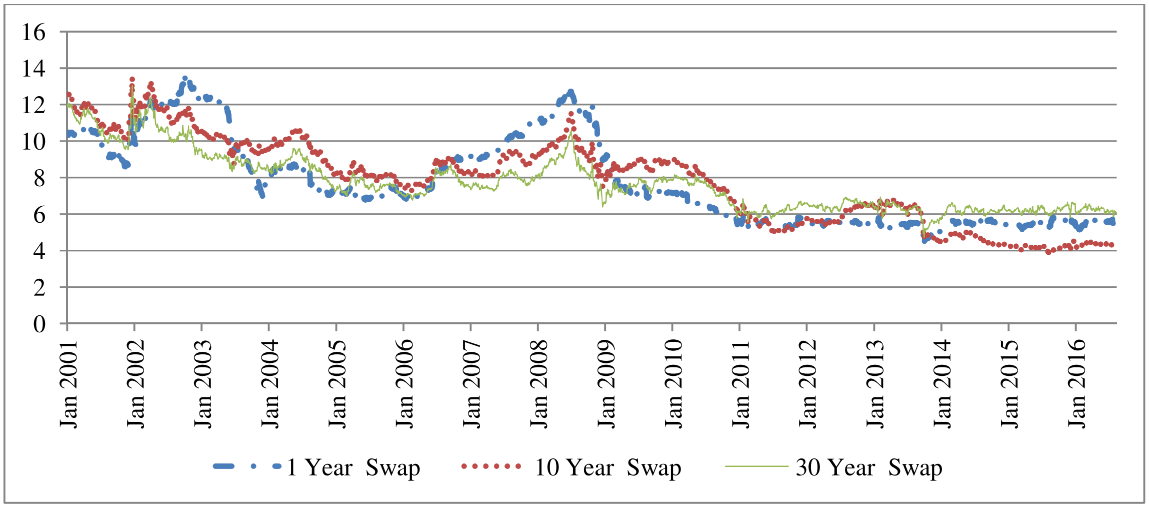
Figure 55: Real-world scenario example 6



Source: Own calculations

In order to assess the hedge program effectiveness practitioners need to consider the specific evolutions one by one. I have randomly chosen to use a single real-world scenario, as shown in Figure 56, for the basis of forecasting the real-yield evolution. One could complete the process outlined in the sections that follow for a number of real world evolutions.

Figure 56: Forecast swap rates example used in modelling forecasts



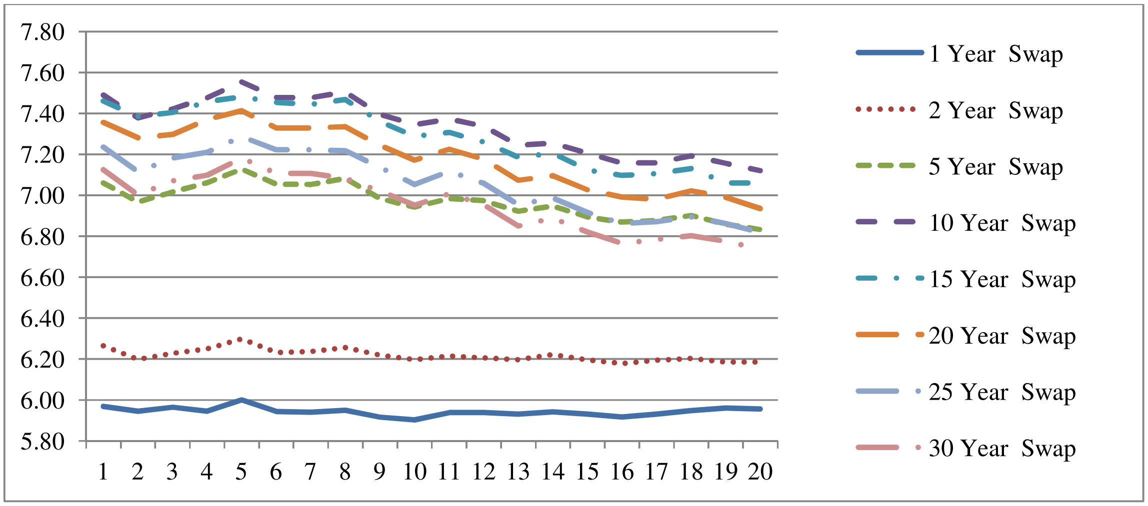
Source: Bloomberg, own calculations

6.6 Demonstration of the effectiveness of periodic rebalancing of a hedge position

This section shows the effectiveness of periodic dynamic hedging on the net of hedge profit and loss position over a 20-day period. Real-world simulations generated in by RMBJN's approach are used to forecast the real-world evolution of traded swap rates and a Black Scholes geometric Brownian motion approach is used to simulate daily equity movements (Rebonato, Mahal et al. 2005). The real-world scenarios simulated in Figure 56 are used over a 20-day forecast period.

Figure 57 shows the evolution of the real-world swap rates over the 20-day forecast period. This shows that this particular scenario models a general decrease in long-term swap rates over the 20-day period.

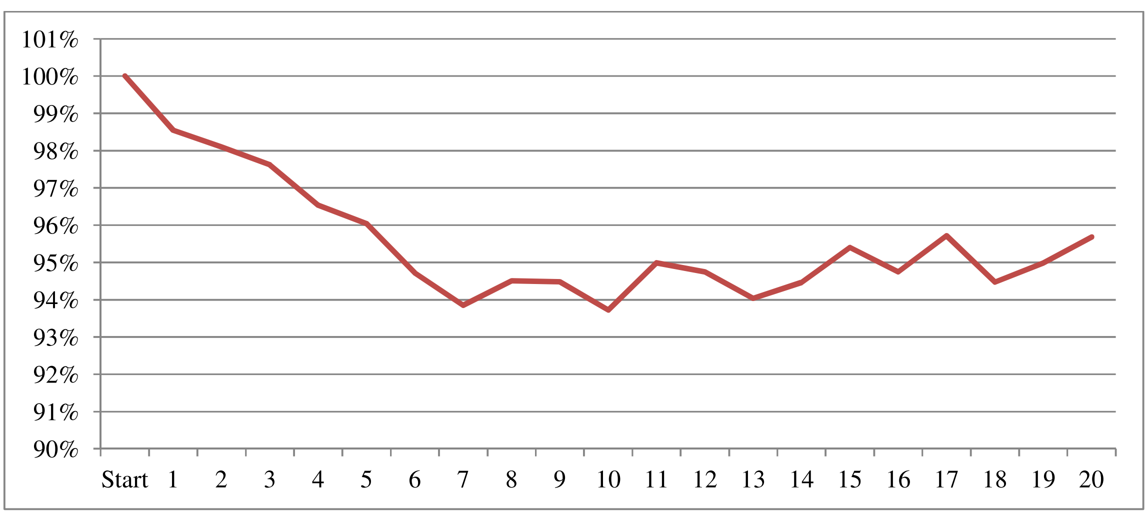
Figure 57: Illustration of the test scenario of real-world evolution of traded swap rates over the 20 day period



Source: Own calculations

Figure 58 illustrates the forecast equity index level over the same 20-day period. The scenario used captures a c.6% fall in equities over the first 7 days and generally flat equity index levels for the remainder of the period.

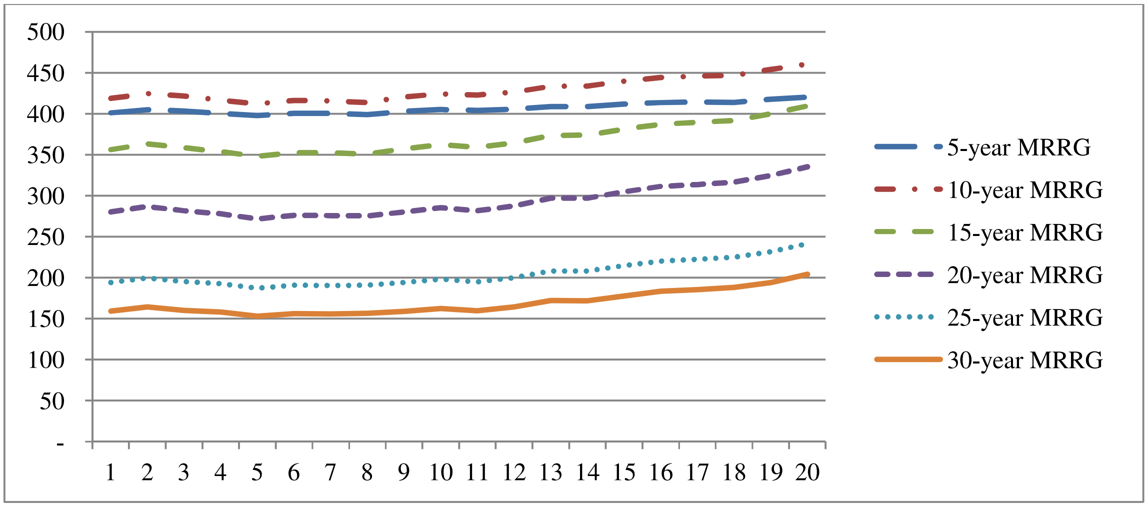
Figure 58: Illustration of the test scenario of the forecast equity index level over the 20-day period



Source: Own calculations

The scenario used results in increasing MRRG prices over the 20-day period. This is to be expected given the falling equity and swap rate yields. The prices of the various terms of MRRG are shown in Figure 59.

Figure 59: Calculations of the prices of the MRRG over the coming 20 days



Source: Own calculations

Table 28 shows the nominal change in the price of each of the various MRRG products under the 20-day forecast period. It can be seen the all terms of MRRG increase in price (this is evident by the losses shown) but the medium- and longer-term MRRG products prices increase more than the 5-year MRRG. The medium- and longer-term MRRG large price changes are proportionately more than in the case of the short term MRRG products.

Table 28: Change in value of the MRRG's of various terms under the 20-day forecast period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	-3	-6	-7	-7	-6	-5
3	2	3	4	5	5	4
4	3	5	5	4	3	2
5	3	4	6	6	5	5
6	-3	-4	-4	-5	-4	-3
7	0	0	0	1	0	1
8	2	2	1	0	0	-1
9	-4	-7	-6	-5	-3	-2
10	-2	-4	-5	-5	-4	-4

11	1	2	3	3	3	3
12	-1	-4	-5	-6	-5	-5
13	-3	-7	-9	-10	-8	-8
14	0	0	0	0	0	0
15	-3	-6	-8	-8	-6	-6
16	-2	-4	-6	-6	-6	-6
17	-1	-2	-2	-3	-2	-2
18	0	-1	-2	-3	-3	-3
19	-3	-7	-9	-8	-7	-6
20	-3	-7	-9	-11	-10	-10
Total	**-19**	**-42**	**-53**	**-55**	**-47**	**-45**

Source: Own calculations

6.6.1 Only hedging interest rate deltas

Hull states that it may be common practice to assume simpler one-factor models for pricing but not only one factor is assumed when hedging (Hull 2003). Hull explains that typical delta (or gamma) sensitivities allow for many yield curve movements, and not just those possible under the chosen model (Hull 2003). This *standard* practice of testing for changes not allowed by the model's parametric form is called outside model hedging (Hull 2003). Hull makes the point that, in reality, a relatively simple one-factor model usually gives a reasonable price if used carefully but a good hedging scheme must explicitly or implicitly assume many factors (Hull 2003).

Table 29 shows the sensitivities of the MRRG prices of various durations under a 1bps decrease in swap rates.

Table 29: Calculations of the -PV01 sensitivities for MRRGs with various terms over the 20-day forecast period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	-0.42	-0.59	-0.67	-0.66	-0.54	-0.50
2	-0.42	-0.60	-0.68	-0.67	-0.55	-0.51
3	-0.42	-0.59	-0.67	-0.66	-0.54	-0.50
4	-0.42	-0.59	-0.67	-0.65	-0.53	-0.49
5	-0.42	-0.58	-0.66	-0.64	-0.52	-0.48
6	-0.42	-0.58	-0.66	-0.65	-0.53	-0.49
7	-0.42	-0.58	-0.66	-0.65	-0.53	-0.49
8	-0.41	-0.58	-0.66	-0.65	-0.53	-0.49
9	-0.42	-0.59	-0.67	-0.65	-0.53	-0.49
10	-0.42	-0.59	-0.67	-0.66	-0.54	-0.50

11	-0.42	-0.59	-0.67	-0.65	-0.54	-0.49
12	-0.42	-0.60	-0.68	-0.67	-0.55	-0.51
13	-0.42	-0.60	-0.69	-0.68	-0.57	-0.53
14	-0.42	-0.61	-0.69	-0.68	-0.57	-0.53
15	-0.42	-0.62	-0.70	-0.70	-0.58	-0.54
16	-0.43	-0.62	-0.71	-0.71	-0.60	-0.56
17	-0.43	-0.63	-0.72	-0.72	-0.61	-0.56
18	-0.43	-0.63	-0.73	-0.72	-0.62	-0.57
19	-0.43	-0.64	-0.74	-0.74	-0.64	-0.59
20	-0.44	-0.65	-0.76	-0.76	-0.67	-0.62

Source: Own calculations

Table 30 shows calculations of the sensitivities of zero-coupon bonds (ZCB) to 1bps reductions in swap rates. The PV01's shown indicate the change in the price of a ZCB (of each term) for a 1bps point change in the swap rates of each term. The ZCB changes shown are in the context of a ZCB which pays R1 nominal face value at maturity.

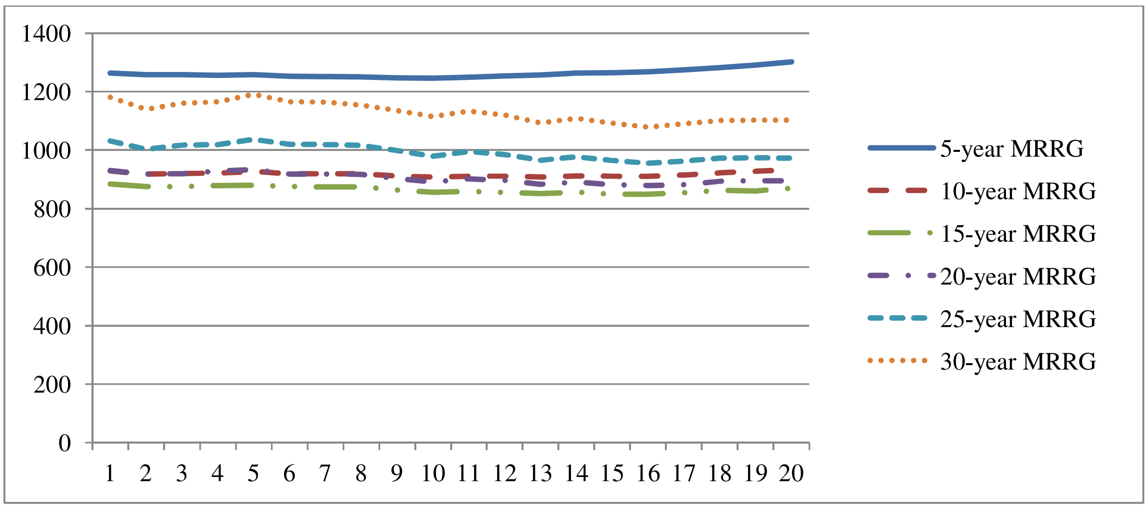
Table 30: Calculation of the PV01 sensitivity of Zero Coupon Bonds with terms matching the MRRG maturities

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	-0.000332	-0.000452	-0.000474	-0.000451	-0.000407	-0.000355
2	-0.000334	-0.000457	-0.000480	-0.000457	-0.000419	-0.000368
3	-0.000333	-0.000455	-0.000478	-0.000456	-0.000412	-0.000361
4	-0.000332	-0.000452	-0.000475	-0.000449	-0.000409	-0.000358
5	-0.000331	-0.000449	-0.000473	-0.000446	-0.000402	-0.000349
6	-0.000332	-0.000452	-0.000475	-0.000453	-0.000408	-0.000357
7	-0.000332	-0.000452	-0.000476	-0.000453	-0.000408	-0.000357
8	-0.000332	-0.000451	-0.000474	-0.000452	-0.000408	-0.000360
9	-0.000333	-0.000456	-0.000481	-0.000460	-0.000417	-0.000366
10	-0.000334	-0.000459	-0.000487	-0.000467	-0.000425	-0.000374
11	-0.000333	-0.000457	-0.000485	-0.000462	-0.000418	-0.000368
12	-0.000334	-0.000459	-0.000489	-0.000467	-0.000424	-0.000373
13	-0.000335	-0.000463	-0.000494	-0.000476	-0.000435	-0.000385
14	-0.000334	-0.000463	-0.000493	-0.000474	-0.000432	-0.000381
15	-0.000335	-0.000465	-0.000499	-0.000481	-0.000439	-0.000388
16	-0.000336	-0.000468	-0.000501	-0.000484	-0.000445	-0.000394
17	-0.000335	-0.000467	-0.000500	-0.000485	-0.000444	-0.000392
18	-0.000335	-0.000466	-0.000498	-0.000481	-0.000442	-0.000390
19	-0.000336	-0.000468	-0.000504	-0.000484	-0.000445	-0.000393
20	-0.000336	-0.000469	-0.000504	-0.000489	-0.000450	-0.000397

Source: Own calculations

In order to hedge against these changes in swap rates the life insurance company should buy (or sell) the appropriate number of ZCB hedge assets to offset the changes in the MRRG prices when swap rates change. Figure 60 shows that the life insurance company is required to hold between c.900 and c.1300 ZCBs (nominal face value of R1) depending on the term of the MRRG in order to hedge the interest rate sensitivity of the MRRGs.

Figure 60: Number of Zero Coupon Bond contracts required to hedge the PV01 at each of the 20 days



Source: Own calculations

If these interest rate hedges are put in place (and rebalanced to these positions each day) then for profit and losses shown in Table 31 would likely be experienced as these ZCB's revalue under the changes in swap rates. What can be seen is that as interest rates fall the hedges increase in value and this assists in offsetting the losses incurred as the prices of the MRRGs increase under decreasing interest rates.

Table 31: Changes in the value of the portfolio of interest rate hedges over the 20-day forecast period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	4	5	3	3	5	5
3	-2	-2	-1	-1	-3	-3
4	-2	-2	-2	-3	-1	-1
5	-3	-3	-1	-2	-3	-3

6	3	3	1	3	3	3
7	0	0	0	0	0	0
8	-1	-1	-1	0	0	1
9	4	5	4	4	3	3
10	2	2	3	3	4	3
11	-2	-1	-1	-2	-3	-2
12	0	1	2	2	2	2
13	2	4	3	4	4	4
14	-1	0	-1	-1	-1	-2
15	2	2	3	3	3	3
16	1	2	1	2	2	2
17	0	0	0	0	0	-1
18	-1	-1	-1	-2	-1	-1
19	2	2	3	1	1	1
20	1	2	0	2	2	2
Total	**10**	**16**	**17**	**18**	**18**	**17**

Source: Own calculations

The net effect of the change in the MRRG price and the ZCB interest rate hedge instruments held is shown in Table 32. It can be seen that by holding a range of ZCBs the net effect on the profit and loss due to changes in interest rates is dampened.

Table 32: Change in MRRG price net of interest rate hedge portfolio changes over the 20-day period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	0	-1	-4	-4	0	0
3	0	1	4	4	2	2
4	1	3	3	1	1	1
5	0	1	5	4	2	2
6	0	-1	-3	-1	-1	0
7	0	0	1	1	0	1
8	1	1	0	0	0	0
9	0	-2	-2	-1	0	1
10	0	-2	-2	-2	-1	-1
11	-1	1	2	1	1	1
12	-1	-2	-3	-4	-3	-3
13	-1	-3	-6	-5	-4	-3
14	-1	-1	-1	-1	-1	-1
15	-1	-4	-4	-5	-3	-3
16	-1	-2	-5	-5	-3	-3
17	-1	-2	-3	-2	-2	-3

18	-1	-2	-3	-4	-4	-4
19	-2	-6	-5	-7	-5	-5
20	-2	-5	-10	-8	-8	-8
Total	**-9**	**-26**	**-36**	**-37**	**-29**	**-28**

Source: Own calculations

6.6.2 Equity underlying deltas

Similarly, the sensitivity of the MRRG prices to changes in equity prices is shown in Table 33. This shows that the life insurer (the MRRG writer) would experience losses (decreasing as the term of the MRRG increases) when equities fell by 10% instantaneously.

Table 33: Calculation of the change in the MRRG prices from a 10% down shock in equities on each of the days of the 20-day forecast period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	-57.64	-30.53	-18.40	-11.27	-6.32	-4.23
2	-57.90	-30.94	-18.66	-11.41	-6.38	-4.37
3	-57.77	-30.70	-18.47	-11.30	-6.32	-4.24
4	-57.56	-30.37	-18.25	-11.24	-6.28	-4.20
5	-57.37	-30.06	-18.02	-11.24	-6.19	-4.12
6	-57.56	-30.33	-18.18	-11.24	-6.24	-4.16
7	-57.55	-30.32	-18.15	-11.24	-6.23	-4.15
8	-57.43	-30.15	-18.11	-11.24	-6.25	-4.17
9	-57.75	-30.59	-18.38	-11.26	-6.28	-4.22
10	-57.90	-30.85	-18.58	-11.36	-6.35	-4.32
11	-57.83	-30.74	-18.47	-11.28	-6.30	-4.23
12	-57.95	-31.02	-18.69	-11.42	-6.38	-4.37
13	-58.24	-31.60	-19.06	-11.69	-6.48	-4.48
14	-58.24	-31.66	-19.07	-11.69	-6.48	-4.48
15	-58.51	-32.16	-19.33	-11.93	-6.63	-4.66
16	-58.69	-32.51	-19.51	-12.22	-6.94	-4.75
17	-58.76	-32.68	-19.61	-12.32	-7.09	-4.75
18	-58.77	-32.74	-19.68	-12.44	-7.27	-4.81
19	-59.08	-33.28	-20.04	-12.71	-7.68	-5.06
20	-59.34	-33.82	-20.42	-12.95	-8.13	-5.22

Source: Own calculations

This exposure is equivalent to holding a long position in equities 10 times greater than this amount. Table 34 shows the equity positions required to offset changes in equity market moves on the price of the MRRG. For example, as the 5-year MRRG price decreases by c.R57.64 if equities drop by 10% on day 1 this is loss would be equivalent to holding a c.R576.40 long position in equities. Therefore, to hedge this position the life insurance company would be required to hold a short position in equities to the same amount, as is shown in Table 34.

Table 34: Equity exposure required at each of the 20 days so that the underlying equity delta is matched

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	-576.41	-305.31	-183.96	-112.72	-63.17	-42.29
2	-579.04	-309.37	-186.56	-114.12	-63.81	-43.74
3	-577.72	-307.00	-184.69	-112.97	-63.18	-42.42
4	-575.63	-303.74	-182.50	-112.41	-62.82	-41.97
5	-573.67	-300.64	-180.21	-112.37	-61.91	-41.24
6	-575.58	-303.30	-181.75	-112.41	-62.40	-41.59
7	-575.54	-303.15	-181.53	-112.40	-62.32	-41.50
8	-574.26	-301.50	-181.09	-112.43	-62.47	-41.71
9	-577.45	-305.91	-183.81	-112.61	-62.83	-42.16
10	-578.97	-308.50	-185.78	-113.57	-63.48	-43.24
11	-578.27	-307.42	-184.69	-112.84	-62.95	-42.31
12	-579.53	-310.19	-186.93	-114.23	-63.77	-43.69
13	-582.35	-316.04	-190.56	-116.90	-64.83	-44.75
14	-582.39	-316.57	-190.73	-116.95	-64.80	-44.76
15	-585.12	-321.59	-193.28	-119.34	-66.32	-46.57
16	-586.91	-325.12	-195.12	-122.17	-69.45	-47.45
17	-587.63	-326.77	-196.11	-123.18	-70.94	-47.53
18	-587.74	-327.39	-196.79	-124.37	-72.65	-48.10
19	-590.83	-332.78	-200.44	-127.14	-76.83	-50.64
20	-593.38	-338.22	-204.21	-129.53	-81.26	-52.22

Source: Own calculations

If such short equity positions are held on each day then the profits and losses on the equity hedge assets held are as shown in Table 35. What this shows is that as equities drop the short positions in equities held as hedges increase in value.

Table 35: Change in the value of the equity hedge portfolio in each day of the 20-day forecast period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	3	1	1	1	0	0
3	3	2	1	1	0	0
4	6	3	2	1	1	0
5	3	2	1	1	0	0
6	8	4	2	2	1	1
7	5	3	2	1	1	0

8	-4	-2	-1	-1	0	0
9	0	0	0	0	0	0
10	5	2	1	1	1	0
11	-8	-4	-3	-2	-1	-1
12	1	1	0	0	0	0
13	4	2	1	1	0	0
14	-3	-1	-1	-1	0	0
15	-6	-3	-2	-1	-1	0
16	4	2	1	1	0	0
17	-6	-3	-2	-1	-1	0
18	8	4	3	2	1	1
19	-3	-2	-1	-1	0	0
20	-4	-2	-1	-1	-1	0
Total	**16**	**8**	**5**	**3**	**2**	**1**

Source: Own calculations

The net effect of holding equity hedges and the MRRG change is shown in Table 36. This table shows that net of hedge profit and loss is dampened compared to the not holding equity hedges.

Table 36: Change in MRRG price net of equity hedge portfolio changes over the 20-day period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	-1	-5	-6	-6	-5	-5
3	5	5	5	6	5	5
4	9	8	7	5	3	2
5	6	6	7	7	6	5
6	5	0	-2	-3	-3	-3
7	5	3	2	2	1	1
8	-2	0	0	-1	-1	-1
9	-4	-7	-6	-5	-3	-2
10	3	-1	-3	-4	-4	-4
11	-7	-3	0	2	2	3
12	0	-3	-5	-5	-5	-5
13	1	-5	-8	-9	-8	-7
14	-3	-2	-1	-1	0	0
15	-9	-9	-9	-9	-7	-6
16	2	-2	-5	-6	-5	-5
17	-7	-5	-4	-4	-3	-2
18	8	4	1	-1	-2	-2
19	-7	-9	-10	-9	-7	-6
20	-7	-9	-11	-12	-10	-10
Total	**-2**	**-33**	**-48**	**-52**	**-46**	**-43**

Source: Own calculations

6.6.3 Overall effectiveness of both interest rate delta and equity delta hedging

The aggregate effect of the change in the MRRG price, the change in the interest rate hedge instruments prices and the change in the equity hedge instruments prices are shown in Table 37.

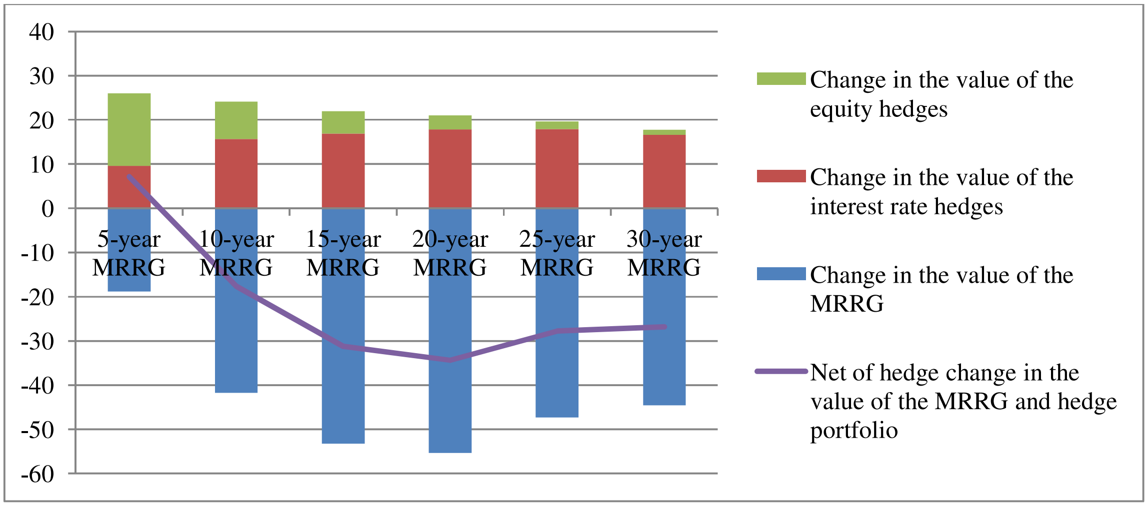
Table 37: Change in the MRRG price net of interest rate and equity hedge portfolio changes over the 20-day period

Day	5-year MRRG	10-year MRRG	15-year MRRG	20-year MRRG	25-year MRRG	30-year MRRG
1	0	0	0	0	0	0
2	3	0	-3	-3	0	1
3	2	3	5	5	2	2
4	7	6	5	2	2	1
5	3	3	6	5	3	2
6	8	4	-1	0	0	0
7	5	3	2	2	1	1
8	-4	-1	-1	-1	-1	0
9	0	-2	-2	-1	0	1
10	4	1	0	-1	0	-1
11	-9	-4	-1	0	0	0
12	0	-1	-3	-3	-2	-3
13	3	-1	-5	-4	-3	-3
14	-4	-2	-2	-2	-2	-2
15	-7	-7	-6	-6	-4	-4
16	3	0	-3	-4	-3	-3
17	-7	-5	-5	-3	-3	-3
18	7	2	0	-3	-3	-3
19	-5	-7	-7	-8	-6	-5
20	-6	-8	-11	-9	-9	-9
Total	**7**	**-18**	**-31**	**-34**	**-28**	**-27**

Source: Own calculations

Figure 61 shows this aggregate effect graphically. This demonstrates that by purchasing appropriate interest rate and equity hedge assets the profit and losses incurred as MRRG prices change can be dampened.

Figure 61: Cumulative effect of changes in the net of hedge profit and loss over the 20-day period



Source: Own calculations

7 Research conclusion

This dissertation has tackled the current life insurance industry challenge to price and hedge minimum rate of return guarantees (MRRG) embedded in recurring-contribution life insurance contracts in a practical manner. As the quantification and projection of the impact of dynamic hedging strategies appears to remain one of the top implementation challenges I focused the analysis on developing the literature on this area.

Chapter 1 introduced the concept of the MRRG benefit and showed that the price of the MRRG written over a recurring-contribution life insurance contract is path dependent with a payoff based on a stochastically-weighted average of the prices of the underlying unit-linked investment fund at different time points. These characteristics make the pricing problem somewhat more complex in that Monte Carlo simulation techniques are required in order to find a solution. The literature specific to the recurring contribution MRRG case is limited but the significant contribution of Schrager and Pelsser was apparent and their work formed the basis of much of the analysis which followed. In Chapter 2 I concluded that ongoing debate as to the appropriate approaches to value long-term guarantees remains active. A chronological analysis of the fast four decades showed that the industry and the professional bodies which work in it have, at times, had strong differences of opinion as to the appropriateness of various assumptions and modelling approaches. Increasingly, financial economics approaches are used to price complex life insurance options and guarantees but still some question the validity of the assumptions behind these relative pricing tools when used to calculate the market-consistent price of very long-dated life insurance guarantees in incomplete market conditions. The life insurance industry is required to look beyond this debate and explore practical approaches to manage the keys risks embedded in such products, and in turn meet likely corporate objectives, such as income statement volatility management.

Schrager and Pelsser's analysis of the variability of the underlying fund value at maturity, over which the MRRG is written, are shown in Chapter 3. Their conclusions provide clear explanations for the sources of volatility in the MRRG prices and therefore assist us in considering what hedging assets many be appropriate. They conclude that from the valuation date and up until the time a contribution is invested into the fund the volatility is confined to only interest rate risk. From the time the contributions are invested until the maturity the variation in the MRRG price is described by the quadratic covariance between the forward stock and forward bond prices. And finally, the volatility in the price between the contribution date and some later time point is confined to forward stock price risk, as at this later time point the past unit prices are all known. This suggested that early in the contracts term, when the majority of contributions were yet to be received, that interest rate risk would drive the MRRG price. On the other hand, as the MRRG contracts approached the benefit maturity date equity pure equity price risk associated to the past contributions already invested in equity would dominate the MRRG price changes. What this showed us is that no one hedging strategy is appropriate for the entire MRRG term, rather the significance of different hedging activities will change over the term of the MRRG.

To move forward and better quantify the sensitivity of the MRRG prices to both the forward bond and forward equity process a practical model to capture these effects was required. The Black Scholes Hull-White (BSHW) model was therefore used as it is complex enough to capture these items, while remaining mathematically tractable and can be calibrated, exactly, to the yield curve and therefore achieves market-consistent yield curve pricing (Chapter 4).

Sensitivity runs performed under the BSHW approach showed that MRRG prices increase as swap rates fall with the effect being more pronounced in the case of longer MRRG terms. As interest rate volatility increased the cost of the MRRGs decreased. As equity volatility increased the cost of MRRGs increased and there is an inverse relationship between moneyness level and MRRG price. Pelsser and Schrager showed that the effect of stochastic interest rates on the fund value volatility increases with maturity (Schrager and Pelsser 2004). This was an important finding in their work as it suggested that the choice of interest rate model, and more specifically the manner in which cross correlations between different time points on the yield curve were dealt with, could have material effect on the MRRG price calculated. This is of particular importance in due to the very long terms associated with typical life insurance MRRGs.

In Chapter 5 I discussed the current state of hedging activity amongst the largest North American life insurers. In reality, not all life insurers are hedging, and those that are focussed on the key first order interest rate and equity underlying risks. In this context and with the objective of addressing the industry challenge of projection and quantification of the effectiveness of specific dynamic hedging approaches the focus remained on outlining a simple approach under a robust model while recognising its limitations. By performing a batch of sensitivity runs on the MRRG prices allowed us to calculate the exposure of the MRRG's to small changes in the underlying economic variables and in turn it allows us to calculate the number of each hedge instrument needed to try offset profit and losses arising from revaluation of the MRRG price each day. Real-world models for the evolution of the term structure of interest rates are required to quantify and project the impact of dynamic hedging programs.

Chapter 6 showed that, under a real-world evolution model, relatively simple dynamic hedging of swap rate changes with zero-coupon bonds and by shorting equity underlying units can dampen the volatility of the profit and loss effects of movements in the MRRG prices over time. Thus, this dissertation has shown that it is possible to outline a practical approach to price and hedge the minimum rate of return guarantees embedded in recurring contribution life insurance contracts.

8 Limitations and areas for future research

8.1.1 Difficulty in unpacking the term structure of interest rate risk

The use of the Nelson-Siegel yield curve fitting method is a key limiting factor in understanding the nuances of the term structure of the interest rate risk our the MRRG benefits. This is because the Nelson-Siegel method focuses on smooth yield curve calibrations and has limited degrees of freedom with which small changes in the yield curve can be captured. As a result we have had to apply judgement as to which interest rate stresses can be interpreted as accurate rather than spurious accuracy emanating from yield curve fitting methods.

8.1.2 Limitations on the interest rate volatility captured in the Hull-White Model

The trade off of the mathematical tractability of the Hull White model is that the model can only capture a limited range of interest rate volatilities. As Schrager and Pelsser showed, the effects of stochastic interest rates have a material effect on the calculated price of the MRRG. It is therefore important to bear in mind the limitations of the Hull-White model when interpreting results. A key finding in Schrager and Pelssers' research was that the effects of stochastic interest rates on the volatility of the simulated underlying fund value increase with the time to maturity of the MRRG. Thus, different interest rate models, with different abilities to model stochastic interest rates are likely to lead to different simulated fund values, and in turn, MRRG prices.

Another limitation to BSHW approach is the difficulty in calibrating the Hull-White Model to observed interest rate options. In practice much of this calibration would be an "eye-balling" of the general characteristics of the stochastic interest rate volatility implied by market traded instruments. In the Libor Market Model (LMM) for example, calibration to traded instruments is direct, thus removing the noise introduced by "eye-balling" processes. This is of great value if users are to understand the sensitivity of the MRRG prices to small changes in interest rate volatility as, under LMM, the extent of calibration noise will be greatly reduced.

A key area of potential future research is to reproduce the analysis above on alternative choices of interest rate models and the LMM in particular.

8.1.3 Focussed on simple first-order hedging

This dissertation showed that MRRG prices exhibit positive convexity to interest rates and increased sensitivity to the underlying unit fund price as unit fund prices fall. In short, MRRG are non-linear to their key underlying economic risks. However, I have acknowledged these risks but only structured a simple hedging program to address the key first-order risks to interest rate and underlying unit fund prices. This has been done in the context of the majority of the North American life insurance industry only concentrating hedge activity on such practice and to keep focus on the research objective to project and quantify a hedge programs effectiveness. An extension to the analysis would be to include a portfolio of appropriately sensitive interest rate and equity options with low tracking error to the underlying fund units so as to attempt to better manage these non-linear dynamics.

Other areas of future potential research focus on the optimisation of hedging program design decisions. This dissertation has not delved into the relative merits of: move-based or time-based rebalancing; cost benefit analysis or any practical considerations such as borrowing costs and regulatory rules on short-selling of underlying unit funds, etc.

8.1.4 Not considering the interaction between policyholder behaviour and economic market variables.

In practice MRRG's are sold to individuals as rider benefits to their medium- and long-term savings policies. These individuals will take decisions whether to keep these products or surrender them depending on their own financial circumstances as well as on the value they see in them. There is an argument to state that policyholders are less likely to surrender guaranteed products (such as MRRG benefits) if they believe they are likely to claim from the benefit, for example of equity returns have been poor and the guarantee is in the money. While I treated policyholder behaviour as independent in this analysis, a potential area of future research could be to assess the additional cost associated to the life insurer if behaviour was linked to economic conditions.

9 Appendices:

9.1 Detailed calculations of 3-year policy fund value and guarantee build-up

Table 38: Detailed calculation of 3-year policy fund value and 0% rate of return guarantee value build-up from 1999 to 2011

Date	Fund index (J200T)	Guarantee return index	Rand contribution to fund	Fund units purchased	Fund units held	Value of fund units at end of 3-year terms	Guarantee fund units purchased	Guarantee fund units held	Value of guarantee fund units at end of 3-year terms	Guarantee payment top-up	Fund annual return	Fund 3-year rolling annualized return
01/01/96	411.38	100.00	1000.00	2.43			10.00					
01/01/97	447.12	100.00	1000.00	2.24			10.00				8.7%	
01/01/98	398.97	100.00	1000.00	2.51			10.00				-10.8%	
01/01/99	368.73	100.00	1000.00	2.71	7.17	2645.20	10.00	30.00	3000.00	354.80	-7.6%	-4%
03/01/00	660.65	100.00	1000.00	1.51	7.45	4925.14	10.00	30.00	3000.00	0.00	79.2%	14%
01/01/01	682.79	100.00	1000.00	1.46	6.73	4596.62	10.00	30.00	3000.00	0.00	3.4%	20%
01/01/02	909.65	100.00	1000.00	1.10	5.69	5176.14	10.00	30.00	3000.00	0.00	33.2%	35%
01/01/03	807.06	100.00	1000.00	1.24	4.08	3290.83	10.00	30.00	3000.00	0.00	-11.3%	7%
01/01/04	914.46	100.00	1000.00	1.09	3.80	3477.66	10.00	30.00	3000.00	0.00	13.3%	10%
03/01/05	1143.51	100.00	1000.00	0.87	3.43	3924.45	10.00	30.00	3000.00	0.00	25.0%	8%
02/01/06	1673.83	100.00	1000.00	0.60	3.21	5368.15	10.00	30.00	3000.00	0.00	46.4%	28%
01/01/07	2358.35	100.00	1000.00	0.42	2.57	6050.29	10.00	30.00	3000.00	0.00	40.9%	37%
01/01/08	2805.72	100.00	1000.00	0.36	1.90	5319.53	10.00	30.00	3000.00	0.00	19.0%	35%
01/01/09	2144.23	100.00	1000.00	0.47	1.38	2954.48	10.00	30.00	3000.00	45.52	-23.6%	9%
01/01/10	2824.62	100.00	1000.00	0.35	1.25	3521.76	10.00	30.00	3000.00	0.00	31.7%	6%
03/01/11	3333.02	100.00	1000.00	0.30	1.18	3922.34	10.00	30.00	3000.00	0.00	18.0%	6%

Source: Inet, own calculations

9.2 Nelson-Siegel yield curve fitting

Figure 62: Nelson-Siegel parameter solution for fitting eight bond yields

Long-run levels of interest rates	β_0	**6.4%**	
Short-run component	β_1	-0.9%	
Medium-term component	β_2	5.6%	
Decay parameter 1	τ_1	9.941	
Decay parameter 2	τ_2	5.766	
Components of the Nelson Siegel Spot Rate spot rate			
	Component 1	6.358%	β_0
	Component 2	-0.857%	β_1*((1-EXP(-m/τ_1))/(m/τ_1))
	Component 3	0.435%	β_2*((1-EXP(-m/τ_2))/(m/τ_2)-EXP(-m/τ_2))

Source: Own calculations

Figure 63: Calculations of the least squares minimisation process for fitting a Nelson-Siegel yield curve

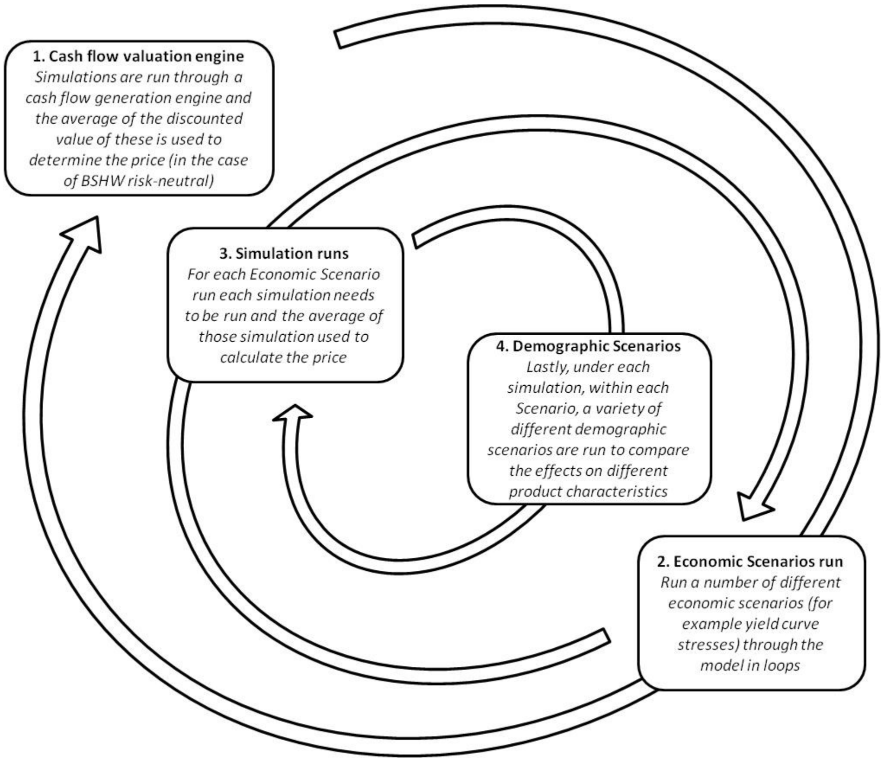
	1 Year Swap	**2 Year Swap**	**5 Year Swap**	**10 Year Swap**	**15 Year Swap**	**20 Year Swap**	**25 Year Swap**	**30 Year Swap**	
	Y1	Y2	Y5	Y10	Y15	Y20	Y25	Y30	
Swap rate	5.97	6.27	7.06	7.49	7.46	7.36	7.24	7.13	
P(0,t) market	0.94366	0.88556	0.71099	0.48565	0.33986	0.24185	0.17442	0.12685	
P(0,t) model	0.94397	0.88468	0.71142	0.48670	0.33907	0.24141	0.17444	0.12712	
Spot rate model	0.05935	0.06318	0.07047	0.07467	0.07477	0.07365	0.07234	0.07117	
Difference model to market	-0.03488	0.05288	-0.01303	-0.02336	0.01654	0.00990	-0.00053	-0.00761	**Least squares minimize cell J11**

Difference model to market ^2	0.00122	0.00280	0.00017	0.00055	0.00027	0.00010	0.00000	0.00006	0.00516

Source: Own calculations

9.3 Outline of scenario and simulation looping allowing for comparison between different demographic and economic scenarios

Figure 64: Economic and demographic scenario looping with simulations



Source: Own calculations

9.4 Explanation of the Excel models used

9.4.1 Excel file 1: RB Rice MSc Random Normal Distribution Generator.xlsm

This sheet is used to generate two sets of independent Normally distributed random variables for each required future time step. These random variables are used in the simulation of the forward rate and underlying risky equity return processes respectively.

It contains *MACRO 1: Generate two independent sets of Normally distributed random variables for quarterly and fortnightly time steps*. The outputs sheets are dN(ti,n) forward rates and dN(ti,n) for the equity underlying. The outputs are provided in fortnightly and quarterly time steps.

9.4.2 Excel file 2: RB Rice MSc Simulations – BSHW.xlsm

This file generates Black Scholes Hull-White (BSHW) simulation outputs for a given set of input assumptions.

The sheet runs under *MACRO 2: Copy the two sets of independent standard normally distributed random numbers and paste these as values.* What this macro does is copies the data generated in the separate random number generation file (Excel file 1) and pastes these as values. This step was done to enhance the overall speed of calculation as well as to hold allow randomness to be held constant between while a further calculations were being performed.

In order for the BSHW simulations to be calculated a range of input assumptions are required. These are:

1. The bootstrapped yield curve until a 30-year duration.
2. The calibration parameters for the Hull-White Model. These are given by two parameters, alpha and sigma, and time-dependent parameter theta(t).
3. A representation of volatility for the underlying (equity return) process
4. An assumption as to the time-dependent drift parameters applicable to the equity return process.
5. The correlation parameter between the equity return and forward rate process.

All five of these input assumptions are sourced from Excel File 3, the GMAB Cashflow Valuation Spreadsheet. They are captured on the "Input information" sheet in Excel File 2.

The quarterly and fortnightly time step outputs generated in this sheet are shown on the two output sheets: Forwards(ti,n) and Underlying returns (ti,n). 2000 simulations are shown.

9.4.3 Excel file 3: RB Rice MSc GMAB Cashflow Valuation Spreadsheet

This file calculates cash flows in the future time periods across the input number of simulations. The maximum number of simulations is set at 2000 so as to allow for the lengthy times with which cash flow generation takes. The simulation source file is Excel file 2, Simulations – BSHW.

These cash flows are calculated under asset of economic and demographic scenarios. The purpose of this structure is two-fold.

Firstly, it allows for recalculation of cash flows under a range of stressed economic scenarios for the same set of simulations. For example, I use a base economic scenario under the current fitted yield curve and then I use another economic scenario to recalculate cash flows under an input yield curve of 100 basis points lower.

Secondly, a range of different demographic variables can be tested. For example, by entering a number of demographic scenarios users can price different return guarantee percentages (such as 0% p.a., 3% p.a. and 5% p.a., say) simultaneously. Other key demographic factors which can be flexed are contract term, recurring premium escalation rates and monthly contribution size.

The manner in which the Excel file manages these inputs is by way of input fields which limit which Economic Scenario's, which Demographic Scenarios should be run.

Excel file 3 also allows the cashflow forecasts calculations to be calculated on multiple days in the future. This procedure allows the user to calculate a new set of cash flows off which to price the option at future dates.

To perform this function the cash flow generation file looks up the real-world simulation of the traded swap rates based on the number of days into the future the calculation relates to. These swap rates are then used to fit another Nelson-Siegel curve and calibrated Hull-White parameters for the future date in question. To do this the Nelson-Siegel and Hull-White fitting and calibration process is built into Excel file 3.

The processing of the functions in this sheet is controlled by *MACRO 4: Run all Economic Scenarios for each demographic model point input and capture the output in the "Scen - Output Table for all demographics file"*.

9.4.4 Excel file 4: RB Rice MSc Market Data hard coded

This is an output of the Rebonato method application to South Africa for real world swap rate forecasting. The reason this sheet has hard-coded values is to all the use of the same set of future real-world yield curves.

10 References

Bacinello, A. R. and F. Ortu (1993). "Pricing equity-linked life insurance with endogenous minimum guarantee." Insurance: Mathematics and Economics **12**: 245-257.

Bacinello, A. R. and F. Ortu (1993). "Pricing equity-linked life insurance with endogenous minimum guarantees: A corrigendum." Insurance Mathematics and Economics **13**: 303-304.

Benjamin, S., A. Ford, et al. (1980). "Maturity Guarantees Working Party (MGWP)." Journal of the Institute of Actuaries **107**: 103-209.

Black, F. and M. J. Scholes (1973). "The Pricing of Options and Corporate Liabilities." Journal of Political Economy **81**: 637-654.

Boyle, P., M. Hardy, et al. (2007). Chapter 18 Financial Engineering: Applications in Insurance. Handbooks in Operations Research and Management Science, Elsevier. **Volume 15:** 763-786.

Boyle, P. and E. S. Schwartz (1977). "Equilibrium prices of guarantess under equity-linked contracts." Journal of Risk and Insurance **44 (4)**: 639-660.

Boyle, P. P. and M. R. Hardy (1997). "Reserving for maturity guarantees: Two approaches." Insurance: Mathematics and Economics **21**(2): 113-127.

Brennan, M. J. and E. S. Schwartz (1976). "The pricing of equity-linked life insurance policies with an asset value guarantee." Journal of Financial Economics **3**: 195-213.

Brennan, M. J. and E. S. Schwartz (1979). Pricing and Investment Strategies for Guaranteed Equity-Linked Life Insurance, The S.S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania.

Chan, T. (1999). "Pricing contingent claims on stocks driven by Levy processes." The Annals of Applied Probability **9**(2): 504-528.

Coleman, T. F., Y. Li, et al. (2006). "Hedging guarantees in variable annuities under both equity and interest rate risks." Insurance: Mathematics and Economics **38**(2): 215-228.

Consiglio, A. and D. Giovanni (2007). "Evaluation of insurance products with guarantee in incomplete markets." Insurance: Mathematics and Economics **42**: 332-342.

Cox, J., J. E. Ingersoll, et al. (1985). "A Theory of the Term Structure of Interest Rates." Econometrica **53**: 385-407.

Dahl, M. (2004). "Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts." Insurance Mathematics and Economics **35**: 113-136.

Fagen, J. C. (1977). Maturity Guarantees under Investment-Linked Contracts, Presented to the Society of Actuaries in Ireland.

Finkelstein, G., E. McWilliam, et al. (2003). Guarantee and embedded options, Ernst & Young.

Follmer, H. and Schweizer, Eds. (1991). Hedging of contingent claims under incomplete markets. Applied Stochastic Analysis, Stochastic Monographs. New York, London, Gordon and Breach.

Follmer, H. and D. Sondermann, Eds. (1986). Hedging of non-redundant contingent claims. In: Contributions to Mathematical Finance., North-Holland.

Gilbert, C. L., K. Ravindran, et al. (2007). Results of the survey on variable annuity hedging programs for life insurance companies, Society of Actuaries, ALM Institute.

Girsanov, I. (1960). "On transforming a certain class of stochastic process by absolutely continuous substitution of measures." Theory of Probability Applications **5**(285-301).

Haastrecht, A. v., R. Lord, et al. (2008). Pricing long-maturity equity and FX derivatives with stochastic interest rates and stochastic volatility.

Harrison, M. J. and D. M. Kreps (1979). "Martingales and Arbitrage in Multiperiod Securities Markets." Journal of Economic Theory **20**: 381-408.

Heath, D., R. A. Jarrow, et al. (1989). Bond pricing and the term structure of interest rates: a new methodology., Cornell University.

Hill, T., D. Visser, et al. (2008). Stochastic Pricing for Embedded Options in Life Insurance and Annuity Products, Milliman, Inc Society of Actuaries.

Ho, T. S. Y. and S. B. Lee (1986). "Term structure movements and pricing interest rate contingent claims." Journal of Finance **41**: 1011-28.

Hughston, L. and D. C. Brody (2000). Modern theory of interest rates, with extensions to foreign exchange, inflation and credit, Kings College.

Hull, J. and A. White (1990). "Pricing interest rate securities." Review of Financial Studies **3**: 573-92.

Hull, J. a. W., A. (2000). "Forward Rate Volatilities, Swap Rate Volatilities and the implementation of the LIBOR Market Model." Journal of Fixed Income **10**(3): 46-62.

Hull, J. C. (2003). Options, Futures, and Other Derivatives, Prentice Hall.

Jaimungal, S. and V. R. Young (2005). "Pricing equity-linked pure endowements with risky assets that follow Levy processes." Insurance: Mathematics and Economics **36**: 329-346.

Levy, E (1992). "Pricing European average currency options." Journal of International Money Finance **11**: 474-491

Lin, X. and K. S. Tan (2003). "Valuation of equity-indexed annuities under stohastic interest rate." North American Actuarial Journal **7 (4)**: 72-91.

Longstaff, F. A. and E. S. Schwartz (1992). "Interest Rate Volatility and the Term Stucture: A Two-Factor General Equilibrium Model." Journal of Finance (47): 1259-1282.

Mahayni, A. B. and K. Sandmann (2006). Return Guarantees with Delayed Payment.

Melnikov, A. and Y. Romanyuk (2006). Efficient Hedging and Pricing of Equity-Linked Life Insurance Contracts on Several Risky Assets, Bank of Canada.

Merton, R. (1973). "Theory of rational option pricing." Bell Journal of Economic Management Science **4**: 141-183.

Merton, R. (1977). "On the pricing of contingent caims and the Modigliani-Miller theorem." Journal of Financial Economics **5**: 241-249.

Miltersen, K. R. and S.-A. Persson (1999). "Pricing rate of return guarantees in a Heath-Jarrow-Morton framework." Insurance: Mathematics and Economics **25**(3): 307-325.

Moeller, T. (1998). "Risk minimizing hedging strategies for unit-linked life insurance contracts." ASTIN Bulletin **28**: 17-47.

Moeller, T. (2001). "Risk minimizing hedging strategies for insurance payment processes." Finance and Stochastics **5**(4): 419-446.

Nelson, C.R. and Siegel, A.F. (1987). "Parsimonious Modeling of Yield Curves" Journal of Business **60**: 473-489

Nielsen, A. J. and K. Sandmann (1995). "Equity-linked life insurance: A model with stochastic interest rates." Insurance: Mathematics and Economics **16**(3): 225-253.

Nielsen, J. A. and K. Sandmann (1996). "Uniqueness of the fair premium for equity-linked life insurance contracts." Geneva Papers Risk Insurance Theory **21**: 65-102.

Nielsen, J. A. and K. Sandmann (2002). "The fair premium of an equity-linked life and pension insurance." Schonbucher, P., Sandmann, K (Eds.), Advances in Finance and Stochastics: Essays in Honor of Dieter Sondermann. Springer Verlag, Heidelberg.

Nyholm, K. and R. Rebonato (2007). "Long-Horizon Yield Curve Forecasts: Comparison of Semi-Parametric and Parametric Approaches."

Patel, N. (2006). Life's perilous journey; Difficulty of hedging guaranteed variable annuity products. Risk. **March 2006:** 23-26.

Rebonato, R., S. Mahal, et al. (2005). "Evolving Yield Curves in the Real-World Measure: a Semi-Parametric Approach." Journal of Risk.

Riesner, M. (2006). "Hedging life insurance contracts in a Levy process financial market." Insurance: Mathematics and Economics **38**: 599-608.

Ross, S. A. (1976). "The arbitrage theory of capital asset pricing." Journal of Economic Theory **13**: 341-360.

Schrager, D. F. and A. A. J. Pelsser (2004). "Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance." Insurance: Mathematics and Economics **35**(2): 369-398.

Schweizer, M. (1996). "Approximation pricing and the variance-optimal Martingale measure." Annals of Probability **24**: 206-236.

Vanmaele, M. and N. Vandaele (2008). "A locally risk-minimizing hedging strategy for unit-linked life insurance contracts in a Levy process financial market." Insurance: Mathematics and Economics **42**: 1128-1137.

Vasicek, O. (1977). "An Equilibrium Characterization of the Term Structure." Journal of Financial Economics **5**: 177-188.