Finite element approximations for fluid flows governed by nonlinear slip boundary conditions of friction type: from theory to computations

by

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Declaration

I, Mbehou Mohamed declare that the thesis, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:

DATE:

To

Angeline Mbehou, my love and my best friend

&

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Abstract

This thesis is divided in three main chapters devoted to the study of finite element approximations of fluid flows with special nonlinearities coming from boundary conditions.

In Chapter 1, we consider the finite element approximations of steady Navier-Stokes and Stokes equations driven by threshold slip boundary conditions. After re-writing the problems in the form of variational inequalities, a fixed point strategy is used to show existence of solutions. Next we prove that the finite element approximations for the Stokes and Navier Stokes equations converge respectively to the solutions of each continuous problem. Finally, Uzawa's algorithm is formulated and convergence of the procedure is shown, and numerical validation tests are achieved.

Chapter 2 is concerned with the finite element approximation for the stationary power law Stokes equations driven by slip boundary conditions of "friction type". It is shown that by applying a variant of Babuska-Brezzi's theory for mixed problems, convergence of the finite element approximation formulated is achieved with classical assumptions on the regularity of the weak solution. Solution algorithm for the mixed variational problem is presented and analyzed in details. Finally, numerical simulations that validate the theoretical findings are exhibited.

In Chapter 3, we are dealing with the study of the stability for all positive time of Crank-Nicolson scheme for the two-dimensional Navier-Stokes equation driven by slip boundary conditions of "friction type". We discretize these equations in time using the Crank-Nicolson scheme and in space using finite element approximation. We prove that the numerical scheme is stable in L^2 and H^1-norms with the aid of different versions of discrete Grownwall lemmas, under a CFL-type condition.

Introduction

Significant progress has been achieved in the analysis of the motion of incompressible fluid models of differential type using finite element methods. This work is concerned with the finite element approximation of the boundary value problems for the motion of incompressible fluid governed by the Stokes/Navier-Stokes equations, or by the non-Newtonian Stokes equation with certain nonlinear slip boundary conditions. Since these classes of nonlinear slip boundary conditions include the subdifferential property, the variational formulations are variational inequality problems.

So far extensive study has been done for the motion of incompressible fluid which is governed by the Stokes/Navier-Stokes equation, or by the non-Newtonian Stokes/Navier-Stokes equation in hydrodynamics as well as in mathematics. As to the boundary condition, almost all of these works have dealt with the adhesive boundary condition to the surface of a rigid body, namely, with the Dirichlet boundary condition (see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). This is of course reasonable from or consistent with the nature of such fluids and walls. However, there are phenomena, whose mathematical analysis seems to require introduction of some non-routine boundary conditions which might allow non-trivial motion of fluid on or across the boundary, for instance, slip or leak of fluid at the boundary. As examples, we can refer to flow through a drain or canal with its bottom covered by sherbet of mud and pebbles, flow of melted iron coming out from a smelting furnace, avalanche of water and rocks, blood flow in a vein of an arterial sclerosis patient, flow through a net or sieve, water flow in purification plant, etc. This observation is consistent with the hypothesis that the velocity at the wall is not zero. Several studies have been made and showed not only that slip takes place when a threshold is reached [12] but also it's the origin of many defects and instabilities in the polymer injection process [13, 14].

The inadequacy of the adherence condition is also evident from experimental observations (e.g.[15, 16, 17]) which show that non-Newtonian fluids such as polymer melts often exhibit macroscopic wall slip, and that in general this is governed by a nonlinear and nonmonotone relation between the slip velocity and the traction. This may be an important factor in sharkskin, spurt and hysteresis effects; see [18, 19, 20] for a detailed discussion and additional references. Moreover, fluids that exhibit boundary slip have important technological applications. For example, the polishing of artificial heart valves and internal cavities in a variety of manufactured parts is achieved by imbedding such fluids with abrasives [21].

A more important class of slip laws are those in which the magnitude of the tangential stress must reach some critical value, here called the slip yield stress, before slip occurs. These problems are especially interesting because the part of the boundary where slip occurs is not known and may vary with time. In fact, some experiments show that the onset of slip and the slip velocity may also depend on the normal stress at the boundary [15, 17, 22]. Not surprisingly, the theory and the numerical analysis for flow problems of this kind is equally limited. But since the last two decades, a remarkable progress has been achieved in the field of computational fluid dynamics with slip boundary conditions.

For a stationary Stokes problem, Fujita [23] introduced the following slip law.

$$\left.\begin{aligned} &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{w}_{\boldsymbol{\tau}}, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{w}_{\boldsymbol{\tau}}\ , \ \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\frac{(\boldsymbol{u}-\boldsymbol{w})_{\boldsymbol{\tau}}}{|(\boldsymbol{u}-\boldsymbol{w})_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S, \qquad (0.1)$$

where the notations will be explained later.

Fujita et al. [23, 24] have studied the existence of steady solution to the Stokes problem with slip condition (0.1) which they call slip of the "friction type", and with an analogous leak condition and later on Li et al. [25] for Navier-Stokes equations with the slip condition (0.1) . The regularity and the solvability of the solution for Stokes and Navier-Stokes equations have been carried out in [26, 27, 28, 29, 30] and for non-Newtonian Stokes equations by [31]. Regarding the numerical analysis for Stokes and Navier-Stokes equations with this slip conditions in terms of finite element method, [32] proposed an iterative algorithm of Uzawa type and gave some numerical examples. Recently Kashiwabara [33] presented the framework of finite element method including existence, uniqueness, error analysis and implementation. Error estimates for the Stokes problem with such slip are obtained in [34]. Based on the penalty method, in [35, 36], Li et al. proposed a finite element approximation combined with penalty method and error estimates with strong regularity assumption on the velocity. Low-order finite elements, such as the $P1/P1$ element with stabilized terms, are applied in [37, 38, 39]. Another approach by the $P1+/P1$ element, based on a saddle-point formulation is found in [40]. In our knowledge, no work has been done in finite element methods regarding the non-Newtonian Stokes equations with slip boundary condition (0.1).

C. Leroux [41] introduced the following slip boundary condition in Stokes equations and later on in Navier-Stokes equations with A. Tani [42] where they have studied the wellposedness of the steady solution.

$$\left.\begin{aligned} &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{w}_{\boldsymbol{\tau}}, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{w}_{\boldsymbol{\tau}}\ ,\ \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = (g + h(|(\boldsymbol{u}-\boldsymbol{w})_{\boldsymbol{\tau}}|))\frac{(\boldsymbol{u}-\boldsymbol{w})_{\boldsymbol{\tau}}}{|(\boldsymbol{u}-\boldsymbol{w})_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S. \tag{0.2}$$

The threshold slip boundary condition (0.2) arise in the modeling of flows of polymer melts during extrusion (where the slip threshold may depend on the normal stress at the boundary) and flows of yield-stress fluids [43, 44, 45]. Regarding the numerical analysis, in our knowledge, there is no work with this slip condition in Stokes and Navier-Stokes equations or in Non-newtonian Stokes equations dealing with finite element methods.

0.1 Thesis overview and our contributions

The thesis is divided into three main research chapters. Each of these chapters represents scientific contributions (in form of published, accepted or submitted journal papers). As such, each chapter is intended to be self-contained and can be read independently of the other chapters. Note also that the notation in each chapter is therefore slightly different.

Chapter 1 is devoted to the study of finite element analysis for Stokes and Navier-Stokes equations driven by threshold slip boundary conditions of type (0.2) defined in [41]. The principal goal is to analyze from the numerical analysis viewpoint the solvability, stability and convergence of the resulting variational inequalities of such problems. In this chapter, after re-writing the problems in the form of variational inequalities, a fixed point strategy is used to show existence of solutions. The finite element formulation for both Stokes and Navier-Stokes equations are derived and we establish the convergence of the finite element solutions to the continuous solutions of each problems. For Stokes, we consider a scheme related to the variational formulation of second kind and for Navier-Stokes, we consider a scheme related to the Oseen problem and show that their solution respectively converges to the finite element solution of the Stokes and Navier-Stokes equations. We formulate and show the convergence of the Uzawa's algorithm and finally, present some numerical experiments to verify the feasibility of our algorithm.
This chapter has been the object of the papers [46, 47] and is the first work on finite element approximation dealing with slip boundary condition of type (0.2).

Chapter 2 is dealing with finite element approximation of the stationary power-law Stokes equations driven by boundary conditions of "friction type" (0.1). The theoretical analysis of this chapter is based on the paper of Han and Reddy [48], where sufficient conditions for existence and uniqueness are derived for the kind of weak formulations we analyzed here. It is shown that by applying a variant of Babuska-Brezzi's theory for mixed problems, convergence of the finite element

approximation formulated is achieved with classical assumptions on the regularity of the weak solution. We also present the implementation of the nonlinear saddle point problem formulated by adopting a particular algorithm based on vanishing viscosity approach and long time behavior of an initial value problem. Finally, the predictions observed by the theory developed are validated by numerical experiments presented.
This chapter has been the object of the paper [49] and is also the first work on finite element approximation for power-law Stokes equations with slip boundary conditions of "friction type".

Chapter 3 reports on the long-time stability of the Crank-Nicolson scheme for the 2D Navier-Stokes equations driven by the slip boundary conditions of "friction type" (0.1). In the case of the Navier-Stokes equations with Dirichlet boundary conditions, establishing the H^1-stability for all time using Crank-Nicolson scheme on time has be proven in [50]. Our object here is to extend the results to the case of slip boundary condition. We discretize these equations using the Crank-Nicholson scheme in time and in space the finite element approximation. We establish its well-posedness and stability of the numerical scheme on L^2-norm and H^1-norm for all positive time.
This chapter has been the object of the paper [51].

0.2 Generalities on variational inequality and finite element approximation

We will present preliminary material from functional analysis that will be used subsequently. Most of these theorems are stated without proofs, since they are standard and can be found in references. We start with a review of definitions of several functional spaces (p-integrable functions, Sobolev spaces, etc.), this material can be found in many books on functional analysis, e.g., [52, 53, 54, 55] among others. We then recall some standard results on variational inequalities that will be applied repeatedly in proving existence and uniqueness results for problems dealing with slip boundary conditions of type (0.1) or (0.2). A list of some books and surveys on variational inequalities and nonlinear partial differential equations include [56, 57, 58, 59, 60]. Finally, basic notions on finite element related to variational inequalities are presented. More details on theoretical analysis of the finite element methods can be found in [61, 62, 63, 64, 65].

0.2.1 Function spaces

The spatial domain Ω is assumed to be an open, bounded, and connected subset of $\mathbb{R}^d$ with boundary Γ. A point in $\mathbb{R}^d$ is denoted by $\boldsymbol{x} = (x_1, \cdots, x_d)^T$. A multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ is an ordered collection of d nonnegative integers α_i. Denote $|\alpha| = \sum_{i=1}^{d} \alpha_i$. If v is an m-times differentiable real-value function defined on Ω, then for any α with $|\alpha| \leq m$,

$$\partial^\alpha v(\boldsymbol{x}) = \frac{\partial^{|\alpha|} v(\boldsymbol{x})}{\partial x_1^{\alpha_1}, \cdots, \partial x_d^{\alpha_d}}$$

denotes the αth partial derivative of v.
The space $L^r(\Omega)$ with $1 \leq r \leq \infty$, is the Banach space of Lebesgue measurable functions $v : \Omega \to \mathbb{R}$ such that $\|v\|_{L^r} < \infty$, where the norm $\|\cdot\|_{L^r}$ is given by

$$\|v\|_{L^r} = \begin{cases} \left(\int_\Omega |v(\boldsymbol{x})|^r dx \right)^{1/r} & \text{if} \quad r \in [1, \infty), \\ ess \sup\limits_{\boldsymbol{x} \in \Omega} |v(\boldsymbol{x})| & \text{if} \quad r = \infty. \end{cases}$$

In particular, the space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(\boldsymbol{x}) v(\boldsymbol{x}) dx \quad \text{for all} \quad u,\, v \in L^2(\Omega).$$

$\|\cdot\|_{L^2}$ is usually denoted by $\|\cdot\|$.
We denote by

$$L_0^r(\Omega) = \{v \in L^r(\Omega),\, (v, 1) = 0\}.$$

We say that a subdomain Ω' is compactly included in Ω, and denote it by $\Omega' \subset\subset \Omega$, if $\overline{\Omega'} \subset \Omega$.

Definition 0.2.1 *Let $1 \leq r < \infty$. A function $v : \Omega \to \mathbb{R}$ is said to be locally r-integrable, written as $v \in L^r_{loc}(\Omega)$, if for every $\boldsymbol{x} \in \Omega$ there is an open neighborhood Ω' of $\boldsymbol{x}$ such that $\Omega' \subset\subset \Omega$ and $v \in L^r(\Omega')$.*

In Sobolev spaces, derivatives are understood to be in the following weak sense.

Definition 0.2.2 *Let Ω be a nonempty open set in $\mathbb{R}^d$, and v, $w \in L^1_{loc}(\Omega)$. Then w is called an αth weak derivative of v if*

$$\int_\Omega v(\boldsymbol{x})\partial^\alpha \phi(\boldsymbol{x})dx = (-1)^{|\alpha|}\int_\Omega w(\boldsymbol{x})\phi(\boldsymbol{x})dx \quad \text{for all} \quad \phi \in C_0^\infty(\Omega). \tag{0.3}$$

If v is m-times continuously differentiable on Ω, then for each α with $|\alpha| \leq m$, the classical partial derivative $\partial^\alpha v$ is also the αth weak derivative of v. Thus, the usual derivative, when it exists, is also a weak derivative and so we use the notation $\partial^\alpha v$ also for the αth weak derivative of v.
Let m be a nonnegative integer and let $r \in [1, \infty]$. The Sobolev space

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega);\ \partial^\alpha v \in L^r(\Omega) \ \text{ for all } \quad |\alpha| \leq m\},$$

is equipped with the norm

$$\|v\|_{m,r} = \begin{cases} \left(\displaystyle\sum_{|\alpha|\leq m}\int_\Omega |\partial^\alpha v|^r \mathrm{dx}\right)^{1/r} & \text{if} \quad r \in [1,\infty), \\ \displaystyle\max_{|\alpha|\leq m} \|\partial^\alpha v\|_{L^\infty} & \text{if} \quad r = \infty, \end{cases} \tag{0.4}$$

and seminorm

$$|v|_{m,r} = \begin{cases} \left(\displaystyle\sum_{|\alpha|= m}\int_\Omega |\partial^\alpha v|^r \mathrm{dx}\right)^{1/r} & \text{if} \quad r \in [1,\infty), \\ \displaystyle\max_{|\alpha|= m} \|\partial^\alpha v\|_{L^\infty} & \text{if} \quad r = \infty. \end{cases} \tag{0.5}$$

When $r = 2$, $W^{m,r}(\Omega)$ is the Hilbert space $H^m(\Omega)$ with the scalar product

$$((v,w))_m = \sum_{|\alpha|\leq m} (\partial^\alpha v, \partial^\alpha w).$$

We denote the norm of $H^m(\Omega)$ by $\|\cdot\|_m$ and its seminorm by $|\cdot|_m$. The closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{m,r}$ is a closed subspace of $W^{m,r}(\Omega)$, denoted by $W_0^{m,r}(\Omega)$. When $r = 2$ we use the notation $H_0^m(\Omega) \equiv W_0^{m,2}(\Omega)$. It can be shown that the seminorm $|\cdot|_{m,r}$ is a norm on $W_0^{m,r}(\Omega)$ and is equivalent to $\|\cdot\|_{m,r}$ on $W_0^{m,r}(\Omega)$. For system of equations, we will use the product space and will be indicated by boldface type letters ($\boldsymbol{L}^r(\Omega) = [L^r(\Omega)]^d$, $\boldsymbol{W}^{m,r}(\Omega) = [W^{m,r}(\Omega)]^d$, etc.).

We need function spaces on the boundary or a part of it. The related Lebesgue and Sobolev spaces can be defined through the use of a smooth partition of unity. Details can be found in [66, chap 6]. In this context, we will restrict our attention to the boundary of a polygonal or polyhedral domain. Then, it is quite straightforward to define the related Lebesgue and Sobolev spaces. Suppose $\Gamma_0 \subset \Gamma$, where Γ_0 is a union of Γ^i with $1 \leq i \leq i_0$ and each Γ^i is straight or planar. Then $v \in L^r(\Gamma)$ if and only if $v \in L^r(\Gamma^i)$, $1 \leq i \leq i_0$ and we use the norm

$$\|v\|_{L^r(\Gamma_0)} = \left(\sum_{i=1}^{i_0} \|v\|_{L^r(\Gamma^i)} \right)^{1/r}.$$

In a Sobolev space, it is possible to define the notion of generalized boundary values, i.e., the notion of the trace of a function on the boundary. The trace of a function that is continuous up to the boundary coincides with the value of the function on the boundary.

Theorem 0.2.1 *Assume that Ω is an open, bounded, Lipschitz domain in $\mathbb{R}^d$ with boundary Γ and $1 \leq r < \infty$. Then there exists a continuous linear operator $\gamma : W^{1,r}(\Omega) \to L^r(\Gamma)$ such that $\gamma v = v|_\Gamma$ if $v \in W^{1,r}(\Omega) \cap C(\overline{\Omega})$. Moreover, the mapping $\phi : W^{1,r}(\Omega) \to L^r(\Gamma)$ is compact, i.e., for any bounded sequence $\{v_n\}$ in $W^{1,r}(\Omega)$, there is a subsequence $\{v_{n'}\} \subset \{v_n\}$ such that $\{\gamma v_{n'}\}$ is convergent in $L^r(\Gamma)$.*

The operator γ is called the trace operator, and γv is the trace or generalized boundary value of $v \in W^{1,r}(\Omega)$. For the sake of simplicity, when no ambiguity may occur, we write v instead of γv. It follows from the continuity of γ that there exists a constant $c > 0$ such that

$$\|\gamma v\|_{L^r(\Gamma)} \leq c\|v\|_{1,r} \quad \text{for all} \quad v \in W^{1,r}(\Omega). \tag{0.6}$$

Generally, the trace operator is neither an injection nor a surjection from $W^{1,r}(\Omega)$ to $L^r(\Gamma)$ (the only exception is when $r = 1$: the trace operator is surjective from $W^{1,r}(\Omega)$ onto $L^r(\Gamma)$). Let $H^{1/2}(\Gamma)$ denote the range of the trace operator on the space $H^1(\Omega)$; it can be shown to be a Hilbert space and we denote by $(\cdot,\cdot)_{1/2}$ its inner product. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with boundary Γ. Then the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ is linear, continuous, and surjective. The kernel of the trace operator on $H^1(\Omega)$ is $H_0^1(\Omega)$. We denote by $H^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$ and by $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. This duality pairing is an extension of the $L^2(\Gamma)$ inner product and if $\xi' \in L^2(\Gamma)$, then $\xi' \in H^{-1/2}(\Gamma)$ and

$$\langle \xi', \xi\rangle_\Gamma = \int_\Gamma \xi'\xi da \quad \text{for all} \quad \xi \in H^{1/2}(\Gamma).$$

0.2.2 Elements of nonlinear analysis

In this subsection, we review some standard results on nonlinear operators defined on Banach or Hilbert spaces, including the well-known Banach fixed point theorem.

Definition 0.2.3 *Let X be a normed space, X' its dual and $\langle\cdot,\cdot\rangle$ the duality pairing between X' and X. Let $A : X \to X'$ be an operator. The operator A is said to be monotone if*

$$\langle Au - Av, u - v\rangle \geq 0 \quad \textit{for all} \quad u,\, v \in X.$$

The operator A is strictly monotone if

$$\langle Au - Av, u - v\rangle > 0 \quad \textit{for all} \quad u,\, v \in X, \quad u \neq v,$$

and strongly monotone if there exists a constant $m > 0$ *such that*

$$\langle Au - Av, u - v \rangle \geq m\|u - v\|_X^2 \quad \text{for all} \quad u,\, v \in X.$$

The operator A *is coercive if*

$$\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle}{\|u\|_X} = \infty.$$

The operator A *is nonexpansive if*

$$\|Au - Av\|_{X'} \leq \|u - v\|_X \quad \text{for all} \quad u,\, v \in X.$$

The operator A *is Lipschitz continuous if there exists* $L_A > 0$ *such that*

$$\|Au - Av\|_{X'} \leq L_A\|u - v\|_X \quad \text{for all} \quad u,\, v \in X.$$

The operator A *is said to be hemicontinuous if the real function* $t \mapsto \langle A(u+tv), w \rangle$ *is continuous on* $[0, 1]$ *for all* $u,\, v,\, w \in X$.

Definition 0.2.4 *Let* $f : X \to \overline{\mathbb{R}}$. *The function* f *is said to be proper if* $f(v) > -\infty$ *for all* $v \in X$ *and* $f(u) < \infty$ *for some* $u \in X$.
The function f *is convex if*

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v) \quad \text{for every} \quad u,\, v \in X \quad \text{and} \quad t \in (0, 1). \tag{0.7}$$

The function f *is strictly convex if the inequality (0.7) is strict for* $u \neq v$ *and all* $t \in (0, 1)$.

Definition 0.2.5 *A function* $f : X \to \overline{\mathbb{R}}$ *is said to be lower-semicontinuous, written l.s.c., at* $u \in X$ *if*

$$\liminf_{n \to \infty} f(u_n) \geq f(u) \tag{0.8}$$

for each sequence $\{u_n\} \subset X$ *converging to* $u \in X$. *The function* f *is l.s.c. on a subset of* X *if it is l.s.c. at each point of the subset. We say that* f *is l.s.c. if it is l.s.c. on* X. *When inequality (0.8) holds for every sequence* $\{u_n\} \subset X$ *converging weakly to* u, *the function* f *is said to be weakly lower-semicontinuous written as weakly l.s.c., at* u*; weakly l.s.c. on the subset; and weakly l.s.c., respectively.*

If f is a continuous function, then it is also l.s.c. However, the converse of this statement is not true since lower semicontinuity does not imply continuity. Since strong convergence in X implies weak convergence, it follows that a weakly lower semicontinuous function is lower-semicontinuous. Moreover, it can be shown that a proper convex function $f : X \to \overline{\mathbb{R}}$ is lower-semicontinuous if and only if it is weakly lower-semicontinuous.

The notion of the subdifferential (e.g. [67]) is very useful in describing various mechanical laws and constraints that arise in problems dealing with slip boundary conditions, and other models as well.

Definition 0.2.6 *Let X be a real normed space with dual X', and let $f : X \to \overline{\mathbb{R}}$. Assume that $u \in X$ is such that $f(u) \neq \pm\infty$. Then, the subdifferential of f at u is the set*

$$\partial f(u) = \{w \in X' : \ f(v) - f(u) \geq \langle w, v-u \rangle \text{ for all } v \in X\}. \tag{0.9}$$

Each $w \in \partial f(u)$ is called a subgradient of f at u. The function f is said to be subdifferentiable at $u \in X$ if $\partial f(u) \neq \emptyset$.

In the case where $X = \mathbb{R}$, Definition 0.2.6 is equivalent to

Definition 0.2.7 *Let $\psi : \mathbb{R} \to (-\infty, \infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below (ψ is not identical with $+\infty$). Let $a \in \mathbb{R}$, the sub-differential set $\partial\psi(a)$ is the set;*

$$\partial\psi(a) = \{b \in \mathbb{R} : \ \psi(c) - \psi(a) \geq b(c-a) \text{ for all } c \in \mathbb{R}\}. \tag{0.10}$$

Theorem 0.2.2 *(**Banach fixed-point theorem**)[68]. Let K be a nonempty and closed set in a Banach space X. Assume that $\Lambda : K \to K$ is a contraction mapping, with contraction constant $\alpha \in [0, 1)$, i.e.,*

$$\|\Lambda u - \Lambda v\|_X \leq \alpha \|u - v\|_X \quad \text{for all} \quad u, v \in K. \tag{0.11}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$, i.e., Λ has a unique fixed point in K.

0.2.3 Standard results on variational inequalities

In this subsection, we review some standard existence and uniqueness results for elliptic variational inequalities.

Definition 0.2.8 *Let X be a real Hilbert space with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Assume that $a : X \times X \to \mathbb{R}$ is a bilinear form. $a(\cdot,\cdot)$ is said to be continuous or bounded if there exists a number $M > 0$ such that*

$$|a(u,v)| \leq M\|u\|_X\|v\|_X \quad \textit{for all} \quad u, v \in X. \tag{0.12}$$

The form $a(\cdot,\cdot)$ is said to be X-elliptic or coercive if there is a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha\|u\|_X^2 \quad \textit{for all} \quad u \in X. \tag{0.13}$$

We have the following results. The proof of Theorem 0.2.3 can be found in [69, 63] whereas Theorem 0.2.4 was proved by Reddy [70], and Han and Reddy [48].

Theorem 0.2.3 *Let X be a Hilbert space. Assume that $a : X \times X \to \mathbb{R}$ is a continuous and X-elliptic bilinear form, $J : X \to \mathbb{R} \cup \infty$ is convex, lower semi-continuous and proper and $l : X \to \mathbb{R}$ is a linear continuous functional. Then there exists a unique solution to the elliptic variational inequality of second kind:*

$$u \in X, \quad a(u, v-u) + J(v) - J(u) \geq l(u-v) \quad \textit{for all} \quad v \in X.$$

Theorem 0.2.4 *Let Ψ and N be two Hilbert spaces, and A a nonlinear operator from Ψ to its dual Ψ', $b : \Psi \times N \to \mathbb{R}$ a bilinear form, and $j : \Psi \to \overline{\mathbb{R}}$ a functional. The bilinear form $b(\cdot,\cdot)$ is assumed to have the following properties:*

(i) *$b(\cdot,\cdot)$ is continuous, so that there exists a bounded linear operator B defined by $B : \Psi \to N^*$,*
$\langle B\psi, m\rangle = b(\psi, m)$, for all $\psi \in \Psi$ and $m \in N$.

(ii) *$b(\cdot,\cdot)$ is inf-sup stable, that is there exists a constant $\beta \geq 0$ such that*

$$\beta\|m\|_{N/kerB^T} \leq sup_{\psi\in\Psi}\frac{b(\psi,m)}{\|\psi\|_\Psi} \tag{0.14}$$

The operator A is assumed to be coercive, monotone, hemi-continuous and bounded. The functional j is assumed to be proper, convex and l.s.c.
Then, for each $f \in \Psi'$, the problem:

$$\begin{cases} \textit{Find} \quad (\phi, m) \in \Psi \times N \quad \textit{such that} \\ \langle A\phi, \psi - \phi\rangle - b(\psi - \phi, m) + j(\psi) - j(\phi) \geq \langle f, \psi - \phi\rangle \qquad \textit{for all} \ \ \psi \in \Psi \\ b(\phi, q) = 0 \qquad \textit{for all} \ \ q \in N \ , \end{cases} \tag{0.15}$$

has a unique solution.

Note that since the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (0.14), problem (0.15) is equivalent to the following problem.

$$\begin{cases} \text{Find} \quad \phi \in \Psi \quad \text{such that} \\ \langle A\phi, \psi - \phi\rangle + j(\psi) - j(\phi) \geq \langle f, \psi - \phi\rangle \quad \text{for all} \ \ \psi \in \Psi. \end{cases} \tag{0.16}$$

0.2.4 Preliminaries on finite element approximations

There are some basic steps in the construction of finite element functions. First, we need a partition or a triangulation of the domain of the differential equation into subdomains called the elements. To each partition we associate a finite element space, and then we choose its basis functions. For practical considerations it is desirable that the basis functions have small supports. For definiteness and for the sake of simplicity, we assume that Ω is a polygonal domain which is partitioned into a finite number of elements $K \in \mathfrak{T}_h$, where the discretization parameter h is defined as $h = \max_{K \in \mathfrak{T}_h} diam(K)$.
For an arbitrary element K, we denote

$$h_K = diam(K) = max\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x}, \boldsymbol{y} \in K\}$$

and

$$\rho_K = \text{diameter of the largest sphere inscribed in K.}$$

$(\mathfrak{T}_h)_h$ is a regular family of triangulation in the sense of Ciarlet [62] if the followings are satisfied.

- For all $K_1,\ K_2 \in \mathfrak{T}_h$ such that $K_1 \neq K_2$, $K_1 \cap K_2$ is a side, a vertex or the empty set.
- $\bigcup_{K \in \mathfrak{T}_h} K = \overline{\Omega}$.
- There exists a constant $\rho \geq 1$ such that

$$\frac{h_K}{\rho_K} \leq \rho \quad \text{for all} \quad K \in \mathfrak{T}_h, \quad \text{for all} \quad h.$$

Denote by $\{\boldsymbol{x}_i\}_{i=1}^{N_h} \subset \overline{\Omega}$ the set of vertices of the elements in the partition $\mathfrak{T}_h$. For each node $\boldsymbol{x}_i$, a corresponding finite element basis function ϕ_i is such that

- $\phi_i|_K \in \mathbb{P}_k$ for all K, where $\mathbb{P}_k$ is a space of polynomials with degree less than or equal to k,
- $\phi_i(\boldsymbol{x}_j) = \delta_{ij}$, for all $1 \leq j \leq N_h$,
- let $\widetilde{K_i}$ be the patch of the elements K which contain $\boldsymbol{x}_i$ as a node, then ϕ_i is nonzero only on $\widetilde{K_i}$.

The finite element interpolant of a continuous function $v \in C(\overline{\Omega})$ is given by

$$\Pi^h v = \sum_{i=1}^{N_h} v(\boldsymbol{x}_i)\phi_i.$$

If the space $H^1(\Omega)$ is to be approximated, then the corresponding piece-wise function space is

$$X_h = span\{\phi_i,\ 1 \leq i \leq N_h\}.$$

Most boundary value problem involve essential boundary conditions and therefore need special finite element spaces. As an example, suppose that $\Gamma_1 \subset \Gamma$ is a relatively closed subset of the boundary Γ. Consider the space

$$\boldsymbol{X}^{\Gamma_1} = \{\boldsymbol{v} \in (H^1(\Omega))^d : \ v_n \leq 0 \quad \text{on} \quad \Gamma_1\},$$

where v_n is the normal component of $\boldsymbol{v}$. We define the finite element space to be $\boldsymbol{X}_h^{\Gamma_1} = X_h^d \cap \boldsymbol{X}^{\Gamma_1}$. In other words,

$$\boldsymbol{X}_h^{\Gamma_1} = \{\boldsymbol{v}_h \in (X_h)^d : \ v_{n\,h} \leq 0 \quad \text{on} \quad \Gamma_1\}.$$

Chapter 1

Finite element analysis on steady Navier-Stokes and Stokes equations driven by threshold slip boundary conditions

1.1 Introduction

This chapter is devoted to the finite element analysis of the Stokes and Navier-Stokes equations driven by threshold slip boundary conditions. The Stokes systems of equations for stationary flows of incompressible Newtonian fluids we considered satisfies

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

we assume the homogeneous Dirichlet boundary condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{1.3}$$

with the impermeability boundary condition

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } S, \tag{1.4}$$

and the slip boundary condition [41, 42]

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ , \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{array}\right\} \text{ on } \quad S. \tag{1.5}$$

Here $\Omega \subset \mathbb{R}^d$ (d=2,3) is a bounded domain, with boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is made of two components S, and Γ with $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, and $S \cap \Gamma = \emptyset$. ν is a positive quantity representing the viscosity coefficient, k is the "friction" coefficient assumed to be positive, and $g : S \to (0, \infty)$ is the barrier or threshold function. The velocity of the fluid is $\boldsymbol{u}$ and p stands for the pressure, while $\boldsymbol{f}$ is the external force. Furthermore, $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$ of Ω, $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - u_n\boldsymbol{n}$ is the tangential component of the velocity $\boldsymbol{u}$, and $(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = \boldsymbol{\sigma n} - (\boldsymbol{n} \cdot \boldsymbol{\sigma n})\boldsymbol{n}$ is the tangential traction. Of course, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{D}(\boldsymbol{u})$ is the Cauchy stress tensor, where $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$. It should quickly be mentioned that (1.5) is equivalent to

$$(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} + (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)|\boldsymbol{u}_{\boldsymbol{\tau}}| = 0 \quad \text{on } S, \tag{1.6}$$

following [69] which is rewritten with the use of sub-differential as

$$-(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \in (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)\partial|\boldsymbol{u}_{\boldsymbol{\tau}}| \quad \text{on } S, \tag{1.7}$$

where the symbol $\partial|.|$ is the sub-differential of the real value function $|.|$, with $|\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$.

The Stokes system can be considered as a simplification of the Navier-Stokes equations when convection is negligible. That is (1.1) is replaced by

$$-\nu\,\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.8}$$

with (1.2),(1.3), (1.4) and (1.5) unchanged, and the nonlinear term in (1.8) is the convection term given as

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \sum_{i=1}^{d} u_i \frac{\partial \boldsymbol{u}}{\partial x_i}.$$

Over the past few years a remarkable progress has been achieved in the field of computational contact mechanics. One of the key ingredients in this phenomenal growth is attributed to the better mathematical understanding of problems. The formulation by means of variational inequalities (see [69, 71, 57, 58, 72, 73, 74]) and the finite element method have contributed to the development of reliable frameworks for the numerical treatment of such problems. Despite such advances in the modeling and numerical treatment of contact problems with friction, it should be mentioned that most works reported in the literature are still restricted to solid mechanics. The numerical analysis works dealing with fluids flow are concerned with the standard Amontons-Coulomb law of perfect friction [75, 76, 49, 35, 77, 37, 78, 40, 33], replacing (1.5) by

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \mathbf{0}\ , \ \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{array}\right\} \text{ on } \quad S. \tag{1.9}$$

As pointed out by C. Leroux [41], such a theory can represent only a limited range of possible situations. The purpose of this chapter is to numerically analyze by means of finite element approximation equations (1.1)–(1.5), and (1.2)–(1.5),(1.8). At this juncture, it is important to recall that this type of nonlinear slip boundary conditions as far as fluid flows are concerned was first introduced by Fujita in [23, 24]. This is in continuation of a series of investigations aimed at the analysis of Stokes and Navier-Stokes equations driven by nonlinear slip boundary conditions of friction type (see [75, 76, 49]). In order to provide a background for a better mathematical understanding of the problems, we shall introduce in Section 2.2 some needed tools, and quickly indicate how the problems are solvable. At this step, we recall that in C. Leroux and Tani [41, 42] a fixed point argument is used to establish the solvability of a class of problems similar to what we want to study. It is re-introduced here because of its usefulness in the finite element analysis. Hence one can see a sort of "continuum" between the continuous and discrete analysis. The finite element formulations for both Stokes and Navier-Stokes equations are derived in Section 1.3. The finite elements are defined on conforming triangular mesh as introduced in [62], and in each triangle the velocity and pressure are taken so that the Babuska-Brezzi's condition [79, 80] is satisfied. Here, we do not use penalty method, or pressure stabilized method to enforce the incompressibility condition. Instead we use a direct method and sufficient conditions of existence of solutions are employed to derive a priori error estimates in Section 1.3. In Section 1.4, Uzawa's algorithm is formulated and analyzed for solving the Stokes and Navier-Stokes finite element discretization. It is shown that the Uzawa's algorithm converges. In Section 1.5 numerical simulations that confirm the predictions of the theory are exhibited.

1.2 Preliminaries and Variational Formulations

In this section, we introduce notation and some results that will be used throughout this chapter. We also formulate various weak formulations and discuss (recall) some existence results.

1.2.1 Notations and Preliminaries

For the analysis of (1.1)–(1.5) and (1.2)–(1.8), we introduce

$$\boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega)^d, \ \ \boldsymbol{v}|_\Gamma = 0, \ \boldsymbol{v}\cdot\mathbf{n}|_S = 0\}, \quad \boldsymbol{V}_0 = \boldsymbol{H}_0^1(\Omega),$$
$$\boldsymbol{V}_{div} = \{\boldsymbol{v} \in \boldsymbol{V}, \ \operatorname{div}\boldsymbol{v} = 0\}, \quad M = L_0^2(\Omega).$$

From Poincaré-Fredrichs's inequality, there exists a positive constant C, such that

$$\int_\Omega |\boldsymbol{v}|^2 \mathrm{dx} \le C\int_\Omega |\nabla\boldsymbol{v}|^2\mathrm{dx} \ \text{ for all } \ \boldsymbol{v}\in\boldsymbol{V}, \tag{1.10}$$

which implies that on $\boldsymbol{V}$, the semi-norm (0.5) defines a norm which is equivalent to the norm in (0.4). Also, of importance in this work is the Korn's inequality which reads; there exists a positive constant C, such that

$$\int_\Omega |\nabla\boldsymbol{v}|^2 \mathrm{dx} \le C\int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^2\mathrm{dx} \ \text{ for all } \ \boldsymbol{v}\in\boldsymbol{V}, \tag{1.11}$$

which implies that we can equip $\boldsymbol{V}$ with $\|\cdot\|_V = \|D(\cdot)\|$ which is equivalent to $\|\cdot\|_1$. We now recall classical operators associated with the formulation of the

Stokes problem (1.1)–(1.5), and Navier-Stokes problem (1.2)–(1.8) (see [79, 80]). We first introduce bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined as follows

$$a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad a(\boldsymbol{u},\boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) dx$$

$$b : \boldsymbol{V} \times M \to \mathbb{R} \quad \text{with} \quad b(\boldsymbol{u},p) = \int_\Omega p \operatorname{div} \boldsymbol{u} dx.$$

Let $d(\cdot,\cdot,\cdot)$ be the trilinear form defined as follows

$$d : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} \, dx.$$

The trilinear form $d(\cdot,\cdot,\cdot)$ is continuous on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$, i.e., there exists a positive constant C_d such that

$$|d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq C_d \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}$$

Moreover, for all $\boldsymbol{u} \in \boldsymbol{V}_{div}$ and $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}$

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -d(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \tag{1.12}$$

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0. \tag{1.13}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition, i.e., there exists a positive constant β such that

$$\beta\|p\| \leq \sup_{\boldsymbol{u}\in\boldsymbol{V}} \frac{b(\boldsymbol{u},p)}{\|\boldsymbol{u}\|_V} \qquad \text{for all} \quad p \in L_0^2(\Omega). \tag{1.14}$$

As a readily obtainable consequence of Korn's inequality (1.11), $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}$, that is

$$a(\boldsymbol{v},\boldsymbol{v}) = 2\nu\|\boldsymbol{v}\|_V^2 \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{1.15}$$

The coercivity of $a(\cdot,\cdot)$ will allow us to apply the classical existence and uniqueness result (see Theorem 0.2.3) for elliptic variational inequalities of the second kind.

1.2.1.1 Mixed Variational formulation of (1.1)–(1.5)

Suppose that $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $g \in L^2(S)$ with $g \geq 0$ on S. We multiply the equation (1.1) by $\boldsymbol{v} - \boldsymbol{u}$ for all $\boldsymbol{v} \in \boldsymbol{V}$ and integrate the resulting equation over Ω. After application of Green's formula, we obtain

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) - \int_S \boldsymbol{\sigma} n \cdot (\boldsymbol{v} - \boldsymbol{u})\, ds = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx. \tag{1.16}$$

Next, we briefly recall that since

$$\boldsymbol{\sigma n} = \boldsymbol{\sigma}_N \boldsymbol{n} + \boldsymbol{\sigma}_{\boldsymbol{\tau}}, \quad \boldsymbol{v} - \boldsymbol{u} = (v_N - u_N)\boldsymbol{n} + (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}),$$

then we have

$$\int_S \boldsymbol{\sigma n} \cdot (\mathbf{v} - \mathbf{u})\, ds \;=\; \int_S \boldsymbol{\sigma}_{\boldsymbol{\tau}} (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\, ds, \quad \text{since} \quad v_N - u_N|_\Gamma = 0. \tag{1.17}$$

On the other hand, it follows from boundary conditions (1.5) which are equivalent to (1.7) after using Definition 0.2.7 that

$$\int_S (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)(|\boldsymbol{v}_{\boldsymbol{\tau}}| - |\boldsymbol{u}_{\boldsymbol{\tau}}|) ds \geq - \int_S \boldsymbol{\sigma}_{\boldsymbol{\tau}} (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}) \mathrm{ds}. \tag{1.18}$$

We now define the functional

$$J : \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to [0, \infty) \quad \text{with} \quad J(\boldsymbol{u}, \boldsymbol{v}) = \int_S (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)|\boldsymbol{v}_{\boldsymbol{\tau}}| dx. \tag{1.19}$$

Together with (1.16)-(1.19), the following weak formulation is obtained: Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ such that

$$\begin{aligned} &\text{for all} \;\; \boldsymbol{v}, q \in \boldsymbol{V} \times M, \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + J(\boldsymbol{u}, \boldsymbol{v}) - J(\boldsymbol{u}, \boldsymbol{u}) \;&\geq\; \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle \\ b(\boldsymbol{u}, q) \;&=\; 0. \end{aligned} \tag{1.20}$$

Note that since the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (1.14), then the variational inequality problem (1.20) is equivalent to

$$\begin{cases} \text{Find} \;\; \boldsymbol{u} \in \boldsymbol{V}_{\text{div}} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{w} - \boldsymbol{u}) + J(\boldsymbol{u}, \boldsymbol{w}) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u} \rangle \;\; \text{for all} \;\; \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}. \end{cases} \tag{1.21}$$

Proposition 1.2.1 *The functional J satisfies:*

(a) for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, $J(\boldsymbol{v}, \cdot)$ is convex, nonnegative and continuous on $\boldsymbol{H}^1(\Omega)$.

(b) for all $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2 \in \boldsymbol{H}^1(\Omega)$, there exists C_0 such that

$$J(\boldsymbol{v}_1, \boldsymbol{\zeta}_2) - J(\boldsymbol{v}_1, \boldsymbol{\zeta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\zeta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\zeta}_2) \leq C_0 k \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_V. \tag{1.22}$$

Proof (a) is readily obtained. For (b), note that:

$$\begin{aligned} J(\boldsymbol{v}_1, \boldsymbol{\zeta}_2) - J(\boldsymbol{v}_1, \boldsymbol{\zeta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\zeta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\zeta}_2) &= \int_S k(|\boldsymbol{v}_1| - |\boldsymbol{v}_2|)(|\boldsymbol{\zeta}_2| - |\boldsymbol{\zeta}_1|)ds \\ &\leq \int_S k(|\boldsymbol{v}_1 - \boldsymbol{v}_2|)(|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|)ds \\ &\leq C_0 k \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_V. \end{aligned}$$

The main result of this subsection is the following

Theorem 1.2.1 *Suppose that*

$$0 < \frac{C_0 k}{2\nu} < 1. \tag{1.23}$$

Then the mixed variational problem (1.20) admits a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$, which satisfies the following bound

$$\begin{aligned} \|\boldsymbol{u}\|_V &\leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}), &(1.24) \\ \|p\| &\leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}). &(1.25) \end{aligned}$$

It is clear from our condition (1.23), that we either need; a large enough viscosity or a small friction coefficient.
The proof of Theorem 1.2.1 is based on fixed point arguments and established in two steps.
First, let $\boldsymbol{v} \in \boldsymbol{V}_{\text{div}}$ and the functional $J_{\boldsymbol{v}}$ defined on $\boldsymbol{H}^1(\Omega)$ by $J_{\boldsymbol{v}}(\boldsymbol{w}) = J(\boldsymbol{v}, \boldsymbol{w})$. By Proposition 1.2.1, $J_{\boldsymbol{v}}$ is convex, non negative and continuous. We consider the following variational inequality.

$$\begin{cases} \text{Find } \boldsymbol{\eta_v} \in \boldsymbol{V}_{\text{div}} \text{ such that} \\ a(\boldsymbol{\eta_v}, \boldsymbol{w} - \boldsymbol{\eta_v}) + J_{\boldsymbol{v}}(\boldsymbol{w}) - J_{\boldsymbol{v}}(\boldsymbol{\eta_v}) \geq \langle \mathbf{f}, \boldsymbol{w} - \boldsymbol{\eta_v} \rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}} . \end{cases} \tag{1.26}$$

It follows from Theorem 0.2.3 that

Lemma 1.2.1 *there exists a unique solution $\boldsymbol{\eta_v} \in \boldsymbol{V}_{\text{div}}$ to the problem (1.26).*

Next, let us consider an operator $\Phi : \boldsymbol{V}_{\text{div}} \to \boldsymbol{V}_{\text{div}}$ defined by

$$\Phi(\boldsymbol{v}) = \boldsymbol{\eta_v},$$

where $\boldsymbol{\eta_v}$ is a unique solution of problem (1.26).

Lemma 1.2.2 *Suppose that $0 < \frac{C_0 k}{2\nu} < 1$, then the operator Φ has a unique fixed point $\boldsymbol{u} \in \boldsymbol{V}_{div}$.*

Proof Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \boldsymbol{V}_{\text{div}}$ and set $\boldsymbol{\eta}_1 = \Phi(\boldsymbol{v}_1)$, $\boldsymbol{\eta}_2 = \Phi(\boldsymbol{v}_2)$. Then, we have

$$a(\boldsymbol{\eta}_1, \boldsymbol{w} - \boldsymbol{\eta}_1) + J_{\boldsymbol{v}1}(\boldsymbol{w}) - J_{\boldsymbol{v}1}(\boldsymbol{\eta}_1) \geq \langle \mathbf{f}, \boldsymbol{w} - \boldsymbol{\eta}_1 \rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}} , \tag{1.27}$$

and

$$a(\boldsymbol{\eta}_2, \boldsymbol{w} - \boldsymbol{\eta}_2) + J_{\boldsymbol{v}2}(\boldsymbol{w}) - J_{\boldsymbol{v}2}(\boldsymbol{\eta}_2) \geq \langle \mathbf{f}, \boldsymbol{w} - \boldsymbol{\eta}_2 \rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}} . \tag{1.28}$$

Taking $\boldsymbol{w} = \boldsymbol{\eta}_2$ in (1.27) and $\boldsymbol{w} = \boldsymbol{\eta}_1$ in (1.28) and add the resultant equations, we obtain

$$a(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \leq J(\boldsymbol{v}_1, \boldsymbol{\eta}_2) - J(\boldsymbol{v}_1, \boldsymbol{\eta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\eta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\eta}_2).$$

It follows from (3.15) and (1.22) that

$$2\nu \|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_1^2 \leq C_0 k \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_1 \|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_1,$$

we have

$$\|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_1 \leq \frac{C_0 k}{2\nu} \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_1,$$

that is

$$\|\Phi(\boldsymbol{v}_2) - \Phi(\boldsymbol{v}_1)\|_1 \leq M\|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_1,$$

with $0 < M := \frac{C_0 k}{2\alpha} < 1$. Then Φ is a contraction of the Hilbert space $\boldsymbol{V}_{\text{div}}$ and has a unique fixed point $\boldsymbol{u} \in \boldsymbol{V}_{\text{div}}$. A unique fixed point $\boldsymbol{u}$ is a unique solution of the variational problem (1.21).

To derive the a priori estimate (1.24), let $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{w} = 2\boldsymbol{u}$ in (1.21), one has

$$a(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}, \boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u} \rangle,$$

which from (3.15), and Theorem 0.2.1 gives

$$2\nu\|\boldsymbol{u}\|_V \leq \|\boldsymbol{f}\|_{-1} + C\|g\|_{L^2(S)}.$$

Next, we derive the a priori bound for the pressure. For that purpose, let $\boldsymbol{w} \in \mathbf{V}_0$, and replace $\boldsymbol{v}$ in (1.20) successively by $\boldsymbol{u} + \boldsymbol{w}$ and $\boldsymbol{u} - \boldsymbol{w}$, and observe that $J(\boldsymbol{u}, \boldsymbol{v}) = J(\boldsymbol{u}, \boldsymbol{u} \pm \boldsymbol{w}) = J(\boldsymbol{u}, \boldsymbol{u})$. Then one obtains

$$a(\boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{w}, p) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \quad \text{for all} \quad \boldsymbol{w} \in \boldsymbol{V}_0. \tag{1.29}$$

Next, from the compatibility condition (1.14) and (1.29), one has

$$\begin{aligned} \beta\|p\| &\leq \sup_{\boldsymbol{w} \in \boldsymbol{V}_0} \frac{b(\boldsymbol{w}, p)}{\|\boldsymbol{w}\|_V} = \sup_{\boldsymbol{w} \in \boldsymbol{V}_0} \frac{|a(\boldsymbol{u}, \boldsymbol{w}) - \langle \mathbf{f}, \boldsymbol{w} \rangle|}{\|\boldsymbol{w}\|_V} \\ &\leq 2\nu\|\boldsymbol{u}\|_V + \|\boldsymbol{f}\|_{-1}, \end{aligned}$$

and the use of the bound on $\boldsymbol{u}$ leads to the desired estimate.

1.2.1.2 Mixed Variational formulation (1.2)–(1.5) and (1.8)

Suppose that $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $g \in L^2(S)$ with $g \geq 0$ on S. We multiply (1.8) by $\boldsymbol{v} - \boldsymbol{u}$ for all $\boldsymbol{v} \in \boldsymbol{V}$, integrate the resulting equation over Ω, and apply Green's formula to obtain

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) - \int_S \boldsymbol{\sigma} \cdot (\boldsymbol{v} - \boldsymbol{u})\, ds = \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle.$$

According to the relations (1.17), (1.18) and (1.19), the weak formulation of (1.2)-(1.8) can be written as follows: Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ such that

$$\begin{array}{rcl} & \text{for all} \quad (\boldsymbol{v}, q) \in \boldsymbol{V} \times M & \\ a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u}, p) + J(\boldsymbol{u}, \boldsymbol{v}) - J(\boldsymbol{u}, \boldsymbol{u}) & \geq & \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u} \rangle \\ b(\boldsymbol{u}, q) & = & 0. \end{array} \tag{1.30}$$

Since the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (1.14), then the variational inequality problem (1.30) is equivalent to

$$\begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V}_{\text{div}} \text{ such that for all } \quad \boldsymbol{w} \in \boldsymbol{V}_{\text{div}} \\ a(\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}) + J(\boldsymbol{u}, \boldsymbol{w}) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{w}-\boldsymbol{u} \rangle. \end{cases} \tag{1.31}$$

By the contraction mapping principle, we can prove the following existence and uniqueness theorem.

Theorem 1.2.2 *If the following conditions hold:*

$$0 < \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{\nu^2} < 1, \tag{1.32}$$

$$0 < \frac{C_0 k}{2\nu} < 1 - \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}, \tag{1.33}$$

then the mixed variational problem (1.31) admits a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{K}_{\text{div}} \times M$, *which satisfies the following bound*

$$\|\boldsymbol{u}\|_V \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}), \tag{1.34}$$

$$\|p\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)} + \|\boldsymbol{f}\|^2_{-1} + \|g\|^2_{L^2(S)}). \tag{1.35}$$

where C_1 *satisfies*

$$\left| \langle \boldsymbol{f}, \boldsymbol{v} \rangle - \int_S g |\boldsymbol{v}_{\boldsymbol{\tau}}| ds \right| \leq C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}) \|\boldsymbol{v}\|_V \qquad \textit{for all} \quad \boldsymbol{v} \in \boldsymbol{V}$$

and

$$\boldsymbol{K}_{\text{div}} = \{\boldsymbol{v} \in \boldsymbol{V}_{\text{div}} \ , \quad \|\boldsymbol{v}\|_V \leq \frac{C_1}{\nu} \left(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)} \right) \}.$$

Proof. The proof of Theorem 1.2.2 follows the same lines as the proof of Theorem 1.2.1, but it is more involved because of the additional nonlinear convection term.

First, for a fixed $\boldsymbol{v} \in \boldsymbol{K}_{\text{div}}$, consider the following variational inequality problem:

$$\begin{cases} \text{Find } \boldsymbol{\eta_v} \in \boldsymbol{K}_{\text{div}} \text{ such that for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}, \\ a(\boldsymbol{\eta_v}, \boldsymbol{w} - \boldsymbol{\eta_v}) + d(\boldsymbol{\eta_v}, \boldsymbol{\eta_v}, \boldsymbol{w} - \boldsymbol{\eta_v}) + J_{\boldsymbol{v}}(\boldsymbol{w}) - J_{\boldsymbol{v}}(\boldsymbol{\eta_v}) \geq \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{\eta_v} \rangle. \end{cases} \tag{1.36}$$

Lemma 1.2.3 *Assume that the condition (1.32) holds, then there exists a unique solution* $\boldsymbol{\eta_v} \in \boldsymbol{K}_{\text{div}}$ *to the problem (1.36).*

The proof of this Lemma can be found in [78, Theorem 2.1, P553] where similar condition is needed.
Next, let us consider the mapping $\Phi : \boldsymbol{K}_{\text{div}} \to \boldsymbol{K}_{\text{div}}$ defined as follows

$$\Phi(\boldsymbol{v}) = \boldsymbol{\eta_v}$$

where $\boldsymbol{\eta_v}$ is a unique solution of problem (1.36). It is obvious that the fixed point of Φ if exists will be the solution of (1.31).

Lemma 1.2.4 *Under the assumption of Theorem 1.2.2, the operator* Φ *will be a contraction on* $\boldsymbol{K}_{\text{div}}$.

Proof Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \boldsymbol{K}_{\text{div}}$ and set $\boldsymbol{\eta}_1 = \Phi(\boldsymbol{v}_1)$, $\boldsymbol{\eta}_2 = \Phi(\boldsymbol{v}_2)$ then, we have

$$a(\boldsymbol{\eta}_1, \boldsymbol{w}-\boldsymbol{\eta}_1)+d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{w}-\boldsymbol{\eta}_1)+J_{\boldsymbol{v}1}(\boldsymbol{w})-J_{\boldsymbol{v}1}(\boldsymbol{\eta}_1) \geq \langle \mathbf{f}, \boldsymbol{w}-\boldsymbol{\eta}_1 \rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}, \tag{1.37}$$

and

$$a(\boldsymbol{\eta}_2, \boldsymbol{w}-\boldsymbol{\eta}_2)+d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{w}-\boldsymbol{\eta}_2)+J_{\boldsymbol{v}2}(\boldsymbol{w})-J_{\boldsymbol{v}2}(\boldsymbol{\eta}_2) \geq \langle \mathbf{f}, \boldsymbol{w}-\boldsymbol{\eta}_2 \rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}. \tag{1.38}$$

Taking $\boldsymbol{w} = \boldsymbol{\eta}_2$ in (1.37) and $\boldsymbol{w} = \boldsymbol{\eta}_1$ in (1.38) and add the resultant equations, we obtain

$$\begin{aligned} a(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \;\; &\leq \;\; d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) - d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) + J(\boldsymbol{v}_1, \boldsymbol{\eta}_2) \\ &- \;\; J(\boldsymbol{v}_1, \boldsymbol{\eta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\eta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\eta}_2) \end{aligned}$$

On the other hand, since from (1.13) $d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) = 0$,

$$\begin{aligned} d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) - d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) &= d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) + d(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \\ &= d(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1), \end{aligned}$$

then

$$a(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \leq d(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) + J(\boldsymbol{v}_1, \boldsymbol{\eta}_2) - J(\boldsymbol{v}_1, \boldsymbol{\eta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\eta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\eta}_2).$$

It follows from (3.15), (1.22) and the continuity of $d(\cdot, \cdot, \cdot)$ that

$$2\nu \|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_V^2 \leq C_d \|\boldsymbol{\eta}_1\|_V \|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_V^2 + C_0 k \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_V \|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_V,$$

and due to the fact that $\boldsymbol{\eta}_1 \in \boldsymbol{K}_{div}$, $\|\boldsymbol{\eta}_1\|_V \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})$,
we have

$$\|\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1\|_V \leq \frac{C_0 k / 2\nu}{\left(1 - \dfrac{C_d C_1 (\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)} \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_V,$$

that is

$$\|\Phi(\boldsymbol{v}_2) - \Phi(\boldsymbol{v}_1)\|_V \leq L \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_V,$$

with $0 < L := \dfrac{C_0 k / 2\nu}{\left(1 - \dfrac{C_d C_1 (\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)} < 1.$

The derivation of the a priori estimates for the velocity and pressure are similar to the one obtained for the Stokes equations and will not be repeated here. □

Remark 1.2.1

(a) *It should be noted that (1.32) is only needed for (1.36), while (1.32) and (1.33) are required for (1.31).*

(b) *It is manifest that in both conditions (1.32), and (1.33), we need smallness of the applied forces or large enough kinematic viscosity. In fact such requirement are not new, and are similar to those needed for Navier-Stokes equations with classical Dirichlet boundary conditions [79, 81, 82].*

1.3 Finite element approximations

We assume that $\mathcal{T}_h$ is a regular partition of Ω in the sense introduced by Ciarlet [62]. The diameter of an element $K \in \mathcal{T}_h$ is denoted by h_K, and the mesh size h is defined by $h = \max_{K\in\mathcal{T}_h} h_K$. Let introduce the following subspaces:

$$\begin{aligned}
M_h &= \{q_h \in M \cap \mathcal{C}(\Omega), \quad q_h|_K \in \mathcal{P}_l(K) \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{V}_h &= \{\boldsymbol{v}_h \in \boldsymbol{V} \cap \mathcal{C}(\Omega)^d, \quad \boldsymbol{v}_h|_K \in \mathcal{P}_k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{W}_h &= \{\boldsymbol{v}_h \in \boldsymbol{V}_h\ , \quad b(\boldsymbol{v}_h, q_h) = 0 \quad \text{for all} \quad q_h \in M_h\},\\
\boldsymbol{V}_{0h} &= \boldsymbol{V}_h \cap \boldsymbol{V}_0,
\end{aligned}$$

$\mathcal{P}_l(K)$ the space of polynomial functions of two variables in K with degree less than or equal to l. In fact the integers k, l are such that the discrete counterpart of the inf-sup condition (1.14) holds with its constant β_h independent of h. For instance, the choice of $\mathcal{P}_k/\mathcal{P}_{k-1}$ elements with $k \geq 2$ satisfies this inf-sup condition (see [83, Theorem 8.1,P77]). We can also cite [79, 80] for more discussions.

1.3.1 Finite element approximation of the variational inequality (1.20)

1.3.1.1 Existence and uniqueness of solution

With the finite dimensional spaces $\boldsymbol{V}_h$ and M_h introduced, the finite element discretization of the variational inequality (1.20) reads: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned}
&\text{for all} \quad (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h,\\
a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) - b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) &\geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h\rangle,\\
b(\boldsymbol{u}_h, q_h) &= 0,
\end{aligned} \tag{1.39}$$

which is equivalent to

$$\begin{cases}
\text{Find} \quad \boldsymbol{u}_h \in \boldsymbol{W}_h \text{ such that for all } \quad \boldsymbol{w}_h \in \boldsymbol{W}_h\\
a(\boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + J(\boldsymbol{u}_h, \boldsymbol{w}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{w}_h - \boldsymbol{u}_h\rangle.
\end{cases} \tag{1.40}$$

The existence of solutions of (1.40) follows the same procedure as the existence result for (1.20), and thus it holds that

Theorem 1.3.1 *Suppose that* $0 < \dfrac{C_0 k}{2\nu} < 1$. *Then the mixed variational problem (1.39) admits a unique solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$, *which satisfies the following bound*

$$\|\boldsymbol{u}_h\|_V + \|p_h\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}). \tag{1.41}$$

1.3.1.2 A priori error estimate

One of the main contributions of this chapter is the following result

Theorem 1.3.2 *Suppose that* $0 < \dfrac{C_0 k}{2\nu} < 1$. *Let* $(\boldsymbol{u}, p)$ *be the unique solution of (1.20), and* $(\boldsymbol{u}_h, p_h)$ *the unique solution of (1.39). Then there exists a generic positive constant* C *independent on* h *such that for all* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *and* $q_h \in M_h$,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}, \tag{1.42}$$

$$\|p - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}. \tag{1.43}$$

Proof For $\boldsymbol{w} \in \boldsymbol{V}_0$, we replace $\boldsymbol{v}$ in (1.20) by $\boldsymbol{u}+\boldsymbol{w}$ and $\boldsymbol{u}-\boldsymbol{w}$, and putting together the resulting equations, one gets

$$a(\boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{w}, p) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \quad \text{for all } \boldsymbol{w} \in \boldsymbol{V}_0. \tag{1.44}$$

Likewise with (1.39) and $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$, one arrives at

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \quad \text{for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}. \tag{1.45}$$

Let $\boldsymbol{w} = \boldsymbol{w}_h$, then (1.44) and (1.45) give

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p - p_h) = 0 \quad \text{for all} \quad \boldsymbol{w}_h \in \boldsymbol{V}_{0h},$$

which is re-written as

$$b(\boldsymbol{w}_h, p_h - q_h) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h).$$

Now, the equality together with the discrete version of the inf-sup condition (1.14) and the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ gives

$$\begin{aligned}\beta\|p_h - q_h\| \leq \sup_{\boldsymbol{w}_h\in\boldsymbol{V}_{0h}} \frac{b(\boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_V} &= \sup_{\boldsymbol{w}_h\in\boldsymbol{V}_{0h}} \frac{|a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h)|}{\|\boldsymbol{w}_h\|_V} \\ &\leq (2\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p - q_h\|),\end{aligned}$$

so that,

$$\|p - p_h\| \leq \|p - q_h\| + \|q_h - p_h\| \leq C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p - q_h\|\right\}. \tag{1.46}$$

Next, let $\boldsymbol{v}_h \in \boldsymbol{V}_h$, replacing successively $\boldsymbol{v}$ in $(1.20)_1$ by $\boldsymbol{v} = \boldsymbol{u}_h$ and $\boldsymbol{v} = 2\boldsymbol{u} - \boldsymbol{v}_h$ and putting together the resulting inequalities, yields

$$a(\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{v}_h) - b(\boldsymbol{u}_h-\boldsymbol{v}_h, p) + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) + J(\boldsymbol{u},\boldsymbol{u}_h) - 2J(\boldsymbol{u},\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle. \tag{1.47}$$

Note that the inequality $(1.39)_1$ can be recast as

$$-a(\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h) + b(\boldsymbol{u}_h-\boldsymbol{v}_h, p_h) + J(\boldsymbol{u}_h,\boldsymbol{v}_h) - J(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq -\langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle. \tag{1.48}$$

Next, (1.47)+(1.48) yields

$$\begin{aligned}a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h) - b(\boldsymbol{u}_h-\boldsymbol{v}_h, p - p_h) + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) \quad + \\ J(\boldsymbol{u},\boldsymbol{u}_h) - 2J(\boldsymbol{u},\boldsymbol{u}) + J(\boldsymbol{u}_h,\boldsymbol{v}_h) - J(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq 0.\end{aligned} \tag{1.49}$$

By linearity of $a(\cdot,\cdot)$,

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h) = a(\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h) - a(\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h). \tag{1.50}$$

Using $(1.20)_2$ and $(1.39)_2$, one has

$$\begin{aligned}b(\boldsymbol{u}_h-\boldsymbol{v}_h, p - p_h) &= b(\boldsymbol{u}_h-\boldsymbol{u}, p - q_h) + b(\boldsymbol{u}_h-\boldsymbol{u}, q_h - p_h) + b(\boldsymbol{u}-\boldsymbol{v}_h, p - p_h) \\ &= b(\boldsymbol{u}_h-\boldsymbol{u}, p - q_h) + b(\boldsymbol{u}-\boldsymbol{v}_h, p - p_h).\end{aligned} \tag{1.51}$$

Returning to (1.49) with (1.50) and (1.51), we obtain

$$2\nu\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V^2 \leq a(\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h) \leq I_1 + I_2, \tag{1.52}$$

where

$$I_1 = a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{u}, p - q_h) - b(\boldsymbol{u} - \boldsymbol{v}_h, p - p_h), \quad (1.53)$$

$$I_2 = J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h). \quad (1.54)$$

By simple algebra manipulation, we obtain

$$I_2 = J(\boldsymbol{u}, \boldsymbol{u}_h) - J(\boldsymbol{u}, \boldsymbol{v}_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) + J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) \\ -2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}, \boldsymbol{v}_h),$$

which from (1.22), and the application of the triangle's inequality gives

$$I_2 \leq C_0 k \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + 2(\|g\|_{L^2(S)} + C_0 k \|\boldsymbol{u}\|_V) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}. \quad (1.55)$$

Next, applying the continuity of both bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we get

$$I_1 \leq 2\nu \|\boldsymbol{u} - \boldsymbol{v}_h\|_V \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + \|p - q_h\| \|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\| \|\boldsymbol{u} - \boldsymbol{v}_h\|_V,$$

together with (1.55) and (1.52) gives

$$2\nu \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V^2 \leq \left\{ \begin{array}{c} 2\nu \|\boldsymbol{u} - \boldsymbol{v}_h\|_V \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + \|p - q_h\| \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \\ + \|p - p_h\| \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + C_0 k \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V \\ +2(\|g\|_{L^2(S)} + C_0 k \|\boldsymbol{u}\|_V) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)} \end{array} \right\},$$

which together with Young's inequality, the a priori estimate (1.34), (1.46), and the triangle inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|\boldsymbol{v}_h - \boldsymbol{u}_h\|_V,$$

gives the desired bound (1.42), whereas (1.43) is a consequence of (1.42) and (1.46). □

1.3.2 Finite element approximation of the variational inequality (1.30)

1.3.2.1 Existence and uniqueness of solution

The finite element approximation of the variational inequality (1.30) reads: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{cases} \text{for all} \quad \boldsymbol{v}_h, q_h \in \boldsymbol{V}_h \times M_h \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) - b(p_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \\ +J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle \\ b(\boldsymbol{u}_h, q_h) = 0, \end{cases} \tag{1.56}$$

which is equivalent to: Find $\boldsymbol{u}_h \in \boldsymbol{W}_h$ such that

$$\begin{cases} \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle. \end{cases} \tag{1.57}$$

As far as the existence of solutions of (1.56) is concerned, we claim that

Theorem 1.3.3 *If the following conditions hold:*

$$0 < \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{\nu^2} < 1 \tag{1.58}$$

$$0 < \frac{C_0 k}{2\nu} < 1 - \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}. \tag{1.59}$$

Then the mixed finite variational problem (1.56) admits a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{K}_h \times M_h$, *which satisfies the following bound*

$$\|\boldsymbol{u}_h\|_V \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}) \tag{1.60}$$

$$\|p_h\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)} + \|\boldsymbol{f}\|_{-1}^2 + \|g\|_{L^2(S)}^2). \tag{1.61}$$

where

$$\boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{W}_h, \quad \|\boldsymbol{v}_h\|_1 \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\}.$$

The proof goes along the same lines as the proof of Theorem 1.2.2, and hence will not be repeated here.

1.3.2.2 A priori error estimate

Theorem 1.3.4 *If conditions (1.58) and (1.59) are satisfied with* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$*,* $g \in L^2(S)$*, and* $g > 0$*, then there exists a generic positive constant* C *independent of* h *such that for all* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *and* $q_h \in M_h$*,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}, \tag{1.62}$$

$$\|p - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}. \tag{1.63}$$

Proof Let $\boldsymbol{w} \in \boldsymbol{V}_0$. Replacing $\boldsymbol{v}$ in (1.30) by $\boldsymbol{u} - \boldsymbol{w}$ and $\boldsymbol{u} + \boldsymbol{w}$ and adding the resulting equations gives

$$a(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{w}, p) = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \quad \text{for all } \boldsymbol{w} \in \boldsymbol{V}_0.$$

Next, let $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$, and replace $\boldsymbol{v}_h$ in (1.56) by $\boldsymbol{u}_h - \boldsymbol{w}_h$ and $\boldsymbol{u}_h + \boldsymbol{w}_h$, adding the resulting equations, one gets

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \quad \text{for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}.$$

Putting together the former and later equations for $\boldsymbol{w} = \boldsymbol{w}_h$, gives

$$a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{w}_h)+d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h)-d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h)-b(\boldsymbol{w}_h, p-p_h) = 0 \quad \text{for all} \quad \boldsymbol{w}_h \in \boldsymbol{V}_{0h}. \tag{1.64}$$

From the linearity

$$\begin{aligned} b(\boldsymbol{w}_h, p_h - q_h) &= b(\boldsymbol{w}_h, p_h - p) + b(\boldsymbol{w}_h, p - q_h), \\ d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) &= d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h), \end{aligned}$$

which together with the inf-sup on $b(\cdot,\cdot)$ and (1.64) gives

$$\begin{aligned} &\beta\|p_h - q_h\| \leq \sup_{\boldsymbol{w}\in\boldsymbol{V}_{0h}} \frac{b(\boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_V} \\ = &\sup_{\boldsymbol{w}_h\in\boldsymbol{V}_{0h}} \frac{|a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h)|}{\|\boldsymbol{w}_h\|_V} \\ \leq &(2\nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}\|_V\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|) \\ \leq &(2\nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \frac{2C_dC_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|) \\ &\text{since} \quad \boldsymbol{u} \in \boldsymbol{K}_{div} \quad \text{and} \quad \boldsymbol{u}_h \in \boldsymbol{K}_h. \end{aligned}$$

Hence

$$\|p - p_h\| \leq \|p - q_h\| + \|q_h - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|\right\}. \tag{1.65}$$

Next, we take $\boldsymbol{v}_h \in \boldsymbol{V}_h$, replacing successively $\boldsymbol{v}$ in $(1.30)_1$ by $\boldsymbol{v} = \boldsymbol{u}_h$ and $\boldsymbol{v} = 2\boldsymbol{u} - \boldsymbol{v}_h$, one gets

$$a(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) - b(\boldsymbol{u}_h - \boldsymbol{u}, p) + J(\boldsymbol{u}, \boldsymbol{u}_h) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{u}\rangle, \tag{1.66}$$

and

$$a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}_h) - b(\boldsymbol{u} - \boldsymbol{v}_h, p) + J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{v}_h\rangle. \tag{1.67}$$

(1.66)+(1.67) yields

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{v}_h, p) \\ &+J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle. \end{aligned} \tag{1.68}$$

Note that the inequality $(1.56)_1$ can be written as

$$\begin{aligned} -a(\boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + b(\boldsymbol{u}_h - \boldsymbol{v}_h, p_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) \\ -J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq -\langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle, \end{aligned}$$

which together with (1.68) leads to

$$\begin{aligned} &a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{v}_h, p - p_h) \\ + \;&J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0. \end{aligned} \tag{1.69}$$

Note that

$$d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) = d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h). \tag{1.70}$$

Substituting equations (1.50), (1.51) and (1.70) into (1.69) yields

$$\begin{aligned} 2\nu\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V^2 &\leq a(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) \\ &\leq a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{u}, p - q_h) - b(\boldsymbol{u} - \boldsymbol{v}_h, p - p_h \\ &\quad +d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) \\ &\quad +J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h). \end{aligned} \tag{1.71}$$

Using standard inequalities and (1.55), (1.71) becomes

$$2\nu\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V^2 \leq \left\{ \begin{array}{c} 2\nu\|\boldsymbol{u} - \boldsymbol{v}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + \|p - q_h\|\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \\ +\|p - p_h\|\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + C_d\|\boldsymbol{u}\|_V\|\boldsymbol{u} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V \\ +C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + C_0k\|\boldsymbol{u} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V \\ +2(\|g\|_{L^2(S)} + C_0k\|\boldsymbol{u}\|_V)\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)} \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} 2\nu\|\boldsymbol{u} - \boldsymbol{v}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + \|p - q_h\|\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \\ +\frac{2C_dC_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{u} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V \\ +C_0k\|\boldsymbol{u} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_V + \|p - p_h\|\|\boldsymbol{u} - \boldsymbol{v}_h\|_V \\ +2(\|g\|_{L^2(S)} + \frac{C_0kC_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}))\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)} \end{array} \right\}$$

$$\text{since} \quad \boldsymbol{u} \in \boldsymbol{K}_{div} \quad \boldsymbol{u}_h \in \boldsymbol{K}_h$$

Hence using the triangle inequalities, the Young's inequality, and the relations (1.58), (1.59) and (1.65), we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V^2 \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V^2 + \|p - q_h\|^2 + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}\right\},$$

which automatically gives (1.62), while (1.63) is a consequence of (1.62) and (1.65).
□

Remark 1.3.1 *It should be mentioned that specific choice of $\boldsymbol{V}_h$ and M_h leads derivation of particular rate of convergence in Theorem 1.3.2 and Theorem 1.3.4. (see [79, 80]).*

1.4 Numerical Algorithm

In this section, we present and analyze the algorithms for the implementation of (1.39) and (1.56). Next, we present some numerical computations related to the algorithms described.

1.4.1 Numerical algorithm for Stokes variational inequality (1.39)

Let us consider the following problem: Given $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned} &\text{for all} \quad \boldsymbol{v}_h, q_h \in \boldsymbol{V}_h \times M_h \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) - b(\boldsymbol{v}_h - \boldsymbol{u}_h^n, p_h^n) + J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &\geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle \\ b(\boldsymbol{u}_h^n, q_h) &= 0, \end{aligned} \tag{1.72}$$

which is also equivalent to; given $\boldsymbol{u}_h^0 \in \boldsymbol{W}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{W}_h$ such that

$$\begin{aligned} &\text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{W}_h \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &\geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle. \end{aligned} \tag{1.73}$$

About the convergence of the algorithm (1.72), one can claim the following

Theorem 1.4.1 *Suppose that* $0 < \dfrac{C_0 k}{2\nu} < 1$, *problem (1.72) admits a unique solution* $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$. *Moreover let* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ *be the solution of problem (1.39). Then the iterative solution* $(\boldsymbol{u}_h^n, p_h^n)$ *converges to* $(\boldsymbol{u}_h, p_h)$ *in* $\boldsymbol{V}_h \times M_h$ *as* $n \to \infty$. *More precisely,*

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \left(\frac{C_0 k}{2\nu}\right)^n \|\boldsymbol{u}_h^0 - \boldsymbol{u}_h\|_V \tag{1.74}$$

$$\|p_h^n - p_h\| \leq C\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \ . \tag{1.75}$$

Proof. For the solvability of (1.72), note that knowing $\boldsymbol{u}_h^{n-1} \in \boldsymbol{V}_h$, computing $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ in (1.72) is to solve the variational inequality of the second kind with g replaced by $g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|$.
Next, let $\boldsymbol{v}_h = \boldsymbol{u}_h^n$ in (1.40) and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (1.73), adding the resulting equations, we obtain:

$$a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \leq J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h, \boldsymbol{u}_h^n) - J(\boldsymbol{u}_h, \boldsymbol{u}_h). \tag{1.76}$$

(3.59) is treated using the coercivity (3.15) on the left, whereas its right hand side is bounded using the inequality (1.22). We then obtain

$$2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V^2 \leq C_0 k\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V,$$

which gives

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \frac{C_0 k}{2\nu}\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V.$$

By induction one has (1.74).
For the convergence of p_h^n, let $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$ and replace $\boldsymbol{v}_h$ in $(1.72)_1$ successively by $\boldsymbol{u}_h^n + \boldsymbol{w}_h$ and $\boldsymbol{u}_h^n - \boldsymbol{w}_h$. Observe that $J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n \pm \boldsymbol{w}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)$, then

$$a(\boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}. \tag{1.77}$$

Likewise, one obtains

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h},$$

which together with (1.77) gives

$$a(\boldsymbol{u}_h - \boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h - p_h^n) = 0 \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}.$$

That relation together with the discrete inf-sup condition on $b(\cdot,\cdot)$ leads to,

$$\beta\|p_h^n - p_h\| \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, p_h - p_h^n)}{\|\boldsymbol{v}_h\|_V} = \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a(\boldsymbol{u}_h - \boldsymbol{u}_h^n, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_1} \leq 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V,$$

and the proof is terminated using (1.74). □

Remark 1.4.1 *Knowing* $\boldsymbol{u}_h^{n-1} \in \boldsymbol{V}_h$, *computing* $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ *in (1.72) is to solve the variational inequality of the second kind with* g *replaced by* $g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|$ *which can be solved numerically using Uzawa iteration method (see [71, 77, 33, 79]).*

Then we construct the following Uzawa iteration algorithm to solve (1.39) via (1.72)
Algorithm 1:

$$\boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \quad \lambda_h^1 \in \Lambda_h \quad \text{arbitrary given} \tag{1.78}$$

where $\Lambda = \{\lambda \in L^2(S) : \ |\lambda(x)| \leq 1 \ a.e. \text{ on } S\}$ and $\Lambda_h \subset \Lambda$ is the finite element space.

Step 1: knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n) \in \boldsymbol{V}_h \times \Lambda_h$, compute $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ by

$$\begin{cases} a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - \int_S \lambda_h^n (g + k|\boldsymbol{u}_{\boldsymbol{\tau} h}^{n-1}|) \boldsymbol{v}_{\boldsymbol{\tau} h} ds \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h^n, q_h) = 0 \quad \text{for all} \quad q_h \in M_h \ , \end{cases} \tag{1.79}$$

Step 2: Renew $\lambda_h^{n+1} \in \Lambda_h$

$$\lambda_h^{n+1} = P_{\Lambda_h}(\lambda_h^n + \rho(g + k|\boldsymbol{u}_{\boldsymbol{\tau} h}^{n-1}|)\boldsymbol{u}_{\boldsymbol{\tau} h}^n) \tag{1.80}$$

where $P_{\Lambda_h}(\mu) = \sup(-1, \inf(1, \mu))$ for all $\mu \in L^2(S)$ and $\rho > 0$.

Remark 1.4.2 *The unique existence of $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ satisfying (1.79) is guaranteed by the discrete inf-sup condition on $b(\cdot, \cdot)$.*

1.4.2 Numerical algorithm for Navier-Stokes variational inequality (1.56)

Let us consider the following problem: Given $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned} &\text{for all} \quad \boldsymbol{v}_h, q_h \in \boldsymbol{V}_h \times M_h \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) - b(\boldsymbol{v}_h - \boldsymbol{u}_h^n, p_h^n)& \\ +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &\geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle, \\ b(\boldsymbol{u}_h^n, q_h) &= 0, \end{aligned} \tag{1.81}$$

which is equivalent to

$$\begin{cases} \text{Knowing} \ \ \boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \ \ \text{Find} \ \ \boldsymbol{u}_h^n \in \boldsymbol{W}_h \text{ such that} \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) \\ +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle \ \ \text{for all} \ \ \boldsymbol{v}_h \in \boldsymbol{W}_h. \end{cases} \tag{1.82}$$

About the convergence of the algorithm (1.81), we claim that

Theorem 1.4.2 *Assume (1.32), and (1.33). Then the problem (1.81) admits a unique solution $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{K}_h \times M_h$. Moreover let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ be* *the solution of problem (1.56). Then the iterative solution* $(\boldsymbol{u}_h^n, p_h^n)$ *converges to* $(\boldsymbol{u}_h, p_h)$ *in* $\boldsymbol{V}_h \times M_h$ *as* $n \to \infty$. *More precisely,*

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)^n \|\boldsymbol{u}_h^0 - \boldsymbol{u}_h\|_V \quad (1.83)$$

$$\|p_h^n - p_h\| \leq C(\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + \|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V) \ . \quad (1.84)$$

Proof. We start by proving that $\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})$ i.e, $\boldsymbol{u}_h^n \in \boldsymbol{K}_h$. Let $\boldsymbol{w}_h = \boldsymbol{0}$ and $\boldsymbol{w}_h = 2\boldsymbol{u}_h^n$ in (1.82), since $d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n) = 0$, one has

$$\begin{aligned} 2\nu\|\boldsymbol{u}_h^n\|_V^2 \leq \int_S k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}||\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds + a(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) &= \langle \boldsymbol{f}, \boldsymbol{u}_h^n\rangle - \int_S g|\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds \\ &\leq |\langle \boldsymbol{f}, \boldsymbol{u}_h^n\rangle - \int_S g|\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds| \\ &\leq C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{u}_h^n\|_V, \end{aligned}$$

hence

$$\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{2\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}) \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}).$$

Next, setting $\boldsymbol{v}_h = \boldsymbol{u}_h^n$ in (1.57) and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (1.82) and adding the resulting equations, we obtain:

$$\begin{aligned} -a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) - d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h, \boldsymbol{u}_h^n) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0. \end{aligned}$$

Note that since $d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) = 0$,

$$d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) - d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) = -d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h),$$

thus

$$\begin{aligned} 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V^2 &\leq a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ &\leq -d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ &+ J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h, \boldsymbol{u}_h^n) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \\ &\leq C_d\|\boldsymbol{u}_h^n\|_V\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + C_0 k\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V, \end{aligned}$$

which together with $\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})$, yields

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V,$$

and (1.83) follows by induction.
For the convergence of p_h^n, let $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$ and replace $\boldsymbol{v}_h$ in $(1.81)_1$ successively by $\boldsymbol{u}_h^n + \boldsymbol{w}_h$ and $\boldsymbol{u}_h^n - \boldsymbol{w}_h$. Observing that $J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n \pm \boldsymbol{w}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)$, one has

$$a(\boldsymbol{u}_h^n, \boldsymbol{w}_h) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}. \quad (1.85)$$

Likewise, one has

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h},$$

which when combined with (1.85) gives

$$\begin{aligned} a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h) \\ -b(\boldsymbol{w}_h, p_h^n - p_h) = 0 \text{ for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}. \end{aligned}$$

That relation together with the discrete inf-sup condition on $b(\cdot,\cdot)$ gives

$$\begin{aligned} \beta\|p_h^n - p_h\| &\leq \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_{\sigma h}} \frac{b(\boldsymbol{w}_h, p_h^n - p_h)}{\|\boldsymbol{w}_h\|_V} \\ &\leq \sup \frac{|a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h)| + d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_V} \\ &\leq 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h^n\|_V\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V, \end{aligned}$$

therefore since $\boldsymbol{u}_h \in \boldsymbol{K}_h$ and $\boldsymbol{u}_h^n \in \boldsymbol{K}_h$, we claim (1.84). □

Remark 1.4.3 *(a) The convergence factor* $\left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)^n$ *is strictly less than one as one can see from (1.33).*

(b) It should be observed that similar condition is obtained in [84] for Navier-Stokes equations under Dirichlet boundary conditions.

As in Stokes formulation (1.39), we construct the following Uzawa iteration algorithm to solve (1.56) via (1.81) .

Algorithm 2:

$$\boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \quad \lambda_h^1 \in \Lambda_h \quad \text{arbitrary given} \tag{1.86}$$

Step 1: knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n) \in \boldsymbol{V}_h \times \Lambda_h$, compute $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ by

$$\begin{cases} a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - \int_S \lambda_h^n (g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|) \boldsymbol{v}_{\boldsymbol{\tau}h} ds \\ \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h^n, q_h) = 0 \quad \text{for all} \quad q_h \in M_h, \end{cases} \tag{1.87}$$

Step 2: Renew $\lambda_h^{n+1} \in \Lambda_h$

$$\lambda_h^{n+1} = P_{\Lambda_h}(\lambda_h^n + \rho(g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|)\boldsymbol{u}_{\boldsymbol{\tau}h}^n) \tag{1.88}$$

where $P_{\Lambda_h}(\mu) = \sup(-1, \inf(1, \mu))$ for all $\mu \in L^2(S)$ and $\rho > 0$.

Remark 1.4.4 *The unique existence of $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ satisfying (1.87) is guaranteed by the inf-sup condition (1.14).*
The initialization of the flow defined by (1.79) and (1.87) is important. Let us observe that since one has well-posedness of (1.39) and (1.56), in order to consolidate the convergence of (1.39) and (1.56), we suggest the solution of Stokes equations

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h, q_h) = 0, \quad \forall q_h \in M_h \ , \end{cases} \tag{1.89}$$

as initial condition for our algorithms (1.79) and (1.87).

1.5 Numerical experiments

Let us explain our numerical experiments. We assume $\Omega = (0,1)^2$, the boundary of which consists of two portions Γ and S given by:

$$\Gamma = \{(0,y)/0 < y < 1\} \cup \{(x,0)/0 < x < 1\} \cup \{(1,y)/0 < y < 1\} \tag{1.90}$$

$$S = \{(x,1)/0 < x < 1\} \tag{1.91}$$

For the triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, we employ a uniform $N \times N$ mesh, where N denotes the division number of each side of the domain. The implementation is done by extending the Matlab code developed in [85, 86]. In all the examples presented, the velocity and pressure will be approximated by $P2 - P1$ element.
We recall that the different steps of our algorithm are as follows: Choosing the parameter ρ (here we choose $\rho = 0.5$),

(a) Starting with $\boldsymbol{u}_h^0$, solution of (2.65) and $\lambda_h^1 = 1$,

(b) knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n)$, compute $(\boldsymbol{u}_h^n, p_h^n, \lambda_h^{n+1})$ solution of (1.79) or (1.87).

The stopping criteria for iteration is

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\| \leq 10^{-6}.$$

Let us consider

$$\begin{cases} u_1(x,y) = 20x^2(1-x)^2 y(1-2y) \\ u_2(x,y) = -20x(1-x)(1-2x)(1-y)^2 y^2 \\ p(x,y) = (2x-1)(2y-1) \end{cases} \tag{1.92}$$

1.5.1 Numerical examples for Stokes problem (1.1)-(1.5)

$(\boldsymbol{u}, p)$ defined by (1.92) turns out to be the solution of the problem (1.1)-(1.5) under the appropriate choice g where $\nu = 1$ and $\boldsymbol{f}(= \boldsymbol{f}_{stokes})$ is given by

$$\begin{cases} f_1(x,y) = 80x^2(1-x)^2 - 20(2+12x^2-12x)y(1-2y) + 2(2y-1); \\ f_2(x,y) = 20(12x-6)y^2(1-y)^2 + 20x(1-2x)(1-x)(2+12y^2-12y) + 2(2x-1) \end{cases} \tag{1.93}$$

It is easy to verify that the solution $\boldsymbol{u}$ satisfies $\boldsymbol{u} = \boldsymbol{0}$ on Γ, $\boldsymbol{u} \cdot \boldsymbol{n} = u_2 = 0,\ u_1 \neq 0$ on S. By direct computations, we have

$$\begin{aligned} \boldsymbol{\sigma_\tau} &= -60x^2(1-x)^2 \quad \text{on} \quad S \\ \boldsymbol{u_\tau} &= 20x^2(1-x)^2 \quad \text{on} \quad S \end{aligned} \tag{1.94}$$

and

$$\max_S |\boldsymbol{\sigma_\tau}| = 3.75. \tag{1.95}$$

On the other hand, from the slip boundary conditions (1.5), we have

$$|\boldsymbol{\sigma_{\tau}}| \leq g + k|\boldsymbol{u_{\tau}}| \qquad \text{on} \quad S \tag{1.96}$$

then we find from (1.96) that with g constant:

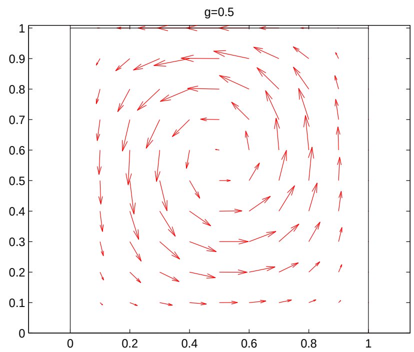
$$\begin{aligned} g + k|\boldsymbol{u_{\tau}}| &\geq 3.75 \Rightarrow (1.92) \text{ remains a solution.} \\ g + k|\boldsymbol{u_{\tau}}| &< 3.75 \Rightarrow (1.92) \text{ is no longer a solution and a non-trivial slip occurs.} \end{aligned}$$

Indeed it is observable in Figures 1.1, slip and non-slip condition on the boundary. In fact in Figure 1.1-a and Figure 1.1-b, $g + k|\boldsymbol{u_{\tau}}| < 3.75$ and we see the manifestation of the slip due to the adherence of the flow at the boundary, whereas in Figure 1.1-c, $g + k|\boldsymbol{u_{\tau}}| \geq 3.75$ and no slip occurs. In addition, we find that

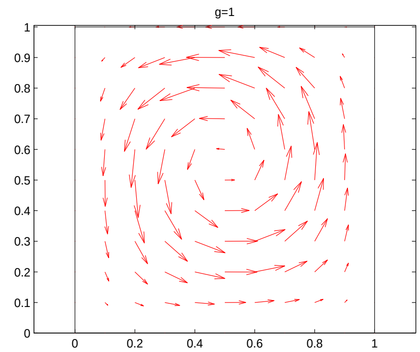
(a) as the threshold g of tangential stress increases, the more difficult it becomes for a non-trivial slip to occur,

(b) the smaller the threshold g of tangential stress becomes, the easier it becomes for a non-trivial slip to occur,

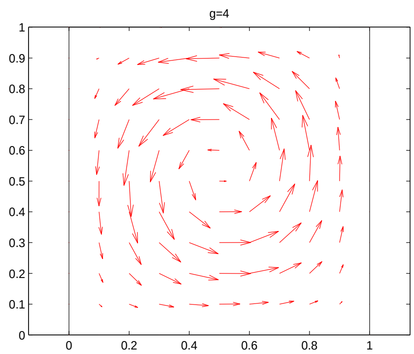
which is in agreement with the predicted outcome.
For all the numerical results here, we set $\nu = 1$, $k = 10^{-1}$, $\rho = 0.5$ and g is indicated on the pictures.



(a)



(b)



(c)

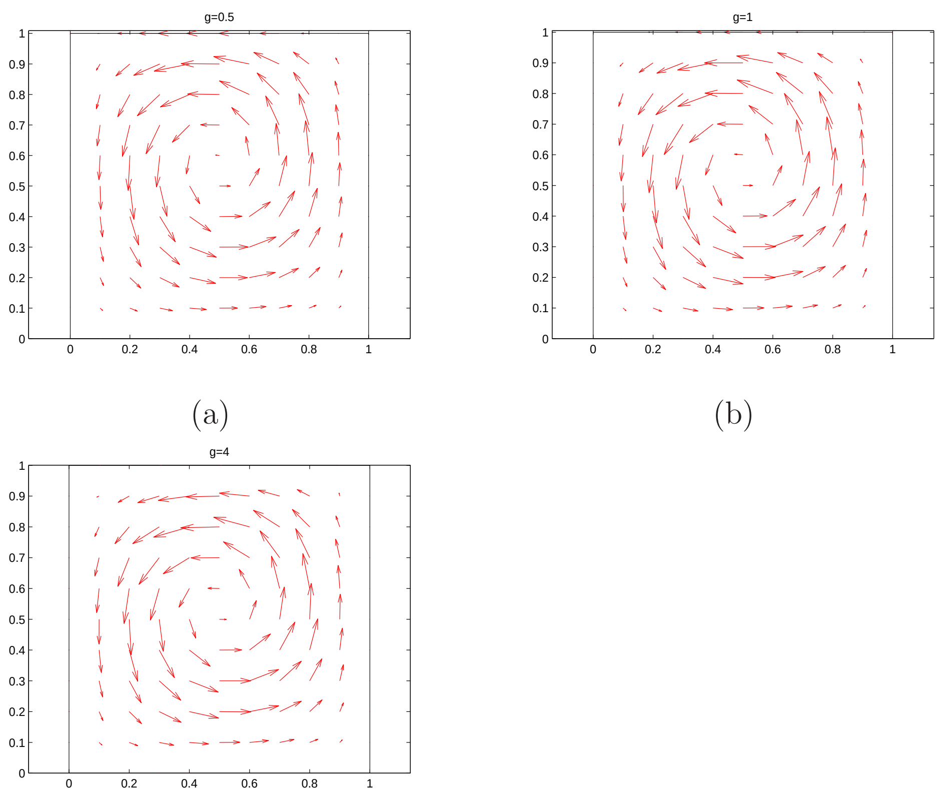
Figure 1.1: Velocity field respectively for $g = 0.5$, $g = 1$, $g = 4$

1.5.2 Numerical examples for Navier-Stokes problem (1.2)–(1.5),(1.8)

For Navier-Stokes problem, we consider the same solution (1.92) as in Stokes problem with appropriate choice of g , and $\boldsymbol{f}$ given by

$$\boldsymbol{f} = \boldsymbol{f}_{stokes} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$$

We observe similar pattern as commented for Figure 1.1. In our computations we did not observe a major difference between Stokes and Navier-Stokes as far as the driven cavity is concerned. Of course as it was expected, the simulations with Navier-Stokes system is more time involved than the one of Stokes equations.



(a) (b)

(c)

Figure 1.2: Velocity field respectively for $g = 0.5$, $g = 1$, $g = 4$

1.5.3 Numerical accuracy check

We evaluate the error between approximate solutions and exact ones as the division number N increased. Since we do not know the explicit exact solution when $g = 1$, we employ the approximate solutions with $N = 60$ as the reference solutions $(\boldsymbol{u}_{ref}, p_{ref})$, and we compute the H^1-norm and L^2-norm respectively for velocity and pressure of the difference of the reference solution and the approximate solution $(\boldsymbol{u}_h, p_h)$. The results are presented in Table 1.1 for Stokes problem and Table 1.2 for Navier-Stokes problem.

Table 1.1: convergence results for Stokes problem

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	rate H^1	$\|p - p_h\|$	rate L^2
1/6	1.150e-3		1.308e-2	
1/10	7.814e-4	0.756	7.071e-3	1.204
1/12	6.863e-4	0.711	5.657e-3	1.223
1/15	5.783e-4	0.767	4.243e-3	1.289
1/20	4.790e-4	0.654	2.928e-3	1.289
1/30	6.185e-4	0.630	1.814e-3	1.180

Table 1.2: convergence results for Navier-Stokes problem

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	rate H^1	$\|p - p_h\|$	rate L^2
1/6	1.103e-2		1.208e-2	
1/10	8.262e-3	0.566	7.171e-3	1.021
1/12	7.499e-3	0.531	5.957e-3	1.017
1/15	6.714e-3	0.495	4.743e-3	1.021
1/20	7.896e-3	0.563	3.528e-3	1.028
1/30	6.446e-3	0.500	2.414e-3	0.935

Chapter 2

Finite element analysis of the stationary power-law Stokes equations driven by friction boundary conditions

2.1 Introduction

We devote this chapter to the finite element approximation of the power law Stokes flow governed by the partial differential equations

$$\begin{aligned} -\nu \operatorname{div}(|\boldsymbol{D}(\mathbf{u})|^{r-2}\boldsymbol{D}(\mathbf{u})) + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega\,, \end{aligned} \tag{2.1}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^2$. The motion of our incompressible fluid is described by the velocity $\boldsymbol{u}(\boldsymbol{x}) = (u_1, u_2)$ and pressure $p(\boldsymbol{x})$, the external force per unit volume is $\boldsymbol{f}$, while the positive parameter ν is the viscosity of the fluid. Of course, $\boldsymbol{D}$ is the deformation tensor given as

$$\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)\,.$$

The motion of the fluid at the boundary, say, $\partial\Omega$ is characterized by the presence of the Tresca type conditions which is described below. First, we assume that $\partial\Omega$ is made of two components, S (say the outer wall) and Γ (the inner wall), and it is required that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma \, . \tag{2.2}$$

We have chosen to work with homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma (see [79], Chapter 4, Lemma 2.3). In order to describe the motion of the fluid on S, we first assume the impermeability condition, that is

$$u_N = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } S \, , \tag{2.3}$$

where $\boldsymbol{n} = (n_1, n_2)$ is the outward unit normal on the boundary $\partial\Omega$, and u_N is the normal component of the velocity, while $\boldsymbol{u_\tau} = \boldsymbol{u} - u_N\boldsymbol{n}$ is its tangential component. In addition to (2.3) we also impose on S a threshold slip condition [41, 23] which is the particularity of this work. The threshold slip condition can be formulated with the knowledge of a positive function $g : S \longrightarrow (0, \infty)$ which is called barrier or threshold function and the tangential part of the traction force acting on S glue together in the following way

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u_\tau} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u_\tau} \neq \boldsymbol{0}, \ \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u_\tau}}{|\boldsymbol{u_\tau}|} \end{array}\right\} \text{ on } \quad S. \tag{2.4}$$

Of course $(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}$ is the tangential component of the traction force $\boldsymbol{\sigma n}$ acting on the boundary S, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor given by $\boldsymbol{\sigma} = -p\boldsymbol{I} + \nu|\boldsymbol{D}(\boldsymbol{u})|^{r-2}\boldsymbol{D}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix. It should quickly be mentioned that (2.4) is equivalent to (see [69])

$$(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u_\tau} + g|\boldsymbol{u_\tau}| = 0 \text{ on } \quad S,$$

which is re-written with the use of sub-differential as

$$-(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \in \ g\partial|\boldsymbol{u_\tau}| \quad \text{on } S, \tag{2.5}$$

where $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$ with $|\boldsymbol{w}|^2 = \boldsymbol{w}\cdot\boldsymbol{w}$. It should be mentioned that different boundary conditions describe different physical phenomena. The slip boundary conditions of friction type (2.5) can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important. The class of boundary condition (2.5) was introduced by Fujita in [23], where he studied some hydrodynamics problems, such as the blood flow in a vein of an arterial sclerosis patient and the avalanche of water and rocks. Subsequently, many studies have focused on the properties of the solution of the resulting boundary value problem, for example, existence, uniqueness, regularity, and continuous dependence on data, for Stokes, Navier-Stokes and Brinkman-Forchheimer equations under such boundaries condition. Details can be found in [41, 23, 28, 40, 42, 87, 31, 27, 29, 88, 89, 24, 32, 90] among others. But the combination of (2.5) with the p-Laplacian has not yet been considered in the literature, and in this work we give a detailed mathematical analysis on the existence and uniqueness of weak solution. The aim of this study is to contribute to the numerical analysis of flows problem driven by non-conventional boundary conditions. Hence, our main focus is to analyze numerically (2.1)—(2.5) via finite element approximations. That is to establish the convergence of the finite element solution. It is manifest that (2.1)—(2.5) has many numerical challenges among others; the nonlinear operator (p-Laplacian), the incompressible condition and the related pressure, and the nontrivial boundary condition (2.5) which brings a non-differentiable expression into the variational formulation of the problem. Hence, our second contribution here is to formulate and analyze an algorithm well adapted and easy to implement for the numerical challenges mentioned.

Even though many researches have been done for the approximations of variational inequalities [91, 63, 48, 72, 71] (just to mention a few), not much research in theoretical numerical analysis has been done for the kind of problem described by (2.1)—(2.5). Li and Li [35] proposed a penalty finite element approximation method for the Stokes equation with nonlinear slip boundary conditions (2.5). They proved the optimal order error estimate provided that the velocity is H^2 up to the boundary, however, no numerical simulations are exhibited. An and Li [78] proposed a penalty finite element method for the steady Navier-Stokes equations. The Mathematical analysis of this chapter borrows heavily on the contribution of

Reddy [70] and Han and Reddy [48], where sufficient conditions for existence and uniqueness are derived for the kind of weak formulations we analyze here, while the solution procedure we propose is divided in three steps. The first step is based on some works of R. Glowinski [91] in that, we associated to a steady problem an evolution problem in which only the long time effect is taken into consideration. Next, because of the incompressibility condition, and the non differentiable term appearing in the variational problem to solve, we approximated then the problem by a sequence of penalized/regularized "better behaved" variational equations and justify the approximations by some convergence results. Thirdly, to improve the performance of our scheme, we add to the problem obtained in step 2 a viscosity term and show that the new "perturbed" problem converges to the original variational formulation. All these theoretical results are supported by numerical simulations indicating the robustness of our algorithm. The rest of this chapter is organized as follows. We give some notations, formulate the variational models in section 2.2 and indicate how existence of weak solution is obtained. In section 2.3, we formulate the finite element procedure, explain how existence and uniqueness of solution is obtained and derive error estimates. Section 2.4 is concerned with the algorithm, while section 2.5 deals with numerical simulations.

2.2 Variational Formulations

In this section, we formulate variational models associated to problem (2.1)-(2.5). We also indicate how existence and uniqueness of solution is obtained.

2.2.1 Notation

We introduce the following spaces

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{W}^{1,r}(\Omega),\ \mathbf{v}|_\Gamma = 0,\ \mathbf{v}\cdot\mathbf{n}|_S = 0\}, \quad \mathbf{V}_0 = \mathbf{W}_0^{1,r}(\Omega),$$
$$\mathbf{V}_{div} = \{\mathbf{v} \in \mathbf{V},\ \operatorname{div}\mathbf{v} = 0\}, \quad M = L_0^{r'}(\Omega)\ .$$

We can equip $\mathbf{V}$ by $\|\cdot\|_{\mathbf{V}} = \|\boldsymbol{D}(\cdot)\|_{L^r}$ because $\|\boldsymbol{D}(\cdot)\|_{L^r}$ is equivalent to $\|\cdot\|_{1,r}$. Next, we define the following operators A and $b(\cdot,\cdot)$ as follows

$$A : \mathbf{W}_0^{1,r}(\Omega) \to \mathbf{W}^{-1,r'}(\Omega) \quad \text{with} \quad \langle A\mathbf{u}, \mathbf{v}\rangle = \nu \int_\Omega |\boldsymbol{D}(\mathbf{u})|^{r-2}\boldsymbol{D}(\mathbf{u}) : \boldsymbol{D}(\mathbf{v})\, dx$$

$$b : \mathbf{V} \times M \to \mathbb{R} \quad \text{with} \quad b(\mathbf{u}, p) = \int_\Omega p \operatorname{div} \mathbf{u}\, dx \ .$$

2.2.2 Mixed variational formulation

Given $\mathbf{f} \in \mathbf{W}^{-1,r'}(\Omega)$ and $g \in L^{r'}(S)$ with $g \geq 0$ on S, we multiply the first equation in (2.1) by $\mathbf{v} - \mathbf{u}$ for all $\mathbf{v} \in \mathbf{V}$, integrate the resulting equation over Ω, after application of Green's formula, we obtain

$$\langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle - b(\mathbf{v} - \mathbf{u}, p) - \int_S \boldsymbol{\sigma} \cdot (\mathbf{v} - \mathbf{u})\, ds = \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx \,. \tag{2.6}$$

Next, we briefly recall that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_N \mathbf{n} + \boldsymbol{\sigma}_{\boldsymbol{\tau}} \ , \quad \mathbf{v} - \mathbf{u} = (v_N - u_N)\mathbf{n} + (v_{\boldsymbol{\tau}} - u_{\boldsymbol{\tau}}) \ ,$$

then we have

$$\int_S \boldsymbol{\sigma} \cdot (\mathbf{v} - \mathbf{u})\, ds \ = \ \int_S \boldsymbol{\sigma}_{\boldsymbol{\tau}} (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\, ds \ , \quad \text{since} \quad v_N - u_N|_\Gamma = 0 \ .$$

On the other hand, according to the definition (0.2.7),

$$\int_S g(|\boldsymbol{v}_{\boldsymbol{\tau}}| - |\boldsymbol{u}_{\boldsymbol{\tau}}|)\mathrm{ds} \geq - \int_S \boldsymbol{\sigma}_{\boldsymbol{\tau}} (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\mathrm{ds},$$

which together with (2.6) leads to the following weak formulation of (2.1)-(2.5): Find $(\mathbf{u}, p) \in \mathbf{V} \times M$ such that

$$\begin{array}{rcll} \langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle - b(\mathbf{v} - \mathbf{u}, p) + j(\boldsymbol{v}_{\boldsymbol{\tau}}) - j(\boldsymbol{u}_{\boldsymbol{\tau}}) & \geq & \langle \mathbf{f}, \mathbf{v} - \mathbf{u}\rangle & \text{for all } \ \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) & = & 0 & \text{for all } \ q \in M \ , \end{array} \tag{2.7}$$

where $j(\eta) = \int_S g|\eta|\mathrm{ds}$. Note that because the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (1.14), the variational inequality problem (2.7) is equivalent to

$$\begin{cases} \text{Find } \ \mathbf{u} \in \mathbf{V}_{\text{div}} \text{ such that} \\ \langle A\mathbf{u}, \mathbf{w}-\mathbf{u}\rangle + j(\boldsymbol{w_\tau}) - j(\boldsymbol{u_\tau}) \geq \langle \mathbf{f}, \mathbf{w}-\mathbf{u}\rangle \ \text{ for all } \ \mathbf{w} \in \mathbf{V}_{\text{div}} \ , \end{cases} \tag{2.8}$$

which is also equivalent to the following optimization problem

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_{\text{div}} \text{ such that} \\ Q(\mathbf{u}) \leq Q(\mathbf{w}) \ \text{ for all } \ \mathbf{w} \in \mathbf{V}_{\text{div}} \ , \end{cases} \tag{2.9}$$

where

$$Q(\mathbf{w}) = \frac{\nu}{r}\|\nabla \mathbf{w}\|_{L^r}^r + j(\boldsymbol{w_\tau}) - \langle \mathbf{f}, \mathbf{w}\rangle \, .$$

To show the existence and uniqueness of solution of (2.7) it suffices to show that the assumptions of Theorem 0.2.4 (see [70, 48]) are satisfied. The coercivity, monotonicity and boundedness of A are proved by Barrett and Liu [92], see also Chow [93]. In fact one has; for all $\mathbf{u}$, $\mathbf{v}$ in $\mathbf{W}_0^{1,r}(\Omega)$

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|_{1,r}^2 &\leq C\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle \left(\|\mathbf{u}\|_{1,r} + \|\mathbf{v}\|_{1,r}\right)^{2-r}, \\ \|A(\mathbf{u}) - A(\mathbf{v})\|_{-1,r'} &\leq C\|\mathbf{u}-\mathbf{v}\|_{1,r}^{r-1}, \quad 1 < r \leq 2; \end{aligned} \tag{2.10}$$

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|_{1,r}^r &\leq C\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle, \\ \|A(\mathbf{u}) - A(\mathbf{v})\|_{-1,r'} &\leq C\|\mathbf{u}-\mathbf{v}\|_{1,r} \left(\|\mathbf{u}\|_{1,r} + \|\mathbf{v}\|_{1,r}\right)^{r-2}, \quad 2 \leq r < \infty, \end{aligned} \tag{2.11}$$

where $C > 0$ denotes a generic constant independent of $\mathbf{u}$ and $\mathbf{v}$.
The inf-sup condition (1.14) has been proved by Baranger and Najib [6], Amrouche and Girault [94]. The functional $j(\cdot)$ is easily shown to be convex, nonnegative continuous. However j is not differentiable.
We thus conclude this section with the following result.

Lemma 2.2.1 *The mixed variational problem (2.7) admits a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$*, which satisfies the following bound*

$$\|\boldsymbol{u}\|_{1,r} \leq C(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)})^{1/(r-1)}, \tag{2.12}$$
$$\|p\|_{L^{r'}} \leq C(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)}) \, . \tag{2.13}$$

Proof Let $\mathbf{w} = \mathbf{0}$ and $\mathbf{w} = 2\mathbf{u}$ in (2.8), one has

$$\langle A\mathbf{u}, \mathbf{u}\rangle + j(\boldsymbol{u_\tau}) = \langle \mathbf{f}, \mathbf{u}\rangle,$$

then, we have by (2.11),

$$\begin{aligned} \|\mathbf{u}\|_{1,r}^r &\leq C\langle A\mathbf{u}, \mathbf{u}\rangle = C(\langle \mathbf{f}, \mathbf{u}\rangle - j(\boldsymbol{u_\tau})), \\ &\leq C(\|\mathbf{f}\|_{-1,r'}\|\mathbf{u}\|_{1,r} + \|g\|_{L^{r'}(S)}\|\mathbf{u}\|_{1,r}), \quad \text{for} \quad 2 \leq r < \infty. \end{aligned}$$

Similarly, by (2.10),

$$\begin{aligned} \|\mathbf{u}\|_{1,r}^2 \leq C\|\mathbf{u}\|_{1,r}^{2-r}\langle A\mathbf{u}, \mathbf{u}\rangle &= C\|\mathbf{u}\|_{1,r}^{2-r}(\langle \mathbf{f}, \mathbf{u}\rangle - j(\boldsymbol{u_\tau})) \\ &\leq C\|\mathbf{u}\|_{1,r}^{2-r}(\|\mathbf{f}\|_{-1,r'}\|\mathbf{u}\|_{1,r} + \|g\|_{L^{r'}(S)}\|\mathbf{u}\|_{1,r}), \\ \text{for} \quad 1 < r \leq 2. \end{aligned}$$

Thus the two relations above give the same result (2.12).
Next, we derive the *a priori* bound for the pressure. For that purpose, let $\mathbf{w} \in \mathbf{V}_0$, and replace $\boldsymbol{v}$ in (2.7) successively by $\mathbf{u} + \mathbf{w}$ and $\mathbf{u} - \mathbf{w}$, and observe that

$$j(\boldsymbol{v_\tau}) = j(\boldsymbol{u_\tau} \pm \boldsymbol{w_\tau}) = j(\boldsymbol{u_\tau}) .$$

We now obtain

$$\langle A\mathbf{u}, \mathbf{w}\rangle - b(\mathbf{w}, p) = \langle \mathbf{f}, \mathbf{w}\rangle \quad \text{for all} \quad \mathbf{w} \in \mathbf{V}_0 . \tag{2.14}$$

Next, from the compatibility condition (1.14) and (2.14), one has

$$\begin{aligned} \beta\|p\|_{L^{r'}} &\leq \sup_{\mathbf{w}\in\mathbf{V}_0} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{1,r}} = \sup_{\mathbf{w}\in\mathbf{V}_0} \frac{|\langle A\mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{f}, \mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,r}} \\ &\leq \|A\mathbf{u}\|_{-1,r'} + \|\mathbf{f}\|_{-1,r'} . \end{aligned} \tag{2.15}$$

Thirdly from (2.10) and (2.11), there holds that for all $r > 1$,

$$\|A\mathbf{u}\|_{-1,r'} \leq C\|\mathbf{u}\|_{1,r}^{r-1},$$

which when combined with (2.15) and (2.12) leads to result announced in (2.13) .
□

2.3 Finite element approximation of the variational inequality (2.7)

2.3.1 Preliminaries and existence of solution

In this section, we analyze the finite element discretization of the variational inequality (2.7). We assume that $\mathcal{T}_h$ is a regular partition of Ω in the sense introduced in Cialert [62]. The diameter of an element $K \in \mathcal{T}_h$ is denoted by h_K, and the mesh size h is defined by $h = \max_{K \in \mathcal{T}_h} h_K$.
Let $\mathbf{V}_h \subset \mathbf{V}$ and $M_h \subset M$ be two conforming finite element spaces that will be made precise later. Let introduce the following subspaces:

$$\mathbf{V}_{\sigma h} = \{\mathbf{v}_h \in \mathbf{V}_h \ , \quad b(\mathbf{v}_h, q_h) = 0 \quad \text{for all} \quad q_h \in M_h\} \ ,$$

$$\mathbf{V}_{0h} = \mathbf{V}_h \cap \mathbf{V}_0 \ .$$

The mixed weak formulation for the finite element discretization of the variational inequality (2.7) reads:

$$\begin{cases} \text{Find } (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h \text{ such that} \\ \langle A\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle - b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + J(\boldsymbol{v}_h) - J(\boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle, \\ b(\boldsymbol{u}_h, q_h) = 0, \\ \text{for all } \ (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \end{cases} \tag{2.16}$$

For the existence and uniqueness of solution of (2.16), we apply Theorem 0.2.4, with the special requirement among others that the constant in the discrete counterpart of the inf-sup condition (1.14) be independent of h. Indeed, the reader can consult [95, 79, 80] where many examples of elements pair which satisfies the discrete version of (1.14) are given. In summary, we have the following result.

Lemma 2.3.1 *The finite element formulation (2.16) has a unique solution $(\boldsymbol{u}_h, p_h)$ which moreover satisfies;*

$$\|\boldsymbol{u}_h\|_{1,r} \leq C(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)})^{1/(r-1)}, \tag{2.17}$$

$$\|p_h\|_{L^{r'}} \leq C(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)}) \ . \tag{2.18}$$

where $C > 0$ is a generic constant independent of h .

Proof The proof follows the same lines as the proof of Lemma 2.2.1 .
Let $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v}_h = 2\mathbf{u}_h$ in $(2.16)_1$. Using $(2.16)_2$, one has

$$\langle A\mathbf{u}_h, \mathbf{u}_h\rangle + j(\boldsymbol{u}_{h\boldsymbol{\tau}}) = \langle \mathbf{f}, \mathbf{u}_h\rangle,$$

then, we conclude (2.17) by (2.11) and (2.10) .
To show the bound (2.18), for all $\mathbf{w}_h \in \mathbf{V}_{0h}$, let $\mathbf{v}_h = \mathbf{u}_h + \mathbf{w}_h$ and $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$ in (2.16) and since $j(\boldsymbol{v}_{h\boldsymbol{\tau}}) = j(\boldsymbol{u}_{h\boldsymbol{\tau}} \pm \boldsymbol{w}_{h\tau}) = j(\boldsymbol{u}_{h\boldsymbol{\tau}})$, one has

$$\langle A\mathbf{u}_h, \mathbf{w}_h\rangle - b(\mathbf{w}_h, p_h) = \langle \mathbf{f}, \mathbf{w}_h\rangle \quad \text{for all} \quad \mathbf{w}_h \in \mathbf{V}_{0h}. \tag{2.19}$$

Then from the inf-sup condition (1.14), we conclude (2.18) by using (2.17). □

2.3.2 A priori error estimate

To start with, we recall the following result which will be useful for this subsection

$$|\sum_{i=1}^{m} a_i|^\beta \le C(m,\beta)\sum_{i=1}^{m} |a_i|^\beta \quad \text{for all } \ a_i \ge 0 \quad \text{ and for all } \quad \beta \ge 0\ , \tag{2.20}$$

where $C(m,\beta)$ is a positive constant depending on m and β. The main result of this paragraph can be stated as follows.

Theorem 2.3.1 *Suppose that Ω is a bounded convex domain in $\mathbb{R}^2$ and assume that $\boldsymbol{f} \in \boldsymbol{W}^{-1,r'}(\Omega)$. Let $(\boldsymbol{u}, p)$ be the unique solution of (2.7) and $(\boldsymbol{u}_h, p_h)$ the unique solution of (2.16). Then there exists a generic positive constant C independent on h such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $q_h \in M_h$, there hold;*
• *For $1 < r \le 2$*

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}^2 \;&\le\; C\left\{\begin{array}{c}
\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^2 \\
+\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^r + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^{2(r-1)} \\
+\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^{\frac{2}{(3-r)}} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)}^{2/r} \\
+\|p-q_h\|_{L^{r'}}^{\frac{2}{(r-1)}} + \|p-q_h\|_{L^{r'}}^2
\end{array}\right\}, \\
\|p-p_h\|_{L^{r'}} \;&\le\; C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}^{(r-1)} + \|p-q_h\|_{L^{r'}}\right\}\ .
\end{aligned} \tag{2.21}$$

- *For* $2 \leq r < \infty$

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^r_{1,r} \leq C \left\{ \begin{array}{l} \|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)} + \|\boldsymbol{u}-\boldsymbol{v}_h\|^2_{1,r} + \|\boldsymbol{u}-\boldsymbol{v}_h\|^r_{1,r} \\ +\|\boldsymbol{u}-\boldsymbol{v}_h\|^{r'}_{1,r} + \|p-q_h\|^2_{L^{r'}} + \|p-q_h\|^{r'}_{L^{r'}} \end{array} \right\}$$

$$\|p-p_h\|_{L^{r'}} \leq C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r} + \|p-q_h\|_{L^{r'}}\right\} . \tag{2.22}$$

Proof Subtracting (2.14) from (2.19) with $\mathbf{w} = \mathbf{w}_h$ we obtain

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}\rangle - b(p-p_h, \mathbf{w}_h) = 0 \quad \text{for all} \quad \mathbf{w}_h \in \mathbf{V}_{0h}.$$

From the relation above, we have

$$b(p_h - q_h, \mathbf{w}_h) = \langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}_h\rangle + b(p-q_h, \mathbf{w}_h) \ ,$$

which together with the discrete version of the inf-sup condition (1.14), gives

$$\begin{aligned} \beta\|p_h - q_h\|_{L^{r'}} &\leq \sup_{\mathbf{w}_h \in \mathbf{V}_{0h}} \frac{|\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}_h\rangle + b(\mathbf{w}_h, p-q_h)|}{\|\mathbf{w}_h\|_{1,r}}, \\ &\leq C(\|A\mathbf{u} - A\mathbf{u}_h\|_{-1,r'} + \|p-q_h\|_{L^{r'}}) \ , \end{aligned}$$

so that,

$$\|p-p_h\|_{L^{r'}} \leq \|p-q_h\|_{L^{r'}} + \|q_h - p_h\|_{L^{r'}} \leq C\|A\boldsymbol{u} - A\boldsymbol{u}_h\|_{-1,r'} + C\|p-q_h\|_{L^{r'}} \ .$$

Note that by the property (2.10),

$$\|A\boldsymbol{u} - A\boldsymbol{u}_h\|_{-1,r'} \leq C\|\boldsymbol{u}-\boldsymbol{u}_h\|^{r-1}_{1,r}, \quad \text{for} \quad 1 < r \leq 2; \tag{2.23}$$

and by (2.11)) together with (2.12) and (2.17) we have

$$\|A\boldsymbol{u} - A\boldsymbol{u}_h\|_{-1,r'} \leq C\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}, \quad \text{for} \quad 2 \leq r < \infty. \tag{2.24}$$

Therefore

$$C\|p-p_h\|_{L^{r'}} \leq \left\{ \begin{array}{ll} \|\boldsymbol{u}-\boldsymbol{u}_h\|^{r-1}_{1,r} + \|p-q_h\|_{L^{r'}} & 1 < r \leq 2 \\ \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r} + \|p-q_h\|_{L^{r'}} & 2 \leq r < \infty. \end{array} \right. \tag{2.25}$$

Next, replacing successively $\boldsymbol{v}$ in $(2.7)_1$ by $\mathbf{v} = \mathbf{u}_h$ and $\mathbf{v} = 2\mathbf{u} - \mathbf{v}_h$, one gets

$$\langle A\mathbf{u}, \mathbf{u}_h - \mathbf{u}\rangle - b(\mathbf{u}_h - \mathbf{u}, p) + j(\boldsymbol{u}_{h\boldsymbol{\tau}}) - j(\boldsymbol{u}_{\boldsymbol{\tau}}) \geq \langle \mathbf{f}, \mathbf{u}_h - \mathbf{u}\rangle \tag{2.26}$$

and

$$\langle A\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}_h\rangle - b(\boldsymbol{u} - \boldsymbol{v}_h, p) + j(2\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{v}_{h\boldsymbol{\tau}}) - j(\boldsymbol{u}_{\boldsymbol{\tau}}) \geq \langle \mathbf{f}, \mathbf{u} - \mathbf{v}_h\rangle. \tag{2.27}$$

Summing the inequalities (2.26) and (2.27) yields

$$\langle A\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle - b(\boldsymbol{u}_h - \boldsymbol{v}_h, p) + j(2\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{v}_{h\boldsymbol{\tau}}) + j(\boldsymbol{u}_{h\boldsymbol{\tau}}) - 2j(\boldsymbol{u}_{\boldsymbol{\tau}}) \geq \langle \mathbf{f}, \boldsymbol{u}_h - \boldsymbol{v}_h\rangle. \tag{2.28}$$

Note that the inequality $(2.16)_1$ can be written as

$$-\langle A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle + b(\mathbf{u}_h - \mathbf{v}_h, p_h) + j(\boldsymbol{v}_{h\boldsymbol{\tau}}) - j(\boldsymbol{u}_{h\boldsymbol{\tau}}) \geq -\langle \mathbf{f}, \mathbf{u}_h - \mathbf{v}_h\rangle. \tag{2.29}$$

Summing the inequalities (2.28) and (2.29) yields

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle - b(\mathbf{u}_h - \mathbf{v}_h, p - p_h) + j(2u_\tau - v_{h\tau}) + j(\boldsymbol{v}_{h\boldsymbol{\tau}}) - 2j(\boldsymbol{u}_{\boldsymbol{\tau}}) \geq 0. \tag{2.30}$$

Note that

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle = \langle A\mathbf{u} - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle - \langle A\mathbf{u}_h - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle. \tag{2.31}$$

Also using $(2.7)_2$ and $(2.16)_2$, one has

$$\begin{aligned} b(\mathbf{u}_h - \mathbf{v}_h, p - p_h) &= b(\mathbf{u}_h - \mathbf{u}, p - q_h) + b(\mathbf{u}_h - \mathbf{u}, q_h - p_h) + b(\mathbf{u} - \mathbf{v}_h, p - p_h) \\ &= b(\mathbf{u}_h - \mathbf{u}, p - q_h) + b(\mathbf{u} - \mathbf{v}_h, p - p_h). \end{aligned} \tag{2.32}$$

Substituting the equalities (2.31) and (2.32) into (2.30) yields

$$\begin{aligned} \langle A\mathbf{u}_h - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle \leq\ & \langle A\mathbf{u} - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle \\ & - b(\mathbf{u}_h - \mathbf{u}, p - q_h) - b(\mathbf{u} - \mathbf{v}_h, p - p_h) \\ & + j(2\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{v}_{h\boldsymbol{\tau}}) + j(\boldsymbol{v}_{h\boldsymbol{\tau}}) - 2j(\boldsymbol{u}_{\boldsymbol{\tau}}). \end{aligned} \tag{2.33}$$

Using Holder inequality, we have

$$\langle A\mathbf{u} - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle \leq \|A\mathbf{u} - A\mathbf{v}_h\|_{-1,r'} \|\mathbf{u}_h - \mathbf{v}_h\|_{1,r}, \tag{2.34}$$

$$j(2\boldsymbol{u_\tau} - \boldsymbol{v}_{h\boldsymbol{\tau}}) + j(\boldsymbol{v}_{h\boldsymbol{\tau}}) - 2j(\boldsymbol{u_\tau}) \le C\|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)},$$

and using the continuity of $b(\cdot,\cdot)$,

$$-b(\mathbf{u}_h - \mathbf{u}, p - q_h) - b(\mathbf{u} - \mathbf{v}_h, p - p_h) \le C\left\{\|\mathbf{u} - \mathbf{u}_h\|_{1,r}\|p - q_h\|_{L^{r'}} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|p - p_h\|_{L^{r'}}\right\}.$$

Then (2.33) becomes

$$\begin{aligned}\langle A\mathbf{u}_h - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle &\le \|A\mathbf{u} - A\mathbf{v}_h\|_{-1,r'}\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r} \\ &+ C\left\{\|\mathbf{u} - \mathbf{u}_h\|_{1,r}\|p - q_h\|_{L^{r'}} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|p - p_h\|_{L^{r'}}\right. \\ &\left.+\|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)}\right\} \qquad (2.35)\end{aligned}$$

Therefore, using $(2.10)_1$ to the left hand side of (2.35), we get

for $1 < r \le 2$

$$\begin{aligned}&\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r}^2 \\ &\le C\left(\|\mathbf{u}_h\|_{1,r} + \|\mathbf{v}_h\|_{1,r}\right)^{2-r}\left\{\begin{array}{c}\|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r} + \|p - q_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ +\|p - p_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{v}_h\|_{1,r} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)}\end{array}\right\}. \qquad (2.36)\end{aligned}$$

Similarly using $(2.11)_1$ to the left hand side of (2.35), we get
for $2 \le r < \infty$

$$\begin{aligned}&\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r}^r \\ &\le \left\{\begin{array}{c}C(\|\mathbf{u}_h\|_{1,r} + \|\mathbf{v}_h\|_{1,r})^{r-2}\|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r} + \|p - q_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ +\|p - p_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{v}_h\|_{1,r} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)}\end{array}\right\}. \qquad (2.37)\end{aligned}$$

Using the triangle inequalities, the relation (2.20), the bounded relations (2.12) and (2.17) and $(2.25)_1$ to (2.36), we obtain for $1 < r \le 2$

$$\begin{aligned}C\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^2 &\le \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^r + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ &+ \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|p - q_h\|_{L^{r'}} + \|\mathbf{u} - \mathbf{u}_h\|_{1,r}\|p - q_h\|_{L^{r'}} \\ &+ \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^{r-1} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} \\ &+ \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} + \|p - q_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{3-r} \\ &+ \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2-r}\|p - q_h\|_{L^{r'}}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \qquad (2.38) \\ &+ \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2-r}\|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{3-r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^{r-1},\end{aligned}$$

which with Young's inequality, leads to the following inequalities

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,r}^2 \le C \left\{ \begin{array}{c} \|\mathbf{u}-\mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^r \\ +\|\mathbf{u}-\mathbf{v}_h\|_{1,r}^{2(r-1)} + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^{\frac{2}{(3-r)}} + \|\mathbf{u}-\mathbf{v}_h\|_{L^r(S)}^{2/r} \\ +\|p-q_h\|_{L^{r'}}^{\frac{2}{(r-1)}} + \|p-q_h\|_{L^{r'}}^2 \end{array} \right\}.$$

Likewise, when $2 \le r < \infty$, using the triangle inequalities, the relation (2.20), the bounded relations (2.12) and (2.17) and $(2.25)_2$ to (2.37), we obtain

$$\begin{aligned} C\|\mathbf{u}-\mathbf{u}_h\|_{1,r}^r \le\ & \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^r + \|\mathbf{u}-\mathbf{v}_h\|_{L^r(S)} \\ +\ & \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u}-\mathbf{u}_h\|_{1,r} + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}\|p-q_h\|_{L^{r'}} \\ +\ & \|\mathbf{u}-\mathbf{u}_h\|_{1,r}\|p-q_h\|_{L^{r'}} + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}\|\mathbf{u}-\mathbf{u}_h\|_{1,r}\,, \end{aligned} \tag{2.39}$$

which again with the help of Young's inequality yields for $2 \le r < \infty$

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,r}^r \le C \left\{ \begin{array}{c} \|\mathbf{u}-\mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u}-\mathbf{v}_h\|_{1,r}^r \\ +\|\mathbf{u}-\mathbf{v}_h\|_{1,r}^{r'} + \|p-q_h\|_{L^{r'}}^2 + \|p-q_h\|_{L^{r'}}^{r'} \end{array} \right\}.$$

□

2.3.3 Rate of convergence

In this paragraph, we derive rate of convergence by considering classical assumptions on regularity of the solution $(\boldsymbol{u}, p)$, and adopting well known finite element spaces $\boldsymbol{V}_h$ and M_h.

We first consider finite element approximations defined in [92]. We state the following assumptions for an integer $m \ge 1$ and any $\alpha \in [1, \infty]$.

(H1): Approximation property of $\mathbf{V}_h$

There is a continuous linear operator $\pi_h : W_0^{1,\alpha}(\Omega)^2 \to \mathbf{V}_h$ such that for $k = 0, \cdots, m$

$$\|\mathbf{w}-\pi_h\mathbf{w}\|_{1,\alpha} \le Ch^k\|\mathbf{w}\|_{k+1,\alpha} \qquad \forall \mathbf{w} \in (W^{k+1,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega))^2. \tag{2.40}$$

(H2): Approximation property of M_h

There is a continuous linear operator $\rho_h : L^\alpha(\Omega) \to M_h$ such that for all $k = 0, \cdots, m$

$$\|q-\rho_h q\|_{L^\alpha} \le Ch^k\|q\|_{k,\alpha} \qquad \forall q \in W^{k,\alpha}(\Omega). \tag{2.41}$$

(H3): The spaces M_h and $\mathbf{V}_h$ satisfy the inf-sup condition (1.14) .
Then the following result hold.

Theorem 2.3.2 *Let $(\boldsymbol{u}, p)$ be the unique solution of (2.7) and $(\boldsymbol{u}_h, p_h)$ the unique solution of (2.16). Then if (H1), (H2) and (H3) hold for $k = 0, \cdots, m$ we have:*
For $1 < r \leq 2$

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,r} &\leq C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) h^{k \min\{\frac{1}{r},(r-1)\}} \\ \|p - p_h\|_{r'} &\leq C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) h^{k \min\{\frac{r-1}{r},(r-1)^2\}} \end{aligned} \tag{2.42}$$

For $2 \leq r < \infty$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,r} + \|p - p_h\|_{r'} \leq C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) h^{\frac{k}{r}} \tag{2.43}$$

where $C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)})$ is generic constant depending on $\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,r}(\Omega)}$, $\|p\|_{W^{k,r'}(\Omega)}$ and independent on h.

Proof Let $\mathbf{w}_h = \pi_h \mathbf{u}$ and $q_h = \rho_h u$ in (2.21),(2.22), and applying the usual trace theorem, (2.40) and (2.41), we obtain:
For $1 < r \leq 2$

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,r}^2 &\leq C(\|\mathbf{u}\|_{\mathbf{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) \left\{h^k + h^{2k} + h^{kr}\right. \\ &+ \left. h^{2k(r-1)} + h^{2k/(3-r)} + h^{2k/(r-1)} + h^{2k/r}\right\} \\ &\leq C(\|\mathbf{u}\|_{\mathbf{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) \left\{h^{2k(r-1)} + h^{\frac{2k}{r}}\right\} \\ &\leq C(\|\mathbf{u}\|_{\mathbf{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) h^{2k \min\{\frac{1}{r},(r-1)\}} , \end{aligned}$$

where Young's inequality has been used.
For $2 \leq r < \infty$

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,r}^r &\leq C(\|\mathbf{u}\|_{\mathbf{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) \left\{h^k + h^{2k} + h^{kr} + h^{kr'}\right\} \\ &\leq C(\|\mathbf{u}\|_{\mathbf{W}^{k+1,r}(\Omega)}, \|p\|_{W^{k,r'}(\Omega)}) h^k \end{aligned}$$

where the last expression is obtained by the same technique as above. □

2.4 Numerical Algorithm

In this section, we formulate and analyze the algorithm for the implementation of (2.16). We first regularize the formulation (2.16) by replacing the non differentiable functional j by a "better behaved" approximation j_ε where ε is a small positive parameter (see B.D. Reddy [70]) . It should be mentioned that the introduction of the new functional j_ε transforms the variational inequality problem into a variational equation. The next step in our strategy consists of eliminating the incompressibility constraint by penalizing the regularized problem by adding a coercive-like term in the form $\eta(p, q)$, of course η is a small positive parameter. We recall that the transformed problem is very close to the original one in the sense that when ε, η tend to zero, one recovers the original problem. Finally, the perturbed problem with parameters is solved by considering the numerical solution of the long time behavior of an appropriate initial value problem in $\mathbf{V}_h \times M_h$. One of the advantages of using this approach is that a linear scheme can be formulated for a nonlinear problem. We next present the details of our approach.

The non-differentiable functional j is replaced in (2.16) by the regularized functional j_ε defined by

$$j_\varepsilon(\boldsymbol{v}) = \int_S g\sqrt{|\boldsymbol{v}|^2 + \varepsilon^2}\,\mathrm{ds}. \tag{2.44}$$

Note that j_ε satisfies the following properties:

(i) j_ε is convex and differentiable, with Gateaux derivative and

$$\langle j'_\varepsilon(\boldsymbol{v}), \boldsymbol{w}\rangle = \int_S g\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\sqrt{|\boldsymbol{v}|^2 + \varepsilon^2}}\,\mathrm{ds}, \tag{2.45}$$

(ii)

$$0 \le j_\varepsilon(\boldsymbol{w}) - j(\boldsymbol{w}) \le C_1\varepsilon \quad \text{for all} \quad \boldsymbol{w}\ , \tag{2.46}$$

(iii)

$$\|j'_\varepsilon(\boldsymbol{w})\| \le C_2\|g\|_{L^{r'}(S)} \quad \text{for all} \quad \boldsymbol{w}\ . \tag{2.47}$$

The constants C_1 and C_2 are both independent of ε, and furthermore C_2 is independent of ψ.
With the introduction of the smoother functional j_ε, the regularized problem reads

$$\begin{cases} \text{Find } (\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \boldsymbol{V}_h \times M_h \text{ such that} \\ \langle A\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_h^\varepsilon\rangle - b(\boldsymbol{v}_h - \boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) + j_\varepsilon(\boldsymbol{v}_h) - j_\varepsilon(\boldsymbol{u}_h^\varepsilon) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^\varepsilon\rangle, \\ b(\boldsymbol{u}_h^\varepsilon, q_h) = 0, \\ \text{for all } \ (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \end{cases} \tag{2.48}$$

The solution of (2.48) is related to the one of (2.16) by the following result.

Lemma 2.4.1 *Let $(\boldsymbol{u}_h, p_p)$ be the solution of (2.16), and $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ the solution of (2.48), then there is positive constant C, independent of ε such that*

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r} \leq C \begin{cases} \varepsilon^{1/2} & \text{if} \quad 1 < r \leq 2 \\ \varepsilon^{1/r} & \text{if} \quad r \leq r < \infty. \end{cases}$$

Proof. For $\boldsymbol{v}_h = \boldsymbol{u}_\varepsilon^\varepsilon$ in (2.16), $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (2.48) and $q_h = p_h^\varepsilon - p_h$, one obtains

$$j_\varepsilon(\boldsymbol{u}_h^\varepsilon) - j(\boldsymbol{u}^\varepsilon) + \langle A\boldsymbol{u}_h^\varepsilon - A\boldsymbol{u}_h, \boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h\rangle \leq j_\varepsilon(\boldsymbol{u}_h) - j(\boldsymbol{u}_h).$$

Now using (2.46), and the properties of the operator A (see (2.10) and (2.11), one obtains the desired results.

Remark 2.4.1 *The convergence result in Lemma 2.4.1 as $\varepsilon \to 0$ ensures that one can approximate the solution of (2.16) by the one of (2.48). Moreover since j_ε is differentiable, it can be shown (see [69]) that (2.48) is equivalent to*

$$\begin{cases} \textit{Find } (\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \boldsymbol{V}_h \times M_h \textit{ such that} \\ \langle A\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h\rangle - b(\boldsymbol{v}_h, p_h^\varepsilon) + \langle j_\varepsilon'(\boldsymbol{u}_h^\varepsilon), \boldsymbol{v}_h\rangle = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, \\ b(\boldsymbol{u}_h^\varepsilon, q_h) = 0, \\ \textit{for all } \ (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \end{cases} \tag{2.49}$$

Next, to eliminate the incompressibility condition, we introduce a penalization term by considering

$$b(\boldsymbol{u}_h^{\epsilon,\eta}, q_h) + \eta\, c(p_h^{\epsilon,\eta}, q_h) = 0,$$

where $c(p,q) = \int_\Omega pq dx$. Then we "approximate" the problem (2.49) by

$$\begin{cases} \text{Find } (\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}) \in \mathbf{V}_h \times M_h \text{ such that} \\ \langle A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{v}_h\rangle - b(p_h^{\varepsilon,\eta}, \mathbf{v}_h) + \langle j'_\varepsilon(\boldsymbol{u}_{h\boldsymbol{\tau}}^{\varepsilon,\eta}), \boldsymbol{v}_{h\boldsymbol{\tau}}\rangle = \langle \mathbf{f}, \mathbf{v}_h\rangle \quad , \\ b(\mathbf{u}_h^{\varepsilon,\eta}, q_h) + \eta\, c(p_h^{\varepsilon,\eta}, q_h) = 0 \quad , \\ \text{for all } \ (\boldsymbol{v}_h, q_h) \in M_h \times \mathbf{V}_h \ . \end{cases} \tag{2.50}$$

Remark 2.4.2 *Following [48, 70], (2.50) admits a unique solution* $(\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}) \in \boldsymbol{V}_h \times M_h$ *which converges to* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$*, solution of (2.16), as* (ε, η) *goes to* 0*.*

The numerical resolution of (2.50) remains quite challenging because of the presence of the nonlinear expressions $\langle A\boldsymbol{u}_h^{\varepsilon,\eta}, \boldsymbol{v}_h\rangle$, and $\langle J'_\varepsilon(\boldsymbol{u}_h^{\varepsilon,\eta}), \boldsymbol{v}_h\rangle$ among other. We next introduce a solution strategy of (2.50) by adopting a suitable time evolution in which the long term behavior of the solution of the later problem will be "close enough in some sense" to solution of the former problem (see the pioneering work [63]). Hence the following steps will be adopted to solve (2.50)

Step 1 associate to the weak formulation (2.50) an initial value problem in $\boldsymbol{V}_h \times M_h$.

Step 2 time discretize the initial value problem formulated in step 1.

Applying the above methodology, we obtain step 1 by associating to (2.50) the following initial value problem: Given $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$, and assuming that $\boldsymbol{f} \in \boldsymbol{V}^*$ is independent of time, we consider the following problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \boldsymbol{V}_h \times M_h \text{ such that} \\ \left(\dfrac{d}{dt}\boldsymbol{u}_h^{\varepsilon,\eta}(t), \boldsymbol{v}_h\right) + \langle A\boldsymbol{u}_h^{\varepsilon,\eta}(t), \boldsymbol{v}_h\rangle - b(p_h^{\varepsilon,\eta}(t), \boldsymbol{v}_h) \\ +\langle J_\varepsilon'(\boldsymbol{u}_h^{\varepsilon,\eta}(t)), \boldsymbol{v}_h\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ b(\boldsymbol{u}_h^{\varepsilon,\eta}(t), q_h) + \eta\, c(p_h^{\varepsilon,\eta}(t), q_h) = 0, \\ \boldsymbol{u}_h^{\varepsilon,\eta}(0) = \boldsymbol{0}, \\ \text{for all } (q_h, \boldsymbol{v}_h) \in M_h \times \boldsymbol{V}_h. \end{cases} \tag{2.51}$$

Following J.L. Lions [96] and Theorem 0.2.4, it can be shown that the initial value problem (2.51) admits a unique solution $(\mathbf{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \mathbf{V}_h \times\ M_h$ which is bounded independently on time.
The next result tells us why it is important to consider only the long time behavior of the solution $(\mathbf{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t))$ of (2.51).

Theorem 2.4.1 *The evolution problem (2.51) admits a unique solution $(\boldsymbol{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \mathbf{V}_h \times M_h$ which is bounded independently on time. Furthermore $\boldsymbol{u}_h^{\varepsilon,\eta}(t)$ converges to $\boldsymbol{u}_h^{\varepsilon,\eta}$ solution of (2.50) exponentially as t goes to infinity. More precisely, we have:*

$$\|\boldsymbol{u}_h^{\varepsilon,\eta}(t) - \boldsymbol{u}_h^{\varepsilon,\eta}\|_r^2 \le \|\boldsymbol{u}_0 - \boldsymbol{u}_h^{\varepsilon,\eta}\|_{1,r}^2 e^{-2C\nu t}, \quad \textit{for all } \ t \ge 0 \tag{2.52}$$

where C is a generic positive constant independent of ϵ and η.

Proof. Note that the equation $(2.51)_1$ is equivalent to

$$\langle \partial_t \mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)\rangle + \langle A\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)\rangle - b(p_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)) \tag{2.53}$$
$$+j_\varepsilon(\boldsymbol{v}_{h\boldsymbol{\tau}}) - j_\varepsilon(\boldsymbol{u}_{h\boldsymbol{\tau}}^{\varepsilon,\eta}(t)) \ge \langle \mathbf{f}, \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)\rangle \ \ \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Note also that $(2.48)_1$ and $(2.50)_1$ are equivalent. Let $\mathbf{v}_h = \mathbf{u}_h^{\varepsilon,\eta}(t)$ in $(2.48)_1$ and $\mathbf{v}_h = \mathbf{u}_h^{\varepsilon,\eta}$ in (2.53) and adding the resulting inequalities, it follows that

$$\langle \partial_t \mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{u}_h^{\varepsilon,\eta} - \mathbf{u}_h^{\varepsilon,\eta}(t)\rangle - \langle A\mathbf{u}_h^{\varepsilon,\eta}(t) - A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}\rangle + b(p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}, \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}) \ge 0. \tag{2.54}$$

Subtracting $(2.50)_2$ from $(2.51)_2$ and taking $q_h = p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}$ ones obtains:

$$b(\mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) = -\eta\, c(p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) \le 0.$$

Let us set $\mathbf{w}_h(t) = \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}$, using the fact that

$$-b(\mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) \geq 0$$

and the monotonicity (2.10) and (2.11), then we have from (2.54) that:
for $1 < r \leq 2$

$$\frac{d}{dt}\|\mathbf{w}_h(t)\|_r^2 + 2C\nu(\|\mathbf{u}_h^{\varepsilon,\eta}(t)\|_{1,r} + \|\mathbf{u}_h^{\varepsilon,\eta}\|_{1,r})^{r-2}\|\mathbf{w}_h(t)\|_{1,r}^2 \leq 0,$$

for $2 \leq r < \infty$

$$\frac{d}{dt}\|\mathbf{w}_h(t)\|_r^2 + 2C\nu\|\mathbf{w}_h(t)\|_{1,r}^r \leq 0$$

Note that $\|\mathbf{u}_h^{\varepsilon,\eta}\|_{1,r} \leq C$ and $\|\mathbf{u}_h^{\varepsilon,\eta}(t)\|_{1,r} \leq C$ when t goes to infinity, then for $1 < r \leq 2$

$$\frac{d}{dt}\|\mathbf{w}_h(t)\|_r^2 + 2C\nu\|\mathbf{w}_h(t)\|_{1,r}^2 \leq 0.$$

We obtain the result via Gronwall's lemma. □

Finally, concerning Step 2, we first consider the following discrete linear scheme:

Let $N \in \mathbb{N}^*$ and set $k = T/N$. Given $(\mathbf{u}_h^{\varepsilon,\eta,0}, p_h^{\varepsilon,\eta,0})$ which is a suitable approximation of $(\mathbf{u}_0, p_0)$. Knowing $(\mathbf{u}_h^{\varepsilon,\eta,m-1}, p_h^{\varepsilon,\eta,m-1})$, compute $(\mathbf{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ in $\mathbf{V}_h \times M_h$, solution of:

$$\begin{cases} \frac{1}{k}(\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}, \mathbf{v}_h) + \nu((|\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2})\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) \\ -b(p_h^{\varepsilon,\eta,m}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - \langle j'_\varepsilon(\boldsymbol{u}_{h\boldsymbol{\tau}}^{\varepsilon,\eta,m-1}), \boldsymbol{v}_{h\boldsymbol{\tau}}\rangle \ \ \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h^{\varepsilon,\eta,m}, q_h) + \eta\, c(p_h^{\varepsilon,\eta,m}, q_h) = 0, \ \ \forall q_h \in M_h \ . \end{cases} \tag{2.55}$$

But in order to obtain better numerical results, we instead adopted the following scheme

$$\begin{cases} \frac{1}{k}(\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}, \mathbf{v}_h) + \nu((|\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2})\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) + \\ \mu(k)(\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) - b(p_h^{\varepsilon,\eta,m}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - \langle j'_\varepsilon(\boldsymbol{u}_{h\boldsymbol{\tau}}^{\varepsilon,\eta,m-1}), \boldsymbol{v}_{h\boldsymbol{\tau}}\rangle \ \ \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h^{\varepsilon,\eta,m}, q_h) + \eta\, c(p_h^{\varepsilon,\eta,m}, q_h) = 0, \ \ \forall q_h \in M_h, \end{cases} \tag{2.56}$$

where $\mu(k)$ should be regarded as artificial viscosity, and given as follows

$$0 < \mu(k) < 1 \qquad \text{such that} \qquad \lim_{k \to 0} \mu(k) = 0 \ . \tag{2.57}$$

Remark 2.4.3 *(i) Since (2.56) is a linear system of equations, by rearranging terms, one sees that it is a square linear system in finite dimension; hence uniqueness implies existence of solution. Uniqueness of $\boldsymbol{u}_h^{\varepsilon,\eta,m}$ follows from the energy estimate (2.58), while uniqueness of $p_h^{\varepsilon,\eta,m}$ is the consequence of the discrete version of the inf-sup condition (1.14).*

(ii) The introduction of a coercive term $\mu(k)(\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m}, \nabla \boldsymbol{v}_h)$ has the effect of bringing more stability/smoothness to the system. The results of our numerical computations support that intuition.

Theorem 2.4.2 *Suppose that $\boldsymbol{u}_h^{\varepsilon,\eta,0}$ is chosen to satisfy*

$$\|\boldsymbol{u}_h^{\varepsilon,\eta,0}\| \leq C \|\boldsymbol{u}_0\|,$$

where C denote a constant independent of k and h. Let $(\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta})$ be the solution of (2.51), then the iterative solution $(\boldsymbol{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ of (2.56) converges to $(\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta})$ as m tends to infinity.

Proof. For that, we follow Girault and Gonzalez [97]. The proof is obtained in two steps. First, one obtains some energy estimates, next we use compactness results and pass to the limit.

Step 1: energy estimates

Lemma 2.4.2 *There exists $C > 0$ and δ which verify $0 < \delta^2 < \mu(k) < 1$ such that:*

$$\sup_{0 \leq m \leq N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|^2 \leq C, \qquad k(\mu(k) - \delta^2) \sum_{m=1}^{N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \leq C, \tag{2.58}$$

$$\sum_{m=1}^{N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m} - \boldsymbol{u}_h^{\varepsilon,\eta,m-1}\|^2 \leq C, \tag{2.59}$$

$$k \sum_{m=1}^{N} \int_\Omega |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m}|^2 dx \leq C. \tag{2.60}$$

Proof Take $\boldsymbol{v}_h = \boldsymbol{u}_h^{\varepsilon,\eta,m}$, $q_h = p_h^{\varepsilon,\eta,m}$ in (2.56), use (2.47) and the fact

$$-b(\boldsymbol{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m}) = \eta c(p_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m}) \geq 0,$$

and $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ we have:

$$\begin{aligned} \|\mathbf{u}_h^{\varepsilon,\eta,m}\|^2 &- \|\mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 + \|\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 \\ &+ 2\nu k \int_\Omega |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \mathbf{u}_h^{\varepsilon,\eta,m}|^2 dx + 2k\mu(k) \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \\ &\leq 2k(\boldsymbol{f}, \boldsymbol{u}_h^{\varepsilon,\eta,m}) - 2k\langle j'_\varepsilon(\boldsymbol{u}_h^{\varepsilon,\eta,m-1}), \boldsymbol{u}_h^{\varepsilon,\eta,m}\rangle \\ &\leq 2k\|\boldsymbol{f}\|_{-1,r'} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r} + 2k\|g\|_{L^{r'}(S)} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r} \\ &\leq k\frac{\|\boldsymbol{f}\|_{-1,r'}^2}{\delta^2} + k\frac{\|g\|_{L^{r'}(S)}^2}{\delta^2} + 2k\delta^2 \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|^2 &- \|\boldsymbol{u}_h^{\varepsilon,\eta,m-1}\|^2 + \|\mathbf{u}_h^{\varepsilon,\eta,m} - \boldsymbol{u}_h^{\varepsilon,\eta,m-1}\|^2 \\ &+ 2\nu k \textstyle\int_\Omega |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m}|^2 dx + 2k(\mu(k) - \delta^2) \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \\ &\leq \frac{k}{\delta^2} \left(\|\boldsymbol{f}\|_{-1,r'}^2 + \|g\|_{L^{r'}(S)}^2 \right). \end{aligned} \tag{2.61}$$

Summing (2.61) for $m = 1, ..., N$ and using the fact that $\sum_{m=1}^N k = T$ yields

$$\begin{aligned} \|\boldsymbol{u}_h^{\varepsilon,\eta,N}\|^2 &+ \sum_{m=1}^N \|\boldsymbol{u}_h^{\varepsilon,\eta,m} - \boldsymbol{u}_h^{\varepsilon,\eta,m-1}\|^2 + 2\nu k \sum_{m=1}^N \int_\Omega |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \mathbf{u}_h^{\varepsilon,\eta,m}|^2 dx \\ &+ 2k(\mu(k) - \delta^2) \sum_{m=1}^N \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \leq \frac{T}{\delta^2} \left(\|\boldsymbol{f}\|_{-1,r'}^2 + \|g\|_{L^{r'}(S)}^2 \right) + \|\boldsymbol{u}_h^{\varepsilon,\eta,0}\|^2, \end{aligned}$$

therefore we obtain the second relation in (2.58), (2.59) and (2.60). The first inequality in (2.58) is readily obtained by summing (2.61) for $m = 1, ..., n$ and using the fact that $\sum_{m=1}^n k \leq \sum_{m=1}^N k = T$.

Step 2: weak convergence/passage to the limit

Let $\boldsymbol{u}_{hk}^{\varepsilon,\eta} \in \mathcal{C}^0([0,T], \boldsymbol{V})$ be affine in each subinterval $[t_{m-1}, t_m]$ with $\boldsymbol{u}_{hk}^{\varepsilon,\eta}(t_m) = \boldsymbol{u}_h^{\varepsilon,\eta,m}$ for $1 \leq m \leq N$. Let $\boldsymbol{u}_{hk}^{\varepsilon,\eta\, r}$, $\boldsymbol{u}_{hk}^{\varepsilon,\eta l}$ be the piecewise constant function such that

$$\boldsymbol{u}_{hk}^{\varepsilon,\eta\, r}|_{[t_{m-1},t_m[} = \boldsymbol{u}_h^{\varepsilon,\eta,m} \quad , \quad \boldsymbol{u}_{hk}^{\varepsilon,\eta\, l}|_{]t_{m-1},t_m]} = \boldsymbol{u}_h^{\varepsilon,\eta,m-1}.$$

Using *a priori* estimate obtained in Lemma 2.4.2,

$$\begin{aligned} &\boldsymbol{u}_{hk}^{\varepsilon,\eta},\ \boldsymbol{u}_{hk}^{\varepsilon,\eta\, r},\ \boldsymbol{u}_{hk}^{\varepsilon,\eta\, l} \text{ remain in a bounded set of } L^\infty(\boldsymbol{L}^2(\Omega)) \text{ and,}\\ &\boldsymbol{u}_{hk}^{\varepsilon,\eta\, r} \text{ remains in a bounded set of } L^2(\boldsymbol{W}_0^{1,r}(\Omega)). \end{aligned} \tag{2.62}$$

Furthermore, from (2.59), we have

$$\|\boldsymbol{u}_{hk}^{\varepsilon,\eta\, r} - \boldsymbol{u}_{hk}^{\varepsilon,\eta\, l}\|_{L^2(\boldsymbol{L}^2(\Omega))} \le Ck^{1/2} \textit{ and } \|\boldsymbol{u}_{hk}^{\varepsilon,\eta} - \boldsymbol{u}_{hk}^{\varepsilon,\eta\, r}\|_{L^2(\boldsymbol{L}^2(\Omega))} \le Ck^{1/2}. \tag{2.63}$$

Then we can extract a subsequence $k' \subset k$ still denoted k such that

$$\begin{aligned} \boldsymbol{u}_{hk}^{\varepsilon,\eta\, r} &\to \boldsymbol{u}_{h}^{\varepsilon,\eta\, r} \text{ weakly* in } L^\infty(\boldsymbol{L}^2(\Omega)),\\ \boldsymbol{u}_{hk}^{\varepsilon,\eta\, l} &\to \boldsymbol{u}_{h}^{\varepsilon,\eta\, l} \text{ weakly* in } L^\infty(\boldsymbol{L}^2(\Omega)),\\ \boldsymbol{u}_{hk}^{\varepsilon,\eta} &\to \mathbf{u}_{h}^{\varepsilon,\eta} \text{ weakly* in } L^\infty(\boldsymbol{L}^2(\Omega)),\\ \boldsymbol{u}_{hk}^{\varepsilon,\eta\, r} &\to \boldsymbol{u}_{h}^{\varepsilon,\eta\, r} \text{ weakly in } L^2(\boldsymbol{W}_0^{1,r}(\Omega)), \end{aligned}$$

and from (2.63), one obtains $\boldsymbol{u}_{h}^{\varepsilon,\eta\, r} = \boldsymbol{u}_{h}^{\varepsilon,\eta\, l} = \boldsymbol{u}_{h}^{\varepsilon,\eta}$.
Note that $\boldsymbol{u}_{hk}^{\varepsilon,\eta\, r}$, $\boldsymbol{u}_{hk}^{\varepsilon,\eta\, l}$ and $\boldsymbol{u}_{hk}^{\varepsilon,\eta}$ verify:

$$\begin{aligned} &\left(\frac{d}{dt}\boldsymbol{u}_{hk}^{\varepsilon,\eta}, \boldsymbol{v}_h\right) + \nu((|\nabla \mathbf{u}_{hk}^{\varepsilon,\eta\, l}|^{r-2})\nabla \mathbf{u}_{hk}^{\varepsilon,\eta\, r}, \nabla \boldsymbol{v}_h) + \mu(k)(\nabla \boldsymbol{u}_{hk}^{\varepsilon,\eta\, r}, \nabla \boldsymbol{v}_h)\\ &+\frac{1}{\eta}\left(\rho_h(\nabla\cdot \boldsymbol{u}_{h}^{\varepsilon,\eta\, r}), \rho_h(\nabla\cdot \boldsymbol{v}_h)\right) = (\boldsymbol{f}, \boldsymbol{v}_h) - \langle J_\varepsilon'(\boldsymbol{u}_{hk}^{\varepsilon,\eta\, l}), \boldsymbol{v}_h\rangle \;\; \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned} \tag{2.64}$$

where ρ_h is the orthogonal projection of $L^2(\Omega)$ onto M_h. The weak convergence above allows us to pass to the limit in all bilinear terms in (2.64). But as far as the nonlinear term is concerned, we advice the reader to see the results in [96], where similar expression has been analyzed.

Remark 2.4.4 *Note that establishing convergence of the pressure is more delicate because it involves convergence of the time derivative of the velocity whose proof is fairly long and intricate; cf. Lions [96] and Temam [4].*
The initialization of the flow defined by (2.51) and of its time discrete counterpart defined by (2.56) is important. Let us observe that since one has well-posedness of (2.50) for all values of r in $[1, \infty)$, in order to consolidate the convergence of (2.50), we suggest the solution of Stokes equations

$$\begin{cases} \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \;\; \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\\ b(\boldsymbol{u}_h, q_h) + \eta\, c(p_h, q_h) = 0, \;\; \forall q_h \in M_h, \end{cases} \tag{2.65}$$

as initial condition for our algorithm.

2.5 Numerical experiments

In this section our goal is to test numerically the convergence and rate of convergence obtained in Section 2.3, and to illustrate the performance of our algorithm via benchmark examples. The implementation is done by extending the Matlab code developed in [85]. In all the examples presented, the velocity and pressure will be approximated by the continuous $P2-P1$ element.
We recall that the different steps of our algorithm are as follows:

(a) compute the initial flow by solving (2.65).

(b) knowing $(\mathbf{u}_h^{\varepsilon,\eta,m-1}, p_h^{\varepsilon,\eta,m-1})$, compute $(\mathbf{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ solution of (2.56).

2.5.1 Numerical accuracy check

We consider $\nu = 0.4$, $k = 1/100$, $\mu(k) = 1/500$, $\eta = \varepsilon = 1/1000$, the exterior force to be unity and $g = 0.1$.
The convergence result obtained in Section 2.3, is tested by computing the rate of convergence using (2.56). In this first example, we take $\Omega = (0,1)^2$, with $\partial\Omega = \Gamma \cup S$ where

$$\begin{aligned}\Gamma &= \{(x,0)\ ,\ 0<x<1\} \cup \{(0,y)\ ,\ 0<y<1\}\\ S &= \{(1,y)\ ,\ 0<y<1\} \cup \{(x,1)\ ,\ 0<x<1\}\ .\end{aligned}$$

Since we do not know the exact solution, we employ the approximate solutions with $N = 60$ as the reference solutions $(\mathbf{u}_{ref}, p_{ref})$, and we compute the L^r-norm and $\mathbf{W}_{1,r}$-norm of the difference of the reference solution and the approximate solution $(\mathbf{u}_h, p_h)$. The results are presented in Tables 2.1, 2.2 and 2.3. The predicted convergence rate $O(h^{1/r})$ for $W^{1,r}$ is noted.

Table 2.1: Velocity convergence results for $r = 3$

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{L^r}$	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.6696E-5	4.4945E-4		
1/10	7.9862E-6	3.6173E-4	1.4437	0.4351
1/12	6.1507E-6	3.3163E-4	1.4324	0.4765
1/15	4.5180E-6	2.9887E-4	1.3825	0.4661
1/20	2.9500E-6	2.4827E-4	1.4368	0.5430

Table 2.2: Velocity convergence results for $r = 7/2$

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{L^r}$	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.7653E-5	4.8807E-4		
1/10	8.6259E-6	4.0641E-4	1.4019	0.3584
1/12	6.6953E-6	3.7694E-4	1.3897	0.4129
1/15	4.9884E-6	3.4468E-4	1.3188	0.4010
1/20	3.3116E-6	2.9126E-4	1.4241	0.4854

Table 2.3: Velocity convergence results for $r = 4$

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{L^r}$	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.8528E-5	5.2119E-4		
1/10	9.2218E-6	4.4538E-4	1.3659	0.3077
1/12	7.2046E-6	4.1679E-4	1.3542	0.3639
1/15	5.4342E-6	3.8538E-4	1.2636	0.3511
1/20	3.6620E-6	3.4006E-4	1.3720	0.4349

Next, to see the effect of the "stabilizing term" added, we repeat the same exercise with $\mu(k) = 0$. The contribution of the added term is made visible (see Table 2.4). Indeed, the convergence is much faster when $\mu(k)$ is not zero.

Table 2.4: Velocity convergence results for $r = 3$ and $\mu(k) = 0$

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{L^r}$	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.8428E-5	5.2009E-4		
1/10	9.3218E-6	4.4538E-4	1.3341	0.3036
1/12	7.4046E-6	4.2080E-4	1.2629	0.3114
1/15	5.534E-6	3.9438E-4	1.3049	0.2906
1/20	3.7620E-6	3.6120E-4	1.3417	0.3151

2.5.2 Driven cavity

Driven cavity is a benchmark test problem that has been considered by many researchers [98, 99, 5] among others. We assume $\Omega = (0,1)^2$, the boundary of which consists of two portions Γ and S is given by

$$\Gamma = \{(0,y)/0 < y < 1\} \cup \{(x,0)/0 < x < 1\} \tag{2.66}$$

$$S_1 = \{(x,1)/0 < x < 1\}, \qquad S_2 = \{(1,y)/0 < y < 1\},$$

$$S = S_1 \cup S_2. \tag{2.67}$$

For the triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, we employ a uniform $N \times N$ mesh, where N denotes the division number of each side of the domain.
Let us consider

$$\begin{cases} u_1(x,y) = -x^2y(x-1)(3y-2) \\ u_2(x,y) = xy^2(y-1)(3x-2) \\ p(x,y) = (2x-1)(2y-1), \end{cases} \tag{2.68}$$

which turns out to be the exact solution of the problem (2.1)-(2.5) under the appropriate choice of $\mathbf{f}$ and g. It is easy to verify that the exact solution $\mathbf{u}$ satisfies $\mathbf{u} = \mathbf{0}$ on Γ, $\mathbf{u} \cdot \mathbf{n} = u_1 = 0$, $u_2 \neq 0$ on S_2, and $u_1 \neq 0$, $\mathbf{u} \cdot \mathbf{n} = u_2 = 0$ on S_1. Thus

$$\begin{aligned} \boldsymbol{\sigma}_{\boldsymbol{\tau}} &= -2\mu x^2(x-1)[2(x^2-2x)^2 + 8x^4(x-1)^2]^{(r-2)/2} \qquad \text{on} \quad S_1 \\ \boldsymbol{\sigma}_{\boldsymbol{\tau}} &= -2\mu y^2(y-1)[2(y^2-2y)^2 + 8y^4(y-1)^2]^{(r-2)/2} \qquad \text{on} \quad S_2 \end{aligned} \tag{2.69}$$

On the other hand, from the slip boundary conditions (2.5), we have

$$|\boldsymbol{\sigma}_{\boldsymbol{\tau}}| \leq g \qquad \text{on} \quad S = S_1 \cup S_2$$

then we find that with g constant,

$$\begin{aligned} g &\geq \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| \Rightarrow (2.68) \text{ remains a solution.} \\ g &\leq \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| \Rightarrow (2.68) \text{ is no longer a solution and a non-trivial slip occurs.} \end{aligned}$$

We indeed observe some of the above mentioned phenomena in our numerical computation, as indicated in the plots of the velocity field shown in Figures 2.1-2.5.
In addition, we find that

(a) the bigger the threshold g of tangential stress becomes, the more difficult it becomes for a non-trivial slip to occur,

(b) the smaller the threshold g of tangential stress becomes, the more easier it becomes for a non-trivial slip to occur,

which is in agreement with the predicted outcome.
For all the numerical results which follow, $\nu = 0.4$, $k = 1/100$, $\mu(k) = 1/500$, $\eta = \varepsilon = 1/1000$, and g is indicated on the pictures.
To be more precise, one observes in Figure 2.1, that in some region of the boundaries, the fluid slips (see the right and top part of the boundary for right picture), where as in the left hand side, the fluid adheres at the boundaries. Similar pattern are observed in Figures 2.2—2.5 below.

- $r = 3/2 \quad \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.1035$

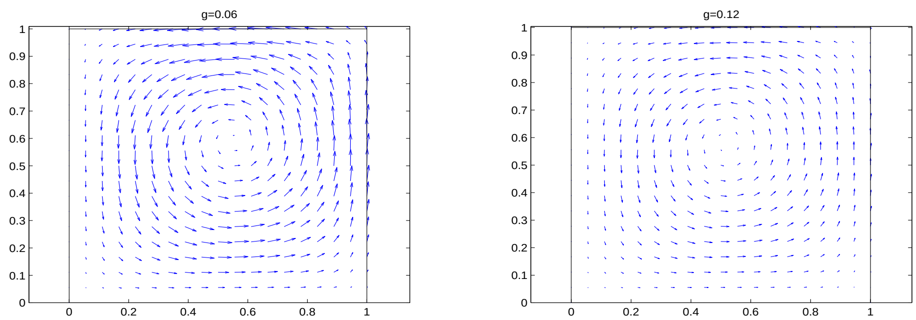


Figure 2.1: Velocity field for $r = 3/2$

- $r = 2 \quad \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.1185$

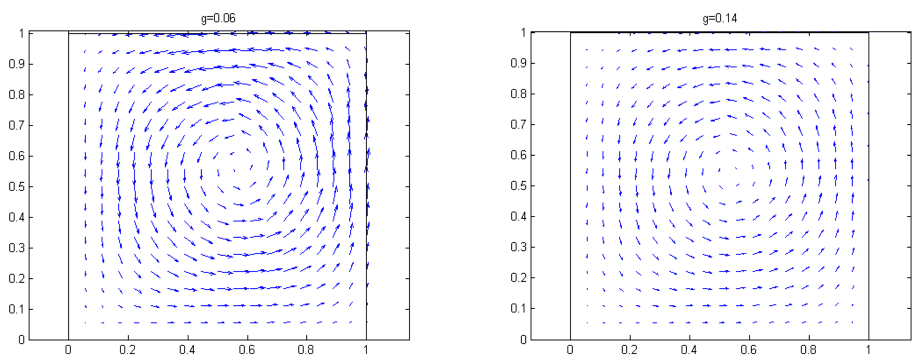


Figure 2.2: Velocity field for $r = 2$

- $r = 3 \quad \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.1591$

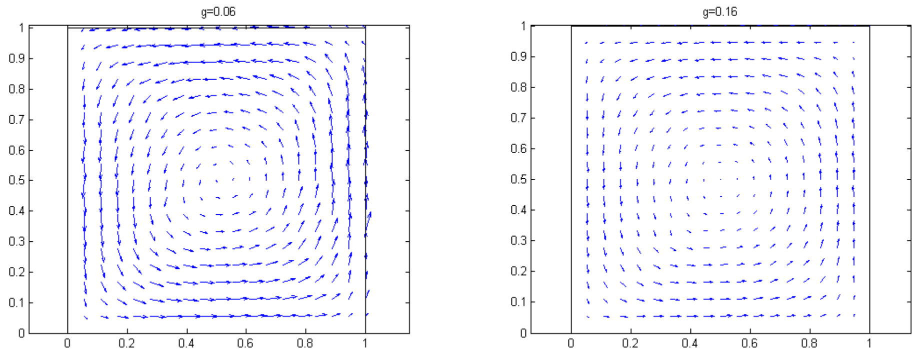


Figure 2.3: Velocity field for $r = 3$

- $r = 7/2 \quad \max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.1855$

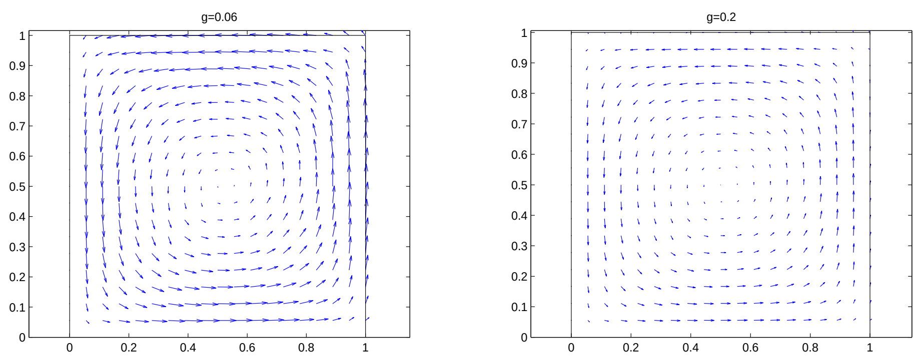


Figure 2.4: Velocity field for $r = 7/2$

- $\mathbf{r = 4}$ $\max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.2170$

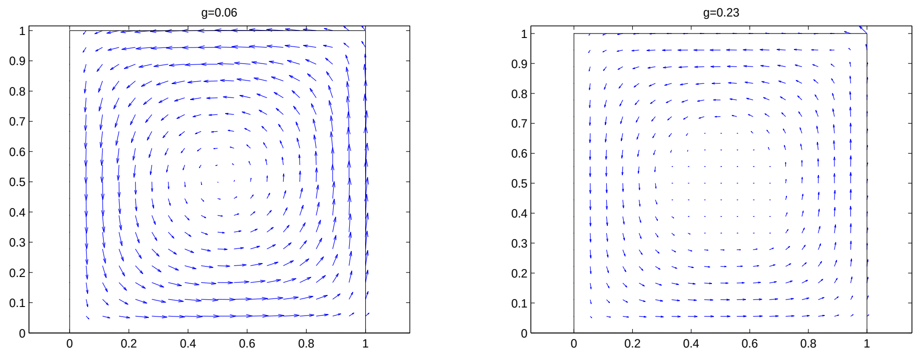


Figure 2.5: Velocity field for $r = 4$

Lastly, the role of the "stabilizing term" is also justified in this test problem. Indeed, when $\mu(k)(\nabla \mathbf{u}_h^{\eta,\varepsilon,m}, \nabla \mathbf{v}_h)$ is neglected, that is $\mu(k) = 0$ the slip is more pronounced in Figure 2.3 than Figure 2.6.

- $r = 3$ $\max_S |\boldsymbol{\sigma}_{\boldsymbol{\tau}}| = 0.1591$

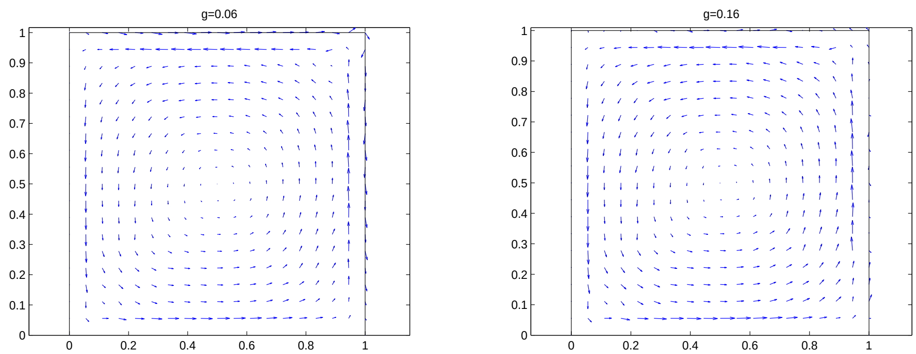


Figure 2.6: Velocity field for $r = 3$ with $\mu(k) = 0$

Chapter 3

On the long-time stability of the Crank-Nicolson scheme for the 2D Navier-Stokes equations driven by threshold slip boundary conditions

3.1 Introduction

We consider the Navier-Stokes equations of viscous incompressible fluids:

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\,\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \boldsymbol{Q} = \Omega\times\mathbb{R}^+, \tag{3.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \boldsymbol{Q}, \tag{3.2}$$

with the impermeability boundary condition

$$u_n = \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } S\times\mathbb{R}^+ \tag{3.3}$$

and the slip boundary condition [23, 28]

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ , \ \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{array}\right\} \text{ on } \quad S\times(0,\infty). \tag{3.4}$$

On the remaining part of the boundary, Γ, we assume Dirichlet boundary condition, i.e,

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times \mathbb{R}^+ \tag{3.5}$$

Finally, the initial condition is given by

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{on } \overline{\Omega}. \tag{3.6}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain, with boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is made of two components S, and Γ with $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, and $S \cap \Gamma = \emptyset$. ν is a positive quantity representing the viscosity coefficient, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^2$ is the initial velocity, and $g : S \times (0, \infty) \to (0, \infty)$ is the barrier or threshold function. The velocity of the fluid is $\boldsymbol{u}$ and p stands for the pressure, while $\boldsymbol{f}$ is the external force. Furthermore, $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$ of Ω, $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - u_n\boldsymbol{n}$ is the tangential component of the velocity $\boldsymbol{u}$, and $(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = \boldsymbol{\sigma n} - (\boldsymbol{n} \cdot \boldsymbol{\sigma n})\boldsymbol{n}$ is the tangential traction. Of course, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\varepsilon(\boldsymbol{u})$ is the Cauchy stress tensor, where $\boldsymbol{I}$ is the identity matrix, and $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$.

There are numerous works devoted to the development of efficient schemes for the nonstationary Navier-Stokes problem dealing with Dirichlet or periodic boundary conditions, some works can be found in [100, 50, 101, 102, 103, 104]. A few work has been done with the slip boundary conditions of friction type (3.4), we can mention the work of Djoko [75] on the Stokes equation with slip boundary conditions of friction type (3.4).
The subject of the present chapter is to approximate the two-dimensional problem (3.1)-(3.4) using the Crank-Nicholson scheme in time and the finite element approximation in space. We also establish its well-posedness and stability of the numerical scheme on L^2-norm and H^1-norm for all positive time. This follows the study done by Tone [50], in which the same question is answered for fluid flows governed by Dirichlet boundary condition. It is immediate that our work differs from that of Tone, in the sense that the resulting variational structure we are dealing with is in the form of inequality, and obtaining H^1-estimate is more involved because of the presence of the non-differentiable term appearing on the boundary where slip occurs.

The chapter is organized as follows. In Section 3.2, We give some notations, formulate the variational models and indicate how existence of weak solution is obtained. Section 3.3 is concerned with numerical scheme while in Section 3.4, is devoted for the well-posedness and L^2-stability of the scheme. Finally, in Section 3.5 we present the H^1-stability.

3.2 Preliminaries and Variational formulation

In this section, we introduce notations and some results that will be used throughout our work. We also derive various weak formulations and discuss (recall) some existence results. For the mathematical setting of the problem, we need to introduce the following spaces:

$$\boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega)^2,\ \boldsymbol{v}|_\Gamma = 0,\ \boldsymbol{v}\cdot\mathbf{n}|_S = 0\} \tag{3.7}$$

$$\boldsymbol{V}_\sigma = \{\boldsymbol{v} \in \boldsymbol{V},\quad \operatorname{div}\boldsymbol{v} = 0\} \tag{3.8}$$

$$\boldsymbol{H} = \{\boldsymbol{v} \in Ł^2(\Omega)^2,\ \operatorname{div}\boldsymbol{v} = 0,\ \boldsymbol{v}|_{\partial\Omega} = 0\}, \tag{3.9}$$

$$M = L_0^2(\Omega). \tag{3.10}$$

The space $\boldsymbol{H}$ is endowed with the scalar product and the norm of $L^2(\Omega)^2$, denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively and $\boldsymbol{V}$ is endowed with the scalar product

$$((\boldsymbol{u},\boldsymbol{v})) = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_i}{\partial x_j}(\boldsymbol{x})\frac{\partial u_j}{\partial x_i}(\boldsymbol{x})dx, \tag{3.11}$$

and the corresponding norm

$$\|\boldsymbol{u}\|_1 = ((\boldsymbol{u},\boldsymbol{u}))^{1/2}.$$

We will use the following bilinear and trilinear forms (see [79, 80, 1])

$$b : \boldsymbol{V}\times M \to \mathbb{R} \quad \text{with} \quad b(\boldsymbol{u},p) = (\operatorname{div}\boldsymbol{u},p).$$

$$a : \boldsymbol{V}\times\boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad a(\boldsymbol{u},\boldsymbol{v}) = \nu(\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v})) = 2\nu((\boldsymbol{u},\boldsymbol{v}))$$

$$d : \boldsymbol{V}\times\boldsymbol{V}\times\boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}).$$

We introduce the linear continuous operator A from $\boldsymbol{V}$ into $\boldsymbol{V}'$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = ((\boldsymbol{u}, \boldsymbol{v})) \quad \text{for all} \quad \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}\ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $\boldsymbol{V}$ and $\boldsymbol{V}'$. We denote by B a bilinear operator from $\boldsymbol{V} \times \boldsymbol{V}$ into $\boldsymbol{V}'$ such that

$$\langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle = d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \quad \text{for all} \quad \boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in \boldsymbol{V}\ .$$

Note that we introduce the Stokes operator A by following the approach adopted in [105, 106]. We denote by $\mathcal{P} : L^2(\Omega)^2 \to \boldsymbol{H}$ the Helmholtz projection operator, which is bounded projection associated to the Helmholtz decomposition of $L^2(\Omega)^2$. We define the Stokes operator as follows $A : \boldsymbol{V} \to \boldsymbol{V}'$ such that $A = -\mathcal{P}\Delta$, with domain given as follows, $D(A) = \{\boldsymbol{v} \in \boldsymbol{V},$ such that $A\boldsymbol{v} \in \boldsymbol{H}\}$. Now, assuming that Γ is C^2 and S is C^3, then $D(A) \subset H^2(\Omega)^2$ since $\|\boldsymbol{w}\|_2 \leq C\|A\boldsymbol{w}\|$

From Poincaré-Fredrichs's inequality, we obtain

$$\lambda_1 \int_\Omega |\boldsymbol{v}|^2 \mathrm{dx} \leq \int_\Omega |\nabla \boldsymbol{v}|^2 \mathrm{dx} \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{3.12}$$

Also, of importance in this work is the Korn's inequality which reads;

$$\lambda_1 \int_\Omega |\nabla \boldsymbol{v}|^2 \mathrm{dx} \leq \int_\Omega |\varepsilon(\boldsymbol{v})|^2 \mathrm{dx} \quad \text{for all} \quad \boldsymbol{v} \in D(A). \tag{3.13}$$

where λ_1 is the first eigenvalue of the Stokes operator A.
The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup conditions, i.e., there exists a positive constant β such that

$$\beta\|p\| \leq \sup_{\boldsymbol{u} \in \boldsymbol{V}} \frac{b(\boldsymbol{u}, p)}{\|\boldsymbol{u}\|_1} \qquad \text{for all} \quad p \in L_0^2(\Omega). \tag{3.14}$$

As a readily obtainable consequence of Korn's inequality (3.13), there exists a positive constant α such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|_1^2 \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{3.15}$$

The trilinear form $d(\cdot,\cdot,\cdot)$ is continuous on $H^1(\Omega)^3$ and enjoys the following properties:

$$\begin{aligned}
|d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\leq c_d\|\boldsymbol{u}\|^{1/2}\|A\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\| \\
&\qquad \text{for all } \boldsymbol{u}\in D(A),\ \boldsymbol{v}\in\boldsymbol{V},\ \boldsymbol{w}\in\boldsymbol{H}, && (3.16)\\
|d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\leq c_d\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_1^{1/2}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|^{1/2}\|\boldsymbol{w}\|_1^{1/2} \quad \text{for all } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{V} && (3.17)\\
d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) &= 0 \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}_\sigma, && (3.18)\\
d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= -d(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{V}_\sigma. && (3.19)
\end{aligned}$$

We will make reference to the following:

$$2(u-v,u) = \|u\|^2 - \|v\|^2 + \|u-v\|^2 \quad \text{for all} \quad u,\, v\in L^2(\Omega), \tag{3.20}$$

$$ab \leq \frac{\epsilon}{p}a^p + \frac{1}{q\epsilon^{q/p}}b^q \quad \text{for all} \quad a,\, b,\, \epsilon>0 \quad \text{and} \quad \frac{1}{p}+\frac{1}{q}=1. \tag{3.21}$$

We assume that $\boldsymbol{f}\in L^\infty(0,\infty;L^2(\Omega)^2)$, $\boldsymbol{u}_0\in L^2(\Omega)^2$ and we set $\|\boldsymbol{f}\|_\infty := \|\boldsymbol{f}\|_{L^\infty(0,\infty;L^2(\Omega)^2)}$. With these notations, we introduce the following variational formulation for (3.1)-(3.6): Find $(\boldsymbol{u}(t),p(t))\in\boldsymbol{V}\times M$ such that

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \quad \text{in} \quad \Omega, \tag{3.22}$$

and for a.e. t, with $t\geq 0$

$$\begin{cases}
\text{for all } \ (\boldsymbol{v},q)\in\boldsymbol{V}\times M\\
\langle\boldsymbol{u}'(t),\boldsymbol{v}-\boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t)) + d(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))\\
-b(\boldsymbol{v}-\boldsymbol{u}(t),p(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t)),\\
b(\boldsymbol{u}(t),q)=0\ ,
\end{cases} \tag{3.23}$$

where $J(\boldsymbol{v}) = (g,|\boldsymbol{v}_{\boldsymbol{\tau}}|)_S$.
Note that since the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (3.14), the variational inequality problem (3.23) is equivalent to the following:
Find $\boldsymbol{u}(t)\in\boldsymbol{V}_\sigma$ such that

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \quad \text{in} \quad \Omega, \tag{3.24}$$

and for a.e. t, with $t \geq 0$

$$\begin{cases} \text{for all } \ \boldsymbol{v} \in \boldsymbol{V}_\sigma \\ \langle \boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + d(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\ +J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)), \end{cases} \tag{3.25}$$

The problem of existence and uniqueness of (3.24)-(3.25) can be stated as follows and has been proved in Kashiwabara [30].

Theorem 3.2.1 *Assume:*

$\boldsymbol{f} \in H^1(\mathbb{R}^+, L^2(\Omega)^2)$,
$g \in H^1(\mathbb{R}^+, L^2(S)^2)$ *with* $g(0) \in H^1(S)$,
$\boldsymbol{u}_0 \in H^2(\Omega)^2 \cap \boldsymbol{V}_\sigma$, *and slip boundary condition (3.4) is satisfied at* $t = 0$, *i.e.,*
$|\boldsymbol{\sigma}_{\boldsymbol{\tau}}(\boldsymbol{u}_0)| \leq g(0)$ *and* $\boldsymbol{\sigma}_{\boldsymbol{\tau}}(\boldsymbol{u}_0)\boldsymbol{u}_{0\boldsymbol{\tau}} + g|\boldsymbol{u}_{0\boldsymbol{\tau}}| = 0$ *a.e. on* S.

Then there exists a unique solution $\boldsymbol{u}$ *of problem (3.24)-(3.25) such that*

$$\boldsymbol{u} \in L^\infty(\mathbb{R}^+, \boldsymbol{V}_\sigma), \quad \text{and} \quad \boldsymbol{u}' \in L^\infty(\mathbb{R}^+, L^2(\Omega)^2) \cap L^2(\mathbb{R}^+, \boldsymbol{V}_\sigma). \quad \square$$

In this chapter, we consider a time discretization of (3.24)-(3.25) using the Crank-Nicolson scheme, Find $\boldsymbol{u}^n \in \boldsymbol{V}$ such that

$$\boldsymbol{u}^0 = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \tag{3.26}$$

and for all $n \geq 1$

$$\begin{cases} \text{for all } \ \boldsymbol{v} \in \boldsymbol{V}_\sigma \\ (\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{k}, \boldsymbol{v} - \boldsymbol{u}^n) + \frac{1}{2}a(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}, \boldsymbol{v} - \boldsymbol{u}^n) + \frac{1}{4}d(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}, \boldsymbol{u}^n + \boldsymbol{u}^{n-1}, \boldsymbol{v} - \boldsymbol{u}^n) \\ +J(\boldsymbol{v}) - J(\boldsymbol{u}^n) \geq (\boldsymbol{f}^n, \boldsymbol{v} - \boldsymbol{u}^n), \end{cases} \tag{3.27}$$

where $\boldsymbol{f}^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} \boldsymbol{f}(t)dt$. We want to show that the solution $\boldsymbol{u}^n$ of (3.26)-(3.27) is uniformly bounded for all $n \geq 0$, both in the L^2- and H_0^1-norms. In what follows, we discretize in space and derive such a result assuming some kind of stability condition.

3.3 Numerical scheme

For the spatial discretization, we introduce the general framework as in, e.g.,[1, 107, 108]. We consider a family of finite element spaces $\boldsymbol{V}_{\sigma h} \subset L^2(\Omega)^2$, each of which is endowed with two scalar products, $(\cdot,\cdot)_h$ and $((\cdot,\cdot))_h$, with the corresponding norms, $\|\cdot\|_h$ and $\|\cdot\|_{1,h}$ which mimic the L^2- and H_0^1-norms. These norms are related as follows:

$$\|\boldsymbol{u}_h\|_h \leq K_1 \|\boldsymbol{u}_h\|_{1,h} \quad \text{for all} \quad \boldsymbol{u}_h \in \boldsymbol{V}_{\sigma h}, \tag{3.28}$$

$$\|\boldsymbol{u}_h\|_{1,h} \leq S(h) \|\boldsymbol{u}_h\|_h \quad \text{for all} \quad \boldsymbol{u}_h \in \boldsymbol{V}_{\sigma h}, \tag{3.29}$$

where K_1 is independent of h and $S(h)$ is such that

$$S(h) \to \infty \quad \text{as} \quad h \to 0. \tag{3.30}$$

We assume that the operator A satisfies the same properties on $\boldsymbol{V}_{\sigma h}$ as on $\boldsymbol{V}$ that is:

$$(A\boldsymbol{u}_h, \boldsymbol{v}_h)_h = ((\boldsymbol{u}_h, \boldsymbol{v}_h))_h, \quad \text{for all} \quad \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_{\sigma h}. \tag{3.31}$$

We also assume that a trilinear continuous form $d(\cdot,\cdot,\cdot)$ enjoys the same properties on $\boldsymbol{V}_{\sigma h} \times \boldsymbol{V}_{\sigma h} \times \boldsymbol{V}_{\sigma h}$ with the constant c_d independent of h.
We introduce the so-called restriction operators $r_h : \boldsymbol{V}_\sigma \to \boldsymbol{V}_{\sigma h}$ and assume that, if $\boldsymbol{u}_0 \in \boldsymbol{V}_\sigma \cap C^1(\overline{\Omega})^2$, then

$$\|r_h \boldsymbol{u}_0\|_h \leq K_2 \|\boldsymbol{u}_0\|_{C^1(\overline{\Omega})^2}, \tag{3.32}$$

with the constant K_2 being independent of h (see, e.g.,[1, 108]).
As for the temporal discretization, we consider the following scheme, a discrete version of (3.24)-(3.25): Find $\boldsymbol{u}_h^n \in \boldsymbol{V}_{\sigma h}$ such that

$$\boldsymbol{u}_h^0 = r_h \boldsymbol{u}_0, \tag{3.33}$$

and for all $n \geq 1$

$$\begin{cases} \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_{\sigma h} \\ (\frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{k}, \boldsymbol{v}_h - \boldsymbol{u}_h^n)_h + \frac{1}{2} a(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + \frac{1}{4} d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h - \boldsymbol{u}_h^n) \\ +J(\boldsymbol{v}_h) - J(\boldsymbol{u}_h^n) \geq (\boldsymbol{f}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n)_h. \end{cases} \tag{3.34}$$

Remark 3.3.1 *For existence and uniqueness of the solution of (3.33)-(3.34), one observes that the variational inequality (3.33)-(3.34) can be seen as a case of following modified variational formulation associated to the stationary Navier-Stokes equations with slip boundary conditions type.*

$$\begin{cases} \textit{Find} \quad \boldsymbol{v}_h \in \boldsymbol{V}_{\sigma h} \\ \mathbb{T}(\boldsymbol{v}_h, \boldsymbol{w}_h - \boldsymbol{v}_h) + \mathbb{D}(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{w}_h - \boldsymbol{v}_h) + j(\boldsymbol{w}_h) - j(\boldsymbol{v}_h) \geq (\boldsymbol{F}, \boldsymbol{w}_h - \boldsymbol{v}_h) \\ \quad \textit{for all} \quad \boldsymbol{w}_h \in \boldsymbol{V}_{\sigma h}, \end{cases} \tag{3.35}$$

where $\mathbb{T}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) + \frac{k}{2} a(\boldsymbol{u}, \boldsymbol{v})$, $\mathbb{D}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \frac{k}{4} d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, $j(v) = kJ(v)$ and $\boldsymbol{F} = k\boldsymbol{f}_h^n$. Following [42, 36, 109], (3.35) admits a unique solution $\boldsymbol{v}_h \in \boldsymbol{V}_{\sigma h}$. □

3.4 The $(\boldsymbol{V}_h, \|\cdot\|_h)-$ stability

We start this section by performing the stability analysis of the scheme (3.33)-(3.34) in $(\boldsymbol{V}_{\sigma h}, \|\cdot\|_h)$ and show that the solution is uniformly bounded, provided that a stability CFL-type condition is satisfied.

Lemma 3.4.1 *Let*

$$M \geq K_1^2 \sqrt{2} \|\boldsymbol{f}\|_\infty / \nu$$

be arbitrarily fixed and assume that

$$\|\boldsymbol{u}_0\| \leq M$$

and

$$kS^2(h) \leq \frac{4\nu}{15(c_d^2 M^2 + \nu^2)} \ . \tag{3.36}$$

Then, for any integer $n \geq 1$, we have

$$\|\boldsymbol{u}_h^n\|_h^2 \leq (1+Ck)^{-n}\|\boldsymbol{u}_0\|^2 + C\|\boldsymbol{f}\|_\infty^2[1-(1+Ck)^{-n}], \tag{3.37}$$

$$\|\boldsymbol{u}_h^n\|_h \leq M, \tag{3.38}$$

$$J(\boldsymbol{u}_h^n) \leq \frac{K_1^2}{2\nu}\|\boldsymbol{f}\|_\infty^2, \tag{3.39}$$

$$k\sum_{j=i}^{n}\|\boldsymbol{u}_h^j\|_{1,h}^2 \leq C(M^2 + (n-i+1)k\|\boldsymbol{f}\|_\infty^2) \quad \textit{for all} \quad i = 1, \cdots, n, \tag{3.40}$$

$$\sum_{i=1}^{n}\|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_h^2 \leq C(M^2 + nk\|\boldsymbol{f}\|_\infty^2). \tag{3.41}$$

Proof We first establish the relation (3.48) below and next use it to handle the proof by induction. First, let $\boldsymbol{v}_h = \boldsymbol{0}$ and $\boldsymbol{v}_h = 2\boldsymbol{u}_h^n$ in (3.34), one has

$$\begin{aligned}-\frac{1}{k}(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_h - \frac{1}{2}a(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &- \frac{1}{4}d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)\\ &- J(\boldsymbol{u}_h^n) \geq -(\boldsymbol{f}_h^n, \boldsymbol{u}_h^n)_h,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{k}(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_h + \frac{1}{2}a(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &+ \frac{1}{4}d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)\\ &+ J(\boldsymbol{u}_h^n) \geq (\boldsymbol{f}_h^n, \boldsymbol{u}_h^n)_h.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{k}(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_h + \frac{1}{2}a(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &+ \frac{1}{4}d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)\\ &+ J(\boldsymbol{u}_h^n) = (\boldsymbol{f}_h^n, \boldsymbol{u}_h^n)_h,\end{aligned}$$

which is

$$\begin{aligned}2(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_h + ka(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &+ \frac{k}{2}d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)\\ &+ 2kJ(\boldsymbol{u}_h^n) = 2k(\boldsymbol{f}_h^n, \boldsymbol{u}_h^n)_h.\end{aligned} \tag{3.42}$$

Using the relation (3.20), we have

$$2(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)_h = \|\boldsymbol{u}_h^n\|_h^2 - \|\boldsymbol{u}_h^{n-1}\|_h^2 + \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2. \tag{3.43}$$

Using Cauchy-Schwarz inequality, (3.29) and (3.21), we write

$$\begin{aligned} ka(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &= 2\nu k \|\boldsymbol{u}_h^n\|_{1,h}^2 + ka(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h^n, \boldsymbol{u}_h^n) \\ &\geq 2\nu k \|\boldsymbol{u}_h^n\|_{1,h}^2 - \nu k \|\boldsymbol{u}_h^n\|_{1,h} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{1,h} \\ &\geq 2\nu k \|\boldsymbol{u}_h^n\|_{1,h}^2 - \nu k S(h) \|\boldsymbol{u}_h^n\|_{1,h} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h \qquad (3.44) \\ &\geq 2\nu k \|\boldsymbol{u}_h^n\|_{1,h}^2 - \frac{1}{6} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 - \frac{3}{2} \nu^2 k^2 S^2(h) \|\boldsymbol{u}_h^n\|_{1,h}^2, \end{aligned}$$

and the right hand side of (3.42) is bounded as follows

$$2k(\boldsymbol{f}_h^n, \boldsymbol{u}_h^n)_h \leq 2K_1 k \|\boldsymbol{f}_h^n\|_h \|\boldsymbol{u}_h^n\|_{1,h} \leq \nu k \|\boldsymbol{u}_h^n\|_{1,h}^2 + \frac{K_1^2}{\nu} k \|\boldsymbol{f}\|_\infty^2. \qquad (3.45)$$

To bound the nonlinear term $d(\cdot,\cdot,\cdot)$ in (3.42), we write it as

$$\frac{k}{2} d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) = kd(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + \frac{k}{2} d(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n).$$

Using properties (3.17)-(3.19), (3.21) and recalling (3.29), we obtain the following bounds:

$$\begin{aligned} kd(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &= kd(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) \\ &\leq c_d k S(h) \|\boldsymbol{u}_h^{n-1}\|_h \|\boldsymbol{u}_h^n\|_{1,h} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h \qquad (3.46) \\ &\leq \frac{1}{6} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 + \frac{3}{2} c_d^2 k^2 S(h)^2 \|\boldsymbol{u}_h^{n-1}\|_h^2 \|\boldsymbol{u}_h^n\|_{1,h}^2, \end{aligned}$$

$$\begin{aligned} \frac{k}{2} d(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &= -\frac{k}{2} d(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}) \\ &\leq c_d k S(h) \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h \|\boldsymbol{u}_h^n\|_{1,h} \|\boldsymbol{u}_h^{n-1}\|_h \qquad (3.47) \\ &\leq \frac{1}{6} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 + \frac{3}{8} c_d^2 k^2 S(h)^2 \|\boldsymbol{u}_h^{n-1}\|_h^2 \|\boldsymbol{u}_h^n\|_{1,h}^2. \end{aligned}$$

Gathering (3.42)-(3.47), we obtain

$$\begin{aligned} \|\boldsymbol{u}_h^n\|_h^2 - \|\boldsymbol{u}_h^{n-1}\|_h^2 + \frac{1}{2} \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 &+ \nu k \left\{ 1 - \frac{15}{8\nu} k S(h)^2 (c_d^2 \|\boldsymbol{u}_h^{n-1}\|_h^2 + \nu^2) \right\} \|\boldsymbol{u}_h^n\|_{1,h}^2 \\ &+ 2kJ(\boldsymbol{u}_h^n) \leq \frac{K_1^2}{\nu} k \|\boldsymbol{f}\|_\infty^2. \qquad (3.48) \end{aligned}$$

Note that according to CFL-condition (3.36), if

$$\|\boldsymbol{u}_h^n\|_h \leq M$$

then

$$0 \leq \left\{1 - \frac{15}{8\nu} k S(h)^2 (c_d^2 \|\boldsymbol{u}_h^{n-1}\|_h^2 + \nu^2)\right\} \leq \frac{1}{2}. \qquad (3.49)$$

We now use the induction. It is clear that (3.37) and (3.38) hold for $n = 0$. Then assuming that (3.37) holds for $n = 0, ..., m-1$, for $m \geq 2$, we see under the assumption of the Lemma 3.4.1 that (3.38) holds for $n = 0, ..., m-1$. Then (3.48), together with (3.38) and (3.36), imply:

$$\|\boldsymbol{u}_h^n\|_h^2 - \|\boldsymbol{u}_h^{n-1}\|_h^2 + \frac{1}{2}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 + \frac{\nu}{2} k \|\boldsymbol{u}_h^n\|_{1,h}^2 + 2kJ(\boldsymbol{u}_h^n) \leq \frac{K_1^2}{\nu} k \|\boldsymbol{f}\|_\infty^2 \quad \text{for all} \quad n = 1, ..., m. \qquad (3.50)$$

If we drop the last term on the left hand side and rewrite the remaining equation with n replaced by j and take the sum with $j = i, ..., n$, for some $1 \leq i \leq n$, we obtain

$$\|\boldsymbol{u}_h^n\|_h^2 + \frac{1}{2}\sum_{j=i}^{n} \|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\|_h^2 + \frac{\nu}{2} k \sum_{j=i}^{n} \|\boldsymbol{u}_h^j\|_{1,h}^2 \leq M^2 + \frac{K_1^2}{\nu}(n-i+1)k\|\boldsymbol{f}\|_\infty^2, \qquad (3.51)$$

and, hence, (3.40) and (3.41) hold for all $n = 1, ..., m-1$.
Now using (3.28), relation (3.50) implies

$$\|\boldsymbol{u}_h^n\|_h^2 \leq \frac{1}{(1 + \frac{\nu}{2K_1^2}k)}\|\boldsymbol{u}_h^{n-1}\|_h^2 + \frac{K_1^2}{(1 + \frac{\nu}{2K_1^2}k)\nu} k \|\boldsymbol{f}\|_\infty^2, \quad \text{for all} \quad n = 1, ..., m. \qquad (3.52)$$

Using recursively (3.52), we obtain

$$\begin{aligned} \|\boldsymbol{u}_h^m\|_h^2 &\leq \frac{1}{(1 + \frac{\nu}{2K_1^2}k)^m}\|\boldsymbol{u}_h^0\|_h^2 + \frac{K_1^2}{\nu} k \|\boldsymbol{f}\|_\infty^2 \sum_{i=1}^{m} \frac{1}{(1 + \frac{\nu}{2K_1^2}k)^i}, \\ &\leq (1 + \frac{\nu}{2K_1^2}k)^{-m}\|\boldsymbol{u}^0\|^2 + C\|\boldsymbol{f}\|_\infty^2 \left[1 - (1 + \frac{\nu}{2K_1^2}k)^{-m}\right]. \qquad (3.53) \end{aligned}$$

Thus, (3.37) holds for $n = m$. □

3.5 The $(\boldsymbol{V}_h, \|\cdot\|_{1,h})$- stability

For proving the uniform bound of $\boldsymbol{u}_h^n$ in $(\boldsymbol{V}_h, \|\cdot\|_{1,h})$ for all $n \geq 1$, we first show that it is bounded on any finite interval of time. Then we extend the result to the infinite time using the discrete uniform Gronwall lemma.

Lemma 3.5.1 *Assume that besides the assumptions of Lemma 3.4.1, k also satisfies*

$$k \leq \frac{4K_1^2}{\nu} := \kappa_1. \tag{3.54}$$

Assume also that for some n the following is true:

$$K_3 M^2 k(L_1 \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2\kappa_1}{\nu}\|\boldsymbol{f}\|_\infty^2) \leq \frac{1}{6}, \tag{3.55}$$

where $L_1 = 2 + 3\frac{c_d^2 M^2}{\nu^2}$ and K_3 will be defined later. Then

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 \left[1 + K_4 k(\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2)\right] + K_5 k \|\boldsymbol{f}\|_\infty^2, \tag{3.56}$$

where K_4 and K_5 are positive constants independent of h and n.

Proof Let us define the operator $\mathbb{A}$ as follows (see [109])

$$\mathbb{A} = \begin{cases} A & \text{if in} \quad \Omega \\ 0 & \text{if in} \quad S. \end{cases}$$

Let $\boldsymbol{v}_h = \boldsymbol{u}_h^n + \mathbb{A}(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})$ and $\boldsymbol{v}_h = \boldsymbol{u}_h^n - \mathbb{A}(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})$ in (3.34), using the fact that

$$J(\boldsymbol{u}_h^n \pm \mathbb{A}(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})) - J(\boldsymbol{u}_h^n) = 0,$$

one obtains

$$\frac{1}{k}(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}))_h + \frac{\nu}{2}\|A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})\|_h^2 +$$
$$\frac{1}{4}d(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})) = (\boldsymbol{f}_h^n, A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}))_h,$$

that is

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 - \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{\nu}{2}k\|A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})\|_h^2$$
$$+\frac{1}{4}kd(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1})) = k(\boldsymbol{f}_h^n, A(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}))_h. \tag{3.57}$$

Using relations (3.16), (3.21) and the uniform bound (3.38), we bound the non-linear term as:

$$\begin{aligned} &\frac{1}{4}kd(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1},A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}))\\ &\leq \frac{1}{4}kc_d\|\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}\|_h^{1/2}\|\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}\|_{1,h}\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^{3/2}\\ &\leq \frac{1}{4}kc_d\sqrt{M}\left\{\|\boldsymbol{u}_h^n\|_{1,h}\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^{3/2}+\|\boldsymbol{u}_h^{n-1}\|_{1,h}\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^{3/2}\right\}\\ &\leq \frac{\nu}{8}k\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^2+K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4+K_3M^2k\|\boldsymbol{u}_h^n\|_{1,h}^4, \end{aligned} \tag{3.58}$$

where $K_3=\frac{27c_d^4}{16\nu^3}$. Using Cauchy-Schwarz inequality and relation (3.21), we have that

$$\begin{aligned} k(\boldsymbol{f}_h^n,A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}))_h &\leq k\|\boldsymbol{f}\|_\infty\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h\\ &\leq \frac{\nu}{8}k\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^2+\frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2. \end{aligned} \tag{3.59}$$

Gathering relations (3.57)-(3.59), we find

$$\|\boldsymbol{u}_h^n\|_{1,h}^2-\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2+\frac{\nu}{4}k\|A(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1})\|_h^2\leq K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4+K_3M^2k\|\boldsymbol{u}_h^n\|_{1,h}^4+\frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2, \tag{3.60}$$

from which we obtain

$$K_3M^2k\|\boldsymbol{u}_h^n\|_{1,h}^4-\|\boldsymbol{u}_h^n\|_{1,h}^2+K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4+\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2+\frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2\geq 0 \quad \text{for all} \quad n\geq 1. \tag{3.61}$$

From (3.61) we obtain either

$$\|\boldsymbol{u}_h^n\|_{1,h}^2\leq \frac{1-\sqrt{\Delta_h^{n-1}}}{2K_3M^2k}, \tag{3.62}$$

or

$$\|\boldsymbol{u}_h^n\|_{1,h}^2\geq \frac{1+\sqrt{\Delta_h^{n-1}}}{2K_3M^2k}, \tag{3.63}$$

where

$$\Delta_h^{n-1}=1-4K_3M^2k(K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4+\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2+\frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2)\geq \frac{1}{3} \quad \text{by (3.54) and (3.55).}$$

Let us show that with our assumption, (3.63) is impossible. Taking $\boldsymbol{v}_h = \boldsymbol{u}_h^{n-1}$ in (3.34), we find

$$\begin{aligned}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 + \frac{\nu}{2}k\|\boldsymbol{u}_h^n\|_{1,h}^2 &- \frac{\nu}{2}k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{1}{4}kd(\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) \\ &+ k(J(\boldsymbol{u}_h^n) - J(\boldsymbol{u}_h^{n-1})) \leq k(\boldsymbol{f}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})_h. \qquad (3.64)\end{aligned}$$

Using (3.28) and (3.21), we bound the right hand side of (3.64) by

$$K_1 k\|\boldsymbol{f}\|\|\boldsymbol{u}_h^n\|_{1,h} + K_1 k\|\boldsymbol{f}\|\|\boldsymbol{u}_h^{n-1}\|_{1,h} \leq \frac{\nu}{12}k\|\boldsymbol{u}_h^n\|_{1,h}^2 + \frac{\nu}{2}k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{7K_1^2}{2\nu}k\|\boldsymbol{f}\|_\infty^2. \qquad (3.65)$$

Since $d(\cdot,\cdot,\cdot)$ is a trilinear form, we can rewrite the nonlinear term in (3.64) as

$$\frac{1}{4}kd(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}) = \frac{1}{2}kd(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) - \frac{1}{2}kd(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}),$$

and using property (3.17), we obtain the following bounds:

$$\begin{aligned}\frac{1}{2}kd(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) &\leq \frac{1}{2}c_d k\|\boldsymbol{u}_h^n\|_h\|\boldsymbol{u}_h^n\|_{1,h}\|\boldsymbol{u}_h^{n-1}\|_{1,h} \\ &\leq \frac{\nu}{12}k\|\boldsymbol{u}_h^n\|_{1,h}^2 + \frac{3}{4\nu}c_d^2 k\|\boldsymbol{u}_h^n\|_h^2\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2, \qquad (3.66)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}kd(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}) &\leq \frac{1}{2}c_d k\|\boldsymbol{u}_h^{n-1}\|_h\|\boldsymbol{u}_h^{n-1}\|_{1,h}\|\boldsymbol{u}_h^n\|_{1,h} \\ &\leq \frac{\nu}{12}k\|\boldsymbol{u}_h^n\|_{1,h}^2 + \frac{3}{4\nu}c_d^2 k\|\boldsymbol{u}_h^{n-1}\|_h^2\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2. \qquad (3.67)\end{aligned}$$

Employing (3.39), we bound the last term of the left hand side of (3.64) by

$$-\frac{K_1^2}{2\nu}k\|\boldsymbol{f}\|_\infty^2 \leq k(J(\boldsymbol{u}_h^n) - J(\boldsymbol{u}_h^{n-1})) \leq \frac{K_1^2}{2\nu}k\|\boldsymbol{f}\|_\infty^2. \qquad (3.68)$$

Gathering (3.64)-(3.68) and recalling (3.38), we obtain

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_h^2 + \frac{\nu}{4}k\|\boldsymbol{u}_h^n\|_{1,h}^2 - \left(\nu + \frac{3}{2\nu}c_d^2 M^2\right)k\|\boldsymbol{u}_h^{n-1}\|_h^2 \leq \frac{8K_1^2}{2\nu}k\|\boldsymbol{f}\|_\infty^2,$$

and hence

$$k\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq 2\left(2 + \frac{3}{\nu^2}c_d^2 M^2\right)k\|\boldsymbol{u}_h^{n-1}\|_h^2 + \frac{16K_1^2}{\nu^2}k\|\boldsymbol{f}\|_\infty^2,$$

from which we find, using (3.55):

$$2K_3M^2k\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq \frac{2}{3} < 1. \tag{3.69}$$

(3.69) contradicts (3.63) and therefore we obtain

$$\begin{aligned}\|\boldsymbol{u}_h^n\|_{1,h}^2 &\leq \frac{1-\sqrt{\Delta_h^{n-1}}}{2K_3M^2k} \\ &= 2\frac{K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2}{1+\sqrt{1-x}},\end{aligned} \tag{3.70}$$

where $x = 4K_3M^2k(K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2)$. Since $x \leq 4/5$ (by (3.55)) and

$$\frac{2}{1+\sqrt{1-x}} \leq 1 + x/2 \quad \text{if} \quad 0 \leq x \leq 4/5,$$

we obtain, using (3.54), (3.55) and the fact that $M \geq K_1^2\sqrt{2}\|\boldsymbol{f}\|_\infty/\nu$,

$$\begin{aligned}\|\boldsymbol{u}_h^n\|_{1,h}^2 &\leq \left(K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2\right) \\ &\times \left[1 + 2K_3M^2k(K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2)\right] \\ &\leq K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|\boldsymbol{f}\|_\infty^2 \\ &\quad + 2K_3M^2k(L_1\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \frac{\kappa_1}{\nu}\|\boldsymbol{f}\|_\infty^2)^2 \\ &\leq \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2\left[1 + K_4k(\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2)\right] + K_5k\|\boldsymbol{f}\|_\infty^2,\end{aligned} \tag{3.71}$$

with appropriate choice of constants K_4 and K_5. □

To prove that the scheme (3.34) is conditionally stable on a finite interval of time, we need the following discrete Gronwall lemma [110].

Lemma 3.5.2 *(***Discrete Gronwall Lemma***).*
Given $k > 0$, an integer $n_\star > 0$ and positive sequences α_n, β_n and γ_n such that

$$\alpha_n \leq \alpha_{n-1}(1 + k\beta_{n-1}) + k\gamma_n \quad \textit{for all} \quad n = 1, ..., n_\star, \tag{3.72}$$

we have

$$\alpha_n \leq \alpha_0 \exp\left(k\sum_{i=0}^{n-1}\beta_i\right) + \sum_{i=1}^{n-1} k\gamma_i \exp\left(k\sum_{j=i}^{n-1}\beta_j\right) + k\gamma_n \quad \textit{for all} \quad n = 2, ..., n_\star. \tag{3.73}$$

Proof Using recursively (3.72), we derive

$$\alpha_n \leq \alpha_0 \prod_{i=0}^{n-1}(1+k\beta_i) + \sum_{i=1}^{n-1} k\gamma_i \prod_{j=i}^{n-1}(1+k\beta_i) + k\gamma_n,$$

and since $1+x \leq \exp x$, for all $x \in \mathbb{R}$, the conclusion of the lemma follows. □

Proposition 3.5.1 *(***estimates on a finite interval of time***).*
Let $T > 0$ and

$$M \geq K_1^2\sqrt{2}\|\boldsymbol{f}\|_\infty/\nu$$

be fixed and let

$$\|\boldsymbol{u}_0\| \leq M.$$

Assume that, besides the CFL-condition (3.36), k also satisfies

$$k \leq \min\left\{\kappa_1, \kappa_2(M, \|\boldsymbol{f}\|_\infty), \kappa_3(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, T)\right\}, \tag{3.74}$$

where

$$\kappa_2(M, \|\boldsymbol{f}\|_\infty) = \frac{1}{12K_3K_6M^2\|\boldsymbol{f}\|_\infty^2}, \tag{3.75}$$

$$\kappa_3(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, T) = \frac{1}{12K_3M^2L_1L_2(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, T)}. \tag{3.76}$$

$L_2(\cdot,\cdot,\cdot,\cdot)$ is a monotonically increasing function in all its arguments and is given in (3.83) below and $K_6 = \frac{8K_1^2}{\nu^2}$.
Then

(a) relation (3.56) holds for all $n = 1, ..., N = \lfloor T/k \rfloor$ (integer part of T/k), and

(b)

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq L_2(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, nk), \quad \textit{for all} \quad n = 1, ..., N = \lfloor T/k \rfloor \tag{3.77}$$

Proof Let $T > 0$ and let h, k be such that (3.36) and (3.74) are satisfied. We will use induction on n. If $n = 1$, assumption (3.74) implies

$$K_3 M^2 k (L_1 \|\boldsymbol{u}_h^0\|_{1,h}^2 + \frac{2\kappa_1}{\nu} \|\boldsymbol{f}\|_\infty^2) \leq \frac{1}{6},$$

thus conclusion (3.56) of lemma 3.5.1 holds for $n = 1$. Now assume that (3.55) holds for $n = 1, ..., m$, for some $m \leq N$. Hence (3.56) holds for $n = 1, ..., m$. If we rewrite (3.56) as (3.72) with

$$\alpha_n = \|\boldsymbol{u}_h^n\|_{1,h}^2, \quad \beta_n = K_4 M^2 (\|\boldsymbol{u}_h^n\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2) \quad \text{and} \quad \gamma_n = K_5 \|\boldsymbol{f}\|_\infty^2$$

and noting that, using (3.40), we have

$$\begin{aligned} k \sum_{j=i}^{m-1} \beta_j &= K_4 M^2 k \sum_{j=i}^{m-1} (\|\boldsymbol{u}_h^j\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2) \\ &\leq K_7 M^2 [M^2 + (m-i) k \|\boldsymbol{f}\|_\infty^2], \end{aligned} \tag{3.78}$$

and therefore

$$\begin{aligned} \sum_{i=1}^{m-1} k\gamma_i \exp\left(k \sum_{j=i}^{m-1} \beta_j\right) &\leq K_5 k \|\boldsymbol{f}\|_\infty^2 \sum_{i=1}^{m-1} \exp(K_7 M^2 [M^2 + (m-i) k \|\boldsymbol{f}\|_\infty^2]) \\ &\leq K_5 \|\boldsymbol{f}\|_\infty^2 \exp(K_7 M^4) mk \exp(K_7 M^2 mk \|\boldsymbol{f}\|_\infty^2). \end{aligned} \tag{3.79}$$

Similarly for $i = 0$, we have

$$\begin{aligned} k \sum_{j=0}^{m-1} \beta_j &= K_4 M^2 k \sum_{j=0}^{m-1} (\|\boldsymbol{u}_h^j\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2) \\ &\leq K_7 M^2 (M^2 + mk \|\boldsymbol{f}\|_\infty^2) + K_4 M^2 k \|\boldsymbol{u}_h^0\|_{1,h}^2. \end{aligned} \tag{3.80}$$

Using (3.74) and recalling that $L_1 \geq 2$, the last term of (3.80) can be bounded as

$$K_4 M^2 k \|\boldsymbol{u}_h^0\|_{1,h}^2 \leq \frac{K_4 \|\boldsymbol{u}_h^0\|_{1,h}^2}{12 K_3 L_1 L_2 (M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, T)} \leq \frac{K_4}{24 K_3}. \tag{3.81}$$

Then Lemma 3.5.2 and relations (3.78)-(3.81) imply

$$\|\boldsymbol{u}_h^m\|_{1,h}^2 \leq L_2 (M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, mk), \tag{3.82}$$

where

$$L_2(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, mk) = \|\boldsymbol{u}_h^0\|_{1,h}^2 \exp(K_7 M^4 + \frac{K_4}{24K_3}) \exp(K_7 M^2 mk \|\boldsymbol{f}\|_\infty^2) + 2K_5 \exp(K_7 M^2) mk \|\boldsymbol{f}\|_\infty^2 \exp(K_7 M^2 mk \|\boldsymbol{f}\|_\infty^2) \quad (3.83)$$

Using (3.82) and recalling assumption (3.74), it is easily check that condition (3.55) holds for $n-1=m$, and by the same Lemma 3.5.2, we have (3.56) holds for $n=m+1$. □

To prove the uniform bound of $\|\boldsymbol{u}_h^n\|_{1,h}$ for all $n \geq 1$, we will repeatedly apply Proposition 3.5.1 on different intervals of time, considering different initial values. To do that, we need the following discrete uniform Gronwall lemma, a generalized version of the discrete uniform Gronwall lemma of Shen [110], whose proof can be found in [100].

Lemma 3.5.3 *(***Discrete Uniform Gronwall lemma***).*
Given $k>0$, positive integers n_1, n_2, $n_\star$ such that $n_1 \leq n_\star$, $n_1+n_2+1 \leq n_\star$, and positive sequences α_n, β_n and γ_n such that

$$\alpha_n \leq \alpha_{n-1}(1+k\beta_{n-1}) + k\gamma_n \quad \textit{for all} \quad n=1,...,n_\star, \quad (3.84)$$

Assume also that for any n' satisfying $n_1 \leq n' \leq n_\star - n_2$

$$\sum_{n=n'}^{n'+n_2} k\beta_n \leq C_1(n_1,n_\star), \quad \sum_{n=n'}^{n'+n_2} k\alpha_n \leq C_2(n_1,n_\star), \quad \sum_{n=n'}^{n'+n_2} k\gamma_n \leq C_3(n_1,n_\star), \quad (3.85)$$

then we have

$$\alpha_n \leq \left(\frac{C_3(n_1,n_\star)}{kn_2} + C_2(n_1,n_\star)\right)\exp(C_1(n_1,n_\star)) \quad \textit{for any} \quad n_1+n_2+1 \leq n \leq n_\star, \quad (3.86)$$

Theorem 3.5.1 *(***Uniform bound of $\|\boldsymbol{u}_h^n\|_{1,h}$ for all $n \geq 1$***).*
Let $\boldsymbol{u}_0 \in \boldsymbol{V}_\sigma \cap C^1(\overline{\Omega})^2$, $\boldsymbol{f} \in \boldsymbol{L}^\infty(\mathbb{R}^+;\boldsymbol{H})$ and assume that

$$\|\boldsymbol{u}_0\| \leq M,$$

where

$$M \ \geq \ K_1^2\sqrt{2}\|\boldsymbol{f}\|_\infty/\nu.$$

Also let $r \geq 4\kappa_1$ *be arbitrarily fixed and assume that, besides the CFL-condition (3.36),* k *also satisfies*

$$k \leq \min\left\{\kappa_1, \kappa_2(M, \|\boldsymbol{f}\|_\infty), \kappa_3(M, K_2\|\boldsymbol{u}_h^0\|_{C^1(\overline{\Omega})^2}, \|\boldsymbol{f}\|_\infty, r), \kappa_3(M, \rho_1, \|\boldsymbol{f}\|_\infty, r)\right\}, \tag{3.87}$$

where κ_1*,* κ_2*,* κ_3 *are defined above and* ρ_1 *is given in (3.92) below.*
Then we have

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq L_3(\|\boldsymbol{u}_h^0\|_{C^1(\overline{\Omega})^2}, \|\boldsymbol{f}\|_\infty), \quad \textit{for all} \quad n \geq 1, \tag{3.88}$$

where $L_3(\cdot,\cdot)$ *is a continuous function defined on* $\mathbb{R}_+^2$*, increasing in both arguments.*
Moreover, there exists an $N > 0$ *such that*

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq L_4(\|\boldsymbol{f}\|_\infty), \quad \textit{for all} \quad n \geq N. \tag{3.89}$$

Proof In order to derive uniform bounds $\|\boldsymbol{u}_h^n\|_{1,h}$ for all $n \geq 1$, we apply Proposition 3.5.1 on successive intervals of time, with different initial values. On each interval considered, we obtain a bound $L_2(\cdot,\cdot,\cdot,\cdot)$ which depends on the norm $\|\boldsymbol{u}_h^0\|_{1,h}$ and on the length of the interval. Using the discrete uniform Gronwall lemma, we bound the norm of the initial values $\|\boldsymbol{u}_h^0\|_{1,h}$ by a constant ρ_1 and recalling the fact that L_2 is an increasing function of its arguments, we obtain a bound independent on the initial value considered.
First using (3.32), (3.87) and since κ_3 is a decreasing function of its arguments, we can apply Proposition 3.5.1 with $T = r$ to obtain

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2\left[1 + K_4M^2k(\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2)\right] + K_5k\|\boldsymbol{f}\|_\infty^2, \tag{3.90}$$

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \leq L_2(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, r), \quad \text{for all} \quad n = 1, ..., N_r := \lfloor r/k \rfloor. \tag{3.91}$$

To extend the bound (3.91) to $n = N_r + 1, ..., 2N_r$, we apply again Proposition 3.5.1, namely, $L_2(M, \|\boldsymbol{u}_h^{N_r}\|_{1,h}, \|\boldsymbol{f}\|_\infty, r)$ depends on the discrete initial value, we want to bound $\|\boldsymbol{u}_h^{N_r}\|_{1,h}$ independently of h and k.

Let us rewrite (3.90) in the form of (3.84) with $\alpha_n = \|\boldsymbol{u}_h^n\|_{1,h}^2$, $\beta_n = K_4 k(\|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2$ and $\gamma_n = K_5 k\|\boldsymbol{f}\|_\infty^2$. Then we apply Lemma 3.5.3 with $n_1 = 1$, $n_2 = N_r - 2$, $n_\star = N_r$ to obtain the bound of $\|\boldsymbol{u}_h^{N_r}\|_{1,h}$. For $n' = 1, 2$, using (3.40), we have

$$k\sum_{n=n'}^{n'+n_2} \beta_n = K_4 M^2 k \sum_{n=n'}^{n'+n_2} (\|\boldsymbol{u}_h^n\|_{1,h}^2 + \|\boldsymbol{f}\|_\infty^2) \le K_8 M^2 (M^2 + r\|\boldsymbol{f}\|_\infty^2),$$

$$k\sum_{n=n'}^{n'+n_2} \gamma_n = K_5 k \sum_{n=n'}^{n'+n_2} \|\boldsymbol{f}\|_\infty^2 \le K_5 r \|\boldsymbol{f}\|_\infty^2,$$

$$k\sum_{n=n'}^{n'+n_2} \alpha_n = k \sum_{n=n'}^{n'+n_2} \|\boldsymbol{u}_h^n\|_{1,h}^2 \le K_9 (M^2 + r\|\boldsymbol{f}\|_\infty^2).$$

Then Lemma 3.5.3, together with the assumption $r \ge 4\kappa_1$, yields

$$\begin{aligned} \|\boldsymbol{u}_h^{N_r}\|_{1,h}^2 &\le [2K_9(M^2/r + \|\boldsymbol{f}\|_\infty^2) + K_5 r\|\boldsymbol{f}\|_\infty^2] \exp(K_8 M^2 (M^2 + r\|\boldsymbol{f}\|_\infty^2)) \\ &:= \rho_1(M, \|\boldsymbol{f}\|_\infty, r). \end{aligned} \tag{3.92}$$

Taking into account the assumption (3.87) on the time step k, relation (3.92) and the fact that $L_2(\cdot,\cdot,\cdot)$ is an increasing function of its arguments, we apply Proposition 3.5.1 with $T = r$ and initial data $\boldsymbol{u}_h^{N_r}$. We obtain that the relation (3.56) holds for all $n = N_r + 1, ..., 2N_r$, and

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \le L_2(M, \|\boldsymbol{u}_h^{N_r}\|_{1,h}, \|\boldsymbol{f}\|_\infty, r) \le L_2(M, \rho_1, \|\boldsymbol{f}\|_\infty, r) \quad \text{for all} \quad n = N_r + 1, ..., 2N_r. \tag{3.93}$$

Applying again Lemma 3.5.3 with $n_1 = N_r + 1$, $n_2 = N_r - 2$ and $n_\star = 2N_r$, we obtain

$$\|\boldsymbol{u}_h^{2N_r}\|_{1,h}^2 \le \rho_1. \tag{3.94}$$

Iterating the above procedure, we find

$$\|\boldsymbol{u}_h^n\|_{1,h}^2 \le L_2(M, \rho_1, \|\boldsymbol{f}\|_\infty, r) := L_3(\|\boldsymbol{f}\|_\infty), \quad \text{for all} \quad n \ge N_r, \tag{3.95}$$

and recalling (3.91), we conclude

$$\begin{aligned} \|\boldsymbol{u}_h^n\|_{1,h}^2 &\le \max\{L_2(M, \|\boldsymbol{u}_h^0\|_{1,h}, \|\boldsymbol{f}\|_\infty, r), L_3(\|\boldsymbol{f}\|_\infty)\} \\ &\le \max\{L_2(M, K_2\|\boldsymbol{u}_0\|_{C^1(\overline{\Omega})^2}, \|\boldsymbol{f}\|_\infty, r), L_3(\|\boldsymbol{f}\|_\infty)\} \quad \text{by (3.32)} \\ &:= L_4(K_2\|\boldsymbol{u}_0\|_{C^1(\overline{\Omega})^2}, \|\boldsymbol{f}\|_\infty) \quad \text{for all} \quad n \ge 1. \end{aligned} \tag{3.96}$$

As for the N beyond which $\|\boldsymbol{u}_h^n\|_{1,h}$ is bounded independent of $\boldsymbol{u}_0$, we can evidently take $N = N_r$ (see (3.95)). This completes the proof of the Theorem. $\square$

Conclusion

The main focus of this work was to develop both theoretically and numerically the finite element approximation for the fluid flows governed by nonlinear slip boundary conditions.

In Chapter 1, introducing a threshold slip boundary conditions (0.2) to the Stokes and Navier Stokes equations, the resulting variational inequalities obtained are analyzed by the means of fixed approach, and a priori error estimates are derived using sufficient conditions for existence of solutions. We next formulated and established the convergence of the Uzawa's algorithm associated to the finite element equations for both the Stokes and Navier-Stokes equations. Finally, some numerical simulations which confirm the predictions of the theory was presented.

In Chapter 2, we formulated and analyzed the finite element approximation for the power law Stokes flow driven by slip boundary condition (0.1). Next, we proceeded to implement a particular algorithm combining vanishing viscosity method and stationary solution of an initial value problem (flow in the dynamical system terminology). The well posedness of the finite element approximation was obtained by using the generalized version of Babuska-Brezzi's theory of mixed formulation introduced in [70, 48]. As far as the implementation of the finite element presented is considered, we adapted the well known methodology consisting to associate to a stationary problem, an initial value problem in which the focus was on the behavior of the solution of the later problem when the time was big enough. But in order to improve the rate of convergence, we added a stabilizing term to the initial value problem (numerical computations confirm the predictions). This approach led naturally to a solution method based on time discretization; it has also an advantage of being easily implementable, but much progress has to be made for a systematic way of choosing the initial flow.

In Chapter 3, we studied the stability for all positive time of Crank-Nicolson scheme for the two-dimensional Navier-Stokes equation driven by slip boundary conditions of friction type (0.1). We discretized these equations in time using the Crank-Nicolson scheme and in space using finite element approximation. We proved that the numerical scheme is stable in L^2 and H^1-norms with the aid of different versions of discrete Grownwall lemmas, under a CFL-type condition.

In our future works, we would like to extend our results on:

- The study of finite element of viscosity-splitting scheme for the Navier-Stokes equations in 2D and 3D. Similar works have been done for instance in [111, 112, 113] with Dirichlet boundary conditions.
- The study of finite element approximation for the power law Stokes flow driven by slip boundary conditions (0.2) formulated by C. Le Roux and A. Tani in [41, 42].
- It will be interesting to see how the works of F. Tone and co-workers [100, 50, 114] are incorporated in our research direction.

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