A study on interest rate basis-risk models after the 2008 liquidity crunch

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work, except where acknowledged in the customary manner, and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Abstract

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In this dissertation we take a look at the rise of interest rate basis spreads in the market following the liquidity and credit crunch of 2008. We show that post 2008 the valuation of all interest rate instruments of a single yield curve for a particular currency is no longer a feasible approach and the assumption of no arbitrage between different tenors is no longer applicable. Following that a closer look is taken into the cause of such widening basis spreads and the impact they have had on the market with a focus on reconstituting the no arbitrage argument and looking at a post crisis multiple curve framework following an axiomatic approach as introduced by Henrard [37] and further explored by Bianchetti and Morini [6, 50]. A bottom-up market approach is taken by Ametrano [2] and the two approaches are shown to be equivalent in result. An analogy is made to quanto style cross currency swap adjustments observed by the aforementioned authors as well as Michaud and Upper [47], and Tuckman and Porfirio [57].

We proceed to look at the approaches taken by authors such as Henrard [36, 37] in extending the Black and Stochastic Alpha Beta Rho models to include basis spreads and Kijima et al. [42] who extend a model introduced by Boenkost and Schmidt [11] and put forward a quadratic Gaussian model and a Vasicek model. Mercurio [46] puts forward an extension to the LIBOR Market Model (also referred to as the Brace-Gatarek-Musiela model) under both forward measures and spot measures.

Finally we consider the rise of using overnight index swaps in construction OIS discount curves and their application in the valuation of interest rate derivatives in the presence of collateral as well as reconciling the spread between OIS and vanilla interest rate swaps with credit risk measures.

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Contents

List of Figures

Nomenclature

Acronyms and Abbreviations	
Acronym/Abbreviation	Description
AxB FRA	A Forward Rate Agreement (see below under FRA) with a the forward period being the period of time starting A months from today and ending B months from today, e.g. a 3x6 FRA has the floating reference period being the three month period starting three months from today and ending six months from today.
bps	Basis points; refers to a percent of one percent or 0.0001.
BBA	British Bankers' Association; an association for the United Kingdom's banking and financial services sector.
CCY	Cross Currency, often used in the context of a cross currency swap as CCY Swap.
CDS	Credit Default Swap; a particular type of swap where a stream of payments is exchanged for a nominal amount should a particular default event occur.
CSA	Credit Support Annex; an annex to an ISDA master agreement defining the placement of collateral (see ISDA).
EUR	Euros; EUR is the International Standards Organization's code for the Euro.
EURIBOR	Euro Interbank Offered Rate which is the equivalent to LIBOR in the Eurozone interbank market (see LIBOR). EURIBOR differs from EUR LIBOR in that EURIBOR is the rate at which Euros could be raised in the Eurozone interbank market as opposed to the London interbank market.
EONIA	Euro Overnight Index Average; weighted average of overnight unsecured lending transactions in the Eurozone interbank market.
FRA	Forward Rate Agreement; an interest rate instrument exchange a single fixed payment at some future date for a floating interest rate payment.
FX	Foreign Exchange.
GBP	Great British Pounds; the ISO code for Great British Pounds, or Pounds Sterling.
ISDA	International Swaps and Derivative Association; a body governing derivative transactions. Referring to "an ISDA" most often refers to an ISDA master agreement, an contract governing the general terms under which derivatives would be transacted under.
ISO	International Standards Organization, an organisation governing the setting of specific standards.
JIBAR	Johannesburg Interbank Agreed Rate; the South African interbank market equivalent of LIBOR (see LIBOR).

Acronym/Abbreviation	Description
LIBOR	London Interbank Offered Rate; the average of the estimated rate at which banks in the London interbank market could borrow from one another, used as a fixing or reset rate for interest rate derivatives. USD LIBOR, GBP LIBOR and EUR LIBOR would each represent the rate at which US Dollars, British Pounds and Euros could be raised respectively.
LMM	LIBOR Market Model; also known as the BGM (Brace-Gaterek-Musiela) after the authors who first introduced it, a model to describe the interest rate markets and the dynamics of forward rates.
OIS	Overnight Indexed Swap; a type of interest rate swap exchanging a floating overnight rate for a fixed rate.
SABOR	South African Benchmark Overnight Rate; the benchmark for rates paid in the South African interbank market.
SABR	Stochastic Alpha Beta Rho; a stochastic volatility model aimed at modelling the volatility smile in derivatives markets.
USD	United States Dollars; USD is the ISO code for United States Dollars, often also abbreviated as US Dollars.
X-BOR	X-Bank Offered Rate; occasionally used to specify a generic fixing rate, such as LIBOR.
ZAR	Zuid-Afrikaanse Rand; the ISO code for the South African Rand.

Symbol	Description
$\in$	Is an element of; denotes when a particular variable or set is an element or subset of another set.
$\forall$	For all; used to denote that a particular equation or expression is true for all values of some variable.
$\mathbb{P}$	Probability; used to denote a probability function. $\mathbb{P}(A\|B)$ denotes the probability of an event A occurring given that an B is true.
t	Time; used to denote a point in time.
t_0	Time zero; used to denote today.
T	Maturity time; used to denote a maturity date or time of an instrument, though not exclusively so.
$\approx$	Approximately equal to; used either when due to rounding or limited space a particular number cannot be fully expressed (e.g. $\pi \approx 3.14$) or when a particular function or equation does not have an exact answer but has an approximate answer determined via some process e.g. Monte Carlo simulation.
$\neq$	Does not equal to; used to express an inequality.
Var	Variance; $Var^Q(X)$ would denote the variance of X under the measure Q.

Software and Market Data

All market data referred to, unless otherwise stated, was sourced from Bloomberg L.P.

All calculations and their respective graphs were performed and generated using Microsoft Excel and VBA.

Chapter 1

Introduction

Beginning in August 2007 and continuing through 2008 the world's financial markets experienced a global liquidity crunch. The financial markets experienced a shortage in market liquidity and funding liquidity (the lack of funding or extension of credit was often referred to as the 2008 credit crunch). This financial crisis and liquidity crunch also brought to light various shortcomings in in financial model's ability to accurately measure liquidity risks as well as certain counterparty credit risks linked to liquidity risk. One of the effects of the 2008 liquidity crunch was the rise of interest rate basis and the breakdown of prevalent interest rate models which introduced arbitrage into what was previously considered an arbitrage free market. This gave rise to increasing basis spreads and introduced basis risk into traditionally basis free models.

The aim of this dissertation is to take a look at the pre and post 2008 interest rate models and interest rate curve construction methodologies and show how the rise of interest rate basis in the wake of the 2008 liquidity crisis led to the pre 2008 models and methodologies no longer being applicable, to explain the causes of this divergence, and to take a look at which new models and methodologies could be proposed and implemented to take into account the increasing basis spreads. Being a relatively new phenomenon there is unfortunately a lack of available long term data as well as limited understanding and research of the subject. One of the benefits of such a study would be to illustrate the loss of no arbitrage assumptions post 2008, collate the various new approaches proposed, see how no arbitrage assumptions could be reclaimed, and how the new approaches and models can be reconciled with other changes arising after the 2008 liquidity crunch such as the rise of CVA (credit valuation adjustments) and of multiple discount curves to take into account counterparty risk and the effect of collateral. We will thus also take a look at the impact of inter-bank credit default risk as well as collateralisation on the valuation and pricing of interest rate derivatives and specifically the use of overnight index swaps and their corresponding basis spreads for the construction of overnight rate curves.

To achieve this we will initially take a look at the pre 2008 crisis models and show how, before the liquidity crunch, there were no significant basis spreads prevalent in constructing the various interest rate curves used to price and value interest rate derivatives. We initially introduce some background theory in Chapters 2 and 3 leading into Chapter 4 in which the common interest rate curve construction methods and forward rate estimation methods before the crisis will be investigated.

This is followed by applying the same methods and models to post 2008 market data to determine if they would still be applicable after the 2008 liquidity crunch and the rise of large interest rate basis spreads quoted in the market. We wish to show that post 2008 the assumptions used in the original models are no longer applicable and would give rise to perceived arbitrage opportunities and to explore some possible explanations to this breakdown of previous no arbitrage assumptions. This is done in Chapter 5 along with exploring some possible new approaches to model this basis risk to reclaim the assumption of no arbitrage and to explain the rise of multiple curve interest rate frameworks. In Chapter 6 we aim to look at some post crisis market models as proposed by various authors and how these models are extensions of existing pre crisis models and frameworks. Of specific interest are extensions to the Black model, the SABR/CEV model, the Vasicek model and the LIBOR Market Model.

The 2008 liquidity crunch also gave rise to the concern that banks and other sources of funding can fail and previously assumed “risk-free” curves could no longer be considered as such. In Chapter 7 we take a look at pricing and modelling of interest rate instruments when banks can default as well as the impact of netting agreements and presence or absence of collateral. We specifically take a look at the rise of overnight index swap curves used for discounting in the presence of collateral and the construction of such curves. Finally we attempt to reconcile overnight index swap curve discounting with basis spreads in the market and other credit risk measures (for example credit default swap spreads).

1.1 Plan of Study

Consider the rise of basis spreads as observed in the interest rate market and to the cause such basis spread. Further to understand the implications of basis spreads on discounting and pricing interest rate derivatives in the interbank market both in the absence and presence of collateral. We wish to take a look at the pre-crisis approach to interest rate derivative pricing and how under the rise of basis spreads for instruments with different tenors the previously held assumption of no arbitrage in a single curve framework would breakdown. We then wish to investigate how such a no arbitrage assumption could be reclaimed and how under such a new framework these basis spreads could be included in some of the pre-crisis prevalent interest rate models. Finally we wish to explore the concept of a "risk-free" curve and whether the current forward rate curve would still be applicable and if not what approach could be taken to approximate such a curve. To do this we would take a look at how the interbank market has reacted to the pricing of interest rate derivatives transacted under daily cash collaterlisation which would greatly mitigate any default risk in the interbank market.

1.2 Literature Review

The literature to be reviewed covers several interlinking topics. The review often covers work done simultaneously by different authors and expanded upon by others.

Tuckman and Porfirio [57] show that interest rate parity conditions require default free rates and that the practice of using interest rates swaps is not appropriate as they inherently do contain an element of default risk. They derive interest rate parity conditions dependent on basis swap spreads and reveal the credit risk inherent in one floating rate versus another. Chibane and Sheldon [17] further set the scene addressing issues surrounding basis swap adjustments and the state of discount curve construction. The authors also extend the framework to the joint existence of single currency swaps, cross currency swaps and money market basis swaps.

Mercurio [46] describes some of the major changes that occurred in the market post subprime mortgage crisis. Starting with former analogies, such as presented by Hull [38] in *Options, Futures and Other Derivatives*, allowing the construction of zero curves [32, 38] the author shows the changes resulting post August 2007. Seeing as the former analogies no longer hold the author then investigates the framework allowing for the existence of different values for same tenor rates. The paper shows how to generalise the main market models to account for the new market practice of using multiple curves. Finally a new LIBOR Market Model is introduced based on the joint evolution of FRA rates under different measures.

Mercurio [46] expands on work that was simultaneously proposed by Morini and which was subsequently expanded upon by Morini in [50]. Morini [50] begins with the widening between the market quotes for FRAs and their standard spot LIBOR replication. The author also looks at the large basis spreads for floating payments of different tenors. The author uses quoted basis spreads to explain the difference between FRAs and spot LIBOR and then explains the market patterns of basis spreads by modelling them as options in the credit worthiness of a counterparty. In Part 2 the author considers a credit risk factor to link spot and forward quotes. An important finding though is that credit risk alone cannot explain the market patterns and thus the author expands upon other elements. The author also observes the analogy between Foreign Exchange and decoupled forward and discount curves introduced by Bianchetti [6].

Ametrano and Bianchetti [2] explore how large basis spreads (as observed in Swap markets) implied that different yield curves are required for forward rate estimation at different tenors. The authors explore current methodologies for creating smooth yield curves. They then review the market practice for pricing and hedging interest rate derivatives and present a double-curve approach. They describe a bootstrapping procedure and review Euro market instruments available for yield curve construction. The authors also investigate the impact of the role played by the, quite fundamental, interpolation scheme adopted. Finally numerical results are depicted for Euro Interbank Offered Rates (EURIBOR) of varying tenor.

Expanding on that work; Bianchetti [6] proposes that single-curve no arbitrage relationships are no longer valid and can be recovered by taking into account forward basis. He also takes into account a "foreign currency analogy" using a quanto adjustment for double curve market-like formulas for various interest rate derivatives and notes that previously the subject matter has been explored mostly in the context of cross currency swap basis. The author cites Wolfram Boenkost and Wolfgang Schmidt [11] whose methodology for pricing cross-currency swaps coincides with the procedure he described in [6].

Fries [24] brings in funding costs and counterparty risk for swap pricing and valuation showing that it has become non-trivial. The widespread use of collateral has changed the effective funding cost for many financial institutions [27].

Fujii, Shimada and Takahashi [28, 25, 26, 27] further elaborate that simple market models no longer reflect the exposure to basis spreads in pricing and hedging. The authors, similar to [24], introduce collateral and explain methods to construct a market model as well as multiple swap curves with and without collateral agreements. They further expand the effect collateral has had when multiple currencies are considered [27]. The basis spreads are considered dynamic and all factors are stochastic. The authors extend the work of Johannes and Sundaresan to include contracts which have differing payment and collateral currencies. Also expanded upon is a general multi currency Heath Jarrow Morton framework in the presence of collateral and stochastic basis spreads.

Of final interest is the spread between Treasury bonds and interest rate swaps (commonly referred to as Swap Spreads). Whilst the literature thus covered is focused primarily on the derivatives market, modelling swap spreads may be of particular interest when considering interest rate basis as in many markets, such as the South African market, these are readily quoted as asset packs (a treasury bond sold together with with a similar maturity interest rate swap hedge). Feldhütter and Lando [23] analyse a six factor model for Treasury bonds, corporate bonds and swap rates; decomposing them into a convenience yield for holding treasury bonds, a credit risk element from the underlying rates and a factor specific to the swap market. They also pose the question of whether treasury yields or swap rates would be closer to the "risk-free" rate. This ties back to the original literature given that the original analogies for risk-free rates and swap curve discounting no longer hold. Related to this; Bhansali, Schwarzkopf and Wise [5] develop an integrated model for the term structure of swap spreads. They also look at the effect of liquidity crisis on the the traditional swap curve and its term structure.

Chapter 2

Great Expectations

In this chapter we shall introduce some probability theory, define some of the necessary probability theory and give an outline of the concepts of equivalent martingale measures and the change of numeraire technique so as to create a more rigorous framework for the market model described in Chapter 3.

2.1 Market Dynamics

It will be useful to discuss some expectation theory as a basis for the various models to be discussed throughout this paper. Björk's *Arbitrage Theory in Continuous Time* [9] and Brigo and Mercurio's *Interest Rate Models, Theory and Practice* [13] give nice outlines in their respective books and some of the definitions below will be from those two books.

Let us first define a trading strategy

Definition 2.1 (Trading Strategy)**.** *A* ***trading strategy*** [9, 13] *is a* $K+1$ *dimensional process* $\phi = \{\phi_t : 0 \leq t \leq T\}$ *whose components are locally bounded and predictable and the* ***value process*** *associated with* ϕ *is defined by*

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k, 0 \leq t \leq T,$$

and the ***gains process*** *associated with* ϕ *is*

$$G_t(\phi) = \int_0^t \phi_t \, \mathrm{d}S_u = \sum_{k=0}^{K} \int_0^t \phi_u^k \, \mathrm{d}S_u^k, \quad 0 \leq t \leq T.$$

Consider the component ϕ_t^k to be the number of units of security k held at time t. Predictability of ϕ_t^k simply means that the value of ϕ_t^k is known immediately before t. $V_t(\phi)$ is thus the value of such a portfolio and $G_t(\phi)$ are the cumulative gains or losses realised until time t by adopting the strategy ϕ. We are primarily concerned with trading strategies which do not require external cashflows which we name a self-financing trading strategy.

Definition 2.2 (Self-financing Trading Strategy)**.** *A* ***self-financing trading strategy*** [9, 13] *is a trading strategy* ϕ *where the following condition holds true*

$$V_t(\phi) = V_0(\phi) + G_t(\phi), 0 \leq t \leq T. \tag{2.1}$$

Thus the value of a self-financing trading strategy is dependant only on the market value of the securities and no additional cash inflows or outflows occur after the initial time. A portfolio is simply a collection of securities or assets used in our trading strategy. In terms of a self-financing trading strategy we can now define a multi-period arbitrage condition.

Definition 2.3 (Arbitrage)**.** *An* ***arbitrage*** *possibility* [9, 13] *is a self financing portfolio* ϕ *with the properties*

$$\begin{aligned} V_t(\phi) &= 0, \\ P(V_T(\phi) \geq 0) &= 1, \\ P(V_t(\phi) > 0) &> 0. \end{aligned} \tag{2.2}$$

The concept of arbitrage and the assumption of no arbitrage in the market will be explored in greater detail in Chapters 3 and 4.

We now have a market filled with securities $k = 0, 1, ..., K$ with values S_t^k for each security k at times t and we shall assume that it is arbitrage free. Let us also consider financial derivatives, or *contingent claims*, which is a contract with a value dependent on the value of some other underlying security or basket of securities.

Definition 2.4 (Contingent Claim)**.** *A simple* ***contingent claim*** [9, 13] *is a stochastic variable* H *of the form*

$$H = \Phi(S_T),$$

where the function Φ*, known as the contract function, is some real valued function.*

This is a simple contingent claim as H depends only on the value of the underlying securities at time $t = T$. One could also consider contingent claims based on the entire path of the price process over the time interval $[0, T]$.

Definition 2.5. *A contingent claim is said to be* ***reachable*** *or* ***attainable*** *if there exists a self-financing strategy ϕ such that $V_T(\phi) = H$.*

We call such a portfolio ϕ a hedging or replicating portfolio.

Definition 2.6. *A financial market is* ***complete*** *if and only if every contingent claim is attainable.*

2.2 Equivalent Martingale Measures

The no arbitrage pricing approach we use postulates that in the market there are no arbitrage opportunities. We would like to link the economic principle of no arbitrage with the mathematical property of probability measures, i.e. risk-neutral measures, which are given the term equivalent martingale measures.

Definition 2.7 (Martingale Measure)**.** *A probability measure Q is called a* ***martingale measure*** [9, 13] *if the following condition holds:*

$$E^Q[S_{t+1}] = \frac{1}{P(t, t+1)} S_t, \tag{2.3}$$

which simply put is that today's price for security S is the discounted value (or present value) of the expected value of tomorrow's price. This is a risk neutral valuation formula, in other words, we use the probability measures under Q instead of objective probabilities to obtain a risk neutral valuation of the stock given the absence of arbitrage. Such probability measures are also called **risk neutral measures** or **risk adjusted measures**.

From this follows the definition of an equivalent martingale measure [13].

Definition 2.8 (Equivalent Martingale Measure)**.** *An* ***equivalent martingale measure*** [9, 13] *Q is a probability measure on the space $(\Omega, \mathcal{F})$ such that*

1. *Q_0 and Q are equivalent measures, that is $Q_0(A) = 0$ if and only if $Q(A) = 0$, for every $A \in \mathcal{F}$;*

2. *the Radon-Nikodym derivative dQ/dQ_0 belongs to $L^2(\Omega, \mathcal{F}, Q_0)$, i.e. it is square integrable with respect to Q_0;*

3. *the discounted asset price process, $P(0,t)S_t$, is an $(\mathbb{F}, Q)$-Martingale, i.e.*

$$E^Q[P(0,t)S_t^k|\mathcal{F}_u] = P(0,u)S_u^k, \quad k = 0, 1, ..., K$$

and

$$0 \leq u \leq t \leq T.$$

It is proposed that the market model described above is arbitrage free if there exists such an equivalent martingale measure Q. Tomas Björk gives an informal proof of this proposition, named the **First Fundamental Theorem** [9].

Theorem 2.1. ***(The First Fundamental Theorem)*** *The market model is arbitrage free essentially if and only if there exists a (local) martingale measure Q.*

Here the term "essentially" means that the model is arbitrage free only in reference to the underlying model and may vary in ways that do not affect the mathematical content and a local martingale refers to a stochastic process which is a martingale, as defined in Definition 2.7, before some stopping time T, i.e. the process is stopped at time T and is a martingale at all times before T.

Equivalent martingale measures are of particular interest to us as they allow us to give a general pricing formula for any attainable contingent claim [33].

Proposition 2.1. *Assume there exists an equivalent martingale measure Q and an attainable contingent claim H. Then, for each time $t, 0 \leq t \leq T$, there exists a unique price π_t associated with H:*

$$\pi_t = E^Q(P(t,T)H|\mathcal{F}_t). \tag{2.4}$$

Proof of the proposition may be found in [33].

2.3 Change of Numeraire

The concept of a Numeraire as a measure is a very important tool as under normal circumstances a stochastic discount factor $df(t_1, t_2)$ complicates the calculation of the expectations of various portfolios. Geman, Karoui and Rochet introduce us to the numeraire as well as the changing or choosing of the numeraire to normalise the other assets in the market [29].

Definition 2.9. *A **numeraire** is any positive non-dividend paying asset.*

The price process for a numeraire, $Z = Z_t, 0 \leq t \leq T$, is thus almost surely strictly positive $\forall\ t \in [0, T]$.

In the context of our self-financing strategy Definition 2.2, with the securities expressed in terms of their present values, we have the following

Theorem 2.2. *Let ϕ be a trading strategy. Then ϕ is self-financing if and only if*

$$P(0,t)V_t(\phi) = V_0(\phi) + \int_0^t \phi_u d(P(0,u)S_u), \tag{2.5}$$

which is proven by Harrison and Pliska in [33]. We then choose a numeraire to act as a reference asset and to normalise all other market prices with respect to this numeraire; say Z. We thus consider normalised prices, $\frac{S^k}{Z}, k = 0, 1, ..., K$ instead of the individual security prices. Theorem 2.2 holds true for any choice of numeraire. A self-financing strategy will remain a self-financing strategy under any numeraire [29].

Our zero coupon bond (defined in Definition 3.4), or cash amount, is a natural choice for a numeraire, however, it is only one of a number of possible choices which may be more convenient for our calculations. Geman et al. [29] generalise Proposition 2.1 to any numeraire.

Proposition 2.2. *Assume there exists a numeraire N and a probability measure Q^N, equivalent to the initial Q_0, such that the price of any traded asset X (without intermediate payments) relative to N is a martingale under Q^N, i.e.*

$$\frac{X_t}{N_t} = E^N \left\{ \frac{X_T}{N_T} | \mathcal{F}_t \right\} \quad 0 \leq t \leq T. \tag{2.6}$$

Let U be an arbitrary numeraire, Then there exists a probability measure Q^U, equivalent to the initial (real-world measure) Q_0, such that the price of any attainable claim Y normalised by U is a martingale under Q^U, i.e.

$$\frac{Y_t}{U_t} = E^U \left\{ \frac{Y_T}{U_T} | \mathcal{F}_t \right\} \quad 0 \leq t \leq T. \tag{2.7}$$

Moreover, the Radon-Nikodym derivative defining the measure Q^U is given by

$$\frac{dQ^U}{dQ^N} = \frac{U_T N_0}{U_0 N_T}. \tag{2.8}$$

We derive Equation (2.8) as follows [13]: By definition of Q^N, we know that for any tradable asset price Z,

$$E^N\left[\frac{Z_T}{N_T}\right] = E^U\left[\frac{U_0}{N_0}\frac{Z_T}{U_T}\right], \tag{2.9}$$

as both would be equal to $\frac{Z_0}{N_0}$ as per Definition 2.8 of an equivalent martingale measure.

The Radon-Nikodym derivative can be defined as follows [55]:

Definition 2.10. *When a measure λ is absolutely continuous with respect to a positive measure ν then it can be written as*

$$\lambda(E) = \int_E f d\nu, \tag{2.10}$$

for a measurable set E. The function f is called the Radon-Nikodym derivative of λ with respect to ν. It may also be denoted as $\frac{d\lambda}{d\nu}$ or $\frac{D\lambda}{D\nu}$.

By this definition we know that for all Z

$$E^N\left[\frac{Z_T}{N_T}\right] = E^U\left[\frac{Z_T}{N_T}\frac{dQ^N}{dQ^U}\right], \tag{2.11}$$

giving us

$$E^U\left[\frac{U_0}{N_0}\frac{Z_T}{U_T}\right] = E^U\left[\frac{Z_T}{N_T}\frac{dQ^N}{dQ^U}\right], \tag{2.12}$$

which can be reduced to the formula in Equation (2.8) by the arbitrariness of Z. Equivalent martingale measures, as defined in Definition 2.8, are absolutely continuous and as such are Radon-Nikodym integrable.

The reader is referred to Brigo and Mercurio (2006) [13] for a *Change of Numeraire Toolkit* some useful formulae and notations on using the numeraire.

2.4 Choosing a Numeraire

Changing the numeraire is a useful technique and in the context of pricing contingent claims is often used to simplify solving the contingent claim. This of course raises the question of what would be a suitable choice for a numeraire for a particular problem. For a contingent claim $H(S_T) = \Phi(S_T)$ the contract or payoff function is $\Phi(S_T)$ and it is dependent on the underlying variable S at the time T. From Proposition 2.1 we would price such a contingent claim by taking the risk neutral expectation of the payoff.

$$H = E_0[df(0,T)\Phi(S_T)], \tag{2.13}$$

with E_0 being the expectation under objective or real world measures.

Let us consider the risk neutral numeraire to be the bank account as described in Definition 3.1.

$$B(t) = \frac{1}{df(0,t)} = \exp\left(\int_0^t r_s \,\mathrm{d}s\right), \tag{2.14}$$

and by Definition 2.8 under a new numeraire N

$$E_0[df(0,T)\Phi(S_T)] = S_0 E^N\left[\frac{\Phi(S_T)}{S_T}\right]. \tag{2.15}$$

We would like to find a numeraire N which would make the above equation simpler to solve, i.e. that $\frac{\Phi(S_T)}{S_T}$ is simple. Ideally we would also like that $S_t N_t$ is the price of a tradeable asset so that $\frac{S_t N_t}{N_t} = S_t$ is a martingale under Q^N [13]. Assuming lognormal martingale dynamics for X

$$\mathrm{d}S_t = \sigma(t) S_t \,\mathrm{d}W_t, \quad Q^N,$$

with distribution

$$\ln S_t \sim \mathcal{N}\left(\ln S_0 - \frac{1}{2}\int_0^t \sigma(s)^2 \,\mathrm{d}s, \int_0^t \sigma(s)^2 \,\mathrm{d}s\right),$$

allowing us to use some of our more familiar tools such as *Itō's Lemma.*

Chapter 3

Interest Rate Market Model

We will now take a look at some of the theory and the framework around which interest rate market models are built. These concepts and definitions will be used later when looking at curve construction and the concepts of interest rate basis. The assumption of no arbitrage is used as a foundation and overarching concept which we attempt to preserve in the various basis models analysed later.

3.1 The Spot Interest Rate

Typically money does not sit as unutilised cash and is rather deposited or lent out in return for compensation for allowing someone else the use of that money; the borrower pays this compensation, which we call interest, in return for being able to utilise that money for a period of time. For the purposes of this section, unless otherwise specified, a deposit with a bank is considered a loan to the bank (for economic purposes this is true, though not necessarily from a legal perspective). Thus a sum of money, say one unit of currency, deposited in a bank account is expected to earn interest and thus we can have $B(t)$ the value of a bank deposit and the process for $B(t)$ defined as follows [13].

Definition 3.1 (Bank Deposit). *$B(t)$, is the value of a* ***bank deposit*** [13] *at a time $t \geq 0$ and it follows the following process:*

$$\mathrm{d}B(t) = i_t B(t)\,\mathrm{d}t \quad \textit{and} \quad B(0) = 1, \tag{3.1}$$

i_t is called the instantaneous spot rate or short rate and is the rate of interest at which the deposit continuously accrues interest giving

$$B(t) = \exp(\int_0^t i_s\,\mathrm{d}s). \tag{3.2}$$

Hence a deposit of 1 at a bank will yield B(t) at time t. If we considered the question of how much one should invest today to receive 1 unit of currency at time t it would lead to the concept of a discount factor.

Definition 3.2. *(Discount Factor) The* ***discount factor*** [13] $df(t_1, t_2)$ *is the amount that would need to be deposited at time* t_1 *to yield an amount of 1 at time* t_2 *given by*

$$df(t_1, t_2) = \frac{B(t_1)}{B(t_2)}, \quad t_1 < t_2. \tag{3.3}$$

i_t is not deterministic but rather a stochastic process, hence $B(t)$ and $df(t_1, t_2)$ are also stochastic. Typically though this is often treated as a deterministic function often justified in that either the money to be paid out is funded at the same rate at which it is borrowed or that for fixed term contracts the rate of interest to be received can be fixed upfront. $df(0, t)$ is often dubbed the **present value** of a unit of 1 amount of currency to be received at time t.

Before continuing we should define the concepts of the year fraction, compounding type, and the daycount basis which would be used in interest rate and present value processes.

Definition 3.3 (Year Fraction)**.** *The* ***year fraction****, or coverage,* $\tau(t_1, t_2)$ *is the period of time between times* t_1 *and* t_2 *expressed as a fraction of a year. Simply put* $\tau(t_1, t_2)$ *is the number of days between* t_1 *and* t_2 *divided by the number of days in the year.*

The choice of how to measure time between two dates and the length of a year, the daycount convention, varies in each market with the typical South African convention being Actual/365. In this case the number of days in the numerator of $\tau(t_1, t_2)$ is considered to be all the days between the two applicable dates and the number of days in the year, the denominator of $\tau(t_1, t_2)$ is always considered to be 365. Numerous daycount conventions exist and some common ones are discussed in Appendix A along with common business day rules (being adjustments to take into account public holidays and even weekends).

Let us now define the zero coupon bond:

Definition 3.4 (Zero Coupon Bond)**.** *A* ***zero coupon bond*** *(ZCB), or pure discount bond, is a contract that provides the holder with the payment of one* *currency unit at time T with no intermediate payments. $P(t,T)$ denotes the value of the zero coupon bond, at time t, for a zero coupon bond maturing at T for $0 \leq t < T$ and by definition $P(T,T) = 1 \,\forall\, T$.*

The zero coupon bond differs from the discount factor defined above in that it is deterministic and may be seen as the expected discount factor $df(t,T)$ which would apply from time t to time T[1]. Analogously $P(0,T)$ is the present value of a zero coupon bond paying an amount of 1 unit of currency at time T.

The interest a zero coupon bond earns may be expressed in different ways depending on the compounding frequency: the frequency at which interest earned is reinvested to earn more interest so that one is indifferent towards leaving a ZCB to mature or continually selling and repurchasing ZCBs so as to reinvest the interest accrued.

Definition 3.5 (NACC - Nominal Annual Compounded Continuously)**.** *The constant rate of interest at which a ZCB, with value $P(t,T)$, accrues to a value of 1 at maturity time T. That is if interest is continuously earned and instantaneously reinvested and may be expressed as $r(t,T)$ such that*

$$P(t,T) = e^{-r(t,T)\cdot\tau(t,T)}. \tag{3.4}$$

This is consistent with our definition of the discount factor above should the interest rate be deterministic.

Definition 3.6 (NACA - Nominal Annual Compounded Annually)**.** *The constant rate of interest, $r_a(t,T)$ at which a ZCB, with value $P(t,T)$, accrues to a value of 1 at maturity time T with interest being reinvested once a year such that*

$$P(t,T) = (1 + r_a(t,T))^{-\tau(t,T)}. \tag{3.5}$$

Also of interest is the simple rate of interest because LIBOR (and the other fixing rates such as JIBAR in South Africa) is just such a rate and is used extensively in the LIBOR Market Model or LMM (also referred to as the Brace, Gatarek and Musiela model or BGM model). This model is further discussed in Section 6.4 but for now let us consider the rate $L(t_1, t_2)$.

Definition 3.7 (Simple Interest Rate)**.** *The **simple interest rate** $L(t_1,t_2)$ is the rate of interest which would apply for a deposit starting at time t_1 and maturing at time t_2 such that,*

$$P(t_1,t_2) = \frac{1}{1 + L(t_1,t_2)\tau(t_1,t_2)}, \tag{3.6}$$

[1] An uppercase T is often used to denote a maturity time for financial contract thought it need not be exclusively so.

and

$$L(t_1, t_2) = \frac{1 - P(t_1, t_2)}{\tau(t_1, t_2)}. \tag{3.7}$$

3.2 Forward Interest Rates

Now let us consider the amount to be placed in a deposit at some time in the future, t_1, so as to receive a unit of currency at some later time, t_2. There are now three time factors to consider: the day which this deposit amount is to be determined on, the date on which this deposit begins (the near maturity) and the date on which the deposit matures (the far maturity). One would need to lock in an interest rate applicable for that future period and this can be done using a Forward Rate Agreement (FRA).

Definition 3.8 (Forward Rate Agreement)**.** *A* ***Forward Rate Agreement****, or FRA, is a contractual agreement between two parties whereby one party (the fixed rate payer) will, at time t_1 (the near maturity date), pay a fixed amount of interest K and the other party (the floating rate payer) will pay an amount of interest being the spot rate $L(t_1, t_2)$ as determined (or fixing) on date t_1 and applicable for a deposit of length $t_2 - t_1$ (where t_2 is sometimes referred to as the far maturity date).*

The use of such a contract therefore allows one to lock in an interest rate to be made in the future. Typically a FRA may be settled as a net of the two payments and conventions may vary as to whether it is settled on the far maturity date or the near maturity date (using the fixing rate to discount the net cashflow so that it is bilaterally accepted). FRAs have prices quoted in the market for various tenors and maturity dates. FRAs of various tenors are often referred to in the market in terms of their effective date, the near maturity date, and their termination date, the far maturity date, as the number of months from the current date. For example a 3x6 FRA would refer to a FRA with an effective date 3 months from today and maturing 6 months from today with a tenor of 3 months, likewise a 3x9 FRA refers to a FRA with an effective date 3 months from today and maturing 9 months from today.

As mentioned previously it is common to reference the LIBOR rate for fixing the rate in FRAs (and other floating interest rate payments). LIBOR stands for London Inter Bank Offer Rate and is published by the British Banker's Association (BBA) daily. It is derived as an average of the various rates at which the member banks could borrow money at if asked “At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am?"[3]. There are other conventions for deposits in other countries and currencies, such as JIBAR the Johannesburg Inter Bank Acceptance Rate, and whilst the technical details may vary they are used in similar ways and this paper will use LIBOR as a universal rate.

Of significant importance in various interest rate frameworks is the economic principle of no-arbitrage. Arbitrage is defined as the being able to invest zero today with a non-zero probability of receiving a positive profit (i.e. a positive cashflow) and a zero probability of making a loss (i.e. a negative cashflow). It can be defined mathematically as follows

Definition 3.9 (Arbitrage Opportunity)**.** *An **arbitrage opportunity** may be defined as the case where:*
$\mathbb{P}(V_t > 0) \geq 0$ *and* $\mathbb{P}(V_t < 0) = 0$ *for a portfolio of value* V_t *time* t *and* $V_0 = 0$. $\mathbb{P}(A)$ *refers to the probability of an event,* A*, occurring. In other words an arbitrage opportunity is the case where there is a possibility of making a future profit, with zero probability of making a loss, at zero cost today.*

The assumption that no-arbitrage opportunities exist in the market leads to the fact that any two portfolios having the same payoff at a future date must have the same value today. This leads to the method of portfolio replication for determining unknown market values; by creating a portfolio containing all known prices which replicates the payoff of another which contains an unknown price (or unknown variable such as a forward interest rate). This allows one to define K as the simple fair forward interest rate which would be applicable to a FRA.

Derivation 3.1 (Fair Forward Rate)**.** *Consider a portfolio* A *consisting of a zero coupon bond with value* $P(0, t_2)$ *maturing at time* t_2 *(and a maturity value of* 1*) and another portfolio* B *consisting of* $P(0, t_2)$ ***worth*** *of zero coupon bonds maturing at time* t_1 *which at maturity is reinvested into another zero coupon bond maturing at time* t_2, $t_2 > t_1$ *(noting that the latter is stochastic as the value of such a zero coupon bond is unknown today under stochastic interest rates) and a FRA applicable with near maturity* t_1 *and far maturity* t_2*. At the far maturity portfolio* A *has a value of 1 by the definition of a zero coupon bond. Thus at the far maturity date* t_2 *portfolio* B *has a* *value, V_B, as follows*

$$\begin{aligned} V_B &= \frac{P(0,t_2)}{P(0,t_1)}\frac{1}{P(t_1,t_2)} + \frac{P(0,t_2)}{P(0,t_1)}\frac{1}{P(t_1,t_2)}(K - L(t_1,t_2))\tau(t_1,t_2) \\ &= \frac{1 + L(0,t_1)\tau(0,t_1)}{1 + L(0,t_2)\tau(0,t2)}(1 + K)\tau(t_1,t_2), \end{aligned} \tag{3.8}$$

and considering that, by the no arbitrage assumptions given above, the only value for K which is fair is one such that the value of portfolio A at far maturity must equal that of portfolio B at far maturity else one could sell portfolio A and purchase portfolio B (or vice versa) to give rise to an arbitrage.

$$K = \left(\frac{1 + L(0,t_2)\tau(t_2)}{1 + L(0,t_1)\tau(t_1)} - 1\right)\frac{1}{\tau(t_1,t_2)}. \tag{3.9}$$

This results in the concept of the fair forward rate used in calculating cash-flows of various floating interest rate instruments.

Definition 3.10 (Simple Compounding Forward Interest Rate)**.** *$F(t;t_1,t_2)$, the forward interest rate using simple compounding as at time t prevailing from the near maturity date t_1 until the far maturity date t_2 is expressed as*

$$F(t;t_1,t_2) = \frac{1}{\tau(t_1,t_2)}\left(\frac{P(t,t_1)}{P(t,t_2)} - 1\right), \quad t_2 > t_1. \tag{3.10}$$

The forward rate $F(t;t_1,t_2)$ is treated as the expectation, based on values observable at time t, of the stochastic future spot rate $L(t_1,t_2)$ and is used to replace the LIBOR rate in valuing a FRA to give the well known formula to value $V_{FRA}(t)$ a FRA with a notional of N at time t [13, 38].

$$V_{FRA}(t,t_1,t_2,K,N) = N \cdot P(t,t_2)\tau(t_1,t_2)(K - F(t;t_1,t_2)). \tag{3.11}$$

The following no-arbitrage relation could also be postulated

Proposition 3.1. *Let $P(t,t_1)$ and $P(t,t_2)$ be the values, at time t, of two zero coupon bonds maturing at times t_1 and t_2 respectively with $t \le t_1 \le t_2$. Let $P(t;t_1,t_2)$ be the value at time t of a zero coupon bond starting at time t_1 and maturing at time t_2, i.e. a forward starting zero coupon bond. The following would hold true:*

$$P(t,t_2) = P(t,t_1)P(t;t_1,t_2), \quad t \le t_1 \le t_2, \tag{3.12}$$

and further, analogous to the fair forward rate above

$$P(t;t_1;t_2) = \frac{P(t,t_2)}{P(t,t_1)} = \frac{1}{1 + F(t;t_1,t_2)\tau(t_1,t_2)}. \tag{3.13}$$

This means that discounting a unit of currency in one step from t_2 to t, or in two steps from t_2 to t_1 and then from t_1 to t should both give rise to the same unique present value. Again we may see $P(t; t_1; t_2)$ as the expectation at time t of the discount factor $df(t_1, t_2)$.

The forward rate may also be expressed in other compounding forms though simple interest is the most widely used and matches the form in which LIBOR is quoted. Much like the instantaneous short rate i_t defined in Definition 3.1 there exists the analogous concept of an instantaneous forward rate f_{t_1,t_2}. By considering what happens as t_2 tends to t_1 from right we obtain

$$f_{t_1,t_2} = \lim_{t_2 \to t_1^+} F(t; t_1, t_2). \tag{3.14}$$

Much of the historical approach to modelling yield curves and the term structure of interest rates was based on modelling the instantaneous short and forward rates. The ubiquitous Heath-Jarrow-Morton framework (HJM framework)[35] is just such an example and is based on the exogenous specification of the evolution of the instantaneous forward rates.

We shall quickly look at another ubiquitous interest rate derivative; the interest rate swap.

Definition 3.11 (Interest Rate Swap)**.** *An **Interest Rate Swap** or often simply a **Swap** is an agreement between two parties to exchange a series of cashflows. One party (the floating rate payer) shall, at regular intervals for a fixed number of intervals, pay variable interest rate payments being the spot rate $L(t_{i-1}, t_i)$ as determined (or fixing) on date t_{i-1} and applicable for a deposit of length $t_i - t_{i-1}$ with t_i, $\quad i = 1, 2, .., n$ being the payment dates. The other party (the fixed rate payer) shall make interest rate payments based on a fixed interest rate K on those same dates. The length of the regular intervals, $t_i - t_{i-1}$ is the* term *of the swap, and the total duration of the swap is called the* maturity*.*

A swap would typically reference LIBOR as the fixing rate, much like a FRA. A Swap may be seen as analogous to a series of FRAs each with the same fixed rate K being exchanged for a floating interest rate payment. Likewise a Swap may be seen as borrowing a unit of currency at a floating interest interest rate, say 3-Month LIBOR, and then depositing a single unit of currency to receive a fixed interest rate of, say, K. Swaps are discussed in most texts on interest rate markets or derivatives such as [38] and [13].

A particular type of swap, the overnight index swap (OIS), is of specific importance, especially when considering interest rate derivatives priced in the presence of collateral (as discussed later in Section 7.3).

Definition 3.12 (Overnight Index Swap)**.** *An **overnight index swap** or **OIS** is an interest rate swap where a floating rate referencing an overnight deposit rate such as the Euro Overnight Index Average (EONIA), a weighted average of overnight unsecured lending transactions, or the Federal Funds Rate is swapped for a fixed payment. The floating interest rate is usually compounded to the maturity date of the swap and settled on that maturity date. For OIS' with tenor less than one year the fixed interest rate is treated as a simple rate of interest and for those with a tenor greater than a year the fixed rate is an annual fixed rate.*

Overnight index swaps are covered in greater detail in Section 7.4.

3.3 Forward Measures

In this section we will now tie together the previous two as well as introducing the expectation theory described in Chapter 2; we will take the concept of the instantaneous spot rate, forward rates and our zero coupon bonds and use them in the context of suitable numeraires and martingale theory. When dealing with interest rates and interest rate derivatives a choice of a zero coupon bond maturing at time T ($P(0,T)$) is a particularly useful and intuitive numeraire.

Definition 3.13. *For a fixed time T the T**-forward measure** Q^T is defined as the martingale measure for the numeraire process $P(t,T)$.*

Q^T can then have the following explicit description [9, 29] with the corresponding proofs in [9],

Proposition 3.2. *If Q denotes the risk neutral martingale measure then the likelihood process*

$$L^T(t) = \frac{dQ^T}{dQ}, \text{ on } \mathcal{F}_t, \; 0 \leq t \leq T,$$

is given by

$$L^T(t) = \frac{P(t,T)}{B(t)P(0,T)}.$$

In particular, if the Q-dynamics of the T-bond are Wiener driven, i.e. of the form

$$\mathrm{d}P(t,T) = r(t)P(t,T)\,\mathrm{d}t \,+\, P(t,T)\nu(t,T)\,\mathrm{d}W_t,$$

where W is a (possibly multidimensional) Q-Wiener process, then the L^T dynamics are given by

$$\mathrm{d}L^T(t) = L^T(t)\nu(t,T)\,\mathrm{d}W_t,$$

i.e. the Girsanov kernel for the transition from Q to Q^T is given by the T-bond volatility $\nu(t,T)$.

As we know $P(T,T) = 1$ we have a corollary to Proposition 2.1 [9]

Proposition 3.3. *For any T-claim X we have*

$$\pi(t;X) = P(t,T)E^T[X|\mathcal{F}_t], \tag{3.15}$$

where E^T denotes integration with regard to Q^T.

We also note that the price $P(t,T)$ is observable in the market and thus does not need to be computed.

Let us also reintroduce the forward interest rate described earlier. $f(t,T)$ is our instantaneous forward rate process describing the rate of return we would have for the *instantaneous* infinitesimal time period $[T, T + \mathrm{d}T]$ if struck at time t. The short rate i_T is the rate of return over the infinitesimal period $[T, T + \mathrm{d}T]$ if struck at time T. We could thus interpret $f(t,T)$ to be an estimate of the future short rate i_T. The **unbiased expectation hypothesis** for forward rates is a common form of expectation hypothesis and asserts that forward rates equal the conditional expectation of future spot rates, or that the current forward rate is an unbiased estimator of the future spot rate.[2]

$$f(t,T) = E[i_t|\mathcal{F}_t]. \tag{3.16}$$

We would expect this hypothesis to hold true in the risk neutral world (if not under the objective measure P) so we can reformulate it as follows [9]

$$f(t,T) = E^Q[i_T|\mathcal{F}_t], \tag{3.17}$$

where Q is the risk neutral martingale measure. We can now follow [6] and other common texts to express the value of contingent claims (in this case a FRA as described earlier) in the context of martingale measures and expectations.

$$V_{FRA}(t;t_1,t_2,K,N) = N \cdot P(t,t_2)\tau(t_1,t_2)(K - F(t;t_1,t_2)). \tag{3.18}$$

[2]Cox, Ingersoll and Ross [20] show that this does not always hold equilibrium and must always give rise to arbitrage. This has been shown to not always be the case though under certain circumstances and under the Heath-Jarrow-Morton framework [53]

Introducing expectations we have

$$
\begin{aligned}
V_{FRA}(t;t_1,t_2,K,N) &= P(t,t_2)E_t^Q[V_{FRA}(t_2;t_1,t_2,K,N)] && (3.19)\\
&= N\cdot P(t,t_2)\tau(t_1,t_2)\left\{K - E_t^Q[L(t_1,t_2)]\right\} && (3.20)\\
&= N\cdot P(t,t_2)\tau(t_1,t_2)\left\{K - F(t;t_1,t_2)\right\}, && (3.21)
\end{aligned}
$$

and Q is the forward measure corresponding to the numeraire $P(t,t_2)$ and E_t^Q is the expectation at time t with regard to measure Q and filtration $\mathcal{F}$ assuming the standard martingale property for forward rates.

Chapter 4

Pre-Crisis Single Curve Framework

In this chapter we shall consider the framework used to create interest rate curves from which various interest rate instruments were priced and valued before the 2008 liquidity crisis had an impact on basis spreads. This is the treatment of interest rates found in most textbooks (or their specific editions) before the crisis such as [9, 13, 38]. We shall also look at how, pre-crisis, under the assumption of no-arbitrage there would be no difference, or spread, arising from using forward rates of different terms.

4.1 Once Upon a Time...

The concept of the **risk-free rate** is prevalent in almost all finance and economics literature. It is the rate of return an investor would expect to receive in an investment with no risk of financial loss. This is often treated in literature as some or other form of bank-account; or more often as an investment in government **Treasury Bills** (or **T-Bills**) or other government issued money-market securities. A T-Bill is similar to our Zero Coupon Bond in that a sum of money will be paid out at a given maturity. A T-Bill is issued by the government and as such is (was!) often seen as risk-free. It was the benchmark for all other investments as anything bearing a greater risk would require a higher potential rate of return. This risk-free rate is also prevalent in the *Capital Asset Pricing Model* (CAPM) and in this context is the compensation for systemic risk which cannot be hedged out or diversified out by the investor.

Tied in with the risk-free rate is the concept of a central bank rate, sometimes called the **official bank rate**, **repo rate** or **minimum lending rate**, the minor details may differ for various central banks but the overall concept is the same, this is the rate at which a central (government) bank will lend money to various banks, usually for a short term such as a week or overnight. Analogous to this is the rate at which banks would lend money to each other. This too has a number of different names in each jurisdiction such as **LIBOR** (London Inter Bank Offer Rate), **JIBAR** (Johannesburg Inter Bank Agreed Rate) and various other **X-BOR** rates. We shall use the term LIBOR generically in this text to be in line with most other literature. LIBOR itself can be quoted for different currencies such as USD LIBOR, GBP LIBOR or even EUR LIBOR which is the rate at which the participant BBA (British Banker's Association) banks will lend those respective currencies to each other.[1] LIBOR is also quoted for different terms such as 1-Month USD LIBOR, 3-Month USD LIBOR or 6-Month USD LIBOR. LIBOR is of particular importance in the context of interest rate derivatives and contingent claims as it is the fixing rate against which FRAs, Swaps, Caps/Floors and many other interest rate derivatives' payoffs are calculated against. Of interest is also the overnight rate such as **EONIA** (Euro OverNight Index Average), **US Federal Reserve overnight rate** or simply the **OIS** (overnight index swap) rate.

Before the credit crunch of August 2007 many of these various rates tracked each other quite closely [50]; FRAs and LIBOR had a precise relationship. Cashflows at different frequencies were considered equivalent (apart from very small basis spreads). In typical interest rate theory swapping out 3-Month LIBOR for an OIS rate had the same present value. In the absence of credit risk floating rate notes at inception had a present value equal to the par nominal regardless of whether they paid out a 3-Month or 6-Month floating rate or whether they had a term of 1-year or 3-years. Henrard 2007 [36] gives an analogy of how different curves would either be used to value different financial instruments or the same instrument with different obligors (having different credit or default risk). According to [36] there is a fundamental problem in the treatment of contingent derivative cashflows and the respective discounting without a lot of attention in the literature (at the time). The cashflows are thus contingent on the obligor being able to pay. Often bonds or similar instruments were quoted as a yield, a number which is part discounting and part default risk (and later, part liquidity).

[1]EUR LIBOR is distinct to EURIBOR which is the rate at which Eurozone member banks would lend EUR to each other at. The intricacies of calculating each X-BOR rate differs for each jurisdiction. The details for LIBOR may be found at http://www.bbalibor.com.

The task of valuing each cashflow as a contingent claim is made simpler as each is treated as a discounted cashflow. Incorrect methods are applied to the cashflows (discounting applicable to non contingent cashflows) with an incorrect rate (the discount rate with a spread) replacing ignorance in method with ignorance on the data. This is addressed in a later section.

4.2 Absence of Arbitrage Between Different Tenors

The assumption of no-arbitrage in the market implies that there exists a relationship between FRA rates of different yet overlapping tenors. By assuming no-arbitrage one should be indifferent between having a 3 month LIBOR deposit and a 3x6 FRA (as defined in Section 3.2) or having a 6 month LIBOR deposit. Likewise one should be indifferent between having a 3x6 FRA and a 6x9 FRA or having 3x9 FRA. Pre-2008 this assumption was consistent with what was observable in the market. As an example let us consider some market quotes from 2005-11-11.

Market Quotes for Interest Rate Derivatives	
3 Month USD LIBOR	4.3400
6 Month USD LIBOR	4.5500
3x6 FRA	4.7165
6x9 FRA	4.8840
3x9 FRA	4.8300

Figure 4.1: Interest rate instrument quotes as at 2005-11-11, quoted as percentages. (source: Bloomberg Finance L.P.)

As Figure 4.1 shows, the 3 Month USD LIBOR rate was 4.34% and 3x6 FRA rate was 4.7165% while the 6 Month LIBOR rate was 4.55%. We can now apply the following to calculate the 6 Month USD LIBOR rate as implied by the 3 Month LIBOR rate and the 3x6 FRA:

$$\begin{aligned}
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= \left(1+\frac{\text{3M LIBOR}}{4}\right)\left(1+\frac{\text{3x6 FRA}}{4}\right), \\
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= \left(1+\frac{4.34\%}{4}\right)\left(1+\frac{4.7165\%}{4}\right), \\
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= 1.022769\ldots, \\
\text{6M LIBOR'} &= 4.5538\ldots\%,
\end{aligned}$$

where 3M LIBOR refers to the 3 Month LIBOR rate and 6M LIBOR' refers to the implied no-arbitrage 6 Month LIBOR rates. Comparing this result to the quoted 6 Month LIBOR rate we see that there is a difference of approximately 0.00384% which is less than half of a basis point (a percent of 1%) and thus negligible.

In the case of FRAs we can calculate an implied 3x9 FRA rate by taking the market quotes for a 3x6 and a 3x9 FRA as follows:

$$\begin{aligned}
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= \left(1+\frac{\text{3x6 FRA}}{4}\right)\left(1+\frac{\text{6x9 FRA}}{4}\right), \\
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= \left(1+\frac{4.7165\%}{4}\right)\left(1+\frac{4.8840\%}{4}\right), \\
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= 1.024145\ldots, \\
\text{3x9 FRA'} &= 4.82904\ldots\%,
\end{aligned}$$

where 3x9 FRA' refers to the implied no-arbitrage 3x9 FRA rate. In this case the difference, or spread, between the implied and quoted rate is 0.1 basis points; again this is a negligible difference[2].

This relationship held true before the 2008 liquidity crisis. To show this we have calculated the spreads between the implied 3x9 FRA rate and the quoted 3x9 FRA rate beginning in 2005-05-03 up until 2005-12-27, which are illustrated in Figure 4.2. This spread is never greater than 2.5 basis points and even this difference can be explained away due to timing mismatches, rate aggregation or business day rules (to account for weekends and public holidays).

[2]Here we have ignored the impact of public holidays on the maturity dates and have applied the rates to a full three month or six month period; this has no material impact on the principle being illustrated.

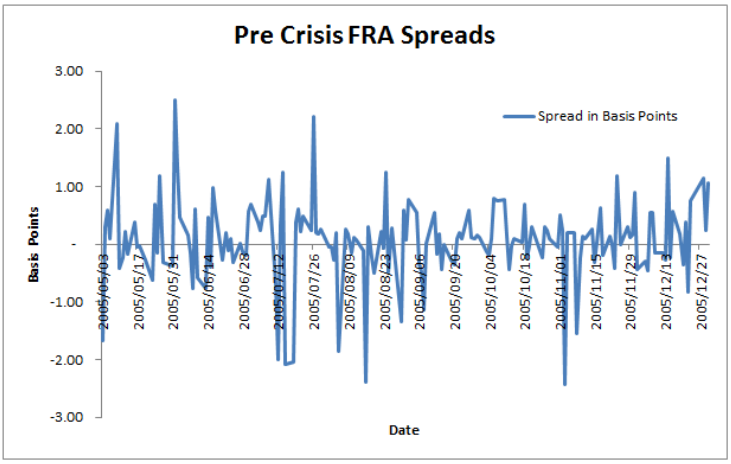

Figure 4.2: Spreads between the quoted 3x9 FRA rate and that implied from 3x6 and 6x9 FRAs. (source: Bloomberg Finance L.P.)

4.3 Simple Single Currency Single Curve Construction

The concept of discounting cashflows is prevalent in much of the literature on the pricing and valuation financial instruments. Using the market quoted prices available a yield curve or zero-coupon curve is the continuous function of interest rates $r(0,t)\,\forall\, t > 0$ prevalent at the time of construction. For each interest rate there is the corresponding discount factor $df(0,t)$. This curve may be expressed using different daycount conventions. Any cashflow arising at time t would thus be discounted using the corresponding discount factor calculated from the interest rate $r(0,t)$ as per the curve. Although considered and often plotted continuously for practical reasons such a curve would have daily points t_i. These points would only be known for specific days as quotes in the market and the rest of the curve would be filled using a suitable interpolation method.

Before the credit crunch and liquidity crisis arose the standard market practice for creating interest rate curves for the pricing and valuation of interest rate derivatives was fairly simple. A single curve for a particular currency was bootstrapped using a finite set of the most liquid quoted vanilla interest rate instruments (viz. short term deposits, forward rate agreements (FRAs), interest rate swaps (vanilla fixed interest rate payments exchanged for those linked to the prevalent standard floating interbank rate such as 3 Month JIBAR) [6].

The methods of constructing/bootstrapping these curves, along with methods of interpolating such curves, are discussed by various authors and of particular interest are the methods discussed by Uri [54], Hagan and West [32] and by Hull [38]. The various technical methods of constructing such curves will not be discussed in depth here but there are some standard procedures used in practice for creating such a curve and calibrating it to the market [6, 32, 54]. Bianchetti [6] summarises the pre-crisis standard market practice quite nicely referencing [32, 54] with the following steps being a short reprise from [6]

First
Select a single finite set of liquid (with observable market prices) vanilla interest rate instruments traded in the market with increasing maturities; for examples short term deposits in the very short end, moving on to FRAs and then Swaps.

Second
Build a single yield curve using the selected instrument and utilising a set of interpolation rules.

Third
Using that same yield calculate forward rates, cashflows and discount factors.

That is not to say that only one curve was ever constructed. Other yield curves would be constructed for specific instruments such as government bonds or T-Bills (using government bonds and money market bills as the benchmark instruments) but for interbank derivatives a single curve was often used for all instruments.

Some important assumptions in this method are that:

- All cashflows were free of default risk
- Liquidity (i.e. availability of cash) was guaranteed, i.e. a frictionless market
- One could borrow at the prevailing rate, i.e. at JIBAR flat

Historically these assumptions were reasonable for interbank lending and the prevailing term rate, such as JIBAR, was, or was reasonably close, to that at which banks would lend to each other. Thus one would be indifferent in having a deposit of, say, three months and then reinvesting for another three months, fixing the rate with a FRA, or simply having a deposit with a term of six months in accordance with the no-arbitrage framework defined above.

It is also of interest to discuss how such a curve, commonly called a swap curve, was treated. In the prolific textbook (at least in the pre 2008 editions) by Hull [38], *Options, Futures and Other Derivatives* [38], the assumption is made that borrowing or lending would be done at LIBOR. This approach is consistent in much of the literature regarding derivatives prior to the 2008 crisis. Further the concept of a risk-free rate is used in many instances for discounting cashflows and pricing; intuitively there would be no reason for different cashflows to be discounted at different rates but the market had previously shown that in certain instances, such as overnight indexed swaps (OIS), separate curves would be used [37].

Deriving the single currency single-curve framework mathematically follows from the previous sections. Let us define M_Z our interest rate market in currency Z with our yield curve Y_Z defined as a continuous term structure of discount factors

$$Y_Z = T \rightarrow P_Z(t_0, T), T \geq t_0 \tag{4.1}$$

with t_0 being time zero, i.e. the reference date used (which may be today, or perhaps a settlement date in, say, 2 days time). $P_Z(t_0, T)$ as before is the price at time t_0 of a zero coupon bond in currency Z maturing at time T as per Definition 3.4.

Let us have a look at an example of a simple method of constructing such a yield curve.

Single Curve Construction					
Instrument	t (in months)	Quote	r(0,t) NACQ	f(0,t-3,t) NACQ	P(0,t)
3 Month USD LIBOR	3	4.3400	4.340%	4.340%	0.989266459
6 Month USD LIBOR	6	4.5500	4.524%	4.709%	0.977756050
6x9 FRA	9	4.8840	4.644%	4.884%	0.965961658
1Y Swap	12	5.0000	4.714%	4.925%	0.954212912

Figure 4.3: Interest rate instrument quotes as at 2005-11-11, quoted as percentages, used to construct a simple yield curve. (source: Bloomberg Finance L.P.)

Figure 4.3 is a table of USD LIBOR, USD FRAs and USD quarterly swap rates. As per the first step outlined above this is a selection of liquid (with observable market prices) vanilla interest rate instruments, of increasing maturities, and traded in the market. $r(0,t)$, $f(0,t-3,t)$ and $P(0,t)$ represent the zero coupon interest rate, the forward interest rate and the zero coupon bond price respectively with t expressed as the number of months from time 0 and and interest rates expressed as quarterly compounded rates. The 3 Month USD LIBOR rate is a simple interest rate which applies for the period $(0,3)$, expressed in number of months from today, and as the corresponding interest rate $r(0,3)$ matches the quote itself. Likewise as it starts at time 0 the forward rate is the same as the zero coupon rate.

The zero coupon bond price for the first 3 month period follows from Definition 3.7 and may be derived as follows:

$$\begin{aligned} P(0,3) &= \frac{1}{1+L(0,3)\tau(0,t)} \\ &= \frac{1}{1+4.34\% \cdot 0.25} \\ &\approx 0.989266459. \end{aligned}$$

Here we have simply substituted the 3 Month LIBOR rate into $L(0,3)$. The same may be done to calculate the zero coupon bond price $P(0,6)$ substituting the 6 Month LIBOR rate into the equation for $L(0,6)$.

The derivation of the forward rate, $f(0,3,6)$, follows from Definition 3.10 and can be calculated as follows:

$$\begin{aligned} f(0,3,6) &= \frac{1}{\tau(3,6)}\left(\frac{P(0,3)}{P(0,6)}-1\right) \\ &= 0.25\left(\frac{0.989266\ldots}{0.977756\ldots}-1\right) \\ &\approx 0.04709. \end{aligned}$$

The forward rate $f(0,6,9)$ is given by the market quote for the 6x9 FRA as per by Derivation 3.1. To derive the zero coupon bond price $P(0,9)$ we can once again use Definition 3.10 solving for $P(0,9)$:

$$\begin{aligned} P(0,9) &= \frac{P(0,6)}{f(0,6,9)\tau(6,9)-1} \\ &= \frac{0.977756\ldots}{0.04884 \cdot 0.25 - 1} \\ &\approx 0.965961658. \end{aligned} \tag{4.2}$$

Finally the one year point (12 months) may be calculated from the quotes 1 Year quarterly swap rate of 5%. As the swap is a series of cashflows based off of a fixed rate in exchange for another series of cashflows based off of the 3 Month USD LIBOR rate one would need to determine the fair forward rate applying to the period $(9, 12)$ such that the value of the swap is zero. This can be done as follows

$$\begin{aligned} \sum_{t=3}^{12} 0.05\tau(t-3,t)P(0,t) &= \sum_{t=3}^{12} f(0,t-3,t)\tau(t-3,t)P(0,t), \\ &\qquad t = 0,3,6,9,12, \\ f(0,9,12)\tau(9,12)P(0,12) &= \sum_{t=3}^{12} 0.05\tau(t-3,t)P(0,t) \\ &\quad - \sum_{t=3}^{9} f(0,t-3,t)\tau(t-3,t)P(0,t) \\ &\qquad t = 0,3,6,9,12, \end{aligned} \tag{4.3}$$

(4.4)

and since

$$P(0,12) = \frac{P(0,9)}{f(0,9,12)\tau(9,12) - 1},$$

from Definition 3.10, we can substitute the right hand side into Equation (4.3) and solve for $f(0,9,12)$ to obtain

$$f(0,9,12) \approx 0.04925,$$

and

$$P(0,12) \approx 0.9542129. \tag{4.5}$$

What is important to note in this example is that the choice of deposit points, both 3 month and 6 month, was based on liquidity and not on tenor. The curve is a mix of 3 month and 6 month instruments and as shown in Section 4.2 and Figure 4.2 one could have elected to use FRAs of other tenors without a significant impact on the construction of the curve.

This is a simplified example to illustrate the concept of yield curve construction. A more rigorous approach would take into account specific business day rules (see Appendix A) for each of the maturity dates. For further points on the yield curve using swaps with maturities greater that 1 year one would apply a choice of an interpolation rule to calculate points without a FRA or deposit maturing on that date. The topic of yield curve bootstrapping techniques and interpolation rules is itself extensive and the reader is referred to Hagan and West's [32] treatment on the topic for a comparison of such techniques and interpolation rules and methods.

4.4 Simple Multiple Currency Multiple Curve Construction

The concept of interest rate basis pre-crisis was most prevalent in the realm of cross currency derivatives and contingent claims with cross currency basis giving rise to either arbitrage or the concept of multiple curves in the context of cross currency derivatives (including quanto derivatives). The concepts for multiple curve construction, such as the interest rate parity and the basis between a forward exchange rate and an implied forward exchange rate are used in the analogies later explored in Section 5.6. Let us first define some multiple currency derivatives.

Definition 4.1 (Forward Exchange Contract)**.** *A **forward exchange contract** or **FEC** (also called a currency forward) is an agreement between two counterparties to exchange a fixed amount of one currency for a fixed amount of another currency at some predetermined date in the future (the maturity date). This rate of exchange, the **forward exchange rate** or **strike price**, is usually agreed upon so that there is no net exchange of currency at inception.*

Definition 4.2 (Cross Currency Swap)**.** *A **cross currency swap** or **CCY Swap** is an agreement between two counterparties to exchange regular floating rate interest rate payments based on a floating rate fixing (such as USD 6-Month LIBOR, possibly with a spread) on a nominal amount in one currency for floating rate interest rate payments based on a floating rate fixing (such as ZAR 3-Month JIBAR, possibly with a spread) on a nominal amount in another currency until some predetermined date in the future (the maturity date). The nominal amount of currency may or may not be exchanged at inception though it is almost always exchanged at maturity. As for a single currency swap these usually agreed upon such that there is no net exchange of money at inception.*

A cross currency swap may of course be combined with a single currency swap so that a floating rate in one currency is exchanged for fixed interest payments in another currency or even to exchange a fixed interest rate in one currency for a fixed interest rate in another currency. Typically though we shall consider the vanilla case being an exchange of two floating interest rates.

Definition 4.3 (Quanto). *A* ***quanto*** *is a contingent claim with an underlying security in one currency (the foreign currency) but settlement in another currency (the domestic currency) at a predetermined exchange rate. At the settlement the derivatives value is calculated in the amount of foreign currency and then converted at a fixed rate into the domestic currency. Essentially, a quanto has an embedded currency forward with a variable notional amount (i.e. that of the payoff based on the payoff formula of the contingent claim). Quanto is short for "quantity adjusting option" as the quantity of the embedded FEC adjusts based on the contingent claim.*

Cross currency derivatives are of importance to us as they are a first introduction to multiple curves used for discounting or for forward rate estimation. Likewise they also gave rise to non-negligible interest rate basis. Further the concept of a *quanto adjustment* analogy shall be explored when considering the post-crisis yield curve frameworks as shown in multiple sources such as [6, 37, 42, 50, 57].

Interest Rate Parity

Interest rate parity is a classic no-arbitrage argument used to derive forward foreign exchange rates for the pricing of forward exchange contracts [38, 57]. Following our typical no arbitrage assumption and the martingale property of interest rates let us give a derivation of such forward exchange rates using interest rate parity.

Derivation 4.1 (Interest Rate Parity).

- *Let S_0 be the amount of currency D (the domestic currency) required, today (i.e. time t_0) to purchase one unit of currency F (the foreign currency) i.e. it is the prevalent foreign exchange rate.*
- *Let $L_D(t_0, T)$ be the prevailing simple rate of interest for a deposit of currency D starting today and maturing at time T. Likewise $L_F(t_0, T)$ is the prevailing simple rate of interest for a deposit of currency F starting today and maturing at time T.*

Consider the self-financing trading strategy ϕ which consists of borrowing S_0 amount of currency D to be repaid at time T with interest of $S_0 \, L_D(t_0, T)\tau(t_0, T)$

which is used to purchase a unit of currency F which is in turn deposited into an account till time T earning a rate of $L_F(t_0, T)$. At the same time consider the contingent claim H an FEC entered into at time t_0 being a contract to buy $[1+L_D(t_0, T)\tau(t_0, T)]S_0$ units of currency D for K amount of currency F.

Let S_T denote the exchange rate at time T. At maturity, time T, the value of ϕ in currency D is

$$[1 + L_F(t_0, T)\tau(t_0, T)]S_T - [1 + L_D(t_0, T)\tau(t_0, T)]S_0,$$

and the value of H is

$$[1 + L_D(t_0, T)\tau(t_0, T)]S_0 - KS_T.$$

As K is deterministic and both H and ϕ_1 require no outlay at time t_0 and for no arbitrage to hold they should neither generate nor require cash at time T if one enters into both ϕ and H simultaneously. Thus

$$\begin{aligned}[1 + L_F(t_0, T)\tau(t_0, T)]S_T &- [1 + L_D(t_0, T)\tau(t_0, T)]S_0 \\ &= [1 + L_D(t_0, T)\tau(t_0, T)]S_0 - KS_T,\end{aligned}$$

$$K = S_0 \frac{1 + L_D(t_0, T)\tau(t_0, T)}{1 + L_F(t_0, T)\tau(t_0, T)}. \tag{4.6}$$

Thus Equation (4.6) gives us the only possible forward exchange rate to be used in an FEC contract (with no exchange of money at inception) consistent with no arbitrage and is referred to as the fair forward exchange rate.

Introducing expectations under a numeraire we obtain from Equation (2.1), under martingale measure Q, with $V_{FEC}(t_0; T, K) = 0$ the value of an FEC as at time t_0 with strike price K at the forward exchange rate and maturing at time T.

$$V_{FEC}(t_0; T, K) = P(t_0, T)E^Q[K - S_T|\mathcal{F}_t], \tag{4.7}$$

$$\text{therefore} \quad E^Q[K - S_T|\mathcal{F}_t] = 0,$$

$$\text{therefore} \quad E^Q[S_T|\mathcal{F}_t] = K,$$

$$\text{therefore} \quad E^Q[S_T|\mathcal{F}_t] = S_0 \frac{1 + L_D(t_0, T)\tau(t_0, T)}{1 + L_F(t_0, T)\tau(t_0, T)}. \tag{4.8}$$

In other words the fair forward exchange rate is the expected value of the exchange rate at some future time T under a risk neutral measure. The forward exchange rate derived from interest rate parity relies on the default free interest rate that can be earned in the foreign and domestic currencies and by extension to the relevant risk-free curves derived for those currencies. Market quoted FEC rates though often differ from those implied by interest rate parity and result in a cross-currency basis. We would thus have that a market quoted price for an FEC is K_M which results in

$$K_M \neq K = S_0 \frac{1 + L_D(t_0, T)\tau(t_0, T)}{1 + L_F(t_0, T)\tau(t_0, T)}.$$

Market practice is typically to quote exchange rates against the USD (though there are numerous other conventions for specific currency pairs) and we shall treat USD in this case as the domestic currency and the "base" currency. Let $K_M(t_i)$ be the market FEC rate for an FEC beginning today and maturing at time t_i[3]. Keeping the domestic interest rate constant we could create an implied foreign interest rate $L_F^M(t_0; t_i)$, which is the interest rate implied by the market rate $K_M(t_i)$ by solving the following equation

$$\begin{aligned} K_M(t_i) &= S_0 \frac{1 + L_D(t_0, t_i)\tau(t_0, t_i)}{1 + L_F^M(t_0, t_i)\tau(t_0, t_i)}, \\ L_F^M(t_0; t_i) &= \left[\frac{S_0}{K_M(t_i)} \left(1 + L_D(t_0, t_i)\tau(t_0, t_i)\right) - 1 \right] \tau(t_0, t_i), \end{aligned} \tag{4.9}$$

which would give us a set of interest rates which together with some interpolation rule would be used to create a continuous implied FEC curve. The spread between this curve and the standard LIBOR curve would be the cross-currency basis. This curve would then be used to value FECs when calculating the expected forward exchange rate.

FECs though are not typically quoted for long tenors. For the long end of this curve, beyond 1 or 2 years, cross-currency swaps would be used in conjunction with FECs as the instruments from which the curve is derived. Again, typically, USD LIBOR would be the base interest rate against which

[3]Foreign exchange and FECs typically trade with a settlement day rule which means that a foreign exchange transaction will settle on a specific "spot date" that is a number of business days in the future from the transaction date. Typically this is in 2 business days or $t + 2$, a notable exception is Canadian Dollars which settle in 1 business day. Likewise an FEC with a tenor of two weeks would settle two weeks after the "spot date" or 12 business days after the transaction date (2 business days for spot plus 10 business days being the tenor of the FEC). For simplicity we shall treat today as the spot date and any curves would be treated as beginning from the spot date.

a cross-currency swap basis spread is quoted. For example 3-Month USD LIBOR would be exchanged for 3-Month JIBAR plus 50 basis points. Under the normal pricing assumptions using only a traditional LIBOR and JIBAR swap curve for valuing such a cross currency swap would imply an arbitrage. One would then calibrate an implied cross currency JIBAR curve to be used for valuing cross currency swaps and other cross-currency instruments. We shall not go into a mathematical derivation of such a curve but simply point out that a multiple curve framework before the liquidity crisis was in place in the context of cross-currency swaps and as such is not a completely foreign concept.

Tuckman and Porfirio [57] give an analogy of a cross-currency swap being a portfolio of three imaginary swaps: a cross-currency basis swap of overnight default-free rates; a money market basis swap of USD LIBOR for USD overnight default-free rates; and a money market swap for foreign LIBOR for foreign default free overnight rates[4]. From this perspective the quoted cross currency swap spread arises from the difference between two local basis spreads. These basis spreads are explored in greater detail in later sections but the market practice of using a different implied curve to value FECs and cross currency swaps is carried through to the post crisis multiple curve framework using multiple curves. This analogy ties in with the overnight curve discussed in Section 7.3.

[4]Overnight default-free curves are discussed in greater detail in Section 7.3.

Chapter 5

Post Crisis Multiple-Curve Framework

We shall show how, after the 2008 liquidity crisis, that the previously assumed no-arbitrage assumptions no longer apply and how the previously used framework breaks down. We then look at how a multiple-curve framework would arise. Two approaches are considered, a top-down axiomatic approach and bottom-up market related approach; under either approach it is shown that in a multiple-curve framework the assumption of no-arbitrage could be reclaimed. Finally we look at a quanto-style cross currency swap analogy and how the multiple curve approach is similar to that observable in the cross-currency swap market.

5.1 Market Divergence: Who let the arbitrage out?

Post the financial crisis much of the previous pre-crisis approach and theory outlined in the previous sections would no longer directly apply without some adjustment. Various prices quoted in the market were no longer consistent with the previously mentioned no-arbitrage assumptions, as shown in Section 4.2. Using the single curve approach to value an interest rate derivative would no longer result in the aforementioned market prices. Let us take a look at some USD LIBOR and FRA prices as at 2009-04-17.

As Figure 5.1 shows, the 3 Month USD LIBOR rate was 1.1019% and 3x6 FRA rate was 1.0890% while the 6 Month LIBOR rate was 1.6363%. Let us now calculate the 6 Month USD LIBOR rate as implied by the 3 Month

Market Quotes for Interest Rate Derivatives	
3 Month USD LIBOR	1.1019
6 Month USD LIBOR	1.6363
3x6 FRA	1.0890
6x9 FRA	1.2300
3x9 FRA	1.6230

Figure 5.1: Interest rate instrument quotes as at 2009-04-17, quoted as percentages. (source: Bloomberg Finance L.P.)

LIBOR rate and the 3x6 FRA:

$$\begin{aligned}
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= \left(1+\frac{\text{3M LIBOR}}{4}\right)\left(1+\frac{\text{3x6 FRA}}{4}\right),\\
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= \left(1+\frac{1.1019\%}{4}\right)\left(1+\frac{1.089\%}{4}\right),\\
\left(1+\frac{\text{6M LIBOR'}}{2}\right) &= 1.005484\ldots,\\
\text{6M LIBOR'} &= 1.096939934\ldots\%,
\end{aligned}$$

where 3M LIBOR refers to the 3 Month LIBOR rate and 6M LIBOR' refers to the implied no-arbitrage 6 Month LIBOR rates. Comparing this result to the quoted 6 Month LIBOR rate we see that there is a difference of approximately 54 basis points. This difference is much larger than those previously experienced in the market; as calculated in Section 4.2.

In the case of FRAs let us calculate the implied 3x9 FRA rate by taking the market quotes for a 3x6 and a 3x9 FRA as follows:

$$\begin{aligned}
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= \left(1+\frac{\text{3x6 FRA}}{4}\right)\left(1+\frac{\text{6x9 FRA}}{4}\right),\\
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= \left(1+\frac{1.089\%}{4}\right)\left(1+\frac{1.23\%}{4}\right),\\
\left(1+\frac{\text{3x9 FRA'}}{2}\right) &= 1.005805\ldots,\\
\text{3x9 FRA'} &= 1.16117\ldots\%,
\end{aligned}$$

where 3x9 FRA' refers to the implied no-arbitrage 3x9 FRA rate. This results in a spread, when compared to the quoted 3x9 FRA rate of 46 basis points;

again a dramatic increase from that previously experienced and illustrated in Figure 4.2.

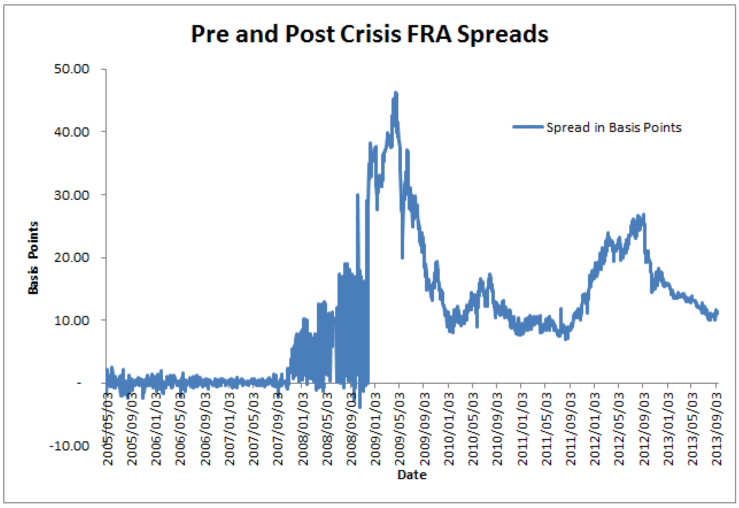

Figure 5.2: Pre and post 2008-crisis spreads between the quoted 3x9 FRA rate and that implied from 3x6 and 6x9 FRAs. (source: Bloomberg Finance L.P.)

The relationship which previously held true no longer did post the 2008 crisis. Figure 5.2 illustrates the spread between the quoted 3x9 FRA rate and that implied from 3x6 and 6x9 FRAs. From around September 2007 some erratic behaviour begins to appear, coinciding with the beginning of the subprime mortgage crisis, up until the end of September 2008, coinciding with the default and collapse of Lehman Brothers Holdings Inc. after which the spreads rapidly rose.

The single curve approach was no longer consistent with widening *basis spreads* quoted for basis swaps [6].

Definition 5.1 (Basis Swap)**.** *A **Basis Swap** is an interest rate swap where one floating interest rate is swapped for another floating interest rate with a different basis. For example 6 Month LIBOR payments could be exchanged for 3 Month LIBOR payments. One leg of the swap may have an added **basis spread** applied to it. For example 6 Month LIBOR could be exchanged for 3 Month LIBOR plus 0.1%.*

Under the assumption of no-arbitrage, using the framework in previous sections, the following derivation would hold.

Derivation 5.1 (Basis Swap Replication)**.** *Under the definition of LIBOR a highly rated bank would be able to lend and borrow at LIBOR. Thus a bank could borrow a unit of currency with repayments at one month frequency and pay 1 Month LIBOR; the same bank could then deposit that unit of currency with the original lender receiving interest at a three month frequency thus receiving 3 Month LIBOR. The cashflows for the bank under this strategy would match those entering into a basis swap where 1 Month LIBOR were exchanged for 3 Month LIBOR using our no-arbitrage assumption and based on Definition 2.5.*

This would imply that a basis swap should trade at zero or very little basis spread. Such basis spreads, though, were no longer negligible under the new market regime. The single curve approach did not account for the various sub areas of the interest market each with different dynamics (such as the short rate process) [6, 50]. Morini [50] evidences this showing how the replication FRA rate using forward rates derived from the single interest rate curve and market FRA rate had an average difference of 0.88bps (basis points, each bp being a percent of a percent, i.e. 0.000088) in the three years prior to the crisis (up to July 2007); the basis gap then exploded to an average of 50bps between August 2007 and May 2009 and at the same time the basis spread to swap 6 Month LIBOR for 12 Month LIBOR also exploded from a few basis points to an average of 40bps.

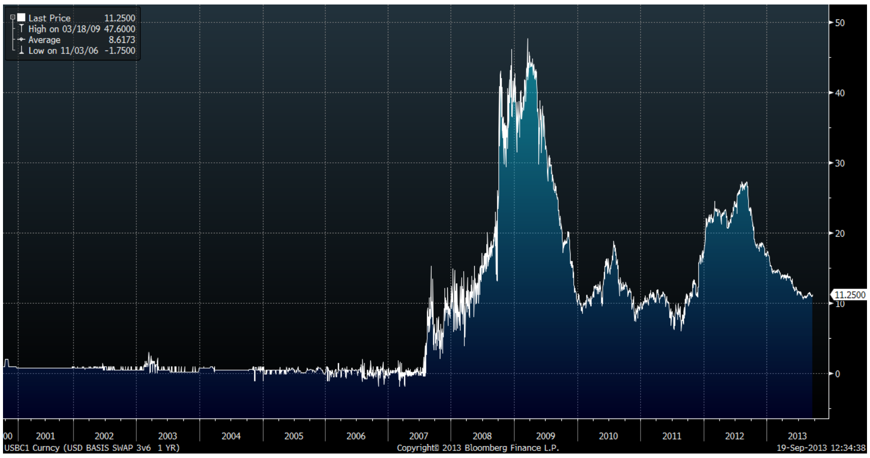

Figure 5.3: Pre- and post-2008 crisis basis swap quotes for 1 year maturity basis swaps exchanging 3 Month USD LIBOR for 6 Month USD LIBOR. (source: Bloomberg Finance L.P.)

This blowout of basis spreads is evident when looking at market quotes for basis swaps (swapping 3 Month USD LIBOR for 6 Month USD LIBOR) pre- and post- crisis as illustrated by Figures 5.3 and 5.4.

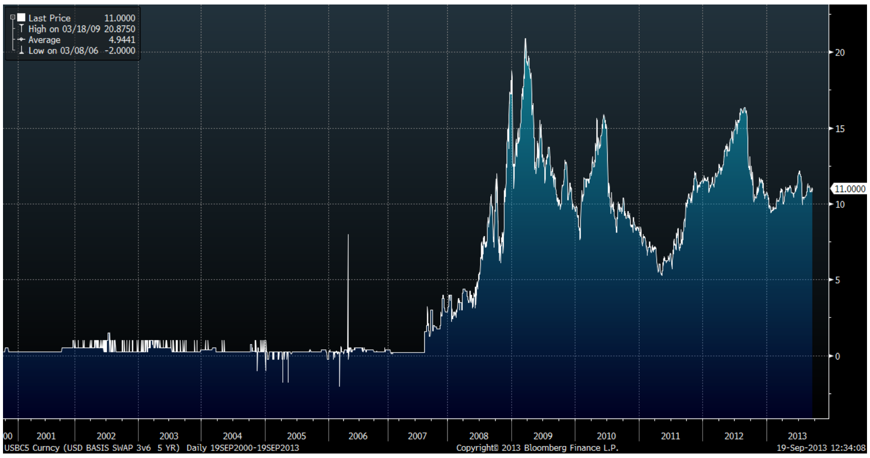

Figure 5.4: Pre- and post- 2008 crisis basis swap quotes for 5 year maturity basis swaps exchanging 3 Month USD LIBOR for 6 Month USD LIBOR. (source: Bloomberg Finance L.P.)

5.2 The cause of widening basis spreads

Various authors have delved into the possible causes and reasons behind these widening basis gaps. Michaud and Upper [47], in the Bank of International Settlements (BIS) quarterly review, advocate liquidity problems and credit concerns amongst banks as drivers of this basis gap. Morini [50] and Acerbi and Scandolo [1] give some insight into what is meant by Liquidity risk. Morini [50] gives the following definitions of Liquidity risk citing [1]:

1 Funding Liquidity Risk: the risk of running short of available funds.

2 Market Liquidity Risk: the risk of not being able to sell or purchase a particular security.

3 Systemic Liquidity Risk: the risk of global crisis where it is difficult or particularly expensive to borrow or acquire funding.

Morini [50] adds that each of the above elements can be overcome if a bank faces them in isolation. For example assets could be sold to raise funding in the presence of 1 but not 2 and 3. These various elements, along with the presence of credit risk, are particularly difficult to disentangle [47]. It is also not difficult to see that a bank's liquidity risk would be strongly correlated to bilateral credit risks (i.e. the risk of both a bank and its counterparties defaulting). The cost of funding and thus the price for liquidity would be directly related to a bank's risk of default (both as a cause and as a consequence [50]. One cannot easily consider or easily measure each risk in isolation though some possible models have been considered [50].

Under our previous replication strategies we have that the fixed rate payable under a FRA, K, should be equal to the implied forward rate for the same term based off our LIBOR or Swap curve as under Derivation 3.1. As shown in Figure 5.2, post-2008, the FRA rates were significantly different to those implied by the fair forward rates. This deviation would imply an arbitrage opportunity as one could enter into a 3x6 FRA to receive 3-month LIBOR and make a fixed payment of 1.1019%, simultaneously one could borrow money at the three month LIBOR rate for three months and deposit it for six months and make a positive gain of 54 basis points, as illustrated in Figure 5.1, without, seemingly, taking on any risk. This is not necessarily the case. As described above; one would face funding liquidity risk, market liquidity risk and systemic liquidity risk, each historically not being particularly large risks though through the crisis the impact of each in isolation and in concert increased to non-negligible levels. The possibility of either being unable to borrow money for another three months after the initial three month period or that the deposit counterparty may have defaulted (the impact of an interbank FRA counterparty defaulting is largely mitigated by collateralisation as under most interbank derivative contracts there would be a Credit Support Annexure or CSA). The divergence between the FRA rate and the implied forward rate may thus be seen as a measure or representation of the potential future credit or liquidity issues [46].

5.3 Defining Basis Spreads

We shall now look at defining some of the basis spreads one could typically encounter in the markets and which we shall deal with in later sections.

Definition 5.2 (Tenor Basis Spread)**.** *The* ***Tenor Basis Spread*** *can be defined as the spread between interest rate swaps with floating interest payments based on different tenors.*

In the absence of liquidity and counterparty risks leading to basis spread risk interest rate swaps with different underlying tenors should trade without any basis spread to preserve the original assumption of the absence of arbitrage.

In other words one would be indifferent between receiving 6-Month LIBOR or 3-Month LIBOR. Interest rate swaps and FRAs are, as mentioned earlier, typically traded under CSA agreements and counterparty risk is thus largely mitigated, they do, however, reference risky term deposit or borrowing rates in the form of unsecured LIBOR deposits [43]. We therefore encounter that the payer of the shorter tenor floating leg of an interest rate basis swap would need to pay large spreads to the payer of the longer term floating rate. When looking at the underlying deposits this is justified in that the lender requires a higher rate to compensate for the additional counterparty risk in lending for a longer period of time and conversely the borrower is willing to pay a higher rate for funding which would not carry the liquidity risk of not being available for extended borrowing should there be another liquidity crunch in the market [57]. To put it in other words: a 6-month deposit can no longer be treated as a 3-month deposit reinvested after the first 3-months with a rate fixed by a FRA. Likewise the expected forward interest rate for a 6-month deposit can no longer be implied by using 3-month FRA rates.

Definition 5.3 (Cross Currency Basis Spread)**.** *The **Cross Currency Basis Spread** is the spread between cross currency swaps with floating interest payments in different currencies.*

As for vanilla basis swaps, cross currency swaps should trade without any spread if both legs of the swap are valued using the same forward and discount curve for that particular currency. In practice though cross currency swaps trade with a spread over one of the reference floating rates.

Definition 5.4 (Bond Swap Basis Spread)**.** *The **Bond Swap Basis** is the spread between "risk-free" bonds (such as a sovereign bond or T-Bill) and interest rate swaps of the same tenor.*

Again if both the bond and the swap rate are assumed to be "risk-free" there should be no basis spread between "risk-free" bond rates and swap rates. Bond-Swap basis is explored in greater detail in Section 6.3.

5.4 Single Currency Multiple-Curve Construction

As the information now implied by basis swap spreads or by derivative (FRA, Swap) spreads when compared to LIBOR and cash instruments is now of material consequence and can no longer be considered as negligible the original single curve construction paradigm is no longer complete. The market is can now be seen as segmented into a number of sub-markets corresponding to different instrument types, counterparty risk assumptions and liquidity risk premia. Different tenors may now be seen as characterised by their own dynamics [6]. Various authors such as Bianchetti [6], Mercurio [46], Michaud [47], Morini [50], Tuckman [57] and others all agree that since cashflows of various interest rate derivative instruments such as FRAs and Swaps are all based off the interbank market with the aforementioned collateralisation being the norm that these cashflows should all be discounted off the same discount curve. This is consistent with an intuitive view that cash payments at future dates each with the same risk of occurring should have the same present value. To preserve the assumption of no-arbitrage two identical cashflows with the same probability of occurring must have the same present value [6]. Having a single discount curve would then imply that these future cashflows should then each be determined using a forward rate curve consistent with that cashflow's reference rate tenor.

To preserve the assumption of no-arbitrage when determining forward rates used to price, value and hedge instruments of a specific tenor they should be based off a curve created using vanilla instruments of the same underlying tenor. Bianchetti gives a brief procedure of how this is to be achieved and the following is a reprise from [6].

First

Build a single discount curve using the most liquid instruments reflective of each time period.

Second

Build multiple distinct forward curves $C_{f_1}, ..., C_{f_2}$ each using distinct sets of interest market rate instruments of the same underlying LIBOR tenor (e.g. 1 Month, 3 Month and 6 month); these can be built using the typical selected bootstrapping and interpolation procedures as described by Hagan and West in [32].

Third

For each relevant period forward rates are determined using the curves as defined in step two using the corresponding yield curve for that tenor.

$$F_f(t; t_1, t_2) = \frac{1}{\tau_f(t_1, t_2)} \left(\frac{P_f(t, t_1)}{P_f(t, t_2)} - 1 \right), \quad t_2 > t_1. \tag{5.1}$$

Fourth Cashflows are calculated based off the corresponding forward curve built using underlyings of the same tenor.

Fifth The relevant discount factors, $df(t_1, t_2)$ are calculated from the single discount curve.

Sixth The derivative's price at a given time is the sum of the discounted expected cashflows.

Seventh Hedging and interest rate sensitivities (deltas) are calculated by calculating the change in value of the instrument as a result of changes to each of the benchmark bootstrapping instruments for the relevant yield curves.

Unfortunately it is still not obvious as to which instruments should be considered as benchmark bootstrapping instruments when creating the discount curve C_d or which to use when creating the various yield curves of homogenous tenor $C_{f_1}, ..., C_{f_2}$. Further liquidity and availability of market rates for these instruments is not always readily available; such as in the South African market where 3-Month JIBAR swaps are the most prevalent but those of other tenors do not trade readily. A further example is that of the overnight yield curve, which may be constructed using overnight index swaps, which is a curve with an increasingly greater importance though the market for such swaps is not always liquid and with market data not always readily available (such as in South Africa where a similar equivalent, the Rand Overnight Deposit Swaps or RODS, is no longer readily traded). Such overnight yield curves have an increasingly important role when considering counterparty basis adjustment in the presence of daily collateralisations and shall be discussed later in the relevant parts of Section 7.2 on pricing with and without collateral.

Multiple curves each with their own market of bootstrapping benchmark instruments create pricing and hedging complexities as each instrument's price and change in price would now be sensitive to more than one source of market risk. Also the driving factors behind each spread adjustment is not always clearly defined or separate containing elements of counterparty risk as well as liquidity or funding risk.

5.5 Basis following an Axiomatic Approach

Henrard [37] has approached the problem of modelling basis using an axiomatic approach. This approach is also explored by Bianchetti and Morini

[6, 50]. The question of how cashflows linked to LIBOR related derivatives, especially in the presence of spreads was presented in [37] and further explored in [37]. Given what we have seen so far the question of what curve to use and how to estimate and discount LIBOR related cashflows had been much debated post the financial crisis. Before this crisis, such questions had been posed and analysed, but the implications had not been fully explored and the market in general was reticent to alter methodologies which would have accounting and legal implications should the standard approach be questioned or changed. Post the crisis though LIBOR had begun to be seen as an ambiguous notion (though it is explored by Michaud and Upper [47]), a rate which was set and fixed, but carried large basis spreads between tenors and was not necessarily a market related rate which a bank would be willing to take or give funding at (or the amount offered could be a rather small amount); this was further put under the spotlight recently with the "LIBOR scandal" which had been brought to light with various settlements being made by Barclays Bank. Further details were published by the press with reports having come in from the Financial Times, BBC, Reuters and others. Whilst there are many references to this scandal the reader is referred to the release (PR6289-12) by the Commodity Futures Trading Commission (CFTC) [18]. This approach attempts to offer a solution to this conundrum.

The approach proposed by Henrard [36, 37] is that of using a unique curve to discount all the cashflows regardless of the tenor and this approach differs somewhat from other multiple curve approaches and frameworks. Similar to other authors such as Pallavicini and Tarenghi [51] the problem is tackled by taking each forward rate as a single asset without individually modelling the dynamics between liquidity and credit risks. This approach is a top-down approach in that it begins by proposing a multiple forward curve, single discount curve framework which is then fitted to observations and trends in the market. Let us consider a discount curve and forward curve defined as:

Definition 5.5 (Discount Curve)**.** *The discount curve C_d is the curve given by the continuous set of discount factors $df(t,t_1)$ as defined in Definition 3.2.*

Definition 5.6 (Forward Curve)**.** *The forward curve C_f is the continuous function $P^j(t_1,t_2)$, analogous to formula 3.6, such that*

$$df(t,t_2)\cdot\left(\frac{P^j(t,t_1)}{P^j(t,t_2)}-1\right), t\leq t_1\leq t_2, \tag{5.2}$$

is the price at time t of the floating interest rate payment with beginning at time t_1 and maturity date t_2.

Henrard [37] notes that this curve definition differs to that of Mercurio [46], Kijima et al. [42], Ametrano and Bianchetti [6] but it is similar to that of Chibane and Sheldon [17].

It is given that the price of an interest rate swap is the discounted value of all the estimated future cashflows with the LIBOR forward rate being the estimation of such cashflows; as shown previously in the discussion of forward measures in Section 3.3. In that context under the risk neutral martingale measure Q then $F(t, t_1, t_2)$ (from Definition 3.10) is unbiased estimator of the future LIBOR rate $L(t_1, t_2)$. We can now define the LIBOR forward rate corresponding to a specific forward curve.

Definition 5.7 (LIBOR Forward Rate)**.**

$$F^j(t, t_1, t_2) = \frac{1}{\tau(t_1, t_2)} \left(\frac{P^j(t, t_1)}{P^j(t, t_2)} - 1 \right), \quad t \leq t_1 \leq t_2, \tag{5.3}$$

is the fair forward rate (or LIBOR forward rate) for the period beginning at time t_1 and maturing at time t_2 as determined at time t and corresponding to the forward curve.

The *cashflow equivalent* approach to pricing interest rate swaps is where a receiving floating rate leg of a swap is treated as paying the notional at inception (or just after a payment date) and receiving it at maturity. The value of this leg at inception or just after a payment was traditionally considered to be equal to the notional. To have a similar result in the new framework the following is defined

$$\beta_t^f(t_1, t_2) = \frac{P^j(t, t_1)}{P^j(t, t_2)} \frac{df(t, t_2)}{df(t, t_1)}, \quad t \leq t_1 \leq t_2, \tag{5.4}$$

which results in the price of a floating interest rate payment, as per the definition above, as

$$\begin{aligned} df(t, t_2) \cdot \left(\frac{fc(t, t_1)}{fc(t, t_2)} - 1 \right) &= df(t, t_2) \left(\beta_t^f(t_1, t_2) \frac{df(t, t_1)}{df(t, t_2)} - 1 \right) \\ &= \beta_t^f(t_1, t_2) df(t, t_1) - df(t, t_2), \end{aligned} \tag{5.5}$$

which is the equivalent of receiving $\beta_t^f(t_1, t_2)$ at time t_1 and paying 1 at time t_2.

Following from Proposition 2.2 from the earlier section on martingale measures, taking the LIBOR related interest rate payment which is an asset with value

$$\beta_t^f(t_1, t_2) df(t, t_1) - df(t, t_2),$$

and dividing it by the numeraire (as per Definition 2.9) $P(t,t_1) = df(t,t_1)$ is a martingale. Dividing $\beta_t^f(t_1,t_2)df(t,t_1) - df(t,t_2)$ by $df(t,t_1)$ gives us $\beta_t^f - 1$ and thus (discarding the deterministic last term) β_t^f is a martingale under $Q^{df(t,t_1)}$ since the price of any attainable claim (in this case a LIBOR payment) normalised by a numeraire U is a martingale under Q^U [29].

Having a separate discount curve and forward curve the swap rate (i.e. the rate at which a vanilla interest rate swap has a value of zero) would now be expressed as

$$S_t^j = \frac{\sum_{i=1}^{n} F^j(t,t_{i-1},t_i)\, df(t,t_i)\tau(t_{i-1},t_i)}{\sum_{i=1}^{n} df(t,t_i)\tau(t_{i-1},t_i)} \tag{5.6}$$

for an interest rate swap with a tenor corresponding to that of curve C_j and the value of a FRA, analogous to Equation (3.11), is

$$V_{FRA}^j = df(t,t_1)\frac{\tau(t_1,t_1+j)(F^j(t,t_1,t_1+j) - K)}{1+\tau(t_i,t_i+j)F^j(t,t_i,t_i+j)} \tag{5.7}$$

which differs slightly from the original formula as in this case the payment is treated as being made in advance at time t_i rather than at the end of the period, i.e. time $t_i + j$ and is thus discounted by the LIBOR rate for that period.

Treating the FRA as a contingent claim its value would give us

$$V_{FRA}^j = N \cdot E^{N_{t_1}^{-1}}\left[\frac{\tau(t_1,t_1+j)(L^j(t,t_1,t_1+j) - K)}{1+\tau(t_i,t_i+j)F^j(t,t_i,t_i+j)}\right] \tag{5.8}$$

for numeraire $N_{t_1}^{-1}$.

The forward curve P^j is not an asset as $df(0,t) = P^j(0,t)$ under any any numeraire N which is contrary to the framework of having multiple curves [36]. Henrard explores the following two propositions under the axiomatic approach in [37].

Proposition 5.1. *The multiplicative coefficient between discount factor ratios $\beta_t^j(u,u+j)$ as defined in Definition 5.4 is independent of the rate $df(t,u)/df(t,u+j)$.*

and the stricter proposition for the pricing of interest rate options.

Proposition 5.2. *The multiplicative coefficient between discount factor ratios $\beta_t^j(u,u+j)$ as defined in Definition 5.4 constant through time such that $\beta_t^j(u,v) = b_0^j(u,v) \quad \forall \quad t$ and u.*

Under these propositions β_t^j is a martingale for any $df(t,u)$ measure. Further, as per the change of numeraire technique described in previous sections and as described by Geman in [29], the changing of the numeraire does not affect this martingale property of β_t^j.

The spread implied by β_t^j is not constant across maturities but deterministic and given by its initial values [36, 37]. The spread is given by

$$\ln(\frac{\beta^j(u,v)}{(u-v)}),$$

under continuous compounding.

Theorem 5.1 (FRA). *Under Proposition 5.1 and that the LIBOR coupon and zero coupon bonds are assets, and thus suitable numeraires, the price of a FRA with tenor j as determined at time t with payment date t_1 and fixed rate K is*

$$\begin{aligned} V^j_{FRA} &= df(t,t_1)\frac{\tau(t_1,t_1+j)(F^j(t,t_1,t_1+j)-K}{1+\tau(t_1,t_1+j)F^j(t,t_1,t_1+j)} \\ &= df(t,t_1+j)\frac{\tau(t_1,t_1+j)(F^j(t,t_1,t_1+j)-K)}{\beta^j(t_1,t_1+j)}, \end{aligned} \tag{5.9}$$

achieved by algebraic substitution and rearranging of the above coefficients as well as the independence Proposition 5.1. Even if the floating rate payment is given under LIBOR and exists as a fixed given LIBOR rate one could still not price a FRA without the adjustment implied by the spread β_t^j. An interest rate swap is no longer a string of FRAs in a portfolio. The Proposition 5.1 justifies the decoupling of the discount curve and the forward curves [37] [46].

An at-the-money FRA, or market FRA, is a FRA where the fixed rate K is such that the FRA has a value of 0 as is usually the case of FRAs traded at their inception date. It thus follows, and referring to the previous derivation of the fair forward rate, Derivation 5.7, that we obtain

$$K = F^j(t_0,t_1,t_2) = \frac{1}{\tau(t_1,t_2)}\left(\beta^j(t_1,t_2)\frac{df(t_0,t_1)}{df(t_0,t_2)} - 1\right), \tag{5.10}$$

which coincides with the FRA fair rate obtained by Mercurio in [46] in the simple top-down multiple curve approach:

$$\beta^j = \frac{1}{R+(1-R)E[Q(t_1,t_2])}. \tag{5.11}$$

This spread can be linked to credit risk measures [24], liquidity risk measures [42, 46] or as is the case in the axiomatic approach the model parameters are fitted to the market curves, i.e. those spreads as quoted in the interbank market. Having defined our market instruments, their market related spreads and their equivalences under martingale measures the goal is to construct the multiple forward curves C_f and their functions $P^j(t_0, t)$ with the discount curve C_d. and its function $df(t_0, t)$ a given single curve. The approach differs to that of Kijima et al. [42] (one factor quadratic Gaussian model) which imposes a parameterised shape to the spread between curves. By taking $P^j(t_0, t_0) = 0$ and for the function $P^j(t_0, t)$ where t_0 is considered the present point in time then, with the current market price/fixing of LIBOR L_0^j the function $P^j(t_0, t)$ is such that

$$L_0^j = \frac{1}{\tau(t, t+j)} \left(\frac{P^j(t_0, t)}{P^j(t_0, t+j)} \right), \tag{5.12}$$

where t is the spot-date at which the LIBOR payment references and equivalent to the time $Spot(t)$ as used by Henrard in [37] and thus the curve would match the current fixing. The difference in this notation due to that in many markets there is a delay between the date on which LIBOR is fixed and the day on which the LIBOR deposit is settled, for example in the USD LIBOR market there is a two business day difference between these days. As such we consider t the initial settlement date and in the context of the South African JIBAR market this could be set to t_0. In theory the curve interval has no impact but in practice not all possible dates are used for curve construction and interpolation is relied upon [32] to fill the missing points there is an impact [37].

The instruments used in curve construction vary in liquidity in each market. In the LIBOR market it was common practice to use Eurodollar Futures in the short end of the curve due to their liquidity. In the South African market though FRAs have typically been much more liquid in the market with JIBAR futures (the South African equivalent of Eurodollar Futures with some minor settlement and quotation differences) being much less liquid. As such the use of FRAs has been the mainstay in South African curve construction and have recently gained greater popularity in overseas markets as a way to obtain information for tenors greater than 3 months [37].

A set of FRAs of tenor j are selected and are typically selected up until the further maturity dates where liquidity decreases and there is increased liquidity in interest rate swaps. A reminder on FRA notation: a 1x4 FRA is the FRA, at time t_0, which references a floating rate beginning in 1 month time and maturing in 4 months time (i.e. having a tenor of 3-months and for the ease of notation we shall set j to be the time in months, i.e. the 1x4 FRA has $j = 3$ which is a tenor of 3 months. These FRAs are sorted in order of increasing maturity with start dates t_i and end date T_i, also referred to as the near maturity and the far maturity. Each of those FRAs would have an equivalent market quoted fair forward rate K_i^j and this is considered as a known value. The value of such a FRA is zero with $K_i^j = F^j(t_0, t_i, T_i)$ and $T_i - t_i = j$.

Following the no-arbitrage assumption and that the value of an initial zero coupon bond of tenor j, $P^j(0, t_i)$, is known we can determine, from the value $V_{FRA}^j(t_0, t_i, T_i)$, the unknown value of a zero coupon bond $P^j(0, T_i)$. This initial zero coupon bond may be our LIBOR deposit and is equivalent to L_0^j.

For maturities usually exceeding 2 years (depending on liquidity of market traded instruments and thus the reliability of market quotes) interest rate swaps are substituted for FRAs in the curve. Typically interest rate swaps are quoted in a single liquid tenor; often referred to as the leading swap. To obtain market information for other tenors basis swaps would need to be used (that is an interest rate swap which exchanges floating interest rate payments of tenor for that of another tenor, i.e. of a different basis). Where there exist liquid swaps of the various tenors those may be preferable. The equivalence of a basis swap in conjunction with that of a vanilla interest rate swap to that of a vanilla interest rate swap with a different tenor is not discussed by many authors but may easily be shown via the following derivation:

Derivation 5.2. *Consider portfolio A consisting of a market at-the-money vanilla interest rate swap (receiving the floating rate) with maturity date T and cashflow dates t_i referencing LIBOR of tenor j such that $t_{i+1} - t_i = j$ and with fixed rate payments K_T^j. Further consider portfolio B which consists of a market at-the-money vanilla interest rate swap (receiving the floating rate) with the same maturity date T but with a tenor of h and payment dates t_k and with payments based on the fixed rate K_T^h; further portfolio B also consists of a market at-the-money basis swap which exchanges floating rate cashflows of tenor j (receiving this rate) for cashflows of tenor h plus some spread S (note that S may be a negative spread though typically we would not have $K^j + S \leq 0$). Thus portfolio A and portfolio B would have the exact same incoming cashflows on each of the dates t_i; typically the payments of the shorter tenor rate would be compounded to match the payment dates to that of the longer tenor (in this case we may arbitrarily assign j to be the longer tenor with corresponding dates t_i. Thus not to have an arbitrage opportunity* *between the two portfolios the present value of each of the payments* K_i^j *and those of* $K^h - S$ *must be equal. Thus if one of the rates* K_T^j *or* K_T^h *is known we can derive what the other rate should be under risk neutral measures and the assumption of no-arbitrage. For either swaps the discount curve* C_f *would be identical but the actual derivation of this curve would be arbitrary as the equivalence would remain (more specifically this curve could be based off the liquid leading tenor or off of overnight rates as is currently being advocated for interest rate swaps in the presence of collateral).*

Also of consideration would be that under a risky counterparty framework where the swaps in either portfolio face a different counterparty we would introduce a new spread, that of the specific counterparty and not just that arising due to the market related factors of each tenor. This is mitigated in the interbank market due to the presence of collateral in the interbank market and for the purposes of the derivation the swaps should be considered trades between equally risky counterparties.

As for the FRAs the swaps are then also sorted by increasing maturities (with overlapping generally occurring between the last few FRAs and the first interest rate swap). The curve is then constructed so that the value of a market at-the-money interest rate swap is zero. The value of such a swap is given by Equation (5.6). Interpolation techniques and bootstrapping techniques discussed by Hagan and West in [32] could then be extended to the multiple curve approach.

This forward curve, C_f is important only in that it gives the ratio between different values of P_t^j for different times t. The value of P_t^j on its own is not a market related rate; it is not a tradeable instrument. The curve up to the first point may appear arbitrary but it is of significant importance as it would most likely be used to interpolate other points in the curve; Henrard highlights this in [37] whilst also noting that this is not broached by Chibane and Sheldon in[17] and Mercurio [46].

Henrard [37] proposes an approach to pricing contingent claims under the Proposition 5.2 of deterministic spreads. This could be extended to the purposes of the risk modelling of these spreads, the β ratios, under well known models. Taking a standard model for the discount curve, such as that of the Brace, Gatarek and Musiela model (the LIBOR Market Model) [12] which is extended in the LogNormal LIBOR Market Model and the Extended LIBOR Market Model of Mercurio in [46]. Henrard does not apply a specific market model or extend any of the existing models. The approach to contingent claims under the axiomatic framework is to extend the cashflow equivalent approach of valuing swaps to value contingent claims on the Swap or FRA rate applying a model which models the term structure of the discount curve C_d; stochasticity in β is not approached but the spreads are considered under the Black model and the SABR/CEV model. These market models are discussed in greater detail in Chapter 6.

5.6 Bottom-up Market Related Approach

Following the above axiomatic approach is an approach which acts as a corollary, it is a bottom-up approach as proposed by Ametrano in [2] using forward basis to recover the no-arbitrage assumption and extended further by Bianchetti in [6] who adds a foreign currency, or quanto, type analogy to prevent implied arbitrage. Curve construction follows the approaches previously described as far as methodology goes in that it uses a discounted cashflow model based off of multiple distinct forward curves, $C_{f_1}, ..., C_{f_n}$. In this approach the cashflows c_i at each time period, t_i, are computed as the expectation at time t_0 of the corresponding interest rate payments $\pi_i(F_f)$ under the forward measure $Q_d^{t_i}$ corresponding to the forward curve C_f and the discount curve C_d associated with the numeraire $P(t_0, t_i)$ which results in

$$c_i = c(t_0, T_i, \pi_i) = E_t^{Q_d^{t_i}}[\pi_i(F_f)]. \tag{5.13}$$

Being a bottom-up approach it begins with the current market practice of using multiple yield curves [2, 57] and extending it to recover the no-arbitrage assumption. This begins with a market segmentation which treats the interest rate market as a set of sub-markets corresponding to instruments of different tenors. Whilst there appears to be a general market consensus on the creation of multiple curves based on the underlying there is no consensus on what should be a unique standard discount curve. Of the various approaches the most encountered are that of either the original pre-crisis approach of multiple instruments based on the liquidity of each tenor there is also the approach of basing this discount curve off of overnight rates such as that based off of EONIA swaps, Overnight Index Swaps (OIS) or in the South African context Rand Overnight Deposit Swaps (unfortunately no longer actively traded). The merits of the overnight type curve are discussed in the section of approaching the pricing of interest rate derivatives under collateral which is typically funded overnight: this still leaves open though the question of what is appropriate for valuing interest rate swaps where there is no collateralising or the complexities where the collateral may be in a currency other than that of the base swap (as would be encountered in South Africa when collateralising interest rate derivatives with a foreign bank which would be done in USD or EUR even for a ZAR based swap). Nonetheless there would exist some discount curve C_d which would need to be the same for each interest rate submarket.

Ametrano and Bianchetti define a set of N yield curves, C_f which we shall carry through from Henrard [37] and which extend to Morini [50] and Kijima et al. [42] and other authors. These curves, in this approach, are treated as a continuous term structure of discount factors,

$$C_f^P = T \rightarrow P_f(t_0, T)T \geq t_0, \tag{5.14}$$

where [2] uses the superscript P to stand for the discount curve, C_d which is the same as that used in the previous section and $P_f(t_0, T)$ is a C_X^P zero coupon bond with price at time t_0, which is today, maturing at time T such that $P_f(T, T) = 1$. We can place this in contrast to the previous approach where P^f was an arbitrary function which fulfilled equality in Definition 5.6. We shall continue to use the superscript to distinguish between these definitions.

Continuously compounded zero coupon rates $z_x(t_0, T)$ and the simple compounded instantaneous forward rates $f_{x,t_0,T}$ (which were previously considered under Equation (3.14) under Proposition 3.1) are such that

$$\begin{aligned} P_f(t_0, T) &= exp[-z_x(t_0, T)\tau(t_0, t)] \\ &= exp\left[-\int_{t_0}^{T} f_{x,t_0,u}\, \mathrm{d}u\right], \end{aligned} \tag{5.15}$$

which [2] extends to the equivalent log notation,

$$\begin{aligned} \ln P_x(t_0, T) &= -z_x(t_0, T)\tau(t_0, T) \\ &= -\int_{t_0}^{T} f_{x,t_0,T}\, \mathrm{d}u. \end{aligned} \tag{5.16}$$

The following observations, based on the above relationships, are taken from [2]:

1. $z_x(t_0, T)$ is the average of $f_{x,t_0,T}$ over $[t_0, T]$;

2. If interest rates are non-negative then the above log function is a monotone non-increasing function of T such that $0 < P(t_0, T) \leq 1 \ \forall T > t_0$;

3. The instantaneous forward curve C_f is the most severe indicator of yield curve smoothness since anything else is obtained through its integration and therefore smoother by construction.

We can now define the following rate curves which are associated to C_f^P

$$\begin{aligned} C_f^z &= \{T \to z_f(t_0, T), T \geq t_0\}, & (5.17) \\ C_f^x &= \{T \to f_{f,t_0,T}, T \geq t_0\}, & (5.18) \end{aligned}$$

the zero coupon yield curve and the instantaneous forward rate curve [1] where

$$\begin{aligned} z_f(t_0, T) &= -\frac{1}{\tau(t_0, T)} \ln P_f(t_0, t), & (5.19) \\ f_{f,t_0,T} &= -\frac{\partial}{\partial t} \ln P_f(t_0, t)|_{t=T} \\ &= z_f(t_0, T) + \frac{\partial}{\partial t} z_f(t_0, t)|_{t=T} \tau(t_0, T). & (5.20) \\ & & (5.21) \end{aligned}$$

To preserve no-arbitrage the following relationship between discount factors (or zero coupon bond prices) hold

$$P_f(t, t_2) = P_f(t, t_1) \cdot P_f(t, t_1, t_2), \quad \forall \quad t_0 \leq t \leq t_1 \leq t_2, \tag{5.22}$$

where $P_f(t, t_1, t_2)$ is as defined in the earlier Proposition 3.1 being the value at time t of a zero coupon bond starting at time t_1 and maturing at time t_2 in this case though this zero coupon bond is of tenor corresponding to the forward curve C_f. Equation (3.13) under Proposition 3.1 could further be extended as

$$P_f(t, t_1, t_2) = \frac{P_f(t, T_2)}{P_f(t, T_1)} = \frac{1}{1 + F_f(t, t_1, t_2)\tau(t_1, t_2)}, \tag{5.23}$$

where now $F_f(t, t_1, t_2)$ is the fair forward rate, as measured at time t for the period beginning at time t_1 and ending at time t_2 now under the submarket corresponding to tenor f and the forward curve C_f. We would thus obtain

$$\begin{aligned} F_f(t, t_1, t_2) &= \frac{1}{\tau(t_2, t_1)} \left[\frac{1}{P_f(t, t_1, t_2)} - 1 \right] \\ &= \frac{P_f(t, t_1) - P_f(t, t_2)}{\tau(t_2, t_1) P_f(t, t_2)}, & (5.24) \end{aligned}$$

[1] Here the superscript and subscript in C_f^x are switched around from that which the author uses in [2]; this is to be consistent with the previous section where f was used as a subscript to denote market curve C_f whereas Ametrano and Bianchetti use x. Here f is used to denote the submarket curve C_f and the superscript x to define it as the instantaneous forward rate curve.

which corresponds to the original definition given for a simple compounding forward interest rate as was defined in Definition 3.10 but now under the submarket corresponding to C_f.

We could thus express the value of a fair swap rate $K_f(t, \mathbf{t}, \mathbf{s}$ of a swap with maturity T with floating rate payment dates $\mathbf{s} = \{s_0, ..., s_m\}$ paying the LIBOR rate corresponding to tenor j on the date s_j with reference LIBOR fixing date s_{j-1} and fixed leg payments paying the fixed rate on payment dates $t_i, \quad i = 0, ..., n$ and where $\mathbf{t} = \{t_0, ...t_n\}$ as

$$\begin{aligned} K_f(t, \mathbf{t}, \mathbf{s}) &= \frac{\sum_{j=1}^{m} P_f(t, s_j)\tau(s_{j-1}, s_j)F_f(t, s_{j-1}, s_j)}{A_f(t, \mathbf{t})} \\ &= \frac{P_f(t, t_0) - P_f(t, t_n)}{A_f(t, \mathbf{t})}, \end{aligned} \tag{5.25}$$

where

$$A_f(t, \mathbf{t}) = \sum_{i=1}^{n} P_f(t, t_i)\tau(t_{i-1}, t_i) \tag{5.26}$$

is an annuity for curve C_f.

Ametrano and Bianchetti [2] note that in Equation (5.25) they use the telescopic property of summation and that this would only hold true if each forward rate end date equals the next forward rate start date with no gaps or overlaps and that this is not always the case due to business day and spot/settlement date conventions in the various X-BOR markets and that in practice the error is small and of order 0.1 basis points. In the JIBAR market there is no spot/settlement day difference between each period so the telescopic property holds true.

The bootstrapping and interpolation of our set of curves follow the methodologies previously outlined; again the main contrast to this approach is the segmentation of the various markets without (yet) reconstituting them nor defining the equivalent spread measure β used in the axiomatic approach. To address this a foreign currency **quanto** (defined in Definition 4.3) analogy is used by Bianchetti in [6], Chibane and Sheldon in [17] and Kijima et al. in [42] and this shall now be addressed.

5.6.1 Reclaiming No-Arbitrage Assumptions

It can now be shown the assumption of no-arbitrage is broken under this framework, the pre-crisis single curve no-arbitrage relations simply do not hold any longer. We can show that, from [6, 17, 42] that now

$$
\begin{aligned}
P_f(t,t_1,t_2) = \frac{P_f(t,t_2)}{P_f(t,t_1)} &= \frac{1}{1+F_f(t,t_1,t_2)\tau(t_1,t_2)} \\
&\neq \frac{1}{1+F_d(t,t_1,t_2)\tau(t_1,t_2)} \\
&= \frac{P_d(t,t_2)}{P_d(t,t_1)} = P_d(t,t_1,t_2). \qquad (5.27)
\end{aligned}
$$

Or simply put the fair forward rate as determined by the market related discount curve C_d does not correspond to that of C_f and unlike the axiomatic approach, where the forward curve is not equivalent to market instruments but rather some arbitrary function, under this approach no assumption is made that the zero coupon bond prices corresponding to C_f are not market instruments. To recover the assumption of no-arbitrage forward basis is to be taken into account via a riskiness measure on the forward counterparty for basis swaps

$$
P_f(t,t_1,t_2) = \frac{1}{1+F_d(t,t_1,t_2)BA_{fd}(t,t_1,t_2)\tau(t_1,t_2)}, \qquad (5.28)
$$

or from the transformation of forward rates [6]

$$
F_f(t,t_1,t_2)\tau(t_1,t_2) = F_d(t,t_1,t_2)\tau(t_1,t_2)BA_{fd}(t,t_1,t_2), \qquad (5.29)
$$

where $BA_{fd}(t,t_1,t_2)$, to use the notation from [6], is a function of the forward basis between C_f and C_d. Rearranging Equation (5.29) we obtain

$$
\begin{aligned}
BA_{fd}(t,t_1,t_2) &= \frac{F_f(t,t_1,t_2)\tau(t_1,t_2)}{F_d(t,t_1,t_2)\tau(t_1,t_2)} \\
&= \frac{P_d(t,t_2)P_f(t,t_1) - P_f(t,t_2)}{P_f(t,t_2)P_d(t,t_1) - P_d(t,t_2)}, \qquad (5.30)
\end{aligned}
$$

which expresses the forward basis as a ratio between forward rates or in terms of discount factors of C_d and C_f. This is similar in concept to the ratio β under the axiomatic approach though the two are not equal. Here the forward basis is a multiplicative term but it could also be defined as an additive term BA'_{fd}:

$$\begin{aligned}
P_f(t,t_1,t_2) &= \frac{1}{1+(F_d(t,t_1,t_2)+BA'_{fd}(t,t_1,t_2))\tau(t_1,t_2)},\\
BA'_{fd}(t,t_1,t_2) &= \frac{[F_f(t,t_1,t_2)-F_d(t,t_1,t_2)]\tau(t_1,t_2)}{\tau(t_1,t_2)}\\
&= \frac{1}{\tau(t_1,t_2)}\left(\frac{P_f(t,t_1)}{P_f(t,t_2)}-\frac{P_d(t,t_1)}{P_d(t,t_2)}\right)\\
&= F_d(t,t_1,t_2)[BA_{fd}(t,t_1,t_2)-1],
\end{aligned} \tag{5.31}$$

and we may note that under the pre-crisis frameworks $BA_{fd}(t,t_1,t_2)=1$ and $BA'_{fd}(t,t_1,t_2)=0$ following from $C_f=C_d$

For the sake of simplicity the year fraction, or coverage from Definition 3.3, $\tau(t_1,t_2)$ is assumed to follow the same daycount convention under both C_d and C_f though this is not a requirement; Bianchetti [6] distinguishes between $\tau_f(t_1,t_2)$ and $\tau_d(t_1,t_2)$.

Curves can now be bootstrapped from a given yield curve plus a given forward basis using the recursive relations:

$$\begin{aligned}
P_d(t,t_i) &= \frac{P_f(t,t_1)BA_{fd}(t,t_{i-1},t_i)}{P_f(t,t_{i-1})-P_f(t,t_i)+P_f(t,t_i)BA_{fd}(t,t_{i-1},t_i)}P_d(t,t_{i-1})\\
&= \frac{P_f(t,t_i)}{P_f(t,t_{i-1})-P_f(t,t_i)BA'_{fd}(t,t_{i-1},t_i)\tau(t_{i-1},t_i)}P_d(t,t_{i-1}),
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
P_f(t,t_i) &= \frac{P_d(t,t_i)}{P_d(t,t_i)+(P_d(t,t_{i-1})-P_d(t,t_i))BA_{fd}(t,t_{i-1},t_i)}P_f(t,t_{i-1})\\
&= \frac{P_d(t,t_i)}{P_d(t,t_i)+P_d(t,t_{i-1})BA'_{fd}(t,t_{i-1},t_i)\tau(t_{i-1},t_i)}P_f(t,t_{i-1}),
\end{aligned} \tag{5.33}$$

which can be used iteratively given a yield curve up to step i and the forward basis for the step $i-1$ to i. A similar algorithm is described by Kijima et al. in [42].

Bianchetti [6] further observes that once a "*smooth and robust*" bootstrapping technique for yield curve construction is used the term structure for forward interest rate basis curve provides a sensitive indicator for the tiny yet observable differences in the different tenor based interest rate sub-markets and provides a tool for the assessment of the degree of liquidity and credit risks contained in the various derivatives' prices.

5.6.2 Quanto Style Cross Currency Swap Analogy

Bianchetti [6], Kijima et al. [42], Michaud and Upper[47] and Tuckman and Porfirio [57] make similar observations regarding interest rate basis; that it can be explained and defined in a manner similar to that observable in a multi-currency model using a quanto-style cross-currency swap analogy, similar to that as described by Boenkost and Schmidt [11]. Let us first consider the generalised FRA price under the multiple curve framework

$$V_{FRA}(t;t_1,t_2,K,N) = Ndf(t,t_2)\tau(t_1,t_2)\left[E_t^{Q_d^{t_2}}\left[L_f(t_1,t_2)-K\right]\right], \quad (5.34)$$

compared to the simple market practice where

$$V_{FRA}(t;t_1,t_2,K,N) = Ndf(t,t_2)\tau(t_1,t_2)\left[F_f(t;t_1,t_2)-K\right], \quad (5.35)$$

where the forward rate F_f is seen as an unbiased estimator of the future LIBOR fixing, L_f, of tenor f.

The forward rate $F_f(t;t_1,t_2)$ is not in general a martingale under $Q_d^{t_2}$ and thus the simple market practice discards adjustments coming from the measure mismatch [6]. To calculate the expectation in Equation (5.34) it would require modelling the dynamic properties of the two interest rate curves under C_f and C_d which is accomplished by introducing the cross-currency swap analogy [6, 42] using a quanto style adjustment used for cross currency swaps as evidenced by Tuckman and Porfirio (using credit risk inherent in a currency as the driver) in [57] and modelled further by Boenkost (using a liquidity premium as the driver) in [11].

Some of the base instruments in a multiple currency framework as well as the concept of interest rate parity and its importance regarding no-arbitrage was discussed in Section 4.4 and those concepts carry through. No-arbitrage in the double currency double curve framework requires the existence of a spot and forward exchange rate (as shown in the definition for a Forward Exchange Contract, or FEC in Definition 4.1) and we have that

$$\begin{aligned} c_d(t) &= x_{fd}(t)c_f(t), \\ X_{fd}(t,t_1)P_d(t,t_1) &= x_{fd}(t)P_f(t,t_1), \end{aligned} \quad (5.36)$$

using the notation from [6] and the subscripts d and f refer to the domestic and foreign currency respectively (and which may be seen as analogous to the discount and forward rate in the single currency framework). $c_d(t)$ is a cashflow in the domestic currency at time t and $c_f(t)$ is the corresponding foreign currency cashflow. $x_{fd}(t)$ is the exchange rate at time t and $X_{fd}(t, t_1)$ is the forward exchange rate at time t_1 as calculated at time t and K in the calculation of an FEC and under interest rate parity in Derivation 4.1 is seen as the fair forward exchange rate at time t_1 (analogous to the fair forward interest rate). The derivation of the two equations follow interest rate parity under Derivation 4.1 and the equations must hold under the assumption of no-arbitrage.

Taking this analogy to the double curve single currency framework we take d and f to refer to the discount and forward curve and that the spot exchange rate must be 1, i.e. $x_{fd}(t) = 1$ resulting in

$$X_{fd} = \frac{P_f(t, t_1)}{P_d(t, t_1)}, \tag{5.37}$$

which is analogous to the forward interest rate basis. This can be substituted into Equation (5.29) which depicts our forward rate basis as a ratio of discount factors off of C_d and C_f and results in [2]

$$BA_{fd}(t, t_1, t_2) = X_{fd}\frac{P_d(t, t_1) - P_d(t, t_2)}{P_d(t, t_1)X_{fd}(t, t_1) - P_d(t, t_2)X_{fd}(t, t_1)}. \tag{5.38}$$

Standard market practice is to assume the lognormal martingale dynamic for the curve, C_f, of foreign currency forward rates

$$\frac{\mathrm{d}F_f(t, t_1, t_2)}{F_f(t, t_1, t_2)} = \sigma_X(t)\,\mathrm{d}W_f^{t_2}(t), \quad t \leq t_1, \tag{5.39}$$

where $\sigma_f(t)$ is the volatility of the process under the probability space $(\Omega, \mathcal{F}^f, Q_f^{t_2})$ with the filtration $\mathcal{F}_t^f$ generated by the Brownian motion $W_f^{t_2}$ under the foreign currency forward t_2 measure $Q_f^{t_2}$ associated to the C_f numeraire $P_f(t, t_2)$ [6].

It follows from Proposition 2.2 that since $X_{fd}(t, t_2)$ is the ratio between the price at time t of a tradeable asset and the C_d numeraire $P_d(t, t_2)$ it is a

[2]Bianchetti in [6] also multiplies the below equation by a ratio $\frac{\tau_f}{\tau_d}$ catering for the case where the two curves use differing daycount conventions.

martingale process under the forward t_2 measure $Q_d^{t_2}$ such that

$$\frac{\mathrm{d}X_{fd}(t,t_2)}{X_{fd}(t,t_2)} = \sigma_X(t)\,\mathrm{d}W_X^{t_2}(t), \quad t \leq t_2, \tag{5.40}$$

where $\sigma_X(t)$ is the volatility of the process and $W_X^{t_2}$ is Brownian motion under $Q_d^{t_2}$ such that [6]

$$\mathrm{d}W_f^{t_2}(t)\,\mathrm{d}W_X^{t_2}(t) = \rho_{fX}(t)\,\mathrm{d}t. \tag{5.41}$$

In calculating FRA rates a cashflow under C_f needs to be transformed to a corresponding cashflow under C_d. This is accomplished by changing the numeraire following Definition 2.10 and the formulae given by [13] in the *Change of Numeraire Toolkit* to obtain the dynamics of $F_f(t,t_1,t_2)$ under $Q_d^{t_2}$

$$\begin{aligned} \frac{\mathrm{d}F_f(t,t_1,t_2)}{F_f(t,t_1,t_2)} &= \mu_f(t)\,\mathrm{d}t + \sigma_f(t)\,\mathrm{d}W^(t_2)_d, \quad t \leq t_1, && (5.42) \\ \mu_f(t) &= -\sigma_f(t)\sigma_X(t)\rho_{fX}(t), && (5.43) \end{aligned}$$

where the forward rate dynamic now has a non-zero drift $\mu_f(t)$ and $F_f(t_1,t_1,t_2)$ is lognormally distributed under $Q_d^{t_2}$ with a mean and variance respectively as

$$\begin{aligned} E_t^{Q_d^{t_2}}\left[\ln\frac{F_f(t_1,t_1,t_2)}{F_f(t,t_1,t_2)}\right] &= \int_t^{t_1}\left[\mu_f(u) - \frac{\sigma_f^2(u)}{2}\right]\mathrm{d}u, && (5.44) \\ Var_t^{Q_d^{t_2}}\left[\ln\frac{F_f(t_1,t_1,t_2)}{F_f(t,t_1,t_2)}\right] &= \int_t^{t_1}\sigma_f^2(u)\,\mathrm{d}u. && (5.45) \end{aligned}$$

This allows us to define our (multiplicative) Quanto Adjustment $QA_{fd}(t,t_1,\sigma_f,\sigma_X,\rho_{fX})$ which we shall denote as $QA_{fd}(t,t_1)$ for ease of notation

$$\begin{aligned} QA_{fd} &= exp\left[\int_t^{t_1}\mu_f(u)\,\mathrm{d}u\right], \\ &= exp\left[-\int_t^{t_1}\sigma_f(u)\sigma_X(u)\rho_{fd}(u)\,\mathrm{d}u\right], && (5.46) \end{aligned}$$

which can be substituted into the expectation of the spot LIBOR or simple interest rate to give the result

$$E_t^{Q_d^{t_2}}\left[F_f(t_1,t_1,t_2)\right] = F_f(t,t_1,t_2)QA_{fd}(t,t_1). \tag{5.47}$$

Parallel to the additive basis adjustment BA'_{fd} we could also define the additive quanto adjustment,

$$QA'_{fd}(t, t_1) = F_f(t, t_1, t_2)\left[QA'_{fd}(t, t_1) - 1\right], \tag{5.48}$$

with the expected spot LIBOR rate as

$$E_t^{Q_d^{t_2}}\left[F_f(t_1, t_1, t_2)\right] = F_f(t, t_1, t_2) + QA'_{fd}(t, t_1). \tag{5.49}$$

By combining Equations (5.47) and (5.49) with the equations for the forward basis adjustment, Equations (5.30) and (5.31), for the multiplicative and additive adjustments respectively the following relationship between the quanto adjustment and the forward adjustment is obtained: [6]

$$\frac{BA_{fd}(t, t_1, t_2)}{QA_{fd}(t, t_1)} = \frac{E_t^{Q_d^{t_2}}[L_d(t_1, t_2)]}{E_t^{Q_d^{t_2}}[L_f(t_1, t_2)]}, \tag{5.50}$$

$$BA'_{fd}(t, t_1, t_2) - QA'_{fd}(t, t_1) = E_t^{Q_d^{t_2}}[L_d(t_1, t_2)] - E_t^{Q_d^{t_2}}[L_f(t_1, t_2)]. \tag{5.51}$$

This analogy allows the computation of the expected value of the forward rates from curve C_f under the measure $Q_d^{t_2}$. The volatilities and correlations arising in Equation (5.46) may be estimated from market data from either quoted volatilties or from historical data. The dynamics of the basis and quanto adjustment are further explored in the next chapter, Chapter 6, which shall introduce two short rate models constructed by a quadratic Gaussian model and by the Vasicek model as proposed by Kijima et al. in [42], a Black Model and a SABR (Stochastic Alpha, Beta, Rho) as discussed by Henrard [37] and Pallavicini and Tarenghi [51], and extensions of the LIBOR Market Model into a LogNormal LIBOR Market Model and an Extended LIBOR Market Model proposed by Mercurio in [46].

Chapter 6

Post-Crisis Market Models

The traditional pre-crisis market model would now need to be adapted to include measures for the aforementioned basis spreads. We shall take a look at the inclusion of the spread factor modelled under a Black Model and under a Stochastic Alpha Beta Rho(SABR) model as proposed by Henrard in [37]. Following that Kijima et al. [42] extend a model introduced by Boenkost and Schmidt [11] to model three different curves in one currency under a stochastic interest rate environment under a no-arbitrage paradigm. Kijima et al.[42] put forward two short rate models to derive closed form formulae for the three cures, the first a quadratic Gaussian model and the second by the Vasicek model which we shall explore. Finally we shall explore an extension to the LIBOR Market Model, as proposed by Mercurio [46], under a multiple curve framework.

6.1 Black Model and SABR/CEV Model for Basis Spreads

Henrard's hypothesis [37], Proposition 5.2, is proposed as a means to link the various curves under a multi-curve framework. This is extended to a number of other spread hypotheses. The market spread hypothesis is proposed as the following

Proposition 6.1. *The spreads or basis,*

$$B^j(t_0, t_1) = S^j_{t_0}(t_0, t_1) - S^M_{t_0}(t_0, t_1), \tag{6.1}$$

are known for every fixing date t_0 and every tenor j.

Where M is defined as the *standard market frequency*, the one for which there is market data [37], and commonly referred to as the *leading swap term* and this spread is known for every fixing. A model for S^M would be extended to cover the swap rates for other tenors via a deterministic shift which can be used under a Black-like model under which if the market rate is log-normal then the other tenors are shifted log-normally [37].

The Black Model used to model forward rates was presented by Fischer Black in [10] and the base equation may be presented as

$$\mathrm{d}S_t^M = \sigma S_t^M \,\mathrm{d}W_t, \tag{6.2}$$

with σ the market volatility given by the market for each tenor, expiry and strike. Similarly the rates for the forward curve should follow a similar equation,

$$\mathrm{d}S_t^j = \sigma S_t^j \,\mathrm{d}W_t, \tag{6.3}$$

for curves of tenor j. Under the Black spread model the following is proposed

Proposition 6.2. *The forward rate follows a Black equation, between time 0 and expiry, with the same Brownian motion as the rate in the market convention.*

Under that approach the spread between the rates $S^M - S^j$ is neither constant nor deterministic [37]. Rather it is a constant proportion of the rate and it increases and reduces with the rate

$$S_t^j = \frac{S_0^j}{S_0^M} S_t^M. \tag{6.4}$$

This is analogous to the model proposed by Mercurio in [46] though no relation is proposed between each rate S^j.

The *Stochastic Alpha Beta Rho*, commonly referred to as the **SABR** model is a stochastic volatility model developed by Hagan, Kumar, Lesniewski and Woodward introduced in [31]. It describes a single forward rate, F, with volatility σ where both F and σ are stochastic variables whose time evolution is given by

$$\mathrm{d}F_t = \sigma_t F_t^\beta \,\mathrm{d}W_t, \tag{6.5}$$

$$\mathrm{d}\sigma_t = \alpha\sigma_t \,\mathrm{d}Z_t, \tag{6.6}$$

with W_t and Z_t two correlated Wiener processes with correlation coefficient ρ and constant parameters α and β such that $0 \leq \beta \leq 1, \alpha \geq 1$ and $0 \leq \rho \leq 1$. Greater detail on the SABR model can be found in [31] and where $\alpha = 0$ it reduces to the *Constant Elasticity of Variance* model, or **CEV** model developed by Cox [19].

Under market basis spreads the following base equations are proposed by Henrard in [37]

$$\mathrm{d}S_t^M = \alpha_t (S_t^M)^\beta \,\mathrm{d}W_t, \tag{6.7}$$
$$\mathrm{d}\alpha_t = \sigma \alpha_t \,\mathrm{d}Z_t, \tag{6.8}$$

under the following proposition

Proposition 6.3. *The forward swap rate follows a SABR equation, between 0 an expiry) with the same parameters and Brownian motion as the rate in the market convention.*

As above, the market convention is the term followed by the most liquid, leading swap. Henrard though points out that in the Black and SABR approaches the spreads would increase when the rates increase though in the recent crises the spreads increased while the rates decreased. A possible reason for this though could be a result of the original lack of spreads in the market with spreads increasing as models attempted to catch up with what was being observed and made evident in the market.

6.2 Quadratic Gaussian Model

We shall now consider the Quadratic Gaussian Model proposed by Kijima et al. [42]. The market is extended to include a third curve, denoted by Kijima et al. as **G**, which is a government yield curve, i.e. that of a yield curve corresponding to prices for various government bonds using the **G** curve to discount the coupons. This can be seen as introducing another basis spread between the discount curve C_d and the government curve G. Of primary interest to us are the discount curve C_d and the market forward curve C_f, the government yield curve, which we shall denote as C_g for consistency may be seen as analogous to a risk-free rate (though this is debatable given the rise of credit risk in the sovereign bond market).

Under the quadratic Gaussian model the short rate spreads are defined by

$$h_f(t) = r_f(t) - r_d(t), \tag{6.9}$$
$$h_g(t) = r_g(t) - r_d(t), \tag{6.10}$$

for the short rates $r_f(t), r_d(t)$ and $r_g(t)$. Kijima et al. [42] interpret the spreads as first, $h_f(t)$ as the cost of LIBOR funding over that of $r_d(t)$ and second, $-h_g(t)$ as a convenience yield for holding the government bond for market participants. Under the quadratic Gaussian model proposed by Pelsser [52] the short rate $r_d(t)$ is assumed to follow the process

$$\begin{aligned} r_d(t) &= (y(t) + \alpha + \beta t)^2, && (6.11) \\ \mathrm{d}y(t) &= -a_d y(t)\,\mathrm{d}t + \sigma_d\,\mathrm{d}W_d(t), && (6.12) \end{aligned}$$

where α, β, a_d and σ_d are constants. Following Pelsser [52] the zero-coupon bond price can be derived from Equations (6.11) and (6.12) as

$$P_d(t,T) = e^{\left[A_d(t,T) - B_d(t,T)y(t) - C_d(t,T)y(t)^2\right]}, \tag{6.13}$$

where

$$\begin{aligned} \gamma &= \sqrt{a_d^2 + 2\sigma_d^2}, \\ F_d(t,T) &= 2\gamma e^{\gamma(T-t)}[(\gamma + a_d)e^{2\gamma(T-t)} + \gamma - a_d]^{-1}, \\ C_d(t,T) &= (e^{2\gamma(T-t)} - 1)[(\gamma + a_d)e^{2\gamma(T-t)} + \gamma - a_d]^{-1}, \\ B_d(t,T) &= 2F_d(t,T)\int_t^T \frac{\alpha + \beta s}{F_d(t,T)}\,\mathrm{d}s, \\ A_d(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_d^s B_d(s,t)^2 - \sigma_d^2 C_d(s,T) - (\alpha + \beta s)^2\right)\mathrm{d}s, \end{aligned} \tag{6.14}$$

as shown in [42]. An explicit formula for the curve C_f can then be derived. As $W_d(t)$ and $W_f(t)$ are independent under the measure Q_d [42] shows that

$$P_f(t,T) = E^{Q_d}\left[\frac{Z_f(T)}{Z_f(t)}|\mathcal{F}_t\right] = P_d(t,T)H_f(t,T), \tag{6.15}$$

where

$$H_f(t,T) = E^{Q_f}\left[e^{-\int_t^T h_f(s)\,\mathrm{d}s}|\mathcal{F}_t\right], \tag{6.16}$$

for numeraire $Z_f(t)$ and tradeable asset $Z_f(T)$ as in Proposition 2.2. Taking $W_f^f(t) = W_f(t) + \lambda_f t$ a standard Brownian motion under Q_f using Girsanov's theorem we can obtain the explicit calculation of $H_f(t,T)$ from [42]

$$dh_f(s) = a_f\left[\left(b_f - \frac{\sigma_f\lambda_f}{a_f}\right) - h_f(s)\right] + \sigma_f\,\mathrm{d}W_f^f(s), \tag{6.17}$$

where it is noted that $h_f(t)$ is the Vasicek process under Q_f. Details on the Vasicek process may be found in [9] and other literature which covers interest rate models. Following the Vasicek process

$$H_f(t,T) = e^{A_f(t,T)+B_f(t,T)h_f(t)}, \tag{6.18}$$

where

$$\begin{aligned} B_f(t,T) &= -\frac{1}{a_f}(1-e^{-a_f(T-t)}), \\ A_f(t,T) &= -\left(B_f(t,T)+(T-t)\right)\left[b_f - \frac{\sigma_f\lambda_f}{a_f} - \frac{\sigma_f^2}{2a_f^2}\right] - \frac{\sigma_f^2 B_f(t,T)^2}{4a_f}. \end{aligned}$$

From Equation (6.15) the spread in the zero rates between the curves C_d, the discount curve and C_f, the LIBOR curve, can be represented as

$$-\ln\frac{P_f(t,T)}{T-t} + \ln\frac{P_d(t,T)}{T-t} = -\ln\frac{H_f(t,T)}{T-t}. \tag{6.19}$$

To calculate the forward LIBOR curve, $C-f$, Kijima et al. utilise the property that the dynamics of $h_f(t)$ are not affected by the change of measure from Q_d to $Q_d^{T_1}$ which follows from Proposition 2.2 and the properties of changing the numeraire. Following from Equation (6.18) we obtain [42]

$$\begin{aligned} \delta L(t,t_1,t_2) &= E^{Q_d^{t_2}}\left[\frac{1}{P_f(t_1,t_2)}|\mathcal{F}_t\right] - 1 \\ &= \frac{P_d(t,t_1)}{P_d(t,t_2)}e^{-A_f(t_1,t_2)-B_f(t_1,t_2)E^{Q_d}[h_f(t_1)|\mathcal{F}_t]+\frac{1}{2}B_f(t_1,t_2)^2 VAR^{Q_d}[h_f(t_1)|\mathcal{F}_t]} - 1. \end{aligned}$$

Which under the conditional normality of $h_f(t_1)$ [42] results in the forward LIBOR rates as

$$L(t,t_1,t_2) = \frac{1}{\delta}\left(\frac{P_d(t,t_1)}{P_d(t,t_2)}K_f(t,t_1,t_2) - 1\right), \tag{6.20}$$

where

$$\begin{aligned} K_f(t,t_1,t_2) = &Exp\{-A_f(t_1,t_2) - B_f(t_1,t_2)(h_f(t)e^{-a_f(t_1,t)} + b_f(1-e^{-a_f(t_1-t)})) \\ &+ \frac{\sigma_f^2 B_f(t_1,t_2)^2}{4a_f}(1-e^{-2a_f(t_1,t)})\}. \end{aligned} \tag{6.21}$$

The same arguments are extended to derive explicit formulae for the curve C_g in [42] but which will not be repeated here.

6.3 Vasicek Model

One of the merits of the Gaussian model is that the short rate $r_d(t)$ remains non-negative. To derive a closed form solution though it was assumed that $r_d(t)$ and the short rate spreads $h_f(t)$ and $h_g(t)$ are mutually independent [42]. When fitting to actual market data though this assumption limits the model as the two processes are correlated significantly. Kijima et al. sacrifice the non-negativity of the short rate and develop a correlated Gaussian model where the short rate $r_d(t)$ is assumed to follow the Vasicek model. The Vasicek model is a model of the evolution of interest rates and is a type of one-factor model of short rate. It was described by Oldrich Vasicek and is an Ornstein-Uhlenbeck stochastic process. Under the the Vasicek model, introducing basis spreads, the short rate $r_d(t)$ is assumed to follow the process

$$dr_d(t) = a_d(b_d - r_d(t))\,\mathrm{d}t + \sigma_d\,\mathrm{d}W_d(t), \tag{6.22}$$

where a_d, b_d and σ_d are constants. The spreads $h_f(t)$ and $h_g(t)$ follow the Vasicek process

$$\begin{aligned} \mathrm{d}h_f(t) &= a_f(b_f - h_f(t))\,\mathrm{d}t + \sigma_f\,\mathrm{d}W_f(t), \\ \mathrm{d}h_g(t) &= a_g(b_g - h_g(t))\,\mathrm{d}t + \sigma_g\,\mathrm{d}W_g(t), \end{aligned} \tag{6.23}$$

and the Brownian motions $W_f(t)$, $W_g(t)$ and $W_d(t)$ are correlated as

$$\begin{aligned} \mathrm{d}W_d(t)\,\mathrm{d}W_f(t) &= \rho_{df}\,\mathrm{d}t, \\ \mathrm{d}W_d(t)\,\mathrm{d}W_g(t) &= \rho_{dg}\,\mathrm{d}t, \\ \mathrm{d}W_f(t)\,\mathrm{d}W_g(t) &= \rho_{fg}\,\mathrm{d}t. \end{aligned}$$

The market prices of risk are assumed to be given by

$$\begin{pmatrix} \lambda_d^d(t) & \lambda_d^f(t) & \lambda_d^g(t) \\ \lambda_f^d(t) & \lambda_f^f(t) & \lambda_f^g(t) \\ \lambda_g^d(t) & \lambda_g^f(t) & \lambda_g^g(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_f & 0 \\ 0 & 0 & \lambda_g \end{pmatrix}, \tag{6.24}$$

following the Vasicek model [42].

Under the conditional normality of the Ornstein-Uhlenbeck model Kijima et al. [42] give the relevant prices and forward rates as

$$\begin{aligned}
P_d(t,T) &= e^{A_d(t,T)+B_d(t,T)r_d(t)}, \\
P_f(t,T) &= E^{Q_d}\left[\frac{Z_f(T)}{Z_f(t)}|\mathcal{F}_t\right] = P_d(t,T)H_f(t,T), \\
P_g(t,T) &= E^{Q_d}\left[\frac{Z_g(T)}{Z_g(t)}|\mathcal{F}_t\right] = P_d(t,T)H_g(t,T), \\
L(t,t_1,t_2) &= \frac{1}{\delta}\left(E^{Q^{t_2}d}\left[\frac{1}{P_f(t_1,t_2)}|\mathcal{F}_t\right] - 1\right) \\
&= \frac{1}{\delta}\left(\frac{P_d(t,t_1)}{P_d(t,t_2)}K_f(t,t_1,t_2) - 1\right), \\
G(t,t_1,t_2) &= \frac{1}{\delta}\left(E^{Q^{t_2}d}\left[\frac{1}{P_g(t_1,t_2)}|\mathcal{F}_t\right] - 1\right) \\
&= \frac{1}{\delta}\left(\frac{P_d(t,t_1)}{P_d(t,t_2)}K_g(t,t_1,t_2) - 1\right),
\end{aligned} \tag{6.25}$$

with K as defined under Derivation 3.1 and where

$$\begin{aligned}
B_k(t,T) &= -\frac{1}{a_k}(1-e^{-a_k(T-t)}), \\
A_k(t,T) &= -(B_k(t,T)-(T-t))\left(b_k - \frac{\sigma_k^2}{2a_k^2}\right) - \frac{\sigma_k^2 B_k(t,T)^2}{4a_k}, \\
B_{dk}(t,T) &= -\frac{1-e^{-(a_d+a_k)(T-t)}}{4a_k}, \\
A_{dj}(t,T) &= \lambda_j\frac{\sigma_j}{a_j}(B_j(t,T)+T-t) + \lambda_j\frac{\rho_{dj}\sigma_d}{a_d}(B_d(t,T)+T-t) \\
&+ \frac{\rho_{dj}\sigma_d\sigma_j}{a_d a_j}(B_d(t,T)+B_j(t,T)-B_{dj}(t,T)+T-t) \\
H_j(t,T) &= e^{A_j(t,T)+A_{dj}(t,T)+B_j(t,T)h_j(t)}, \\
K_j(t,t_1,t_2) &= exp\left[-A_j(t_1,t_2) - B_j(t_1,t_2)(h_j(t)e^{-a_j(t_1-t)} + b_j(1-e^{-a_j(t_1-t)}))\right. \\
&+ \frac{\sigma_j^2 B_j(t_1,t_2)^2}{4a_j}(1-e^{-2a_j(t_1-t)}) - A_{dj}(t_1,t_2) \\
&- \left.\frac{\rho_d\sigma_d\sigma_j}{a_d}B_j(t_1,t_2)(B_d(t,t_1) - B_d d(t,t_1))\right],
\end{aligned} \tag{6.26}$$

for $k = d, f, g$ and $j = f, g$.

The derivation of Kijima et al. [42] is based on distinguishing discount rates from forward rates (i.e. the expectation of deposit rates over a period in the future) and further to distinguish the discount and swap rate from the fixed-coupon bond rate, thus modelling the *bond-swap gap* and treating a bond purchase as swap contract to exchange the fixed-coupon bond against a floating rate bond. The values of at-the-money swaps in the market have a value of zero and the market value of in-the-market differs from the pre-crisis traditional net-present-value model due to the existence of the basis swap spreads; a well-known fact for international institutions [42].

6.4 Extended LIBOR Market Model

The prolific LIBOR Market Model (LMM; also referred to as the BGM model after the original authors) was introduced by Brace, Gątarek and Musiela in [12]. The model models a set of forward rates (or LIBOR rates) as opposed to the previously mentioned models which model the short-rate models (such as the Vasicek model) or instantaneous forward rate models (such as the Heath-Jarrow-Morton framework). The forward LIBOR rates are observable in the market (as FRAs and Swaps) and their volatilities are also observable (where liquid enough, as FRA Options, Swaptions, Caps and Floors). Under the LMM the joint evolutions of consecutive forward LIBOR rates is modelled under its forward measure such as the Black model. The consecutive forward rates are modelled under a common pricing measure which is typically some terminal forward measure of the spot LIBOR measure which corresponds to a set of times which define the set of forward rates. One may see the LMM as a collection of forward LIBOR dynamics for different forward rates and each forward rate matching a Black interest rate caplet formula for its term and maturity. Under the LMM a set of forward rates $L_j, \quad j = 1, ..., n$ where each L_j, being a forward rate for the period $[t_j, t_{j+1}]$, is a lognormal process where

$$dL_j(t) = \sigma_j(t)L_j(t)\, dW^{Q_{t_j}}, \tag{6.27}$$

for the t_j forward measure Q_{t_j}. Under the LMM using the multivariate Girsanov's theorem to model the dynamics of a whole set of forward rates jointly as

$$dW^{Q_{t_j}}(t) = \begin{cases} dW^{Q_{t_i}}(t) - \sum_{k=j+1}^{i} \frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_k(t)\, dt, & j < p, \\ dW^{Q_{t_i}}(t), & j = p, \\ dW^{Q_{t_i}}(t) + \sum_{k=p}^{j-1} \frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_k(t)\, dt, & j > p, \end{cases} \tag{6.28}$$

and

$$\mathrm{d}L_j(t) = \begin{cases} L_j(t)\sigma_j(t)\,\mathrm{d}W^{Q_{t_j}}(t) - L_j(t)\sum_{k=j+1}^{p}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_j(t)\sigma_k(t)\rho_{jk}\,\mathrm{d}t, & j < p, \\ L_j(t)\sigma_j(t)\,\mathrm{d}W^{Q_{t_j}}(t), & j = p, \\ L_j(t)\sigma_j(t)\,\mathrm{d}W^{Q_{t_j}}(t) - L_j(t)\sum_{k=p}^{j-1}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_j(t)\sigma_k(t)\rho_{jk}\,\mathrm{d}t, & j > p, \end{cases} \tag{6.29}$$

Mercurio [46] extends the LMM to a multiple curve framework where the curve used for discounting differs to the curve used for calculating forward interest rates.Under a set of times $\mathbf{T} = 0 < T_0^i, ..., T_M^i$ for a corresponding interest rate curve C_i. It is assumed that each forward rate $L_k^i(t)$ under its forward measure $Q_d^{T_k^i}$ is a driftless geometric Brownian motion process

$$\mathrm{d}L_k^i(t) = \sigma_k(t)L_k^i(t)\,\mathrm{d}Z_k(t), \quad t \leq T_{k-1}^i, \tag{6.30}$$

with a deterministic instantaneous volatility $\sigma_k(t)$ and Z_k is the k-th component of an M-dimensional $Q_d^{T_k^i}$ Brownian motion $\mathbf{Z}$, $\mathrm{d}Z_k(t)\,\mathrm{d}Z_j(t) = \rho_{kj}\,\mathrm{d}t$, with instantaneous correlation matrix $(\rho_{kj})_{k,j=1,...,M}$ [46].
Under a double-curve framework Mercurio models the evolution of rates as

$$F_k^d(t) = F_d(t, T_{k-1}^i, T_k^i) = \frac{1}{\tau_k^d}\left[\frac{P_d(t, t_{k-1}^i)}{P_d(t, T_k^i)} - 1\right], \tag{6.31}$$

where τ_k^i represents the year fraction $\tau(T_{k-1}^i, t_k^i)$ under the curve C_d.
The dynamics of each rate F_h^d under the forward measure $Q_d^{T_h^i}$ is given by

$$\mathrm{d}F_h^d(t) = \sigma_h^d(t)F_h^d(t)\,\mathrm{d}Z_h^d(t), \quad t \leq T_{h-1}^i, \tag{6.32}$$

again with instantaneous volatility $\sigma_h^d(t)$ deterministic and Z_h^d the h-th component of an M-dimensional $Q_d^{T_h^i}$ Brownian motion $\boldsymbol{Z^d}$ with correlations

$$\begin{aligned} \mathrm{d}Z_k^d(t)\,\mathrm{d}Z_h^d(t) &= \rho_{kh}^{dd}\,\mathrm{d}t, \\ \mathrm{d}Z_k(t)\,\mathrm{d}Z_h^d(t) &= \rho_{kh}^{id}\,\mathrm{d}t, \end{aligned}$$

such that the global matrix

$$R := \left[\begin{array}{c|c} \rho & \rho^{id} \\ \hline (\rho^{id})' & \rho^{dd} \end{array}\right]$$

is positive semi-definite.

6.4.1 Forward Measure

The LMM forward rates dynamics can be written under a common pricing measure such as the forward measure or the spot LIBOR measure. Under the forward measure, $Q_d^{T_j^i}$ the dynamics of $L_k^i(t)$ can be derived by changing the measure from $Q_d^{T_k^i}$ to $Q_d^{T_j^i}$ with numeraires the discount curve, C_d, zero-coupon bonds with maturities T_k^i and T_j^i respectively [46]. By applying the change of numeraire techniques relating to the drifts of a given process under measures with known numeraires [13]. The drift of $L_k^i(t)$ under $Q_d^{T_j^i}$ is

$$\textbf{Drift}(L_k^i; Q_d^{T_j^i}) = -\frac{d\langle L_k^i, \ln(P_d(\cdot, T_k^i)/P_d(\cdot, T_j^i))\rangle_t}{\mathrm{d}t}, \tag{6.33}$$

where $\langle X, Y\rangle_t$ denotes the instantaneous covariation between the processes X and Y at time t [46].

Under the case $j < k$ the log ratio of the two numeraires can be written as

$$\begin{aligned} \ln(P_d(t, T_k^i)/P_d(t, T_j^i)) &= \ln\left(\frac{1}{\prod_{h=j+1}^{k}(1+\tau_h^d F_h^d(t))}\right) \\ &= -\sum_{h=j+1}^{k} \ln(1+\tau_h^d F_h^d(t)), \end{aligned} \tag{6.34}$$

from which results

$$\begin{aligned} \textbf{Drift}(L_k^i; Q^{T_j^i} d) &= d\langle L_k^i, \ln(P_d(\cdot, T_k^i)/P_d(\cdot, T_j^i))\rangle_t \\ &= \sum_{h=j+1}^{k} \frac{d\langle L_k^i, \ln(1+\tau_h^d F_h^d)}{\mathrm{d}t} \\ &= \sum_{h=j+1}^{k} \frac{\tau_h^d}{1+\tau_h^d F_h^d(t)} \frac{d\langle L_k^i, F_h^d\rangle_t}{\mathrm{d}t}, \end{aligned} \tag{6.35}$$

and following from Equation (6.32)

$$\textbf{Drift}(L_k^i; Q_d^{T_j^i}) = \sigma_k(t) L_k^i(t) \sum_{h=j+1}^{k} \frac{\rho_{kh}^{id} \tau_h^d \sigma_h^d(t) F_h^d(t)}{1+\tau_h^d F_h^d(t)}. \tag{6.36}$$

For the case $j > k$ the derivation of the drift follows the same process [46].

The forward rates F_k^d, the $Q_d^{T_j^i}$ dynamics are equivalent to those obtained in the single curve case as is shown in various sources, such as [13, 9], as the probabilities are associated with the same curve C_d.
The aim of the extended LMM model under a multiple curve framework is to model the joint evolution of the forward LIBOR rates $L_1^i, ..., L_M^i$ and forward discount rates $F_1^d, ..., F_M^d$ under a common forward measure. Mercurio [46] gives the following proposition

Proposition 6.4. *The dynamics of L_k^i and F_k^d under the forward measure $Q_d^{T_j^i}$ under the three cases $j < k, \quad j = k, \quad j > k$ are, respectively,*

$$j < k, t \leq T_j^i \quad : \quad \begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t) \left[\sum_{h=j+1}^{k} \frac{\rho_{kh}^{id} \tau_h^d \sigma_h^d(t) F_h^d(t)}{1 + \tau_h^d F_h^d(t)} \,\mathrm{d}t + \mathrm{d}Z_k^j(t) \right] \\ \mathrm{d}F_k^d(t) = \sigma_k^d(t) F_k^d(t) \left[\sum_{h=j+1}^{k} \frac{\rho_{kh}^{dd} \tau_h^d \sigma_h^d(t) F_h^d(t)}{1 + \tau_h^d F_h^d(t)} \,\mathrm{d}t + \mathrm{d}Z_k^{jd}(t) \right] \end{cases}$$

$$j = k, t \leq T_{k-1}^i \quad : \quad \begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t) \,\mathrm{d}Z_k^j(t) \\ \mathrm{d}F_k^d(t) = \sigma_k^d(t) = \sigma_k^d(t) F_k^d(t) \,\mathrm{d}Z_k^{jd}(t) \end{cases}$$

$$j > k, t \leq T_{k-1}^i \quad : \quad \begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t) \left[-\sum_{h=k+1}^{j} \frac{\rho_{kh}^{id} \tau_h^d \sigma_h^d(t) F_h^d(t)}{1 + \tau_h^d F_h^d(t)} \,\mathrm{d}t + \mathrm{d}Z_k^j(t) \right] \\ \mathrm{d}F_k^d(t) = \sigma_k^d(t) F_k^d(t) \left[-\sum_{h=k+1}^{j} \frac{\rho_{kh}^{dd} \tau_h^d \sigma_h^d(t) F_h^d(t)}{1 + \tau_h^d F_h^d(t)} \,\mathrm{d}t + \mathrm{d}Z_k^{jd}(t) \right] \end{cases}$$

where Z_k^j and Z_k^{jd} are the k-th components of the M-dimensional $Q_d^{T_j^i}$ Brownian motion $\boldsymbol{Z^j}$ and $\boldsymbol{Z^{jd}}$ with correlation matrix R.

6.4.2 Spot Measure

Mercurio [46] further derives an extended LMM under the spot LIBOR measure, Q_d^T, for times $T = T_0^i, ..., T_M^i$ whose numeraire is the bank account as, defined in Definition 3.1, B_d^T

$$B_d^T(t) = \frac{P_d(t, T_{\beta(t)-1}^i)}{\prod_{j=0}^{\beta(t)-1} P_d(T_{j-1}^i, T_j^i)}, \tag{6.37}$$

where $\beta(t) = m$ if $T_{m-2}^i < t \leq T_{m-1}^i, \quad m \geq 1$, so that $t \in (T_{\beta(t)-2}^i, T_{\beta(t)-1}^i]$, [46].

By applying the change of numeraire technique, as discussed in Section 2.3, Mercurio gives the following proposition [46].

Proposition 6.5. *The dynamics of FRA and forward rates under the spot LIBOR measure* Q_d^T *are given by:*

$$\begin{aligned} \mathrm{d}L_k^i(t) &= \sigma_k(t)L_k^i(t)\sum_{h=\beta(t)}^{k}\frac{\rho_{kh}^{id}\tau_h^d\sigma_h^d(t)F_h^d(t)}{1+\tau_h^dF_h^d(t)}\,\mathrm{d}t+\sigma_k(t)L_k^i(t)\,\mathrm{d}Z_k^l(t), \\ \mathrm{d}F_k^d(t) &= \sigma_k^d(t)F_k^d(t)\sum_{h=\beta(t)}^{k}\frac{\rho_{kh}^{dd}\tau_h^d\sigma_h^d(t)F_h^d(t)}{1+\tau_h^dF_h^d(t)}\,\mathrm{d}t+\sigma_k^d(t)F_k^d(t)\,\mathrm{d}Z_k^{ld}(t), \end{aligned} \tag{6.38}$$

where $Z^l = \{Z_1^l, ..., Z_m^l\}$ *and* $Z^{ld} = \{Z_1^{ld}, ..., Z_M^{ld}\}$ *are M-dimensional* Q_d^T *Brownian motions with correlation matrix* R.

Chapter 7

Risky Markets

In this chapter we take a look at how the liquidity crisis, in the wake of the default by Lehman Brothers Holdings Inc., brings rise to default considerations in the interbank market. LIBOR is no longer a suitable proxy for the risk-free interest rate. We take a look at the discounting of cashflows in the absence and in the presence of collateral and the rise of the overnight index swap (OIS) curve for discounting collateralised cashflows. We then take a look at a simple method for constructing and interpolating such a discount curve and the pricing of interest rate instruments under such a curve in the continuous and discrete time case. We will then attempt to show that the overnight deposit rate forming the underlying interest rate for this OIS curve can be shown to be a better proxy for the risk-free interest rate and that the OIS curve could now be considered a close approximation of the risk-free curve. Finally we will attempt to reconcile the basis spreads between the OIS curve and the traditional LIBOR based curves as a measure of the default risk between interbank counterparties and how that can be reconciled with common default and survival probabilities used in credit derivative pricing models.

7.1 Introduction

One of the major considerations that the credit crisis brought to light was the impact counterparty risk in the interbank market in the absence of collateral and the impact on the cost of funding in the presence of collateral. A prominent example of such a change was the move by Goldman Sachs to price cash-collateralised trades using an overnight index swap (OIS) curve to discount cashflows rather than the more traditional discount curve based on liquid swaps with a longer term; as highlighted by Cameron in [15], the need for this type of pricing may have been identified in the early 2000s. As basis spreads increased and the gaps between various curves widened the difference between the OIS curve and the traditional single-curve pricing approach increased.

Traditional, pre-crisis, pricing was to view swaps as a portfolio of forward contracts and in the single-curve framework all cashflows were discounted at the base discount curve (especially in the interbank market) [41]. Collateralisation of swaps and other interest rate derivatives would lead to daily intermediate payments between the counterparties and large swings in the mark-to-market value of these instruments could lead to a significant posting of cash collateral. This would lead to a cost in funding this collateral and there may be disparity in the rate of raising such funding and the overnight rate received on posting such collateral.

Numerous authors have commented on and investigated the impact of basis risks and counterparty default risks in the interbank market on interest rate derivative pricing and models. Examples include Johannes and Sundaresan [41]; Fujjii, Shimada and Takahashi [25, 28]; Crépey and Grbac [21]; Fries [24]; Castanga [16], and Morini [50] to name a few. Complications also arise depending on the nature of the collateralisation agreement (typically a Credit Support Annexure, or CSA, under a master ISDA agreement) as differences may arise based on the currency used for collateralisation, which could differ from the currency of the instrument being collateralised, threshold amounts where collateralisation occurs only after a certain mark-to-market threshold is breached and netting agreements in the absence of collateral, so that even in the case of a default the amount owed is the net of the mark-to-markets of a portfolio of trades (without such a netting agreement all amounts owed to the defaulted counterparty would need to be paid in full and all amounts owed by the defaulted counterparty would be subject to liquidation and seniority of debt). Morini approaches this problem and discusses the case of a netting "no-fault" rule as well as case of cash collaterlisation [50].

Further considerations to be made are the impact of wrong-way risk, for example when transacting a cross-currency swap with collateral in the base currency of counterparty bank (e.g. transacting a United States Dollar (USD) for Swiss Franc (CHF) cross currency swap with a Swiss bank with CHF cash collateral), the value of the collateral could decline sharply along with the mark-to-market value of the swap should the counterparty bank default causing currency devaluation.

7.2 Pricing and Modelling when banks can default

Due to the short term nature of an overnight deposit, the OIS reference rate, as defined in Definition 3.12, is typically seen to have negligible liquidity and credit risk [50]. Conversely the LIBOR funding rate (or equivalent X-BOR rate) is viewed as an indicator of the cost of funding for banks in the interbank market for longer terms and as such contains elements of both counterparty default and liquidity risk. The spread between the OIS rate[1] and the various LIBOR rates of different terms can be seen as an indicative measure of the credit and liquidity risk inherent in the interbank market. However as not all banks are equal the market quoted OIS spreads while indicative of the market may not be representative of the risk in a trade between two specific counterparties (in the absence of other mitigating circumstances such as collateral).

7.2.1 Defaultable Counterparties without Netting

Morini [50] explores how the mathematical no-arbitrage relationships are modified when market-wide counterparty risk is introduced exploring replication and change of numeraire techniques. As bank counterparties, post the liquidity crunch crisis, can no longer be deemed as risk-free from a counterparty and liquidity risk perspective bond cashflows and bond payoffs of 1 need to be replaced with probabilistic values

$$R + 1_{\{t^B > t_i\}}(1 - R), \tag{7.1}$$

where R is a deterministic recovery rate, t^B is the default time of the bond issuer B and the recovery payments occur at maturity t_i (this assumption differs from reality where recovery payments would be made at a time after default depending on legal process, though for short-term bonds this approximation is acceptable [50]. $1_{\{t^B > t_i\}}$ may be interpreted as "given that the default time is after the maturity time". Taking the price of such a defaultable ZCB as the present value of the expected payoff we obtain [50]

$$\begin{aligned} P^B(t, t_i) : &= E_t[df(t, t_i)(R + 1_{\{t^B > t_i\}}(1 - R))] \\ &= df(t, t_i)R + E[df(t, t_i)(1_{\{t^b > t_i\}})](1 - R). \end{aligned} \tag{7.2}$$

[1]We distinguish between the OIS reference rate, which is the overnight benchmark or reset rate, and the OIS spread which is the spread between the price of an OIS and LIBOR

A similar result is reached by Mercurio [46] who gives the result as

$$\begin{aligned} P^B(t,t_i) &= E\left[e^{-\int_t^{t_i} r(u)\,\mathrm{d}u}(R+(1-R)1_{\{t^B>t_i\}}|\mathcal{F}_t\right] \\ &= R\cdot df(t,t_i)+(1-R)df(t,t_i)E[1_{\{t^B>t_i\}}|\mathcal{F}_t], \end{aligned} \tag{7.3}$$

the difference being that Mercurio treats the discount factor as deterministic, or more specifically, as the price of a default-free zero coupon bond. This result could also be rewritten as

$$P^B(t,t_i)=1\cdot df(t,t_i)\mathbb{P}[t^B>t_i]+R\cdot df(t,t_i)\mathbb{P}[t^B\le t_i], \tag{7.4}$$

where $\mathbb{P}[A]$ is the probability of event A occurring.

Considering a contract with inception time t where counterparty A agrees to pay an amount of 1 at time $t_{i-1}\ge t$ if A has not defaulted, else only a recovery fraction R is paid and where counterparty B pays at time $t_i\ge t_{i-1}$ the amount $1+K\tau(t_{i-1},t)$ if B has not defaulted earlier, else only a recovery fraction R is paid. For there to be no-arbitrage, i.e. for it to be fair, the following needs to hold [50]

$$\begin{aligned} & E_t[df(t,t_{i-1})(R+1_{\{t^A>t_{i-1}\}}(1-R)] \\ =\ & E_t[df(t,t_i)(R+1_{\{t^B>t_i\}}(1-R))(1+K\tau(t_{i-1},t_i))], \end{aligned} \tag{7.5}$$

and taking $t_{i-1}=t$ results in

$$(R+1_{\{t^A>t\}}(1-R))=P^B(t,t_i)(1+K\tau(t,t_i)). \tag{7.6}$$

From Definition 3.8 and Derivation 3.1 we consider the equilibrium rate as

$$\begin{aligned} L^{A,B}(t,t_i) &= \frac{1}{\tau(t,t_i)}\left(\frac{R+1_{\{t^A>t\}]}(1-R)}{P^B(t,t_i)}-1\right) \\ &= \frac{1}{\tau(t,t_i)}\left(\frac{P^A(t,t)}{P^B(t,t_i)}-1\right), \end{aligned} \tag{7.7}$$

where $L^{A,B}$ represents a spot rate related to the credit riskiness of counterparties A and B. Further, if it is assumed that both A and B do not default before the inception date t this may be seen as a zero coupon bond issued by B purchased by A at time t and Equation (7.6) can be interpreted as the relationship between a risky bond and a risky spot rate [50] and leads to the equilibrium rate

$$L^B(t,t_i)=\frac{1}{\tau(t,t_i)}\left(\frac{1}{P^B(t,t_i)}-1\right). \tag{7.8}$$

For the case where $t_{i-1} > t$ the contract may be seen as the equivalent of a FRA with credit-risky counterparties with the fair forward interest rate (which follows from definition of a fair forward interest rate, Definition 3.10, and the proposition of the replication of a FRA, Proposition 3.1) as

$$F^{A,B}(t; t_{i-1}, t_i) = \frac{1}{\tau(t_{i-1}, t_i)} \left(\frac{P^A(t, t_{i-1})}{P^B(t, t_i)} - 1 \right), \tag{7.9}$$

where $K^{A,B}$ represents the fair forward rate for a defaultable FRA between counterparties A and B [50]. Again similar results are obtained by Mercurio [46] who gives the value of the fair FRA rate as

$$F^A = \frac{1}{\tau(t_{i-1}, t_i)} \left(\frac{P(0, t_{i-1})}{P(0, t_i)} \frac{1}{R + (1 - R)E[Q(t_{i-1}, t_i)]} - 1 \right), \tag{7.10}$$

where

$$Q(t_{i-1}, t_i) := E[1_{\{t^A > t_i\}} | \mathcal{F}_{t_{i-1}}], \tag{7.11}$$

which only considers single sided (or unilateral) default risk of counterparty A. For this to be truly equivalent to a FRA counterparty A would also need to pay a floating interest rate payment based on LIBOR to counterparty B at time t_i; the value of the contract would not change since the payment at time t_i can be seen as making a payment of 1 at time t_{i-1} reinvested at the floating interest rate for the time period $\tau(t_{i-1}, t_i)$, or mathematically

$$1 = P^A(t_{i-1}, t_i)(1 + L^A(t_{i-1}, t_i)\tau(t_{i-1}, t_i)). \tag{7.12}$$

However, Morini [50] remarks that for this to be equivalent to a market FRA (i.e. a generic FRA quoted in the inter bank market) then the rate L^A should be very similar to the market LIBOR rate L, and this would lead to $F^{A,B}(t; t_{i-1}, t_i)$ to be equal to $F_M(t; t_{i-1}, t_i)$, the market FRA rate, which cannot be since $F^{A,B}$ corresponds to two specific counterparties while F_M is a unique rate for the whole market and when counterparty risk is no longer negligible it is not trivial to arrive at a single unique equilibrium FRA rate. Under these circumstances it cannot be explained why the quoted FRA rate in the market would diverge from that implied by the forward rates of LIBOR deposits.

These circumstances though are not realistic for the way FRAs are traded in the market. Typical interbank FRAs would have a netting agreement and thus would only require net settlement of the amounts owed (i.e. the difference between the fixed strike rate and the floating reference rate). This case is explored in the next section (Section 7.2.2) and the case where FRAs and other interest rate derivatives are transacted under a CSA (**Credit Support Annexure** under which derivatives are collateralised, or margined, daily based on their mark-to-market value) is considered in Section 7.3. FRAs would also typically be settled on the near maturity date (as defined in Definition 3.8). Mercurio [46] gives an explanation for the discrepancy remarked upon in [50] which follows from Equation (7.10)

$$\begin{aligned} & 0 \leq R \leq 1, \quad 0 < Q(t_{i-1}, t_i) < 1, \\ \text{therefore} \quad & 0 < R + (1-R)E[Q(t_{i-1}, t_i)] < 1, \\ \text{such that} \quad & F^A > \frac{1}{\tau(t_{i-1}, t_i)} \left[\frac{P(0, t_{i-1})}{P(0, t_i)} - 1 \right] \end{aligned} \tag{7.13}$$

and therefore the FRA rate, F^A, for a defaultable counterparty is larger than that implied by the default free bonds $P(0, t_{i-1})$ and $P(0, t_i)$. This can be explained if one considers the OIS curve, which is a swap curve, C_O, bootstrapped using the usual methods and the benchmark instruments being the OISs quoted in the market, as the risk-free curve (a reasonable approximation due to the short term nature of the overnight deposit). Thus the FRA rate F^A can be higher than the corresponding forward OIS rate if the LIBOR rate is considered to be credit-risky and has a default risk implicit in the rate [46].

7.2.2 Defaultable Counterparties with Netting

A more pertinent case in the interbank market would be the treatment of interest rate derivatives concluded under a standard ISDA agreement which allows for the netting of payments under a "no-fault" or "two-way payment rule" [50]. Under such an agreement only the net amount owed, i.e. the difference between the strike rate and the reference floating interest rate, is payable. Thus under a default event, where counterparty A defaults at time t^A, the amount owed to B would be

$$[(NPV_{t^A}^B)^+ - (-NPV_{t^A}^B)^+], \tag{7.14}$$

where NPV_t^X refers to the riskless net present value of the residual deal for counterparty X at time t [50] and where $(X)^+$ means $max(X, 0)$. The amount actually received would be scaled by the recovery rate R. In this scenario there exists bilateral counterparty risk which is consistent with [46]. Following Proposition 3.15 which is that the value of a claim is the expected present value, under some measure Q^T, of the payoff, in conjunction with the valuation of a FRA given by Equations (3.21) and (5.34) the value of a defaultable FRA between two interbank counterparties A and B (from the perspective of B) may be given as

$$\begin{aligned} V^{A,B}_{FRA_{Net}}(t;t_{i-1},t_i,K,1) &= \tau(t_{i-1},t_i)E_t[df(t,t_i)(L(t_{i-1},t_i)-K)] \\ &- \tau(t_{i-1},t_i)E_t[df(t,t_i)1_{\{t^A\leq t_i\}}(L(t_{i-1},t_i)-K)^+] \\ &+ \tau(t_{i-1},t_i)E_t[df(t,t_i)1_{\{t^B\leq t_i\}}(K-L(t_{i-1},t_i))^+], \end{aligned} \tag{7.15}$$

where to simplify the notation Morini sets the recovery rate to 0. The first term considers the event of no default, the second term considers the event that party A defaults before t_i, i.e. that $t^A \leq t_i$, and the third term considers the case where B defaults before t_i, i.e $t^B \leq t_i$.

As a model for credit risk Morini uses a framework based on Jamshidian [40], and Bielecki and Rutkowski [7] and the following is a reprise from Morini [50]:

The filtration $(\mathcal{F}_s)_{(s\geq 0)}$ which is the total market information is divided into two subfiltrations

$$\mathcal{F}_t = \mathcal{H}_t \vee_{J=1}^{n} \mathcal{J}_t^J \tag{7.16}$$

$$\mathcal{J}_t^J = \sigma(\{t^J > u\}, u \leq t), \tag{7.17}$$

where $(\mathcal{J}_s^J)_{(s\geq 0)}$ is the natural filtration of the default time t^J of the J^{th} market participant. $\mathcal{H}_t$ is the no-default information, which is the information up to time t on economic factors which influence default but exclude information on the happening of default. $\mathcal{A}\vee\mathcal{B}$ is the joint σ-algebra generated by the σ-algebras $\mathcal{A}$ and $\mathcal{B}$.

Complete credit risk modelling is beyond the scope of this dissertation, the reader is referred to [7, 40] and numerous other references on the subject. The following two assumptions from [40, 7] respectively, referred to by Morini [50] are included for continuity:

Assumption 7.1 (Martingale Invariance). *Every (square-integrable) $\mathcal{H}$-martingale is also a $\mathcal{F}$-martingale, so that for $\mathcal{F}_T \subseteq \mathcal{H}_T$*

$$E[X_T|\mathcal{H}_T] = E[X_T|\mathcal{F}_t], \quad t \leq T.$$

In this case $\mathcal{H}$ and $\mathcal{F}$ are filtrations with the one a subfiltation of the other and an $\mathcal{F}$-Martingale is the stochastic process $\{X_t, t>0\}$ such that the conditional expectation of X_t given $\mathcal{F}_s$ equals X_s whenever $s<t$, where $\mathcal{F}=\{\mathcal{F}_t, t\geq 0\}$ is an increasing family of sigma algebras.

Assumption 7.2 (Positivity). *The survival probability conditional on no default information is strictly positive*

$$\mathbb{P}(t^J > t|\mathcal{H}_t) > 0, \quad t \geq 0.$$

A defaultable payoff $\mathbf{Y} = 1_{\{t^D>t\}}\mathbf{Y}$, where t^D is the default time of a market participant, can be priced using only the default indicator and no-default information [50]. Further, from [7]

$$E[\mathbf{Y}|\mathcal{F}_t] = \frac{\mathbf{1}_{\{t^D>t\}}}{\mathbb{P}(t^D > t|\mathcal{H}_t)}E[\mathbf{Y}|\mathcal{H}_t]. \tag{7.18}$$

Continuing from Morini [50] it is assumed that the set of potential interbank LIBOR counterparties can be considered homogenous with regard to default probabilities expressed by LIBOR quotes since that is the rate at which they can each trade at amongst each other we can rewrite the value for a FRA with $L_M(t, t_i)$ the market LIBOR rate adapted to the no default information $\mathcal{H}_t$

$$\begin{aligned} V^{A,B}_{FRA_{Net}}(t; t_{i-1}, t_i, K, 1) &= \tau(t_{i-1}, t_i)E_t[df(t, t_i)1_{\{t^A>t_i\}}(L_M(t_{i-1}, t_i) - K)^+] \\ &- \tau(t_{i-1}, t_i)E_t[df(t, t_i)1_{\{t^B>t_i\}}(K - L_M(t_{i-1}, t_i))^+]. \end{aligned}$$

Applying the expectation of a defaultable payment as in Equation (7.18) the value of a FRA can be rewritten as

$$\begin{aligned} &V^{A,B}_{FRA_{Net}}(t; t_{i-1}, t_i, K, 1) \\ &= \frac{1_{\{t^A>t\}}}{\mathbb{P}(t^A > t\ |\mathcal{H}_t)}\tau(t_{i-1}, t_i)E_{\mathcal{H}_t}[df(t, t_i)1_{\{t^A>t_i\}}(L_M(t_{i-1}, t_i) - K)^+] \\ &-\frac{1_{\{t^B>t_i\}}}{\mathbb{P}(t^B > t\ |\mathcal{H}_t)}\tau(t_{i-1}, t_i)E_{\mathcal{H}_t}[df(t, t_i)1_{\{t^B>t_i\}}(K - L_M(t_{i-1}, t_i))^+], \end{aligned}$$

and since $L_M(t, t_i)$ is adapted to the filtration $\mathcal{H}_t$

$$\begin{aligned} &V^{A,B}_{FRA_{Net}}(t; t_{i-1}, t_i, K, 1) \\ &= \frac{1_{\{t^A>t\}}}{\mathbb{P}(t^A > t\ |\mathcal{H}_t)}\tau(t_{i-1}, t_i)E_{\mathcal{H}_t}[df(t, t_i)\mathbb{P}(t^A > t\ |\mathcal{H}_t)(L_M(t_{i-1}, t_i) - K)^+] \\ &-\frac{1_{\{t^B>t\}}}{\mathbb{P}(t^B > t\ |\mathcal{H}_t)}\tau(t_{i-1}, t_i)E_{\mathcal{H}_t}[df(t, t_i)\mathbb{P}(t^B > t\ |\mathcal{H}_t)(K - L_M(t_{i-1}, t_i))^+], \end{aligned} \tag{7.19}$$

and the assumption of the homogeneity of A and B in an interbank LIBOR market leads to [50]

$$1_{\{t^A>t_i\}} = 1_{\{t^B>t_i\}} = 1, \tag{7.20}$$

and also implies that, for the price of a defaultable bond P^X issued by some LIBOR market counterparty X,

$$P^A(t,t_i) = P^B(t,t_i) = P_L(t,t_i) = P^X(t,t_i), \quad X \in \mathbb{L},$$

where $\mathbb{L}$ represents the LIBOR market. Thus for a defaultable bond issued by market participant X and following from value of a bond along with martingale pricing and assuming zero recovery

$$\begin{aligned} P^X(t,t_i) &= E_t[df(t,t_i)1_{\{t^X>t_i\}}] = \frac{1_{\{t^X>t_i\}}}{\mathbb{P}(t^X > t\,|\mathcal{H}_t)} E_{\mathcal{H}_t}[df(t,t_i)\mathbb{P}(t^X > t_i\,|\mathcal{H}_{t_i})] \\ &= E_{\mathcal{H}_t}\left[df(t,t_i)\frac{\mathbb{P}(t^X > t\,|\mathcal{H}_{t_i})}{\mathbb{P}(t^X > t\,|\mathcal{H}_t)}\right], \end{aligned}$$

since $1_{\{t^A>t_i\}} = 1_{\{t^B>t_i\}} = 1$ from Equation (7.20).

Further the assumption of homogeneity leads to [50]

$$\mathbb{P}(t^A > t_i\,|\mathcal{H}_{t_i}) = \mathbb{P}(t^B > t_i\,|\mathcal{H}_{t_i}) = \mathbb{P}(t^X > t_i\,|\mathcal{H}_{t_i}),$$

leading to the FRA price

$$\begin{aligned} V^{A,B}_{FRA_{Net}}(t,t_{i-1},t_i,K,1) &= V^X_{FRA_{Net}}(t,t_{i-1},t_i,K,1) \\ &= \tau(t_{i-1},t_i)E_{\mathcal{H}_t}\left[df(t,t_i)\frac{\mathbb{P}(t^X > t_i\,|\mathcal{H}_{t_i})}{\mathbb{P}(t^X > t\,|\mathcal{H}_t)}(L_M(t_{i-1},t_i) - K)\right]. \end{aligned}$$

Morini thus shows that a FRA and by extension a swap with counterparty risk can be priced by way of a defaultable payoff where the survival probability to use is that of a generic LIBOR participant, X and by replacing $\frac{\mathbb{P}(t^X>t_i\,|\mathcal{H}_{t_i})}{\mathbb{P}(t^X>t\,|\mathcal{H}_t)}$ with $R + \frac{\mathbb{P}(t^X>t_i\,|\mathcal{H}_{t_i})}{\mathbb{P}(t^X>t\,|\mathcal{H}_t)}(1-R)$ it can be extended to account for the recovery rate R.

This gives a general framework using survival probabilities and Morini extends this framework with change of numeraire techniques [50]. We shall choose to focus on the more prevalent case in the market where interbank derivatives would be concluded under an ISDA with a collateral agreement which is covered in the next section.

7.3 Pricing and Modelling in the presence of collateral

In this section we shall consider the case where interest rate derivatives, such as FRAs and Swaps, are transacted under a collateralisation agreement. This agreement is usually in the form of a *Credit Support Annex* (CSA) which is additional agreement to a typical ISDA agreement between counterparties. A typical CSA agreement would function in a relatively simple manner. The net mark-to-market value of instruments traded under the CSA would be calculated daily and collateral, usually in the form of cash, would be placed (or "posted") by the counterparty with the negative mark-to-market value (i.e. the net portfolio of trades under the CSA would be considered a liability, or owed to the other counterparty) with the other counterparty, usually in the form of an overnight deposit, as collateral against default thus largely mitigating counterparty credit risk. While being mostly prevalent in the interbank market CSAs are not restricted to interbank counterparties (i.e. LIBOR[2] market participants) but may be included in ISDA agreements between a bank and any other counterparty. We shall focus primarily on the interbank market where, as Christian Fries [24] notes, collateralisation is not merely some special case. It is indeed considered the norm for many interbank LIBOR type markets. Johannes [41] also explores the prevalence of collateralised trades and the importance of such collateral agreements in pricing swaps noting that 65% of plain vanilla derivatives, especially swaps, are collateralised and more specifically nearly all of swap transactions in the interbank market are collateralised.

7.3.1 OIS Discounting: A new market Standard

Pre-2008 Liquidity Crunch the typical single-curve framework swap curve, considered by most as a good approximation of a *risk-free* curve, was used to discount cashflows in pricing and valuing derivatives. This was not necessarily the universal case pre-2008 (as is shown by [15] it is likely that even before the liquidity crisis Goldman-Sachs and possibly a few other market participants were considering the implications of which discount curve to use under certain circumstances) it was though the most prevalent case and was the case referred to originally in literature such as [38], amongst others. While the original choice of discount curve may have been arbitrary without a significant impact on valuations due to negligible basis spreads, post-2008 the choice is no longer trivial. With bid-offer spreads in the interbank market as tight as a couple of points, a basis spread of even 10 basis points may have a significant impact on swap pricing and trading. Figure 7.1 shows the history of the spread between 3 Month USD LIBOR and the corresponding overnight

[2]As mentioned previously we point out that LIBOR is used generically in this context and may refer to any interbank rate fixing, such as JIBAR, and is sometimes referred to as X-BOR.

index swap. Once again, similar to Figure 5.2, spreads begin widening late 2007 up until the default and collapse of Lehman Brothers Holdings Inc. after which the spreads increased rapidly.

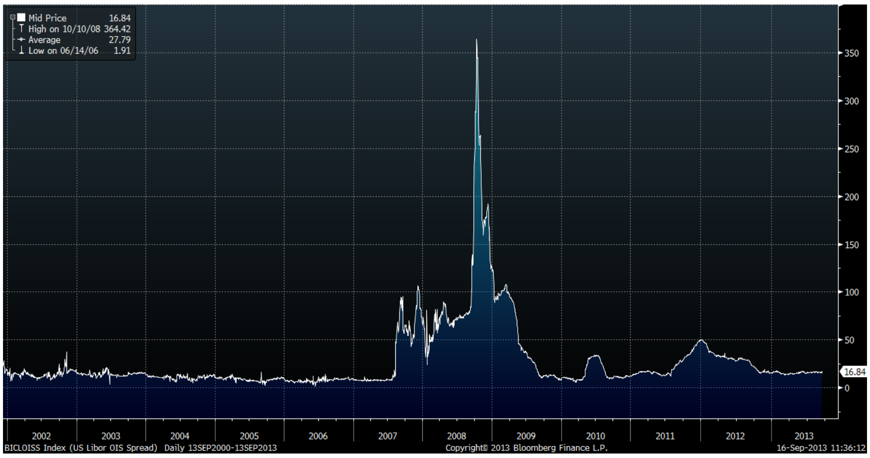

Figure 7.1: Pre- and post- 2008 crisis spread between 3 Month USD LIBOR and 3 Month Overnight Index Swap rates. (source: Bloomberg Finance L.P.)

It appears though that from mid-2009 the spreads tightened to much lower levels, though still quite volatile and reaching peaks of 50 basis points. Whilst seemingly lower than the spreads observed at the height of the liquidity crisis we should also take a look at the level of these basis spreads when compared to the 3 Month LIBOR rate. Figure 7.2 shows the historical level of 3 Month LIBOR over time both before and after the 2008 crisis. The decrease in the LIBOR-OIS spread coincides with the decrease in the 3 Month LIBOR rate. Of interest is how the LIBOR-OIS spread compares to the base underlying LIBOR rate. Figure 7.3 overlays the LIBOR-OIS spread as a percentage of the LIBOR rate and shows that, even though recent spreads are around 15 to 20 basis points, the spread is still considerably large when compared to a base LIBOR rate of 0.25%. As depicted in Figure 7.3 the spreads are currently between 50% and 65% of 3 Month USD LIBOR.

The choice of discount curve for discounting cashflows for collateralised instruments is primarily linked to the choice of collateral and the cost of funding for such collateral and numerous authors, institutions and publications have tackled this question as well as the potential impact of a move to a new discount curve. Whittall [58] highlights the emergence of the Overnight Index Swap curve (OIS) as the new standard curve used to discount collateralised swaps in the interbank market; a move from the pre-crisis assumption that

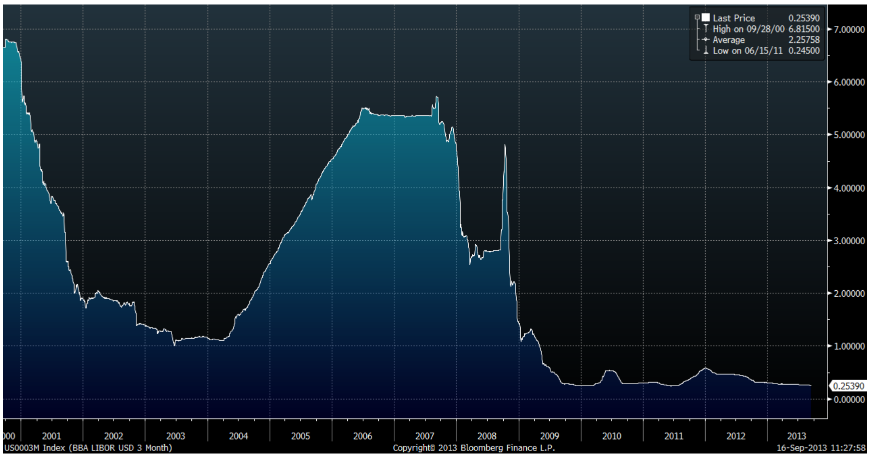

Figure 7.2: Pre- and post- 2008 crisis 3 Month USD LIBOR rates. (source: Bloomberg Finance L.P.)

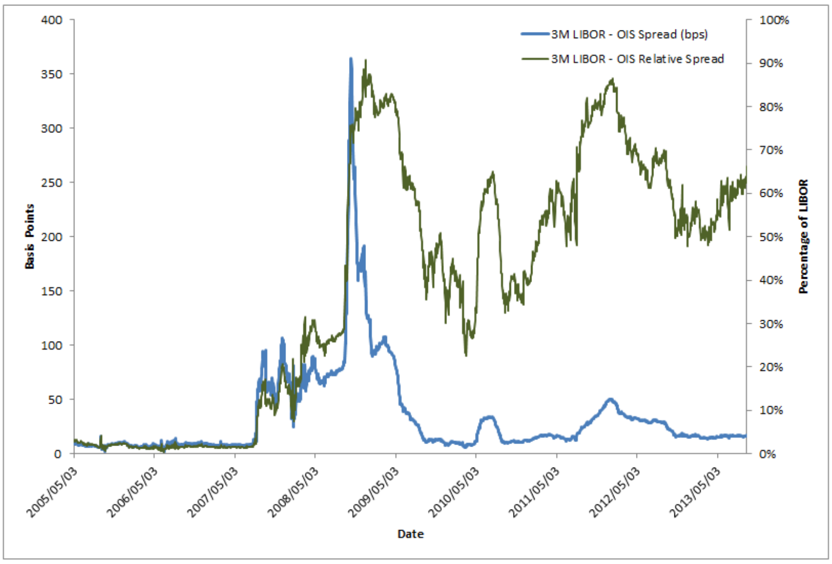

Figure 7.3: Pre and post 2008-crisis spread between 3 Month USD LIBOR and 3 Month Overnight Index Swap rates expressed as basis points as well as a percentage of the LIBOR rate. (source: Bloomberg Finance L.P.)

liquidity was easy to come by and that banks could fund swap cashflows at LIBOR no matter how long the tenor of the swap was. Cameron [15] gives some anecdotal insight into the early adoption of the OIS curve for discounting collateralised trades and how Goldman-Sachs may have had an impact on the traditional approach. As basis spreads were no longer negligible and the cost of funding collateral had a material impact, especially when there may be a large discrepancy between the rate earned on collateral and the cost of funding that collateral, the move to using an OIS curve for discounting had far reaching implications. Accounting implications, for example, were addressed by various accounting firms, such as by *PWC* [8] and pricing has been addressed by Bloomberg L.P. in [45, 44] which discussed the impact on cross currency swaps when comparing OIS curves to cross currency implied basis.

Intuitively it is easy to understand the choice of an OIS curve for collateralised trades; as collateral is posted daily using short term liquidity it is raised and funded at an overnight right. Likewise, being based on a short term rate and the curve created from OIS', which are in fact also collateralised swaps, it may be considered a very close approximation to a *risk-free* rate. As such a choice of an OIS curve for discounting seems logical at best replicating both the cost of funding the collateralised instruments' cashflows as well as accounting for the cost of funding for the collateral.

Unfortunately CSA agreements are not all standardised. Differences arise between choice of collateral with regard to choice of currency as well as the ability to post high-rated bonds, such as sovereign issued bonds, as collateral. CSA agreements are not always fully bilateral with some high-rated counterparties requiring unilateral collateralisation; that is where only one party places collateral. Other CSA agreements only require collateralisation beyond a certain threshold[3] and this threshold is not necessarily at the same level for each counterparty. Again Cameron [15] gives anecdotal evidence on how Goldman-Sachs (and possibly other banks) underwent a lengthy process to go through each CSA to create an infrastructure to develop an individual curve to cater for the intricacies of each CSA. This would have a significant impact on systems with regard to pricing and risk measuring as it would greatly increase the number of curves, and the complexity of each curve, used for discounting and for various simulations.

7.3.2 The South African market

The move to OIS discounting has not been universal and there have been some hurdles to overcome in the many markets where there does not exist a liquid and well developed OIS market from which to obtain OIS rates and

[3]In other words collateral would only be placed with a counterparty once a certain mark-to-market level has been exceeded and the collateral would only be to the amount by which this threshold had been exceeded.

from which to create and calibrate an OIS curve. Historically there existed in South Africa a market for Rand Overnight Deposit Swaps (RODS) which were typically of a short (up to one year) tenor and referenced the South African Futures Exchange (SAFEX) overnight right as the floating rate and operated in a manner similar to Overnight Index Swaps; whereby a floating overnight rate was exchanged for a fixed rate, in the case of a RODS settled at maturity and the overnight rate was averaged monthly and compounded to maturity. The market for RODS' is no longer liquid and market rates no longer published. When it comes to choosing a reference rate, other than the SAFEX overnight rate, the South African Reserve Bank also publishes a South African Benchmark Overnight Rate (SABOR), neither may be seen as entirely representative of the cost of overnight funding for a South African bank as SABOR is just a point of reference and the SAFEX overnight rate represents only a relatively small part of overnight funding in South Africa [14].

There has been, however, a move to create an OIS work group amongst interest rate dealers in South Africa which have been in discussion amongst each other and with the SARB in the creation of an equivalent South African OIS market and to publish relevant rates. In a recent address by Daniel Mminele [48], deputy governor of the South African Reserve Bank, he highlighted that the Fixed Income and Derivatives subcommittee of the Financial Markets Liaison Group (FMLG), a consultative forum between the SARB and market participants, is working with the market participants in the creation of an OIS market, in conjunction with the Money Market Subcommittee of the FMLG who are reviewing the SABOR benchmark rate. Whilst still lacking an OIS market in South Africa the move would certainly appear to be towards creating such a curve and the importance of such a curve was highlighted quite nicely by deputy governor Mminele.

> "*This work will culminate in an industry-wide consultation to deliberate on the format of a rand OIS product that would enable South Africa to be on par with its developed market peers, and to be part of a handful of emerging markets to deliver a more sound derivatives pricing and risk management landscape.*" - Daniel Mminele [48]

Thus this section on the pricing of derivatives in the presence of collateral is still relevant in a South African context where steps are being taken to build and encourage an OIS market and where even though there might not be a South African Overnight Rate curve many transactions are concluded with international counterparties and with which USD or EUR collateral is the norm.

One final comment is that it would also be possible to derive an implied South African overnight curve by using the analogy described in Section 4.4 where a cross currency swap is treated as a swap between two overnight rates in different currencies and then two basis swaps swapping the overnight rates to LIBOR/JIBAR rates. Using the LIBOR-OIS spread, as discussed later in Section 7.6, one could derive an implied South African OIS rate, though this rate could be influenced by cross-currency swap specific factors (such as demand and supply) rather than being a pure view of South African forward overnight rates.

7.4 Bootstrapping and Interpolating the OIS curve

We defined an overnight index swap in Definition 3.12 as an interest rate derivative where one counterparty exchanges a floating rate referencing on overnight deposit rate in return for a fixed interest rate with another counterparty. The floating interest rate is compounded daily to the settlement date and exchanged for the fixed rate of interest. The ISDA standard for the calculation of the floating interest rate is equivalent to interest compounding daily for each business day following the business day convention (or daycount basis) for that particular swap and can be found in numerous supplements to the standard ISDA definitions such as [39]. The nominal interest rate to be paid on the floating leg for a period t_0 to T is expressed as

$$\mathrm{I} = \prod_{i=0}^{n-1} (1 + \tau(t_i, t_i + 1)o_i) - 1, \tag{7.21}$$

where n is the number of days based on the daycount convention in the time period t_0 to T and o_i is the reference floating overnight interest rate set at time t_i[4]. From the **unbiased expectation hypothesis** and following from Equation (3.17) we can express $o(t, t_i)$, the forward overnight interest rate, as at time time t for the business day t_i, which is equivalent to $F(t; t_i, t_{i+1})$ where $t_{i+1} - t_i$ equals one business day, as

$$o(t; t_i) = E^Q[o_i | \mathcal{F}_t], \tag{7.22}$$

[4]This may be seen as the same as a one day LIBOR rate, i.e. $o_i = L(t_i, t_i + 1)$

for the risk neutral martingale measure Q and thus we can rewrite Equation (7.21) but this time as the expected nominal interest to be paid, I as

$$\begin{aligned} I &= E^Q\left[\prod_{i=0}^{n-1}(1+\tau(t_i,t_{i+1})o_i)-1\right] \\ &= \prod_{i=0}^{n-1}\left(1+\tau(t_i,t_{i+i})E^Q[o_i]\right)-1 \\ &= \prod_{i=0}^{n-1}(1+\tau(t_i,t_{i+1})o(t,t_i))-1. \end{aligned} \tag{7.23}$$

Let us define K_T^o the fixed rate for an *at-the-money* OIS, i.e. where the value of the OIS at inception date t is 0. The payoff at maturity for such an OIS would be

$$H=\left[\left(\prod_{i=0}^{n-1}(1+\tau(t_i,t_{i+1})o_i)-1\right)-K_T^o\right]N\tau(t,T), \tag{7.24}$$

and thus from Proposition 2.1 the value of an OIS at inception time t paying the simple interest fixed rate K_T^o and receiving the floating overnight rate for the period $[t,T]$ on a nominal amount N, $V_{OIS}(t;T,K,N)$, is

$$\begin{aligned} V_{OIS}(t;T,K,N) &= E^Q[P(t,T)N\tau(t,T)((\prod_{i=0}^{n-1}(1+\tau(t_i,t_{i+1})o_i)-1)-K_T^o)] \\ &= df_I(t,T)N\tau(t,T)((\prod_{i=0}^{n-1}(1+\tau(t_i,t_{i+1})o(t,t_i))-1)-K_T^o), \end{aligned} \tag{7.25}$$

where the discount factor df_I is taken from the OIS curve C_I.

Now let us look at the construction of this OIS curve C_I, where C_I is the curve given by the continuous set of discount factors $df_I(t,t_i)$. The process for constructing such a curve follows the typical curve construction and interpolation procedures as given by [13, 31, 54] and others substituting the set of market quoted OIS' as the benchmark instruments in place of the typical set of FRAs and Swaps. The curve is constructed to create a set of discount factors such that the value of each market OIS has a value of zero and the following is a brief methodology, the reader is referred to the above mentioned sources for more detailed explanations of interpolation procedures and convexity, seasonality and other adjustments. We shall focus on the base principles.

Let us first define $df_I(t_1, t_2)$ the discount factor based off of the overnight index curve C_I as analogous to Definition 3.2 but in this case our bank deposit $B(t)$ is specifically a bank deposit which pays the overnight reference rate.

Definition 7.1 (Overnight Bank Deposit)**.** *$B_I(t)$ is the value of a deposit at time $t \geq 0$ which pays the overnight reference rate compounded daily and follows the process*

$$B_I(t+1) - B_I(t) = B_I(t)(1 + o_t\, \tau(t+1, t)) \tag{7.26}$$

and $B_I(0) = 1$, o_t is the overnight reference rate for time t and $t+1$ is one business day after t.

Which leads to the following definition of a overnight rate discount factor

Definition 7.2 (Overnight Rate Discount Factor)**.** *The **overnight rate discount factor** $df_I(t_1, t_2)$ is the amount that would need to be deposited at time t_1 in an overnight bank deposit to yield an amount of 1 at time t_2 given by*

$$df_I(t_1, t_2) = \frac{B_I(t_1)}{B_I(t_2)}, \quad t_1 < t_2. \tag{7.27}$$

and thus from Equations (7.26), (7.27) and (7.22) with $B_I(t_0) = 1$ we have that

$$df_I(t_0, T) = \prod_{k=0}^{n-1} [1 + o_k\, \tau(t_k, t_k + 1)]^{-}1, \quad T - t_0 = n \tag{7.28}$$

$$P_I(t_0; t_0, T) = \prod_{k=0}^{n-1} [1 + o(t_0; t_k)\, \tau(t_k, t_k + 1)]^{-}1, \quad T - t_0 = n, \tag{7.29}$$

where $P_I(t; t_1, t_2)$ represents the expected discount factor from time t_2 to t_1 as at time t. We are also able to rewrite o_i as

$$o_i = \tau(t_i, t_i + 1) \left(\frac{df_I(t, t_i)}{df_I(t, t_i + 1)} - 1 \right). \tag{7.30}$$

When creating our OIS curve the first benchmark rate would be the current reference overnight rate, o_0 which is the reference interest rate which applies for a deposit beginning today and maturing the next business day[5].

From the assumption of no-arbitrage and using the typical replication arguments (using an overnight deposit reinvested each day to maturity and a fixed deposit to maturity) one can show that the fixed rate, K_T^o, must be

$$K_T^o = \prod_{i=0}^{n-1}(1 + \tau(t_i, t_{i+1})o(t_0; t_i)) - 1. \tag{7.31}$$

and that the value of such an overnight index swap paying such a fixed rate, at its inception date, must be 0. We thus have quoted in the market, $K_{T_i}^o$, a set of expected geometric averages of the forward overnight rates to each maturity date T_i, $i = 1, ..., q$ quoted in the market for q observable OIS'. This corresponds to I in Equation (7.21). Thus given the first benchmark rate o_0 as deterministic and observable in the market, for the first maturity OIS, maturing at time T_1 we can rewrite I_i the expected nominal interest to be paid as

$$I_i = (1 + \tau(t_0, t_1)o_o)\prod_{k=1}^{n-1}(1 + \tau(t_k, t_k + 1)o(t_0; t_k)) - 1, \tag{7.32}$$

where n is the number of business days between t_0 and T_i.

Finally let us denote $o^p(t; t_i)$ the expected overnight rate for time t_i to $t_i + 1$ following a choice of interpolation rule p based on some choice of $o(t; T_i)$ which gives us

$$I_i = (1+\tau(t_0, t_1)o_o)\left[\prod_{k=1}^{n-2}(1 + \tau(t_k, t_k + 1)o^p(t_0; t_k)) - 1\right](1+\tau(t_{n-1}, T_i)o(t_0; T_i)). \tag{7.33}$$

The final result is a set of equations for each T_i where we set

$$I_i = K_{T_i}^o, \quad i = 1, ..., q. \tag{7.34}$$

We then iteratively select each $o(t; T_i)$, $i = 1, ..., q$, following our interpolation rule p such that Equation (7.34) holds true for every $i = 1, ..., q$. Finally we may use Equations (7.28) and (7.29) to create a set of discount factors

[5]This may easily be adjusted to take into account spot and settlement days where o_i would apply from spot date t_s to settlement day t_{s+1} for the cases where an overnight deposit rate applies to a deposit made beginning in a number of spot business days from today, for example the spot date may be 2 business days from today.

and expected discount factors corresponding to curve C_I.

Many choices can be made for the interpolation rule p and the interpolated rates themselves may affect, recursively, each choice of $o(t;T_i)$. We shall not cover the implications of each possible choice of interpolation rule. Hagan et al. give a good breakdown and formula in [31] for swap curves which one could also apply to the OIS curve which is itself a specific choice of swap curve.

7.5 Pricing of Instruments under OIS discounting

We have shown that using an OIS curve for discounting cashflows in the presence of collateral has become a market accepted practice where possible (depending on the availability of an OIS market) and one would consider how and why this sort of discounting is applicable. Intuitively if a future cashflow is collateralised with cash earning the overnight rate (and this collateral is typically funded at the overnight rate too) then the present value of that cashflow should be that cashflow discounted using the OIS curve. Another intuitive interpretation would be that since the overnight rate is closer to being risk-free than the LIBOR rate it is a better representation of the true risk-free rate and as it represents a one day risk which matched the frequency of the posting of collateral it is a better rate to use when discounting cashflows. These are only intuitive solutions though. As such let us first look at the general case of pricing instruments in the presence of collateral.

7.5.1 General pricing in the presence of collateral

Christian Fries gives a re-interpretation of a collateralised contract with a discount curve in [24] where for simplicity the collateralisation of a single cashflow to be made in the future is considered. Assume that an entity A shall pay an amount of M at time T to counterparty B. For an uncollateralised cashflow, from the perspective of A, at time t would have a present value of

$$-MP^A(t;T), \tag{7.35}$$

keeping our notation from Section 7.2.1 where $P^A(t;T)$ is the present value of a zero coupon bond which is defaultable by counterparty A. Let us also assume that A holds a contract where an entity C will pay an amount $K, K < M$ at time T thus having a value of

$$KP^B(t;T). \tag{7.36}$$

If this second contract held by A were to be passed on to B as collateral against default there is now a portfolio of two contracts where from the perspective of A it can be seen giving away the second contract should the first default. This would result in the net value, N, being

$$N = (KP^C(t;T) - M)P^A(t;T), \tag{7.37}$$

where $P^C(t;T)$ accounts for the fact that C may default on its obligation to pay K. We can rewrite N as

$$N = KP^C(t;T) - MP^A(t;T) + K(P^C(t;T)P^A(t;T) - P^C(t;T)), \tag{7.38}$$

which may be interpreted as a receiving K if C does not default, and paying out M if A does not default, and paying out K if A does default but C does not default. The third term gives the difference in value between the sum of the individual contracts and the collateralised package, which we will denote as R and is expressed as

$$R = K(P^C(t;T)P^A(t;T) - P^C(t;T)). \tag{7.39}$$

From this Fries [24] obtains an implied zero coupon bond rate, P^{A_C} such that

$$-MP^{A_C}(t;T) = -MP^A(t;T) + K(P^C(t;T)P^A(t;T) - P^C(t;T)), \tag{7.40}$$

resulting in

$$P^{A_C}(t;T) = P^A(t;T) - \frac{K}{M}\left(P^C(t;T)P^A(t;T) - P^C(t;T)\right), \tag{7.41}$$

where P^{A_C} is the discount factor for collateralised deals.

This can be extended to the specific case where the collateral is considered risk-free, i.e. cash collateral, and we may set

$$P^C(t;T) = P(t;T), \tag{7.42}$$

where the collateral earns the risk-free zero coupon bond rate from time t to time T. If the present value of the cash collateral is equal to the payment to be made by A, i.e. $K = M$, then Equation (7.38) can be rewritten as

$$\begin{aligned} N &= -MP^A(t;T) + MP(t;T)P^A(t;T) && (7.43)\\ &= -MP^A(t;T)(1 - P(t;T)), && (7.44) \end{aligned}$$

which results in

$$P^{A_C}(t;T) = P^A(t;T) - (P(t;T)P^A(t;T) - P(t;T)) \tag{7.45}$$

7.5.2 Extension to Continuous Time and Overnight Funding

General pricing in the presence of collateral can be extended to consider the case of modelling under continuous time and to take into account the common form of daily collateral. At first no specific assumption will be made on the cost of collateral (the collateral rate) nor the rate at which cash collateral would earn any interest if invested at the risk-free rate. Both Fujii et al. [25, 26], and Johannes and Sundaresan [41] consider the impact of collateralisation on swaps and other interest rate derivatives analysing the case in continuous time. Johannes builds on the Duffie and Singleton approach to valuing defaultable securities while Fujii et al. take a more axiomatic approach and the two results are comparable. The case of pricing in the presence of collateral is also explored by Morini in [50] who uses a martingale approach with a change of numeraire under the LIBOR Market Model using a quanto cross currency style analogy which coincides with the analogy used in Section 5.6.

To consider the continuous time case an initial assumption (which can later be relaxed) is that collateralisation occurs bilaterally, continuously and perfectly with zero thresholds (i.e. to the full mark-to-market value of the collateralised trade and with perfect timing); the market practice is that of daily collateralisation (which conforms to the ISDA standard) earning an overnight rate and the assumption of continuous collateralisation is reasonably close [26] and the results may be extended to a discrete case. Under continuous collateralisation counterparty risk is completely negated; a swap can thus be treated as a set of independently collateralised payments. A portfolio of contingent claims can also be treated as set of independently collateralised payments (or expected payments).

Fujii et al. consider a stochastic process $V(t)$ which represents the collateral account. One can consider this account as a deposit with the counterparty bank which earns continuous interest as does the bank deposit described in Definition 3.1, in the case of overnight interest it would match the overnight bank deposit described in Definition 7.1. We shall denote the collateral rate (i.e. the cost of funding collateral) at time t as $c(t)$ and the rate earned by the collateral at time t as $r(t)$, in [26, 41], the latter rate is assumed to be the risk-free interest rate but this need not necessarily be the case and the results obtained still hold. The difference between these two rates can be denoted by $y(t) = r(t) - c(t)$ and can be seen as a cost (or profit) over time resulting from an interest rate mismatch arising from the cost to raise collateral and the interest earned on placing collateral and can be seen as a carry cost. Thus the process of the collateral account may be given as

$$\mathrm{d}V(s) = y(s)V(s)\,\mathrm{d}s + a(s)\,\mathrm{d}h(s), \tag{7.46}$$

which can be interpreted as the continuous change in the collateral process is equal to the instantaneous carry cost of the collateral plus the instantaneous change in the value of the collateralised instrument, denoted by $h(t)$, multiplied by the size of the position in that instrument, denoted by $a(t)$. With a maturity date T for the instrument denoted by $h(t)$ integrating Equation (7.46) we obtain

$$V(T) = e^{\int_t^T y(u)\,\mathrm{d}u}\,V(t) + \int_t^T e^{\int_s^T y(u)\,\mathrm{d}u}\,a(s)\,\mathrm{d}h(s). \tag{7.47}$$

Earlier on we defined a trading strategy in Definition 2.1. We will now select a trading strategy with the value process and gains process respectively as

$$V(t) \quad = \phi_t S_t \quad = h(t) \tag{7.48}$$

$$a(s) \quad = G_s(\phi) \quad = exp\left(\int_t^s y(u)\,\mathrm{d}u\right), \tag{7.49}$$

which represents an investment in the instrument with value $h(t)$ re-balanced by process $a(t)$ which can be seen as the continuous reinvestment of the change in the collateral amount, or in other words the change in the collateral amount must equal to the change in value of $h(t)$ by the definition of continuous collateral and thus results in a self-financing trading strategy. Applying this trading strategy in accordance with Equation (7.47) results in

$$V(T) = exp\left(\int_t^T y(s)\,\mathrm{d}s\right) h(T), \tag{7.50}$$

which is an intuitive result in that the value of the collateral process must be the value of the collateralised instrument with any interest earned or lost due to the carry cost $y(t)$. The present value of the instrument with maturity payoff $h(T)$ follows from Proposition 2.2 and is given by

$$\begin{aligned} h(t) &= E_t^Q[P(t,T)h(T)|\mathcal{F}_t] \\ &= E_t^Q[e^{-\int_t^T r(s)\,\mathrm{d}s}\,e^{\int_t^T y(s)\,\mathrm{d}s}h(T)] \\ &= E_t^Q[e^{-\int_t^T (r(s)-y(s))\,\mathrm{d}s}h(T)] \\ &= E_t^Q[e^{-\int_t^T c(s)\,\mathrm{d}s}h(T)]. \end{aligned} \tag{7.51}$$

Using the LIBOR curve (i.e. LIBOR discounting) or some other assumed curve not based on the collateral rate is thus not appropriate [26]. The same result is obtained in [41] where the term $r(s) - y(s)$ is likened to the stochastic dividend yield formula (and hence the analogy that $y(t)$ can be seen as a carry cost) with a further analogy that collateralised swaps can be seen as a portfolio of futures contracts (which are indeed margined daily).

To interpret this collateral cost we may consider the case where a contract has a positive mark-to-market value (i.e. there is the expectation to receive a future cashflow). In such a case there would be an immediate receipt of an equivalent cash amount of collateral on which the collateral rate would need to be paid and the whole amount returned at maturity. In other words it can be seen as a loan funded at the collateral rate. The collateral rate used in the market is typically the overnight rate; such as the Federal Funds Rate (or Fed Rate) for US Dollars or the SAFEX Overnight Rate for South African Rands. The use of such a rate makes sense in the context of daily collateral posting and thus the appropriate curve to use from which to obtain $c(t)$ in Equation (7.51) would be the overnight curve or OIS curve being the curve of expected forward overnight rates.

7.5.3 The discrete time case

The results above can be extended to account for discrete time collateralisation and overnight deposit returns. Replacing the integrals with the product of compounding daily simple rates would yield such a result. Johannes and Sundaresan [41] which give a simple interpretation of the mechanics using two time steps. This can be extended to multiple time steps which is the case we shall consider.

Let us first assume that an interest rate swap has a fixed rate K so as to make the market value of all future cashflows 0. At some time t the swap would have an end of day value of $h(t)$ and collateral is posted equal to this amount having the carry cost

$$y(t) = \frac{1}{\tau_t}\left(\frac{1+\tau_t r(t)}{1+\tau_t c(t)} - 1\right), \tag{7.52}$$

where $r(t)$ and $c(t)$ are simple rates compounded daily and τ_t represents the year fraction over time t to $t+$ one business day. $y(t)$ can be seen as representing borrowing at one rate and then investing at another rate.

The collateral account process $V(t)$ can thus be given as

$$\Delta V(t) = (1 + \tau(t+1,t)y(t))V(t) + a(t)\Delta h(t), \tag{7.53}$$

where $\Delta V(t)$ is the change in the collateral process over a single day. This arises by posting collateral to the value of the change in the mark-to-market of the swap represented by $\Delta h(t_i)$ over time t_i to $t_i + 1$ which is one business day. The resulting value at maturity would thus be

$$V(T) = \prod_{u=t}^{n-1} (1 + \tau(u+1,u)y(u))\, V(t) + \sum_{s=t}^{n-1} \left(\prod_{u=t}^{n-1} [1 + \tau(u+1,u)y(u)] \right) a(s)\Delta h(s), \tag{7.54}$$

where the number of days between times T and t is n. Using the same trading strategy as described in the continuous case but adapted for discrete re-balancing would give the value process and gains process respectively as

$$V(t) = h(t) \tag{7.55}$$

$$a(s) = \prod_{u=t}^{n-1} (1 + \tau(u+1,u)y(u)). \tag{7.56}$$

Applying this trading strategy with Equation (7.54) results in

$$V(T) = \prod_{s=t}^{n-1} (1 + \tau(s+1,s)y(s))\, h(T). \tag{7.57}$$

Now $h(t)$ is the present value of a set of, say, k future payoffs at times t_1 to t_k. Each payoff is of the amount

$$H_j = \tau(t_{j-1}, t_j)[L(t_{j-1}, t_j) - K], \tag{7.58}$$

from the perspective of the fixed rate payer where H_j represents the payoff at time t_j, $j = 1, ..., k$. The present value of the swap at time t, taken again using Proposition 2.2 is thus

$$h(t) = \sum_{j=1}^{k} E^Q \left[P^r(t;t_j) H_j | \mathcal{F}_t \right], \tag{7.59}$$

where P^r represents the zero coupon bond taken off our (previously assumed) “risk-free” curve. We can also set

$$P^r(t;t_j) = \prod_{u=1}^{n_j} (1 + \tau(u+1,u)r(u))^{-1}, \tag{7.60}$$

which follows by applying Equations (3.12) and (3.13) iteratively where the number of days between t and t_j is n_j.
Finally by substituting Equation (7.58) into (7.59) and then applying Equation (7.57) and applying the following algebraic relation

$$\frac{1+y}{1+r} = \frac{1+r}{1+c} \cdot \frac{1}{1+r} = \frac{1}{1+c}, \tag{7.61}$$

where r can be seen to represent $\tau(t+1,t)r(t)$, c to represent $\tau(t+1,t)c(t)$ and y to represent $\tau(t+1,t)y(t)$ we obtain the result

$$\begin{aligned} h(t) &= \sum_{j=1}^{k} \left(E^Q \left[\frac{\prod_{u=1}^{n_j} (1+\tau_u r(u))}{\prod_{u=1}^{n_j} (1+\tau_u y(u))} \frac{1}{\prod_{u=1}^{n_j} (1+\tau_u r(u))} H_j \right] \right) \\ &= \sum_{j=1}^{k} \left(E^Q \left[\prod_{u=1}^{n_j} \frac{1}{1+\tau_u c(u)} (\tau_j [L(t_{j-1},t_j) - K]) \right] \right), \end{aligned} \tag{7.62}$$

where we have abbreviated the relevant year fractions as τ_z for simplicity and formatting.

This result is equivalent to that obtained in continuous time and shows the appropriateness of discounting future collateralised cashflows at the collateral rate; which we have determined to be the overnight rate. The above case shows the valuation for a swap though H_j can be substituted for any cashflow or contingent cashflow.

7.6 Reconciling OIS discounting with Basis Spreads and Credit Risk Measures

As shown in the previous sections the LIBOR rate can no longer be considered as reflective of a "risk-free" rate but rather also contains an element of credit and liquidity risk in the interbank market. Under our assumption of homogeneity of LIBOR market banks we assume that each bank participating at LIBOR, and thus able to fund itself at LIBOR for the term of the relevant LIBOR deposit (e.g 6-Month LIBOR), has a similar risk of default and carries similar liquidity risk. Should a particular bank no longer be able to fund itself at (or respectably near) LIBOR it should be treated as no longer being a LIBOR bank. We can extend this assumption of homogeneity to the overnight deposit market where each bank in market is able to fund itself at the overnight rate. Due to the very short-term nature of such an overnight deposit the overnight rate can be seen as a strong indicator of where the theoretical "risk-free" rate should be as it would contain, at a maximum, a single day's worth of credit risk. We can also assume that at any point the overnight reference rate should be lower than the LIBOR rate of a longer term as the longer term LIBOR deposit carries greater credit risk as well as a liquidity premium. Likewise we would expect OIS swaps to trade at a lower fixed rate than a term LIBOR swap with the same maturity date.

Let us consider a theoretical OIS for LIBOR basis swap (as defined in Definition 5.1) with a maturity date of T which can be transacted between two LIBOR banks. This basis swap would be replicated by entering into a vanilla LIBOR swap paying the floating reference rate and receiving the fixed swap right and simultaneously entering into an OIS swap receiving the floating interest rate and paying the fixed swap rate. Under the traditional single curve approach to pricing and valuing interest rate derivatives the fixed rate for the two swaps should be the same or else an arbitrage opportunity would arise. In the market though there is a difference between these two rates referred to as the LIBOR-OIS spread. This **basis spread** is the amount by which the LIBOR swap rate exceeds the OIS swap rate for swaps of the same tenor.

Before the liquidity crisis these spreads were relatively small and stable at around 10 basis points but peaked to around 350 basis points (6-Month LIBOR-OIS Spread) following the collapse of Lehman-Brothers in September 2008 [56].The LIBOR-OIS spread can be interpreted as an indicator of expected market turmoil as well as a measure of the credit riskiness of LIBOR banks. Further, if the OIS curve may be considered as the "risk-free" curve (or a reasonably close approximation thereof) due to its referencing the very short-term overnight rate the LIBOR-OIS spread may be seen as a measure of the credit risk (and built in liquidity risk) of the relevant LIBOR banks.

We will now attempt to reconcile the concept of basis-spreads, and specifically the LIBOR-OIS spread, with credit risk measures and the traditional approach to discounting default contingent cashflows, i.e. CDS pricing. We will also attempt to justify using the LIBOR-OIS spread as a measure of inter bank counterparty risk

Earlier we considered approaches to including basis spreads into the traditional interest rate curve creation and swap and FRA pricing where different forward rate curves are created to estimate expected future LIBOR reference rates. The basis spreads between each tenor can again be seen as a measure of credit risk for the LIBOR bank counterparties when considering interbank deposits of those tenors. Our LIBOR-OIS spread is just another basis spread and can be treated appropriately as such. Under the axiomatic approach in Section 5.5 we obtained a beta coefficient β_t^f as used in Equation (5.10) which is a multiplicative factor used in calculating our forward rates. In the bottom-up market related approach in Section 5.6 we obtained a similar factor to account for the riskiness between terms which was denoted BA_{fd} for the multiplicative case and BA'_{fg} for the additive case which were compared to a Quanto-Style approach. The question raised is that if these basis spreads are measures of bank default risk then how do they reconcile with the approaches used to value credit derivatives such as credit defult swaps.

Let us fist give a simple definition of a credit default swap, or CDS:

Definition 7.3 (Credit Default Swap)**.** *A **credit default swap**, or CDS, is an agreement between two counterparties whereby one counterparty (the CDS buyer) pays a series of fixed payments over a predetermined period of time (i.e. up until a specific maturity date) to the other counterparty (the CDS seller) in exchange for a payment of a notional amount occurring in the event of a default of some third party reference entity during the predetermined period. The CDS can be cash settled where the nominal payment by the second party is the notional amount of the CDS less any recovery rate determined to be applicable to debt (of the same seniority as referenced) of the third party reference entity. Alternatively the CDS can be physically settled whereby the CDS buyer receives the full notional amount of the CDS and returns to the CDS seller a debt instrument (now defaulted) of the same notional amount. The event of default by the third party reference entity is referred to as a credit event and no further fixed rate payments are made by the CDS buyer after this time.*

The CDS buyer is often also referred to as the protection buyer and the CDS seller as the protection seller and the fixed amount paid is referred to as the CDS premium. As is the case for swaps and FRAs an at-the-market CDS is one where the CDS premium is such that the value at inception of the CDS is zero and CDS quotes can be obtained in the market for CDSs of various tenors and reference entities. We shall not go into the details of valuing a CDS but shall give a brief description of the concept of a survival probability and how it is used to value a CDS. As in the pricing of all contingent claims the value of the CDS can be given by the present value of all expected cashflows. As the fixed payments only occur so long as the third party reference entity has not defaulted the expected present value at time t of some credit contingent cashflow of amount p to be made at time T, p_t, can be expressed as

$$\begin{aligned} p_t &= E^Q[P(t;T)p|\mathcal{F}_t] \\ &= P(t;T)sp^A(t;T)p, \end{aligned} \tag{7.63}$$

where $sp^A(t;T)$ is called the survival probability (for entity A) and it denotes the probability, at time t, that some entity A has not defaulted before time T. This is analogous to the term $1_{\{t^A>T\}}$ which was used in Sections 7.2.1 and 7.2.2 where we considered defaultable counterparties. Equation (7.63) assumes that there is no recovery rate on the payment of p. This can be extended to the case of some recovery rate R as follows

$$\begin{aligned} p_t &= P(t;T)\left[p\,sp^A(t;T) + pR\,(1 - sp^A(t;T))\right] && (7.64) \\ &= P(t;T)\left[p\,sp^A(t;T) + pR - pR\,sp^A(t;T)\right] \\ &= P(t;T)\left[pR + p(1-R)sp^A(t;T)\right]. && (7.65) \end{aligned}$$

We can see that Equation (7.65) is analogous to Equation (7.1). In both these equations it is assumed that the recovery payment will be made at time T and not on the default event day. The rate $dp^A(t;T) = 1 - sp^A(t;T)$ is commonly called the default probability. Typically in the market survival probabilities are derived from “hazard rates” or “default intensities” commonly denoted as $\lambda(t)$ via the following relationship which can be found in various sources, for example in Hull [38],

$$sp^A(t;T) = e^{-\int_t^T \lambda(s)\,\mathrm{d}s} = e^{-\bar{\lambda}(T)\,T}, \tag{7.66}$$

where $\bar{\lambda}(T)$ is the average hazard rate between times t and T.

The hazard rate function, or hazard rate curve, $\lambda(t)$, can be obtained by from market prices either by using the market quoted premium rate for a CDS and deriving the hazard rate such that the value of a CDS is zero, or inferred from credit spreads. A credit spread is the rate above some reference risk-free rate (for example a government bond) at which an entity can obtain funding for a certain term. The relationship between hazard rates and credit spreads can also be found in various sources and we refer to the relationships given in Hull [38]

$$\begin{aligned} \int_0^T \lambda(u)\,\mathrm{d}u &= \frac{1}{1-R}\int_0^T s(u)\,\mathrm{d}u, && (7.67) \\ \bar{\lambda}(T) &= \frac{s}{1-R}, && (7.68) \end{aligned}$$

for some assumed recovery rate R. In this case s could be the spread of the reference entities bond yield over the risk-free rate for tenor T. We can reformulate this survival probability, when measured at time 0, as

$$sp^A(0;T) = \frac{e^{-\int_0^T r(u)-\lambda(u)\,\mathrm{d}u}}{e^{-\int_0^T r(u)\,\mathrm{d}u}} \tag{7.69}$$

$$= \frac{e^{-\int_0^T r(u)-\frac{s(u)}{1-R}\,\mathrm{d}u}}{e^{-\int_0^T r(u)\,\mathrm{d}u}}, \tag{7.70}$$

where $r(t)$ is the instantaneous short rate at time t and $s(t)$ the equivalent instantaneous credit spread at time t. We can thus express the survival probability as the ratio between two zero coupon bonds or between two discount factors and the same result can be derived for other forms of discounting.

We can therefore justify the interpretation of LIBOR-OIS spreads and other basis spreads as measures of credit risk and the multiplicative basis adjustments β and BA defined in Sections 5.5 and 5.6 as forward survival probabilities. This also matches the factor H_f used in the Quadratic Gaussian Model in Section 6.2 and used in the Vasicek Model in Section 6.3. Likewise as per Tuckman and Porfirio [57], where the spread between a market FEC rate and the implied FEC rate derived from the single-curve framework swap curve or the basis spread in cross-currency swap rates can be interpreted as a measure of default risk, the factor X_{fd} used in the cross currency swap analogy of Section 5.6, which is also a ratio between two zero coupon bond prices, can also be interpreted as a survival probability.

Chapter 8

Conclusion

The objective of this dissertation was to study the rise of basis spread and basis risk, the cause of these spreads, and the relevant models to model it after the 2008 liquidity crisis. We began by introducing the background theory to interest rate curve construction and forward rate estimation in Chapters 2 and 3. Of importance is the concept of arbitrage and the assumption that no arbitrage opportunity should exist in the market. This assumption gives rise to the common interest rate curve construction and forward rate estimation methods typically used before the 2008 liquidity crisis which is explored in Chapter 4. The main result shown in this chapter is that under the traditional single curve framework, pre 2008, there was no-arbitrage opportunity arising between interest rate instruments of different tenors with the forward interest rates estimate and discounted off the same interest rate curve. This was illustrated by example and the results of taking forward interest rates calculated using instruments of various tenors compared to the market rate as depicted in Figure 4.2. This figure depicted the spread, or difference, in the quoted forward rate for 3x9 USD LIBOR FRAs and the calculated forward rate as implied by 3x6 and 6x9 USD LIBOR FRAs. It was shown that this spread was never greater than 2.5 basis points in the period observed.

We then took a look at just how such a pre-crisis single interest rate curve would be constructed; specifically that instruments were chosen based on liquidity rather than with reference to a specific tenor as the choice of underlying tenor was assumed arbitrary and inconsequential. We then also took a look at a simple case for multiple currency multiple curve construction and the concept of interest rate parity as this would later be used when dealing with the cross currency swap analogy for a multiple curve framework post the 2008 liquidity crunch.

8.1 Multiple Curve Approach

In Chapter 5 we aimed to depict the rise of basis spreads between interest rate instruments of tenors which arose during and after the 2008 liquidity crisis. This was illustrated by taking the same approach and examples from Chapter 4 and applying it to market rates post the 2008 crisis. The result is that there would now appear to be an arbitrage opportunity between instruments of different tenors. Figure 5.2 illustrated the spreads between the quoted 3x9 USD LIBOR FRA and the 3x9 forward rate implied by the single curve approach using instruments of a different tenor. What was observed and depicted is that starting from late 2007 when the subprime mortgage crisis began these spreads started increasing up until the 2008 collapse of Lehman Brothers Holding inc. after which these spreads exploded reaching a peak of 46 basis points. This rise of basis spreads was further illustrated by observing the market quotes of basis swaps exchanging 3 Month USD LIBOR for 6 Month USD LIBOR. Figures 5.3 and 5.4 depict these market quotes for basis swaps with maturities of 1 and 5 years respectively.

What was now clearly evident is that the pre-crisis single-curve fits all approach is no longer suitable and would give rise perceived arbitrage opportunities under such a single curve framework. We wished to explain the cause of these widening basis spreads and the breakdown of the traditional single curve approach and to take a look at what the literature available on the topic explained as stated under our objectives in the introduction. Acerbi and Scandolo [1] defined various sources of liquidity risk and Morini [50] expanded on these risks and it was evident that the various specific elements of basis spreads, together with credit risk which is strongly correlated to liquidity risk, would be difficult to separate. To cite Mercurio [46] the divergence between implied forward rates and the FRA rate may be seen as a measure or representation of the potential future credit or liquidity issues. As the single curve paradigm which considers such liquidity risks as negligible is no longer complete we wished to take a look at how such a paradigm could be expanded so as to preserve the assumption of no-arbitrage. This gives rise to the multiple-curve approach described in Section 5.4. We took a look at the approach taken by various authors [6, 46, 47, 50, 57] who agreed that cashflows with the same probability of occurring (in this case in an assumed homogenous interbank market) on the same date would need to be discounted off of the same curve so as to preserve the assumption of no-arbitrage. This then implied that the future cashflows as estimated by forward interest rates would need to be determined off of different curves or with an adjustment to the rate calculated off of a single curve.

We proceeded to take a look at two general approaches to modelling basis risk. Firstly that of an axiomatic approach as proposed by Henrard [36] and further explored by Binanchetti [6] and Morini [50]. This approach is similar to that of Pallavicini and Tarenghi [51] and tackles the problem by taking each forward rate as a single independent asset without individually modelling the dynamics between liquidity and credit risks. The approach begins by proposing multiple forward rate curves which are then fitted to observations in the market. No specific market model is applied to the spreads by Henrard [36]; we instead took a look at such possible models later in Chapter 6. The second approach considered was that of a bottom-up market related approach as proposed by Ametrano [2]. Under this approach forward basis spreads are used to reclaim the no-arbitrage assumption. This approach was further extended by Bianchetti [6] who explores a foreign currency quanto-style analogy to reclaim the no-arbitrage assumption. This analogy is supported by Kijima et al. [42], Michaud and Upper [47], and Tuckman and Porfirio [57] who make similar observations. We see that under these two approaches, which segment the interest rate market to a number of sub-markets each of different tenors, that the assumption of no-arbitrage can now be reclaimed and that the basis spread, denoted as β, can now be explored and modelled either independently or as multiplicative of additive factor correlated to a base underlying interest rate curve.

8.2 Post Crisis Market Models

Under Chapter 6 we explore the dynamics of the previously mentioned basis or quanto style adjustments. We take a look at a number of models proposed such as modelling basis under the Black model and under the SABR/CEV model as proposed by Henrard [36]. It is shown that under such a model the basis spreads between forward rates is neither constant nor deterministic but rather proportional to the forward rates. We see that this is analogous to the model proposed by Mercurio [46], though under Mercurio no relationship is proposed between the forward rates which are treated independently. Further Henrard [36] explores basis spreads under the SABR/CEV model with forward rates following a SABR equation with an added basis spread factor. It is observed though that under these models the basis spreads increase as the underlying rates increase which was contrary to that observed in the market. A possible explanation proposed was that due to the lack of spreads in the original market models the observed increasing spreads were a result of models attempting to catch up with what was being observed at the time.

We followed by taking a look at a Quadratic Gaussian model as proposed by Kijima et al. which extends the market model to include a third curve, the government yield curve, followed by taking a look at modelling basis spreads under the Vasicek model also proposed by Kijima et al. These approaches are also based on distinguishing discount rates from forward rates and further by distinguishing swap rates from fixed-coupon bond rates. We close this chapter by considering an extension to the prolific LIBOR Market Model as proposed by Mercurio [46] and considering basis under both forward measures and under spot measures. The LIBOR Market Model itself is a computationally intensive method and such an extension for multiple basis spreads for curves of different tenors would greatly increase this intensity.

8.3 Risky Markets

We close this study by exploring the implications of basis spreads and the risks implied to the traditional approaches of valuing interest rate derivatives in the interbank market and the implications to the pre-crisis "risk-free" curve. Under the risks implied by basis spreads the traditional single curve approach to creating a forward rate curve could no longer be considered risk-free. We took a look what various authors such as Johannes and Sundaresan [41], Fujjii et al. [25, 26, 27, 27], Crepey [21] and others had researched with regard to the impact of basis risks and counterparty default risk in the interbank market on interest rate derivative pricing. It was shown that separate approaches would need to be taken for interest rate derivatives transacted in the presence of default risk and those transacted under a collateral agreement under which such risks were mitigates. The approach considered was the use of the overnight index swap curve which has risen as a new market standard curve for the pricing of instruments transacted under daily collaterlisation agreements and we also considered how such a curve could be considered as a proxy for the hypothetical "risk-free" curve. Finally we consider a simple approach to constructing such a curve and showed the equivalence between the LIBOR-OIS spread and credit spreads commonly used in the pricing of credit default swaps and related credit derivative instruments. This enabled us to reconcile the OIS discounting approach which implied the basis spread between OIS curves and LIBOR curves as a measure of default risk in the interbank market and the default and survival probabilities, themselves ratios between two implied interest rates, used in credit derivative pricing models.

8.4 Closing Remarks and Areas of Further Study

Our objective was to study the rise of basis spreads in the interest rate market, depict how the rise of such spreads would cause a breakdown in the traditional no-arbitrage assumption for interest rates of different tenors, and then explore how the no-arbitrage assumption could be reclaimed. We also wished to consider how some prevalent interest rate market models could be extended to include basis spread factors to account for the changes observable in the interest rate market. Finally we wished to reclaim the concept of a "risk-free" curve and show how the rise of the overnight index swap curve could be a proxy for just such a curve and its applicability as a discount curve for risk-free (or close to risk free) cashflows for instruments transacted under daily cash collateralisation.

It was illustrated that the rise of basis spread and basis risk in the market gave rise to a breakdown of the no-arbitrage assumption previously assumed. We found that the interest rate market could be segmented into a multiple-curve framework which could be fitted to market observable rates for interest rate derivatives of different tenors. Following either an axiomatic approach or a bottom-up approach the assumption of no-arbitrage could be reclaimed and the two approaches could be seen as equivalent.Such an approach though relies on the prevalence of liquid market related quotes for interest rate derivatives with tenors matching those we wish to model.

We also found that numerous current interest rate modelling methods, such as the Black model and the Vasicek model, could be extended to such a multiple curve framework; though without making any assumption of modelling the basis spread as an independent factor. We also took a look at how the ubiquitous LIBOR Market Model could be extended, as proposed by Mercurio [46], to include a basis spread factor. We remark though that such an extension leads to even greater computational intensity to what is already a computationally intensive model.

It was also shown that the rise of an overnight index swap curve as a proxy for the "risk-free" curve is indeed a suitable approach and how interest rate derivatives transacted under a collateralisation agreement could be priced using such a curve. We concluded that the LIBOR-OIS spreads between such a curve and the traditional LIBOR swap curve, implying that it is a measure of the default risk inherent in the interbank market, can be reconciled with the credit spread approaches used in the valuation of credit default derivatives.

Further areas of study would be to consider the modelling of basis spread factors as stochastic variables under, for example, a Heston type model. Also of consideration would be how to model the correlation between the basis spread factors of different tenors. Application of such models in the South African interest rate market is currently hampered by the lack of liquid observable market quotes for interest rate derivatives of different tenors though with the proposed introduction of a South African equivalent to the overnight index swap this could be tackled at a later date when there is more observable data. The multiple curve approach could also be extended to include counterparty specific risks and to disentangle the various risks all implied by a single basis spread (e.g. liquidity and default risks) as currently the approach still considers each LIBOR market participant as being mostly homogenous.

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Appendix A

Daycount and Business Day Conventions

A daycount convention determines how interest is calculated over a period of time and varies between currencies and instruments. This section will outline a few common daycount conventions and their use. The term daycount basis is also used to refer to daycount conventions. In this section t_1 and t_2 represent two calendar dates with t_2 falling after t_1. t_1 may be treated as $y_1/m_1/d_1$ and t_2 as $y_2/m_2/d_2$ to represent the year, month and day of each date respectively. This is not an exhaustive set of conventions but covers the ones most commonly used.

A.1 Actual Methods

actual/actual

Also referred to as **act/act** or **act/act (ISDA)**. Under this convention, when calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/act}(t_1, t_2) = \frac{D_1}{365} + \frac{D_2}{366} \tag{A.1}$$

where D_1 is the total actual number of calendar days between t_1 and t_2 which do not fall within a leap year and D_2 is the number of total actual calendar days between t_1 and t_2 which do fall within a leap year.

actual/365

Also referred to as **act/365** or **act/365F** where the F stands for "Fixed". Under this convention, when calculating the year fraction between times t_1

and t_2, the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{365} \tag{A.2}$$

where D is the total actual number of calendar days between t_1 and t_2 and the denominator is always 365. This is the convention typically used in South Africa for interest rate swaps, FRAs and other derivatives.

actual/360

Also referred to as **act/360**. Under this convention, when calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{365} \tag{A.3}$$

where D is the total actual number of calendar days between t_1 and t_2 and the denominator is always 360.

actual/365A

Also referred to as **act/365A**. Under this convention, when calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{Y} \tag{A.4}$$

where D is the total actual number of calendar days between t_1 and t_2 and Y is 366 if the leap day (29th February) falls between t_1 and t_2 else it is 365. This is in contrast to the **actual/actual** convention which takes 366 into the denominator even if the accrual period ends before the leap day.

actual/365L

Also referred to as **act/365L**. Under this convention, when calculating the year fraction between times t_1 and t_2 the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{Y} \tag{A.5}$$

where D is the total actual number of calendar days between t_1 and t_2 and Y is 366 if the accrual period ends in a leap year else it is 365. This is in contrast to the **actual/actual** convention which takes 366 into account in the denominator if the period begins in a leap year but does not end in one.

NL/365

Also referred to as **act/365 NL** or **NL365**. Under this convention, when calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{365} \tag{A.6}$$

where D is the total actual number of calendar days between t_1 and t_2 if there is no leap day in the accrual period otherwise it is the total actual number of calendar days - 1.

BUS/252

Also referred to as **business/252** Under this convention, when calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/365}(t_1, t_2) = \frac{D}{365} \tag{A.7}$$

where D is the total actual number of business days between t_1 and t_2 where a business day is any day not falling on a weekend and not falling on a holiday.

A.2 Fixed Day Methods

In this section the daycount convention is usually represented in the form **d/n** where d represents the denominator and assumes a fixed number of days in a period and n represents the denominator and is usually 360 or 365. The definitions below shall assume d/360 as a standard but 360 can be substituted for any appropriate number as defined by that convention e.g. **20/252** could represent a fixed 20 business days in a month divided by a fixed 252 to represent the number of business days in a year.

30/360

Also referred to as **30/360 ISDA**. Under this convention each month is treated as having only 30 days in it and the year contains 360 days. When calculating the year fraction between times t_1 and t_2 the year fraction is calculated as

$$\tau_{act/360}(t_1, t_2) = \frac{D}{360} \tag{A.8}$$

where D is $30 \cdot (m_2 - m_1) + 360 \cdot (y_2 - y_1) + (d'_2 - d'_1)$ and $d'_1 = 30$ if $d_1 = 31$ else $d'_1 = d_1$ and if $d_1 = 31$ then also set $d'_2 = 30$ else $d'_2 = d_2$.

30E/360

Also referred to as **30/360 European** or **30/360 ISMA**. Under this convention each month is treated as having only 30 days in it and the year contains 360 days. When calculating the year fraction between times t_1 and t_2 the year fraction is calculated as

$$\tau_{act/360}(t_1, t_2) = \frac{D}{360} \quad \text{(A.9)}$$

where D is $30 \cdot (m_2 - m_1) + 360 \cdot (y_2 - y_1) + (d'_2 - d'_1)$ and $d'_1 = 30$ if $d_1 = 31$ else $d'_1 = d_1$ and if $d_2 = 31$ then set $d'_2 = 30$ else $d'_2 = d_2$. This differs to **30/360 ISDA** in that under **30E/360** $d'_2 = 30$ is changed if $d_2 = 31$ as opposed to being changed based on d_1. There is no special treatment for the last day of February (i.e. d_1 or d_2 stay $= 29$).

30U/360

Also referred to as **30/360 US**. Under this convention each month is treated as having only 30 days in it and the year contains 360 days. When calculating the year fraction between times t_1 and t_2, the year fraction is calculated as

$$\tau_{act/360}(t_1, t_2) = \frac{D}{360} \quad \text{(A.10)}$$

where D is $30 \cdot (m_2 - m_1) + 360 \cdot (y_2 - y_1) + (d'_2 - d'_1)$ and if t_1 and t_2 is the last day of February (28 in a non leap year or 29 in a leap year) then $d'_2 = 30$; and if t_1 is the last day of February then $d'_1 = 30$; and if $d_1 = 31$ then $d'_1 = 30$; and if $d_2 = 31$ and $d_1 = 30$ or 31 then $d'_2 = 30$.

A.3 Business Day Conventions

Also known as date rolling convention or business day rules. This is a set of rules for the treatment of interest accrual periods beginning or ending on a weekend or public holiday, i.e. if either t_1 or t_2 fall on a weekend or public holiday. The rule defines what adjustments would be made to t_1 or t_2 before calculating the year fraction as defined by the day count basis. It is possible (though not common) for t_1 and t_2 each to abide by a different rule, as such we shall simply use t_i to define the calculation date in the definitions below.

Actual

No adjustment is made and payments are made and calculated based on the actual date, even if it is a non-business day.

Following

If the reference date t_i falls on a non-business day then t_i is adjusted to be the date of the immediate following business day, i.e. if t_i is a Saturday it is adjusted to Monday's date if Monday is not a holiday, else it will be set to the next business day.

Previous

If the reference date t_i falls on a non-business day then t_i is adjusted to be the date of the first previous business date, i.e. if t_i is a Saturday then it is adjusted to the previous Friday's date if that Friday was not a holiday, else it is set to the previous business day.

Modified Following

This follows the same convention as the **following** rule except if that would cause the adjusted date to fall into the next calendar month, in which case a **previous** rule would apply.

Modified Previous

This follows the same convention as the **previous** rule except if that would cause the adjusted date to fall into the previous calendar month, in which case a **following** rule would apply.