

## STABILITY OF SOLUTAL CONVECTION IN A GRAVITY MODULATED MUSHY LAYER DURING THE SOLIDIFICATION OF BINARY ALLOYS

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### ABSTRACT

The linear stability theory is used to investigate the effects of gravity modulation on solutal convection in mushy layers found in solidifying binary alloys. The gravitational field is modeled to consist of constant part and a sinusoidally varying part. The linear stability results are presented for both the synchronous and subharmonic solutions, and it is demonstrated that up to the transition point between the synchronous and subharmonic regions, increasing the frequency of vibration rapidly stabilizes the solutal convection. However, beyond the transition point, further increases in the frequency tends to destabilize convection. It is also demonstrated that the effect of increasing the ratio of the Stefan number and the solid composition ( $\eta_o$ ) is to destabilize the solutal convection.

### INTRODUCTION

Alloyed components are used in a wide spectrum of engineering applications, such as turbine blades, automotive components etc. A consistent internal structure is therefore paramount to the performance and integrity of a mechanical component. The internal structure of an alloy traces back to its solidification from the melt phase. Alloys are susceptible to the formation of vertical channels of a composition different to the surrounding solid, commonly known as freckles. Copley et al. [1] initially proposed a mechanism for the origin of freckles whilst Chen and Chen [2], and Tait and Jaupart [3] investigated various modes of convection and the relation to freckle formation in their experimental studies. Fowler [4] proposed a model for the mushy layer and analyzed it for the limiting case when the mushy layer behaved like a non-reacting porous layer, i.e. he took the composition difference across the mushy layer to be infinitesimally small as a first approximation. Amberg and Homsey [5] conducted a weak non-linear analysis of convection in a mushy layer and were the first to propose a means of decoupling the mushy layer from the overlying liquid melt and underlying solid layer. Perturbations were used to re-introduce the effects of permeability to higher orders of approximation. Anderson and Worster [6,7] extended the model

of Amberg and Homsey [5], adopting large Stefan number scaling and observed a hitherto unobserved oscillatory mode of convection which was distinctly different to the double diffusive mode observed by Chen et al. [8]. A mushy layer including rotational effects, for a Stefan number of unit order of magnitude, was investigated by Govender and Vadasz [9]. It was observed that rotation stabilizes solutal convection in the mushy layer.

Recently, Govender [10] analysed the stability of free convection in a vertically modulated porous layer subjected to constant vertical stratification, i.e. modulated Rayleigh-Benard convection and thereafter went on to further analyse the actual transition points from synchronous to subharmonic solutions, Govender [11]. The objective of the current work is to analyse the stability of solutal convection in a mushy layer during the solidification of binary alloys subject to gravity modulation.

### NOMENCLATURE

$b_s$	[m]	vibration amplitude.
$C$	[-]	dimensionless composition, equals, $C_s - C_o / \Delta C$ .
$\hat{e}_x$	[-]	unit vector in the $x$ -direction.
$\hat{e}_y$	[-]	unit vector in the $y$ -direction.
$\hat{e}_z$	[-]	unit vector in the $z$ -direction.
$D$	[ ]	solutal diffusivity.
$Da$	[-]	Darcy number, equals $\Pi_o / H_s^2$ .
$Fr$	[-]	Froude number, equals $\lambda_s^2 / (g_s H_s^3)$ .
$g_s$	[ $m/s^2$ ]	gravitational acceleration
$h_f$	[kJ/kg]	latent heat of solidification.
$H_s$	[m]	the height of the mushy layer.
$\Pi_o$	[ $m^2$ ]	characteristic permeability
$k$	[W/mK]	thermal conductivity.
$\Pi_s$	[ $m^2$ ]	variable permeability of the mushy layer.

$l_{\lambda^*}$	[m]	thermal diffusion length, equals $\lambda_{\lambda^*}/V_{f^*}$
$L_m$	[m]	the length of the mushy layer.
$Le$	[-]	Lewis number, equals $\lambda_{\lambda^*}/D_f$
$p$	[-]	dimensionless reduced pressure.
$Pr$	[-]	Prandtl number, equals $\nu_{\lambda^*}/\lambda_{\lambda^*}$ .
$q_s$	[kJ/kgm <sup>3</sup> ]	specific heat per unit volume.
$Ra_m$	[-]	porous media gravitational Rayleigh number, equals, $\Pi_0 \beta_c g_s \Delta C / \nu_{\lambda^*} V_{f^*}$ .
$s$	[-]	convection wavenumber.
$St$	[-]	Stefan number, equals $h_{fs}/q_s \Delta T$
$t_s$	[s]	time.
$T$	[-]	dimensionless temperature, equals, $(T_s - T_L(C_o))/\Delta T$ .
$u$	[m/s]	horizontal x component of the filtration velocity.
$v$	[m/s]	horizontal y component of the filtration velocity.
$w$	[m/s]	vertical z component of the filtration velocity.
$\mathbf{V}$	[-]	dimensionless filtration velocity vector, equals $u\hat{e}_x + v\hat{e}_y + w\hat{e}_z$ .
$V_{f^*}$	[m/s]	Front/Solidification velocity.
$\mathbf{X}$	[m]	space vector, equals $x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ .
$x$	[m]	horizontal length coordinate.
$y$	[m]	horizontal width coordinate.
$z$	[m]	vertical coordinate.
Special characters		
$\alpha$	[-]	a parameter related to the wave number, equals $s^2/\pi^2$ .
$\beta_T$	[1/K]	thermal expansion coefficient.
$\beta_c$	[1/kg]	solubility expansion coefficient.
$\gamma$	[-]	equals $\chi_o/\pi^2$ .
$\delta$	[-]	dimensionless depth of mushy layer, equals $H_m/l_{\lambda^*}$ .
$\Lambda$	[-]	equals, $\kappa Fr \Omega^2$ .
$\Delta C$	[kg]	characteristic composition difference, equals $C_o - C_E$ .
$\Delta T$	[K]	characteristic temperature difference, equals $T_L(C_o) - T_E$ .
$\phi$	[-]	volume fraction of solid dendrites (solid fraction).
$\phi$	[-]	porosity, equals $(1 - \phi)$ .
$\kappa$	[-]	$b_s/H_m$ .
$\lambda_{\lambda^*}$	[m <sup>2</sup> /s]	effective thermal diffusivity.
$\mu_s$	[Pa-s]	fluid dynamic viscosity.
$\rho$	[kg/m <sup>3</sup> ]	density.
$\sigma$	[-]	Growth factor.
$\chi_o$	[-]	modified Darcy-Prandtl number, equals $Pr(l_{\lambda^*}^2/\Pi_0)$
$\chi$	[-]	equals $\delta^2 \chi_o = Da/Pr$
$\nu_{\lambda^*}$	m <sup>2</sup> /s	fluid kinematic viscosity.

$\Pi(\phi)$	[-]	Retardability function, equals $\Pi_0/\Pi_s$
$\eta_o$	[-]	equals $1 + \tilde{S}/C_s$
$\omega_s$	[1/s]	vibration frequency.
$\vartheta_o$	[-]	mobility ratio, equals $l_{\lambda^*}^2/\Pi_0$
$\Omega$	[-]	scaled vibration frequency, equals, $\delta^2 \omega_s^2 \lambda_{\lambda^*} / V_{f^*}^2$ .
$\Gamma$	[K/kg]	slope of liquidus line.
$\zeta$	[-]	composition ratio, equals $(C_s - C_o)/\Delta C$
$\xi$	[-]	scaled growth factor, equals $\sigma/\sqrt{-a}$ .

#### Subscripts

*	dimensional values.
$B$	basic state
$c$	characteristic.
$C$	classic.
$cr$	critical values.
$eff$	effective values
$E$	related to eutectic values.
$l$	related to the liquid melt.
$m$	related to the mushy layer parameters.
$M$	modified.
$o$	related to unmodulated quantities.
$s$	related to the solid layer.

## PROBLEM FORMULATION

A binary alloy melt, cooled from below, subject to vibration parallel to the gravitational field in the vertical direction, is presented in Figure 1. The solidification process results in three distinct regions forming viz. the solid region, of a temperature below the eutectic temperature  $T_E$ , a liquid melt region, with a temperature above the liquidus temperature  $T_L(C_o)$  and a mushy layer sandwiched between the solid layer and the liquid melt.

The composition at the mush-liquid interface is  $C_o$  and the composition at the mush-solid interface is  $C_E$ . We propose that the mush-liquid interface and the mush-solid interface advances at a constant speed  $V_{f^*}$  implying that the binary alloy is directionally cast and the mushy layer is of constant height  $H_m$  and width  $L_m$ , similar to the model of Amberg and Homsey [5] and Govender and Vadasz [9]. This results in the mushy layer having rigid and isothermal upper and lower boundary conditions where the vertical components of velocity is zero, physically isolating and dynamically decoupling the mushy layer from the solid region below and the liquid melt above.

Subject to these conditions, and performing a transformation for the translating frame of reference (for solidification), the following dimensionless set of governing equations for continuity, energy, solute and Darcy, are proposed,

$$\nabla \cdot \mathbf{V} = 0, \quad (1)$$

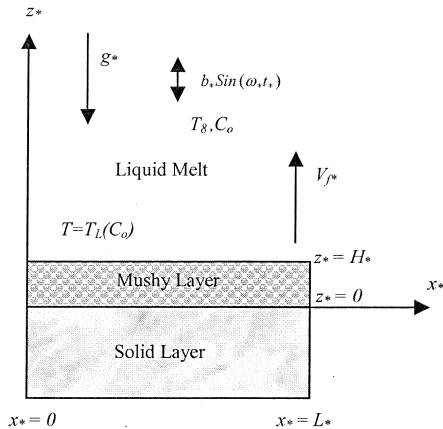


Figure 1: A schematic of the proposed problem showing the imposed boundary conditions.

$$\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) (T - St\phi) + \mathbf{V} \cdot \nabla T = \nabla^2 T, \quad (2)$$

$$\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) ((1 - \phi)T + \zeta\phi) + \mathbf{V} \cdot \nabla T = 0, \quad (3)$$

$$\frac{1}{(1 - \phi)\chi_o} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) \mathbf{V} + \Pi(\phi)\mathbf{V} = -\nabla p - Ra_m [1 + \Lambda \sin(\Omega t)] T \hat{e}_z \quad (4)$$

The Boussinesq approximation is used and consists of setting  $\rho_* = \rho_{o*}$  except in buoyancy terms emanating from body forces in the Darcy equation. The Boussinesq approximation and linear liquidus relation allows us to define a linear density  $\rho_l$  to composition  $C_*$  relation in a similar fashion to Govender and Vadasz [9] with  $\rho_l = \rho_{o*} (1 - \beta_{T*} (T_* - T_E) + \beta_{C*} (C_* - C_o))$ . Note that  $\beta_{C*} \gg \beta_{T*}$  as density is a stronger function of composition than temperature and the density equation can simplify to  $\rho_l - \rho_{o*} = \rho_{o*} \beta_{C*} (C_* - C_o)$ . This allows us to focus on solutal effects on convection without greatly sacrificing accuracy. The linear liquidus relation gives  $C_* - C_o = T\Delta C$  further simplifying the density equation to  $\rho_l - \rho_{o*} = \rho_{o*} \beta_{C*} T\Delta C$ . The scaling variables  $V_{f*}$ ,  $\lambda_* / V_{f*}$ ,  $\lambda_* / V_{f*}^2$ , and  $\lambda_* \mu^* / \Pi_o$  are used to non-dimensionalize the filtration velocity, length, time and pressure components respectively. In the above relations  $V_*$  is the filtration velocity in the mush,  $T_*$  is the dimensional temperature,  $\phi$  is the solid fraction,  $C_*$  is the composition,  $p_*$  is the reduced pressure, and  $\hat{e}_z$  is the unit vector in the direction of gravity. A number of dimensionless parameters emanate from the dimensionless analysis. In Eq.(2),  $St$  is the Stefan number, and represents the ratio of the latent heat ( $h_{fs}$ ) to heat content or internal energy, and is defined as  $St = h_{fs} / (q_{eff} \Delta T)$ . The effective thermal diffusivity  $\lambda_*$  is defined as the ratio of the thermal conductivity and the specific heat per unit volume  $\lambda_* = k_{eff} / q_{eff}$ . The

composition ratio  $\zeta$ , in Eq.(3), relates the differences in the characteristic compositions of the liquid and solid phases with the varying composition of the liquid within the mushy layer, and is defined as  $\zeta = (C_s - C_o) / (C_o - C_E)$ . In the case of solidifying binary alloys the Lewis number,  $Le$ , usually assumes large values and therefore the  $O(1/Le)$  term that normally appears in Eq.(3) is omitted. The specific heat per unit volume ( $q$ ) and the mush thermal conductivity ( $k_m$ ) of the mushy layer are given as  $q_{eff} = \phi q_s + (1 - \phi) q_l$  and  $k_{eff} = \phi k_s + (1 - \phi) k_l$  respectively. Taking  $q_s \approx q_l$  and  $k_s \approx k_l$  yields  $q = q_s = q_l$  and  $k_m = k_s = k_l$ . The linear liquidus relation is defined as  $T = (T_* - T_L(C_o)) / \Delta T$  and  $C = (C_* - C_o) / \Delta C$  where  $\Delta T = T_L(C_o) - T_E$ ,  $\Delta C = C_o - C_E$ ,  $\Delta T = \Gamma \Delta C$ . In Eq.(4), the modified Darcy-Prandtl number (defined as  $\chi_o = Pr \mathcal{G}_o$ ), relevant to solidification-type problems, normally assumes small values for binary alloy mixtures, see Vadasz [12] and is thereby resulting in the retention of the time derivative in the Darcy equation. The mushy layer Rayleigh number  $Ra_m$  is defined as  $Ra_m = \Pi_o \beta_{C*} g_* \Delta C / \nu_* V_{f*}$ . Note that  $\mu_*$  refers to the dynamic viscosity,  $\rho_*$  is the density, and  $g_*$  is the gravitational acceleration.

In Eq.(4), the retardability function, as proposed by Nield [13], is defined as  $\Pi(\phi) = \Pi_o / \Pi_*$  (where  $\Pi_o$  is the characteristic permeability and  $\Pi_*$  is the permeability of the mushy layer), and the dimensional amplitude  $\Lambda$  is defined as  $\Lambda = (\kappa \cdot Fr_M) \cdot \Omega^2$  (where  $\kappa = b_* / H_*$  and  $\Omega = (\delta^2 \omega_* \lambda_*) / V_{f*}^2$ ). Here  $b_*$  and  $\omega_*$  refers to the amplitude and frequency of the imposed vibration. The modified Froude number,  $Fr_M$ , proposed by Govender [10] for solidifying binary alloy systems is given as  $Fr_M = \lambda_* / (H_*^3 g_*) = Fr_C / \delta^2$ , where  $Fr_C$  is the classic Froude Number,  $Fr_C = V_{f*} / (H_* g_*)$  defined in terms of the front velocity, gravitational acceleration and the mushy layer height. For a binary alloy system such as brass (70% Copper-30% Zinc), the thermal diffusivity is  $\lambda_* = 34.2 \times 10^{-6} \text{ m}^2/\text{s}$ , whilst for an ammonium salt-water system ( $NH_4Cl - H_2O$ ), the thermal diffusivity is of the order  $\lambda_* = 0.147 \times 10^{-6} \text{ m}^2/\text{s}$ . If the mushy layer height is taken to be  $H_* = 5 \text{ mm}$ , the modified Froude numbers for the alloy system and the salt-water system may be approximated to be  $Fr_{M, Brass} = 0.000954$ , and  $Fr_{M, Salt-Water} = 1.762 \times 10^{-8}$ . Using the relationship between the classic and modified Froude numbers one may evaluate the solidification front velocities for the alloy and salt-water system (for  $\delta = 0.1$ ) to be  $V_{f*, Brass} = 0.0684 \text{ cm/s}$  and  $V_{f*, Salt-Water} = 6.44 \times 10^{-4} \text{ cm/s}$ . It should be pointed out that Chen et al [8], in their lab experiments, found the solidification front velocity to be of the order  $O(10^{-4}) \text{ cm/s}$ , which is in full

agreement with the velocity magnitude calculated using the two definitions of the Froude number proposed. For  $\kappa \approx 4$ , the grouping  $(\kappa \cdot Fr_M)$  defined in the  $\Lambda$  term above may be calculated for a binary alloy system (Brass), and for the salt-water system ( $NH_4Cl-H_2O$ ) to be  $(\kappa \cdot Fr_M)_{Brass} = 3.82 \times 10^{-3}$  and  $(\kappa \cdot Fr_M)_{Salt-Water} = 7.05 \times 10^{-8}$ . So depending on the type of medium being solidified the parameter grouping  $(\kappa \cdot Fr_M)$  could assume values in the region,  $(\kappa \cdot Fr_M) \in [10^{-3}, 10^{-8}]$ . In the current study we analyse solidifying systems for small Prandtl numbers (or small to moderate  $\gamma$  values) for the parameter grouping  $(\kappa \cdot Fr_M)$  of the order  $(\kappa \cdot Fr_M) = O(10^{-3})$ . The Prandtl number is defined as the ratio of the kinematic viscosity  $\nu_*$  to the thermal diffusivity and is defined as  $Pr = \nu_*/\lambda_*$ . The dimensionless depth of the layer  $\delta$  is the ratio of the height of the mushy layer  $H_*$  to the thermal diffusive length  $l_{\lambda^*}$  and is defined as  $\delta = H_*/l_{\lambda^*}$ .

Following Anderson and Worster [6] and Govender and Vadasz [9] we may scale the dependent variables as in terms of the mushy layer depth  $\delta$  as follows,

$$\mathbf{X} = \delta \tilde{\mathbf{X}}, \quad t = \delta^2 \tilde{t}, \quad R^2 = \delta Ra_m, \quad p = R \tilde{p}, \quad \mathbf{V} = \frac{R}{\delta} \tilde{\mathbf{V}}. \quad (5)$$

Applying the scalings in Eq.(5) to Eqs.(1-4) we obtain the following scaled system of governing equations,

$$\tilde{\nabla} \cdot \tilde{\mathbf{V}} = 0, \quad (6)$$

$$\left( \frac{\partial}{\partial \tilde{t}} - \delta \frac{\partial}{\partial \tilde{z}} \right) (T - St\varphi) + R \tilde{\mathbf{V}} \cdot \tilde{\nabla} T = \tilde{\nabla}^2 T, \quad (7)$$

$$\left( \frac{\partial}{\partial \tilde{t}} - \delta \frac{\partial}{\partial \tilde{z}} \right) ((1-\varphi)T + \zeta\varphi) + R \tilde{\mathbf{V}} \cdot \tilde{\nabla} T = 0, \quad (8)$$

$$\frac{1}{(1-\varphi)\chi} \left( \frac{\partial}{\partial \tilde{t}} - \delta \frac{\partial}{\partial \tilde{z}} \right) \tilde{\mathbf{V}} + \Pi(\varphi)\tilde{\mathbf{V}} = -\tilde{\nabla} \tilde{p} + R[1 + \Lambda \sin(\Omega \tilde{t})] T \hat{e}_z, \quad (9)$$

where  $\chi = \delta^2 \chi_o = Da/Pr$ . When the composition of the melt is close to that of the eutectic composition ( $C_o - C_e \ll 1$ ), a large concentration ratio,  $\zeta$  is obtained. For this so-called near eutectic limit the concentration ratio may be defined as  $\zeta = C_s/\delta$ , where  $C_s$  is the solid concentration and  $\delta \ll 1$ . Worster [14] showed that for large concentration ratios  $\zeta$ , the permeability of the mushy layer is uniform/homogenous, and for this reason we set  $\Pi(\varphi) = 1$ , in the current analysis. We follow Anderson and Worster [7] and define a large Stefan number,  $St$ , according to  $St = \tilde{S}/\delta$ .

## LINEAR STABILITY ANALYSIS

To establish the basic flow we need to analyse the equation set corresponding to the motionless state where the flow velocity is zero, and the temperature and solid fraction is horizontally

uniform. Using an expansion in  $\delta$  where the basic state has expansions:

$$[T_B, \mathbf{V}_B, \varphi_B, p_B] = [T_{B0}, \mathbf{V}_{B0}, \varphi_{B0}, p_{B0}] + \delta [T_{B1}, \mathbf{V}_{B1}, \varphi_{B1}, p_{B1}] + \delta^2 [T_{B2}, \mathbf{V}_{B2}, \varphi_{B2}, p_{B2}] \quad (10)$$

and a motionless state associated with basic flow implies that,  $\mathbf{V}_B = 0$ ,  $\partial T/\partial \tilde{t} = 0$ ,  $\partial T/\partial \tilde{x} = 0$ ,  $\partial T/\partial \tilde{y} = 0$ ,  $\partial \varphi/\partial \tilde{t} = 0$ ,  $\partial \varphi/\partial \tilde{x} = 0$  and  $\partial \varphi/\partial \tilde{y} = 0$ . Substituting Eq.(10) in Eqs.(6-9) yields the motionless basic state solution for the temperature and solid fraction to each order of  $\delta$  subject to the boundary conditions:  $\tilde{z} = 0$ ,  $T_B = -1$  and  $\tilde{z} = 1$ ,  $T_B = 0$ ,  $\varphi_B = 0$ ,

$$T_B = (\tilde{z} - 1) - d \frac{\eta_o}{2} (\tilde{z}^2 - \tilde{z}) + \dots, \quad (11)$$

$$\varphi_B = \delta \tilde{\varphi}_B = \delta \left( \frac{1 - \tilde{z}}{C_s} \right) + \delta^2 \left[ -\frac{(\tilde{z} - 1)^2}{C_s^2} + \frac{\eta_o}{2C_s} (\tilde{z}^2 - \tilde{z}) \right] + \dots, \quad (12)$$

It can be observed from Eq.(12) that to  $O(\delta^0)$ ,  $\varphi_{B0} = 0$ . This result clearly shows that for  $\delta \ll 1$ , a small amount of solid is formed for the near eutectic approximation. Readers are referred to Roberts et al [15] for a more accurate treatment of the fluid - mush interface boundary condition. To analyse the stability of the basic state solution (11-12) we apply small perturbations about of the form,

$$[T, \mathbf{V}, \varphi, p] = [T_B, 0, \varphi_B, p_B] + \varepsilon [T_1, \mathbf{V}_1, \varphi_1, p_1] + \varepsilon^2 [T_2, \mathbf{V}_2, \varphi_2, p_2], \quad (13)$$

where  $\varepsilon \ll 1$ , as required by the linear theory. For the particular case when  $\delta = O(\varepsilon)$ , we note that the basic state solution interacts with the perturbed terms. As a result Eq.(13) may be re-written as follows for the case  $\delta = O(\varepsilon)$ ,

$$[T, \mathbf{V}, \varphi, p] = [T_{B0}, 0, 0, p_{B0}] + \varepsilon [(T_1 + T_{B1}), (\mathbf{V}_1 + \mathbf{V}_{B1}), (\varphi_1 + \varphi_{B1}), (p_1 + p_{B1})] + \varepsilon^2 [(T_2 + T_{B2}), (\mathbf{V}_2 + \mathbf{V}_{B2}), (\varphi_2 + \varphi_{B2}), (p_2 + p_{B2})]. \quad (14)$$

Solving the Equation set (6-8) for the perturbation definition proposed in Eq.(14) to  $O(\varepsilon)$  yields

$$\frac{\partial T_1}{\partial t} + R w_1 - 1 = \frac{1}{\eta_o} \nabla^2 T_1, \quad (15)$$

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] V_1 = -\nabla p_1 - R[1 + \Lambda \sin(\Omega t)] T_1 \hat{e}_z = 0, \quad (16)$$

where  $\eta_o = 1 + \tilde{S}/C_s$ .

Following Govender [10], we apply the curl operator twice on Eq.(16) in order to eliminate the pressure term, and only consider the z-component of the result as follows,

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] \nabla^2 w_1 + R[1 + \Lambda \sin(\Omega t)] \nabla_h^2 T_1 = 0, \quad (17)$$

where  $\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  and  $w_1$  is the perturbation to the vertical component of the filtration velocity. Equation (15) and Eq.(17)

may be decoupled by eliminating  $w_1$  providing a single equation for the temperature perturbation  $T_1$  in the form,

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] \nabla^2 \left[ \frac{\partial}{\partial t} - \frac{1}{\eta_o} \nabla^2 \right] T_1 - R^2 [1 + \Lambda \text{Sin}(\Omega t)] \nabla_H^2 T_1 = 0 \quad (18)$$

Assuming an expansion into the normal modes in the  $x$ - and  $y$ -directions, and a time dependent amplitude  $\theta(t)$  similar to that of Govender [10], we obtain

$$T_1 = \theta(t) e^{i(s_x x + s_y y)} \sin(\pi z) + c.c. \quad (19)$$

where again  $c.c.$  represents the complex conjugate terms and  $s^2 = s_x^2 + s_y^2$ . Substituting Eq.(19) into Eq.(18) yields,

$$\frac{d^2 \theta}{dt^2} + 2p \frac{d\theta}{dt} - F(\alpha) \gamma \left[ (\hat{R} - \hat{R}_o) + \hat{R} \Lambda \text{Sin}(\Omega t) \right] \theta = 0, \quad (20)$$

where  $\alpha = s^2 / \pi^2$ ,  $\gamma = \chi / \pi^2$ ,  $\hat{R} = \tilde{R}^2$ ,  $\tilde{R} = R / \pi^2$ ,  $\hat{R}_o = \tilde{R}_o^2$ ,  $\tilde{R}_o = R_o / \pi^2$ ,  $R_o$  is the un-modulated Rayleigh number defined as  $\tilde{R}_o = \pi^2 (\alpha + 1)^2 / \eta_o \alpha$ ,  $2p = \pi^2 [(\alpha + 1) / \eta_o + \gamma]$  and

$F(\alpha) = \pi^6 \alpha / (\alpha + 1)$ . Using the transformation

$t = (\pi / 2 - 2\tau) / \Omega$ , Eq.(20) may be cast, as indicated by McLachlan [16], into the canonical form of the Mathieu equation:

$$\frac{d^2 X}{d\tau^2} + [a + 2q \cos(2\tau)] X = 0 \quad (21)$$

The solution to Eq.(21) follows the form  $G(\tau) = e^{-\sigma \tau} \chi(\tau)$  where  $G(\tau)$  is a periodic function with a period of  $\pi$  or  $2\pi$  and  $\sigma$  is a characteristic exponent which is a complex number, and is a function of  $a$  and  $q$  respectively (See Govender [10]). In this paper the definitions for  $a$ ,  $q$  and  $\sigma$  are obtained upon transforming Eq.(21) into the canonical form of Mathieu's equation, and are defined as,

$$\frac{2}{\sqrt{-a}} = \frac{\Omega}{[F(\alpha) \gamma (\hat{R} - \eta)]^{1/2}}, \quad \frac{1}{2} q = F(\alpha) \gamma \hat{R} \kappa Fr, \quad (22a,b)$$

$$\sigma = -\frac{2p}{\Omega}, \quad \eta = -\frac{\hat{R}_o \eta_o}{4\gamma(\alpha + 1)} \left[ \frac{\alpha + 1}{\eta_o} - \gamma \right] \quad (23a,b)$$

When  $\sigma = 0$ , the solution to Eq.(21) is defined in terms of Mathieu functions,  $c_e$  and  $d_f$  (where  $e = 1, 2, 3, \dots, E$  and  $f = 1, 2, 3, \dots, F$ ), such that for each Mathieu function,  $c_e$  and  $d_f$ , there exists a relation between  $a$  and  $q$ . This relationship is shown by Govender [11] for the Mathieu function  $d_o, c_1$  and  $d_1$ . One would observe alternating stable and unstable zones if various Mathieu functions,  $c_e$  and  $d_f$ , are plotted on the same set of axes. In the stable regions of Mathieu's equation,  $\sigma$  is complex with a negative real part. Since  $\sigma$  is a function of  $a$  and  $q$ , which are dependent on  $\gamma, \hat{R}, \alpha, \Lambda$  and  $\Omega$ , the stability of the mushy layer is also seen to depend on these variables as well. In addition there are solutions to Eq.(21) for  $a > 0$  and  $a < 0$ ; also,  $q$  may be replaced by  $-q$  with no effect on the solution. In this study for a solidifying binary alloy the numerical values for " $a$ " are less than zero and are defined by Eq.(22a). We also propose

the following definition for the modified characteristic exponent,  $\xi = \sigma / \sqrt{-a}$ . We may now present a relation for the characteristic Rayleigh number in terms of the newly defined parameter  $\xi$ , by substituting  $\xi = \sigma / \sqrt{-a}$  in Eq.(22a), and rearranging to yield,

$$\hat{R} = \frac{\hat{R}_o - \eta}{\xi^2} + \eta. \quad (24)$$

The Mathieu chart, see Govender [11], together with Eqs.(22-23) and Eq.(24) may be used to evaluate the critical Rayleigh number and wave numbers in terms of the frequency,  $\Omega$ , the parameter  $\kappa Fr$  and  $\gamma$ . We proceed, in a similar fashion to Govender [10], by first evaluating the characteristic Rayleigh number versus the frequency for selected values of the wave number according to the following method: (a) select a value of  $\xi$ , (b) evaluate  $\hat{R}$  using Equation (24), (c) compute the value for  $1/2q$  using Equation (23a), (d) read  $2/\sqrt{-a}$  from Mathieu chart, see Govender [11], and (e) evaluate the frequency from Equation (22a). The results of the process outlined in steps 1 to 5 above are depicted in Figure 2 for parameter values  $\gamma = O(1.5)$  and  $(\kappa Fr) = O(10^{-3})$ , see Govender [11]. In Figure 2, for any selected wave number, it is noted that the stable zone is below the curve whilst the unstable zone is above the curve. An example of the location of the unstable and stable regions is shown in Figure 2 for  $\alpha = 1.5$ . Using the results in Figure 2 we may evaluate the critical wave number and Rayleigh number for numerous values of wave number  $\alpha$  across a bandwidth of frequencies.

The effect of the important ratio  $\tilde{S}/C_s$  (embodied in the term  $\eta_o$ ) on the critical Rayleigh number and frequency is shown in Figure 3 for selected  $\eta_o$  values, viz.  $\eta_o = 1, 1.5$ , and  $2$  at  $\gamma = O(1.5)$  and  $(\kappa Fr) = O(10^{-3})$ . It can be observed from Figure 3 that as  $\eta_o$  increases from 1 to 2 the convection becomes destabilised across the indicated bandwidth of frequencies. The setting  $\eta_o = 1$  corresponds to the case when the Stefan number equals zero (i.e. there is no solidification), whilst  $\eta_o = 2$  corresponds to the case  $\tilde{S} = O(C_s)$  (or  $St = O(\zeta)$ ). Calculations were performed for numerous values of  $\eta_o$  and it was found that beyond  $\eta_o = 10$  no solutions exist. The results in general follow similar trends to that generated by Govender [11], but the transition points  $\Omega_{1,i}, \Omega_{2,i}, \Omega_{3,i}$  (from synchronous to subharmonic solutions for each  $\eta_o$  value) occur at significantly lower frequencies. In addition the stabilising effect of vibration occurs over a bandwidth of lower frequency values in comparison to that of the passive porous layer investigated by Govender [10]. This is a welcomed result as high frequency vibration will initiate macro-segregation in the solidifying alloy and defeat the objective of using vibration to stabilise the solutal convection.

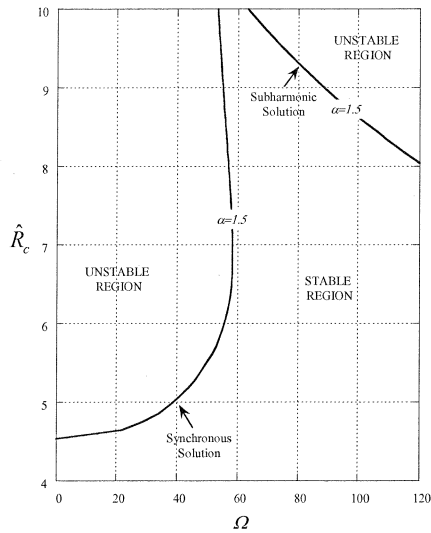


Figure 2: Characteristic Rayleigh number versus the scaled vibration frequency for selected values of wavenumber.

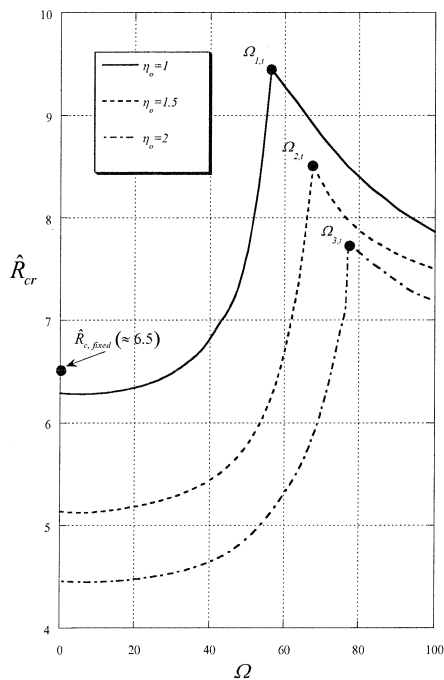


Figure 3: Critical Rayleigh number versus the scaled vibration frequency for various values of the parameter  $\eta_o$  at  $\gamma = O(1.5)$  and  $(\kappa Fr) = O(10^{-3})$ .

Bearing in mind that for solidification to occur we require that  $\eta_o > 1$ , it is quite clear from the curves presented in Figure 3 that the process of solidification (i.e. increasing Stefan number) destabilises the convection.

## CONCLUSION

The current study investigates the effect of gravity modulation on the stability of solutal convection in the mushy layer of a solidifying binary alloy. The study shows that low amplitude gravity modulation stabilizes the solutal convection in solidifying mushy layers thus reducing freckle formation.

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