

SELF-AERATED BOUNDED FLOWS IN SPECIAL HYDRAULIC STRUCTURES. PART 2.**COMPRESSIBILITY EFFECTS ON THE DESIGN OF AERATORS.**

Mutti M.*

*Author for correspondence

Department of Hydraulic and Maritime Constructions and Hydrology,

Politecnico di Milano,

Piazza Leonardo da Vinci n. 32, 20133 Milano,

Italy,

E-mail: matteo.mutti@polimi.it

ABSTRACT

It is well known that common procedures in the design and testing of the air supply systems in special hydraulic structures assume the flow of air through vent ducts to be incompressible ([5]). Several experimental tests on large scale hydraulic models ([2], [6]) however show that this hypothesis quite often may not be realistic. In order to verify when the flow of air may be treated as that of an incompressible fluid, this paper aims at outlining the main mathematical and physical features of a compressible fluid flowing through a vent subjected to specific boundary conditions and evidence distinctions with the flow of the same fluid subjected to identical boundary conditions for which compressibility effects are neglected. Starting from basic equations of compressible flows some general relations are derived and reviewed focusing on design implications for aerators and air vents. It is shown that, when shear stresses are neglected, distinctions may be considered significant when the downstream to upstream pressure ratio in the vent becomes lower than 0.9 which, when the air feeding the vent is at STP, corresponds to a pressure drop of 10'000 Nm⁻² (1 m of water column) and an air velocity larger than 150 ms⁻¹. Much higher pressure ratios and lower velocities would be expected to be suitable however, should singular and distributed friction losses be taken into account. Therefore it is believed that compressible flow equations should always be used when analysing the flow of air through ducts in special hydraulic structures.

NOMENCLATURE

For a complete list of symbols see Part 1 of this paper "Self-aerated bounded flows in special hydraulic structures" also presented at HEFAT 2008.

COMPRESSIBLE FLOW EQUATIONS

The general 3D motion of a compressible fluid can be described through a complex system of partial differential equations. The system can be derived by application of basic physical principles to different control volumes together with the particular equation of state of the fluid and Fourier's law of heat conduction. The fundamental equations therefore result from conservation laws of mass, momentum and energy, the latter being specified by the First and Second Law of Thermodynamics. The attention is now restricted to quasi 1D compressible frictionless flow of ideal and non-heat conducting gas. It is additionally assumed that gas is polytropic and body forces are negligible. Equations are expressed in conservative form.

The mass balance equation gives:

$$\frac{\partial}{\partial t}(Ar) + \frac{\partial}{\partial x}(ArV) = 0 \quad (1)$$

The momentum balance equation reads:

$$\frac{\partial}{\partial t}(ArV) + \frac{\partial}{\partial x}(ArVV) = -\frac{\partial}{\partial x}(Ap) + p \frac{\partial A}{\partial x} \quad (2)$$

The total energy balance equation figures out:

$$\frac{\partial}{\partial t}(ArE) + \frac{\partial}{\partial x}(ArEV) = -\frac{\partial}{\partial x}(ApV) + \dot{q} \quad (3)$$

The kinetic energy balance equation recites:

$$\frac{\partial}{\partial t}(ArK) + \frac{\partial}{\partial x}(ArKV) = -\frac{\partial}{\partial x}(ApV) + p \frac{\partial}{\partial x}(AV) \quad (4)$$

The thermal energy balance equation reports:

$$\frac{\partial}{\partial t}(ArU) + \frac{\partial}{\partial x}(ArUV) = -p \frac{\partial}{\partial x}(AV) + \dot{q} \quad (5)$$

The entropy balance equation states:

$$\frac{\partial}{\partial t}(ArS) + \frac{\partial}{\partial x}(ArSV) = \frac{1}{T} \dot{q} \quad (6)$$

The PDE system is completed by the ideal gas law:

$$\frac{p}{rT} = R \quad (7)$$

with R being the engineering gas constant and by recalling that the total energy per unit mass E , defined as the sum of thermal energy per unit mass U and kinetic energy per unit mass K , for a polytropic gas reduces to:

$$E = U + K = \frac{1}{k-1} \frac{p}{r} + \frac{V^2}{2} \quad (8)$$

with k being the adiabatic coefficient ([1], [3]).

Since it can be shown that balance equation of kinetic energy follows from mass and momentum conservation equations while balance equation of thermal energy can be obtained by subtracting kinetic from total energy balance equation, it is easy to recognise that the PDE system consists of 6 independent equations (balance equations of mass, momentum, total energy and entropy, the equation of state of the gas and the assumption of polytropic gas) with 8 unknown variables (absolute pressure p , density r , absolute temperature T , energy per unit mass E , entropy per unit mass S , pipe area A , flow velocity V , heat transfer per unit time \dot{q}). As a general rule, a solution of the PDE system can then be worked out by specifying 2 additional constraints.

Under steady conditions, these constraints may be identified, for instance, in the system geometry A or the mass flow rate \dot{M} and in a functional relationship between pressure and density of the type:

$$\frac{p}{r^n} = C \quad (9)$$

with n being the polytropic coefficient.

With this choice, the flow of a compressible fluid starting from rest and passing through a nozzle can now be shown to be completely characterised in terms of different boundary conditions by means of simple algebraic equations.

If boundary conditions are expressed in terms of pressure, thermo-dynamical state variables in the nozzle orifice are described from:

$$\frac{r}{\max r} = \left(\frac{p}{\max p} \right)^{\frac{1}{n}} \quad (10)$$

$$\frac{T}{\max T} = \left(\frac{p}{\max p} \right)^{\frac{n-1}{n}} \quad (11)$$

while velocities, mass flow rates and heat transfer are known from:

$$\frac{V}{\max V} = \left(\frac{a}{\max a} \right)^{\frac{1}{2}} \quad (12)$$

$$\frac{\dot{M}}{\max \dot{M}} = \frac{1}{\frac{A}{\min A}} = \left(\frac{b}{\max b} \right)^{\frac{1}{2}} \quad (13)$$

$$\frac{|Q|}{\max |Q|} = \frac{g}{\max g} \quad (14)$$

with:

$$a = 1 - \left(\frac{p}{\max p} \right)^{\frac{n-1}{n}} \quad (15)$$

$$b = \left(\frac{p}{\max p} \right)^{\frac{2}{n}} \left(1 - \left(\frac{p}{\max p} \right)^{\frac{n-1}{n}} \right) \quad (16)$$

$$g = \left| 1 - \left(\frac{p}{\max p} \right)^{\frac{n-1}{n}} \right| \quad (17)$$

Analogous results can be obtained when boundary conditions are expressed in terms of density or absolute temperature.

RESULTS AND DISCUSSION

Many variables characterising the steady compressible flow of air through a nozzle have been computed for a number of thermodynamic processes and boundary conditions.

Figure 1 describes the main gas flow features in adiabatic and isentropic transformations, Figure 2 illustrates endothermic and entropy-increasing processes while Figure 3 demonstrates exothermic and entropy-decreasing processes. Figure 4 instead compares velocity and mass flow rate solutions in adiabatic flows to solutions pertaining to isothermal and incompressible flows. Boundary conditions are always expressed in terms of absolute pressure. In all considered cases absolute pressure, density, absolute temperature, flow area, mass flow rate, velocity, enthalpy, kinetic energy, enthalpy plus kinetic energy and heat transfer of gas in the nozzle are shown. For simpler comparisons, quantities have been scaled by assuming the adiabatic and isentropic solutions as the reference ones.

Continuous smooth lines identify complete mathematical solutions covering sub-sonic and super-sonic flows which are physically admissible only in convergent-divergent nozzles ([1], [3]). Continuous non-smooth lines denote instead partial mathematical solutions involving subsonic flows which are the

only realistic solutions achievable in convergent nozzles ([1], [3]). Both partial theoretical solutions (black solid lines) and complete theoretical solutions (different colour solid lines) are drawn in graphs. Discontinuities in solution derivatives identify critical flow conditions (i.e. transition from sub-sonic to super-sonic flows or from another point of view transition from non-choking to choking conditions). Further details are given in the captions.

The results show that critical flow conditions occur when:

$$\begin{aligned} \frac{p}{\max p} &= \left(\frac{1}{1 + \frac{k}{2} \frac{n-1}{n}} \right)^{\frac{n}{n-1}} \\ \frac{r}{\max r} &= \left(\frac{1}{1 + \frac{k}{2} \frac{n-1}{n}} \right)^{\frac{1}{n-1}} \\ \frac{T}{\max T} &= \left(\frac{1}{1 + \frac{k}{2} \frac{n-1}{n}} \right) \end{aligned} \quad (18)$$

while maximum mass flow rates or minimum flow area take place at:

$$\begin{aligned} \frac{p}{\max p} &= \left(\frac{1}{1 + \frac{n-1}{2}} \right)^{\frac{n}{n-1}} \\ \frac{r}{\max r} &= \left(\frac{1}{1 + \frac{n-1}{2}} \right)^{\frac{1}{n-1}} \\ \frac{T}{\max T} &= \left(\frac{1}{1 + \frac{n-1}{2}} \right) \end{aligned} \quad (19)$$

and is easy to note that these two conditions coincide only for adiabatic and isentropic transformations (i.e. $n = k$).

Some general features of the flow may be deduced from inspection of the graphs. Let us suppose that the upstream and downstream extremities of a nozzle are in a certain state of equilibrium. As the boundary condition (absolute pressure, density, absolute temperature) downstream of the nozzle is decreased below the upstream value, the gas equilibrium breaks so that we may observe in the nozzle:

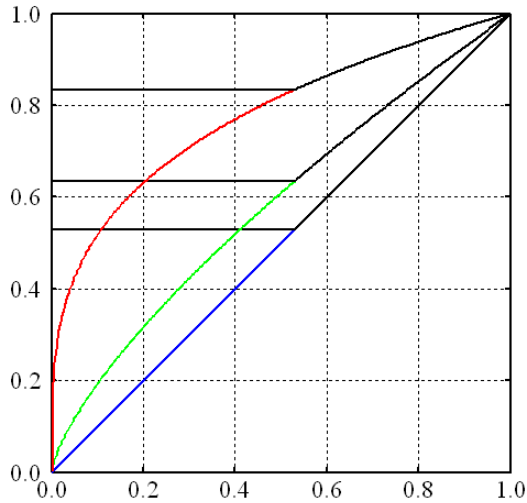
- a decrease in the absolute pressure, density, absolute temperature and enthalpy per unit mass;
- an increase in the flow velocity and kinetic energy per unit mass;
- an increase in the mass flow rate and flow area;

- no heat transfer per unit mass and constancy of the sum of enthalpy and kinetic energy per unit mass when the process is adiabatic and isentropic;
- an increase in the heat transfer per unit mass and in the sum of enthalpy and kinetic energy per unit mass when the process is endothermic and entropy-increasing (i.e. spontaneous process);
- a decrease in the heat transfer per unit mass and in the sum of enthalpy and kinetic energy per unit mass when the process is exothermic and entropy-decreasing (i.e. non-spontaneous process).

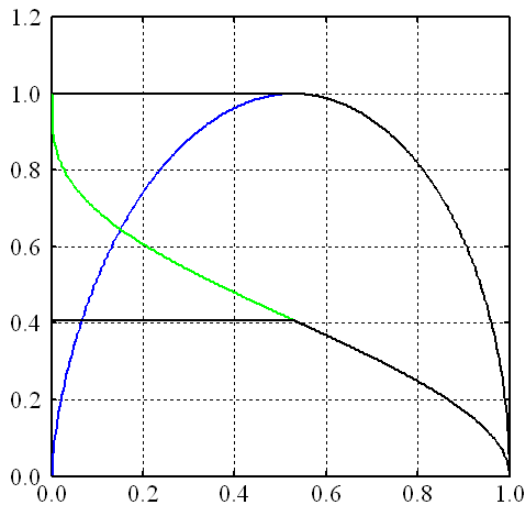
It is to be stressed that in a convergent nozzle thermodynamic and design variables are influenced (either decreasing or increasing) up to the critical point (i.e. subsonic condition) and further decreases of the boundary conditions do not affect anymore the flow within the nozzle (i.e. sonic condition) while in a convergent-divergent nozzle they may be further affected (either reducing or growing up) if the nozzle is accurately shaped for a given mass flow rate (i.e. supersonic condition). Furthermore, from inspection of Figure 4, we may observe that, when the upstream condition and the nozzle geometry is set:

- in an endothermic and entropy-increasing process the mass flow rate is smaller and the velocity is larger than in an adiabatic and isentropic transformation, for any given downstream condition; vice versa, if the mass flow rate passing through the nozzle is specified, downstream conditions are lower for sub-sonic flows and higher for super-sonic flows;
- in an exothermic and entropy-decreasing process the mass flow rate is larger and the velocity is smaller than in an adiabatic and isentropic transformation, for any given downstream condition; vice versa, if the mass flow rate passing through the nozzle is specified, downstream conditions are higher for sub-sonic flows and lower for super-sonic flows;
- as a first rough approximation, discrepancies between compressible and incompressible flows become apparent when the downstream to upstream pressure ratio becomes lower than 0.9 (i.e. $p < 0.9 \max p$); assuming that the air feeding the vent is at STP implies that this limit is reached for velocities of approximately 150 ms^{-1} .

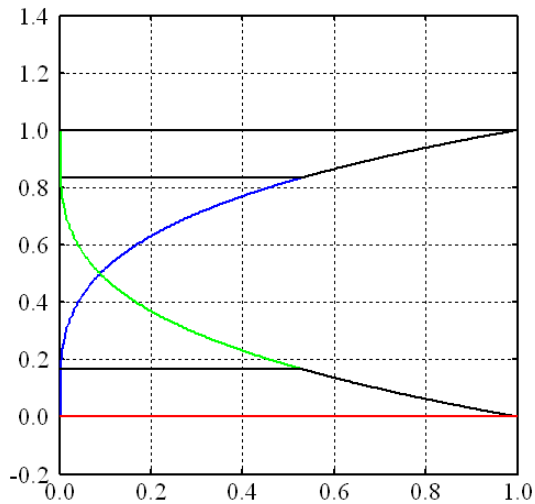
All previous considerations apply however to ideal systems in which friction losses and thermal excursions may be considered negligible. This may not be the case of real hydraulic structures characterised by air supply ducts of finite size, length and roughness, usually designed to work properly under extreme meteorological conditions (external temperature ranging from -30 C° to 40 C°). Substantial modifications of the results are expected, should all the above mentioned factors be properly taken into account. Therefore it is believed that design recommendations ([4], [5]) for dam outlet works to keep the sub-pressure of air at the vent exit in the order of $15'000 \text{ Nm}^{-2}$ (1.5 m of water column) and velocities below 50 ms^{-1} appear to be at the same time optimistic and contrasting and worthy of additional investigation in future work.



1.A.



1.B.



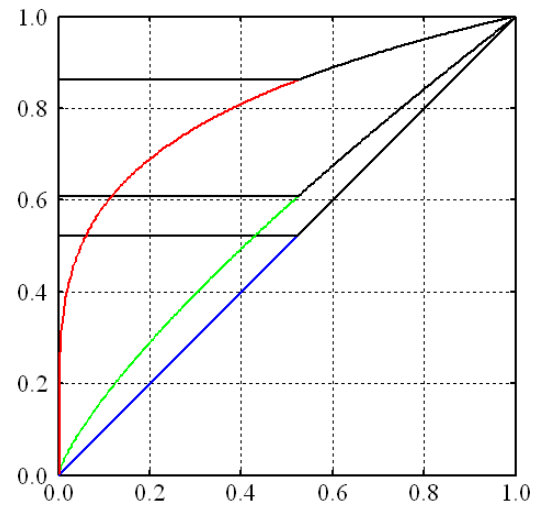
1.C.

Figure 1. Nozzle flow characteristics (on the y-axis) for an adiabatic and isentropic process (i.e. $n = k$) versus boundary conditions in terms of absolute pressure (on the x-axis). Both convergent nozzle solutions (black solid lines) and convergent-divergent nozzle solutions (different colour solid lines) are illustrated. Discontinuities in solution derivatives identify critical flow conditions (i.e. transition from sub-sonic to super-sonic flows or from another point of view transition from non-choking to choking conditions). Variables are non-dimensional.

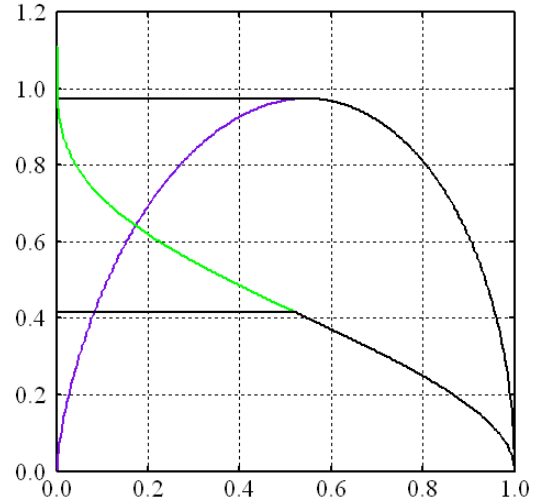
1.A. Absolute pressure (blue-black), density (green-black) and absolute temperature (red-black).

1.B. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black).

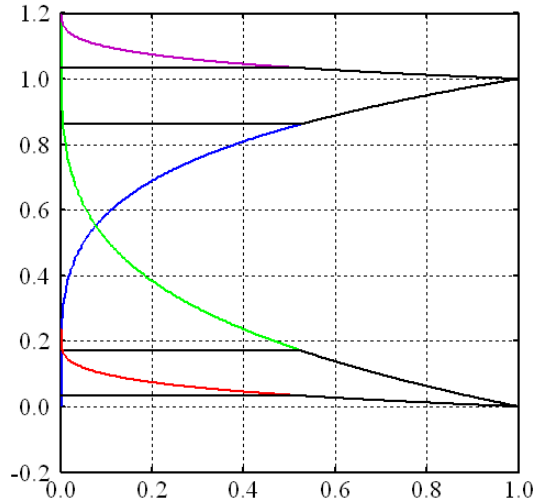
1.C. Enthalpy (blue-black), kinetic energy (green-black), enthalpy plus kinetic energy (magenta-black) and heat transfer (red-black) per unit mass.



2.A.



2.B.



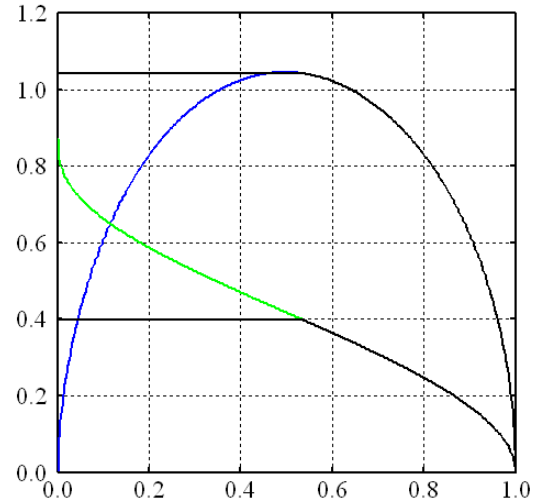
2.C.

Figure 2. Nozzle flow characteristics (on the y-axis) for an endothermic and entropy-increasing process (i.e. $n < k$) versus boundary conditions in terms of absolute pressure (on the x-axis). Both convergent nozzle solutions (black solid lines) and convergent-divergent nozzle solutions (different colour solid lines) are illustrated. Discontinuities in solution derivatives identify critical flow conditions (i.e. transition from sub-sonic to super-sonic flows or from another point of view transition from non-choking to choking conditions). Variables are non-dimensional.

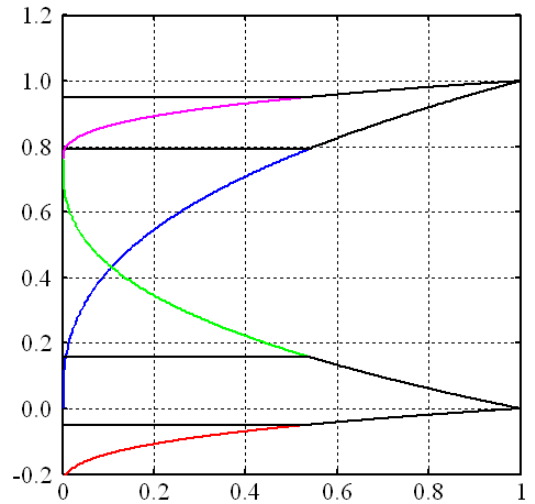
2.A. Absolute pressure (blue-black), density (green-black) and absolute temperature (red-black).

2.B. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black).

2.C. Enthalpy (blue-black), kinetic energy (green-black), enthalpy plus kinetic energy (magenta-black) and heat transfer (red-black) per unit mass.



3.B.



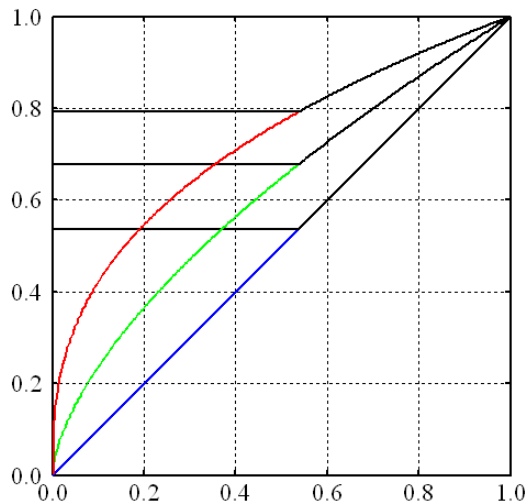
3.C.

Figure 3. Nozzle flow characteristics (on the y-axis) for an exothermic and entropy-decreasing process (i.e. $n > k$) versus boundary conditions in terms of absolute pressure (on the x-axis). Both convergent nozzle solutions (black solid lines) and convergent-divergent nozzle solutions (different colour solid lines) are illustrated. Discontinuities in solution derivatives identify critical flow conditions (i.e. transition from sub-sonic to super-sonic flows or from another point of view transition from non-choking to choking conditions). Variables are non-dimensional.

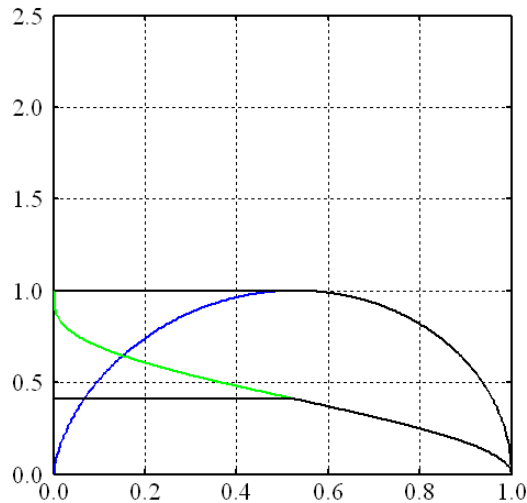
3.A. Absolute pressure (blue-black), density (green-black) and absolute temperature (red-black).

3.B. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black).

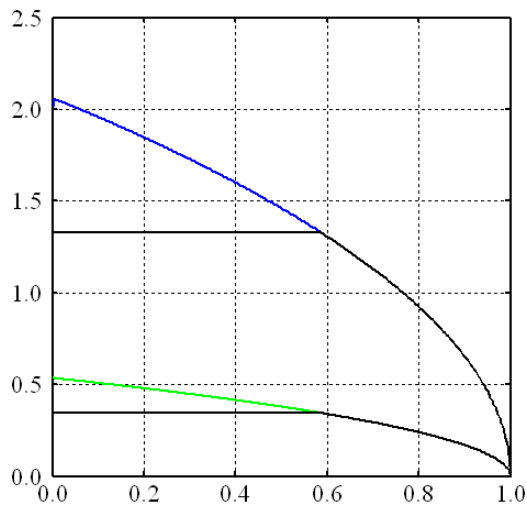
3.C. Enthalpy (blue-black), kinetic energy (green-black), enthalpy plus kinetic energy (magenta-black) and heat transfer (red-black) per unit mass.



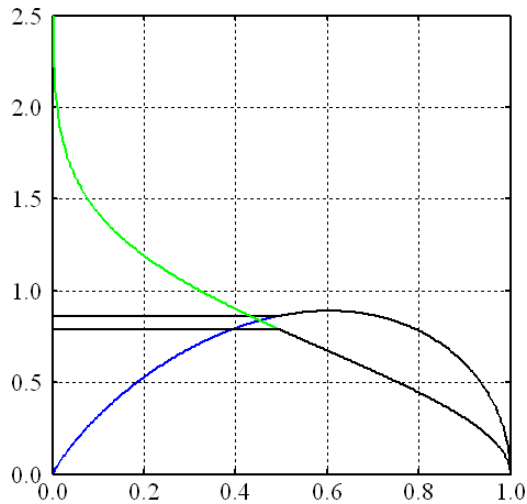
3.A.



4.A.



4.B.



4.C.

Figure 4. Nozzle flow characteristics (on the y-axis) versus boundary conditions in terms of absolute pressure (on the x-axis). Both convergent nozzle solutions (black solid lines) and convergent-divergent nozzle solutions (different colour solid lines) are illustrated. Discontinuities in solution derivatives identify critical flow conditions (i.e. transition from sub-sonic to super-sonic flows or from another point of view transition from non-choking to choking conditions). Variables are non-dimensional.

4.A. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black) for $n = k$ (adiabatic flow).

4.B. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black) for $n \rightarrow \infty$ (incompressible flow).

4.C. Flow area (blue-black), mass flow rate (blue-black) and velocity (green-black) for $n \rightarrow 1$ (isothermal flow).

CONCLUSIONS

It is well known that common procedures in the design and testing of the air supply systems in special hydraulic structures assume the flow of air through vent ducts to be incompressible. To verify to what extent this hypothesis may be considered valid, the main mathematical and physical features of a compressible fluid flowing through a vent subjected to specific boundary conditions have been worked out, enlightening similarities and distinctions with the flow of the same fluid subjected to identical boundary conditions for which compressibility effects are neglected. Starting from basic equations of compressible flows some general relations have been derived and reviewed focusing on design implications for aerators and air vents. It has been shown that differences may become significant when the downstream to upstream pressure ratio is lower than 0.9 which, when the air feeding the vent is at STP, corresponds to a pressure drop of $10'000 \text{ Nm}^{-2}$ (1 m of water column) and an air velocity larger than 150 ms^{-1} . Much higher pressure ratios and lower velocities would be expected to be suitable however, should singular and distributed friction losses be taken into account. Compressible flow equations have been therefore reckoned as necessary to get better insights into the behaviour of air supply systems in special hydraulic structures.

REFERENCES

- [1] Courant R. and Friedrichs K. O., Supersonic flow and shock waves, 1948, *Interscience Publishers*, New York.
- [2] Ruan S.-P. & Wu J.-H. & Wu W.-W. & Xi R.-Z., Hydraulic research of aerators on tunnel spillways, *Journal of Hydrodynamics, Ser. B*, vol. 19, 2007, pp. 330-334.
- [3] Shapiro A. H., The dynamics and thermodynamics of compressible fluid flow, 1954, *The Ronald Press Company*, New York.
- [4] Speerli J. and Hager W. H., Air-water flow in bottom outlets, 2000, *Canadian Journal of Civil Engineering*, vol. 27, pp. 454-462.
- [5] Vischer D. L. and Hager W. H., Dam hydraulics, 1998, *John Wiley and Sons Ltd*, Chichester, United Kingdom.
- [6] Wu J.-H. & Feng S.-R. & Wu W.-W. & Ruan S.-P., Hydraulic characteristics of partial opening on the working gate for a discharge tunnel, *Journal of Hydrodynamics, Ser. B*, vol. 19, 2007, pp. 488-493.