

Investigating stochastic portfolio theory with applications to the South African equity market

by

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*Thesis presented in partial fulfilment of the requirements for
the degree of*

Master of Science in Financial Engineering

*In the Faculty of Natural & Agricultural Sciences
University of Pretoria
Pretoria*

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June 2014

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Abstract

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Stochastic Portfolio Theory (SPT) as a methodology aims to move away from the efficient market hypothesis which was developed mainly as a way of explaining the relationship between risk and returns. SPT attempts to explain stock market behaviour using only the assumption of a logarithmic model of stocks, which is widely used in derivative pricing and hedging. This provides a potential tool for portfolio management and an alternative to the commonly used mean-variance approach of Markowitz. The aim of this dissertation is to provide an overview of the foundations of Stochastic Portfolio Theory, the consequences for portfolio construction and behaviour and apply these concepts to the South African Equity Market.

Acknowledgements

I would like to express my sincere gratitude to Prof. Eben Maré for all his support and assistance.

Dedications

To my parents and my loving wife, Jeanette.

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List of Symbols

Commonly used variables

$X_i(t)$	Stock i price process
$dW_\nu(t)$	Brownian motions describing the ν th source of randomness
$\xi_\nu(t)$	The sensitivity of the ν th source of randomness on a stock price process
$\gamma_i(t)$	Growth rate process for stock i
$\alpha_i(t)$	Arithmetic rate of return process for stock i
$\pi(t)$	Refers to the portfolio with individual weights $\pi_i(t)$
$\gamma_\pi(t)$	Growth rate process for the portfolio with weights π
$\gamma_\pi^*(t)$	Excess growth rate process for the portfolio with weights π
$\alpha_\pi(t)$	Arithmetic rate of return process for the portfolio with weights π
$\sigma_{ij}(t)$	Covariance process between stock i and j
$\tau_{ij}^\pi(t)$	Relative covariance process between stock i and j and relative to the portfolio with weights π
$Z_\pi(t)$	Price process for the portfolio with weights $\pi(t)$
$\delta_i(t)$	Continuous dividend rate process for stock i
$\mu_i(t)$	Weight of stock i in market capitalisation weighted portfolio (the market portfolio)
$\mu(t)$	Refers to the portfolio with individual weights $\mu_i(t)$ i.e. the market portfolio
$Z_\mu(t)$	Price process for the market portfolio
$\mathbf{S}(\mathbf{x})$	A portfolio generating function
$\Theta(t)$	Drift process for a portfolio generating function

List of Acronyms

Commonly used acronyms

<i>APT</i>	Abitrage Pricing Theory
<i>CAPM</i>	Capital Asset Pricing Model
<i>DE</i>	Differential Evolution
<i>GBM</i>	Geometric Brownian Motion
<i>MVO</i>	Mean Variance Optimisation
<i>MPT</i>	Modern Portfolio Theory
<i>SPT</i>	Stochastic Portfolio Theory

Chapter 1

Introduction

The goal of most asset managers is to maximise return whilst adhering to certain constraints. These constraints can come in varied forms, from minimising portfolio variance or risk to limits on asset class weights and liquidity constraints. However, regardless of the restrictions performance is usually relative to some benchmark.

In the case of equity markets, the typical benchmark is simply referred to as the *market portfolio* and is usually a market capitalisation weighted index of some specific stocks. This has benefits when we want to understand the overall market (or specific sector) movements since it weighs the index towards the shares which have a higher market capitalisation. These shares are therefore usually the most liquid shares and constitute a larger part of the overall market. These portfolios, however, may not necessarily be the most optimal portfolios to invest in.

Most quantitative techniques attempt to outperform the market portfolio by selecting stocks along certain factors, see [28] or [42] for an example of this in the South African market. In fact [53] showed that even randomly selected portfolios could outperform the market portfolio on average.

The framework around portfolio construction (especially in light of the market portfolio) is largely a product of Modern Portfolio Theory (MPT). In some respects MPT began with Markowitz [41], who created a framework where the portfolio variance was used as a measure of risk and together with expected returns an efficient frontier could be found. Optimal investments were those investments that returned a specific expected return for the lowest risk (variance).

This also introduced the concept of covariance and its effects on the overall portfolio risk. Tobin [54] extended this further by considering a combination of the risk-free asset and a unique combination of risky assets on the efficient

frontier.

Within this framework we still require the expected future returns, covariances and standard deviations of the assets under review. An attempt to understand the nature of these expected returns led to the widely used and quoted Capital Asset Pricing Model (CAPM) by Sharpe [48]. Under CAPM, excess returns can be expressed into two components, the risk of exposure to the market (systematic risk) and company specific risk (idiosyncratic risk). Since the specific risk can be diversified away, it can be argued that the expected excess return from the specific risk component was necessarily zero and that only the systematic component provided a consistent return.

This systematic risk component or beta (β) is effectively the security's exposure to the market. The CAPM model is effectively a factor model with one factor - the exposure to the market. Since CAPM is a single factor model it is merely a special case of Arbitrage Pricing Theory (APT) [45] which expresses the return of an asset in terms of numerous factors. That is, it is a multi-factor model as opposed to the single-factor of market exposure within the CAPM framework.

Fama and French ([12] and [13]) extended this idea to consider both value or growth and size as additional factors driving the returns of stocks. This, in turn, was extended by Carhart [9] by including momentum as an additional factor. See [2] and [50] for an application of this to the South African market.

In contrast, Stochastic Portfolio Theory (SPT) is descriptive in nature, in that it attempts to describe actual observed behaviours in the market. Beginning with the common Geometric Brownian Motion (GBM) model for stocks, we show in Section 2.3 that, firstly, it is the geometric rates of return which determine long term portfolio (and by extension an individual stock's) behaviour. This has important implications for portfolio optimisation and we show in Chapter 3 the empirical differences in portfolios formed around arithmetic and geometric rates of return.

With an understanding of stock and portfolio long-term behaviour under SPT, we then form portfolios with specific characteristics - the most important of which is the outperformance of a market capitalisation weighted portfolio (market portfolio). This is something we consider in Chapter 4.

Research into SPT has primarily been conducted by Fernholz through a series of published papers which culminated into a book [18]. Further research into aspects of SPT have also been conducted by [16], [24] and [35] to name a few.

Research into SPT has, however, mainly been applied to US equity markets

in general. However, South African markets are markedly different from those in the US.

This difference is most noticeable in the investment universe of the two markets. Consider the S&P500 and even the Russell 2000 index, comprising 500 and 2000 of the largest stocks in the US. In South Africa, the equity market is far smaller and the benchmark All-Share Index contains only between 150 and 170 stocks. Together with the far greater exposure to resources and the fact that our economy is still a developing economy, the structure of the South African market is fundamentally different from that of the US. Our interest is therefore the application of SPT within the South African equity markets.

The aim of this dissertation is, therefore, to provide an overview of key and foundational concepts of Stochastic Portfolio Theory and, most importantly, apply these to the South African market. More specifically:

1. Explain the basic concepts of SPT,
2. Explain the components (under SPT) of portfolio returns and the consequences for portfolio optimisation,
3. Analyse the long-term behaviour and properties of portfolios (and specifically the market portfolio) under SPT,
4. Provide a theoretical overview of functionally generated portfolios which are constructed (under SPT) to outperform the market portfolio over time,
5. Provide empirical results for the key concepts in SPT using historical South African equity market data, and
6. Highlight key inferences of the South African equity market since 1994 through the application of SPT.

The chapters are broken down as follows:

Chapter 2 focuses on the basic concepts of Stochastic Portfolio Theory by considering the common GBM model for stocks and extending this to a value process for portfolios of stocks. We derive growth rate and excess growth rate concepts within this chapter and show how these (specifically the excess growth

rate) relate to portfolio diversification and in turn to portfolio performance.

Chapter 3 applies the concepts considered in Chapter 2 to the South African market by comparing empirical optimisation results for both an arithmetic return target and a growth rate (geometric return) target.

Chapter 4 presents the mathematical properties of portfolios and, specifically, the market portfolio. We show in this chapter how the weights and excess growth rates are related, which further enhances the relationship between the excess growth rate and portfolio diversification as explained in Chapter 2. We also formalise the concept of diversity within this chapter. Chapter 4 also introduces the concept of functionally generated portfolios which can be used to construct portfolios with certain characteristics.

Chapter 5 applies the concepts in Chapter 4 to the South African market by considering the empirical results of certain functionally generated portfolios. The application of the theory from Chapter 4 to the South African market also allows us to draw inferences as to the behaviour of the South African stock market since 1994.

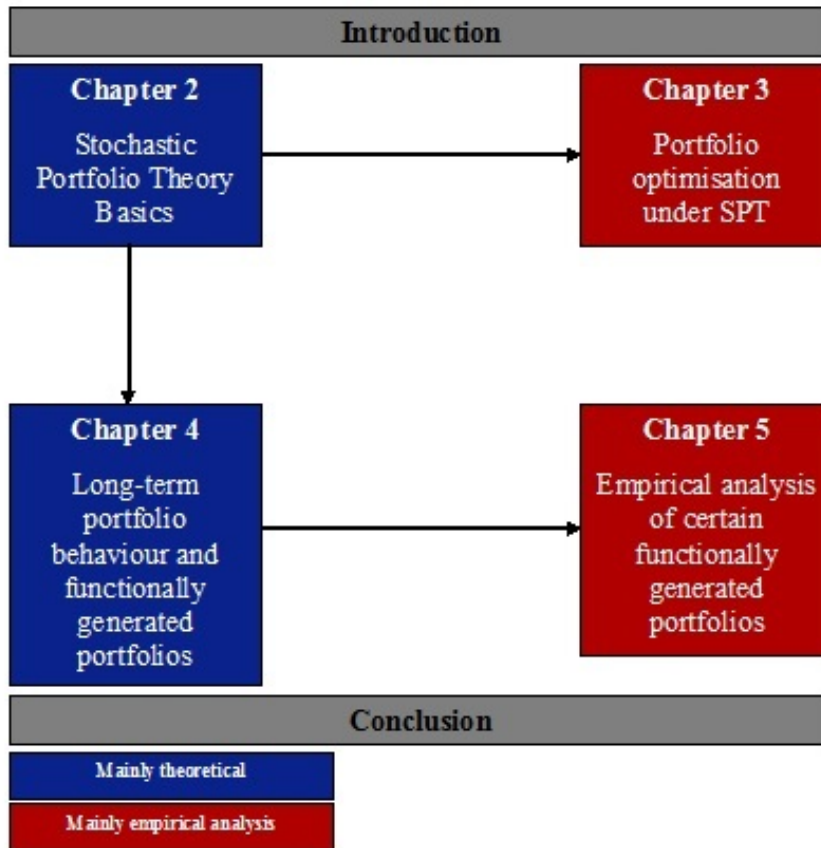


Figure 1.1: Basic structure of this dissertation.

Chapter 2

Stochastic Portfolio Theory basics

In this chapter we provide an explanation of the stock price process and how this translates into the basic concepts of Stochastic Portfolio Theory (SPT). We extend this to consider portfolios of stocks which will provide a foundation for the following chapters. Within this chapter we highlight the implications of SPT on portfolio optimisation and how diversification relates to portfolio growth rates.

2.1 Introduction

We assume a stock $X(t)$ is a function of a drift process, $\alpha(t)$, and some sources of randomness. These sources of randomness are akin to real world volatility in stock prices and we make the further assumption that these sources of randomness follow Brownian motions.

A variable y follows a Brownian motion if it exhibits the following two properties [33]:

Property 1: The change in y , namely Δy , over a small period of time, Δt , is

$$\Delta y = \epsilon \sqrt{\Delta t}, \quad (2.1.1)$$

where ϵ is normally distributed with a mean of zero and a standard deviation of one.

Property 2: The changes in y over two non-overlapping intervals are independent of one another. That is, the change in y over the intervals $(t_1, t_1 + \Delta t)$ and $(t_2, t_2 + \Delta t)$ are independent of one another provided that $t_2 > t_1 + \Delta t$.

It follows from the first property that Δy is also normally distributed with mean zero and variance given by:

$$\begin{aligned} \text{Var}(\Delta y) &= \text{Var}\left(\epsilon\sqrt{\Delta t}\right), \\ &= t\text{Var}(\epsilon), \\ &= t. \end{aligned}$$

Following on from the above, we therefore have a stock price process for $X(t)$ dependent on a drift process, denoted by $\alpha(t)$, and on sources of randomness.

Here we define $dW_\nu(t)$ as the ν -th source of randomness, where $dW_\nu(t)$ are Brownian motions (as described above) for $\nu = 1, 2, \dots, n$.

Furthermore, we define $\xi_\nu(t)$, for $\nu = 1, 2, \dots, n$, as the sensitivity of $X(t)$ on the ν -th source of randomness, $dW_\nu(t)$. We assume that the processes $\alpha(t)$ and $\xi_\nu(t)$ are measurable and adapted processes.

We can now define the price process for the changes in stock $X(t)$ over a short period of time, dt , as,

$$\frac{dX(t)}{X(t)} = \alpha(t)dt + \sum_{\nu=1}^n \xi_\nu(t)dW_\nu(t), \quad t \in [0, \infty), \quad (2.1.2)$$

where $\alpha(t)$, ξ_ν and $dW_\nu(t)$ are defined as above.

Equation (2.1.2) represents the instantaneous return of $X(t)$ and is a function of a drift process and some random disturbances. In this instance, $\alpha(t)$ represents the stock's arithmetic rate of return. Although this rate of return is used in MPT, we will prove that it is instead the geometric rate of return ($\gamma(t)$) and not the arithmetic rate of return that determines long term portfolio behaviour (see Proposition (2.3.1)). For other examples of geometric rates of return within portfolio selection see [16], [11] and [30] amongst others.

We will, therefore, need to transform Equation (2.1.2) from a characterisation of $dX(t)$ into one for $d\log X(t)$. To make the transformation we use Itô's lemma on the function $F(t) = \log X(t)$. This results in the popular logarithmic model for the continuous-time stock price processes:

$$d\log X(t) = \gamma(t)dt + \sum_{\nu=1}^n \xi_\nu(t)dW_\nu(t), \quad (2.1.3)$$

where ξ_ν and $dW_\nu(t)$ are defined as before and $\gamma(t)$ is defined as,

$$\gamma(t) = \left(\alpha(t) - \frac{1}{2} \sum_{\nu=1}^n \xi_\nu^2(t) \right). \quad (2.1.4)$$

$\gamma(t)$ is the geometric rate of return, referred to as the growth rate of $X(t)$ in SPT. As mentioned before, we will show later how the growth rate, and not the arithmetic rate of return, $\alpha(t)$, determines the long term behaviour of a portfolio of stocks. It is for this reason that SPT concerns itself with the growth rate, $\gamma(t)$.

For completeness we present the following definition of the stock price process $X(t)$ along the lines of the same definition in [18].

Definition 2.1.1. *Let n be a positive integer. A stock price process X is a process that satisfies the stochastic differential equation*

$$d\log X(t) = \gamma(t)dt + \sum_{\nu=1}^n \xi_{\nu}(t)dW_{\nu}(t), \quad t \in [0, \infty),$$

where (W_1, \dots, W_n) is a Brownian motion, γ is measurable, adapted and satisfies $\int_0^T |\gamma(t)| dt < \infty$, for all $T \in [0, \infty)$, a.s. Furthermore, the ξ_{ν} , $\nu = 1, \dots, n$ are measurable, adapted and satisfy

1. $\int_0^T (\xi_1^2(t) + \dots + \xi_n^2(t)) dt < \infty$, $T \in [0, \infty)$, a.s.,
2. $\lim_{t \rightarrow \infty} t^{-1} (\xi_1^2(t) + \dots + \xi_n^2(t)) \log \log t = 0$, a.s.,
3. $\xi_1^2(t) + \dots + \xi_n^2(t) > 0$, $t \in [0, \infty)$, a.s..

Note that we can integrate (2.1.3) directly, which yields

$$\log X(t) = \log X(0) + \int_0^t \gamma(s)ds + \int_0^t \sum_{i,\nu=1}^n \xi_{i,\nu}(s)dW_{\nu}(s), \quad t \in [0, \infty).$$

This can also be rearranged into an exponential form

$$X(t) = X(0)\exp\left(\int_0^t \gamma(s)ds + \int_0^t \sum_{i,\nu=1}^n \xi_{i,\nu}(s)dW_{\nu}(s)\right), \quad t \in [0, \infty).$$

2.2 Portfolios of Stocks

A portfolio's value process

In this section we extend the individual stock price process derived in the previous section and consider the value process for a portfolio of stocks. We will then be able to analyse the long-term behaviour of portfolio value processes.

To begin, let us consider a market M of a family of stocks X_1, \dots, X_n , each defined by Equation (2.1.3). A portfolio in the market M is a measurable, adapted vector-valued process π , with $\pi(t) = (\pi_1(t), \dots, \pi_n(t))$, for $t \in [0, \infty)$ and

$$\sum_{i=1}^n \pi_i(t) = 1, \quad t \in [0, \infty).$$

We say a market is nondegenerate if there exists a number $\epsilon_1 > 0$ such that,

$$x\sigma(t)x^T \geq \epsilon_1 \|x\|^2, \quad x \in \mathfrak{R}^n, \quad t \in [0, \infty). \quad (2.2.1)$$

That is, a market is nondegenerate if the variance of the market portfolio is bounded away from zero.

Furthermore, if there exists a number $\epsilon_2 > 0$ such that,

$$x\sigma(t)x^T \leq \epsilon_2 \|x\|^2, \quad x \in \mathfrak{R}^n, \quad t \in [0, \infty), \quad (2.2.2)$$

we say the market M has bounded variance.

Simply put, we have a market M , which has n stocks. A valid portfolio within this market is any combination of the stocks in M such that the weights, $\pi_i(t)$, sum to one.

The process $\pi_i(t)$ therefore represents the proportion of capital invested in the i -th stock. Now, let $Z_\pi(t)$ represent the value of some portfolio with weights π at time t . Obviously, the amount invested in stock X_i is given by,

$$\pi_i(t)Z_\pi(t).$$

Therefore, if the price X_i changes by $dX_i(t)$ then the change in the portfolio value $Z_\pi(t)$ is given by

$$\pi_i(t)Z_\pi(t) \frac{dX_i(t)}{X_i(t)}.$$

and the total change in the portfolio can therefore be expressed by,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{dX_i(t)}{X_i(t)}. \quad (2.2.3)$$

The instantaneous rate of change for the portfolio, $Z_\pi(t)$, is therefore the weighted sum of instantaneous changes of the stocks in the portfolio.

To proceed further, however, we will need a definition of the covariance process.

In Section 2.1 the volatility processes of X_i were given by $\xi_{i\nu}(t)$. We can then define $\xi(t) = (\xi_{i\nu}(t))_{1 \leq i, \nu \leq n}$. That is, $\xi(t)$ is a matrix, where the rows, i , represent each of the n stocks in the portfolio and the columns, ν , represent the sensitivity to each of the n sources of randomness. Furthermore, the covariance process $\sigma(t)$ is defined as $\sigma(t) = \xi(t)\xi^T(t)$.

Then the cross-variance processes for $\log X_i$ and $\log X_j$ is given by

$$\sigma_{ij}(t)dt = d \langle \log X_i, \log X_j \rangle_t = \sum_{\nu=1}^n \xi_{i\nu}\xi_{j\nu}(t), \quad t \in [0, \infty). \quad (2.2.4)$$

For $i = 1, \dots, n$, the process $\sigma_{ii}(t) = d \langle \log X_i \rangle_t$ is called the covariance process of X_i .

We can now use Equation (2.2.3) to derive a price process for the portfolio $Z_\pi(t)$ in differential form. To do this we set out the following proposition as in [21].

Proposition 2.2.1. *Let π be a portfolio and let,*

$$d \log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i, \nu=1}^n \pi_i(t)\xi_{i\nu}(t)dW_\nu(t), \quad (2.2.5)$$

where

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i, j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right). \quad (2.2.6)$$

Then, for any initial value $Z_\pi(0) > 0$, (2.2.5) can be integrated directly to obtain

$$Z_\pi(t) = Z_\pi(0) \exp \left(\int_0^t \gamma_\pi(s)ds + \int_0^t \sum_{i, \nu=1}^n \pi_i(s)\xi_{i\nu}(s)dW_\nu(s) \right) \quad (2.2.7)$$

as a solution of (2.2.3), for $t \in [0, \infty)$.

Proof. To prove this proposition we will begin with Equation (2.2.7) and prove it, and therefore (2.2.5), are equivalent to Equation (2.2.3).

It follows from (2.2.7) that

$$d \log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i, \nu=1}^n \pi_i(t)\xi_{i\nu}(t)dW_\nu(t).$$

We apply Itô's Lemma to $Z_\pi = \exp(\log Z_\pi(t))$ to obtain

$$dZ_\pi(t) = Z_\pi(t)d\log Z_\pi(t) + \frac{1}{2}d\langle \log Z_\pi \rangle_t,$$

and therefore, by substituting in $d\log Z_\pi(t)$,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \gamma_\pi(t)dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)dW_\nu(t) + \frac{1}{2}Z_\pi(t)d\langle \log Z_\pi \rangle_t,$$

where $d\langle \log Z_\pi \rangle_t$ is the covariance process of $d\log Z_\pi$. Since $Z_\pi(t)$ is a portfolio of stocks and given the definition of the covariance process before, we have

$$d\langle \log Z_\pi \rangle_t = \sum_{i,j=1}^n \pi_i(t)\pi_j(t)d\langle \log X_i, \log X_j \rangle_t = \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)dt.$$

Now by definition we have

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right),$$

and therefore,

$$\frac{dZ_\pi(t)}{Z_\pi} = \sum_{i=1}^n \pi_i(t)\gamma_i(t)dt + \frac{1}{2} \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)dt + \sum_{i,\nu=1}^n \pi_i(t)\epsilon_{i\nu}(t)dW_\nu(t).$$

Furthermore, Equation (2.2.4) implies that $\sigma_{ii}(t) = \sum_{\nu=1}^n \epsilon_{i\nu}^2(t)$ and therefore by Equation (2.1.3) we have,

$$dX_i(t) = \left(\gamma_i(t) + \frac{1}{2}\sigma_{ii}(t) \right) X_i(t)dt + X_i(t) \sum_{\nu=1}^n \epsilon_{i\nu}(t)dW_\nu(t), \quad \text{for } t \in [0, \infty).$$

This implies that,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{dX_i(t)}{X_i(t)}.$$

□

The structure of (2.2.5) is, not surprisingly, very similar to the price process for an individual stock in Equation (2.1.3). Once again we have a growth rate process, $\gamma_\pi(t)$, which is called the portfolio growth rate process. Notice in (2.2.6), that $\gamma_\pi(t)$ contains, not only the weighted sum of the individual growth rates, but also a component containing the portfolio's variance.

This component is called the excess growth rate process and is represented by $\gamma_\pi^*(t)$, where

$$\gamma_\pi^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right). \quad (2.2.8)$$

The first summation is a weighted sum of the individual stock variances whilst the second summation represents the portfolio's variance. The portfolio variance will be defined as $\sigma_{\pi\pi}$.

Heuristically, $\gamma_\pi^*(t)$ can be regarded as the benefits of diversification in terms of lowering the variance of the portfolio $Z_\pi(t)$. Although, it is widely known that diversification leads to a lower portfolio variance, it is not widely recognised to have an effect on the growth rate of the portfolio.

As an interesting note, we may want to maximise the portfolio growth rate which would involve maximising, in some part at least, the excess growth rate. That would involve selecting stocks with large volatilities such that their covariance structure allows a portfolio with a much lower overall volatility. In effect, if we consider Equation (2.2.8), we would be maximising the sum of the individual stock volatilities and minimising the resultant portfolio volatility. This difference, according to (2.2.6), would directly contribute to the portfolio's return.

Equation (2.2.5) can be interpreted as the logarithmic instantaneous return of the portfolio $Z_\pi(t)$. However, using Definition (2.1.1) and Equation (2.2.3) we can show that,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \left(\gamma_\pi(t) + \frac{1}{2} \sigma_{\pi\pi}(t) \right) dt + \sum_{i,\nu=1}^n \pi_i(t) \xi_{i\nu}(t) dW_\nu(t). \quad (2.2.9)$$

This can be interpreted, similarly to the single stock case, as the instantaneous rate of return of the portfolio π and we can therefore also define the rate of return process of the portfolio π as,

$$\alpha_\pi(t) = \gamma_\pi(t) + \frac{1}{2} \sigma_{\pi\pi}(t), \quad t \in [0, \infty). \quad (2.2.10)$$

Furthermore, if we use Definition (2.1.1) and Equations (2.2.3) and (2.2.9) we can prove that,

$$d \log Z_\pi(t) = \sum_{i=1}^n \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t) dt. \quad (2.2.11)$$

2.3 Long-term behaviour of portfolios

In this section we prove that the long-term behaviour of a portfolio is determined by the portfolio growth rate, $\gamma_\pi(t)$. This is in contrast to traditional theory which focuses on the rate of return, $\alpha_\pi(t)$. This has important considerations for long-term portfolio performance in practice (as we will illustrate in the next chapter).

To begin, we introduce the concept of martingales as presented in [6]. We refer the reader to [38] for a complete consideration of both brownian motion and martingales, but present some results which will be required.

Consider a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ where \mathcal{F}_t can be regarded as the information of observed events up to time t . We say a process $X(t)$ is adapted to the filtration \mathcal{F}_t if, given the information in \mathcal{F}_t , we can observe the value of $X(t)$ at time t .

Definition 2.3.1. *A stochastic process X is called an (\mathcal{F}_t) -martingale if the following conditions hold:*

1. X is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$
2. $E[|X(t)|] < \infty$ for all $t \geq 0$.
3. $E[X(t)|\mathcal{F}_s] = X(s)$ for all $s, t \geq 0$ and with $s \leq t$.

The second condition requires the expected value of $X(t)$ to be bounded, while the third condition requires that the expected value of X at some future point in time t is X 's current value, given the observed information.

We also introduce the concept of stopping times, as per [38].

Definition 2.3.2. *Consider a measurable space (Ω, \mathcal{F}_t) . A random time T is a stopping time of the filtration \mathcal{F}_t , if the event $\{T \leq t\}$ belongs to the filtration \mathcal{F}_t , for every $t \geq 0$. Here we regard T as the time at which some event occurs. Therefore, if the time at which the event occurs is less than or equal to t , we call T a stopping time of \mathcal{F}_t .*

Further to the above we require some results, the proofs of which can be found in [38]. We present these results here for completeness.

Theorem 2.1 (Time-change theorem for martingales). *Let $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\} \in \mathcal{M}^{c,loc}$ satisfy $\lim_{t \rightarrow \infty} \langle M \rangle_t = \infty$, a.s. Define for each $0 \leq s < \infty$, the stopping time*

$$T(s) = \inf \{t \geq 0; \langle M \rangle_t > s\}.$$

Then the time-changed process

$$B_s \triangleq M_{T(s)}, \quad \mathcal{G}_s \triangleq \mathcal{F}_{T(s)}; \quad 0 \leq s < \infty$$

is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}_s\}$ satisfies the usual conditions and we have a.s.

$$M_t = B_{\langle M \rangle_t}; \quad 0 \leq t < \infty.$$

Furthermore, we reproduce an abridged version of the Law of the Iterated Logarithm from [38], which describes the behaviour of Brownian motion near $t = 0$ and as $t \rightarrow \infty$, as well as the Strong Law of Large Numbers.

Theorem 2.2 (Law of the Iterated Logarithm). *For almost every $\omega \in \Omega$, we have*

$$\lim_{t \rightarrow \infty} \frac{W_t(\omega)}{\sqrt{2t \log \log t}} = 1.$$

Theorem 2.3 (Strong Law of Large Numbers). *Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard, one dimensional Brownina motion on (Ω, \mathcal{F}, P) . Then, we have that*

$$\lim_{t \rightarrow \infty} \frac{W_t}{t} = 0, \text{ a.s.}$$

We will require, Theorems 2.1, 2.2 and 2.3 for the following Lemma (reproduced from [18]), which, in turn, is required for Proposition 2.3.1.

Lemma 2.3.1. *Let M be a continuous local martingale such that*

$$\lim_{t \rightarrow \infty} t^{-2} \langle M \rangle_t \log \log t = 0, \quad \text{a.s.} \quad (2.3.1)$$

Then

$$\lim_{t \rightarrow \infty} t^{-1} M(t) = 0, \quad \text{a.s.}$$

Proof. We can construct a one-dimensional Brownian motion W_0 independent of M , and therefore define

$$M_0(t) = M(t) + W_0(t), \quad t \in [0, \infty).$$

Then M_0 is a continuous local martingale with

$$\langle M_0 \rangle_t = \langle M \rangle_t + t, \quad t \in [0, \infty), \quad a.s. \quad (2.3.2)$$

So we have, by (2.3.1), that

$$\lim_{t \rightarrow \infty} t^{-2} \langle M_0 \rangle_t \log \log t = 0, \quad a.s. \quad (2.3.3)$$

From (2.3.2) we see that,

$$\lim_{t \rightarrow \infty} \langle M_0 \rangle_t = \infty, \quad a.s. \quad (2.3.4)$$

so the time change theorem for local martingales can be applied to show that there exists a Brownian motion B such that

$$B(\langle M_0 \rangle_t) = M_0(t), \quad t \in [0, \infty), \quad a.s. \quad (2.3.5)$$

Due to (2.3.4) we can apply the law of the iterated logarithm for Brownian motion which, along with (2.3.5), implies that

$$\limsup_{t \rightarrow \infty} \frac{|M_0(t)|}{\sqrt{2 \langle M_0 \rangle_t \log \log \langle M_0 \rangle_t}} = 1, \quad a.s. \quad (2.3.6)$$

For (2.3.3) to be correct, it would imply that $\langle M_0 \rangle_t$ increases at a slower rate than t^2 , so we can replace $\log t$ by $\log \langle M_0 \rangle_t$ in (2.3.3), which yields

$$\lim_{t \rightarrow \infty} t^{-2} \langle M_0 \rangle_t \log \log \langle M_0 \rangle_t = 0, \quad a.s.$$

Taking the square root of both sides of this equation we have

$$\lim_{t \rightarrow \infty} t^{-1} \sqrt{\langle M_0 \rangle_t \log \log \langle M_0 \rangle_t} = 0, \quad a.s.$$

Now, taking into account this, we see that for (2.3.6) to hold would require

$$\lim_{t \rightarrow \infty} t^{-1} M_0(t) = 0, \quad a.s.$$

Therefore, as an implication of the strong law of large numbers for Brownian motion, we also have

$$\lim_{t \rightarrow \infty} t^{-1} W_0(t) = 0, \quad a.s.$$

□

We can now proceed to prove, along the same lines as [18], that a portfolio's long-term behaviour is determined by the portfolio's growth rate process.

Proposition 2.3.1. *Let π be a portfolio in the market M with the portfolio price process $Z_\pi(t)$ for $t \in [0, \infty)$. Then,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t) dt \right) = 0, \quad a.s. \quad (2.3.7)$$

Proof. By Proposition (2.2.1) we have,

$$\log \left(\frac{Z_\pi(T)}{Z_\pi(0)} \right) = \int_0^T \gamma_\pi(t) dt + \int_0^T \sum_{i,\nu=1}^n \pi_i(t) \xi_{i,\nu}(t) dW_\nu(t), \quad \text{for } t \in [0, \infty), \quad a.s.$$

Now, for $t \in [0, \infty)$, let

$$\begin{aligned} V(T) &= \log \left(\frac{Z_\pi(T)}{Z_\pi(0)} \right) - \int_0^T \gamma_\pi(t) dt, \\ &= \int_0^T \sum_{i,\nu=1}^n \pi_i(t) \xi_{i,\nu}(t) dW_\nu(t). \end{aligned}$$

So V is a continuous martingale with

$$\langle V \rangle_t = \int_0^t \sigma_{\pi\pi}(s) ds, \quad t \in [0, \infty), \quad a.s. \quad (2.3.8)$$

Now since the portfolio weights π are bounded, condition (2) of Definition (2.1.1) implies that

$$\lim_{t \rightarrow \infty} t^{-2} \sigma_{\pi\pi}(t) \log \log t = 0, \quad a.s.$$

We can now apply Lemma (2.3.1) to the process V . So we have, by Lemma (2.3.1),

$$\lim_{T \rightarrow \infty} \frac{1}{T} V(T) = 0, \quad a.s.$$

Now, substituting in $V(T)$ as defined previously yields

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log \left(\frac{Z_\pi(T)}{Z_\pi(0)} \right) - \int_0^T \gamma_\pi(t) dt \right) = 0, \quad a.s.$$

Since $Z_\pi(0)$ is merely a constant (representing the starting value of the portfolio) and not a function of T , we can express this as,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t) dt \right) = 0, \quad a.s.$$

□

Proposition (2.3.1) shows that the growth rate process of a portfolio determines the long-term behaviour of a portfolio. It is because of this that SPT concerns itself with the portfolio's growth rate process $\gamma_\pi(t)$ and not the instantaneous rate of return, $\alpha_\pi(t)$.

2.4 Relative returns

Most portfolios and/or stocks are considered relative to some benchmark portfolio or index. In this section we, therefore, define the covariance and variance processes of relative returns which will become useful in future sections. Of particular interest are the relative returns of a portfolio versus that of a benchmark, typically a market capitalisation weighted portfolio. This is typically referred to as alpha (over/under-performance), while the variance of alpha (relative returns) is referred to as the tracking error, that is, the volatility of relative returns.

First consider the relative performance of an individual stock, X_i versus a portfolio Z_η . We define the relative return process of X_i versus Z_η as,

$$\log(X_i(t)/Z_\eta(t)), \quad t \in [0, \infty). \quad (2.4.1)$$

Then the cross-variation (covariance) process for the relative return process for stocks X_i and X_j is given by,

$$\begin{aligned} \langle \log(X_i/Z_\eta), \log(X_j/Z_\eta) \rangle_t &= \langle \log(X_i), \log(X_j) \rangle_t - \langle \log(X_i), \log(Z_\eta) \rangle_t \\ &\quad - \langle \log(X_j), \log(Z_\eta) \rangle_t + \langle \log(Z_\eta), \log(Z_\eta) \rangle_t. \end{aligned} \quad (2.4.2)$$

We define $\tau_{ij}^\eta(t)$ as the relative covariance process for stocks X_i and X_j as in (2.4.2) with

$$\tau^\eta(t) = (\tau_{ij}^\eta(t))_{1 \leq i, j \leq n},$$

the relative covariance process $\tau^\eta(t)$ in matrix form.

Furthermore, we define the process $\sigma_{i\eta}(t)$ as,

$$\sigma_{i\eta}(t) = \sum_{j=1}^n \eta_j(t) \sigma_{ij}(t), \quad t \in [0, \infty),$$

and we therefore have that,

$$d \langle \log(X_i), \log(Z_\eta) \rangle_t = \sigma_{i\eta}(t) dt.$$

Now we can write Equation (2.4.2) as,

$$\tau_{ij}^\eta(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t), \quad t \in [0, \infty), \quad (2.4.3)$$

for $i, j = 1, \dots, n$ with $\sigma_{\eta\eta}(t)$ as the variance process of the portfolio η and therefore,

$$\sigma_{\eta\eta}(t) = \eta(t) \sigma(t) \eta^T(t), \quad t \in [0, \infty).$$

Then for all i and j we have,

$$d \langle \log(X_i/Z_\eta), \log(X_j/Z_\eta) \rangle_t = \tau_{ij}^\eta(t) dt, \quad t \in [0, \infty). \quad (2.4.4)$$

Equation (2.4.3) is the relative variance process of two stocks, X_i and X_j , both relative to the portfolio with weights $\eta(t)$.

We now extend this to consider the relative covariance process of a portfolio $\pi(t)$ versus a portfolio $\eta(t)$. Consider first that we have the variance process of a portfolio $\pi(t)$, which is given by,

$$\sigma_{\pi\pi}(t) = \pi(t)\sigma(t)\pi^T(t), \quad t \in [0, \infty).$$

Therefore, to get the relative variance of portfolio $\pi(t)$ versus portfolio $\eta(t)$ we can substitute in the relative covariance of each stock i relative to the portfolio $\eta(t)$. That is instead of using $\sigma(t)$, we use $\tau^\eta(t)$ and we therefore have,

$$\tau_{\pi\pi}^\eta(t) = \pi(t)\tau^\eta(t)\pi^T(t).$$

Furthermore,

$$\pi(t)\tau^\eta(t)\pi^T(t) = (\pi(t) - \eta(t))\sigma(t)(\pi(t) - \eta(t))^T = \eta(t)\tau^\pi(t)\eta^T(t). \quad (2.4.5)$$

This implies that,

$$\tau_{\pi\pi}^\eta = \tau_{\eta\eta}^\pi.$$

Therefore if $\sigma(t)$ is singular then, the relative variance process of two portfolios is zero if and only if the two portfolios are equal.

Note that in the case where $\eta(t)$ is the market portfolio, $\tau_{\pi\pi}^\eta$ is the tracking error of the portfolio $\pi(t)$ relative to the market portfolio.

We can also describe the excess growth rate process $\gamma_\pi^*(t)$ as a function of the tracking error. From Equation (2.2.8) and for a portfolio $\pi(t)$, we have that,

$$\gamma_\pi^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right).$$

However, we also have that $\sigma_{ij}(t) = \tau_{ij}^\eta(t)$ for any portfolio $\eta(t)$ and $i, j = 1, \dots, n$. That is, the covariance between stocks X_i and X_j is equivalent to the covariance between the relative returns of both X_i and X_j to any portfolio $\eta(t)$. $\gamma_\pi^*(t)$ can therefore be expressed as,

$$\gamma_{\pi}^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^{\eta}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \tau_{ij}^{\eta}(t) \right). \quad (2.4.6)$$

We can use any portfolio as a basis for the relative returns in (2.4.6). If we set $\eta(t) = \pi(t)$, that is, consider the covariance of relative returns between stocks X_i and X_j to the portfolio $\pi(t)$ we have

$$\gamma_{\pi}^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^{\pi}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \tau_{ij}^{\pi}(t) \right). \quad (2.4.7)$$

However, the second summation, $\sum_{i,j=1}^n \pi_i(t) \pi_j(t) \tau_{ij}^{\pi}(t)$ is the portfolio π 's relative return covariance to itself. This is naturally zero and we can therefore express the above equation as

$$\gamma_{\pi}^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^{\pi}(t) \right). \quad (2.4.8)$$

Equation (2.4.8) implies that the excess growth rate is half the sum of the weighted individual stocks' tracking error, relative to the portfolio itself. Consider this in the context of diversification. If a stock X_i has a high tracking error relative to the portfolio $\pi(t)$ it would imply that the stock X_i has a (relatively) lower covariance to the stocks forming the remainder of the portfolio $\pi(t)$ and its contribution would therefore increase the growth rate of the portfolio, as implied by (2.4.8).

A stock with a lower tracking error to the portfolio $\pi(t)$ would not have any significant contribution to the portfolio's growth rate (beyond the stocks own weighted growth rate contribution) as it would likely generate returns similar to the portfolio itself.

2.5 Dividends

Up until now, we have not considered dividends. We introduce them here for completeness.

Consider a dividend rate process $\delta_i(s)$. We assume dividends are paid continuously. Then for a stock X_i with dividends, we can define the total return, \widehat{X}_i as,

$$\widehat{X}_i(t) = X_i(t) \exp \left(\int_0^t \delta_i(s) ds \right), \quad t \in [0, \infty). \quad (2.5.1)$$

It follows then that,

$$d\log\widehat{X}_i(t) = d\log X_i(t) + \delta_i(t)dt, \quad t \in [0, \infty). \quad (2.5.2)$$

We can then define the augmented growth rate process for a stock X_i as,

$$\rho_i(t) = \gamma_i(t) + \delta_i(t), \quad t \in [0, \infty).$$

Now suppose we have a portfolio, $\pi(t)$. Then we can define the portfolio dividend process as,

$$\delta_\pi(t) = \sum_{i=1}^n \pi_i(t)\delta_i(t), \quad t \in [0, \infty).$$

Similar to the the individual stock case, we have

$$d\log\widehat{Z}_\pi(t) = d\log Z_\pi(t) + \delta_\pi(t)dt, \quad t \in [0, \infty). \quad (2.5.3)$$

The dividends increase the total return \widehat{Z}_π , but since the dividends are reinvested proportionally to the weights of each individual stock, the weights of the portfolio will remain unaffected.

2.6 Summary

In this section we have introduced the basic concepts of SPT, building up the individual stock price process in (2.1.3) to a portfolio process in (2.2.5). This was done in Proposition (2.2.1) which also yielded a portfolio growth rate process, $\gamma_\pi(t)$ and excess growth rate process, $\gamma_\pi^*(t)$.

The portfolio growth rate is related to the arithmetic rate of return through (2.2.10). It is this arithmetic return that is used within traditional mean-variance optimisation. However, we illustrated through Proposition (2.3.1) that it is the growth rate process, $\gamma_\pi(t)$, and not the arithmetic rate of return $\alpha_\pi(t)$, that determines a portfolio's long term behaviour.

We also noted in this chapter that the excess growth rate process, $\gamma_\pi^*(t)$ is comprised of the difference between the weighted sum of the individual stock variances and the overall portfolio variance. $\gamma_\pi^*(t)$ effectively captures the benefits of diversification. We further illustrated this point by defining the tracking error (in the context of SPT) and showing that the excess growth rate is proportional to the weighted sum of the individual stocks' tracking errors to the portfolio itself.

Chapter 3

Portfolio optimisation under SPT

In this chapter we consider the practical implications of the theory presented in the previous chapter. Here we are concerned with portfolio construction (in the context of SPT) with particular reference to various portfolio optimisations and in particular mean-variance type optimisations.

This chapter is structured as follows:

- We first discuss the implications of the theory presented in the previous chapter and show why the expected geometric growth rate is more accurate in determining the long term growth rate of a stock or portfolio than the arithmetic rate of return.
- We then set out our data and methodology that will be used in the next section and throughout the report.
- We then consider three specific portfolio optimisations as applied to the South African equity market. These are
 1. Traditional mean-variance optimisation.
 2. Optimisation of portfolio growth rates relative to tracking errors.
 3. Optimisation of the excess growth rate.

3.1 Introduction

One of the important findings from the previous chapter was that a portfolio's long term behaviour is determined by the growth rate process $\gamma(t)$. This was proven in Proposition (2.3.1). This has important ramifications for how portfolio behaviour is analysed. In most of traditional modern portfolio theory, a portfolio's arithmetic rate of return $\alpha(t)$ is analysed when, as we have shown in

the previous chapter, it is the geometric rate of return ($\gamma(t)$) that determines the portfolio's long term behaviour. See [5], [11] and [16] for further examples of the application of the geometric rates of return to portfolio optimisation.

To illustrate the difference between the expected arithmetic rate of return and the actual long term arithmetic rate of return consider a stock that generates a return each period of either +25% or -5% with equal probability. Therefore the expected arithmetic return is given by $0.5 \times (0.25 - 0.05) = 10\%$. However, if we simulate this stocks returns for, say 100 periods we get an average compounded periodic growth rate of 8.98%, more than 1% lower than the arithmetic return.

Although we have provided a simulation, the geometric growth rate can be calculated directly as the arithmetic growth rate less one half of the stock's variance. In this example we obtain a growth rate of 8.88%. The difference between the simulated growth rate of 8.98% and the calculated 8.88% growth rate is due to the discrete nature of the simulation versus the continuous time setting in the direct calculation. Notwithstanding this, however, the expected arithmetic growth rate clearly overestimates the actual long term growth rate of the stock.

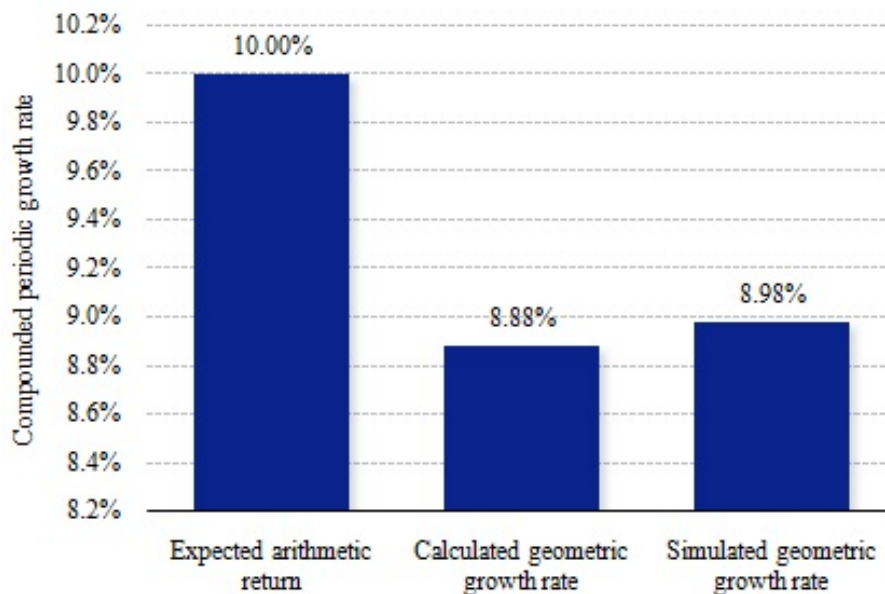


Figure 3.1: Comparison between expected arithmetic, expected geometric and simulated compounded annual returns for a stock which returns either +25% or -5% in each period with an equal probability.

This difference can make a significant impact on the results of portfolio opti-

misation.

3.2 Data and methodology

In this section we set out our data and methodology which will be used in this and future chapters.

We make use of South African listed stocks from December 1994 to September 2013. Data is obtained from Thompson Reuters Datastream on a monthly basis. For our purposes we will only require the following fields:

- Stock closing prices.
- Stock total return indices
- Market capitalisation of individual stocks.

To analyse the accuracy of our data we construct a market capitalisation weighted index over our sample period and compare this to the actual JSE All-Share index from October 2002 (circa formation date of the All-Share index). We find that our reconstructed index closely matches that of the actual All-Share index from October 2002, with annual returns on average within 12 basis points (bps) of each other.



Figure 3.2: Actual and replicated JSE All-Share index from October 2002 to September 2003.

All analyses are conducted using R. The source code for this, and later chapters' analyses, is given in the appendix.

3.3 Mean-Variance optimisation

Under traditional mean-variance optimisation (MVO), introduced by Markowitz [41], we consider reward as the portfolio's expected arithmetic rate of return and a portfolio's risk as the variance of the portfolio. Under MVO we aim to determine a portfolio's weights $\pi_i(t)$ for $i = 1, \dots, n$ and for some $t > 0$ such that we minimise the portfolio's level of risk (portfolio variance) for a minimum level of reward (expected arithmetic return). Alternatively, we could maximise the level of reward for a certain level of risk. In the analysis that follows we will deal with the former objective.

Mathematically, we would like to minimise

$$\sum_{i=1}^n \sum_{j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t), \quad (3.3.1)$$

subject to

$$\sum_{i=1}^n \pi_i(t) \alpha_i(t) \geq \alpha_0, \quad (3.3.2)$$

and

$$\sum_{i=1}^n \pi_i(t) = 1, \quad (3.3.3)$$

with $\pi_i(t) \geq 0$ for $i = 1, \dots, n$. Here $\alpha_0(t)$ is the minimum level of reward (expected arithmetic rate of return) required.

Now consider the same constraints under SPT. Unlike (3.3.2) in traditional MVO, SPT makes use of the expected geometric rate of return, $\gamma_\pi(t)$. In this case, we would replace the constraint in (3.3.2) with

$$\sum_{i=1}^n \pi_i(t) \gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right) \geq \gamma_0. \quad (3.3.4)$$

This can be re-written as

$$\sum_{i=1}^n \pi_i(t) \gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) \right) \geq \gamma_0 + \frac{1}{2} \left(\sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right). \quad (3.3.5)$$

Equations (3.3.4) and (3.3.5) are not linear in π and therefore conventional quadratic programming cannot be used. To optimise this we make use of Differential Evolution (DE) optimisation as set out in [49].

DE is a stochastic population based minimizer. DE begins by randomly generating a population, and then generates a new population vector by using the weighted differences between two population vectors to a third vector called the mutated vector. This mutated vector is then combined with the target vector. If this combined vector yields a lower cost function than the current target vector then it replaces the target vector in the next generation.

We apply the DE algorithm using the *DEoptim* package in R as set out in [1].

To construct a comparison between traditional MVO and MVO under SPT we construct portfolios under the conditions listed above, over our sample period.

Furthermore, we assume perfect foresight in terms of returns over the next 24 months and make use of rolling forward 24 month stock returns to determine expected returns (under MPT) and expected growth rates (under SPT). The covariance matrix is also estimated using 24 month forward stock returns.

The use of forward returns (or assuming perfect foresight) is to only consider the actual methods and eliminate any biases in the data which may distort comparisons. As a result, the actual returns achieved by the portfolios presented here are spurious. The goal of this section is to show that while the constraints are not linear and traditional optimisation cannot be used, there are appropriate tools which can be used.

To construct the SPT optimisation we define a cost function. Given a set of weights π_1, \dots, π_n we define a control variable, $C_1(t)$ using Equation (3.3.5). That is,

$$C_1(t) = \sum_{i=1}^n \pi_i(t) \gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) \right) - \gamma_0 + \frac{1}{2} \left(\sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right). \quad (3.3.6)$$

Given the constraint in (3.3.5), we would like to maximise $C_1(t)$ while also minimising the portfolio variance. To achieve this we define a second control variable, $C_2(t)$, which is given by

$$C_2(t) = \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t) - 1000 \min(C_1(t), 0). \quad (3.3.7)$$

This ensures that if $C_1(t)$ is not positive we increase (or penalise) the portfolio variance (3.3.1) obtained. We multiply by 1000 to ensure that the optimisation routine rejects $C_2(t)$ as a minimum in these cases. Cases where $C_1(t)$ is positive will result in only the portfolio variance being considered by the optimisation algorithm.

Furthermore, for the optimisation results below we have set a target annualised return of 15% p.a., with a maximum weight of 15% for each stock. However, we note that while we present both the MPT and SPT portfolios, their respective returns are not necessarily comparable since our optimisation is based on minimising portfolio variance subject to a minimum return. Both portfolios do achieve this minimum, however, beyond that the returns are spurious.

Our goal is therefore to show that the implementation of an SPT based optimisation is possible.

Figure 3.3 shows the log cumulative performance of both the SPT and MPT portfolios with the universe for both portfolios set at the top 40 stocks by market capitalisation. Both portfolios have a similar behaviour, although the MPT portfolio does achieve a greater return over the full sample period. However, Figure 3.4, showing calendar year returns, highlights that most of this outperformance is generated by the 2007/2008 performance.



Figure 3.3: Log cumulative performance charts of SPT and MPT returns.

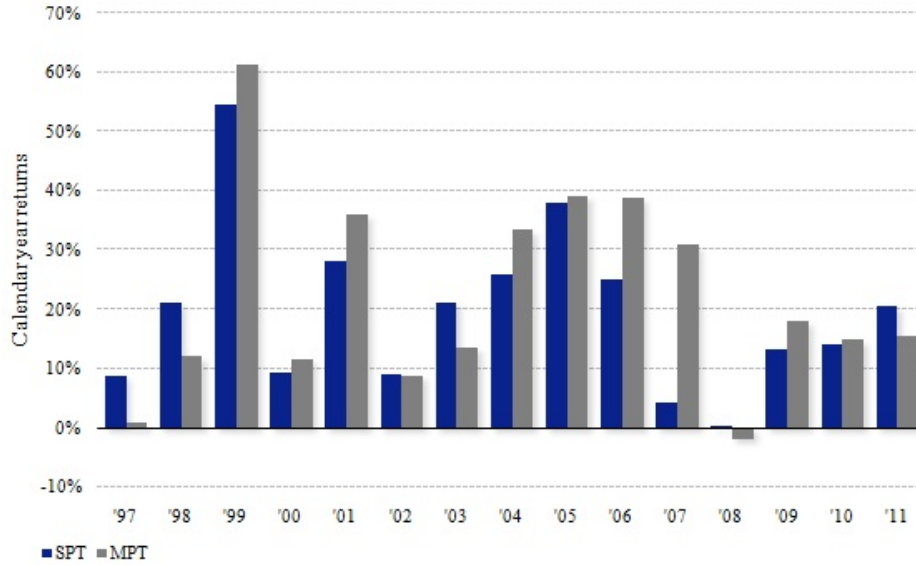


Figure 3.4: Calendar year returns for the SPT and MPT portfolios..

	SPT	MPT
CAGR	19.1%	21.6%
Volatility	11.3%	13.8%
Sharpe ratio	1.68	1.56

Table 3.1: Annualised returns and volatility for the SPT and MPT portfolios.

The SPT portfolio, however, does achieve a higher Sharpe ratio in light of its lower portfolio variance. This is shown in Table 3.3.

3.4 Minimising portfolio tracking error

In Section 2.4 we introduced the concept of relative returns. In practise, most portfolios are managed in relation to some benchmark portfolio, typically the market capitalisation weighted index. Therefore, instead of minimising the portfolio's variance, we may wish to minimise the portfolio's tracking error relative to the benchmark instead.

We therefore, now aim to minimise the portfolio's tracking error,

$$\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t), \quad (3.4.1)$$

subject to

$$\sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t) \right) \geq \gamma_0, \quad (3.4.2)$$

with $\sum_{i=1}^n \pi_i(t) = 1$ and $\pi_i(t) \geq 0$ for $i = 1, \dots, n$.

As in Section 3.3 we can rearrange the constraint in (3.4.2) as,

$$\sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) \right) \geq \gamma_0 + \frac{1}{2} \left(\sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t) \right). \quad (3.4.3)$$

Notice that the last term in (3.4.3) is half the square of the portfolio tracking error. If we target a tracking error of 3% per annum, this term is equal to 0.05% or 5bps per year. This is relatively insignificant, and therefore for a small enough tracking error target (say $\leq 4\%$ per annum, see Figure 3.5) we can ignore this last term.

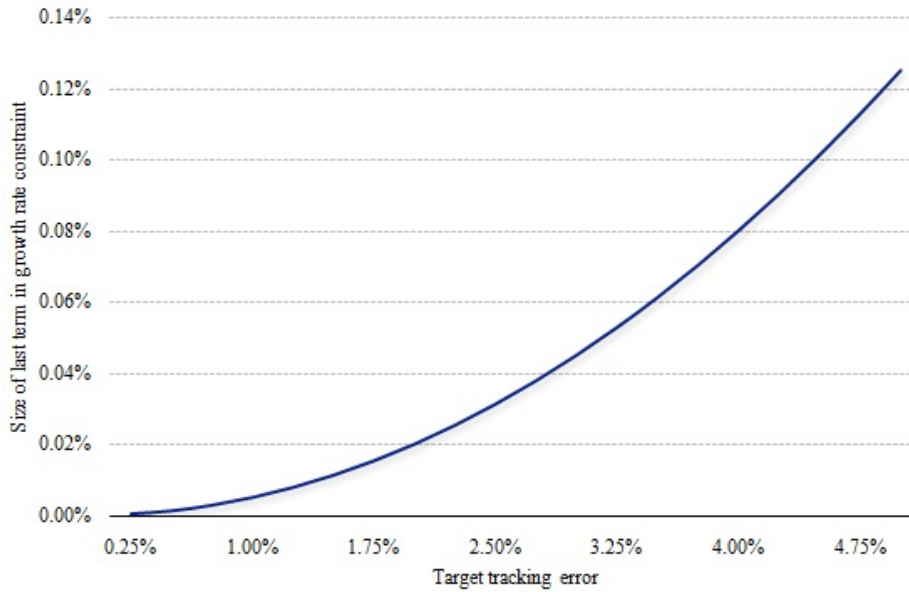


Figure 3.5: Size of last term in tracking error minimisation return constraint.

The return constraint therefore becomes

$$\sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) \right) \geq \gamma_0. \quad (3.4.4)$$

Equation (3.4.4) is now linear and we can, therefore, make use of conventional quadratic programming techniques.

To do this we make use of the *solve.QP* function in R which implements the dual method for solving quadratic programming problems as in [29]. Here our objective is to outperform the market capitalisation weighted portfolio of the top 40 stocks by a certain percentage per annum while minimising the tracking error of our portfolio.

The function *solve.QP* minimises functions of the form,

$$-d^T w + \frac{1}{2} w^T D w,$$

with the constraint

$$A^T b \geq b_0.$$

For this optimisation problem, matrix D represents the relative covariance matrix (relative to the market capitalisation weighted index). We set the matrix d to zero and construct the matrix A and b such that each stock's weight is greater than or equal to zero and all weights sum to one. Furthermore, we also include the criteria (3.4.4) within matrices A and b .

Matrix A therefore takes the form,

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & \gamma_1 - 0.5\tau_{11} \\ 1 & 0 & 1 & \cdots & 0 & \gamma_2 - 0.5\tau_{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & \gamma_n - 0.5\tau_{nn} \end{pmatrix}$$

and matrix b takes the form

$$b = (1, 0, \dots, 0, \gamma_0).$$

Expected growth rates of stocks are notoriously difficult to accurately estimate and we therefore assume perfect foresight over the next 24 month's stock returns to get results which are independent of growth rate estimation. This ensures that our results highlight the effectiveness of the optimisation routine independent of the growth rate estimation process.

We show below the results for portfolios targeting 5% and 10% excess returns together with their respective tracking errors. We first display the log cumulative performance for Portfolio 5.0% and Portfolio 10.0%. These are portfolios optimised to generate 5.0% and 10.0% excess returns, respectively.

We note that while the 5.0% target is achieved, the 10.0% target is not. We ascribe this to the estimation of expected growth rates, while using perfect foresight we still need to make use of a window (we use 24 months) which

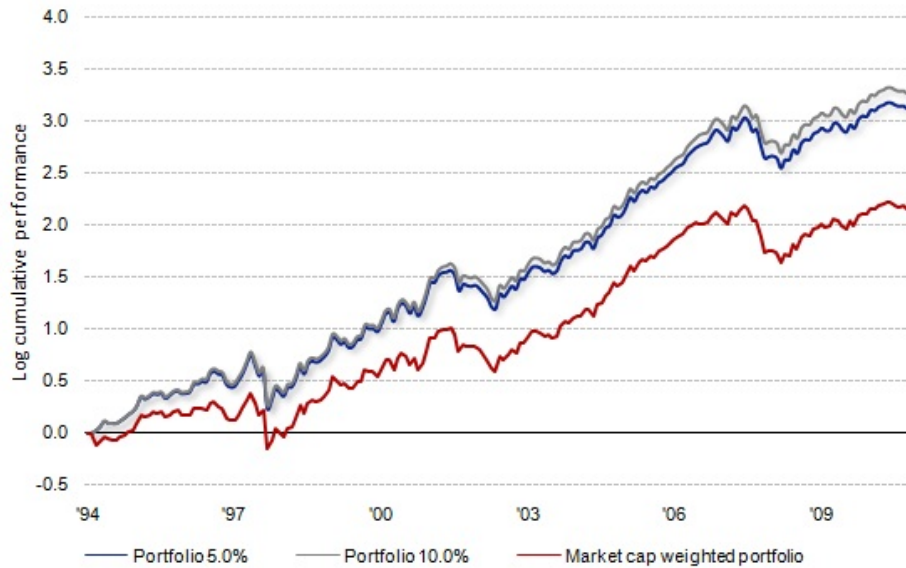


Figure 3.6: Log cumulative performance charts of optimised portfolios while minimising tracking error.

	Portfolio 5.0%	Portfolio 10.0%	Market cap
CAGR	21.18%	22.22%	14.97%
Tracking error	5.49%	5.65%	-

Table 3.2: Compounded annualised growth rate and tracking error for SPT optimised portfolios.

leads to some estimation error in the case of the covariance matrix. Furthermore, a 10.0% outperformance target is reasonably large in the context of minimising the tracking error and the optimisation routine may not be able to both limit the size of the tracking error and achieve the outperformance.

The results above are highly sensitive to the estimated covariance matrix and the estimated growth rates. In our analyse we have used a simple historical estimation method but a more robust method (such as a GARCH based method) would likely lead to better results. Although we have used perfect foresight, covariance processes are likely to change within the sample period itself.

Instead, what we would like to focus on is the difference between using the constraint in Equation (3.4.4) and using only the difference between the arithmetic return of the individuals stocks less that of the benchmark portfolio.

Note that we only change one variable - replacing excess growth rates with arithmetic excess returns. Everything else, including the tracking error matrix

remain the same. That is, this is a direct comparison of the long term difference between the use of geometric and arithmetic returns.

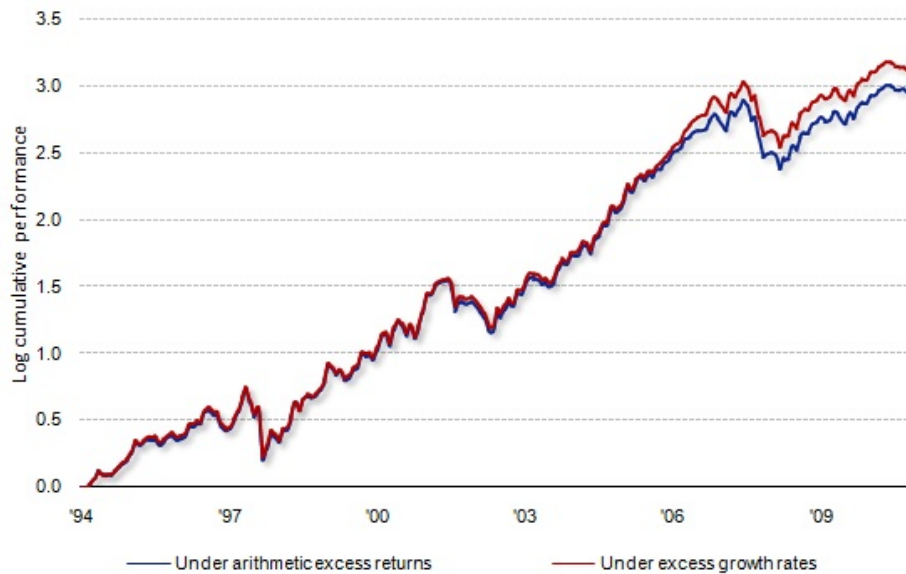


Figure 3.7: Log cumulative performance charts of optimised portfolios while minimising tracking error using arithmetic excess returns and geometric growth rates. Target return of 5% per annum over the benchmark portfolio.

Notice how the portfolio optimised under geometric growth rates outperforms the portfolio under arithmetic expected relative returns even though the same returns and same optimisation routine is used.

In Figure 3.8 we show the calendar year returns for both portfolios. A large portion of the outperformance is generated in 2007, however, even excluding this year's return the geometric portfolio outperforms the arithmetic portfolio by 50bps per year. This is because the use of arithmetic rates of return in the optimisation routine overestimate the portfolio return.

Theoretically, the difference between the two returns is related through Equation (3.4.5), which can also be expressed as

$$\gamma(t) = \alpha(t) - \frac{1}{2}\sigma^2(t). \quad (3.4.5)$$

Here $\alpha(t)$ represents the arithmetic returns and $\gamma(t)$ represents the geometric returns. We show the empirical results versus the expected theoretical results

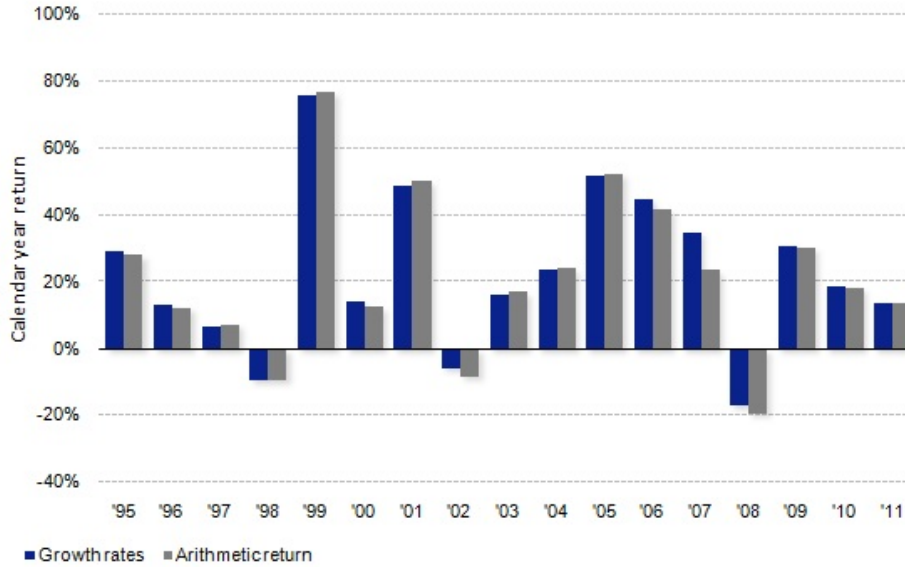


Figure 3.8: Calendar year returns for optimised portfolios under geometric and arithmetic returns

Portfolios maximising	Growth rates γ_{π}	Arithmetic rates of return α_{π}
CAGR	22.22%	19.91%
Volatility	20.48%	20.69%
Growth rate equivalent	-	22.05%
Arithmetic return equivalent	20.12%	-

Table 3.3: Comparison of theoretical and empirical growth rates and arithmetic returns.

in Table 3.3.

3.5 Maximising the excess growth rate

In this section we analyse a portfolio constructed by maximising the excess growth rate, $\gamma_{\pi}^*(t)$.

In Section 2.2 we analysed the portfolio value process and found that the portfolios growth rate $\gamma_{\pi}(t)$ was a combination of the weighted growth rates of each stock and a second term called the excess growth rate, $\gamma_{\pi}^*(t)$. As explained in Section 2.2, this term is can be regarded as the contribution to the return from the diversification present in the portfolio.

In Proposition (2.3.1) we showed that the long term performance of a portfolio is determined by the growth rate process. However, accurately forecasting growth rates for stocks is difficult. The excess growth rate, however, is only a function of the covariance matrix, which is somewhat easier to forecast and arguably more stable than the returns of individual stocks.

We therefore, analyse a portfolio which tries to maximise the portfolio excess growth rate process at the end of each month. We use a very simplistic forecast for the covariance matrix by estimating historical covariances over the previous twenty four months. Furthermore, we also restrict the portfolio to the top 40 stocks by market cap and set a maximum weight per stock as the minimum of fifteen percent or five times the stock's index weight.

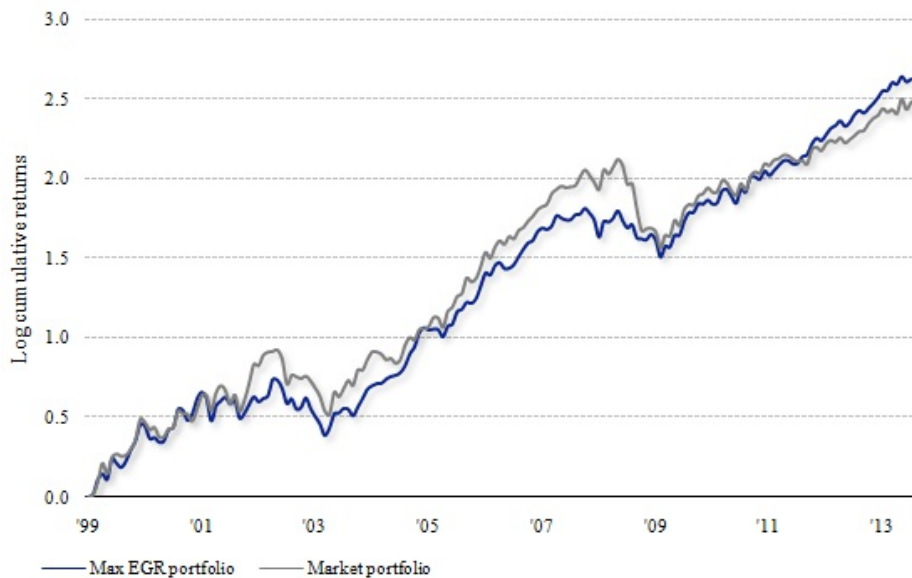


Figure 3.9: Log cumulative performance charts of the market portfolio and a portfolio which maximises excess growth rate

Figure 3.9 shows the log cumulative performance of the market portfolio and the portfolio which maximises the excess growth rate (max EGR portfolio). Although the max EGR portfolio outperforms the market portfolio overall, there are extended periods where the max EGR portfolio underperforms the market portfolio.

The periods of under- and outperformance for the max EGR portfolio are clear in Figure 3.11. The period from 2001 to 2010 show the max EGR portfolio as underperforming the market portfolio. The outperformance of the market

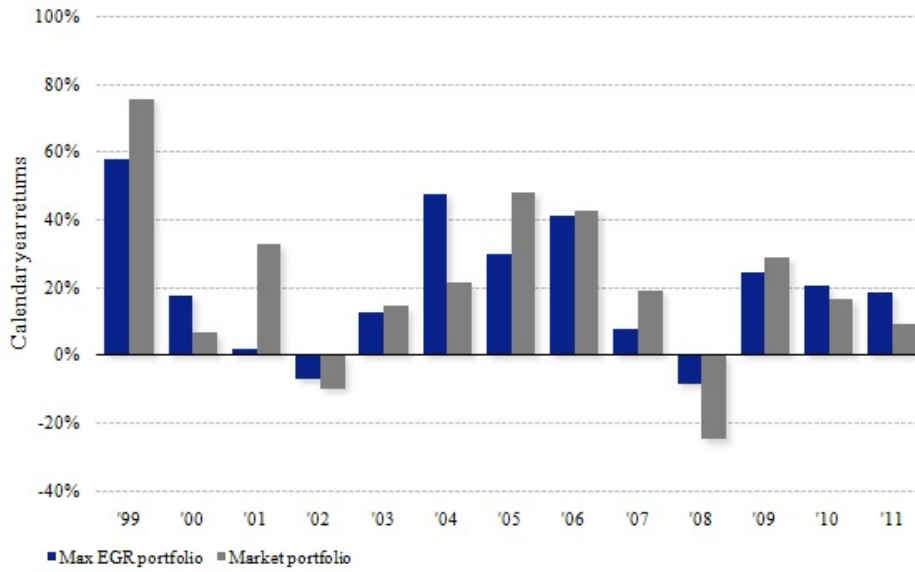


Figure 3.10: Calendar year returns for the market portfolio and a portfolio which maximises excess growth rate

portfolio is largely a result of the post-2008 period.



Figure 3.11: Cumulative return for the portfolio which maximises excess growth rate relative to the market portfolio

Figure 3.11 seems to suggest that while the max EGR portfolio outperforms

the market portfolio over our full sample period, it does so in a cyclical manner. This would imply that the benefits of diversification change over time.

3.6 Summary

In this chapter we applied the foundational results of SPT (presented in Chapter 2) to portfolio optimisations and compared these results to the traditional MVO. The key difference between optimisation under SPT and traditional MVO is the use of geometric rates of return. We showed, at least theoretically, in Chapter 2 that the geometric rates of return determined long term portfolio behaviour and not arithmetic rates of return.

The implication is that arithmetic rates of return overestimate the expected portfolio return and we showed in Section 3.1 with a simple simulation that the geometric growth rate was a much better predictor of future returns over the long term.

The complication, however, is that the constraints under SPT are not always linear in the weights $\pi(t)$ and we therefore cannot use conventional quadratic programming techniques to solve these constraints. In Section 3.3 we made use of the Differential Evolution algorithm to solve the return constraints under SPT. We showed that while the constraints are not linear, optimisation under SPT can, nevertheless, be implemented.

In Section 3.4 we focused on optimisation in relation to the tracking error of a portfolio relative to the market portfolio. We showed that, for small enough tracking error targets, we could reduce the constraints and make use of quadratic programming techniques.

We also highlighted in Section 3.4 how the portfolio under SPT outperformed the portfolio under MPT over the longer term. The only difference between the two portfolios being the use of geometric rather than arithmetic rates of return.

In Section 3.5 we analysed the portfolio which maximises the excess growth rate γ_{π}^* . We showed that this portfolio does outperform the market over our full sample period. However, the relative performance (relative to the market portfolio) is cyclical in nature, implying that the benefits of diversification change over time.

Chapter 4

Long-term portfolio behaviour and functionally generated portfolios

We now turn our attention to the construction of portfolios with specific characteristics. In particular we focus on outperforming the market capitalisation weighted portfolio, a common benchmark for many equity portfolio managers. In the previous chapters we mentioned the stock market or market capitalisation weighted portfolio (hereafter referred to as the market portfolio). However, we did so without explicitly defining (at least mathematically) what we mean by these terms.

Therefore, in this chapter we first formalise our understanding of the stock market and study its long-term behaviour and that of the stocks within the market. This lays the foundation for the remainder of the chapter which focuses on functionally generated portfolios and specifically on generating portfolios which outperform the market portfolio.

This chapter is structured as follows:

- In the first two sections we introduce some basic concepts regarding the market portfolio.
- We also show the relationship between growth rates in a market and the concentration of capital.
- In the third section of this chapter we consider the idea of stock market diversity and its relationship to growth rates.
- In the last section we introduce the concept of functionally generated portfolios.
- Furthermore, we analyse the relative performance of a functionally generated portfolio in functional form. This forms the basis for the empirical analyses in the next chapter.

4.1 Introduction

When we talk about a market M , we are referring to the collection of stocks weighted by market capitalisation. Therefore the market portfolio, Z_μ , is a portfolio formed by weighting each stock in the market M by the relative size of their market capitalisations. More formally let $X_i(t)$ represent the market capitalisation of stock i at time t . Then the stock market portfolio, $Z_\mu(t)$, is a portfolio with weights $\mu(t)$ given by

$$\mu_i(t) = \frac{X_i(t)}{\sum_{i=1}^n X_i(t)}. \quad (4.1.1)$$

Therefore, by Equation (2.2.3) we have,

$$\frac{dZ_\mu(t)}{Z_\mu(t)} = \sum_{i=1}^n \mu_i(t) \frac{dX_i(t)}{X_i(t)}. \quad (4.1.2)$$

Furthermore, by Proposition (2.2.1), we have

$$d\log Z_\mu(t) = \gamma_\mu(t)dt + \sum_{i,\nu=1}^n \mu_i(t)\xi_{i\nu}(t)dW_\nu(t), \quad (4.1.3)$$

where,

$$\gamma_\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^n \mu_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \mu_i(t)\mu_j(t)\sigma_{ij}(t) \right). \quad (4.1.4)$$

Note that, by Equation (4.1.3) and by Definition (2.1.1), we can write

$$d\log \mu_i(t) = (\gamma_i(t) - \gamma_\mu(t)) dt + \sum_{i,\nu=1}^n (\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)) dW_\nu(t). \quad (4.1.5)$$

That is, the change in the market weight of stock i can be expressed as the difference in growth rates and the difference in sensitivities, in relation to the market portfolio.

In order to ensure a logical and orderly market, M , we introduce the definition of a coherent market.

Definition 4.1.1. *The market M is coherent if*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_i(t) = 0, \quad a.s. \quad (4.1.6)$$

for all $i = 1, \dots, n$.

Note that $\log \mu_i(t) < 0$ and therefore Equation (4.1.6) will hold as long as no stock declines too rapidly.

Since $\mu_i(t) = \frac{X_i(t)}{Z_\mu(t)}$, Equation (4.1.6) also implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} (\log X_i(t) - \log Z_\mu(t)) = 0, \quad a.s. \quad (4.1.7)$$

That is, over a long enough time horizon and in a coherent market, the time-weighted average difference between the price process of each stock and the market portfolio will equal zero.

4.2 The stock market

The previous section provided a basis for the concept of a market portfolio with weights $\mu(t)$ and gave a definition for a coherent market through Definition (4.1.1).

In this section we look at the mathematical properties of the stock market M in more detail. We look specifically at what conditions are required for a coherent market as given in Definition 4.1.1 and what this means for the individual stock growth rates, weights and relative variances. We begin with the following proposition from [18] which deals with long-term time weighted average growth rates.

Proposition 4.2.1. *Let M denote the market with stocks X_1, \dots, X_n . Furthermore, the market M is coherent as given in Definition (4.1.1). Then, the following conditions are equivalent to the market's coherence:*

1. $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_\mu(t)) dt = 0$, for $i = 1, \dots, n$
2. $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_j(t)) dt = 0$, for $i, j = 1, \dots, n$

Proof. We begin by proving that coherence implies (1).

From (4.1.6) we have that

$$\lim_{T \rightarrow \infty} \frac{1}{T} (\log X_i(T) - \log Z_\mu(T)) = 0. \quad (4.2.1)$$

However, by Proposition (2.3.1) we also have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t) dt \right) = 0. \quad (4.2.2)$$

Furthermore, assume a portfolio π with a weight of 1 in stock i and zero in all other stocks. Then Proposition (2.3.1) and Equation (2.3.7) implies that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log X_i(T) - \int_0^T \gamma_i(t) dt \right) = 0. \quad (4.2.3)$$

These equations above imply (through substitution) that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_\mu(t)) dt = 0,$$

and therefore coherence implies condition (1).

Furthermore, a portfolio comprising a weight of 1 in stock j and zero in all other stocks instead of the market portfolio. Then Equation (4.2.1) becomes

$$\lim_{T \rightarrow \infty} \frac{1}{T} (\log X_i(T) - \log X_j(T)) = 0, \quad (4.2.4)$$

and therefore the equivalence of condition (2) and (1) follows.

Now we are left to show that condition (2) implies coherence. Therefore, assume condition (2) holds and that we therefore have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_j(t)) dt = 0. \quad (4.2.5)$$

Furthermore, as before, by Equation (4.2.3) we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log X_i(T) - \int_0^T \gamma_i(t) dt \right) = 0. \quad (4.2.6)$$

Using, these two equations and setting $j = 1$ we have,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log X_i(T) - \int_0^T \gamma_1(t) dt \right) = 0. \quad (4.2.7)$$

Since (4.2.7) holds for all $i = 1, \dots, n$, we must also have that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\max_{1 \leq i \leq n} (\log X_i(T)) - \int_0^T \gamma_1(t) dt \right) = 0. \quad (4.2.8)$$

Now, for $t \in [0, \infty)$,

$$X_i(t) \leq X_1(t) + X_2(t) + \dots + X_n(t) \leq n \max_{1 \leq i \leq n} X_i(t),$$

and so,

$$\log X_i(t) \leq \log Z_\mu(t) \leq \log n + \log \left(\max_{1 \leq i \leq n} X_i(t) \right).$$

Dividing the above equation by T , taking the limits and noting that $\lim_{T \rightarrow \infty} \frac{1}{T}n = 0$ we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log X_i(T) \leq \lim_{T \rightarrow \infty} \frac{1}{T} \log Z_\mu(T) \leq \lim_{T \rightarrow \infty} \frac{1}{T} \log \left(\max_{1 \leq i \leq n} X_i(T) \right). \quad (4.2.9)$$

Therefore, using the above Equation (4.2.9), together with Equations (4.2.7) and (4.2.8) we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \gamma_1(t) dt \right) = 0. \quad (4.2.10)$$

Now, using Equations (4.2.10) and (4.2.7) we can write

$$\lim_{T \rightarrow \infty} \frac{1}{T} (\log X_i(T) - \log Z_\mu(T)) = 0. \quad (4.2.11)$$

This is equivalent to (4.1.6) and therefore condition (2) implies that the market M is coherent. □

Proposition (4.2.1) implies that in a coherent market, the time weighted average difference of the market's growth rate and an individual stock's growth rate will tend to zero. As an extension then, the time weighted average difference between each individual stock's growth rates will also tend to zero over time.

We now consider the relationships, firstly, between the relative variances (relative to the market portfolio) and the stock weights and secondly between the stock weights and the individual stock excess growth rates.

Lemma 4.2.1. *Let π be a portfolio in a nondegenerate market. Then there exists an $\epsilon > 0$ such that for $i = 1, \dots, n$*

$$\tau_{ii}^\pi(t) \geq \epsilon (1 - \pi_i(t))^2 \quad t \in [0, \infty), \quad (4.2.12)$$

and by extension,

$$\tau_{ii}^\pi(t) \geq \epsilon (1 - \pi_{max}(t))^2 \quad t \in [0, \infty), \quad (4.2.13)$$

where

$$\pi_{max}(t) = \max_{1 \leq i \leq n} \pi_i(t), \quad t \in [0, \infty).$$

Proof. As per [18], we begin by proving Equation (4.2.12).

Let

$$x(t) = (\pi_1(t), \dots, \pi(i) - 1, \dots, \pi_n(t)), \quad (4.2.14)$$

for $1 \leq i \leq n$ and $t \in [0, \infty)$. Now choose an ϵ as in (2.2.1) such that

$$x\sigma(t)x^T \geq \epsilon\|x\|^2, \quad x \in \mathfrak{R}^n, \quad t \in [0, \infty). \quad (4.2.15)$$

Now, by Equation (2.4.3) we have,

$$\tau_{ii}^\pi(t) = \sigma_{ii}(t) - 2\sigma_{i\pi}(t) + \sigma_{\pi\pi}(t) = x\sigma(t)x^T \geq \epsilon\|x\|^2. \quad (4.2.16)$$

However,

$$\begin{aligned} \|x\|^2 &= \pi_1^2 + \pi_2^2 + \dots + (\pi_i - 1)^2 + \dots + \pi_n^2, \\ &\geq (\pi_i - 1)^2, \\ &\geq (1 - \pi_i)^2. \end{aligned}$$

Since $\epsilon > 0$,

$$\epsilon\|x\|^2 \geq \epsilon(1 - \pi_i)^2,$$

and therefore, by Equation (4.2.16),

$$\tau_{ii}^\pi(t) \geq \epsilon(1 - \pi_i)^2.$$

Since we can choose any weight $\pi_i(t)$ in Equation (4.2.14), we set $x(t)$ such that $x(t) = (\pi_1(t), \dots, \pi(i) - 1, \dots, \pi_n(t))$, but where $\pi_i(t) = \pi_{\max}(t)$.

It then follows (from the proof above) that,

$$\tau_{ii}^\pi(t) \geq \epsilon(1 - \pi_{\max}(t))^2 \quad t \in [0, \infty).$$

□

We now have a lower bound for the relative variances (to a portfolio), which is a function of the respective weights and more importantly the largest weight in the portfolio. If the maximum weight is 1 then the lower bound is zero - there is only one stock in the portfolio and therefore the relative variance to that portfolio of that stock is zero. The smaller the maximum weight (a wider and more diverse portfolio), the larger the lower bound of the relative variances.

We can extend the relation further to consider the relationships between the weights in a portfolio and the portfolio's returns, specifically the excess return of a portfolio π , $\gamma_\pi^*(t)$.

Lemma 4.2.2. *Let π be a portfolio with nonnegative weights in a nondegenerate market. Then there exists an $\epsilon > 0$ such that*

$$\gamma_{\pi}^*(t) \geq \epsilon (1 - \pi_{\max}(t))^2 \quad t \in [0, \infty). \quad (4.2.17)$$

Proof. We prove Lemma 4.2.2 as per [18]. If we select ϵ^* as in the last part of Lemma 4.2.1 so that,

$$\tau_{ii}^{\pi}(t) \geq \epsilon^* (1 - \pi_{\max}(t))^2 \quad t \in [0, \infty),$$

then, since $\pi_i(t) \geq 0$ we have,

$$\frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^{\pi}(t) \geq \frac{1}{2} \epsilon^* (1 - \pi_{\max}(t))^2 \quad t \in [0, \infty).$$

However, by Equation (2.4.8) we have that,

$$\gamma_{\pi}^*(t) = \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^{\pi}(t),$$

and therefore,

$$\gamma_{\pi}^*(t) \geq \epsilon (1 - \pi_{\max}(t))^2 \quad t \in [0, \infty),$$

where $\epsilon = \frac{\epsilon^*}{2}$. □

Lemma (4.2.2) implies that if the maximum weight is bounded away from one then the excess growth rate is bounded away from zero. This makes sense if we consider that the excess growth rate of a portfolio is heuristically similar to a measure of diversification (as described in Section 2.2). The more concentrated a portfolio is in a few stocks, the less diversification is present in the portfolio and therefore the lower the excess growth rate will be.

In the next lemma, we show that the converse is also true. That is, in a market with bounded variance, if $\gamma_{\pi}^*(t)$ is bounded away from zero, then the maximum weight is bounded away from one.

Lemma 4.2.3. *Let π be a portfolio in a market with bounded variance such that for $i = 1, \dots, n$, $0 \leq \pi_i(t) \leq 1$, for all $t \in [0, \infty)$. Then there exists a number $\epsilon > 0$ such that*

$$\pi_{\max}(t) \leq 1 - \epsilon \gamma_{\pi}^*(t), \quad t \in [0, \infty). \quad (4.2.18)$$

Proof. This proof is reproduced from [18].

Since the market has bounded variance, we have by (2.2.2),

$$x\sigma(t)x^T \leq M\|x\|^2, \quad x \in \mathfrak{R}^n, \quad t \in [0, \infty), \quad (4.2.19)$$

Therefore, for any specific stock $1 \leq i \leq n$,

$$\sigma_{ii}(t) \leq M, \quad t \in [0, \infty). \quad (4.2.20)$$

Now define a portfolio η with nonnegative weights η_1, \dots, η_n which, for any integer k , $1 \leq k \leq n$, is defined as follows

$$\eta_i(t) = \begin{cases} \pi_i(t)/(1 - \pi_k(t)) & \text{if } i \neq k, \\ 0 & \text{if } i = k, \end{cases} \quad (4.2.21)$$

for $t \in [0, \infty)$, $i = 1, \dots, n$.

This is basically a portfolio with a weight of zero in stock k , and weights rebalanced so that the sum of the remaining weights is 1. (4.2.20) implies that,

$$\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t) \leq \sum_{i=1}^n \eta_i\sigma_{ii}(t) \leq M. \quad (4.2.22)$$

Let

$$x = (\eta_1(t), \dots, \eta_{k-1}(t), -1, \eta_{k+1}(t), \dots, \eta_n(t)). \quad (4.2.23)$$

Then $\|x\|^2 \leq 2$, and by (4.2.19), for $k = 1, \dots, n$, we have

$$\sigma_{kk}(t) - 2\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t) = x\sigma x^T \leq 2M, \quad (4.2.24)$$

for $t \in [0, \infty)$.

Now we use the formula for the excess growth rate of portfolio π , as given by (2.2.8), and separate the k^{th} stock's components from the summations using the weights η as defined in (4.2.21). After some manipulation we are able to identify each of the inequalities we derived above. This is shown below.

$$\begin{aligned}
 2\gamma_{\pi}^*(t) &= \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t), \\
 &= \pi_k(t)\sigma_{kk}(t) + (1 - \pi_k(t)) \sum_{i=1}^n \eta_i(t)\sigma_{ii}(t), \\
 &\quad - \pi_k^2(t)\sigma_{kk}(t) - 2\pi_k(t)(1 - \pi_k(t)) \sum_{i=1}^n \eta_i(t)\sigma_{ik}(t), \\
 &\quad - (1 - \pi_k(t))^2 \sum_{i,j=1}^n \eta_i(t)\eta_j(t)\sigma_{ij}(t), \\
 &= (\pi_k(t) - \pi_k^2(t)) (\sigma_{kk}(t) - 2\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t)), \\
 &\quad + (1 - \pi_k(t)) \left(\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t) \right), \\
 &\leq (1 - \pi_k(t)) (2M + M). \tag{4.2.25}
 \end{aligned}$$

(4.2.25) follows from the inequalities in (4.2.22) and (4.2.24).

Since (4.2.25) holds for any k , $1 \leq k \leq n$, we can simply choose k as the largest weight, $\pi_{\max}(t)$. Rearranging (4.2.25) we then obtain,

$$\pi_{\max}(t) \leq 1 - \epsilon\gamma_{\pi}^*(t), \quad t \in [0, \infty),$$

with $\epsilon = \frac{2}{3M}$. □

The previous two lemmas highlight the relationship between a portfolio's excess growth rate and the largest weight in the portfolio. The more concentrated a portfolio is in one stock the lower the lower bound is on the portfolio's excess growth rate (by Lemma (4.2.2)).

Conversely, the higher the excess growth rate, the lower the upper bound on the maximum weight. This ties in with our formula for the excess growth rate and our interpretation of it as a measure of diversification. Diversification, therefore, forms an important part of SPT and in the next section we define it more formally.

4.3 Stock market diversity

Although we have discussed portfolio diversification and its relationship to the portfolio's excess growth rate, we have yet to formally define the concept of diversity. In this section we formally define stock market diversity and show its relationship to the excess growth rate of the market.

Definition 4.3.1. *The market M is diverse if there exists a number $\delta > 0$ such that*

$$\mu_{\max}(t) \leq 1 - \delta, \quad t \in [0, \infty). \quad (4.3.1)$$

M is weakly diverse if

$$\frac{1}{T} \int_0^T \mu_{\max}(t) dt \leq 1 - \delta. \quad (4.3.2)$$

Therefore, a market is diverse if there is never a time where the stock market capitalisation is concentrated in a single stock. The market is weakly diverse if this is true on average over time.

We saw in Section 4.2 and specifically in Lemmas (4.2.2) and (4.2.3) how the excess growth rate and the level of diversification in a portfolio are related. The next proposition, as set out in [18], provides a more direct comparison between the two, given our definitions above.

Proposition 4.3.1. *If the market M is nondegenerate and diverse, then there is a $\delta > 0$ such that*

$$\gamma_{\mu}^*(t) \geq \delta, \quad t \in [0, \infty). \quad (4.3.3)$$

Conversely, if M has bounded variance and there exists a $\delta > 0$ such that (4.3.3) holds, then M is diverse.

Proof. Suppose M is nondegenerate and diverse, so (from Equation (4.3.1)) there is a $\delta^* > 0$ such that

$$\mu_{\max}(t) \leq 1 - \delta^*, \quad t \in [0, \infty). \quad (4.3.4)$$

Since M is nondegenerate, by Lemma 4.2.2 we have, for any $\epsilon > 0$,

$$\gamma_{\mu}^*(t) \geq \epsilon(1 - \mu_{\max}(t))^2, \quad t \in [0, \infty). \quad (4.3.5)$$

Therefore, by rearranging Equation (4.3.1) we have

$$\delta^* \leq 1 - \mu_{\max}, \quad (4.3.6)$$

and therefore,

$$\gamma_{\mu}^*(t) \geq \epsilon(1 - \mu_{\max}(t))^2 \geq \epsilon(\delta^*)^2. \quad (4.3.7)$$

Therefore,

$$\gamma_{\mu}^*(t) \geq \delta, \quad (4.3.8)$$

with $\delta = \epsilon(\delta^*)^2$.

Now suppose that M has bounded variance and there exists $\delta > 0$ such that (4.3.3 holds). Since M has bounded variance, Lemma 4.2.3 implies that we can choose $\epsilon > 0$ such that,

$$\mu_{\max}(t) \leq 1 - \epsilon\gamma_{\mu}^*(t) \leq 1 - \epsilon\delta. \quad (4.3.9)$$

Therefore, by Definition (4.3.1), M is diverse.

□

Given the expression for the excess growth rate, γ_{μ}^* , it should come as no surprise that there is such a direct relationship between the level of diversification in the market and the excess growth rate.

4.4 Portfolio generating functions

In the previous sections we have set out definitions of the stock market and conditions needed for a coherent and diverse market. We also derived some implications these conditions have on stock market weights and the excess growth rate of the market.

In this section we now focus on portfolio generating functions, that is, functions that systematically generate portfolio weights. Furthermore, within the context of SPT we are able to analyse the behaviour of these functions and the portfolio's they generate. In particular we will focus on the implications of portfolio generating functions which rely on some measure of stock market diversity, given the importance of diversity as presented in the previous sections.

We begin with a general definition of a portfolio generating function $\mathbf{S}(\mathbf{x})$ following along the lines of [18].

Definition 4.4.1. *Let \mathbf{S} be a continuous function defined on Δ^n and let π be a portfolio. Then \mathbf{S} generates π if there exists a measurable process of bounded variation Θ such that,*

$$\log(Z_{\pi}/Z_{\mu}) = \log \mathbf{S}(\mu(t)) + \Theta(t), \quad t \in [0, T]. \quad (4.4.1)$$

Where Δ^n is the set given by,

$$\{x \in \mathbb{R}^n : x_1 + \dots + x_n = 1, \quad 0 < x_i < 1, \quad i = 1, \dots, n\}.$$

The function \mathbf{S} in Definition 4.4.1 is called the generating function of π , while Θ is called the drift process of \mathbf{S} . We can also express Equation 4.4.1 in differential form,

$$d\log(Z_\pi/Z_\mu) = d\log\mathbf{S}(\mu(t)) + d\Theta(t), \quad t \in [0, T]. \quad (4.4.2)$$

It is important to notice from (4.4.2) that \mathbf{S} operates in a relative space, in this case relative to the market portfolio μ . Furthermore, it is not unreasonable to assume that the portfolio generating function \mathbf{S} is bounded on Δ^n . This implies that the long-term relative performance of the portfolio π generated by \mathbf{S} is determined by the behaviour of $\Theta(t)$. In particular, if $\Theta(t)$ is increasing the portfolio π will outperform the market portfolio in the long-run.

Equation (4.4.2) excludes the impact of dividends. However, if we consider the definition of the portfolio process including a continuous dividend rate δ_π in Equation (2.5.3), (4.4.2) becomes

$$d\log\left(\widehat{Z}_\pi/\widehat{Z}_\mu\right) = d\log\mathbf{S}(\mu(t)) + \int_0^t (\delta_\pi(s) - \delta_\mu(s)) ds + d\Theta(t), \quad t \in [0, T]. \quad (4.4.3)$$

Therefore, including dividends, a functionally generated portfolio's performance relative to the market portfolio is dependent on the generating function, drift process and the difference in dividend rates.

Given the portfolio generating function \mathbf{S} , we require firstly the resulting weights and secondly the drift process $\Theta(t)$ to determine the relative performance of the generated performance. The following theorem characterises these components and is reproduced from [22] and [18].

Theorem 4.1. *Let \mathbf{S} be a positive C^2 function defined on a neighbourhood U of Δ^n such that for $i = 1, \dots, n$, $x_i D_i \log\mathbf{S}(x)$ is bounded on Δ^n . Then \mathbf{S} generates the portfolio π with weights*

$$\pi_i(t) = \left(D_i \log\mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log\mathbf{S}(\mu(t)) \right) \mu_i(t), \quad t \in [0, \infty), \quad (4.4.4)$$

for $i = 1, \dots, n$, and drift process

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t), \quad t \in [0, \infty). \quad (4.4.5)$$

Proof. Recall that the weight process $\mu_i(t)$ is given by $X_i(t)/Z_\mu(t)$, and therefore we can represent the relative covariance process $\tau_{ij}(t)$ as given in Section 2.4 as,

$$d\langle \log\mu_i, \log\mu_j \rangle_t = \tau_{ij}(t)dt, \quad t \in [0, \infty).$$

Now, applying Itô's Lemma to $\mu_i(t) = \exp(\log\mu_i(t))$ we have,

$$d\mu_i(t) = \mu_i(t)d\log\mu_i(t) + \frac{1}{2}\mu_i(t)\tau_{ii}(t)dt. \quad (4.4.6)$$

and

$$d\langle \mu_i, \mu_j \rangle_t = \mu_i(t)\mu_j(t)\tau_{ij}dt, \quad t \in [0, \infty). \quad (4.4.7)$$

Furthermore, if we apply Itô's Lemma to \mathbf{S} and using (4.4.7) we get,

$$d\log\mathbf{S}(\mu(t)) = \sum_{i=1}^n D_i \log\mathbf{S}(\mu(t))d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij} \log\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt.$$

Now,

$$D_{ij} \log\mathbf{S}(\mu(t)) = \frac{D_{ij}\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(t))} - D_i \log\mathbf{S}(\mu(t))D_j \log\mathbf{S}(\mu(t)),$$

so,

$$\begin{aligned} d\log\mathbf{S}(\mu(t)) &= \sum_{i=1}^n D_i \log\mathbf{S}(\mu(t))d\mu_i(t) \\ &+ \frac{1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\ &- \frac{1}{2} \sum_{i,j=1}^n D_i \mathbf{S}(\mu(t))D_j \mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt. \end{aligned} \quad (4.4.8)$$

For (4.4.2) to hold, the martingale components of $\log\mathbf{S}(\mu(t))$ and $\log(Z_\pi/Z_\mu)$ must be equal. Equation (2.2.11) implies that,

$$d\log(Z_\pi(t)/Z_\mu(t)) = \sum_{i=1}^n \pi_i(t) d\log(X_i(t)/Z_\mu(t)) + \gamma_\pi^*(t) dt \quad (4.4.9)$$

$$= \sum_{i=1}^n \pi_i(t) d\log\mu_i(t) + \gamma_\pi^*(t) dt \quad (4.4.10)$$

$$= \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} d\mu_i(t) - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \tau_{ij}(t) dt, \quad (4.4.11)$$

by Equation (2.2.8), Equation (2.4.3) and the fact that $\tau_{\pi\pi}^{\mu(t)} = \pi(t)\tau^\mu(t)\pi^T(t)$ as shown in Section 2.4.

Now suppose that

$$\pi_i(t) = (D_i \log \mathbf{S}(\mu(t)) + \varphi(t)) \mu_i(t), \quad (4.4.12)$$

where $\varphi(t)$ is chosen such that $\sum_{i=1}^n \pi_i(t) = 1$. Then,

$$\begin{aligned} \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} d\mu_i(t) &= \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t)) d\mu_i(t) + \varphi(t) \sum_{i=1}^n d\mu_i(t) \\ &= \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t)) d\mu_i(t), \end{aligned} \quad (4.4.13)$$

since $\sum_{i=1}^n d\mu_i(t) = 0$. Hence the martingale components $\log \mathbf{S}(\mu(t))$ and $\log(Z_\pi/Z_\mu)$, that is in Equations (4.4.9) and (4.4.11), respectively are equal.

Furthermore, if we choose $\varphi(t)$ such that,

$$\varphi(t) = 1 - \sum_{i=1}^n \mu_j(t) D_j \log \mathbf{S}(\mu(t)),$$

$\sum_{i=1}^n \pi_i(t) = 1$ is satisfied. Therefore (4.4.4) is proved and

$$\pi_i(t) = \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log \mathbf{S}(\mu(t)) \right) \mu_i(t), \quad t \in [0, \infty).$$

If $\pi_i(t)$ satisfies (4.4.12) then,

$$\begin{aligned}
 \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t) &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad (4.4.14) \\
 &+ 2\varphi(t) \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) + \varphi^2(t) \sum_{i,j=1}^n \mu_i(t) \mu_j(t) \tau_{ij}(t) \\
 &= \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t),
 \end{aligned}$$

since $\mu(t)$ is in the null space of $\tau(t)$ by Equation (2.4.5). Hence,

$$d \log (Z_\pi / Z_\mu) = \sum_{i=1}^n D_i \log \mathbf{S}(\mu(t)) d\mu_i(t) - \frac{1}{2} \sum_{i,j=1}^n D_i \log \mathbf{S}(\mu(t)) D_j \log \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) dt.$$

This equation and (4.4.9) imply that,

$$d \log (Z_\pi / Z_\mu) = d \log \mathbf{S}(\mu(t)) - \frac{1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t).$$

Comparing this to Equation (4.4.2) we see that,

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t), \quad t \in [0, \infty),$$

and therefore, (4.4.5) is proved. \square

Theorem 4.1 allows us to characterise the drift process which, for a given generating function, allows us to determine long-term behaviour for a corresponding portfolio relative to the market portfolio.

4.5 Summary

In this chapter we have formalised the concept of a coherent and diverse market and showed that the level of diversification in the market has a direct influence on the excess growth rate of the market. This is intuitive given the components of the excess growth rate (see Equation (2.2.8)).

We also introduced the concept of portfolio generating functions and showed that the relative performance of a portfolio generated by a function $\mathbf{S}(\mu(t))$ is a function of changes in,

- the portfolio generating function itself,

- differences in dividend rates between the resulting portfolio and the market portfolio, and
- the drift function of the portfolio.

Furthermore, Theorem 4.1 provided us with a way to determine the resulting weights from a portfolio generating function and how to calculate the drift process to determine long-run behaviour of the portfolio generated by a function $S(\mu(t))$.

Chapter 5

Portfolio generating functions: Empirical performance

In this chapter we consider the empirical performance of specific portfolio generating functions relative to the market portfolio. Given the direct relationship between market diversity and the growth rate, we specifically consider equal weighted portfolios and portfolios constructed using measures of market diversity such as the entropy function.

The data used in this chapter is the same as that set out in Section 3.1. Portfolios are balanced at the end of each month. All total return indices include dividends and the effects of any corporate actions (such as special dividends, spin-offs, etc). We exclude any transaction costs in our analyses.

5.1 Equal weighted portfolios

The equal weighted portfolio is generated by the function $\mathbf{S}(\mu) = (\mu_1 \cdots \mu_n)^{\frac{1}{n}}$, with the individual weights and drift process given by:

$$\pi_i = \frac{1}{n}$$

and,

$$d\Theta(t) = \gamma_{\pi}^*(t)dt$$

Now since $\gamma_{\pi}^*(t) \geq 0$, we have an increasing drift function and would therefore expect the equal weighted portfolio to outperform the market portfolio over time. However, this ignores the difference in dividends between the two portfolios and any changes in the portfolio generating function itself.

While, the drift is positive, its level can change significantly as $\gamma_{\pi}^*(t)$ increases or decreases. When analysing the relative performance of portfolios we will therefore consider the three components in Equation (4.4.3). That is,

- Changes in the portfolio generating function $\mathbf{S}(\mu(t))$,
- Differences in dividends between the two portfolios, and
- The changes in the drift process.

We analyse the performance of two portfolios consisting of the top 40 stocks ranked by market capitalisation at the beginning of each month. The first portfolio contains equal weights of 2.5%, while the second mimicks the market portfolio by using market capitalisation weights. The differences in the average weights over our sample period are shown in Figure 5.1.

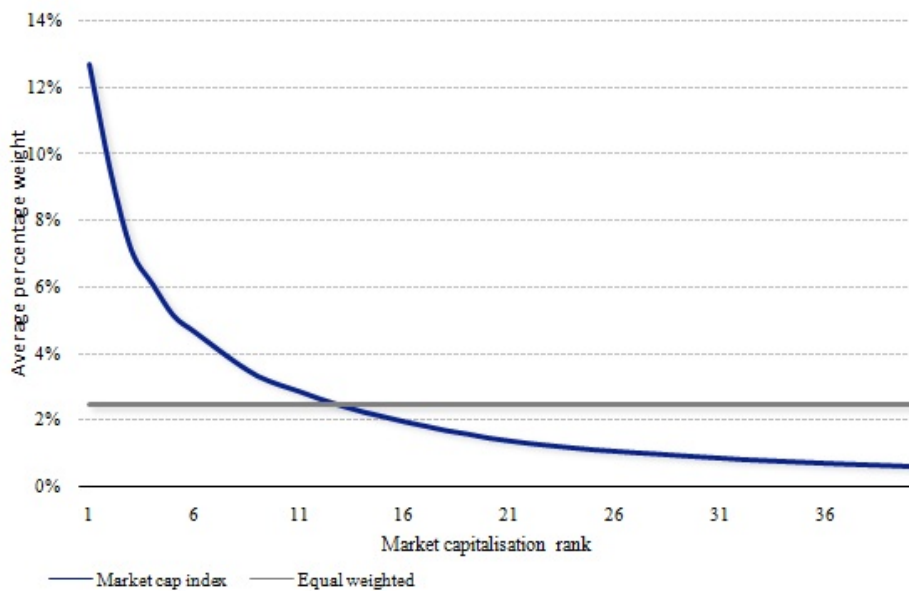


Figure 5.1: Average weight for each ranked market capitalisation position from December 1994 - September 2013

The South African equity market is dominated by only a handful of stocks and this is evident in Figure 5.1. In fact, the top 10 stocks by market cap have, on average, accounted for 50.2% of the South African equity market.

Figure 5.2 shows the differences in the cumulative performance from December 1994 of both portfolios (on a log scale). The equal weighted portfolio marginally outperforms the market portfolio with a CAGR of 15.2% versus

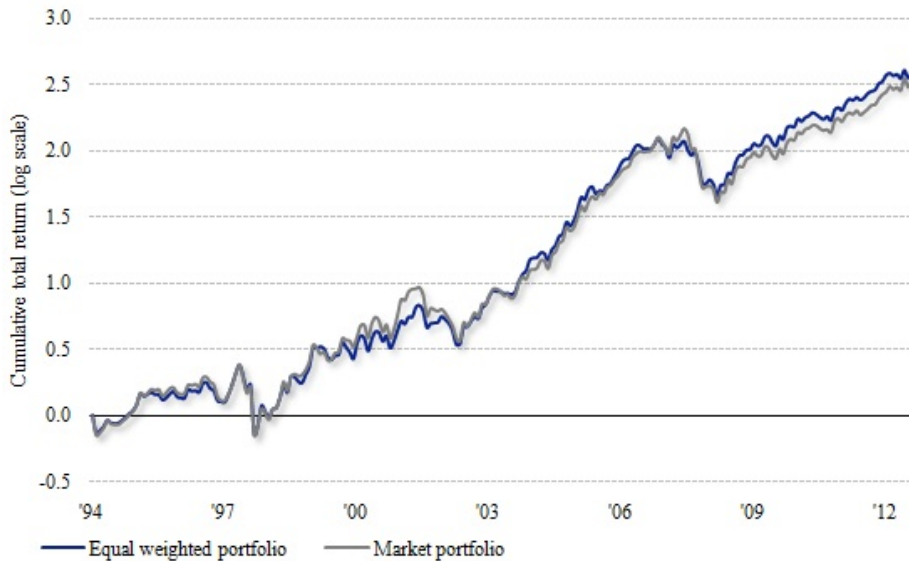


Figure 5.2: Log cumulative performance of an equal weighted portfolio versus that of the market capitalisation weighted portfolio. Both portfolios contain the top 40 stocks by market capitalisation.

14.8% over the period considered here.

Although the cumulative performance appears similar overall there are extended periods of under- and outperformance by the equal weighted portfolio over the market portfolio as shown in Figure 5.3. The equal weighted portfolio seems to suffer during latter stage bull markets (2000-2001 and 2006-2008), while outperforming during market crashes and the subsequent recovery period.

Figure 5.4 highlights the calendar year returns. This illustrates the periods of market portfolio outperformance over the equal weighted portfolio. The equal weighted portfolio performs better during periods of negative returns (1997, 1998, 2002 and 2008).

Equation (4.4.3) allows us to decompose the relative return into three components. Although measuring these components (specifically the estimation of the excess growth rate) is not perfect, it does give us a general idea of where the excess returns come from.

Within the context of equal weighted portfolios we interpret the three components as follows:



Figure 5.3: Relative log cumulative performance of the equal weighted and market portfolios.

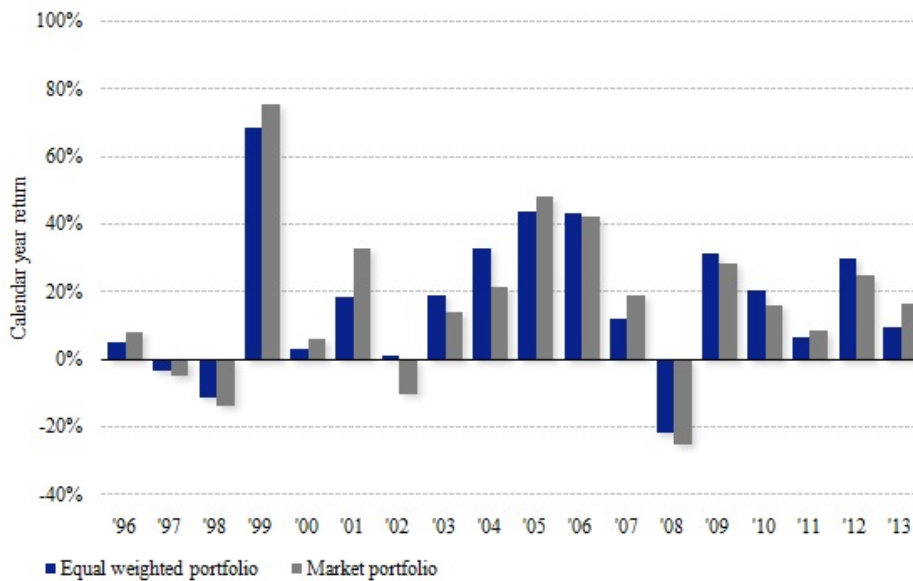


Figure 5.4: Calendar year returns for the equal weighted and market portfolios.

- Relative dividends
 We would expect larger stocks to pay out higher amounts of dividends relative to smaller stocks. This is because smaller stocks should invest higher amounts back into the company, decreasing the dividend payout ratio.
- Changes in the portfolio generating function $\mathbf{S}(\mu(t))$
 $\mathbf{S}(\mu(t))$ is effectively the n th root of the product of the market capitalisation weights. Therefore, $\mathbf{S}(t)$ is a product of weights which are less than one. If we have n small weights, as is the case with equal weights of 1% we obtain a smaller value for $\mathbf{S}(\mu(t))$ than if we had a few large weights with smaller weights in the remaining stocks. This is due to the product of the weights which are smaller than one.

Consider for example a 100 stock portfolio with equal weights. Here $\mathbf{S}(\mu(t)) = 0.01$. However, if we assume the top five stocks each have a weight of 10% with the remaining 50% spread equally amongst the remaining 95 stocks we obtain a value for $\mathbf{S}(\mu(t))$ of 0.01122. Therefore, $\mathbf{S}(\mu(t))$ increases as the market becomes more concentrated in a few stocks. However, in Equation (4.4.3) we consider the log of $\mathbf{S}(\mu(t))$ and since $\mathbf{S}(\mu(t))$ is always smaller than one (since the weights are less than one), we have that $d\log\mathbf{S}(\mu(t))$ actually decreases (since $\log\mathbf{S}(\mu(t)) < 0$) as the market becomes more concentrated in fewer stocks.

That is, the relative performance of the equal weighted portfolio actually decreases as the market becomes more concentrated. This makes intuitive sense since, if the market becomes more concentrated in a few stocks, it implies that those few stocks outperform the rest of the market. The equal weighted portfolio is underweight these stocks relative to the market portfolio and will therefore (all else being equal) underperform the market portfolio.

- Changes in the drift process
 Firstly, the excess growth rate (the drift process for the equal weighted portfolio) is always greater than zero and therefore always positively contributes to the relative performance of the equal weighted portfolio. However, what is in question is the level of contribution, especially in the face of our previous point on the changes in the portfolio generating function.

The excess growth rate $\gamma_{\pi}^*(t)$, as described in Section 2.2 and in Equation (2.2.8) is heuristically the level of benefits obtained from diversification

in the respective portfolio. Now if all stocks tend to behave in a very similar manner (as was the case in the financial crisis) then there are less benefits available from diversification.

In general, the excess growth rate will also be a function of the specific portfolio weights. However, since the equal weights remain constant no matter the market capitalisation of the stock, changes in the excess growth rate are more likely to be a function of changes in the level of diversification available in the market.

We now highlight the 12 month changes in the three measures discussed above. We begin with the relative dividends of the two portfolios. Now since the equal weighted portfolio is underweight the larger stocks, we can view this as the relative dividends of small stocks to large stocks. In Figure 5.5 we show the dividend return of the equal weighted portfolio relative to the market portfolio.

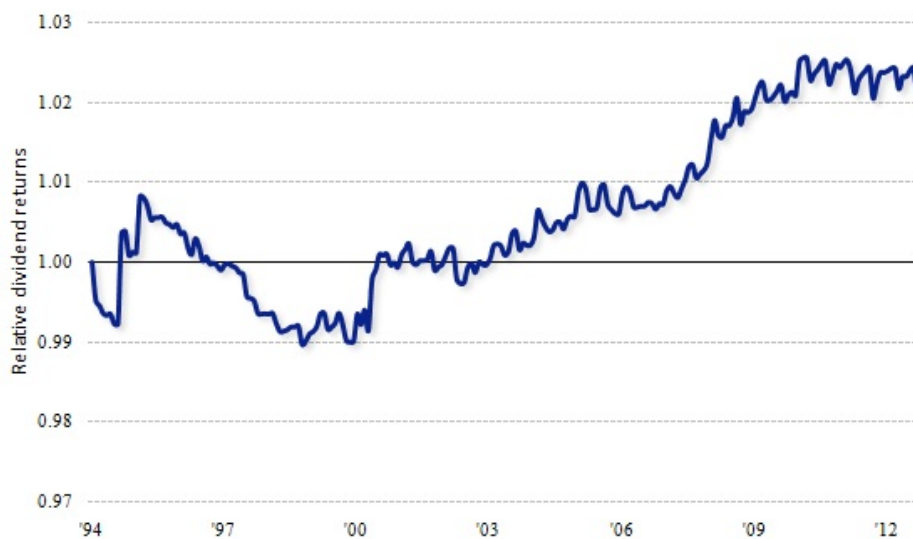


Figure 5.5: Dividend return of the equal weighted portfolio relative to the market portfolio.

The equal weighted portfolio has returned approximately 2.3% more in dividends than the market portfolio. This would imply that smaller stocks have paid relatively more dividends than the larger stocks over our sample period. Most of this difference is due to the 2001 to 2009 period.

Therefore, the dividend differential, in the South African equity market at least, has actually contributed a net positive to the relative return of the equal weighted portfolio.

We next highlight the rolling 12 month changes in the log of the portfolio generating function. Most of these changes are relatively small, with rolling 12 month changes oscillating between -4% and $+3\%$. However, we highlight the 1998-2001 period, where the log of the portfolio generating function declines significantly. Over the is period the equal weighted portfolio, unsurprisingly, underperforms the market portfolio. That is, in bull markets the South African equity market appears to become concentrated in fewer stocks.

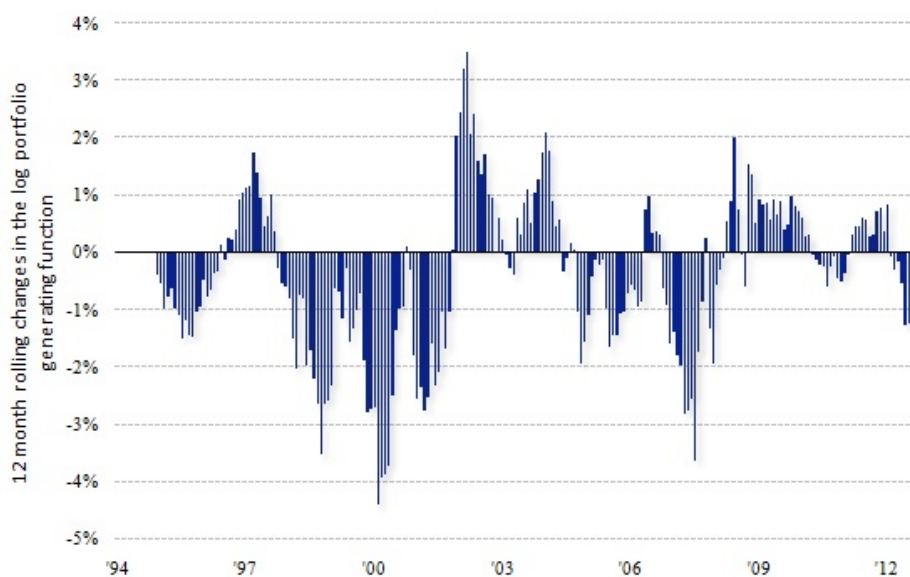


Figure 5.6: Rolling 12 month change in the log of the portfolio generating function of the equal weighted portfolio.

We now consider the changes in the drift process, or more specifically the changes in the excess growth rate, as highlighted in Figure 5.7. We estimate the excess growth rate using 12 month historic covariance matrices at each month.

Interestingly, Figure 5.7 shows some cyclicity, much like the relative performance of the market and equal weighted portfolios. Furthermore, the excess growth rate shows an overall declining trend over our sample period. This may explain why the equal weighted portfolio has not outperformed the market portfolio by much over our sample period. That is, the South African equity market has shown less opportunities or benefits from the diversification

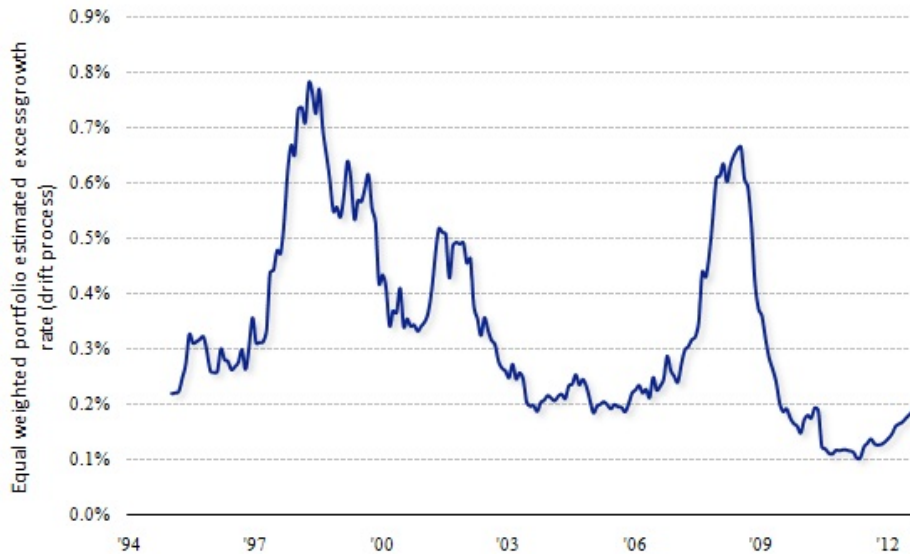


Figure 5.7: Equal weighted portfolio excess growth rate (drift process for the equal weighted portfolio) using rolling 12 month historical covariance matrices.

offered by the equal weighted portfolio over time.

	Equal weighted	Market portfolio
CAGR	15.2%	14.8%
Volatility	19.0%	20.2%
Downside deviation	14.8%	15.4%
Sharpe ratio	0.8	0.73
Sortino ratio	1.01	0.96

Table 5.1: Summary statistics for the equal weighted and market portfolios.

Table 5.1 highlights the marginal improvement in return and risk-adjusted returns that the equal weighted portfolio delivers over the market portfolio. The benefits of SPT, is that we are able to decompose the portfolio generated by functions into return components by Equation (4.4.3).

Firstly, does Equation (4.4.3) explain the overall return differentials between the equal weighted and market portfolio? In this respect, we find that, given the difficulty of measuring some of the components, that Equation 4.4.3 does relatively well at explaining the difference in market and equal weighted returns. The average difference of 0.12% per month is largely a result of the 2008 period, where constantly changing market conditions make estimation of the

excess growth rate difficult. The difference is, therefore, more likely a result of measurement error than a problem with Equation (4.4.3).



Figure 5.8: Comparison of the equal weighted and market portfolio expected and actual total returns.

Figure 5.9 shows the volatility of the three components of relative returns and highlights that the changes in the log of the portfolio generating function has been significantly more volatile than the drift process in particular. We believe this is due to the South African equity market showing less diversification opportunities (as highlighted in Figure 5.7) and becoming more concentrated in fewer stocks.

In our analysis we have excluded transaction costs, but they may have a significant impact especially for equal weighted portfolios. An equal weighted portfolio requires constant rebalancing as some stocks outperform the general market and others underperform. The outperformers need to be sold and the underperformers need to be bought to bring the portfolio back in to line. This complication is not an issue for a market portfolio as the weights, by definition, are a direct function of price movements. An analysis of both transaction costs and the frequency of rebalancing would be a useful extension of our research.

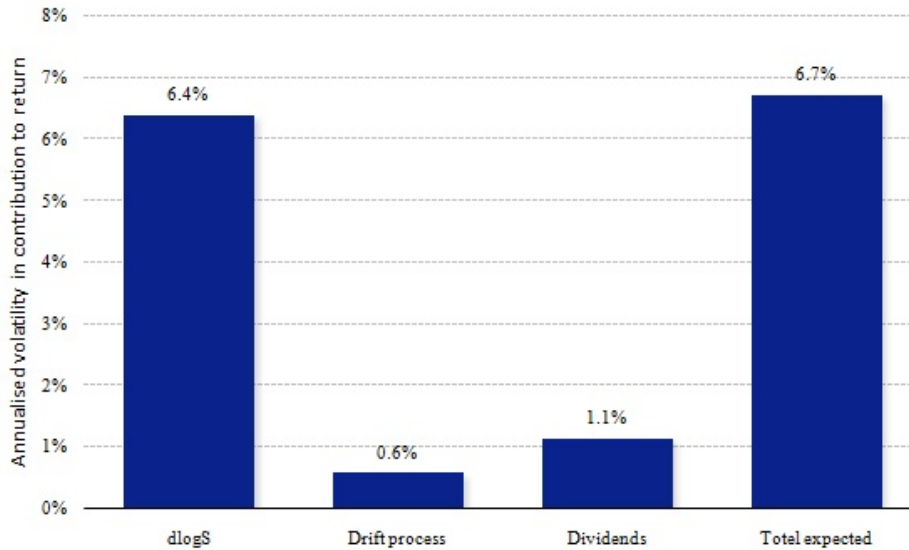


Figure 5.9: Annualised volatilities of the three components of relative return.

5.2 A large stock biased portfolio

In Section 5.1 we considered the equally weighted portfolio and showed how this portfolio has only marginally outperformed the market portfolio due to a declining excess growth rate over time. These equally weighted portfolios effectively decreased the weight in large stocks and increased the weight in smaller stocks. In this section we consider a portfolio (hereafter referred to as the large cap portfolio) which actually increases the weight in large stocks and further downweights the smaller stocks. Theoretically at least, this portfolio should benefit from the declining excess growth rate.

This section implements and analyses the weighted average capitalisation portfolio given in [18]. The large cap portfolio has a portfolio generating function given by,

$$\mathbf{S}(x) = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}, \quad (5.2.1)$$

which yields portfolio weights,

$$\pi_i(t) = \frac{\mu_i^2}{\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2}. \quad (5.2.2)$$

This portfolio has the following, negative, drift process

$$d\Theta(t) = -\gamma_{\pi}^*(t)dt. \quad (5.2.3)$$

Equation (5.2.3) suggests that the large cap portfolio should underperform the market portfolio.

Firstly, we present the difference in average weights per market capitalisation rank over our sample period (Figure 5.10). The increased weights in the larger stocks are clearly evident, with the top five stocks accounting for 71% of the portfolio weights.

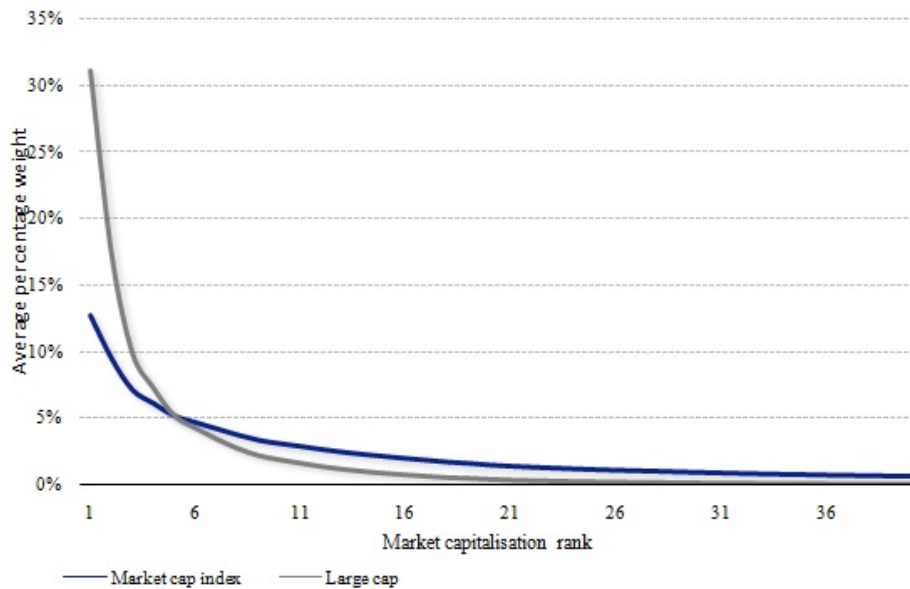


Figure 5.10: Average weight for each ranked market capitalisation position from December 1994 - September 2013. 40 stock portfolio.

As expected, given the implications of a negative drift process (Equation (5.2.3)), the large cap portfolio does indeed underperform the market portfolio.

The large cap portfolio represents something of an antithesis to the equal weighted portfolio with the relative returns (Figure 5.12) appearing to be a negative mirror of those for the equal weighted portfolio. The large cap portfolio, however, does appear to have a negative overall trend to its relative return chart. This is consistent with the expectations that a large cap portfolio would underperform the market portfolio over time.

Figure 5.12 also shows the periods of outperformance which coincide with the underperformance of the equal weighted portfolio. This further illustrates that large caps have tended to outperform in latter stage bull markets.

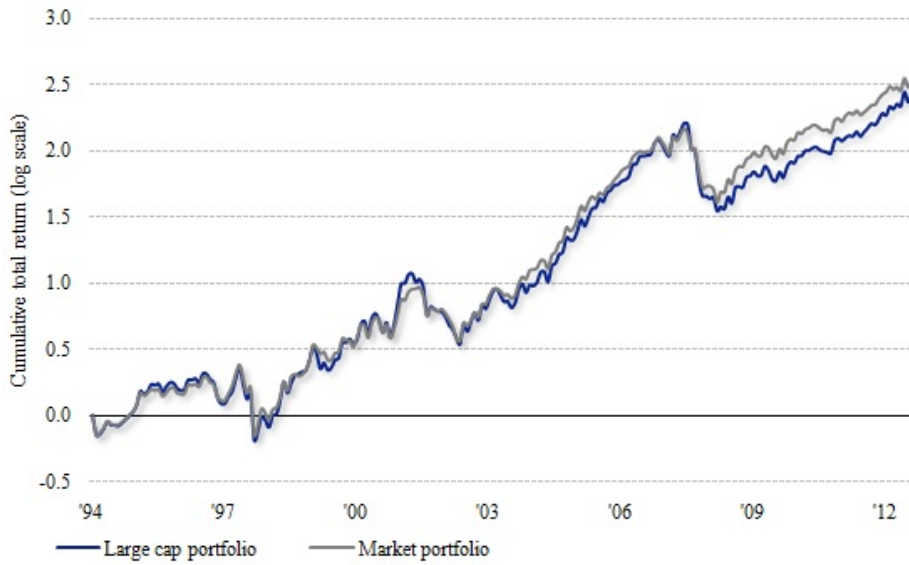


Figure 5.11: Log cumulative performance of the large cap portfolio versus that of the market capitalisation weighted portfolio. Both portfolios contain the top 40 stocks by market capitalisation.



Figure 5.12: Relative log cumulative performance of the large cap and market portfolios.

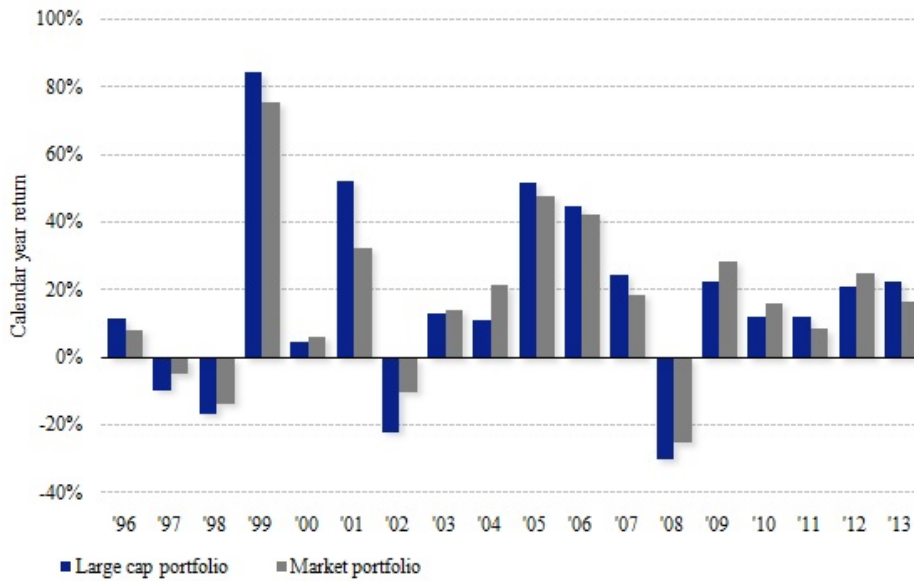


Figure 5.13: Calendar year returns for the large cap and market portfolios.

Figure 5.13, showing calendar year returns, also illustrates the underperformance of large caps during negative years.

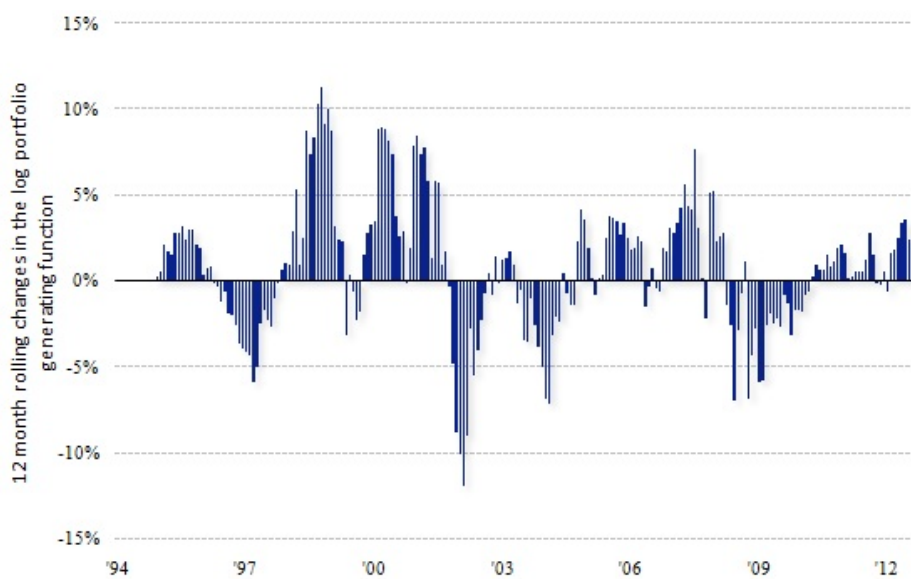


Figure 5.14: Rolling 12 month changes in the log of the portfolio generating function.

The changes in the log of the portfolio generating function are shown in Figure

5.14. This chart is almost an exact opposite of the Figure 5.6 and also illustrates the extended periods where the market becomes concentrated in fewer stocks (1998-2001 for example). This contributes positively to the large cap portfolio and negatively to the equal weighted portfolio.

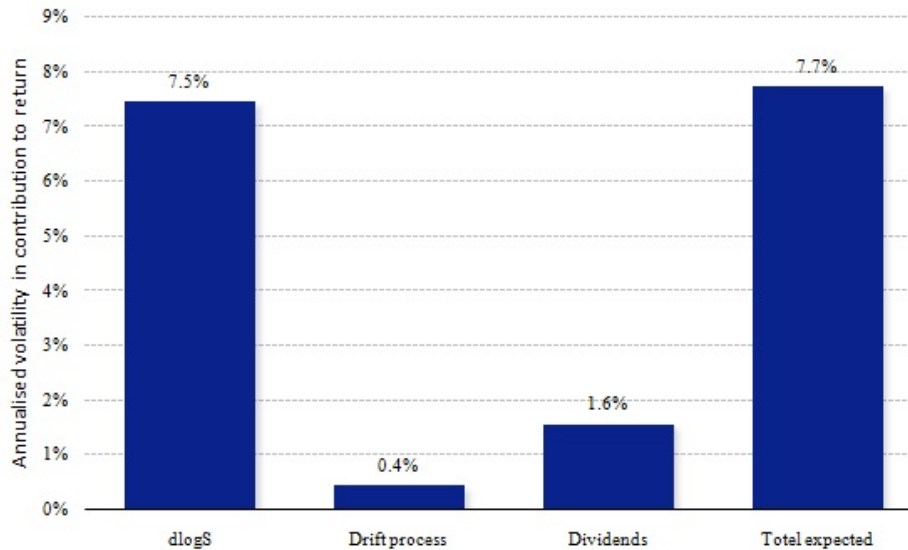


Figure 5.15: Annualised volatilities of the three components of relative return.

Figure 5.15 shows the volatility of the different components of return. We find, as in the case of the equal weighted portfolio, that the changes in the log of the portfolio generating function are the most volatile of the components.

Comparing Figures 5.16 (which shows the negative drift process) and 5.12 highlights that when the excess growth rate increases significantly (as in 2008), the large cap portfolio tends to underperform the market portfolio, consistent with Equation (5.2.3).

	Large cap portfolio	Market portfolio
CAGR	14.0%	14.8%
Volatility	22.9%	20.2%
Downside deviation	17.1%	15.4%
Sharpe ratio	0.61	0.73
Sortino ratio	0.82	0.96

Table 5.2: Summary statistics for the large cap and market portfolios.

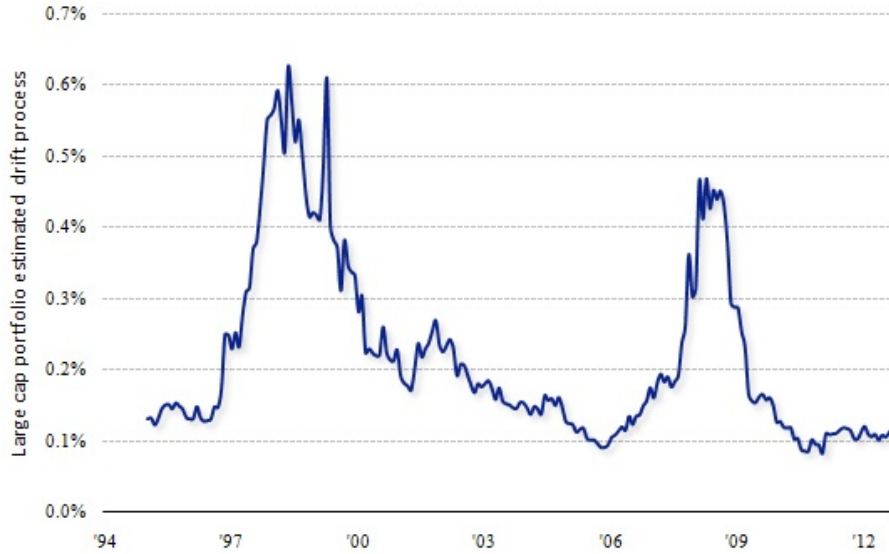


Figure 5.16: Large cap portfolio portfolio excess growth rate (negative drift process for the large cap portfolio) using rolling 12 month historical covariance matrices.

Overall we find that the large cap portfolio does underperform the market portfolio (and by extension the equal weighted portfolio) both on a return and especially a risk-adjusted return basis.

5.3 The entropy weighted portfolio

In Section 4.3 we discussed stock market diversity and showed how the level of diversity in a market is directly related to the excess growth rate (Proposition (4.3.1)). Given this relationship it is natural to consider portfolio generating functions which are related to measures of diversity.

The first of these is the entropy function, a concept used in information theory (amongst other areas) and introduced by Shannon [47]. For our purposes, entropy can be thought of as the spread of capital among the stock's in our universe. Low entropy occurs when capital is concentrated in a few stocks, and vice versa for high entropy.

The entropy function, and consequently the portfolio generating function $\mathbf{S}(x)$, is given by

$$\mathbf{S}(x) = - \sum_{i=1}^n x_i \log x_i, \quad (5.3.1)$$

where we replace the x_i with the market weights μ_i .

This yields weights

$$\pi_i(t) = -\frac{\mu_i(t)\log\mu_i(t)}{\mathbf{S}(\mu(t))}, \quad (5.3.2)$$

and drift process

$$d\Theta(t) = \frac{\gamma_\mu^*(t)}{\mathbf{S}(\mu(t))} dt. \quad (5.3.3)$$

Firstly, since both $\gamma_\mu^*(t)$ and $\mathbf{S}(\mu(t))$ are positive, the drift process of the entropy weighted portfolio is positive. Therefore, excluding dividends and changes in $\mathbf{S}(\mu(t))$, the entropy weighted portfolio should outperform the market portfolio.

Secondly, it is interesting to note that the drift process of the entropy portfolio is a function of the excess growth rate of the market portfolio ($\gamma_\mu^*(t)$) and not the excess growth rate of the entropy portfolio itself ($\gamma_\pi^*(t)$), as was the case in the equal weighted and large cap portfolios.

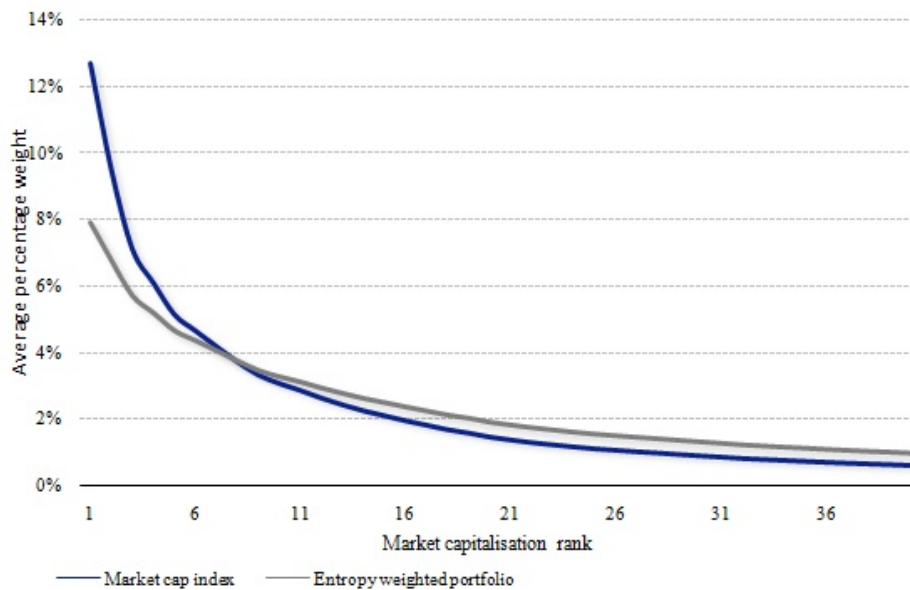


Figure 5.17: Average weight for each ranked market capitalisation position from December 1994 - September 2013. 40 stock portfolio.

The entropy portfolio provides a balance between the equal weight and market portfolio. The portfolio downweights larger stocks and increases the weights on smaller stocks but not at the same level as the equal weighted portfolio.

This provides better implementation in the case of the entropy portfolio versus both the equal weighted and large cap portfolios.

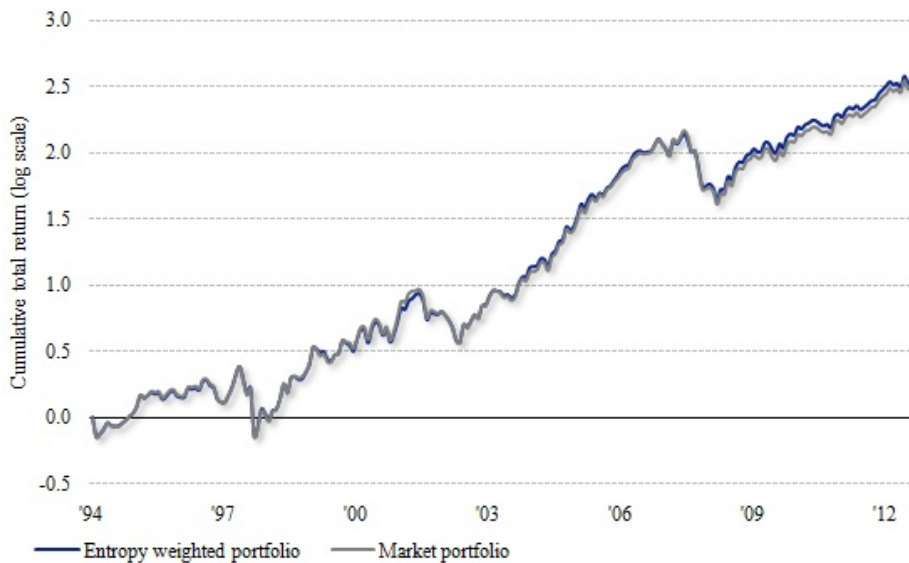


Figure 5.18: Log cumulative performance of the entropy portfolio versus that of the market capitalisation weighted portfolio. Both portfolios contain the top 40 stocks by market capitalisation.

Figure 5.18 highlights the performance of the entropy portfolio against that of the market portfolio. As is the case with the equal weighted portfolio, the entropy portfolio does outperform the market portfolio over this time period but only marginally.

The relative outperformance of the entropy portfolio, shown in Figure 5.19, shows a very cyclical performance relative to the market portfolio, with a slight upward trend and overall marginal outperformance by the entropy weighted portfolio.

The drift process of the entropy portfolio, shown in Figure 5.20, has a similar shape to both the equal weighted and large cap portfolios. The drift process of the entropy weighted portfolio is merely a scaled version of the excess growth rates, but once again, shows that the opportunities or benefits of diversification have largely declined over time.

The entropy function itself, that is Equation (5.3.1), highlights the level of diversity in a stock market. Figure 5.21 shows the entropy function for the South



Figure 5.19: Relative log cumulative performance of the entropy and market portfolios.

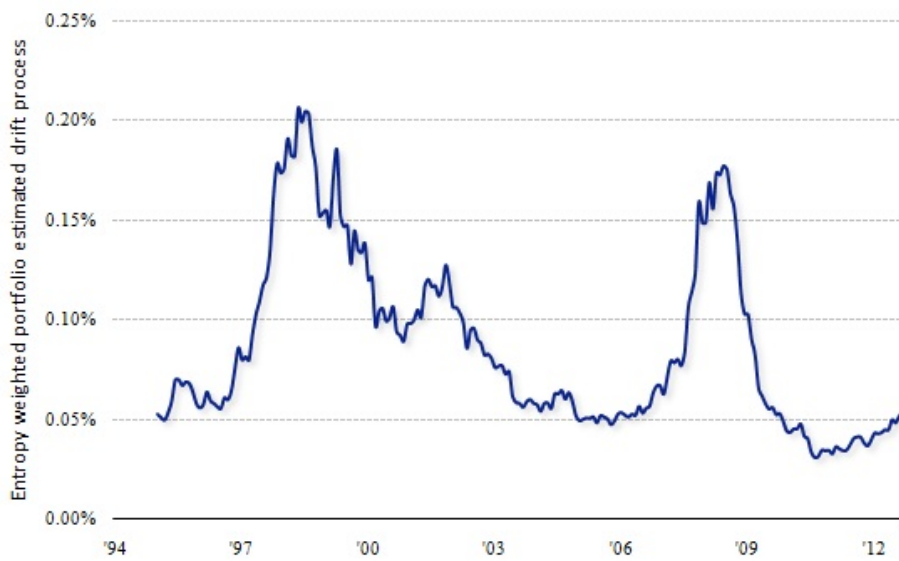


Figure 5.20: Drift process for the entropy portfolio.

African equity market. While the excess growth rates we have shown previously highlight the declining level of benefits from diversification, the entropy function highlights a more direct consequence. That is, that more weight is being held in fewer stocks in the South African market and this works against the relative performance of the entropy weighted portfolio. Therefore, the entropy weighted portfolio has not been able to generate significant excess return.

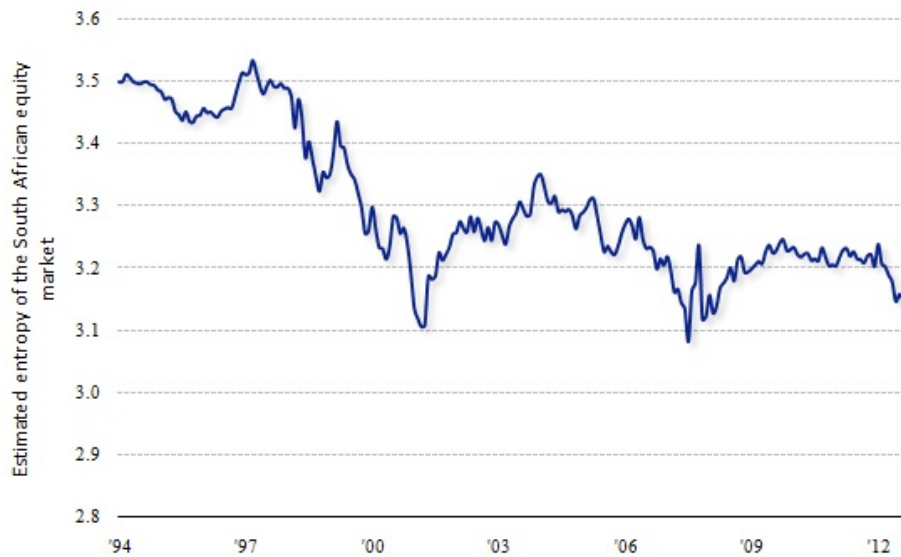


Figure 5.21: Estimated entropy function for the top 40 stocks in the South African equity market.

Table 5.3 highlights that the outperformance of the entropy weighted portfolio over the market portfolio is only marginal both on a return and risk-adjusted return basis.

	Entropy weighted	Market portfolio
CAGR	15.0%	14.8%
Volatility	19.6%	20.2%
Downside deviation	15.0%	15.4%
Sharpe ratio	0.77	0.73
Sortino ratio	1.00	0.96

Table 5.3: Summary statistics for the entropy weighted and market portfolios.

5.4 The diversity weighted portfolio

In the previous section we analysed the entropy function and the resulting portfolio generating function. We showed that over time the South African equity market has become less diverse. This has had a negative impact on the performance of the entropy weighted portfolio. The entropy portfolio, however, is not very flexible and we therefore present, a second measure of diversity taken from [22].

This function is given as

$$\mathbf{D}_p(x) = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}, \quad (5.4.1)$$

for $0 < p < 1$. This function generates weights given by

$$\pi_i(t) = \frac{\mu_i^p(t)}{(\mathbf{D}_p(\mu(t)))^p}, \quad (5.4.2)$$

and with drift process

$$d\Theta(t) = (1 - p)\gamma_{\pi}^*(t)dt. \quad (5.4.3)$$

We are now able to vary p in Equation (5.4.1). Values for p closer to zero will converge to an equal weight portfolio, while p values closer to one converging to the market portfolio. This is show in Figure 5.22.

Unfortunately, all the portfolios suffer from the same problem as the entropy weighted portfolio. That is, returns are diminished due to the declining diversity in the South African equity market. Figures 5.23 and 5.24 highlight relative cumulative returns of the various portfolios.

Although the $p = 0.05$ portfolio does the best overall, the excess return is only marginal and is highly volatile. In fact at some points the $p = 0.05$ portfolio is the worst performing portfolio (2001 and 2007 for example). This highlights that while portfolios weighted away from the market portfolios may outperform overall, there are extended periods of significant underperformance.

The effects of the declining diversity and lack of diversification benefits are best illustrated in Figure 5.25 which shows the 12 month rolling changes in $\log \mathbf{S}(\mu(t))$ for the portfolio with $p = 0.25$. These large negative changes directly impact relative returns through Equation (4.4.3). On average the rolling 12 month change is -1.01% per annum, which over our sample period of just over 18 years could accumulate to over 22%.

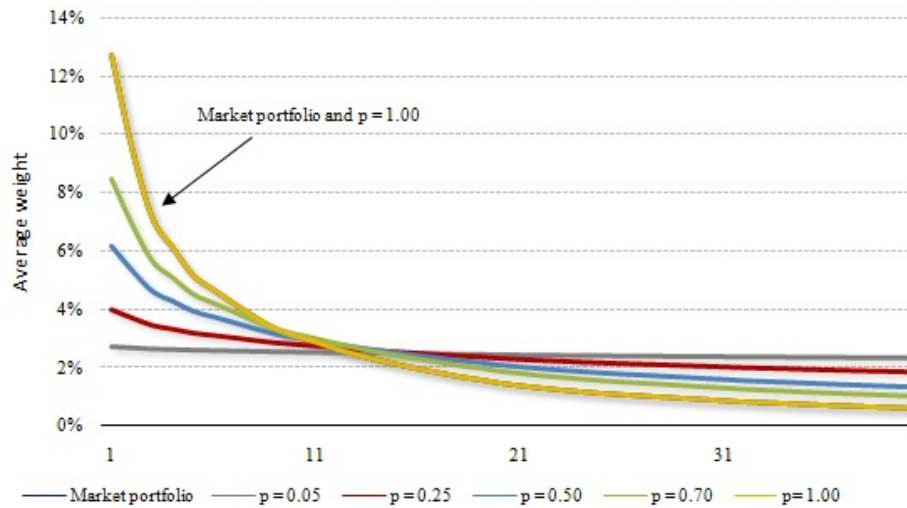


Figure 5.22: Comparison of average weights for different values of p . $p = 1$ is clearly the market portfolio, while the smaller p becomes, the closer the weights become equal.

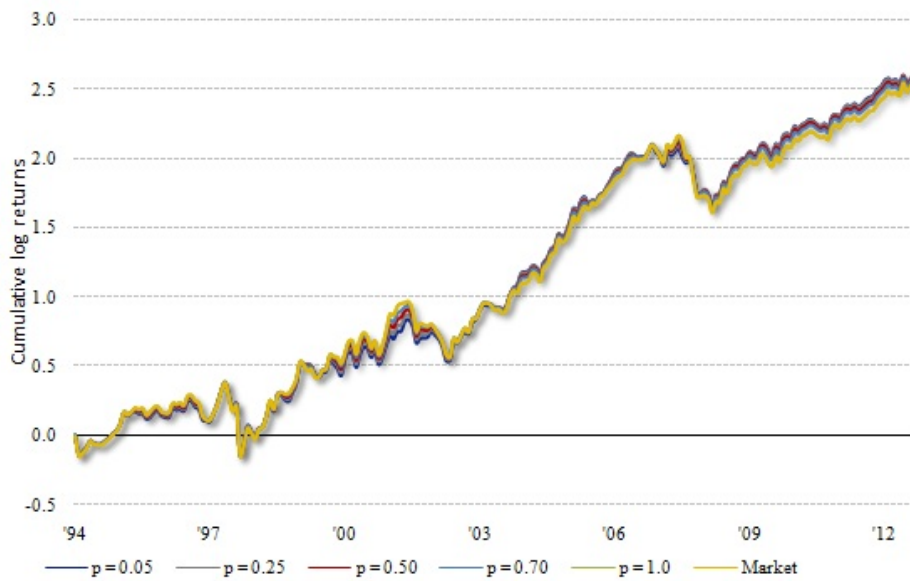


Figure 5.23: Cumulative log-scale returns of portfolios with various values for p .

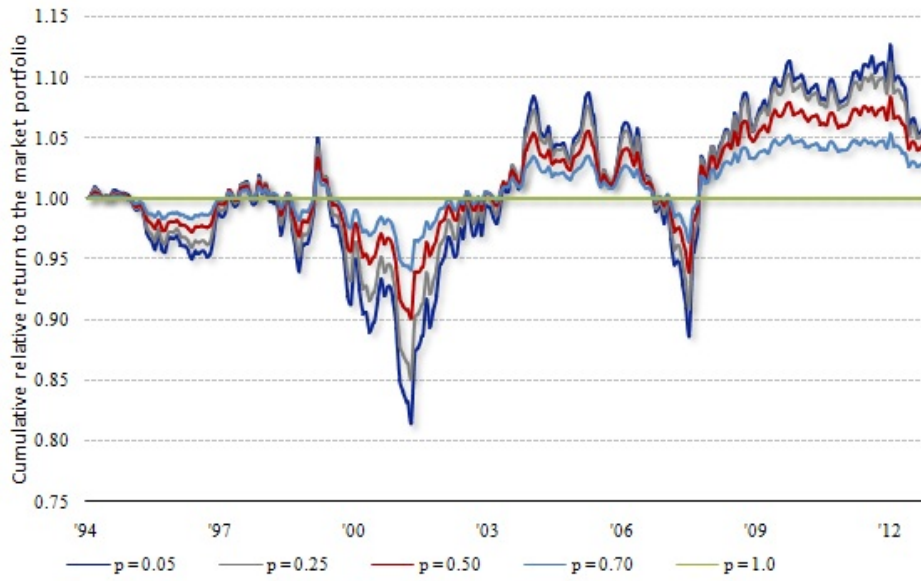


Figure 5.24: Cumulative relative returns of portfolios with various values for p . Returns are relative to the market portfolio.

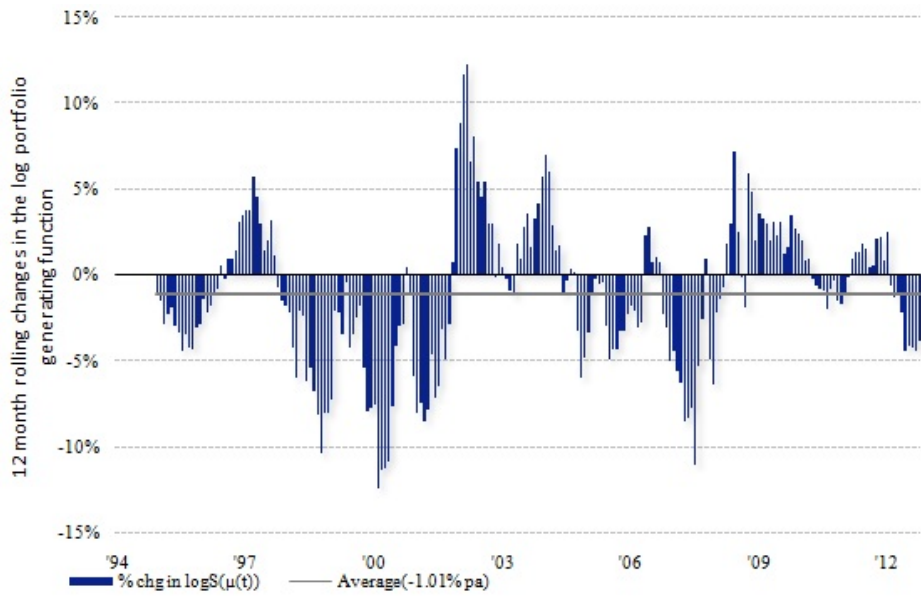


Figure 5.25: Rolling 12 month change in $\log S(\mu(t))$ for $p = 0.25$.

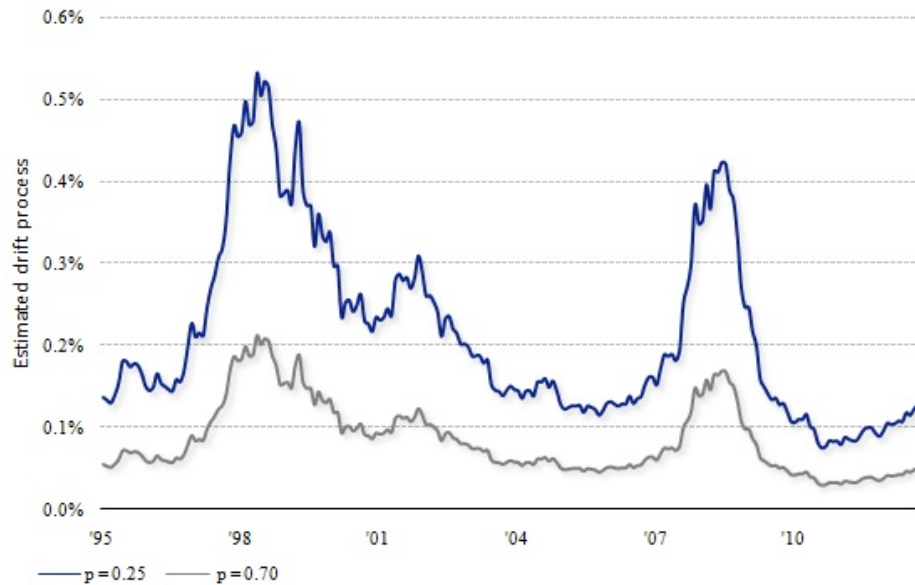


Figure 5.26: Comparison of estimated drift process values for portfolios with $p = 0.25$ and $p = 0.70$.

While the negative changes in $\log S(\mu(t))$ reduce returns, the lower p valued portfolios still outperform the higher ones. This is as a result of the higher drift process which counteracts the changes in $\log S(\mu(t))$. We highlight the differences in the drift processes in Figure 5.26. The drift process is 2.5 times larger for $p = 0.25$ than for $p = 0.70$.

5.5 Summary

In this chapter we analysed the empirical performance of portfolio generating functions, a concept introduced in Chapter 4. Furthermore, Equation (4.4.2), allowed us to classify the excess return over the market portfolio as a function of

- the changes in the log of the portfolio generating function,
- the difference in the dividend process of each portfolio, and
- the drift process as defined in Theorem 4.1.

We found that portfolios that increase the weights in smaller stocks (and therefore decrease the weights in the larger stocks) outperform the market (as the theory would suggest) but only marginally. This is as a result of the declining entropy and diversity in the market over our sample period, as measured through both the entropy function in Equation (5.3.1) and the diversity function in Equation (5.4.1).

This implies that smaller stocks have not necessarily outperformed the market significantly since 1994. This result is supported by research by [2], [50] and [52], amongst others, which show that the size premium is not significant in the South African equity market. Our research shows that the outperformance of a portfolio which places higher weights on smaller stocks (relative to the portfolio) is cyclical. There are periods of strong outperformance of the market and periods of strong underperformance which coincide with changes in the levels of entropy and diversity in the market.

Further to this, we find that the equal weighted, entropy weighted and diversity weighted (with lower values for p) portfolios have very similar (but marginal) levels of outperformance of the market portfolio. However, although these portfolios only marginally outperform the market portfolio, a portfolio which places larger weights in the large stocks performs significantly worse than these portfolios and the market portfolio.

We find that the South African equity market has become concentrated in fewer stocks over the period 1994 - 2013 and that this limits the outperformance of the market portfolio by the portfolio generating functions considered here.

Chapter 6

Conclusion

The aim of this dissertation was to investigate and explain the basic concepts of Stochastic Portfolio Theory (SPT) and to apply these results to the South African equity market.

In Chapter 2 we set out the basic concepts of SPT and analysed the long-term behaviour and properties of portfolios. At its core SPT assumes a lognormal model for stock returns and extends this to consider the value process of a portfolio of stocks. Proposition (2.2.1) shows that the value process of a portfolio of stocks is similar to that of individual stocks, comprising of a portfolio growth rate process and sensitivities to various price disturbances, which were modelled as Brownian motions. Furthermore, in Section 2.3, we showed that a portfolio's growth rate determines its long-term behaviour.

This result is in contrast to traditional portfolio optimisations which consider the arithmetic rate of return and not the geometric rate of return (i.e. the growth rate process). In Section 3.1, we showed in a simulation that expected geometric rates of return were more accurate in estimating long term returns. Arithmetic rates of return were shown to over-estimate rates of return.

The constraints in mean-variance optimisation under the excess growth rate results in constraints which are not linear and we are therefore unable to apply traditional portfolio optimisation techniques. However, we showed that it is possible using, for example, the Differential Evolution technique (implemented in R through the *DEoptim* library) to find solutions to these complex optimisations.

When minimising tracking errors, we found that for small tracking errors constraints could be reduced to a linear form and therefore traditional quadratic programming could be utilised. We also showed that, by only changing the target return from an arithmetic rate of return to a geometric rate of return, the portfolio's long-term return increased by approximately one half the port-

folio's variance. This is consistent with the relationship between arithmetic and geometric rates of return as shown in Equation (2.1.4).

Furthermore, in Chapter 2 we showed that a portfolio's growth rate process, $\gamma_\pi(t)$, is a function of the weighted individual stock growth rates processes and a term referred to as the portfolio excess growth rate, $\gamma_\pi^*(t)$. This portfolio excess growth rate is given in Equation (2.2.8) and is the difference between the weighted variances of the individual stocks and the resulting portfolio's variance.

Heuristically, the excess growth rate can be seen as the benefits of diversification and the inclusion of the excess growth rate in the determination of the portfolio growth rate implies that diversification has a direct impact on the portfolio's return. Of interest, therefore, is the maximisation of the excess growth rate in a portfolio in portfolio construction.

In Section 3.5 we analysed a portfolio which attempts to maximise the excess growth rate of a portfolio. The covariance matrix estimated over the previous 24 months was used as a forecast for the next period. We showed that this portfolio outperforms the market portfolio over the long term. However, the portfolio did show extended periods of underperformance relative to the market portfolio. Outperformance of the market portfolio appeared to be cyclical in nature, consistent with results of the equal weighted, entropy weighted and diversity weighted portfolios in Chapter 5.

A further aim of this dissertation was to provide a theoretical overview of portfolio generating functions and apply these to the South African equity market. Portfolio generating functions were introduced in Chapter 4 where we showed that the relative performance of these portfolios (relative to the market portfolio) is a function of three components,

- the changes in the log of the portfolio generating function,
- the difference in the dividend process of each portfolio, and
- the drift process as defined in Theorem 4.1.

In Sections 4.2 and 4.3 we highlighted the importance that the level of market diversity has on the market's portfolio growth rate. Therefore, given the importance of diversity we analyse portfolio generating functions which make use of market diversity to generate weights. These portfolio generating functions allowed us to draw various inferences on the behaviour of the South African

equity market since 1994.

We found that portfolios which place higher weights on smaller stocks (and therefore lower weights on the larger stocks) only marginally outperformed the market portfolio. These portfolios were the equal weighted, entropy weighted and diversity weighted portfolios as defined in Sections 5.1, 5.3 and 5.4 respectively.

We found that the relative performance of these three portfolios (relative to the market portfolio) behaved in a cyclical manner with periods of underperformance and periods of outperformance. This coincides with the findings in Section 3.5 where we considered the portfolio which optimises the excess growth rate.

This relative performance was found to act in concert with changing levels of both market entropy (Equation (5.3.1)) and market diversity (Equation (5.4.1)). In this regard we found that, since 1994, the market entropy and diversity in the South African equity market has declined and (similar to the respective portfolios) has shown cyclical behaviour over the past 10 years.

The declining levels of entropy and diversity in the South African equity market imply that the market has become more concentrated in fewer stocks since 1994. This has led to those portfolios which overweight smaller stocks only marginally outperforming the market as the decline in diversity acts as a headwind to outperformance through the changes (declines in this case) in the portfolio generating function. Comparing these portfolios with the portfolio in Section 3.5, we find that the portfolio which optimises the excess growth rate shows better outperformance of the market portfolio.

Although the market diversity has declined since 1994, a portfolio which increases weights in the larger stocks does not outperform the market. In fact such a portfolio, as shown in Section 5.2, significantly underperforms the market portfolio. We did, however, note that the large cap portfolio performed significantly ahead of the market portfolio during the later stages of a bull market.

Our findings allow for various possible avenues for further research.

- Although we have investigated and analysed numerous aspects of SPT, we have not looked at ranked effects and modelling the distribution of capital. That is, modelling the behaviour of the actual ranks (by market capitalisation) of stocks and constructing portfolios generated by these models as in [23] and applying these to the South African equity market.

- In Section 5.1 and specifically in Figure 5.5 we highlighted that dividend returns for the equal weighted portfolio were higher than that of the market portfolio, implying larger stocks pay lower dividends than smaller stocks. An investigation into this phenomenon may be useful.
- Both the entropy (Section 5.3) and diversity weighted (Section 5.4) portfolios showed cyclicity with regards to their returns relative to the market portfolio. This was linked to the market level of entropy or diversity which showed very similar cyclicity in recent years. A possible further area of research would be to optimise the values of p (in the case of the diversity weighted portfolio) dependent on a forecast of future levels of market diversity (assuming this can be modelled). This may tie in with models of market capitalisation ranks referred to previously.
- In Section 3.5, we optimised the portfolio's growth rate and showed such a portfolio outperformed the market portfolio. However, in doing so we used a simplistic historic method for estimating the covariance matrix. There are other, arguably more accurate, means of estimating the covariance matrix and combining this with the optimisation shown in Section 3.5 would be another area of interest.
- We have shown a clear trend in the South African equity market, in both Section 3.5 and Chapter 5, which highlights that the market has become concentrated in fewer stocks since 1994. This has negatively impacted the performance of equal weighted, entropy and diversity weighted portfolios over this period. Of interest may be an international study investigating whether this is the case worldwide or only in South Africa as a result of the market structure.

Appendices

Appendix A

R source code

A.1 Importing data

This code imports the data from csv files.

```
1 #import all data
2
3 #*****
4 #last price
5 #use names(array) to get headers
6 outpath = paste0(filepath,"PX_LAST.csv")
7 px <- read.csv(outpath,header=TRUE)
8 dates = as.matrix(px[,1]) #strip out dates column
9 px = px[,2:ncol(px)]
10 px = fixCols(px)
11
12 #*****
13 #Adjusted px
14 outpath = paste0(filepath,"AdjPX_LAST.csv")
15 adj.px <- read.csv(outpath,header=TRUE)
16 adj.px = adj.px[,2:ncol(adj.px)]
17 adj.px = fixCols(adj.px)
18 adj.px= adj.px[,colnames(px)] #rearrange columns to align with px
    matrix
19
20
21
22 #*****
23 #Turnover
24 outpath = paste0(filepath,"TURNOVER.csv")
25 turnover <- read.csv(outpath,header=TRUE)
26 turnover = turnover[,2:ncol(turnover)]
27 turnover = fixCols(turnover)
28 turnover = turnover[,colnames(px)]
29
30 #*****
31 #Market Cap
32 outpath = paste0(filepath,"CUR_MKT_CAP.csv")
```

```

33 mkt.cap <- read.csv(outpath,header=TRUE)
34 mkt.cap = mkt.cap[,2:ncol(mkt.cap)]
35 mkt.cap = fixCols(mkt.cap)
36 mkt.cap = mkt.cap[,colnames(px)]
37
38 #*****
39 #Control variables
40 outpath = paste0(filepath,"ConPX_LAST.csv")
41 con.px <- read.csv(outpath,header=TRUE)
42 con.px = con.px[,2:ncol(con.px)]
43
44 #*****
45 #Index Members
46 outpath = paste0(filepath,"Index.csv")
47 index <- read.csv(outpath,header=TRUE)
48
49 #*****
50 #redoes dates
51 ts.dates = as.Date(dates,"%m/%d/%Y")
52
53 #*****
54 #calculates total returns for stocks
55 #calculates total returns for stocks
56
57 tot.returns = array(0,dim=dim(adj.px))
58 colnames(tot.returns) = colnames(adj.px)
59
60 for (k in 1:ncol(tot.returns)){
61
62   temp.px = as.matrix(adj.px[,k])
63   temp.pxlag =as.matrix(Lag(temp.px,k=1))
64   tot.returns[,k] = temp.px/temp.pxlag-1
65
66 }
67
68 #any returns greater than 1 are set to zero
69 tot.returns[tot.returns > 0.5] = 0
70
71 rm(temp.px)
72 rm(temp.pxlag)

```

A.2 Section 3.1 simulation

This code generates the simulation in Section 3.1.

```

1 #This program illustrates the difference between arithmetic and
   geometric returns
2
3
4 prob.up = 0.5
5 prob.down = 0.5

```

```

6 return.up = 0.25
7 return.down = -0.05
8 expected.arithmetic = prob.up*return.up + prob.down*return.down
9 expected.geometric = expected.arithmetic - 0.5*(0.5*var(c(return.
  up,return.down)))
10
11 number.runs = 1000000
12 cagr.out = array(0,dim=c(1,number.runs))
13
14 for (k in 1:number.runs){
15   x = runif(100,0,1)
16   temp.returns = array(0,dim=c(1,length(x)))
17   temp.returns[x<=prob.up] = return.up
18   temp.returns[x>prob.up] = return.down
19   temp.returns = 1+temp.returns
20   cumulative.return = prod(temp.returns)^(1/length(temp.returns))
  -1
21   cagr.out[k] = cumulative.return
22 }
23
24 var.outpath = "H:\\MSc Financial Engineering Dissertation\\Data
  analysis\\Geometric vs Arithmetic\\"
25
26 hist.breaks = as.matrix(hist(cagr.out)$breaks[1:length(hist.
  density)])
27 hist.density = as.matrix(hist(cagr.out)$density)
28 hist.out = cbind(hist.breaks,hist.density)
29 write.table(hist.out,paste0(var.outpath,"CAGR Hist.csv"),sep="\t")
30
31 mean(cagr.out)
32 expected.arithmetic
33 expected.geometric

```

A.3 Section 3.3 MVO optimisation

This code generates the optimisation in section 3.3.

```

1 # *****
2 #This program runs the mean-variance optimization as in section
  3.3
3 #Assume only a set of risky assets
4 #look to achieve a certain level of return and minimising risk
5
6 # *****
7
8 #for testing
9 #ret.mat = data.set
10 #cov.mat = m.cov
11

```

```

12 #This function will be used as an objective function for the SPT
    optimisation
13 SPT.obj <- function(wp,cov.mat,ret.mat,target){
14   if (sum(wp) == 0){
15     wp = wp + 1e-2
16   }
17   wp=wp/sum(wp)
18
19   wp=as.matrix(wp)
20
21   exp.gRates = as.matrix(colMeans(ret.mat)-0.5*diag(cov.mat)) #
    assume mean returns will be achieved for each stock
22
23   control1 = sqrt(t(wp) %*% cov.mat %*% wp) #we want to minimise
    this
24   control2.1 = t(wp) %*% exp.gRates + 0.5*(t(wp) %*% as.matrix(
    sqrt(diag(cov.mat)))-control1)
25   control2.2 = target + 0.5*control1 #subject to a minimum return
26   control2 = control2.1 - control2.2 #therefore we want this to be
    positive
27
28   #so we introduce a penalty (sufficiently large) in this function
29   out = control1 - min(control2,0)*1e3
30   return(out)
31 }
32
33 #perform returns on last 24 months of data
34 start.pos = 24
35
36 #Initialise some matrices and parameters
37 MPTweights = array(0,dim=dim(tot.returns))
38 SPTweights = array(0,dim=dim(tot.returns))
39 colnames(MPTweights) = colnames(tot.returns)
40 colnames(SPTweights) = colnames(tot.returns)
41 MPTreturns = array(0,dim=c(nrow(tot.returns),2))
42 MPTreturns[,2] = 1
43 SPTreturns = array(0,dim=c(nrow(tot.returns),2))
44 SPTreturns[,2] = 1
45
46 ann.target = 0.15
47 monthly.target = (1 + ann.target)^(1/12)-1
48 maxweight=0.15
49
50 for (k in 24:(nrow(tot.returns)-start.pos+1)){
51
52   #get weights for next period using next 24 months
53   #determine covariance matrix
54   data.set = tot.returns[k:(k+start.pos-1),]
55   data.set = data.set[,in.universe[k,] == 1] #check if in universe
56   data.set = data.set[,colSums(!is.na(data.set))==nrow(data.set) ]
    #remove NAs
57

```



```

58 m.cov = makePositiveDefinite(cov(data.set, use="pairwise.complete
    .obs"))
59
60 #need to remove NAs
61 m.cov = m.cov[rowSums(is.na(m.cov))!=ncol(m.cov), ]
62 m.cov = m.cov[, colSums(is.na(m.cov))!=nrow(m.cov) ]
63
64 #max weight = minimum of maximum or 5*market cap weight
65 temp.maxw = t(as.matrix(index.weight[k, colnames(m.cov)]))
66 maxw = pmin(5*temp.maxw, rep(maxweight, ncol(temp.maxw)))
67
68 #run optimization until a target is reached
69 #this is traditional mvo
70 eret = as.matrix(colMeans(data.set[, colnames(m.cov)]))
71
72
73 Amat = array(1, dim=c(nrow(m.cov), 1))
74 Amat = cbind(Amat, diag(1, nrow(Amat), ncol(m.cov)))
75 Amat = cbind(Amat, diag(-1, nrow(Amat), ncol(m.cov)))
76 Amat = cbind(Amat, eret)
77
78 dvec = array(0, dim=c(nrow(m.cov), 1))
79
80 bvec = array(0, dim=c(1, 2*nrow(m.cov)+2))
81 bvec[1, 1] = 1
82 bvec[1, (nrow(m.cov)+2):(ncol(bvec)-1)] = -maxw
83 bvec[1, ncol(bvec)] = monthly.target
84
85 #use solve.qp to solve for traditional mvo
86 #lower target if it cannot be reached
87 mvol = try(solve.QP(m.cov, dvec, Amat, bvec, 1), silent=TRUE)
88 if(inherits(mvol, "try-error")){
89   temp.target = monthly.target
90   while(inherits(mvol, "try-error")){
91     temp.target = temp.target - 0.01/100 #lower target slowly
92     bvec[1, ncol(bvec)] = temp.target
93     mvol = try(solve.QP(m.cov, dvec, Amat, bvec, 1), silent=TRUE)
94     print(temp.target)
95   }
96 }
97
98 #extract weights here and save in weights matrix
99 temp.w = as.matrix(mvol$solution)
100 temp.w[temp.w<1e-4] = 0
101 temp.w = temp.w/sum(temp.w)
102 temp.w = t(temp.w)
103 colnames(temp.w) = colnames(m.cov)
104 MPTweights[k, colnames(temp.w)] = temp.w
105
106 #this is SPT version
107 minw = 0
108 N = ncol(m.cov)

```

```

109 lower = rep(minw,N)
110 upper = maxw
111 rp = array(0,dim=c(10*ncol(temp.w),ncol(temp.w)))
112 rp[,] = temp.w
113 controlDE = list(initialpop = rp, retol = 0.00000001,itermax
  =5000,trace=FALSE, strategy=6,c=0.8,p=0.25, parallelType=1, NP
  =10*ncol(m.cov))
114 mvo2 = DEoptim(fn = SPT.obj,lower,upper,control=controlDE,cov.
  mat = m.cov,ret.mat = data.set,target = monthly.target)
115
116 #extract weights
117 temp.w = as.matrix(mvo2$optim$bestmem)
118 temp.w[temp.w<1e-4] = 0
119 temp.w = temp.w/sum(temp.w)
120 temp.w = t(temp.w)
121 colnames(temp.w)= colnames(m.cov)
122 SPTweights[k,colnames(temp.w)] = temp.w
123
124 #determine portfolio returns here
125 #need to use previous months weights with current returns
126 if (k > (start.pos+1)){
127
128   #MPT returns
129   temp.w = as.matrix(MPTweights[k-1,])
130   temp.ret = as.matrix(tot.returns[k,])
131
132   temp.w[is.na(temp.w)] = 0
133   temp.ret[is.na(temp.ret)] = 0
134
135   temp.portret = t(temp.w) %*% temp.ret
136   MPTreturns[k,1] = temp.portret
137   MPTreturns[k,2] = MPTreturns[k-1,2] * (1+temp.portret)
138
139   #SPT returns
140   temp.w = as.matrix(SPTweights[k-1,])
141   temp.ret = as.matrix(tot.returns[k,])
142
143   temp.w[is.na(temp.w)] = 0
144   temp.ret[is.na(temp.ret)] = 0
145
146   temp.portret = t(temp.w) %*% temp.ret
147   SPTreturns[k,1] = temp.portret
148   SPTreturns[k,2] = SPTreturns[k-1,2] * (1+temp.portret)
149
150   spt.ann = SPTreturns[k,2]^(12/length(which(SPTreturns[,1]!=0))
  )-1
151   mpt.ann = MPTreturns[k,2]^(12/length(which(MPTreturns[,1]!=0))
  )-1
152
153   print(paste("SPT:",round(spt.ann*100,digits=3)))
154   print(paste("MPT:",round(mpt.ann*100,digits=3)))
155

```

```

156   plot (ts (MPTreturns[24:(k-1),2])
157         lines (ts (SPTreturns[24:(k-1),2]), col="red ")
158
159   }
160
161   print (paste0 (round (100*k/nrow (tot.returns),1), "%"))
162
163 }
164
165 #CAGR's and portfolio variance
166 temp.ret = MPTreturns [MPTreturns [,1]!=0,]
167 temp.cagr = temp.ret [nrow (temp.ret),2]^(12/nrow (temp.ret))-1
168 temp.var = sd (temp.ret [,1]) *sqrt (12)
169 print ("MPT output ")
170 temp.cagr
171 temp.var
172
173 temp.ret = SPTreturns [SPTreturns [,1]!=0,]
174 temp.cagr = temp.ret [nrow (temp.ret),2]^(12/nrow (temp.ret))-1
175 temp.var = sd (temp.ret [,1]) *sqrt (12)
176 print ("SPT output ")
177 temp.cagr
178 temp.var
179
180 #plot reconstructed index vs. reported total return index
181 plot (ts (MPTreturns [,2]))
182 par (new=T)
183 plot (ts (SPTreturns [,2]), col="red ")
184
185
186 wb.path = paste0 (outPath, " MVO optimisation.xlsx ")
187 if (file.exists (wb.path)) {file.remove (wb.path)}
188
189 wb = loadWorkbook (wb.path, create=TRUE)
190
191 #output returns and ordered weights
192 createSheet (wb, "Returns ")
193 writeWorksheet (wb, SPTreturns, "Returns", startRow = 1, startCol = 1)
194 writeWorksheet (wb, MPTreturns, "Returns", startRow = 1, startCol = 5)
195
196 saveWorkbook (wb)

```

A.4 Section 3.4 tracking error optimisation

This code runs the tracking error optimisation in Section 3.4.

```

1 # *****
2 #This code minimises a portfolio's tracking error relative to
3 #the market cap weighted index as in Section 3.4
4 # *****
5

```

```

6 #perform returns on the next 24 months of data
7 start.pos = 24
8
9 #Initialise some matrices and parameters
10 SPTweights = array(0,dim=dim(tot.returns))
11 colnames(SPTweights) = colnames(tot.returns)
12 SPTreturns = array(0,dim=c(nrow(tot.returns),2))
13 SPTreturns[,2] = 1
14
15 ann.target = 0.1 #outperform market by x% per annum
16 monthly.target = (1 + ann.target)^(1/12)-1
17
18 test = array(0,dim=c(nrow(SPTreturns),1))
19
20 for (k in 2:(nrow(SPTweights)-start.pos+1)){
21
22   #get weights for next period using prev 12 months
23   #determine covariance matrix
24   data.set = tot.returns[k:(k+start.pos-1),]
25   data.set = data.set[,in.universe[k,] == 1] #check if in universe
26   data.set = data.set[,colSums(!is.na(data.set))==nrow(data.set) ]
27   #remove NAs
28   data.set = data.set[,colSums(!is.infinite(data.set))==nrow(data.
29   set) ] #remove infinities
30
31   #now determine relative returns
32   temp.indexret = as.matrix(index.returns[k:(k+start.pos-1),1])
33   excess_returns = function(rvec) rvec-temp.indexret
34   m.excessret = apply(data.set,2,excess_returns)
35
36   #tracking error and covariance matrix for stocks
37   m.trackerror = makePositiveDefinite(cov(m.excessret,use="
38   pairwise.complete.obs"))
39   m.cov = makePositiveDefinite(cov(data.set,use="pairwise.complete
40   .obs"))
41
42   gamma = as.matrix(colMeans(data.set)) - 0.5*diag(m.cov)
43   gamma.market = as.matrix(mean(temp.indexret) - 0.5*var(temp.
44   indexret))
45
46   constraint = gamma+0.5*diag(m.trackerror) #this is a geometric
47   returns
48   #constraint = rowMeans(m.excessret) # this is an arithmetic
49   return
50
51   #run optimization using solve.QP
52   #no shorts, weights to sum to 1
53   Amat <- cbind(1, diag(nrow(m.trackerror)),constraint)
54   bvec <- c(1, rep(0, nrow(m.trackerror)),monthly.target)
55   meq <- 1
56
57 }

```

```

50 qp = tryCatch(solve.QP(m.trackerror, rep(0,nrow(m.trackerror)),
  Amat, bvec, meq), error = function(e) e)
51 #mpt = tryCatch(portfolio.optim(t(constraint), pm=monthly.target,
  riskless=FALSE, shorts=FALSE, covmat=m.trackerror), error=
  function(e) e)
52
53 #extract weights
54 #err = mpt$message
55 err = qp$message
56 if (length(err) > 0) {
57   SPTweights[k,] = SPTweights[k-1,]
58   print("Error occurred")
59 } else {
60   temp.w = as.matrix(qp$solution)
61   test[k] = qp$value
62   #temp.w = as.matrix(mpt$pw)
63   temp.w[temp.w<1e-7] = 0
64   temp.w = temp.w/sum(temp.w)
65   temp.w = t(temp.w)
66   colnames(temp.w) = colnames(m.cov)
67   SPTweights[k, colnames(temp.w)] = temp.w
68 }
69
70 print(paste0(round(100*k/nrow(tot.returns),1),"%"))
71
72 }
73
74 SPTreturns = CalcReturnSeries(SPTweights)
75
76 #CAGR's and portfolio variance
77 temp.ret = SPTreturns[SPTreturns[,1]!=0,]
78 temp.cagr = temp.ret[nrow(temp.ret),2]^(12/nrow(temp.ret))-1
79 temp.var = sd(temp.ret[,1])*sqrt(12)
80 print("SPT output")
81 temp.cagr
82 temp.var
83
84 temp.ret = index.returns[index.returns[,1]!=0,]
85 temp.cagr = temp.ret[nrow(temp.ret),2]^(12/nrow(temp.ret))-1
86 temp.var = sd(temp.ret[,1])*sqrt(12)
87 print("Index output")
88 temp.cagr
89 temp.var
90
91 #plot reconstructed index vs. reported total return index
92 plot(ts(SPTreturns[,2]))
93 lines(ts(index.returns[,2]), col="red")
94
95
96 relreturns = SPTreturns[,1] - index.returns[,1]
97 sd(relreturns)*sqrt(12)
98

```

```

99 write.table(SPTreturns, paste0(outPath, "\\SA Market\\
    TE_Optimise_Returns.csv"), sep="\t")
100 write.table(index.returns, paste0(outPath, "\\SA Market\\
    TE_Index_Returns.csv"), sep="\t")

```

A.5 Section 3.5 maximising the excess growth rate

This code runs the excess growth rate optimisation in Section 3.5.

```

1 # *****
2 #This program runs maximises the excess growth rate with no
3 #minimum required growth rate
4 # *****
5
6 #for testing
7 #ret.mat = data.set
8 #cov.mat = m.cov
9
10 #perform returns on last 24 months of data
11 start.pos = 24
12
13 #Initialise some matrices and parameters
14 SPTweights = array(0,dim=dim(tot.returns))
15 colnames(SPTweights) = colnames(tot.returns)
16 SPTreturns = array(0,dim=c(nrow(tot.returns),2))
17 SPTreturns[,2] = 1
18
19
20 ann.target = 0.15
21 monthly.target = (1 + ann.target)^(1/12)-1
22 maxweight=0.15
23
24 for (k in 50:(nrow(tot.returns))){
25
26   #get weights for next period using next 24 months
27   #determine covariance matrix
28   data.set = tot.returns[(k-start.pos+1):k,]
29   data.set = data.set[,in.universe[k,] == 1] #check if in universe
30   data.set = data.set[,colSums(!is.na(data.set))==nrow(data.set)]
31   #remove NAs
32
33   m.cov = makePositiveDefinite(cov(data.set, use="pairwise.complete
34   .obs"))
35
36   #need to remove NAs
37   m.cov = m.cov[rowSums(is.na(m.cov))!=ncol(m.cov), ]
38   m.cov = m.cov[,colSums(is.na(m.cov))!=nrow(m.cov) ]

```

```

37
38 #max weight = minimum of maximum or 5*market cap weight
39 temp.maxw = t(as.matrix(index.weight[k,colnames(m.cov)]))
40 maxw = pmin(5*temp.maxw,rep(maxweight,ncol(temp.maxw)))
41
42 #run optimization until a target is reached
43 #this is traditional mvo
44 eret = as.matrix(colMeans(data.set[,colnames(m.cov)]))
45
46
47 Amat = array(1,dim=c(nrow(m.cov),1))
48 Amat = cbind(Amat,diag(1,nrow(Amat),ncol(m.cov)))
49 Amat = cbind(Amat,diag(-1,nrow(Amat),ncol(m.cov)))
50
51 dvec = -0.5*diag(m.cov)
52
53 bvec = array(0,dim=c(1,(2*nrow(m.cov)+1)))
54 bvec[1,1] = 1
55 bvec[1,(nrow(m.cov)+2):(ncol(bvec))]=maxw
56
57 #use solve.qp to solve for traditional mvo
58 mvol = solve.QP(m.cov,dvec,Amat,bvec,1)
59
60 #extract weights here and save in weights matrix
61 temp.w = as.matrix(mvol$solution)
62 temp.w[temp.w<1e-4] = 0
63 temp.w = temp.w/sum(temp.w)
64 temp.w = t(temp.w)
65 colnames(temp.w)= colnames(m.cov)
66 SPTweights[k,colnames(temp.w)] = temp.w
67
68
69 #determine portfolio returns here
70 #need to use previous months weights with current returns
71 if (k > (start.pos+1)){
72
73   #SPT returns
74   temp.w = as.matrix(SPTweights[k-1,])
75   temp.ret = as.matrix(tot.returns[k,])
76
77   temp.w[is.na(temp.w)] = 0
78   temp.ret[is.na(temp.ret)] = 0
79
80   temp.portret = t(temp.w) %*% temp.ret
81   SPTreturns[k,1] = temp.portret
82   SPTreturns[k,2] = SPTreturns[k-1,2] * (1+temp.portret)
83
84   #print progress
85
86   SPT.ann = SPTreturns[k,2]^(12/length(which(SPTreturns[,1]!=0))
87   )-1

```

```

88     print(paste("SPT:", round(SPT.ann*100, digits=3)))
89
90     plot(ts(SPTreturns[24:(k-1), 2]))
91
92   }
93
94   print(paste0(round(100*k/nrow(tot.returns), 1), "%"))
95
96 }
97
98 #CAGR's and portfolio variance
99 temp.ret = SPTreturns[SPTreturns[, 1] != 0, ]
100 temp.cagr = temp.ret[nrow(temp.ret), 2]^(12/nrow(temp.ret))-1
101 temp.var = sd(temp.ret[, 1])*sqrt(12)
102 print("SPT output")
103 temp.cagr
104 temp.var
105
106 wb.path = paste0(outPath, " max excess growth rate.xlsx")
107 if(file.exists(wb.path)){file.remove(wb.path)}
108
109 wb = loadWorkbook(wb.path, create=TRUE)
110
111 #output returns and ordered weights
112 createSheet(wb, "Returns")
113 writeWorksheet(wb, SPTreturns, "Returns", startRow = 1, startCol = 1)
114 writeWorksheet(wb, index.returns, "Returns", startRow = 1, startCol =
115                 5)
116 saveWorkbook(wb)

```

A.6 Section 5.1 equal weighted portfolio

This code generates the data for the equal weight portfolio in Section 5.1.

```

1 # *****
2 #           Forms the equal weighted portfolio
3 # *****
4
5
6 # *****
7 #Calculate weights in equal portfolio
8 # *****
9 equal.weight = array(0, dim=dim(mkt.cap))
10 colnames(equal.weight) = colnames(index.weight)
11
12 S = array(0, dim=c(nrow(mkt.cap), 2))
13 colnames(S) = c("S(t)", "Change")
14
15 equal.order = array(NA, dim=dim(mkt.cap))
16 colnames(equal.order) = seq(1, ncol(mkt.cap), 1)

```



```

17
18
19 for (row.num in 1:nrow(index.mktcap)){
20
21   #determine equal weights
22   temp.weight = t(as.matrix(in.universe[row.num,]))
23   temp.weight[is.infinite(temp.weight)]=0
24   temp.weight[temp.weight > 0] = 1
25
26   equal.weight[row.num,] = temp.weight/sum(temp.weight)
27
28   #ranks vs weight comparison
29   equal.order[row.num,order.index[row.num,]] = equal.weight[row.
    num,]
30
31   #determine S
32   temp.weights = index.weight[row.num,]
33   temp.weights = temp.weights[temp.weights > 0]
34   S.temp = t(as.matrix(log(temp.weights)))
35   S[row.num,1] = mean(S.temp)
36   if(row.num > 1){S[row.num,2] = S[row.num,1] - S[row.num-1,1]}
37
38 }
39
40 rm(temp.weight)
41 rm(S.temp)
42
43
44 #*****
45 #Calculate portfolio returns
46 #*****
47 equal.returns = array(0,dim =c(nrow(tot.returns),2))
48 equal.returns[1,2] = 1
49
50 equal.div = array(0,dim =c(nrow(tot.returns),2))
51 equal.div[1,2] = 1
52
53 EXGRATELAG = 12
54 ex.grate = array(0,dim =c(nrow(tot.returns),2))
55
56 for (k in 2:nrow(index.returns)){
57
58   #total returns
59   temp.weights = as.matrix(equal.weight[k-1,])
60   temp.returns = as.matrix(tot.returns[k,])
61   temp.returns[is.na(temp.returns)] = 0
62   temp.returns[is.infinite(temp.returns)] = 0
63
64   equal.returns[k,1] = t(temp.returns) %*% temp.weights
65   equal.returns[k,2] = (1+equal.returns[k,1]) * equal.returns[k
    -1,2]
66

```

```

67 #dividend return
68 temp.weights = as.matrix(equal.weight[k-1,])
69 temp.returns = as.matrix(oth.returns[k,])
70 temp.returns[is.na(temp.returns)] = 0
71 temp.returns[is.infinite(temp.returns)] = 0
72
73 equal.div[k,1] = t(temp.returns) %*% temp.weights
74 equal.div[k,2] = (1+equal.div[k,1]) * equal.div[k-1,2]
75
76 var.temp = array(0,dim=c(1,ncol(equal.weight)))
77 covar.temp = array(0,dim=c(ncol(equal.weight),ncol(equal.weight)
  ))
78
79 #determine excess growth rate
80 if(k > EXGRATELAG) {
81
82   #determine weight variance
83   var.temp = t(as.matrix(apply(tot.returns[(k-EXGRATELAG+1):k
  ],2,function(x) var(x,na.rm=TRUE))))
84   covar.temp = cov(tot.returns[(k-EXGRATELAG+1):k], use="
  everything")
85
86   covar.temp[is.na(covar.temp)] = 0
87   var.temp[is.na(var.temp)] = 0
88
89   simple = var.temp %*% temp.weights
90   port.covar = t(temp.weights) %*% covar.temp %*% temp.weights
91
92   ex.grate[k,1] = 0.5*(simple-port.covar)
93
94   if(k > EXGRATELAG + 1){ex.grate[k,2] = ex.grate[k,1] - ex.
  grate[k-1,1]}
95 }
96
97 }
98 rm(temp.returns)
99 rm(temp.weights)
100
101 #total relative outperformance
102 out.perf = S[,2] + ex.grate[,2] + (equal.div[,1] - index.div[,1])
103 mon.outperf = equal.returns[,1] - index.returns[,1]
104
105 #get average order weights
106 equal.avgOrder = t(as.matrix(colMeans(equal.order)))
107
108 #output into excel file
109 wb.path = paste0(outPath,"equal weighted portfolio.xlsx")
110 if(file.exists(wb.path)){file.remove(wb.path)}
111
112 wb = loadWorkbook(wb.path,create=TRUE)
113
114 #output returns and ordered weights

```

```

115 createSheet(wb, "Returns")
116 writeWorksheet(wb, equal.returns, "Returns", startRow = 1, startCol =
    1)
117 writeWorksheet(wb, index.returns, "Returns", startRow = 1, startCol =
    5)
118
119 createSheet(wb, "Market portfolio - Order")
120 writeWorksheet(wb, mktcap.avgOrder, sheet = "Market portfolio -
    order")
121
122 createSheet(wb, "Equal portfolio - Order")
123 writeWorksheet(wb, equal.avgOrder, sheet = "Equal portfolio - order
    ")
124
125 #output components of return
126 createSheet(wb, "Components of return")
127 writeWorksheet(wb, "S", "Components of return", startRow = 1, startCol
    = 1)
128 writeWorksheet(wb, S, "Components of return", startRow = 1, startCol =
    2)
129 writeWorksheet(wb, "Excess growth rate", "Components of return",
    startRow=1, startCol=5)
130 writeWorksheet(wb, ex.grate, "Components of return", startRow=1,
    startCol=6)
131 writeWorksheet(wb, "Dividends", "Components of return", startRow=1,
    startCol=9)
132 writeWorksheet(wb, equal.div, "Components of return", startRow=1,
    startCol=10)
133 writeWorksheet(wb, index.div, "Components of return", startRow=1,
    startCol=13)
134
135 saveWorkbook(wb)

```

A.7 Section 5.2 large cap portfolio

This code generates the data for the large cap portfolio in Section 5.2.

```

1 # *****
2 #           Forms the large cap portfolio
3 # *****
4
5
6 # *****
7 #Calculate weights in the large cap portfolio
8 # *****
9 largeCap.weight = array(0, dim=dim(mkt.cap))
10 colnames(largeCap.weight) = colnames(index.weight)
11
12 S = array(0, dim=c(nrow(mkt.cap), 2))
13 colnames(S) = c("S(t)", "Change")
14

```

```

15 largeCap.order = array(NA,dim=dim(mkt.cap))
16 colnames(largeCap.order) = seq(1,ncol(mkt.cap),1)
17
18
19 for (row.num in 1:nrow(index.mktcap)){
20
21   #determine large cap weights
22   temp.weight = t(as.matrix(index.weight[row.num,]))
23   temp.weight = temp.weight^2
24
25   largeCap.weight[row.num,] = temp.weight/sum(temp.weight)
26
27   #ranks vs weight comparison
28   largeCap.order[row.num,order.index[row.num,]] = largeCap.weight[
     row.num,]
29
30   #determine S
31   temp.weights = index.weight[row.num,]
32   temp.weights = temp.weights[temp.weights>0]
33   S.temp = t(as.matrix(temp.weights^2))
34   S[row.num,1] = 0.5*log(sum(S.temp))
35   if (row.num > 1){S[row.num,2] = S[row.num,1] - S[row.num-1,1]}
36
37 }
38
39 rm(temp.weight)
40 rm(S.temp)
41
42
43 #*****
44 #Calculate portfolio returns
45 #*****
46 largeCap.returns = array(0,dim =c(nrow(tot.returns),2))
47 largeCap.returns[1,2] = 1
48
49 largeCap.div = array(0,dim =c(nrow(tot.returns),2))
50 largeCap.div[1,2] = 1
51
52 EXGRATELAG = 12
53 ex.grate = array(0,dim =c(nrow(tot.returns),2))
54
55 for (k in 2:nrow(index.returns)){
56
57   #total returns
58   temp.weights = as.matrix(largeCap.weight[k-1,])
59   temp.returns = as.matrix(tot.returns[k,])
60   temp.returns[is.na(temp.returns)] = 0
61   temp.returns[is.infinite(temp.returns)] = 0
62
63   largeCap.returns[k,1] = t(temp.returns) %*% temp.weights
64   largeCap.returns[k,2] = (1+largeCap.returns[k,1]) * largeCap.
     returns[k-1,2]

```

```

65
66 #dividend return
67 temp.weights = as.matrix(largeCap.weight[k-1,])
68 temp.returns = as.matrix(oth.returns[k,])
69 temp.returns[is.na(temp.returns)] = 0
70 temp.returns[is.infinite(temp.returns)] = 0
71
72 largeCap.div[k,1] = t(temp.returns) %*% temp.weights
73 largeCap.div[k,2] = (1+largeCap.div[k,1]) * largeCap.div[k-1,2]
74
75 var.temp = array(0,dim=c(1,ncol(largeCap.weight)))
76 covar.temp = array(0,dim=c(ncol(largeCap.weight),ncol(largeCap.
  weight)))
77
78 #determine excess growth rate
79 if(k > EXGRATELAG) {
80
81   #determine weight variance
82   var.temp = t(as.matrix(apply(tot.returns[(k-EXGRATELAG+1):k
  ],2,function(x) var(x,na.rm=TRUE))))
83   covar.temp = cov(tot.returns[(k-EXGRATELAG+1):k], use="
  everything")
84
85   covar.temp[is.na(covar.temp)] = 0
86   var.temp[is.na(var.temp)] = 0
87
88   simple = var.temp %*% temp.weights
89   port.covar = t(temp.weights) %*% covar.temp %*% temp.weights
90
91   ex.grate[k,1] = 0.5*(simple-port.covar)
92
93   if(k > EXGRATELAG + 1){ex.grate[k,2] = ex.grate[k,1] - ex.
  grate[k-1,1]}
94 }
95
96 }
97 rm(temp.returns)
98 rm(temp.weights)
99
100 #total relative outperformance
101 out.perf = S[,2] + ex.grate[,2] + (largeCap.div[,1] - index.div
  [,1])
102 mon.outperf = largeCap.returns[,1] - index.returns[,1]
103
104 #get average order weights
105 largeCap.avgOrder = t(as.matrix(colMeans(largeCap.order)))
106
107 #output into excel file
108 wb.path = paste0(outPath,"largeCap weighted portfolio.xlsx")
109 if(file.exists(wb.path)){file.remove(wb.path)}
110
111 wb = loadWorkbook(wb.path, create=TRUE)

```

```

112
113 #output returns and ordered weights
114 createSheet(wb,"Returns")
115 writeWorksheet(wb,largeCap.returns,"Returns",startRow = 1,
    startCol = 1)
116 writeWorksheet(wb,index.returns,"Returns",startRow = 1, startCol =
    5)
117
118 createSheet(wb,"Market portfolio - Order")
119 writeWorksheet(wb,mktcap.avgOrder,sheet = "Market portfolio -
    order")
120
121 createSheet(wb,"largeCap portfolio - Order")
122 writeWorksheet(wb,largeCap.avgOrder,sheet = "largeCap portfolio -
    order")
123
124 #output components of return
125 createSheet(wb,"Components of return")
126 writeWorksheet(wb,"S","Components of return",startRow = 1,startCol
    = 1)
127 writeWorksheet(wb,S,"Components of return",startRow = 1,startCol =
    2)
128 writeWorksheet(wb,"Excess growth rate","Components of return",
    startRow=1,startCol=5)
129 writeWorksheet(wb,ex.grate,"Components of return",startRow=1,
    startCol=6)
130 writeWorksheet(wb,"Dividends","Components of return",startRow=1,
    startCol=9)
131 writeWorksheet(wb,largeCap.div,"Components of return",startRow=1,
    startCol=10)
132 writeWorksheet(wb,index.div,"Components of return",startRow=1,
    startCol=13)
133
134 saveWorkbook(wb)

```

A.8 Section 5.3 entropy weighted portfolio

This code generates the data for the entropy weighted portfolio in Section 5.3.

```

1 # *****
2 #           Forms the entropy weighted portfolio
3 # *****
4
5
6 # *****
7 #Calculate weights in entropy portfolio
8 # *****
9 entropy.weight = array(0,dim=dim(mkt.cap))
10 colnames(entropy.weight) = colnames(index.weight)
11
12 S = array(0,dim=c(nrow(mkt.cap),2))

```

```

13 colnames(S) = c("S(t)", "Change")
14
15 entropy.order = array(NA, dim=dim(mkt.cap))
16 colnames(entropy.order) = seq(1, ncol(mkt.cap), 1)
17
18
19 for (row.num in 1:nrow(index.mktcap)){
20
21   #determine entropy weights
22   temp.weight = t(as.matrix(index.weight[row.num,]))
23   log.temp.weight = log(temp.weight)
24   temp.S = (temp.weight*log.temp.weight)
25   temp.S[is.nan(temp.S)] = 0
26
27   entropy.weight[row.num,] = (temp.S)/sum(temp.S)
28
29   #ranks vs weight comparison
30   entropy.order[row.num, order.index[row.num,]] = entropy.weight[
     row.num,]
31
32   #determine S
33   S[row.num,1] = -sum(temp.S)
34   if (row.num > 1){S[row.num,2] = S[row.num,1] - S[row.num-1,1]}
35
36 }
37
38 rm(temp.weight)
39 rm(temp.S)
40
41
42 #*****
43 #Calculate portfolio returns
44 #*****
45 entropy.returns = array(0, dim =c(nrow(tot.returns), 2))
46 entropy.returns[1,2] = 1
47
48 entropy.div = array(0, dim =c(nrow(tot.returns), 2))
49 entropy.div[1,2] = 1
50
51 EXGRATELAG = 12
52 ex.grate = array(0, dim =c(nrow(tot.returns), 2))
53
54 for (k in 2:nrow(index.returns)){
55
56   #total returns
57   temp.weights = as.matrix(entropy.weight[k-1,])
58   temp.returns = as.matrix(tot.returns[k,])
59   temp.returns[is.na(temp.returns)] = 0
60   temp.returns[is.infinite(temp.returns)] = 0
61
62   entropy.returns[k,1] = t(temp.returns) %*% temp.weights

```

```

63 entropy.returns[k,2] = (1+entropy.returns[k,1]) * entropy.
    returns[k-1,2]
64
65 #dividend return
66 temp.weights = as.matrix(entropy.weight[k-1,])
67 temp.returns = as.matrix(oth.returns[k,])
68 temp.returns[is.na(temp.returns)] = 0
69 temp.returns[is.infinite(temp.returns)] = 0
70
71 entropy.div[k,1] = t(temp.returns) %*% temp.weights
72 entropy.div[k,2] = (1+entropy.div[k,1]) * entropy.div[k-1,2]
73
74 var.temp = array(0,dim=c(1,ncol(entropy.weight)))
75 covar.temp = array(0,dim=c(ncol(entropy.weight),ncol(entropy.
    weight)))
76
77 temp.index.weights = as.matrix(index.weight[k-1,])
78
79 #determine drift
80 if(k > EXGRATELAG) {
81
82   #determine weight variance
83   var.temp = t(as.matrix(apply(tot.returns[(k-EXGRATELAG+1):k
    ],2,function(x) var(x,na.rm=TRUE))))
84   covar.temp = cov(tot.returns[(k-EXGRATELAG+1):k], use="
    everything")
85
86   covar.temp[is.na(covar.temp)] = 0
87   var.temp[is.na(var.temp)] = 0
88
89   simple = var.temp %*% temp.index.weights
90   port.covar = t(temp.index.weights) %*% covar.temp %*% temp.
    index.weights
91
92   ex.grate[k,1] = 0.5*(simple-port.covar)/S[k,1]
93
94   if(k > EXGRATELAG + 1){ex.grate[k,2] = ex.grate[k,1] - ex.
    grate[k-1,1]}
95 }
96
97 }
98 rm(temp.returns)
99 rm(temp.weights)
100
101 #total relative outperformance
102 out.perf = S[,2] + ex.grate[,2] + (entropy.div[,1] - index.div
    [,1])
103 mon.outperf = entropy.returns[,1] - index.returns[,1]
104
105 #get average order weights
106 entropy.avgOrder = t(as.matrix(colMeans(entropy.order)))
107

```



```

108 #output into excel file
109 wb.path = paste0(outPath,"entropy weighted portfolio.xlsx")
110 if (file.exists(wb.path)){file.remove(wb.path)}
111
112 wb = loadWorkbook(wb.path,create=TRUE)
113
114 #output returns and ordered weights
115 createSheet(wb,"Returns")
116 writeWorksheet(wb,entropy.returns,"Returns",startRow = 1, startCol
    = 1)
117 writeWorksheet(wb,index.returns,"Returns",startRow = 1, startCol =
    5)
118
119 createSheet(wb,"Market portfolio - Order")
120 writeWorksheet(wb,mktcap.avgOrder,sheet = "Market portfolio -
    order")
121
122 createSheet(wb,"entropy portfolio - Order")
123 writeWorksheet(wb,entropy.avgOrder,sheet = "entropy portfolio -
    order")
124
125 #output components of return
126 createSheet(wb,"Components of return")
127 writeWorksheet(wb,"S","Components of return",startRow = 1,startCol
    = 1)
128 writeWorksheet(wb,S,"Components of return",startRow = 1,startCol =
    2)
129 writeWorksheet(wb,"Excess growth rate","Components of return",
    startRow=1,startCol=5)
130 writeWorksheet(wb,ex.grate,"Components of return",startRow=1,
    startCol=6)
131 writeWorksheet(wb,"Dividends","Components of return",startRow=1,
    startCol=9)
132 writeWorksheet(wb,entropy.div,"Components of return",startRow=1,
    startCol=10)
133 writeWorksheet(wb,index.div,"Components of return",startRow=1,
    startCol=13)
134
135 saveWorkbook(wb)

```

A.9 Section 5.4 diversity weighted portfolio

This code generates the data for the diversity weighted portfolio in Section 5.4.

```

1 # *****
2 #           Forms the Dp weighted portfolio
3 # *****
4
5 #p=1
6

```

```

7 #*****
8 #Calculate weights in Dp portfolio
9 #*****
10 Dp.weight = array(0,dim=dim(mkt.cap))
11 colnames(Dp.weight) = colnames(index.weight)
12
13 S = array(0,dim=c(nrow(mkt.cap),2))
14 colnames(S) = c("S(t)","Change")
15
16 Dp.order = array(NA,dim=dim(mkt.cap))
17 colnames(Dp.order) = seq(1,ncol(mkt.cap),1)
18
19
20 for (row.num in 1:nrow(index.mktcap)){
21
22   #determine Dp weights
23   temp.weight = t(as.matrix(index.weight[row.num,]))
24
25
26   temp.Dp = (temp.weight)^(p)
27   temp.Dp[is.nan(temp.Dp)] = 0
28   temp.Dp = sum(temp.Dp)^(1/p)
29
30   Dp.weight[row.num,] = (temp.weight^p)/(temp.Dp^p)
31
32   #ranks vs weight comparison
33   Dp.order[row.num,order.index[row.num,]] = Dp.weight[row.num,]
34
35   #determine S
36   S[row.num,1] = temp.Dp
37   if (row.num > 1){S[row.num,2] = S[row.num,1] - S[row.num-1,1]}
38
39 }
40
41 rm(temp.weight)
42 rm(temp.Dp)
43
44
45 #*****
46 #Calculate portfolio returns
47 #*****
48 Dp.returns = array(0,dim =c(nrow(tot.returns),2))
49 Dp.returns[1,2] = 1
50
51 Dp.div = array(0,dim =c(nrow(tot.returns),2))
52 Dp.div[1,2] = 1
53
54 EXGRATELAG = 12
55 ex.grate = array(0,dim =c(nrow(tot.returns),2))
56
57 for (k in 2:nrow(index.returns)){
58

```

```

59 #total returns
60 temp.weights = as.matrix(Dp.weight[k-1,])
61 temp.returns = as.matrix(tot.returns[k,])
62 temp.returns[is.na(temp.returns)] = 0
63 temp.returns[is.infinite(temp.returns)] = 0
64
65 Dp.returns[k,1] = t(temp.returns) %*% temp.weights
66 Dp.returns[k,2] = (1+Dp.returns[k,1]) * Dp.returns[k-1,2]
67
68 #dividend return
69 temp.weights = as.matrix(Dp.weight[k-1,])
70 temp.returns = as.matrix(oth.returns[k,])
71 temp.returns[is.na(temp.returns)] = 0
72 temp.returns[is.infinite(temp.returns)] = 0
73
74 Dp.div[k,1] = t(temp.returns) %*% temp.weights
75 Dp.div[k,2] = (1+Dp.div[k,1]) * Dp.div[k-1,2]
76
77 var.temp = array(0,dim=c(1,ncol(Dp.weight)))
78 covar.temp = array(0,dim=c(ncol(Dp.weight),ncol(Dp.weight)))
79
80 temp.index.weights = as.matrix(index.weight[k-1,])
81
82 #determine drift
83 if(k > EXGRATELAG){
84
85   #determine weight variance
86   var.temp = t(as.matrix(apply(tot.returns[(k-EXGRATELAG+1):k
87   ],2,function(x) var(x,na.rm=TRUE))))
88   covar.temp = cov(tot.returns[(k-EXGRATELAG+1):k], use="
89   everything")
90
91   covar.temp[is.na(covar.temp)] = 0
92   var.temp[is.na(var.temp)] = 0
93
94   simple = var.temp %*% temp.index.weights
95   port.covar = t(temp.index.weights) %*% covar.temp %*% temp.
96   index.weights
97
98   ex.grate[k,1] = 0.5*(simple-port.covar)*(1-p)
99
100   if(k > EXGRATELAG + 1){ex.grate[k,2] = ex.grate[k,1] - ex.
101   grate[k-1,1]}
102 }
103 }
104 rm(temp.returns)
105 rm(temp.weights)
106
107 #total relative outperformance
108 out.perf = S[,2] + ex.grate[,2] + (Dp.div[,1] - index.div[,1])
109 mon.outperf = Dp.returns[,1] - index.returns[,1]

```

```

107
108 #get average order weights
109 Dp.avgOrder = t(as.matrix(colMeans(Dp.order)))
110
111 #output into excel file
112 dp.name = paste0("Dp-output\\Dp - ",p)
113 dp.name = paste0(dp.name," - weighted portfolio.xlsx")
114 wb.path = paste0(outPath,dp.name)
115 if(file.exists(wb.path)){file.remove(wb.path)}
116
117 wb = loadWorkbook(wb.path,create=TRUE)
118
119 #output returns and ordered weights
120 createSheet(wb,"Returns")
121 writeWorksheet(wb,Dp.returns,"Returns",startRow = 1, startCol = 1)
122 writeWorksheet(wb,index.returns,"Returns",startRow = 1, startCol =
    5)
123
124 createSheet(wb,"Market portfolio - Order")
125 writeWorksheet(wb,mktcap.avgOrder,sheet = "Market portfolio -
    order")
126
127 createSheet(wb,"Dp portfolio - Order")
128 writeWorksheet(wb,Dp.avgOrder,sheet = "Dp portfolio - order")
129
130 #output components of return
131 createSheet(wb,"Components of return")
132 writeWorksheet(wb,"S","Components of return",startRow = 1,startCol
    = 1)
133 writeWorksheet(wb,S,"Components of return",startRow = 1,startCol =
    2)
134 writeWorksheet(wb,"Excess growth rate","Components of return",
    startRow=1,startCol=5)
135 writeWorksheet(wb,ex.grate,"Components of return",startRow=1,
    startCol=6)
136 writeWorksheet(wb,"Dividends","Components of return",startRow=1,
    startCol=9)
137 writeWorksheet(wb,Dp.div,"Components of return",startRow=1,
    startCol=10)
138 writeWorksheet(wb,index.div,"Components of return",startRow=1,
    startCol=13)
139
140 saveWorkbook(wb)
141
142 #*****
143 # Provides loop for Dp portfolio
144 #*****
145
146 stepSize = 0.05
147
148 loopVec = seq(stepSize,1,by=stepSize)
149

```

```
150 for (k in loopVec){  
151   p=k  
152   source(paste0(sourcePath,"SPT - Dp weighted portfolio.R"))  
153   print(paste0(p," completed."))  
154 }
```

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