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A UNIQUENESS THEOREM IN FLUID TURBULENCE

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ABSTRACT

Only the linear viscous fluid is considered herein along with the Reynolds decomposition. A uniqueness theorem is presented for the mean motion equations which shows that the Reynolds tensor is not uniquely defined. The mean pressure field is also not unique. Some implications of this non–uniqueness for the construction of turbulence models are discussed. In particular, the non–uniqueness allows a gauge field to be introduced. One such field is Beltrami.

BACKGROUND

Practical engineering computations of turbulent flows are based upon the mean motion equations and a turbulence model. As pointed out in [1] there is no reason to assume that the mean motion equations, along with a turbulence model, comprise a system of equations with nice mathematical properties. In particular, does such a system possess a solution? The present study sets out some of the consequences of the Reynolds decomposition in terms of the non–uniqueness inherent in the field equations. The non-uniqueness present in the mean motion equations revolves around the construction of a gauge field and some properties of that field are explored. The main constraint on the gauge field being its divergence free nature.

It is well known that there are no complete uniqueness, existence and regularity results for the instantaneous Navier–Stokes equations. The papers contained in the four volumes of [2] summarize many properties of these equations. However, the properties of the Reynolds averaged Navier–Stokes equations have been less well studied. It can be noted that the large eddy simulation technique (see [3]) has been studied in more detail and some of those results and methods apply herein (but the large eddy simulation method, *per se*, is not the subject of current interest).

The construction presented below speaks to the concept of a turbulence model since, if the Reynolds tensor is not uniquely defined, there is more flexibility in the development of a turbulence model. This additional flexibility suggests a mechanism for modeling turbulence control. The process for generating turbulence control is illustrated, in principle, for the Boussinesq turbulence model. The mechanical systems required for the implementation of flow control are not discussed.

An application is examined which discusses the question: can a standard turbulence model be modified to account for the effects of turbulence control?

NOMENCLATURE

$\mathbf{x} \in \mathbb{R}^3$	coordinate in space of simultaneity.
$t \in \mathbb{R}$	time and time axis and reals, $\mathbb{R}$.
d/dt	material derivative.
$\mathbf{v} \in \mathbb{R}^3$	instantaneous velocity field.
$\mathbf{V} \in \mathbb{R}^3$	mean velocity field.
$\mathbf{V}_T \in \mathbb{R}^3$	constant boost velocity in $\mathbb{G}_a$.
$\mathbf{u} \in \mathbb{R}^3$	fluctuating velocity field.
$P \in \mathbb{R}$	pressure normalized by density.
$\overline{P},\ P' \in \mathbb{R}$	mean and fluctuating pressure.
$Q \in \mathbb{R}$	gauge pressure.
$\mathcal{D} \subset \mathbb{R}^3$	spatial domain.
$\langle \mathbf{a}, \mathbf{b} \rangle = a_i b_i \in \mathbb{R}$	vector inner product.
$\langle \mathbf{a}, \mathbf{b} \rangle_g \equiv \int_{\mathcal{D}} \langle \mathbf{a}, \mathbf{b} \rangle dV$	global inner product.
$\langle\!\langle \mathbf{A}, \mathbf{B} \rangle\!\rangle = trace(\mathbf{A}\mathbf{B}^T)$	tensor inner product.
$\lVert \mathbf{A} \rVert^2 = \langle\!\langle \mathbf{A}, \mathbf{A} \rangle\!\rangle$	tensor norm.
$\mathbf{R} = \mathbf{v} \otimes \mathbf{v}$	instantaneous kinetic tensor.
$\overline{\mathbf{R}} = \mathbf{V} \otimes \mathbf{V}$	mean kinetic tensor.
$\mathbf{w}$	gauge velocity.
$\mathbf{G}$	gauge tensor.
$\lvert \mathbf{a} \rvert^2 \equiv \langle \mathbf{a}, \mathbf{a} \rangle = a_i a_i \in \mathbb{R}$	vector norm.
$\lvert \mathbf{a} \rvert_g^2 = \int_{\mathcal{D}} \langle \mathbf{a}, \mathbf{a} \rangle dV$	global norm.
$\boldsymbol{\mathcal{R}} = \mathcal{E}(\mathbf{u} \otimes \mathbf{u})$	The Reynolds tensor.
$\mathcal{E}$	a mean value operator.
$\mathbb{W}_e = \{(\mathbf{x}, t)\}$	classical space–time.
$\mathbb{L}_n$	set of linear operators on $\mathbb{R}^n$.
$\mathbb{O}_n$	group of orthogonal transformations on $\mathbb{R}^n$.

$\mathbf{L} = \boldsymbol{\nabla}(\mathbf{v}) \in \mathbb{L}_3$	velocity gradient.
$\mathbf{D}$ and $\mathbf{W}$	symmetric and skew parts of $\mathbf{L}$.
$\mathbf{f} \in \mathbb{R}^3$	body force.
$\mathbf{Q} \in \mathbb{SO}_3$ or $\mathbb{O}_3$	rotation operator.
Λ	an arbitrary inertial frames.
$\mathbb{G}_a$	Galilean group on space–time.

REYNOLDS DECOMPOSITION

The simple uniqueness theorem discussed in [4] can be extended to give some information about the uniqueness of the mean motion equations used in turbulence computations. This theorem in [4] assumes regularity in the sense that the instantaneous velocity gradient tensor, $\mathbf{L} = \boldsymbol{\nabla}(\mathbf{v})$, must have bounded eigenvalues. This same constraint will be found below but now for the mean motion velocity gradient, $\overline{\mathbf{L}} = \boldsymbol{\nabla}(\mathbf{V})$. At this point in the development, nothing need be said about the Reynolds tensor $\boldsymbol{\mathcal{R}} = \mathcal{E}(\mathbf{u} \otimes \mathbf{u})$ derived from the fluctuating velocity $\mathbf{u}(\mathbf{x}, t)$. It is important to note that no turbulence model is required in the uniqueness theorem. The theorem is simply a property of the mean motion equations which involves both the mean velocity vector and properties of the velocity fluctuations. From a cursory inspection, it would appear that it is the simple fact of introducing a Reynolds decomposition that induces non–uniqueness. A slightly deeper consideration, however, shows that there is another interpretation which leads directly to the introduction of a gauge field (the properties of which are considered in more detail in [5]). The uniqueness theorem for the mean motion equations would appear to be more general than the corresponding result for the instantaneous equations. This is not, however, the case as will emerge below. The uniqueness theorem demands that the mean velocity field be unique but that non–uniqueness may be admitted in the Reynolds tensor and in the mean pressure field (as it is for the instantaneous equations).

Write the instantaneous field equations in an arbitrary inertial frame Λ on $\mathbb{W}_e$. Let $(\mathbf{x}, t)$ be the associated spatial coordinate and time respectively. Then the instantaneous equations of motion take the form:

$$\frac{\partial v_i}{\partial x_i} = 0; \ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i \quad (1a, b)$$

if $\mathbf{f} = \{f_i\}$ denotes a body force per unit mass. Equation (1a) implies that the term $v_j\, \partial v_i / \partial x_j \equiv \partial R_{ij}/\partial x_j$ if $R_{ij} = v_i v_j$ is the kinetic tensor in equation (1b). Equations (1a,b) represents a boundary value problem over the bounded fluid domain, $\mathcal{D}$, of interest; the specific form of this boundary, and the conditions imposed thereon, are not of concern when energy estimates are the main interest. The energy estimate for equation (1b) provides the inequality (as shown in Foias, et al. [6], for example):

$$\frac{\partial}{\partial t} |\mathbf{v}|_g^2 + \nu L_v^2 |\mathbf{v}|_g^2 \leq |\mathbf{f}|_g^2 / \nu L_v^2 \qquad (1c)$$

for a *periodic* domain $\mathcal{D}$. Here $|\mathbf{v}|_g^2 = \int_{\mathcal{D}} \langle \mathbf{v}, \mathbf{v} \rangle dV$ is the global norm of the instantaneous velocity field. This differential inequality can be integrated via the Gronwall lemma (for which see details in [7]) to give the inequality:

$$|\mathbf{v}|_g^2(t) \leq E\, |\mathbf{v}|_g^2(0) + |\mathbf{f}|_g^2 \left[1 - E\right] / (\nu L_v^2)^2 \qquad (1d)$$

where $E = exp[-\nu L_v^2 t]$ with $E \to 0$ as $t \to \infty$. Hence the long time limit in equation (1d) gives the estimate:

$$|\mathbf{v}|_g^2(t) \leq |\mathbf{f}|_g^2 / (\nu L_v^2)^2 \quad \text{as } t \to \infty \qquad (1e)$$

Any body force, $\mathbf{f}(\mathbf{x})$, present in the equations of motion is assumed to be independent of time. It is also assumed, of course, that $\mathbf{f}(\mathbf{x})$ is a bounded function over space. Without a body force being present, there is an exponential decay of the velocity norm:

$$|\mathbf{v}|_g^2(t) \leq exp[-\nu L_v^2 t]\, |\mathbf{v}|_g^2(t = 0)$$

as $t \to \infty$ and reflects the viscous dissipation due to the diffusion term $\nu \nabla^2(\mathbf{v})$. It can be noted that the energy estimate for the velocity field is independent of the pressure field. This finding is directly due to the imposition of periodic boundary conditions. The artificial assumption of periodic boundary conditions removes any real processes at the physical boundaries. In equation (1c) L_v denotes the constant in a Poincaré inequality of the form $\|\mathbf{L}\|_g^2 \geq L_v^2 |\mathbf{v}|_g^2$. The above discussion holds for the instantaneous Navier Stokes equations while the equivalent result for the mean velocity field is given below. Equation (1c) and its integral, equation (1d), apply to both laminar and turbulent flow.

The norm estimate is a global result that only predicts the time evolution of a norm ($|\mathbf{v}|_g$ in this case). No local information is obtained from these estimates so that all the details of any turbulence that may be present in the flow are lost.

Equation (1d) only provides an *upper bound* to the norm $|\mathbf{v}|_g^2$ and does not imply that laminar and turbulent flows have the same norm decay rate. As noted above, the local flow structure is lost in the definition of the global norm. Predictions that are limited to upper bound estimates, only, makes the comparison with similar estimates for turbulence models somewhat uncertain. At best there is the inequality:

$$|\mathbf{v}|_g^2 \leq |\mathbf{V}|_g^2 + |\mathbf{u}|_g^2$$

but no norm estimates are made herein for the velocity fluctuations $\mathbf{u}$.

The set of equations (1a,b) are covariant under the Galilean group with $\mathbf{v} \mapsto \mathbf{Q}\mathbf{v}$ and $P \mapsto P$. The kinetic tensor, $\mathbf{R} = \{v_i v_j\}$, then transforms under $\mathbb{G}_a$ as $\mathbf{R} \mapsto \mathbf{Q}\mathbf{R}\mathbf{Q}^T$. Any decomposition of equations (1a,b) must respect this covariance.

Introduce the Reynolds decomposition in the standard form:

$$\mathbf{v} \mapsto \mathbf{V} + \mathbf{u}; \quad \mathbf{V} = \mathcal{E}(\mathbf{v}) \text{ and } \mathcal{E}(\mathbf{u}) = 0 \qquad (2)$$

if $\mathbf{u}(\mathbf{x}, t)$ denotes the fluctuating velocity field and $\mathbf{V}(\mathbf{x}, t)$ is the mean velocity field. Then equation (2) implies that the instantaneous kinetic tensor decomposes as $\mathbf{R} \mapsto \overline{\mathbf{R}} + \boldsymbol{\mathcal{R}}$ if $\overline{\mathbf{R}} = \mathbf{V} \otimes \mathbf{V}$ is the mean kinetic tensor. In the equation (2), $\mathcal{E}$ represents some mean value operator whose specific form is not required herein and $\boldsymbol{\mathcal{R}} = \mathcal{E}(\mathbf{u} \otimes \mathbf{u})$ defines the Reynolds tensor with $trace(\boldsymbol{\mathcal{R}}) = \mathcal{E}(\langle \mathbf{u}, \mathbf{u} \rangle) = 2k$ if k is the turbulence kinetic energy. The mean velocity was allowed to be a function of time in the present formulation (as discussed in [8]). The mean motion equations follow from equations (1a,b) when equation (2) is introduced. This gives:

$$\frac{\partial V_i}{\partial x_i} = 0 \qquad (3a)$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\overline{R}_{ij} + \mathcal{R}_{ij}\right] + \frac{\partial \overline{P}}{\partial x_i} = \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} + f_i \qquad (3b)$$

and the lack of closure is evident in that no equations exist to define the components, $\mathcal{R}_{ij}$, of the tensor $\boldsymbol{\mathcal{R}}$. It is again assumed that the body force, $\mathbf{f}(\mathbf{x})$, is independent of time. Here, $\overline{P}$ denotes the mean pressure as a scalar field over domain $\mathcal{D}$. Introduce a statement of the form:

$$\mathbf{M}(\mathbf{V}, \boldsymbol{\mathcal{R}}, \boldsymbol{\alpha}) = \mathbf{0} \qquad (4)$$

as a *turbulence model* which is designed to close the set of equations (3a,b). The vector $\boldsymbol{\alpha}$ in the definition (4) represents a set of *model constants*. It was the properties of system (3a,b) and (4) that occupied [1]. In particular energy estimates were presented for the familiar $k \sim \epsilon$ turbulence model considered as a special case of equation (4).

In order to be consistent with Newtonian mechanics, equation (4) must be covariant under the Galilean group where $\mathbf{u} \mapsto \mathbf{Q}\mathbf{u}$ and so:

$$\boldsymbol{\mathcal{R}} \mapsto \mathbf{Q}\boldsymbol{\mathcal{R}}\mathbf{Q}^T \qquad \text{while} \qquad \mathbf{V} \mapsto \mathbf{Q}\mathbf{V} + \mathbf{V}_T$$

model constants in the vector $\boldsymbol{\alpha}$ should be independent of the coordinate frame.

From the epistemological point of view, the decomposition in equation (2) is unsatisfactory since the equations (3a,b) are not closed. That is, additional information must be provided before the Reynolds decomposition can be meaningful. This information is no longer part of basic fluid mechanics (which led to equations (1a,b) — which are closed) and requires knowledge of the velocity fluctuations. The definition $\mathbf{v} = \mathbf{V} + \mathbf{u}$ does not provide this information and only detailed experimental study can repair the epistemological deficit. It is, of course, this lack of information that necessitates the introduction of the turbulence model in equation (4). $\mathbf{u}(\mathbf{x}, t)$ contains an infinite amount of information, which can never be recovered from the turbulence model in equation (4).

THE THEOREM

First introduce a lemma which will be required in the theorem. That is (from [4]):

Lemma *If* $div(\mathbf{h}) = 0$ *and if* $\mathbf{h}$ *is a smooth vector field over the domain* $\mathcal{D}$*, and if either* $\mathbf{h} \equiv \mathbf{0}$ *or* $\xi \equiv 0$ *on the boundary* $\partial\mathcal{D}$ *(with* ξ *a smooth field over* $\mathcal{D}$*). Then:*

$$\int_{\mathcal{D}} \langle \mathbf{h}, \boldsymbol{\nabla}(\xi) \rangle dV = 0$$

PROOF: From the identity:

$$div(\xi\, \mathbf{h}) = \xi\, div(\mathbf{h}) + \langle \mathbf{h}, \boldsymbol{\nabla}(\xi) \rangle \equiv \langle \mathbf{h}, \boldsymbol{\nabla}(\xi) \rangle$$

as $div(\mathbf{H}) \equiv 0$. Hence:

$$\int_{\mathcal{D}} \langle \mathbf{h}, \boldsymbol{\nabla}(\xi) \rangle dV = \int_{\partial\mathcal{D}} \langle \xi\, \mathbf{h}, \mathbf{n} \rangle dA \equiv 0$$

by Gauss and the given boundary conditions. ∎

With this lemma as background, the uniqueness theorem for the constant density mean motion equations for the linear viscous fluid can be given. Specifically:

Theorem *Let* $(\mathbf{V}_1, \boldsymbol{\mathcal{R}}_{\mathbf{1}}, \overline{P}_1)$ *and* $(\mathbf{V}_2, \boldsymbol{\mathcal{R}}_{\mathbf{2}}, \overline{P}_2)$ *be two solutions of the same turbulent flow problem (with the same boundary conditions and body force). Then the velocity field is unique:* $\mathbf{V}_1 = \mathbf{V}_2$*. This condition holds provided that:*

$$\boldsymbol{\mathcal{R}}_{\mathbf{1}} + \overline{P}_1\mathbf{I} = \boldsymbol{\mathcal{R}}_{\mathbf{2}} + \overline{P}_2\mathbf{I} + \mathbf{A}(\mathbf{x}, t)$$

with the constraint that: $div(\mathbf{A}) \equiv \mathbf{0}$.

PROOF: The proof follows directly from that given in [4]. Consider the follows:

1). Place:

a). $$\mathbf{U} = \mathbf{V}_1 - \mathbf{V}_2$$

b). $$\mathbf{A} = (\boldsymbol{\mathcal{R}}_{\mathbf{1}} + \overline{P}_1\mathbf{I}) - (\boldsymbol{\mathcal{R}}_{\mathbf{2}} + \overline{P}_2\mathbf{I})$$

Then, from the boundary conditions it follows that $\mathbf{U} \equiv \mathbf{0}$ on the boundary, $\partial\mathcal{D}$, of the domain $\mathcal{D}$. In addition $div(\mathbf{U}) \equiv 0$ over the whole of $\mathcal{D}$ as a constraint on the velocity difference field $\mathbf{U}$.

Now write the mean motion equations for both $(\mathbf{V}_1, \boldsymbol{\mathcal{R}}_{\mathbf{1}}, \overline{P}_1)$ and $(\mathbf{V}_2, \boldsymbol{\mathcal{R}}_{\mathbf{2}}, \overline{P}_2)$ and subtract to give an equation in the $\mathbf{U}$ field:

$$\partial\mathbf{U}/\partial t + \boldsymbol{\nabla}(\mathbf{V}_1)\, \mathbf{V}_1 - \boldsymbol{\nabla}(\mathbf{V}_2)\, \mathbf{V}_2 = \nu\nabla^2(\mathbf{U}) - div(\mathbf{A})$$

which can be re–written as:

$$\partial \mathbf{U}/\partial t + \boldsymbol{\nabla}(\mathbf{U})\,\mathbf{V}_1 + \boldsymbol{\nabla}(\mathbf{V}_2)\,\mathbf{U} = \nu\nabla^2(\mathbf{U}) - div(\mathbf{A}) \tag{5}$$

since the equality:

$$\boldsymbol{\nabla}(\mathbf{V}_1)\mathbf{V}_1 \equiv \boldsymbol{\nabla}(\mathbf{U})\,\mathbf{V}_1 + \boldsymbol{\nabla}(\mathbf{V}_2)\,\mathbf{V}_1.$$

holds true.

2). Take the inner product of equation (5) with the vector $\mathbf{U}$ to obtain the equality:

$$\langle\partial\mathbf{U}/\partial t, \mathbf{U}\rangle + \langle\boldsymbol{\nabla}(\mathbf{U})\mathbf{V}_1, \mathbf{U}\rangle + \langle\boldsymbol{\nabla}(\mathbf{V}_2)\mathbf{U}, \mathbf{U}\rangle = \nu\langle\nabla^2\mathbf{U}, \mathbf{U}\rangle - \langle div(\mathbf{A}), \mathbf{U}\rangle \tag{A}$$

Then, as the following three equalities can be verified directly:

$$\langle\mathbf{U}, \nabla^2\mathbf{U}\rangle + trace[\boldsymbol{\nabla}(\mathbf{U})(\boldsymbol{\nabla}(\mathbf{U}))^T] = div[\boldsymbol{\nabla}(\mathbf{U})^T\,\mathbf{U}]$$

$$\langle\mathbf{U}, \boldsymbol{\nabla}(\mathbf{U})\mathbf{V}_1\rangle = \langle\mathbf{V}_1, \boldsymbol{\nabla}(U^2)/2\rangle$$

$$\langle\mathbf{U}, \boldsymbol{\nabla}(\mathbf{V}_2)\mathbf{U}\rangle = \langle\mathbf{U}, \overline{\mathbf{D}}_2(\mathbf{U})\rangle$$

where $\overline{\mathbf{D}}_2 \in \mathbb{S}^3_{sym}$ is the symmetric part of the $\mathbf{V}_2$ velocity gradient tensor, $\partial\mathbf{V}_2/\partial\mathbf{x}$. Equation (A) now becomes (if $U^2 = \langle\mathbf{U}, \mathbf{U}\rangle$):

$$\begin{aligned}\partial(U^2/2)/\partial t + \langle\mathbf{V}_1, \boldsymbol{\nabla}(U^2/2)\rangle + \langle\mathbf{U}, \overline{\mathbf{D}}_2(\mathbf{U})\rangle &= \nu\, div[(\boldsymbol{\nabla}(\mathbf{U}))^T\,\mathbf{U}] - \langle\mathbf{U}, div(\mathbf{A})\rangle \\ &\quad - \nu\, trace[\boldsymbol{\nabla}(\mathbf{U})(\boldsymbol{\nabla}(\mathbf{U}))^T]\end{aligned} \tag{B}$$

which is an exact result. At this point, it is convenient to establish an inequality by removing terms that are known to be non–negative. The process is started by passing to the norm. That is define the global norm of the velocity difference $\mathbf{U}(\mathbf{x}, t)$:

$$\|\mathbf{U}\|^2_g(t) \equiv \textstyle\int_{\mathcal{D}_t} U^2(t)\,dV$$

3). Next, form an equation in norm by integrating equation (B) over the domain $\mathcal{D}$. Use the above lemma to show that the integral:

$$\textstyle\int_{\mathcal{D}} \langle\mathbf{V}_1, \boldsymbol{\nabla}(U^2/2)\rangle dV \equiv 0$$

However: $\int_{\mathcal{D}} \langle\mathbf{U}, div(\mathbf{A})\rangle dV = 0$ only if $div(\mathbf{A}) \equiv \mathbf{0}$ (the condition $div(\mathbf{A}) \perp \mathbf{U}$ at all points in $\mathcal{D}$ is not meaningful).

At the same time:

$$\textstyle\int_{\mathcal{D}} div[(\boldsymbol{\nabla}(\mathbf{U}))^T\,\mathbf{U}]dV \equiv 0$$

by Gauss and the boundary conditions. Finally, as:

$$\|\boldsymbol{\nabla}(\mathbf{U}\|^2_g \equiv \textstyle\int_{\mathcal{D}} trace[\boldsymbol{\nabla}(\mathbf{U})(\boldsymbol{\nabla}(\mathbf{U}))^T]dV \geq 0$$

it follows that the integrated form of equation (B) reduces to:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{U}\|^2_g + \int_{\mathcal{D}} \langle\mathbf{U}, \overline{\mathbf{D}}_2\mathbf{U}\rangle\, dV \leq 0 \tag{C}$$

which is still exact (up to inequality).

4). Using the estimate from the inequality given below (with $\lambda_{min} = -\gamma;\ \gamma > 0$):

$$\langle\mathbf{U}, \overline{\mathbf{D}}_2\mathbf{U}\rangle \geq -\gamma\, U^2 \geq -\gamma\, U^2/2$$

there is from equation (C):

$$\frac{d}{dt}\|\mathbf{U}\|^2_g - \gamma\int_{\mathcal{D}} U^2\, dV \equiv \frac{d}{dt}\|\mathbf{U}\|^2_g - \gamma\|\mathbf{U}\|^2_g \leq 0$$

which integrates to give:

$$\|\mathbf{U}\|^2_g(t) \leq \|\mathbf{U}\|^2_g(t=0)\, exp[\lambda_{min} t]$$

and so must vanish by the initial data ($\mathbf{V}_1 = \mathbf{V}_2 \Rightarrow \mathbf{U} = \mathbf{0}$ at $t = 0$). Hence $\mathbf{U}(\mathbf{x}, t) = \mathbf{0}$ for all $\mathbf{x}$ and all t. Hence:

$$\mathbf{V}_1(\mathbf{x}, t) \equiv \mathbf{V}_2(\mathbf{x}, t) \quad \text{for all } \mathbf{x} \text{ and all } t$$

That is, the mean velocity field, $\mathbf{V}$, is a unique field over space for all time.

5). It follows from the above proof that the constraint $div(\mathbf{A}) \equiv \mathbf{0}$ implies that $\mathbf{A}$ is any symmetric second order tensor that has zero divergence. Hence the Reynolds tensor, $\boldsymbol{\mathcal{R}}$, is not required to be uniquely determined nor is the mean pressure, $\overline{P}$. The latter condition is consistent with the result for the instantaneous Navier Stokes equations given in [4]. ∎

It is only $div(\boldsymbol{\mathcal{R}})$ that is significant in the computation of turbulent flows and the gauge field term $\mathbf{A}$ with $div(\mathbf{A}) \equiv \mathbf{0}$ does not enter into any computations. Two turbulence models for $\boldsymbol{\mathcal{R}}$ that differ by a divergence free symmetric second order tensor give the same velocity field.

A simple uniqueness result has been obtained for the mean motion as an extension of that for the instantaneous motion. Here it has to be assumed that the second order tensor $\mathbf{A}$ is arbitrary up to the constraint $div(\mathbf{A}) = \mathbf{0}$. An alternative statement of this constraint is the need for $trace(\overline{\mathbf{L}}\boldsymbol{\mathcal{R}}) \equiv 0$. In this sense, the result is less satisfactory than the result given in [4]. However, some insight has been obtained into the structure of the Reynolds decomposition. Both results depend upon a regularity assumption specified in the following inequality (see [4]).

An Inequality *Let* $\mathbf{D} \in \mathbb{S}^3_{sym}$, *and* $trace(\mathbf{D}) \equiv 0$, *then:*

$$\langle\mathbf{z}, \mathbf{D}\mathbf{z}\rangle \geq -\gamma\,\langle\mathbf{z}, \mathbf{z}\rangle$$

where $-\gamma$ *(with* $\gamma > 0$*) is the smallest eigenvalue,* λ_{min}, *of* $\mathbf{D}$.

PROOF: Since $\mathbf{D} \in \mathbb{S}^3_{sym}$, it follows that a similarity transformation, $\mathbf{Q}\mathbf{B}\mathbf{Q}^T = \mathbf{D}$ places $\mathbf{D}$ in diagonal form $\mathbf{B}$. So for any bounded vector $\mathbf{z} \in \mathbb{R}^3$:

$$\begin{aligned}\langle\mathbf{z}, \mathbf{D}\mathbf{z}\rangle = \langle\mathbf{z}, \mathbf{Q}\mathbf{B}\mathbf{Q}^T\mathbf{z}\rangle &= \langle\mathbf{Q}^T\mathbf{z}, \mathbf{B}\mathbf{Q}^T\mathbf{z}\rangle \\ &\equiv \langle\mathbf{v}, \mathbf{B}\mathbf{v}\rangle = \Sigma(b_{ii}v_iv_i)\end{aligned}$$

where $\mathbf{v} = \mathbf{Q}^T\mathbf{z}$ and as $\mathbf{B}$ is diagonal. Hence:

$$\langle \mathbf{z}, \mathbf{Dz} \rangle = -\gamma v_1^2 + (\Gamma_1 - \gamma) v_2^2 + (\Gamma_2 - \gamma) v_3^2$$
$$\equiv -\gamma v_i v_i + \Gamma_1 v_2^2 + \Gamma_2 v_3^2 \geq -\gamma v_i v_i$$
as $-\gamma$ is the smallest eigenvalue of $\mathbf{D}$ (so that the $\Gamma_i > 0$). This inequality is only meaningful if the constraint $|\gamma| < \infty$ is in place. Finally:
$$\langle \mathbf{z}, \mathbf{Dz} \rangle \geq -\gamma \langle \mathbf{v}, \mathbf{v} \rangle = -\gamma \langle \mathbf{z}, \mathbf{z} \rangle$$
as $\mathbf{Q} \in \mathbb{SO}_n$ is an isometry. ∎

The development of the above inequality demands the restriction $|\gamma| < \infty$ and hence that the constraints $|\langle \mathbf{u}, \mathbf{Du} \rangle| < \infty$ and so $\|\mathbf{D}\| < \infty$ also apply (in order for the condition $|\mathbf{u}| < \infty$ to apply). This inequality imposes the regularity condition on the theorem that was mentioned above.

If the condition $\int_{\mathcal{D}} \langle \mathbf{U}, div(\mathbf{A}) \rangle dV = 0$ is not imposed upon the vector $div(\mathbf{A})$ then the theorem does not hold and no meaning can be given to the Reynolds decomposition. Of course, $\int_{\mathcal{D}} \langle \mathbf{U}, div(\mathbf{A}) \rangle dV = 0$ if $\mathbf{A} \equiv \mathbf{O}$ and then there is total uniqueness in that:
$$\boldsymbol{\mathcal{R}}_1 + \overline{P}_1 \mathbf{I} = \boldsymbol{\mathcal{R}}_2 + \overline{P}_2 \mathbf{I}$$
so that if $\overline{P}_1 = \overline{P}_2$ then $\boldsymbol{\mathcal{R}}_1 = \boldsymbol{\mathcal{R}}_2$. The Reynolds tensor is unique if the mean pressure field is unique. The present interest lies in exploring the implications of the restricted uniqueness implied by the condition $div(\mathbf{A}) = \mathbf{0}$.

MEANING OF THE THEOREM

The uniqueness theorem in the form presented above (with $div(\mathbf{A}) = \mathbf{0}$) has a connection to the non–uniqueness inherent in the mass invariance constraint. Start from the fluctuating velocity field $\mathbf{u}(\mathbf{x}, t)$ when it follows that there exists a solenoidal gauge field $\mathbf{w}$ such that $|\mathbf{w}| < \infty$ and:
$$div(\mathbf{u}) = 0 \quad \Rightarrow \quad div(\mathbf{u} + \mathbf{w}) = 0 \quad \textit{iff} \quad div(\mathbf{w}) = 0$$
That is, the fluctuating velocity field $\mathbf{u}$ also associates with a gauge field, $\mathbf{w}(\mathbf{x}, t)$, resulting from the condition $div(\mathbf{u}) = 0$. It is clear that the set $\mathbb{W} = \{\mathbf{w}_i\}$ of all such gauge fields is an additive group. Here all $\mathbf{w}_i$ are finite and solenoidal.

There is no physical meaning attached to the gauge vector field $\mathbf{w}(\mathbf{x}, t)$. Indeed, any divergence free vector field can be adopted for the quantity $\mathbf{w}(\mathbf{x}, t)$. Nothing more about the vector $\mathbf{w}$ need be said at this point.

It is natural to construct the tensor gauge field:
$$\mathbf{G} = 2\mathbf{w} \otimes \mathbf{w} - \langle \mathbf{w}, \mathbf{w} \rangle \mathbf{I} \in \mathbb{L}_3 \tag{6}$$
from the velocity field $\mathbf{w}$. It follows from the definition that:
$$div(\mathbf{G}) = 2\boldsymbol{\zeta} \times \mathbf{w} \quad \text{with} \quad \boldsymbol{\zeta} = curl(\mathbf{w}) \tag{6a}$$
It then follows that the condition $div(\mathbf{G}) \equiv \mathbf{0}$ holds for any Beltrami field $\mathbf{w}$: the constraint $div(\mathbf{G}) = \mathbf{0}$ is not trivial. Further, it can be noted that equation (6) produces the equality $trace(\mathbf{G}) = -\langle \mathbf{w}, \mathbf{w} \rangle$ and is directly related to the kinetic energy of the gauge velocity field. The group $\mathbb{W}$ does not directly induce a group structure for the tensor gauge $\mathbf{G}$. However, the set $\mathbb{G} = \{\mathbf{G}\}$ has its own additive group structure. More details are given in [5].

It is required that the gauge fields be frame indifferent under the Galilean group $\mathbb{G}_a$. That is $\mathbf{w} \mapsto \mathbf{Qw}$ under $\mathbb{G}_a$ where $\mathbf{Q} \in \mathbb{SO}_3$ is a coordinate rotation. Now $div(\mathbf{w})$ is invariant under coordinate rotations. Then $\mathbf{G} \mapsto \mathbf{QGQ}^T$ defines the transformation of the gauge tensor under $\mathbb{G}_a$.

The same situation occurs with the mean motion equations since these equations contain the term $div(\boldsymbol{\mathcal{R}})$ (as in equation (3b)). A gauge field, $\mathbf{G}$, arises naturally such that:
$$div(\boldsymbol{\mathcal{R}}) = \Gamma \quad \Rightarrow \quad div(\boldsymbol{\mathcal{R}} + \mathbf{G}) = \Gamma \quad \textit{iff} \quad div(\mathbf{G}) = 0$$
for some function $\Gamma(\mathbf{x}, t)$ which represents the rest of equation (3b). That is, the non–uniqueness found in the above theorem can be interpreted as a gauge field. In this sense the Reynolds tensor is not unique since only $div(\boldsymbol{\mathcal{R}})$ enters the mean linear momentum equation. The only restriction on the tensor $\mathbf{G}$ is that $div(\mathbf{G})$ is well defined and identically zero. A boundedness condition $|\mathbf{w}| < \infty \Rightarrow \|\mathbf{G}\| < \infty$ is also assumed.

Both $\mathbf{G}$ and $\boldsymbol{\mathcal{R}}$ are symmetric second order tensors so that they both have real eigenvalues. However, the tensor $\mathbf{G}$, being a tensor product, has the spectrum $\sigma(\mathbf{G}) = \{0, 0, \langle \mathbf{w}, \mathbf{w} \rangle\}$ with the corresponding eigenvectors $\mathbf{X}_1, \mathbf{X}_2 \in span\{\mathbf{w}\}^\perp$; $\mathbf{X}_1 \perp \mathbf{X}_2$ and $\mathbf{X}_3 \in span\{\mathbf{w}\}$. There is no such simple structure for the eigensystem of $\boldsymbol{\mathcal{R}}$.

The function $\mathbf{A}(\mathbf{x}, t)$, with $div(\mathbf{A}) = 0$, from the above theorem can be decomposed in two components:
$$\mathbf{A}(\mathbf{x}, t) = \mathbf{G}(\mathbf{x}, t) + Q(\mathbf{x}, t)\mathbf{I} \tag{7}$$
where $Q(\mathbf{x}, t) \in \mathbb{R}$ represents a gauge pressure field and $\mathbf{G}(\mathbf{x}, t) \in \mathbb{L}_3$ a Reynolds stress gauge function. Since $div(\mathbf{A}) \equiv \mathbf{0}$ there must be:
$$div(\mathbf{G}) + \boldsymbol{\nabla}(Q) = \mathbf{0}$$
as a constraint upon the gauge fields. For the specific application herein, that is in equation (5), express the tensor $\mathbf{G}$ in the form specified in equation (6):
$$\mathbf{G} = 2\mathbf{w} \otimes \mathbf{w} - \langle \mathbf{w}, \mathbf{w} \rangle \mathbf{I} \;\equiv\; 2\mathbf{H} - |\mathbf{w}|^2 \mathbf{I}$$
with $\mathbf{H} = \mathbf{w} \otimes \mathbf{w}$ a simple tensor product. In addition, to complete this example, assume that the gauge pressure is a function of time only (as it was for the instantaneous Navier–Stokes equations in [4]). Then $\boldsymbol{\nabla}(Q) = \mathbf{0}$ and equation (7) shows that the equality $div(\mathbf{G}) = \mathbf{0}$ must hold. The result in equation (6a) then applies and the gauge velocity field, $\mathbf{w}(\mathbf{x}, t)$, must

be Beltrami.

There are, of course, alternative forms for the gauge tensor $\mathbf{G}$ that also have a vanishing divergence, $\partial G_{ij}/\partial x_j = 0$, but these will not be discussed herein. Finally, note that:

$\|\mathbf{G}\|_g^2 = \int_{\mathcal{D}} trace(\mathbf{G}\mathbf{G}^T) dV \equiv 3\int_{\mathcal{D}} \langle \mathbf{w}, \mathbf{w}\rangle^2 dV = 3\|\mathbf{H}\|_g^2$ as a function of the gauge velocity $\mathbf{w}$ only.

Example I *A Beltrami Flow*

It is well known that Beltrami flows can give rise to chaotic motion. For example (see [9] for details) the so–called (Arnold–Beltrami–Childress) *ABC* flows have this characteristic for specific values of the coefficients. These flows also have a large measure of periodicity across space since the velocity field is given by:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} A\, sin[x_3] + C\, cos[x_2] \\ B\, sin[x_1] + A\, cos[x_3] \\ C\, sin[x_2] + B\, cox[x_1] \end{pmatrix} \equiv \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}$$

for constants A, B and C. This periodicity is consistent with the periodic boundary conditions assumed in the energy estimates. Certain *ABC* flows are directly integrable (the case $C = 0$, for example, has a solution expressible in terms of Jacobi elliptic functions). △

The main interest in this example is the recognition that Beltrami flows need not be trivial.

THE ENERGY ESTIMATE

In order to study how a turbulence model interacts with the mean motion, it is instructive to construct an energy estimate from equations (3a,b). Introduce the vector norms:

$$|\mathbf{V}|^2 = \langle \mathbf{V}, \mathbf{V}\rangle; \qquad |\mathbf{V}|_g^2 = \int_{\mathcal{D}} \langle \mathbf{V}, \mathbf{V}\rangle dV$$

along with the tensor norms:

$$\|\boldsymbol{\mathcal{R}}\|^2 = trace[\boldsymbol{\mathcal{R}}\,\boldsymbol{\mathcal{R}}^T]; \quad \|\boldsymbol{\mathcal{R}}\|_g^2 = \int_{\mathcal{D}} trace[\boldsymbol{\mathcal{R}}\,\boldsymbol{\mathcal{R}}^T]\, dV$$

with $|\mathbf{V}|_g^2$, essentially, the total kinetic energy of the mean motion.

Start by taking the inner product of equation (3b) with the mean velocity. Then integrate over a periodic domain $\mathcal{D}$ to generate the inequality:

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + \langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}}\rangle\!\rangle_g + \nu L_V^2 |\mathbf{V}|_g^2 \leq \langle \mathbf{f}, \mathbf{V}\rangle_g$$

where L_V denotes the Poincaré constant for the mean velocity gradient. Or, utilizing a Young inequality allows transformation to the inequality:

$$\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + 2\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}}\rangle\!\rangle_g + \nu L_V^2 |\mathbf{V}|_g^2 \leq \frac{1}{\nu L_V^2}|\mathbf{f}|_g^2 \qquad (8)$$

further progress can be made from equation (8) once a turbulence model is introduced to identify the inner product $\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}}\rangle\!\rangle_g$ in terms of that model. Except for the term involving the Reynolds tensor, equation (8) is identical in form to equation (1c) for the evolution of the instantaneous velocity norm. Further reduction produces the differential inequality:

$$\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + (\nu - \zeta) L_V^2 |\mathbf{V}|_g^2 \leq \frac{2}{\nu L_V^2}|\mathbf{f}|_g^2 + \frac{1}{\zeta}\|\boldsymbol{\mathcal{R}}\|_g^2 \quad (8a)$$

Here $\zeta \in \mathbb{R}$ is an arbitrary constant subject to the constraint that $(\nu - \zeta) > 0$ (to retain dissipation of kinetic energy in the absence of a body force). The Gronwall lemma then produces the estimate:

$$|\mathbf{V}|_g^2(t) \leq |\mathbf{V}|_g^2(0)\, E_R + [1 - E_R]\frac{|\mathbf{f}|_g^2}{\nu(\nu - \zeta)L_V^4} + R(\zeta)$$

if $E_R = exp[-(\nu - \zeta)L_V^2 t]$. The quantity $R(\zeta)$ has the form:

$$R(\zeta) = \zeta^2 \int_0^t \|\boldsymbol{\mathcal{R}}\|_g^2(\tau)\, exp[-(\nu - \zeta)L_V^2(t - \tau)]\, d\tau$$

No further reduction is possible until a statement is made about the form of the Reynolds tensor $\boldsymbol{\mathcal{R}}$: a turbulence model (from equation (4)) is required for that.

It can again be noted that the pressure field does not contribute to the evolution of the norm (here $|\mathbf{V}|_g$). Again, this is a direct consequence of restricting the domain $\mathcal{D}$ to being periodic. The second moment equation does allow estimates to be made for $\|\boldsymbol{\mathcal{R}}\|$ without a turbulence model (see [10] for a discussion) but then there are several other unknown correlations that enter the equation and so reduce its usefulness.

A SIMPLE TURBULENCE MODEL

As a simple example, consider the Boussinesq model (see the discussion in [10] for example) wherein the scalar eddy viscosity, ϵ, is taken to be a global positive constant:

Example II *The Boussinesq model:* $\boldsymbol{\mathcal{R}} = \epsilon\overline{\mathbf{D}}$

so that $\|\boldsymbol{\mathcal{R}}\|_g = \epsilon\|\overline{\mathbf{D}}\|_g$ and

$$\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}}\rangle\!\rangle_g = \epsilon\langle\!\langle \overline{\mathbf{D}}, \overline{\mathbf{L}}\rangle\!\rangle_g \equiv \epsilon\|\overline{\mathbf{D}}\|_g^2$$

on using both the Cauchy Schwarz inequality and the fact that $\langle\!\langle \overline{\mathbf{D}}, \overline{\mathbf{W}}\rangle\!\rangle_g \equiv 0$ if $\overline{\mathbf{W}}$ is the skew part of the velocity gradient $\overline{\mathbf{L}}$. Now $\|\overline{\mathbf{D}}\|_g = \|\overline{\mathbf{L}}\|_g/\sqrt{2}$, (on using the Korn inequalities with $div(\mathbf{V}) = 0$, see [11]).

a). Without the gauge field present.

Here the estimate for $|\mathbf{V}|_g$ becomes (directly from equation (8)):

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + \nu\|\overline{\mathbf{L}}\|_g^2 + \epsilon\|\overline{\mathbf{L}}\|_g^2 \leq \frac{1}{\nu L_V^2}|\mathbf{f}|_g \qquad (9)$$

As in equation (1c) relate $\|\overline{\mathbf{L}}\|_g$ to $|\mathbf{V}|_g$ with the Poincaré inequality (of constant L_V) to find:

$$\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + \epsilon_T L_V^2 |\mathbf{V}|_g^2 \leq \frac{1}{\nu L_V^2}|\mathbf{f}|_g^2$$

If $|\mathbf{f}|_g \neq 0$ (and is not a function of time) Gronwall gives the final inequality for this model:

$$|\mathbf{V}|_g(t) \leq |\mathbf{V}|_g(0)\, E_T(t) + \frac{|\mathbf{f}|_g}{\epsilon_T \nu L_V^4}[1 - E_T(t)] \qquad (10)$$

and defines an absorbing set diameter as $t \to \infty$. Here $E_T(t) = exp[-\epsilon_T L_V^2 t]$. This absorbing set diameter, $|\mathbf{V}|_g(t)|_{t\to\infty} \leq |\mathbf{f}|_g/(\epsilon_T L_V)$, depends upon the modified viscosity $\epsilon_T \equiv (\epsilon + \nu)$ as well as $|\mathbf{f}|_g$ and is smaller than the corresponding one for a laminar flow (assuming that $\epsilon > 0$). This result is consistent with the expectation that the dissipation rate is larger for turbulent flow that for laminar flow. The inadequacy of the Boussinesq model is also shown in equation (10) since nothing more that the viscosity coefficient is modified: nothing like turbulence is modeled.

The result in equation (10) is of the same form as that obtained in equation (1d) with the natural viscosity ν replaced by the modified viscosity ϵ_T. Equation (10) returns to equation (1d) when $\epsilon \equiv 0$ (as, indeed, it should).

b). Including the gauge field.

In this case place $\boldsymbol{\mathcal{R}} = \epsilon\overline{\mathbf{D}} + \beta\mathbf{G}$ (where $\beta \in \mathbb{R}$ is some scalar constant) so that the inner product $\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}} \rangle\!\rangle_g$ has the bound:

$$\langle\!\langle \boldsymbol{\mathcal{R}}, \overline{\mathbf{L}} \rangle\!\rangle_g = \epsilon\|\overline{\mathbf{D}}\|_g^2 + \beta\langle\!\langle \mathbf{G}, \overline{\mathbf{L}} \rangle\!\rangle_g \leq \epsilon\|\overline{\mathbf{L}}\|_g^2/2 + \beta\|\mathbf{G}\|_g\|\overline{\mathbf{L}}\|_g$$

Adopt the form in equation (6) for the gauge tensor $\mathbf{G}$ assuming that the velocity gauge field $\mathbf{w}$ is known. Then equation (8) extends to become:

$$\frac{\partial}{\partial t}|\mathbf{V}|_g^2 + \epsilon_B L_V^2 |\overline{\mathbf{V}}|_g^2 \leq \frac{1}{\nu L_V^2}|\mathbf{f}|_g^2 + 6\beta^2\|\mathbf{H}\|_g^2 \qquad (11)$$

The parameter β is arbitrary (within the constraint $\epsilon - \beta > 0$) in equation (11) and can be selected for convenience. The constant $\epsilon_B = \nu + \epsilon - \beta > 0$ was introduced in equation (11). Now, application of the Gronwall lemma to equation (11) gives:

$$|\mathbf{V}|_g^2(t) \leq |\mathbf{V}|_g^2(0)\, E_B + [1 - E_B]\frac{|\mathbf{f}|_g^2}{\nu\epsilon_B L_V^4} + F(\beta) \qquad (12)$$

if $E_B = exp[-\epsilon_B L_V^2 t]$. The quantity $F(\beta)$ in equation (12) has the form:

$$F(\beta) = 3\beta^2 \int_0^t \|\mathbf{H}\|_g^2(\tau)\, exp[-\epsilon_B L_V^2 (t - \tau)]\, d\tau \quad (12a)$$

It remains to select the gauge velocity field $\mathbf{w}(\mathbf{x}, t)$ to meet the needs of the application of interest. Since it was assumed that the gauge velocity was time dependent: $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ it follows that the norm $\|\mathbf{H}\|_g$ must be also and the temporal rate of change of $\mathbf{w}(\mathbf{x}, t)$ determines the decay in time of the function $F(\beta)$ in equation (12a). △

This example is of little *practical* interest since the basic turbulence model has little value. However, it does illustrate the effect that gauge fields may have on turbulence models and potential applications. There are now additional parameters β and $\mathbf{w}$ to enhance the flexibility of the model. The constant β, via the inequality $\epsilon_B = \nu + \epsilon - \beta > 0$, determines the global norm decay rate in the function E_B. For example, in [12] it was shown how turbulence control could be modeled by means of such a gauge function.

APPLICATION TO TURBULENCE CONTROL

Consider a wall mounted microelectromechanical device (MEMS) which gives an output vector $\mathbf{w}$ corresponding to some near–wall turbulence velocity (see [13], [14]). For example, this could be a wall jet such that $\mathbf{w} = (0, 0, w_3)^T$ where w_3 is that imposed vertical velocity at the wall. For this velocity to be consistent with the above development there is the constraint $div(\mathbf{w}) = 0$ so that $\partial w_3/\partial x_3 = 0$. Hence take $w_3 = w_s(t)$ only.

Standard studies of turbulence control adopt variational methods to obtain a minimum value for some cost function (see [15] for example). The present study has a different intent: to what extent can a turbulence model determine the properties of a control mechanism? A turbulence control strategy is not put forward but, rather, the gauge field is introduced in the turbulence model as a means of predicting the effects of such control. This is a very different requirement to that of finding an optimal control mechanism.

Interest then centers around the possibility of this wall velocity, $\mathbf{w}$, acting to control the turbulence in some appropriate way. One possible constraint, in the context of the present study, is to require that $|\mathbf{V}|_g$ be forced equal to $|\mathbf{v}|_g$. Placing $|\mathbf{V}|_g = |\mathbf{v}|_g$ does not, in any sense, force $\mathbf{V}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t)$ locally as a function of space and time.

The mean motion equations with the turbulence model of example II and the gauge field of equation (6) had the norm estimate given above (from equation (12)). Combine the mean motion equation (11) with the corresponding equation for the instantaneous motion. That is, repeat equation (1c) as:

$$\frac{\partial}{\partial t}|\mathbf{v}|_g^2 + \nu L_v^2 |\mathbf{v}|_g^2 \leq |\mathbf{f}|_g^2/\nu L_v^2$$

Assume that both the Poincaré constants L_v and L_V

are identical then, upon subtraction of these two equations, there is:

$$\frac{\partial \Gamma}{\partial t} + \epsilon_B L_v^2 \Gamma \leq 6\beta^2 \|\mathbf{H}\|_g^2 + (\beta - \epsilon) L_v^2 |\mathbf{v}|_g^2(t)$$

where the definition $\Gamma(t) \equiv |\mathbf{V}|_g^2 - |\mathbf{v}|_g^2$ has been introduced and is the algebraic difference between the two velocity norms. The body force term is included in this equation through the velocity norm in equation (1d). As above, $\|\mathbf{H}\|_g^2$ is a global property of the gauge velocity field $\mathbf{w}(\mathbf{x}, t)$. From this equation, an estimate for the decay of the function $\Gamma(t)$ follows from the Gronwall lemma in the form:

$$\Gamma(t) \leq \Gamma(0) E_D + \int_0^t \left[6\beta^2 \|\mathbf{H}\|_g^2(\tau) + K\right] \; E_D(t - \tau)\, d\tau$$

where $E_D(t) = exp[-(\epsilon_B - \nu) L_v^2 t]$ defines the decay rate for the function $\Gamma(t)$. Here $K = (\beta - \epsilon) L_v^2 |\mathbf{v}|_g^2$.

As $t \to \infty$ the first term vanishes and the limit of the second term depends upon the selection of the tensor $\mathbf{H}$ as a function of time. That is:

$$\Gamma(\infty) \leq 6\beta^2 \lim_{t\to\infty} \int_0^t \|\mathbf{H}\|_g^2(\tau) E_D(t-\tau)\, d\tau + \Xi(\infty) \quad (13)$$

which can be evaluated once the gauge velocity vector has been fixed to specify $\mathbf{H}$. The function $\Xi(t)$ depends upon the velocity and body force norms from equation (1c) and decays to zero, $\Xi(t) \to 0$, as $t \to \infty$.

Example III *The gauge field* $\mathbf{w} = const$

In this case, $\|\mathbf{H}\|_g^2$ is a constant (equal to H, say) and equation (13) integrates to become:

$$\Gamma(\infty) \leq 6\beta^2 H / (\epsilon_B - \nu) L_v^2$$

and $\Gamma(\infty)$ is bounded by a constant. △

For this example, $|\mathbf{V}|_g^2(\infty) = \Gamma(\infty) + |\mathbf{v}|_g^2(\infty)$ and is independent of time (after the transients have decayed). The condition $|\mathbf{V}|_g^2 = |\mathbf{v}|_g^2$ as $t \to \infty$ cannot be satisfied in this example (except in the trivial case $\beta = 0$) when the gauge field is totally absent. Of course, the use of a simple Boussinesq turbulence model in this example limits its usefulness.

FINAL COMMENTS

The study herein has made comments concerning a uniqueness theorem for the well–known mean motion equations associated with the Reynolds decomposition $\mathbf{v} \mapsto \mathbf{V} + \mathbf{u}$. This theorem allowed the introduction of a gauge field for the Reynolds tensor $\boldsymbol{\mathcal{R}}$; that is $\boldsymbol{\mathcal{R}} \mapsto \boldsymbol{\mathcal{R}} + \mathbf{G}$ for some some second order divergence–free tensor $\mathbf{G}$.

It was shown that this additional gauge function gives more flexibility to the Boussinesq turbulence model in that more free model constants are available for forcing predictions to match experimental data.

As noted above the study needs extending to include a more useful turbulence model (such as the $k \sim \epsilon$ model) for example. This extension will be reported elsewhere.

REFERENCES

[1] Moulden, T. H: *On Properties of Turbulence Models,* Advances in Fluid Mechanics, vol. VIII, (Ed. Rahman, M. and Brebbia, C. A.), WIT Press, pp 15–26. 2010

[2] Friedlander, S. and Serre, D. (Eds): *Handbook of Mathematical Fluid Mechanics,* 4 vols, Elsevier. 2002, 2003, 2004 and 2007

[3] Berselli, L. C; Iliescu, T. and Layton, W. J: *Mathematics of Large Eddy Simulation of Turbulent Flows,* Springer. 2006

[4] Gurtin, M. E: *An Introduction to Continuum Mechanics,* Academic Press. 1981

[5] Moulden, T. H: *On Gauge Fields in Fluid Turbulence,* To Appear. 2012

[6] Foias, C, Manley, O. and Temam, R: *Modeling of the interaction of small and large eddies in two–dimensional turbulence, Math. Modeling and Num. Anal.*, **22**, pp 93–118 . 1988

[7] Flavin, J. N. and Rionero, S: *Qualitative Estimates for Partial Differential Equations,* CRC Press. 1996

[8] Monin A. S. and Yaglom, A. M: *Statistical Fluid Mechanics,* Nauku, Moscow, 1965. (English translation edited by Lumey, J. L., MIT Press). 1971

[9] Dombre, T; Frisch, U; Greene, J. M; Hénon, M; Mehr, A. and Soward, A.M: *Chaotic Streamlines of the ABC Flows,* JFM vol. 167, pp 353–391. 1986

[10] Moulden, T. H: *Estimates for Turbulence Models,* IJPAM, vol. 43, # 4. pp 551–562. 2008

[11] Horgan, C. O: *Korn's Inequalities and Their Application in Continuum Mechanics,* SIAM Review, vol. 37, pp 491–511. 1995

[12] Moulden, T. H: *On Turbulence Models,* Paper at 27th ISTS Conference, Tsukuba, Japan. 2009

[13] Kasagi, N; Suzuki, Y. and Fukagata, K: *Micro–electromechanical Systems–Based Feedback Control of Turbulence for Skin Friction Reduction, Annual Review of Fluid Mechanics* (Eds. Davis, S. H. and Moin, P), vol 41, pp 231–251. 2009

[14] Yoshino, T; Suzuki, Y. and Kasagi, N: *Feedback Control of Turbulence Air Channel Flow with Distributed Micro Sensors and Actuators., J. Fluid Sci. Tech.* **3**, pp 137–148. 2008

[15] Kim, J. and Bewley, T. R: *A Linear Systems Approach to Flow Control, Annual Review of Fluid Mechanics* (Eds. Davis, S. H. and Moin, P), vol. 39, pp 383–417. 2007