Common fixed points for multivalued generalized contractions on partial metric spaces

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Abstract We establish some common fixed point results for multivalued mappings satisfying generalized contractive conditions on a complete partial metric space. The presented theorems extend some known results to partial metric spaces. We motivate our results by some given examples and an application for finding the solution of a functional equation arising in dynamic programming.

Keywords Common fixed point · Partial metric space · Partial Hausdorff metric · Weak contraction

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1 Introduction and preliminaries

Since the appearance of Banach's contraction principle, a variety of generalizations, extensions and applications of this principle have been obtained; see Rhoades [30] for a complete survey of this subject. Nadler [27] was the first who combined the ideas of multivalued mappings and contractions. He proved some remarkable results for multivalued contractions. Afterwards, several generalizations of Nadler's fixed point theorem, mainly by modifying the contractive condition, are obtained (see for example, Dube and Singh [18], Iseki [21], Ray [29], Itoh and Takahashi [22], Aubin and Sigel [3], Hu [20], and references mentioned therein; see also [2]). The theory of multivalued mappings has many applications in economics, convex optimizations, optimal control theory and differential inclusions. On the other hand, Banach's contraction principle is broadly applicable in proving the existence of solutions to operator equations, including the ordinary differential equations, partial differential equations and integral equations. This principle has been generalized in many directions. For instance, Matthews [26] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of Banach's contraction principle, more suitable in this context. Many authors followed his idea and gave their contributions in that sense, see for example [5–7,9,10,16,17,24,25,28,32]. Recently, Aydi, Abbas and Vetro [8] introduced the concept of a partial Hausdorff metric and extended the well known Nadler's fixed point theorem to such spaces.

Following are some definitions and known results needed in the sequel.

Definition 1.1 [26] A partial metric on a nonempty set X is a mapping $p : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y, z \in X$, the following conditions are satisfied:

(p1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A nonempty set X equipped with a partial metric p is called partial metric space. We shall denote it by a pair (X, p).

If $p(x, y) = 0$, then (p1) and (p2) imply that $x = y$, but the converse does not hold always. If p is a partial metric on X, then the mapping $p^s : X \times X \longrightarrow \mathbb{R}^+$ (set of all non-negative real numbers) given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

is a metric on X.

Definition 1.2 [4,26] Let (X, p) be a partial metric space. Then a sequence $\{x_n\}$ is called:

(i) convergent if there exists some point x in X such that $p(x, x) = \lim_{n \longrightarrow +\infty} p(x, x_n)$;
(ii) Cauchy sequence if there exists (and is finite) $\lim_{n,m \longrightarrow +\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is $p(x, x) = \lim_{n,m \longrightarrow +\infty} p(x_n, x_m)$.

Lemma 1.3 [4,26] *Let (X, p) be a partial metric space. Then*

(i) *A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(ii) (X, p) *is complete if and only if the metric space* (X, p^s) *is complete. Furthermore,* $\lim_{n\longrightarrow+\infty} p^s(x_n, x) = 0$ *if and only if*

$$p(x, x) = \lim_{n\longrightarrow+\infty} p(x_n, x) = \lim_{n,m\longrightarrow+\infty} p(x_n, x_m).$$

Consistent with Aydi et al. [8], we state the following:

Let $\mathcal{CB}^p(X)$ be the collection of all nonempty closed bounded subsets of X with respect to the partial metric p. For $C \in \mathcal{CB}^p(X)$, we define

$$p(a, C) = \inf\{p(a, x),\ \ x \in C\}.$$

For $A, B \in \mathcal{CB}^p(X)$, set

$$\delta_p(A, B) = \sup\{p(a, B),\ \ a \in A\},$$

$$\delta_p(B, A) = \sup\{p(b, A),\ \ b \in B\}.$$

Also, for $A, B \in \mathcal{CB}^p(X)$, define

$$H_p(A, B) = \max\{\delta_p(A, B),\ \delta_p(B, A)\}.$$

Proposition 1.4 [8] *Let* (X, p) *be a partial metric space. For all* $A, B, C \in CB^p(X)$*, we have the following:*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$*;*
(ii) $\delta_p(A, A) \leq \delta_p(A, B)$*;*
(iii) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$*;*
(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c\in C} p(c, c)$.

Proposition 1.5 [8] *Let* (X, p) *be a partial metric space. For all* $A, B, C \in CB^p(X)$*, we have the following:*

(h1) $H_p(A, A) \leq H_p(A, B)$*;*
(h2) $H_p(A, B) = H_p(B, A)$*;*
(h3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$.

Corollary 1.6 [8] *Let* (X, p) *be a partial metric space. For* $A, B \in CB^p(X)$ *the following holds:*

$$H_p(A, B) = 0 \textit{ implies that } A = B.$$

From Proposition 1.5 and Corollary 1.6, we call the mapping $H_p : CB^p(X)\times CB^p(X) \to [0, +\infty)$, a partial Hausdorff metric induced by p.

It is well known from [4] that for any $A \in \mathcal{CB}^p(X)$

$$p(x, A) = p(x, x) \Longleftrightarrow x \in \bar{A} = A.$$

Definition 1.7 An element x in X is said to be a fixed point of a multivalued mapping $T : X \longrightarrow CB^p(X)$ if $x \in Tx$. An element $x \in X$ is called a common fixed point of two multivalued mappings $T, S : X \longrightarrow CB^p(X)$ if $x \in Tx \cap Sx$.

In [8], Aydi et al. proved the following result.

Theorem 1.8 [8] *Let* (X, p) *be a complete partial metric space. If* $T : X \longrightarrow CB^p(X)$ *is a multivalued mapping such that for all* $x, y \in X$*, we have*

$$H_p(Tx, Ty) \leq k\, p(x, y)$$

where $k \in (0, 1)$*, then* T *has a fixed point.*

In 1973, Wong [33] extended the result of Hardy and Rogers [19] by proving existence of a common fixed point of two self-mappings on a complete metric space, satisfying a contractive type condition. Confirming the interest for partial metric spaces [1,23], in this paper we extend the result of Wong to the case of two multivalued mappings that satisfy a generalized contractive condition in the framework of partial Hausdorff metric spaces. We also prove a common fixed point result for a hybrid pair of single valued and multivalued mappings satisfying a weak contractive condition. The presented theorems extend well known results in the literature to partial metric spaces. Some examples and an application are presented to validate and make effective our obtained results.

2 Main results

The following lemma will be essential in the proof of the main theorems. One may find its analogous for the metric case in [27].

Lemma 2.1 *Let $A, B \in \mathcal{CB}^p(X)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists a point $b \in B$ such that $p(a, b) \leq H_p(A, B) + \varepsilon$.*

Proof We argue by contradiction. Suppose there exists $\varepsilon > 0$, such that for any $b \in B$ we have

$$p(a, b) > H_p(A, B) + \varepsilon.$$

Then,

$$p(a, B) = \inf\{p(a, b),\ \ b \in B\} \geq H_p(A, B) + \varepsilon \geq \delta_p(A, B) + \varepsilon,$$

which is a contradiction. Hence, there exists $b \in B$ such that $p(a, b) \leq H_p(A, B) + \varepsilon$. □

Our first main result is the following theorem.

Theorem 2.2 *Let (X, p) be a complete partial metric space and $T, S : X \longrightarrow \mathcal{CB}^p(X)$ be two multivalued mappings satisfying, for all $x, y \in X$, the following condition*

$$H_p(Tx, Sy) \leq \alpha\, M(x, y), \tag{2.1}$$

where $\alpha \in [0, 1)$ and

$$M(x, y) = \max\left\{p(x, y),\ p(x, Tx),\ p(y, Sy),\ \frac{p(x, Sy) + p(y, Tx)}{2}\right\}. \tag{2.2}$$

Then, T and S have a common fixed point. Moreover, if T or S is single valued, then the common fixed point is unique.

Proof Let $\varepsilon > 0$ be such that $\beta = \alpha + \varepsilon < 1$. Let $x_0 \in X$ and $x_1 \in Sx_0$. Clearly, if $M(x_1, x_0) = 0$ then $x_1 = x_0$ and x_0 is a common fixed point of T and S. Assume $M(x_1, x_0) > 0$, by Lemma 2.1, there exists $x_2 \in Tx_1$ such that $p(x_2, x_1) \leq H_p(Tx_1, Sx_0) + \varepsilon M(x_1, x_0)$. Similarly, assume $M(x_2, x_1) > 0$. Now, using again Lemma 2.1, there exists $x_3 \in Sx_2$ such that $p(x_3, x_2) \leq H_p(Sx_2, Tx_1) + \varepsilon M(x_2, x_1)$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$ and $M(x_{n+1}, x_n) > 0$ with

$$p(x_{2n+1}, x_{2n}) \leq H_p(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(x_{2n}, x_{2n-1})$$

and

$$p(x_{2n+2}, x_{2n+1}) \leq H_p(Tx_{2n+1}, Sx_{2n}) + \varepsilon M(x_{2n+1}, x_{2n}).$$

By (2.1) and the fact that $\beta = \alpha + \varepsilon$, we get

$$\begin{aligned} p(x_{2n+1}, x_{2n}) &\leq H_p(Tx_{2n-1}, Sx_{2n}) + \varepsilon M(x_{2n-1}, x_{2n}) \\ &\leq (\alpha + \varepsilon) M(x_{2n-1}, x_{2n}) \\ &= \beta M(x_{2n-1}, x_{2n}) \\ &= \beta \max \Big\{ p(x_{2n-1}, x_{2n}),\ p(x_{2n-1}, Tx_{2n-1}),\ p(x_{2n}, Sx_{2n}), \\ &\quad \frac{p(x_{2n-1}, Sx_{2n}) + p(x_{2n}, Tx_{2n-1})}{2} \Big\} \\ &\leq \beta \max \Big\{ p(x_{2n-1}, x_{2n}),\ p(x_{2n-1}, x_{2n}),\ p(x_{2n}, x_{2n+1}), \\ &\quad \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2} \Big\} \\ &= \beta \max \Big\{ p(x_{2n-1}, x_{2n}),\ p(x_{2n}, x_{2n+1}) \Big\}. \end{aligned} \tag{2.3}$$

Now, if $p(x_{2n}, x_{2n+1}) > p(x_{2n}, x_{2n-1})$, then by (2.3) we have

$$p(x_{2n}, x_{2n+1}) < p(x_{2n}, x_{2n+1}),$$

that is a contradiction and hence $p(x_{2n-1}, x_{2n}) \geq p(x_{2n}, x_{2n+1})$.
From (2.3) we get

$$p(x_{2n+1}, x_{2n}) \leq \beta p(x_{2n}, x_{2n-1}). \tag{2.4}$$

Using a similar argument, we obtain

$$p(x_{2n+2}, x_{2n+1}) \leq \beta p(x_{2n+1}, x_{2n}). \tag{2.5}$$

From (2.4) and (2.5), we conclude that

$$p(x_{n+1}, x_n) \leq \beta p(x_n, x_{n-1}),$$

for all $n \in \mathbb{N}$. Moreover, by induction, one finds

$$p(x_{n+1}, x_n) \leq \beta^n p(x_1, x_0).$$

For any $k \in \mathbb{N}$, we have

$$\begin{aligned} p^s(x_n, x_{n+k}) \leq 2p(x_n, x_{n+k}) &\leq 2p(x_n, x_{n+1}) + 2p(x_{n+1}, x_{n+2}) + \ldots + 2p(x_{n+k-1}, x_{n+k}) \\ &\leq 2\beta^n p(x_1, x_0) + 2\beta^{n+1} p(x_1, x_0) + \ldots + 2\beta^{n+k-1} p(x_1, x_0) \\ &= 2\beta^n (1 + \beta + \ldots + \beta^{k-1}) p(x_1, x_0) \\ &\leq 2 \frac{\beta^n}{1 - \beta} p(x_1, x_0) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \end{aligned}$$

This yields that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, then (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $v \in X$, that is, $\lim_{n \longrightarrow +\infty} p^s(x_n, v) = 0$. Moreover, we have

$$p(v, v) = \lim_{n \longrightarrow +\infty} p(x_n, v) = \lim_{n \longrightarrow +\infty} p(x_n, x_n) = 0. \tag{2.6}$$

Also, we get

$$\begin{aligned} p(x_{2n+2}, Sv) &\leq H_p(Tx_{2n+1}, Sv) \\ &\leq \alpha M(x_{2n+1}, v) \\ &= \alpha \max \left\{ p(x_{2n+1}, v),\ p(x_{2n+1}, Tx_{2n+1}),\ p(v, Sv),\ \frac{p(x_{2n+1}, Sv) + p(v, Tx_{2n+1})}{2} \right\} \\ &\leq \alpha \max \left\{ p(x_{2n+1}, v),\ p(x_{2n+1}, x_{2n+2}),\ p(v, Sv),\ \frac{p(x_{2n+1}, Sv) + p(v, x_{2n+2})}{2} \right\}. \end{aligned} \tag{2.7}$$

We know that

$$p(x_n, Sv) \leq p(x_n, v) + p(v, Sv), \quad \text{and} \quad p(v, Sv) \leq p(v, x_n) + p(x_n, Sv).$$

Using (2.6) in the above two inequalities, we get

$$\lim_{n \longrightarrow +\infty} p(x_n, Sv) = p(v, Sv).$$

Letting $n \longrightarrow +\infty$ and using the last limit in (2.7), we obtain

$$p(v, Sv) \leq \alpha p(v, Sv).$$

As $\alpha \in [0, 1)$, therefore, $p(v, Sv) = 0 = p(v, v)$; this implies that $v \in Sv$, since Sv is closed Analogously, we get $v \in Tv$ and so T and S have a common fixed point. Now, we show that the common fixed point is unique if T is a single valued mapping. Assume that $u \in X$ is another common fixed point of T and S, then by (2.1) we have

$$\begin{aligned} p(u, v) &\leq H_p(\{u\}, Sv) \\ &= H_p(\{Tu\}, Sv) \\ &\leq \alpha M(u, v) \\ &= \alpha \max \left\{ p(u, v),\ p(u, Tu),\ p(v, Sv),\ \frac{p(u, Sv) + p(v, Tu)}{2} \right\} \\ &\leq \alpha \max \left\{ p(u, v),\ p(u, u),\ p(v, v),\ \frac{p(u, v) + p(v, u)}{2} \right\} \\ &= \alpha p(u, v). \end{aligned}$$

Since $\alpha \in [0, 1)$, it follows $p(u, v) = 0$ and so $u = v$. This completes the proof of Theorem 2.2. □

Corollary 2.3 *Let (X, p) be a complete partial metric space and $T, S : X \longrightarrow \mathcal{CB}^p(X)$ be two multivalued mappings satisfying, for all $x, y \in X$, the following condition*

$$H_p(Tx, Sy) \leq N(x, y),$$

where

$$N(x, y) = a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Sy) + a_4[p(x, Sy) + p(y, Tx)],$$

and a_1, a_2, a_3 and a_4 are non negative real numbers with $a_1 + a_2 + a_3 + 2a_4 < 1$. Then, T and S have a common fixed point. Moreover, if T or S is single valued, then the common fixed point is unique.

Corollary 2.4 *Let (X, p) be a complete partial metric space and $T, S : X \longrightarrow \mathcal{CB}^p(X)$ be two multivalued mappings satisfying, for all $x, y \in X$, the following condition*

$$H_p(Tx, Sy) \leq \alpha\, p(x, y), \tag{2.8}$$

where $\alpha \in [0, 1)$. Then, T and S have a common fixed point. Moreover, if T or S is single valued, then the common fixed point is unique.

If we take $T = S$ in Theorem 2.2, then we obtain the following corollary which generalizes Theorem 3.1 of [8].

Corollary 2.5 *Let (X, p) be a complete partial metric space and $T : X \longrightarrow \mathcal{CB}^p(X)$ be a multivalued mapping satisfying, for all $x, y \in X$, the following condition*

$$H_p(Tx, Ty) \leq \alpha \max\left\{p(x, y),\ p(x, Tx),\ p(y, Ty),\ \frac{p(x, Ty) + p(y, Tx)}{2}\right\},$$

where $\alpha \in [0, 1)$. Then, T has a fixed point. Moreover, if T is single valued, then the fixed point is unique.

Remark 2.6 Theorem 2.2 is a generalization of Theorem 1.8. Also, Theorem 2.2 extends to the setting of partial (Hausdorff) metric spaces Theorem 3.1 of Rouhani and Moradi [31], which is itself an extension of Nadler's [27] and Daffer-Kaneko's [15] theorems to two multivalued mappings, without assuming $x \longrightarrow p(x, Tx)$ to be lower semicontinuous.

The following example shows that the extension of Theorem 3.1 of Rouhani and Moradi [31] to Theorem 2.2 in the setting of partial metric spaces is proper.

Example 2.7 Let $X = \{0, 1, 2\}$ be endowed with the partial metric $p : X \times X \longrightarrow \mathbb{R}^+$ defined by

$$\begin{aligned} p(0, 0) &= p(1, 1) = 0,\ \ p(2, 2) = \frac{4}{9}, \\ p(0, 1) &= p(1, 0) = \frac{1}{3}, \\ p(0, 2) &= p(2, 0) = \frac{11}{24}, \\ p(1, 2) &= p(2, 1) = \frac{1}{2}. \end{aligned}$$

Define the mappings $T, S : X \longrightarrow \mathcal{CB}^p(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x \in \{0, 1\} \\ \{0, 1\} & \text{if } x = 2 \end{cases} \quad \text{and} \quad Sx = \begin{cases} \{0\} \text{ if } x \in \{0, 1\} \\ \{1\} \text{ if } x = 2. \end{cases}$$

Note that Tx and Sx are closed and bounded for all $x \in X$ under the given partial metric p. We shall show that, for all $x, y \in X$, (2.1) is satisfied with $\alpha = \frac{3}{4}$. For this, we distinguish the following cases:

(i) If $x, y \in \{0, 1\}$, then $H_p(Tx, Sy) = H_p(\{0\}, \{0\}) = 0$, and (2.1) is satisfied obviously.
(ii) If $x = 0,\ y = 2$, then

$$\begin{aligned} H_p(T(0), S(2)) &= H_p(\{0\}, \{1\}) \\ &= \max\{p(0, \{1\}), p(1, \{0\})\} \\ &= \frac{1}{3} \leq \frac{11}{24}\alpha = \alpha p(0, 2) \leq \alpha M(0, 2). \end{aligned}$$

(iii) If $x = 1$, $y = 2$, then

$$\begin{aligned} H_p(T(1), S(2)) &= H_p(\{0\}, \{1\}) \\ &= \max\{p(0, \{1\}), p(1, \{0\})\} \\ &= \frac{1}{3} \leq \frac{1}{2}\alpha = \alpha p(1,2) \leq \alpha M(1,2). \end{aligned}$$

(iv) If $x = 2$, $y = 0$, then

$$\begin{aligned} H_p(T(2), S(0)) &= H_p(\{0,1\}, \{0\}) \\ &= \max\{\delta_p(\{0,1\}, \{0\}), \delta_p(\{0\}, \{0,1\})\} \\ &= \max\{\sup\{p(0,0), p(1,0)\}, p(0, \{0,1\})\} \\ &= \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3} \leq \frac{11}{24}\alpha = \alpha p(2,0) \leq \alpha M(2,0). \end{aligned}$$

(v) If $x = 2$, $y = 1$, then

$$\begin{aligned} H_p(T(2), S(1)) &= H_p(\{0,1\}, \{0\}) \\ &= \max\{\delta_p(\{0,1\}, \{0\}), \delta_p(\{0\}, \{0,1\})\} \\ &= \max\{\sup\{p(0,0), p(1,0)\}, p(0, \{0,1\})\} \\ &= \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3} \leq \frac{1}{2}\alpha = \alpha p(2,1) \leq \alpha M(2,1). \end{aligned}$$

(vi) If $x = y = 2$, then

$$\begin{aligned} H_p(T(2), S(2)) &= H_p(\{0,1\}, \{1\}) \\ &= \max\{\delta_p(\{0,1\}, \{1\}), \delta_p(\{1\}, \{0,1\})\} \\ &= \max\{\sup\{p(0,1), p(1,1)\}, p(1, \{0,1\})\} \\ &= \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3} = \frac{4}{9}\alpha = \alpha p(2,2) \leq \alpha M(2,2). \end{aligned}$$

Thus, all the conditions of Theorem 2.2 are satisfied and $x = 0$ is a common fixed point of T and S in X.

On the other hand, the metric p^s induced by the partial metric p is given by

$$\begin{aligned} p^s(0,0) &= p^s(1,1) = p^s(2,2) = 0, \\ p^s(0,1) &= p^s(0,1) = \frac{2}{3}, \\ p^s(0,2) &= p^s(2,0) = \frac{17}{36}, \\ p^s(2,1) &= p^s(1,2) = \frac{5}{9}. \end{aligned}$$

Now, it is easy to show that Theorem 2.2 is not applicable for H_{p^s} (where H_{p^s} is the Hausdorff metric associated to the metric p^s). Indeed, for $x = 0$ and $y = 2$, we have

$$\begin{aligned} H_{p^s}(T(0), S(2)) &= H_{p^s}(\{0\}, \{1\}) \\ &= \max\{p^s(0,1), p^s(1,0)\} \\ &= \frac{2}{3} \nleq \frac{41}{72}\alpha = \alpha M_s(0,2), \end{aligned}$$

for any $\alpha \in [0, 1)$, where

$$M_s(0,2) = \max\left\{p^s(0,2),\ p^s(0,T(0)),\ p^s(2,S(2)),\ \frac{p^s(0,S(2)) + p^s(2,T(0))}{2}\right\} = \frac{41}{72}.$$

The second main result of the paper is the following theorem for a hybrid pair of mappings.

Theorem 2.8 *Let (X, p) be a complete partial metric space. Let $T : X \longrightarrow X$ and $S : X \longrightarrow \mathcal{CB}^p(X)$ be two mappings such that, for all $x, y \in X$, we have*

$$H_p(\{Tx\}, Sy) \leq M(x,y) - \varphi(M(x,y)), \tag{2.9}$$

where $M(x, y)$ is given by (2.2) and $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a lower semicontinuous (l.s.c) function such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for any $t > 0$. Then, T and S have a unique common fixed point.

Proof Let $x_0 \in X$ and $x_1 \in Sx_0$. Also in this proof, if $M(x_1, x_0) = 0$ then $x_1 = x_0$ and x_0 is a common fixed point of T and S. Assume $M(x_1, x_0) > 0$. Let $x_2 := Tx_1$. If $M(x_2, x_1) = 0$ then $x_2 = x_1$ and x_1 is a common fixed point of T and S. Assume $M(x_2, x_1) > 0$, by Lemma 2.1, there exists $x_3 \in Sx_2$ such that

$$p(x_3, x_2) \leq H_p(Sx_2, \{Tx_1\}) + \frac{1}{2}\varphi(M(x_2, x_1)).$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{2n} = Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$ and $M(x_{2n}, x_{2n-1}) > 0$ with, by Lemma 2.1,

$$p(x_{2n+1}, x_{2n}) \leq H_p(Sx_{2n}, \{Tx_{2n-1}\}) + \frac{1}{2}\varphi(M(x_{2n}, x_{2n-1})). \tag{2.10}$$

Adopting the approach in [31], we split the proof into four steps.
Step 1. We prove that $\lim_{n \longrightarrow +\infty} p(x_{n+1}, x_n) = 0$.
By (2.9) and (2.10), we have

$$\begin{aligned} p(x_{2n+1}, x_{2n}) &\leq H_p(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{1}{2}\varphi(M(x_{2n-1}, x_{2n})) \\ &\leq M(x_{2n-1}, x_{2n}) - \frac{1}{2}\varphi(M(x_{2n-1}, x_{2n})), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} p(x_{2n-1}, x_{2n}) &\leq M(x_{2n-1}, x_{2n}) \\ &= \max\left\{p(x_{2n-1}, x_{2n}),\ p(x_{2n-1}, Tx_{2n-1}),\ p(x_{2n}, Sx_{2n}),\ \frac{p(x_{2n-1}, Sx_{2n}) + p(x_{2n}, Tx_{2n-1})}{2}\right\} \\ &\leq \max\left\{p(x_{2n-1}, x_{2n}),\ p(x_{2n-1}, x_{2n}),\ p(x_{2n}, x_{2n+1}),\ \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2}\right\} \\ &= \max\left\{p(x_{2n-1}, x_{2n}),\ p(x_{2n}, x_{2n+1})\right\} \\ &= p(x_{2n}, x_{2n-1}) \quad \text{(by (2.11))}. \end{aligned} \tag{2.12}$$

It follows that $M(x_{2n}, x_{2n-1}) = p(x_{2n}, x_{2n-1})$. Then, by (2.11), we find

$$p(x_{2n+1}, x_{2n}) \leq p(x_{2n}, x_{2n-1}). \tag{2.13}$$

Also, we get

$$\begin{aligned} p(x_{2n+2}, x_{2n+1}) &= p(Tx_{2n+1}, x_{2n+1}) \\ &\leq H_p(\{Tx_{2n+1}\}, Sx_{2n}) \\ &\leq M(x_{2n+1}, x_{2n}) - \varphi(M(x_{2n+1}, x_{2n})), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} p(x_{2n+1}, x_{2n}) &\leq M(x_{2n+1}, x_{2n}) \\ &= \max\left\{p(x_{2n+1}, x_{2n}),\ p(x_{2n+1}, Tx_{2n+1}),\ p(x_{2n}, Sx_{2n}),\ \frac{p(x_{2n+1}, Sx_{2n}) + p(x_{2n}, Tx_{2n+1})}{2}\right\} \\ &\leq \max\left\{p(x_{2n+1}, x_{2n}),\ p(x_{2n+1}, x_{2n+2}),\ p(x_{2n}, x_{2n+1}),\ \frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}\right\} \\ &= \max\left\{p(x_{2n+1}, x_{2n}),\ p(x_{2n+2}, x_{2n+1})\right\} \\ &= p(x_{2n}, x_{2n+1}) \quad (\text{by } (2.14)). \end{aligned}$$

It yields that $M(x_{2n}, x_{2n+1}) = p(x_{2n}, x_{2n+1})$. Then, again by (2.14), we find

$$p(x_{2n+2}, x_{2n+1}) \leq p(x_{2n+1}, x_{2n}). \tag{2.15}$$

Using (2.13) and (2.15), we conclude that

$$p(x_{n+1}, x_n) \leq p(x_n, x_{n-1}),$$

for any $n \geq 1$. Thus, the sequence $\{p(x_{n+1}, x_n)\}$ is monotone nonincreasing and bounded below and hence there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow +\infty} p(x_{n+1}, x_n) = \lim_{n \longrightarrow +\infty} M(x_{n+1}, x_n) = r.$$

Using the fact that φ is l.s.c, we have

$$\varphi(r) \leq \liminf_{n \longrightarrow +\infty} \varphi(M(x_{n-1}, x_n)) \leq \liminf_{n \longrightarrow +\infty} \varphi(M(x_{2n-1}, x_{2n})).$$

Now, by (2.11), we get

$$r \leq r - \frac{1}{2}\varphi(r),$$

that implies $\varphi(r) = 0$. It follows that $r = 0$.

Step 2. $\{x_n\}$ is a bounded sequence.

Suppose to the contrary that $\{x_n\}$ is unbounded, so that by Step 1 the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are unbounded. By Step 1, there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have $p(x_{k+1}, x_k) < \frac{1}{4}$. Now, we can choose a sequence $\{n(k)\}_{k=1}^{+\infty}$ such that $n(1) \geq N_0$ is odd, $n(2) > n(1)$ is even and minimal in the sense that $p(x_{n(2)}, x_{n(1)}) > 1$ and $p(x_{n(2)-2}, x_{n(1)}) \leq 1$, and similarly $n(3) > n(2)$ is odd and minimal in the sense that $p(x_{n(3)}, x_{n(2)}) > 1$ and $p(x_{n(3)-2}, x_{n(2)}) \leq 1$,..., $n(2k) > n(2k-1)$ is even and minimal in the sense that $p(x_{n(2k)}, x_{n(2k-1)}) > 1$ and $p(x_{n(2k)-2}, x_{n(2k-1)}) \leq 1$, and $n(2k+1) > n(2k)$ is odd and minimal in the sense that $p(x_{n(2k+1)}, x_{n(2k)}) > 1$ and $p(x_{n(2k+1)-2}, x_{n(2k)}) \leq 1$. Clearly, $n(k) \geq k$ for any $k \in \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$, we have $n(k+1) - n(k) \geq 2$ and

$$\begin{aligned} 1 &< p(x_{n(k+1)}, x_{n(k)}) \\ &\leq p(x_{n(k+1)}, x_{n(k+1)-1}) + p(x_{n(k+1)-1}, x_{n(k+1)-2}) + p(x_{n(k+1)-2}, x_{n(k)}) \\ &\leq p(x_{n(k+1)}, x_{n(k+1)-1}) + p(x_{n(k+1)-1}, x_{n(k+1)-2}) + 1. \end{aligned}$$

This shows that

$$\lim_{k \longrightarrow +\infty} p(x_{n(k+1)}, x_{n(k)}) = 1.$$

On the other hand

$$\begin{aligned} 1 &< p(x_{n(k+1)}, x_{n(k)}) \\ &\leq p(x_{n(k+1)}, x_{n(k+1)+1}) + p(x_{n(k+1)+1}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{n(k)}) \\ &\leq p(x_{n(k+1)}, x_{n(k+1)+1}) + p(x_{n(k+1)+1}, x_{n(k+1)}) + p(x_{n(k+1)}, x_{n(k)}) \\ &\quad + p(x_{n(k)}, x_{n(k+1)}) + p(x_{n(k)+1}, x_{n(k)}) \\ &\leq 2p(x_{n(k+1)}, x_{n(k+1)+1}) + p(x_{n(k+1)}, x_{n(k)}) + 2p(x_{n(k)+1}, x_{n(k)}). \end{aligned}$$

This yields that

$$\lim_{k \to +\infty} p(x_{n(k+1)+1}, x_{n(k)+1}) = 1.$$

Then, if $n(k+1)$ is odd, we get

$$\begin{aligned} p(x_{n(k+1)+1}, x_{n(k)+1}) &\leq H_p(\{Tx_{n(k+1)}\}, Sx_{n(k)}) \\ &\leq M(x_{n(k+1)}, x_{n(k)}) - \varphi(M(x_{n(k+1)}, x_{n(k)})), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} 1 &< p(x_{n(k+1)}, x_{n(k)}) \leq M(x_{n(k+1)}, x_{n(k)}) \\ &= \max\Big\{ p(x_{n(k+1)}, x_{n(k)}),\ p(x_{n(k+1)}, Tx_{n(k+1)}),\ p(x_{n(k)}, Sx_{n(k)}), \\ &\quad \frac{p(x_{n(k+1)}, Sx_{n(k)}) + p(x_{n(k)}, Tx_{n(k+1)})}{2} \Big\} \\ &\leq \max\Big\{ p(x_{n(k+1)}, x_{n(k)}),\ p(x_{n(k+1)}, x_{n(k+1)+1}),\ p(x_{n(k)}, x_{n(k)+1}), \\ &\quad \frac{p(x_{n(k+1)}, x_{n(k)+1}) + p(x_{n(k)}, x_{n(k+1)+1})}{2} \Big\} \\ &\leq \max\Big\{ p(x_{n(k+1)}, x_{n(k)}),\ p(x_{n(k+1)}, x_{n(k+1)+1}),\ p(x_{n(k)}, x_{n(k)+1}), \\ &\quad \frac{2p(x_{n(k+1)}, x_{n(k)}) + p(x_{n(k)+1}, x_{n(k)}) + p(x_{n(k+1)+1}, x_{n(k+1)})}{2} \Big\}. \end{aligned}$$

This implies that

$$\lim_{k \longrightarrow +\infty} M(x_{n(k+1)}, x_{n(k)}) = 1.$$

Since φ is l.s.c and (2.16) holds, we have $1 \leq 1 - \varphi(1)$. Therefore $\varphi(1) = 0$, that is a contradiction.

Step 3. $\{x_n\}$ is Cauchy.

Let $C_n = \sup\{p(x_i, x_j),\ i, j \geq n\}$. Since $\{x_n\}$ is bounded, $C_n < +\infty$ for all $n \in \mathbb{N}$. Obviously $\{C_n\}$ is decreasing and hence there exists $C \geq 0$ such that $\lim_{n \longrightarrow +\infty} C_n = C$. We will show that $C = 0$. For every $k \in \mathbb{N}$, there exist $n(k), m(k) \in \mathbb{N}$ such that $m(k) > n(k) \geq k$ and

$$C_k - \frac{1}{k} \leq p(x_{m(k)}, x_{n(k)}) \leq C_k. \tag{2.17}$$

Using (2.17), we conclude that

$$\lim_{k \longrightarrow +\infty} p(x_{m(k)}, x_{n(k)}) = C. \tag{2.18}$$

From Step 1 and (2.18), we have

$$\begin{aligned} \lim_{k \longrightarrow +\infty} p(x_{m(k)+1}, x_{n(k)+1}) &= \lim_{k \longrightarrow +\infty} p(x_{m(k)+1}, x_{n(k)}) \\ &= \lim_{k \longrightarrow +\infty} p(x_{m(k)}, x_{n(k)+1}) = \lim_{k \longrightarrow +\infty} p(x_{m(k)}, x_{n(k)}) = C. \end{aligned} \tag{2.19}$$

Therefore we may assume that for every $k \in \mathbb{N}$, $m(k)$ is odd and $n(k)$ is even. Then, we have

$$\begin{aligned} p(x_{m(k)+1}, x_{n(k)+1}) &= p(Tx_{m(k)}, x_{n(k)+1}) \\ &\leq H_p(\{Tx_{m(k)}\}, Sx_{n(k)}) \\ &\leq M(x_{m(k)}, x_{n(k)}) - \varphi(M(x_{m(k)}, x_{n(k)})), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} p(x_{m(k)}, x_{n(k)}) &\leq M(x_{m(k)}, x_{n(k)}) \\ &= \max\Big\{ p(x_{m(k)}, x_{n(k)}),\ p(x_{m(k)}, Tx_{m(k)}),\ p(x_{n(k)}, Sx_{n(k)}), \\ &\qquad \frac{p(x_{m(k)}, Sx_{n(k)}) + p(x_{n(k)}, Tx_{m(k)})}{2} \Big\} \\ &\leq \max\Big\{ p(x_{m(k)}, x_{n(k)}),\ p(x_{m(k)}, x_{m(k)+1}),\ p(x_{n(k)}, x_{n(k)+1}), \\ &\qquad \frac{p(x_{m(k)}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)+1})}{2} \Big\}. \end{aligned} \tag{2.21}$$

Using (2.19) in (2.21) and letting $k \longrightarrow +\infty$, we get

$$\lim_{k \longrightarrow +\infty} M(x_{m(k)}, x_{n(k)}) = C. \tag{2.22}$$

Since φ is l.s.c and (2.20) holds, we find $C \leq C - \varphi(C)$. Hence, $\varphi(C) = 0$ and then $C = 0$. It follows that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p).
Step 4. T and S have a common fixed point.
Since (X, p) is complete and $\{x_n\}$ is Cauchy, then there exists $u \in X$ such that

$$p(u, u) = \lim_{n \longrightarrow +\infty} p(x_n, u) = \lim_{n \longrightarrow +\infty} p(x_n, x_n) = 0, \tag{2.23}$$

using Step 1. For any $n \in \mathbb{N}$, we get

$$\begin{aligned} p(x_{2n+2}, Su) &= p(Tx_{2n+1}, Su) \\ &\leq H_p(\{Tx_{2n+1}\}, Su) \\ &\leq M(x_{2n+1}, u) - \varphi(M(x_{2n+1}, u)), \end{aligned} \tag{2.24}$$

where

$$M(x_{2n+1},u) = \max\left\{p(x_{2n+1},u),\ p(x_{2n+1},Tx_{2n+1}),\ p(u,Su),\ \frac{p(x_{2n+1},Su)+p(u,Tx_{2n+1})}{2}\right\}$$
$$\leq \max\left\{p(x_{2n+1},u),\ p(x_{2n+1},x_{2n+2}),\ p(u,Su),\ \frac{p(x_{2n+1},Su)+p(u,x_{2n+2})}{2}\right\}. \tag{2.25}$$

Using (2.23) in (2.25) and letting $k \longrightarrow +\infty$, we get

$$\lim_{n\longrightarrow+\infty} M(x_{2n+1},u) = p(u,Su). \tag{2.26}$$

By (2.26) and the fact that φ is l.s.c, (2.24) leads to

$$p(u,Su) \leq p(u,Su) - \varphi(p(u,Su)),$$

so $\varphi(p(u,Su)) = 0$ and then $p(u,Su) = 0 = p(u,u)$, that is, $u \in Su$ since Su is closed. Also we have

$$p(Tu,u) \leq H_p(\{Tu\},Su) \leq M(u,u) - \varphi(M(u,u)), \tag{2.27}$$

where

$$M(u,u) = \max\left\{p(u,u),\ p(u,Tu),\ p(u,Su),\ \frac{p(u,Su)+p(u,Tu)}{2}\right\} = p(u,Tu).$$

From (2.27), we get

$$p(u,Tu) \leq p(u,Tu) - \varphi(p(u,Tu)).$$

It follows easily that $p(u,Tu) = 0 = p(u,u)$ and then $Tu = u$.
The uniqueness of the common fixed point follows, after routine calculation, from (2.9) and so to avoid repetitions, we omit the details. Then, the proof of Theorem 2.8 is finished. □

Remark 2.9 Theorem 2.8 extends to partial metric spaces Theorem 4.1 of Rouhani and Moradi [31], which is itself an extension of Zhang and Song's theorem [34] to the case where one of the mappings is multivalued.

Finally, we illustrate Theorem 2.8 by the following two examples, where Theorem 4.1 of Rouhani and Moradi [31] is not applicable.

Example 2.10 Let $X = [0,1]$ be endowed with the partial metric $p : X \times X \longrightarrow \mathbb{R}^+$ defined by

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}, \text{ for all } x,y \in X.$$

Note that $p^s(x,y) = |x-y|$ and so (X,p^s) is a complete metric space. Therefore, by Lemma 1.3, (X,p) is a complete partial metric space.

Also define the mappings $T : X \longrightarrow X$ and $S : X \longrightarrow CB^p(X)$ by

$$Tx = 0 \quad \text{and} \quad Sx = \left[\frac{x}{4},\frac{x}{3}\right], \text{ for all } x \in X,$$

and the function $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ by $\varphi(t) = \frac{5}{8}t$ for any $t \geq 0$.

It is clear that for all $x \in X$, the set Sx is bounded and closed with respect to the topology τ_p. We shall show that (2.9) holds for all $x, y \in X$. First

$$H_p(\{Tx\}, Sy) = H_p\left(\{0\}, \left[\frac{y}{4}, \frac{y}{3}\right]\right) = \frac{y}{4}.$$

Next, we distinguish the following cases:
• Case 1: $x \leq y$. A simple calculation gives that

$$M(x, y) = \max\left\{\frac{3}{4}y - \frac{1}{4}x, \frac{3}{4}x, \frac{2}{3}y, \frac{1}{2}\left[\frac{3}{4}y + p\left(x, \left[\frac{y}{4}, \frac{y}{3}\right]\right)\right]\right\}.$$

In all possible cases: $(0 \leq x \leq \frac{1}{4}y)$, $(\frac{1}{4}y \leq x \leq \frac{7}{12}y)$ and $(\frac{7}{12}y \leq x \leq \frac{y}{3})$, we get that $M(x, y) = \frac{3}{4}y - \frac{1}{4}x$. Therefore

$$H_p(\{Tx\}, Sy) = \frac{y}{4} \leq \frac{3}{8}\left(\frac{3}{4}y - \frac{1}{4}x\right) = \frac{3}{8}M(x, y) = M(x, y) - \varphi(M(x, y)).$$

While if $(\frac{y}{3} \leq x \leq \frac{8}{9}y)$, we find that $M(x, y) = \frac{2}{3}y$, so

$$H_p(\{Tx\}, Sy) = \frac{y}{4} = \frac{3}{8}\frac{2}{3}y = \frac{3}{8}M(x, y) = M(x, y) - \varphi(M(x, y)),$$

Also, if $(\frac{8}{9}y \leq x \leq y)$, we find that $M(x, y) = \frac{3}{4}x$. Thus,

$$H_p(\{Tx\}, Sy) = \frac{y}{4} \leq \frac{3}{8}\frac{3}{4}x = \frac{3}{8}M(x, y) = M(x, y) - \varphi(M(x, y)),$$

• Case 2: $x > y$. Similarly

$$M(x, y) = \max\left\{\frac{3}{4}x - \frac{1}{4}y, \frac{3}{4}x, \frac{2}{3}y, \frac{1}{2}\left(\frac{2}{3}y + \frac{3}{4}x\right)\right\} = \frac{3}{4}x.$$

Then

$$H_p(\{Tx\}, Sy) = \frac{y}{4} \leq \frac{3}{8}\frac{3}{4}y \leq \frac{3}{8}\frac{3}{4}x = M(x, y) - \varphi(M(x, y)).$$

Remark that in all cases (2.9) is satisfied.
Applying Theorem 2.8, the mappings T and S have a unique common fixed point, which is $u = 0$.

Now, take the standard metric $D : X \times X \longrightarrow X$ given by $D(x, y) = 1$ if $x \neq y$ and $D(x, y) = 0$ if $x = y$. Let H_D be the Hausdorff metric associated to the metric D. For $x = 0$ and $y = 1$, we have

$$H_D(\{T(0)\}, S(1)) = H_D\left(\{0\}, \left[\frac{1}{4}, \frac{1}{3}\right]\right) = 1 > \frac{3}{8} = 1 - \varphi(1) = M(0, 1) - \varphi(M(0, 1)),$$

that is, we could not apply Theorem 4.1 of Rouhani and Moradi [31].

Example 2.11 Let $X = \{0, 1, 2\}$ be endowed with the partial metric $p : X \times X \longrightarrow \mathbb{R}^+$ defined by

$$p(0, 0) = p(1, 1) = 0,\ p(0, 1) = p(1, 0) = \frac{1}{4},\ p(2, 2) = \frac{1}{3},$$
$$p(0, 2) = p(2, 0) = \frac{2}{5},\ p(1, 2) = p(2, 1) = \frac{13}{20}.$$

Also define the mappings $T : X \longrightarrow X$ and $S : X \longrightarrow CB^p(X)$ by

$$Tx = \begin{cases} 0 & \text{if } x \in \{0, 1\} \\ 1 & \text{if } x = 2 \end{cases} \quad \text{and} \quad Sx = \begin{cases} \{0\} & \text{if } x \neq 2 \\ \{0, 1\} & \text{if } x = 2 \end{cases}$$

and the function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ by $\varphi(t) = \frac{t}{9}$ for any $t \geq 0$.
Note that Sx is closed and bounded for all $x \in X$ under the given partial metric p. We shall show that (2.9) holds for all $x, y \in X$. We distinguish the following cases:

(i) If $x, y \in \{0, 1\}$, then $H_p(\{Tx\}, Sy) = 0$ and (2.9) is satisfied obviously.

(ii) If $x = 0$, $y = 2$, then

$$\begin{aligned} M(0,2) - \varphi(M(0,2)) &= \max\left\{p(0,2),\ p(0,T(0)),\ p(2,S(2)),\ \frac{p(0,S(2)) + p(2,T(0))}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(0,2),\ p(0,T(0)),\ p(2,S(2)),\ \frac{p(0,S(2)) + p(2,T(0))}{2}\right\}\right) \\ &= \max\left\{p(0,2),\ p(0,0),\ p(2,0),\ \frac{p(0,0) + p(2,0)}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(0,2),\ p(0,0),\ p(2,0),\ \frac{p(0,0) + p(2,0)}{2}\right\}\right) \\ &= \max\left\{\frac{2}{5}, 0, \frac{2}{5}, \frac{0 + \frac{2}{5}}{2}\right\} - \varphi\left(\max\left\{\frac{2}{5}, 0, \frac{2}{5}, \frac{0 + \frac{2}{5}}{2}\right\}\right) \\ &= \frac{2}{5} - \frac{2}{45} = \frac{16}{45} \geq \frac{1}{4} = H_p(\{0\}, \{0, 1\}) = H_p(\{T(0)\}, S(2)). \end{aligned}$$

(iii) If $x = 2$, $y = 0$, then

$$\begin{aligned} M(2,0) - \varphi(M(2,0)) &= \max\left\{p(2,0),\ p(2,T(2)),\ p(0,S(0)),\ \frac{p(2,S(0)) + p(0,T(2))}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(2,0),\ p(2,T(2)),\ p(0,S(0)),\ \frac{p(2,S(0)) + p(0,T(2))}{2}\right\}\right) \\ &= \max\left\{\frac{2}{5}, \frac{13}{20}, 0, \frac{\frac{2}{5} + \frac{1}{3}}{2}\right\} - \varphi\left(\max\left\{\frac{2}{5}, \frac{13}{20}, 0, \frac{\frac{2}{5} + \frac{1}{3}}{2}\right\}\right) \\ &= \frac{13}{20} - \frac{13}{180} = \frac{26}{45} \geq \frac{1}{4} = H_p(\{1\}, \{0\}) = H_p(\{T(2)\}, S(0)). \end{aligned}$$

(iv) If $x = 2$, $y = 1$, then

$$\begin{aligned} M(2,1) - \varphi(M(2,1)) &= \max\left\{p(2,1),\ p(2,T(2)),\ p(1,S(1)),\ \frac{p(2,S(1)) + p(1,T(2))}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(2,1),\ p(2,T(2)),\ p(1,S(1)),\ \frac{p(2,S(1)) + p(1,T(2))}{2}\right\}\right) \\ &= \max\left\{\frac{13}{20}, \frac{13}{20}, \frac{1}{4}, \frac{\frac{2}{5} + 0}{2}\right\} - \varphi\left(\max\left\{\frac{13}{20}, \frac{13}{20}, \frac{1}{4}, \frac{\frac{2}{5} + 0}{2}\right\}\right) \\ &= \frac{13}{20} - \frac{13}{180} = \frac{26}{45} \geq \frac{1}{4} = H_p(\{1\}, \{0\}) = H_p(\{T2\}, S1). \end{aligned}$$

(v) If $x = 1$, $y = 2$, then

$$\begin{aligned} M(1,2) - \varphi(M(1,2)) &= \max\left\{p(1,2),\ p(1,T(1)),\ p(2,S(2)),\ \frac{p(1,S(2)) + p(2,T(1))}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(1,2),\ p(1,T(1)),\ p(2,S(2)),\ \frac{p(1,S(2)) + p(2,T(1))}{2}\right\}\right) \\ &= \max\left\{\frac{13}{20}, \frac{1}{4}, \frac{2}{5}, \frac{\frac{1}{4}+\frac{2}{5}}{2}\right\} - \varphi\left(\max\left\{\frac{13}{20}, \frac{1}{4}, \frac{2}{5}, \frac{\frac{1}{4}+\frac{2}{5}}{2}\right\}\right) \\ &= \frac{13}{20} - \frac{13}{180} = \frac{26}{45} \geq \frac{1}{4} = H_p(\{0\},\{0,1\}) = H_p(\{T(1)\}, S(2)). \end{aligned}$$

(vi) If $x = y = 2$, then

$$\begin{aligned} M(2,2) - \varphi(M(2,2)) &= \max\left\{p(2,2),\ p(2,T(2)),\ p(2,S(2)),\ \frac{p(2,S(2)) + p(2,T(2))}{2}\right\} \\ &\quad -\varphi\left(\max\left\{p(2,2),\ p(2,T(2)),\ p(2,S(2)),\ \frac{p(2,S(2)) + p(2,T(2))}{2}\right\}\right) \\ &= \max\left\{\frac{1}{3}, \frac{13}{20}, \frac{2}{5}, \frac{\frac{2}{5}+\frac{13}{20}}{2}\right\} - \varphi\left(\max\left\{\frac{1}{3}, \frac{13}{20}, \frac{2}{5}, \frac{\frac{2}{5}+\frac{13}{20}}{2}\right\}\right) \\ &= \frac{13}{20} - \frac{13}{180} = \frac{26}{45} \geq \frac{1}{4} = H_p(\{1\},\{0,1\}) = H_p(\{T(2)\}, S(2)). \end{aligned}$$

Thus, all the conditions of Theorem 2.8 are satisfied and $x = 0$ is a common fixed point of T and S in X.

On the other hand, the metric p^s induced by the partial metric p is given by

$$\begin{aligned} &p^s(0,0) = p^s(1,1) = p^s(2,2) = 0,\ p^s(0,1) = p^s(1,0) = \frac{1}{2}, \\ &p^s(1,2) = p^s(2,1) = \frac{29}{30},\ p^s(0,2) = p^S(2,0) = \frac{7}{15}. \end{aligned}$$

Now, it is easy to show that Theorem 2.8 is not applicable for H_{p^s}. Indeed, for $x = 0$ and $y = 2$, we have

$$H_{p^s}(\{T(0)\}, S(2)) = H_{p^s}(\{0\},\{0,1\}) = \frac{1}{2}$$

and

$$M^s(0,2) = \max\left\{p^s(0,2),\ p^s(0,T(0)),\ p^s(2,S(2)),\ \frac{p^s(0,S(2)) + p^s(2,T(0))}{2}\right\} = \frac{7}{15}.$$

Hence for any l.s.c function φ such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for any $t > 0$, we have

$$H_{p^S}(\{T0\}, S2) \nleq M^s(0,2) - \varphi(M^s(0,2)).$$

3 An application to a dynamical process

Generally, the basic description of a dynamical process consists of a state space and a decision space, where:

- the state space is the set of the initial state, actions and transition model of the process;
- the decision space is the set of possible actions that are allowed for the process.

It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well. In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation

$$q(x) = \sup_{y \in D}\{f(x, y) + G(x, y, q(\tau(x, y)))\}, \quad x \in W,$$

which further can be reformulated as

$$q(x) = \sup_{y \in D}\{g(x, y) + G(x, y, q(\tau(x, y)))\} - b, \quad x \in W, \tag{3.1}$$

where $\tau : W \times D \longrightarrow W$, $f, g : W \times D \longrightarrow \mathbb{R}$, $G : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$, $b > 0$, $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, U and V are Banach spaces.
In this section, we study the existence and uniqueness of the bounded solution of the functional equation (3.1). If necessary, the reader can refer to [11–14] for a more detailed explanation of the background of the problem.

Let $B(W)$ denote the set of all bounded real-valued functions on W and, for an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Clearly, $(B(W), \|\cdot\|)$ endowed with the metric d defined by

$$d(h, k) = \sup_{x \in W} |h(x) - k(x)|$$

for all $h, k \in B(W)$, is a Banach space. Precisely, the convergence in the space $B(W)$ with respect to $\|\cdot\|$ is uniform and so, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, the sequence $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded. Thus $h^* \in B(W)$.

Now, for all $h, k \in B(W)$, $x \in W$ and $b > 0$, we consider the partial metric p given by

$$p(h, k) = d(h, k) + b \tag{3.2}$$

and the mapping $T : B(W) \longrightarrow B(W)$ given by

$$T(h)(x) = \sup_{y \in D}\{g(x, y) + G(x, y, h(\tau(x, y)))\} - b, \tag{3.3}$$

that is well-defined if the functions g and G are bounded.

We will prove the following result:

Theorem 3.1 *Assume that the following condition holds:*

$$|G(x, y, h(x)) - G(x, y, k(x))| \leq \alpha M_1(h, k)$$

with

$$M_1(h, k) = \max\left\{p(h, k), p(h, T(h)), p(k, T(k)), \frac{p(h, T(k)) + p(k, T(h))}{2}\right\}$$

where $x \in W$, $y \in D$, $\alpha \in [0, 1)$, $T : B(W) \longrightarrow B(W)$ is given by (3.3), the functions $G : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : W \times D \longrightarrow \mathbb{R}$ are bounded. Then the functional equation (3.1) has a unique bounded solution.

Proof Since $(B(W), d)$ is complete and $p^s(h, k) = 2d(h, k)$ for all $h, k \in B(W)$ and $x \in W$, by Lemma 1.3 we deduce that $(B(W), p)$ is a complete partial metric space. Let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$T(h_1)(x) < g(x, y_1) + G(x, y_1, h_1(\tau(x, y_1))) - b + \lambda, \tag{3.4}$$
$$T(h_2)(x) < g(x, y_2) + G(x, y_2, h_2(\tau(x, y_2))) - b + \lambda, \tag{3.5}$$
$$T(h_1)(x) \geq g(x, y_2) + G(x, y_2, h_1(\tau(x, y_2))), \tag{3.6}$$
$$T(h_2)(x) \geq g(x, y_1) + G(x, y_1, h_2(\tau(x, y_1))). \tag{3.7}$$

Now, from (3.4) and (3.7) it follows that

$$\begin{aligned} T(h_1)(x) - T(h_2)(x) &< G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1))) - b + \lambda \\ &\leq |G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1)))| - b + \lambda \\ &\leq \alpha M_1(h_1, h_2) - b + \lambda. \end{aligned}$$

Then we get

$$T(h_1)(x) - T(h_2)(x) < \alpha M_1(h_1, h_2) - b + \lambda. \tag{3.8}$$

Similarly, from (3.5) and (3.6) we obtain

$$T(h_2)(x) - T(h_1)(x) < \alpha M_1(h_1, h_2) - b + \lambda. \tag{3.9}$$

Therefore, from (3.8) and (3.9) we have

$$|T(h_1)(x) - T(h_2)(x)| < \alpha M_1(h_1, h_2) - b + \lambda, \tag{3.10}$$

that is,

$$p(T(h_1), T(h_2)) < \alpha M_1(h_1, h_2) + \lambda.$$

Since the above inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrary, then we conclude immediately that

$$p(T(h_1), T(h_2)) \leq \alpha M_1(h_1, h_2)$$

and so Corollary 2.5 is applicable in this case. Consequently, the mapping T has a unique fixed point, that is, the functional equation (3.1) has a unique bounded solution. □

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