GLOBAL FINITE-TIME OBSERVERS FOR A CLASS OF NONLINEAR SYSTEMS

by

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SUMMARY

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The contributions of this thesis lie in the area of global finite-time observer design for a class of nonlinear systems with bounded rational and mixed rational powers imposed on the incremental rate of the nonlinear terms whose solutions exist and are unique for all positive time. In the thesis, two different kinds of nonlinear global finite-time observers are designed by employing of finite-time theory and homogeneity properties with different methods. The global finite-time stability of both proposed observers is derived on the basis of Lyapunov theory.

For a class of nonlinear systems with rational and mixed rational powers imposed on the nonlinearities, the first global finite-time observers are designed, where the global finite-time stability of the observation systems is achieved from two parts by combining asymptotic stability and local finite-time stability. The proposed observers can only be designed for the class of nonlinear systems with dimensions greater than 3. The observers have a dynamic high gain and two homogenous terms, one homogeneous of degree greater than 1 and the other of degree less than 1. In order to prove the global finite-time stability of the proposed results, two homogeneous Lyapunov functions are provided, corresponding with the two homogeneous items. One is homogeneous of degree greater than 1, which makes the observation error systems converging into a spherical area around the origin, and the other is of degree less than 1, which ensures local finite-time stability.

The second global finite-time observers are also proposed based on the high-gain technique, which does not place any limitation on the dimension of the nonlinear systems. Compared with the first global finite-time observers, the newly designed observers have only one homogeneous term and a new gain update law where two new terms are introduced to dominate some terms in the nonlinearities and ensure global finite-time stability as well. The global finite-time stability is obtained directly based on a sufficient condition of finite-time stability and only one Lyapunov function is employed in the proof.

The validity of the two kinds of global finite-time observers that have been designed is illustrated through some simulation results. Both of them can make the observation error systems converge to the origin in finite-time. The parameters, initial conditions as well as the high gain do have some impact on the convergence time, where the high gain plays a stronger role. The bigger the high gain is, the shorter the time it needs to converge. In order to show the performance of the two kinds of observers more clearly, two examples are provided and some comparisons are made between them. Through these, it can be seen that under the same parameters and initial conditions, although the amplitude of the observation error curve is slightly greater, the global finite-time observers with a new gain update law can make the observation error systems converge much more quickly than the global finite-time observers with two homogeneous terms. In the simulation results, one can see that, as a common drawback of high gain observers, they are noise-sensitive. Finding methods to improve their robustness and adaptiveness will be quite interesting, useful and challenging.

OPSOMMING

GLOBALE EINDIGETYDWAARNEMERS VIR'N KLAS NIE-LINIÊRE STELSELS

deur

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Die bydra van hierdie tesis lê op die gebied van die ontwerp van eindigetydwaarnemers vir'n klas nie-liniêre stelsels met begrensde rasionele en gemengde rasionele magte opgelê aan die toenemende tempo van die nie-liniêre terme waarvan die oplossings bestaan en uniek is vir alle positiewe tyd. In die tesis word twee verskillende soorte nie-liniêreglobale eindigetydwaarnemers ontwerp deur die aanwending van eindigetydteorie en homegeniteiteienskappe, deur verskillende metodes. Die globale eindigetydstabiliteit van albei voorgestelde waarnemers word afgelei op die basis van Lyapunov-teorie.

Vir'n klas nie-liniêre stelsels met rasionele en gemengde rasionele magteopgelê op die nie-lineariteite, word die eerste globale eindigetydwaarnemers ontwerp deurdat die globale eindigetydstabiliteit van die waarnemingstelsels tweeledig bereik word deur'n kombinasie van asimptotiese stabliteit en lokale eindigetydstabiliteit. Die voorgestelde waarnemers kan slegs bereik word vir die klas nie-liniÍžre stelsels met'n dimensie groter as 3. Die waarnemers het'n hoë dinamiese wins en twee homogene terme, een homogeen met'n graad bo 1 en die ander met'n graad onder 1. Om die globale eindigetydstabiliteit van die voorgestelde resultate te bewys, word twee homogene Lyapunov-funksies voorsien wat ooreenstem met die twee homogene items. Een is homogeen met'n graad hoër as 1, wat die waarnemingfoutstelsel laat konvergeer na'n sferiese gebied om die oorsprong en die ander met'n graad laer as 1, wat lokale eindigetydstabiliteit verseker.

Die tweede globale eindigetydwaarnemers word ook voorgestel op die basis van die hoêwinstegniek, wat nie enige beperking plaas op die dimensie van die nie-liniêre stelsels nie. Vergeleke met die eerste globale eindigetydwaarnemers het die nuut ontwerpte waarnemers een homogene term en'n nuwe winsbywerkwet waardeur twee nuwe terme ingevoer word wat sowel sommige terme in die nie-lineariteite oorheers as globale eindigetydstabiliteit verseker. Die globale eindigetydstabiliteit is direk gebaseer op'n voldoende toestand van eindigetydstabileit en net een Lyapunov-funksie word in die bewysgebruik.

Die geldigheid van die twee ontwerpte soorte eindigetydwaarnemers word geïllustreer deur sommige van die simulasieresultate. Albei kan die waarnemersfoutstelsels laat konvergeer tot die oorsprong in eindige tyd. Die parameters, aanvanklike toestande en hoê wins het wel'n invloed op die konvergensietyd; die hoê wins speel'n meer belangrike rol. Hoe groter die hoê wins is, hoe korter is die tyd wat nodig is vir konvergensie. Om die werkverrigting van die twee soorte waarnemers duideliker te illustreer, word twee voorbeelde gegee en'n aantal vergelykings tussen hulle word gemaak. Daardeur is dit duidelik dat, alhoewel die amplitude van die waarnemingsfoutkurwe effens groter is, onder dieselfde parameters en aanvanklike toestande die globale eindigetydwaarnemers met'n nuwe winsbywerkwet die waarnemingsfoutstelsels baie vinniger kan laat konvergeer as die globale eindigetydwaarnemers met twee homogene terme. Uit die simulasieresultate is dit duidelik dat'n algemene nadeel van hoêwinswaarnemers is dat hulle sensitief is vir geraas. Pogings om hulle robuustheid en aanpassingsvermoê te verbeter mag heel interessant, nuttig en uitdagend wees.

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CHAPTER 1

INTRODUCTION

This thesis contributes to the area of global finite-time observers for a class of nonlinear systems. In this chapter, some recent works on nonlinear asymptotic observers in Section 1.1 and nonlinear finite-time observers in Section 1.2 are presented. In Section 1.3, an introduction to nonlinear systems with an incremental rate imposed on the nonlinear terms is given. Finally, the contributions of the thesis are summarized and its outline is given in Section 1.4.

Observer design for nonlinear systems has received a great deal of attention since the formal introduction of the concept and the Lyapunov-based approach of design as proposed in [1]. Unlike in the case of linear systems, the observability of a nonlinear system depends on the inputs of the system [2], [3]. Perhaps for this reason, quite a number of early works have been devoted to establishing the link between nonlinear observability [4] and the existence of nonlinear observers [5], [6], [7] with the development of the nonlinear observability theory in the differential geometric framework [8]. Thereafter, a lot of research has been done in an attempt to design nonlinear observers through the linearization of nonlinear systems [9], [10], [11], [12]. Since the definition of uniform observability or observability for any input was proposed by [13], a series of nonlinear observers by different methods have been developed based on uniform observability. For example, an extended Luenberger observer for nonlinear systems is proposed in [14]; [15] designs a nonlinear observer using Lyapunov's auxiliary theorem; high-gain nonlinear observers are designed in [16] for nonlinear systems with a triangular structure, while high-gain observers in the presence of measurement noise in [17] are employed to the output feedback control problem through a switched-gain approach, then, in [18], a method is presented for the design of observers for nonlinear systems based on backstepping, where the method is broadly applicable and the observer error is exponentially convergent if the initial error is not too large, among others. Among them the Lyapunov-based approach [19], [1], the observer canonical form approach [20], the recursive design approach [21], [22], and the high-gain approach [23], [16], [17] make great contributions to two major nonlinear observers: nonlinear asymptotically convergent observers and nonlinear finite-time convergent observers.

1.1 NONLINEAR ASYMPTOTIC OBSERVERS AND NONLINEAR FINITE-TIME OBSERVERS

Over the years, nonlinear asymptotically stable observers have been investigated in a number of works such as [13], [24], [25], [26], [27], [28], [23], where a lot of attention has been paid to Lipschitz nonlinear systems [13], [29], [26], [30]. There are some results [13], [29], [31], [24], [32] dealing with asymptotic observers for SISO Lipschitz nonlinear systems: A high-gain asymptotic observer with exponential convergence has been designed by [13], [31] proposes a reduced-order observer and a new observer in [24] is carried out using H_∞ optimization. Then, a sufficient and necessary condition on the stability matrix is derived in [29]. However, the approach in [29] can only be achieved when the Lipschitz constant is small. Subsequently, [32] has made some improvements by proposing a robust nonlinear observer that does not impose small-Lipschitz constant on the system nonlinearity. Nonlinear asymptotic observers are also investigated for MIMO Lipschitz systems, as discussed in the following works: Earlier, based on a block triangular observer normal form, exponential observers were designed in [33] by using methods from the linear case, while [25] developed an observer design through an injective map procedure. In [26], semi-global nonlinear observers were constructed based on some saturation functions. Then a simple observer with an explicit gain was proposed by [30], which can be applied to many physical systems.

Moreover, some progress was made on asymptotic observers for nonlinear systems with other structures [27], [34], [28], [23]: A high-gain observer was proposed for a class of MIMO non-triangular nonlinear systems involving some uncertainties in [27]. Based on a time-scaled block observer form [34], an observer was designed for uncontrolled nonlinear multi-output continuous-time systems. Then [28] presented a globally asymptotically stable observer for nonlinear systems with an output-dependent incremental rate. Later, global high-gain-based observers were developed [23] for nonlinear systems with generalized output-feedback canonical form including output-dependent upper diagonal terms.

The convergence of observation errors of asymptotic observers to zero is always asymptotic with time. Finite-time observers can make the observation error systems converge to zero in finite time with Lyapunov stability. There exist a series of observer design methods which can achieve finite-time convergence. Among them, sliding mode observers are a typical family of observers which can ensure the error systems converge robustly in finite time: for example, sliding mode observer-based feedback control is studied for flexible joints manipulator in [35], and sliding mode observers are constructed [36] for switched mechanical systems. Then, based on two-time-scale approach, adaptive sliding mode observer [37] is proposed for induction motor and sliding mode high-gain observers are developed for a class of uncertain nonlinear systems in [38], etc. However, the sliding mode observers have some problems: such as discontinuity, chattering phenomenon and so on. Thus, it is very important to seek for some new methods of designing continuous finite-time observers.

Recently, since systems with finite-settling-time dynamics possess better disturbance rejection and robustness properties [39], finite-time convergent observers of nonlinear systems have become an active subject with the advance in finite-time stability and stabilization theory [40], [41], [42], [43], [44]. Finite-time stability conditions are studied in [40] for non-autonomous continuous systems and in [43] for a class of nonlinear quadratic systems. Furthermore, stochastically finite-time attractiveness is defined [41] for a class of stochastic nonlinear systems, while in [41], the problem of finite-time stabilization for nonlinear systems with a lower-triangular structure is investigated. [44] studies the finite-time stability of nonlinear autonomous systems, where the finite-time stability of the origin of the system is defined if it is finite-time convergent and Lyapunov stable. Based on finite-time stability theory, in the past several years a number of nonlinear finite-time observers [45], [46], [47], [48] have been proposed: Earlier, finite-time observers existed for partially observed deterministic control systems in [45] and for a class of second order nonlinear systems in [46]. Then, finite-time continuous observers are designed for nonlinear systems that could be transformed into the observer canonical form [47] and for a class of nonlinear systems with bounded trajectories, [48] presented a semi-global finite-time observer. A more detailed overview of nonlinear finite-time observers will be given in the appropriate chapters of the thesis.

In this thesis, based on finite-time stability and homogeneity theory, two kinds of global finite-time observers will be designed for a class of nonlinear systems by different methods.

1.2 NONLINEAR SYSTEMS WITH INCREMENTAL RATE IN NONLINEAR TERMS

It is well known that Lipschitz nonlinear systems, which are defined as those nonlinear terms satisfying Lipschitz conditions, are the most common nonlinear systems. Systems in many applications, such as a flexible link robot in [49], mechanical oscillators in [50], etc., belong to Lipschitz nonlinear systems. During the past decades, a great deal of research has been done on Lipschitz nonlinear systems regarding the output feedback control problem [51], [52], [53] as well as the observer design problem [29], [24], [54], [55].

Compared with Lipschitz nonlinear systems, both the output feedback control problem and the observer design problem for nonlinear systems with nonlinear terms, including the incremental rate in the nonlinearities, are much more difficult and challenging. For the output feedback control problem, in the case of linear systems, stabilization by state feedback plus observability imply stabilization by output feedback, which is the so-called separation principle. Unfortunately, the separation principle does not hold for the problem of output feedback control of nonlinear systems. Earlier, output feedback control for a family of nonlinear systems with nonlinear terms subject to the linear growth condition was studied in [56] through a nonseparation principle. Then, for a class of uncertain nonlinear systems where the nonlinear dynamics have higher order growing conditions in the unmeasurable states, [57] dealt with the problem of semi-global stabilization by linear output feedback and the global stabilization problem was solved via output feedback by a generalized homogeneous domination approach in [58]. Later, global output feedback stabilization was achieved [59] for nonlinear systems with low-order and high-order nonlinearities based on a dual observer. Over the years, some results were also obtained on the observer design for nonlinear systems with incremental rate. An asymptotic observer is employed in the asymptotic stabilization via output feedback [28] for nonlinear systems where the nonlinear terms admit an output-dependant incremental rate. Then, for the same kind of nonlinear systems, [60] proposes a global finite-time observer through a high-gain approach. By exploiting homogeneity in the bi-limit [22], a class of global asymptotic high-gain observers incorporating a gain update law and nonlinear output injection terms [61] is designed for nonlinear systems with output incremental rate as well as bounded rational powers in the nonlinearities through a recursive procedure. Motivated by this result, based on Lyapunov theory, semi-global finite-time observers [62] are designed for nonlinear systems with rational and mixed rational powers in the nonlinearities, where the lower bounds of the rational and mixed powers depend on the homogeneity degree of the homogeneous terms in the proposed observers.

This thesis deals with the problem of global finite-time observer design for a class of nonlinear systems where the nonlinear terms have bounded ration powers and mixed rational powers in the nonlinear terms. Compared with the systems in [62], the lower bounds of both the rational and mixed rational powers of the nonlinearities of the nonlinear systems in this thesis are less than those in [62], which are derived based on the best possible lower bound of the homogeneity degree [63].

1.3 OUTLINE AND CONTRIBUTION OF THE THESIS

The thesis deals with the problem of global finite-time observer design for a class of nonlinear systems. In the thesis, the attention is restricted to estimating the states only for those nonlinear systems whose solutions exist globally and are unique for all positive time. The thesis is divided into five chapters. Both **Chapter 3** and **Chapter 4** address the problem of global finite-time observers for the same class of nonlinear systems, but with different methods. The outline and contributions of the thesis are as follows:

Chapter 2 gives some basic definitions and useful lemmas, especially the definitions and properties of finite-time stability and homogeneous systems. Moreover, for the same kind of nonlinear systems where the nonlinearities are associated with a bigger lower bound of the rational and mixed rational powers, the semi-global finite-time observers designed in [62] are reviewed.

In **Chapter 3**, global finite-time observers are designed for a class of nonlinear systems with incremental rate on the nonlinearities. The design method can be for only nonlinear systems with dimension $n \geq 3$.

- The global finite-time observers are designed for a class of nonlinear systems with both bounded rational and bounded mixed rational powers imposed on the nonlinear terms. The global finite-time stability of the two cases is derived from the combination of global asymptotic stability and local finite-time stability.

- The designed global finite-time observers have two homogeneous terms and an updated high gain. Of the two homogeneous terms, one is of degree greater than 1, which results in the trajectories of the observation error systems asymptotically converging into a spherical area around the origin, and the other is of degree less than 1, which ensures local finite-time stability.

As for the updated high gain, besides the fact that it can dominate the incremental rate in the nonlinearities, the proper selection of the parameters in the high gain also makes a contribution to global finite-time stability.

Chapter 4 proposes global finite-time observers for the same class of nonlinear systems in **Chapter 3**, with both bounded rational and bounded mixed rational powers imposed on the incremental rate of the nonlinearities. The global finite-time stability is derived in only one step, which is based on the result in [44].

- The proposed global finite-time observers have one homogeneous term and a new dynamic high gain where two new homogeneous items are introduced, compared with the high gain in **Chapter 3**.
- There is no restriction of the dimension of the nonlinear systems. The proposed result can be applied to nonlinear systems with any dimension.

In **Chapter 5**, examples are provided to demonstrate the designed two different kinds of global finite-time observers for the class of nonlinear systems with both bounded rational powers and bounded mixed rational powers in the nonlinearities. In the simulation results, some comparisons are made between the two kinds of global finite-time observers, which show that the observers designed in **Chapter 4** can make the observation error systems converge more quickly than those in **Chapter 3**.

Finally, **Chapter 6** summarizes the results of the thesis.

Appendix A includes the explicit proof of the useful lemma which proposes a new homogeneous Lyapunov function and gives some key inequalities for a class of nonlinear homogeneous systems based on this Lyapunov function.

CHAPTER 2

PRELIMINARIES

To deal with the problem of designing global finite-time observers, the idea of homogeneity and the properties of finite-time stability and homogeneous systems (refer to [44], [64] and [65] for details) will be employed. In this chapter, the corresponding definitions of finite-time stability, homogeneous systems as well as some properties of these will be reviewed. Moreover, a lemma is given for a class of nonlinear homogeneous systems where a new Lyapunov function is proposed and some useful inequalities are obtained based on the new proposed Lyapunov function.

Moreover, with respect to the global finite-time observer design for a class of nonlinear systems in the thesis, the semi-global finite-time observers designed in [62] for the same class of nonlinear systems but with different conditions imposed on the linearities will be reviewed in this chapter.

2.1 FINITE-TIME STABILITY

In paper [44], finite-time stability is studied for a class of autonomous systems, in which it is shown that finite-time stability can lead to better rejection of perturbations. Moreover, some useful results for finite-time stability are given, based on Lyapunov theory.

In the following, the definition and a sufficient condition of finite-time stability will be reviewed.

Consider the following nonlinear system,

$$\dot{x}(t) = f(x(t)),\ f(0) = 0,\ x \in \mathscr{R}^n,\ x(0) = x_0, \tag{2.1}$$

where $f : \mathscr{D} \to \mathscr{R}^n$ is continuous on an open neighborhood $\mathscr{D}$ of the origin $x = 0$.

Definition 1 ([44])**.** *The zero solution of (2.1) is said to be finite-time convergent if there is an open neighborhood $\mathscr{U} \subset \mathscr{D}$ of the origin and a function $T : \mathscr{U} \setminus \{0\} \to (0,\infty)$, called the settling-time function, such that $\forall x_0 \in \mathscr{U}$, the solution $\psi(t,x_0)$ of system (2.1) is defined on $[0,T(x_0)]$, $\psi(t,x_0) \in \mathscr{U} \setminus \{0\}$ for $t \in [0,T(x_0)]$ and $\lim_{t\to T(x_0)} \psi(t,x_0) = 0$. If the zero solution of (2.1) is finite-time convergent, the set of point x_0 such that $\psi(t,x_0) \to 0$ is called the domain of attraction of the solution. The zero solution of (2.1) is said to be a finite-time stable equilibrium if it is Lyapunov stable and finite-time convergent. Moreover, the zero solution is said to be a globally finite-time stable equilibrium if it is a finite-time stable equilibrium with $\mathscr{U} = \mathscr{D} = \mathscr{R}^n$.*

Remark 2 ([65])**.** *From the definition of global finite-time stability, it is obvious that global asymptotic stability and local finite-time stability imply global finite-time stability.*

Lemma 3 ([44])**.** *Suppose there exists a positive definite continuous function $\widetilde{V} : \mathscr{D} \longrightarrow \mathscr{R}$ and open neighborhood $\mathscr{U} \subset \mathscr{D}$ of the origin, such that*

$$\dot{\widetilde{V}}(x) + c\widetilde{V}(x)^{\widetilde{\alpha}} \leq 0, x \in \mathscr{U} \setminus \{0\},$$

where $c > 0$ is a real number, $\widetilde{\alpha} \in (0,1)$. Then, the origin is a finite-time stable equilibrium of system (2.1). If in addition $\mathscr{D} = \mathscr{R}^n$, $\widetilde{V}$ is proper and $\dot{\widetilde{V}}$ takes negative value on $\mathscr{R}^n \setminus \{0\}$, then the origin is a globally finite-time stable equilibrium of system (2.1).

The following lemma gives a useful inequality, which will be used in Chapter 5.

Lemma 4 ([66])**.** *For any $x,\ y \in \mathscr{R}$, if $p \leq 1$ is an odd integer or is a fraction number with the numerator and denominator both being odd integers, then*

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|.$$

The objective of the thesis is to design global finite-time observers for a class of nonlinear systems. The definition of global finite-time observers is given in the following.

Consider the following nonlinear system

$$\begin{cases} \dot{x} &= g(x,u), \\ y &= x_1, \end{cases} \tag{2.2}$$

where $u \in \mathscr{R}^m$, $x = (x_1,\ldots,x_n)^T \in \mathscr{R}^n$, $y \in \mathscr{R}$ are the input, state and output of the system.

The following high-gain observer is designed for nonlinear system (2.2):

$$\begin{cases} \dot{\hat{x}} &= g^*(\hat{x},\psi,u), \\ \dot{\psi} &= h(\hat{x},\psi,u). \end{cases} \tag{2.3}$$

Definition 5. *Denote the solutions of systems (2.2), (2.3) with respect to the corresponding input functions and passing through* x_0 $(x_0 = x(t_0))$ *and* $\hat{x}_0$ $(\hat{x}_0 = \hat{x}(t_0))$ *as* $x(t)$ *and* $\hat{x}(t)$*, respectively. Let* $\mathscr{U} = \{(x(t), \hat{x}(t), \psi(t), u) : \; x_0 = x(t_0), \; \hat{x}_0 = \hat{x}(t_0), \; \psi_0 = \psi(t_0)\} \subset \mathscr{R}^{2n+m+1}$*. If the following two conditions are satisfied:*

- *If there exists a function* $\bar{T}(x, \hat{x}, \psi, u) : \; \mathscr{U} \setminus \{0\} \longrightarrow (0, \infty)$ *such that* $x(t) - \hat{x}(t) \longrightarrow 0$ *when* $t \longrightarrow \bar{T}(x_0, \hat{x}_0, \psi_0, u)$.

- *Moreover, for any open neighborhood* $\mathscr{U}_\varepsilon$ *of* 0*, there exists an open subset* $\mathscr{U}_\delta$ *containing* 0 *such that, for every* $x(t_0) - \hat{x}(t_0) \in \mathscr{U}_\delta \setminus 0$*,* $x(t) - \hat{x}(t) \in \mathscr{U}_\varepsilon$ *for all* $t \in [0, \bar{T}(x_0, \hat{x}_0, \psi_0, u))$.

Then, system (2.3) with high gain ψ *is called a global finite-time observer for nonlinear system (2.2).*

2.2 HOMOGENEITY

The definitions and some properties of homogeneous systems are investigated in paper [64], where the link between homogeneity and finite-time stability is studied: a homogeneous system is finite-time stable if and only if it is asymptotically stable and is homogeneous of a negative degree. In what follows, the definitions as well as some properties of homogeneous systems will be reviewed.

The definitions of the homogeneous function and homogeneous vector field are given in the following:

Definition 6 ([64])**.** *A function* $g(x) : \mathscr{R}^n \longrightarrow \mathscr{R}$ *is homogeneous of degree* d *with respect to the weights* $\{r_i\}$ $(1 \le i \le n)$*,* $r_i > 0$*, if for any* $\lambda > 0$ *one has*

$$g(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^d g(x_1, \ldots, x_n).$$

Definition 7 ([64])**.** *A vector field* $h(x) : \mathscr{R}^n \longrightarrow \mathscr{R}^n$*,* $x \in \mathscr{R}^n$*,* $n \ge 2$*, is said to be homogeneous of degree* d *with respect to the weights* $\{r_i\}$ $(1 \le i \le n)$*,* $r_i > 0$*, if for any* $\lambda > 0$ *the following equation is established*

$$h_i(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^{d+r_i} h_i(x_1, \ldots, x_n).$$

The lemma below gives a comparison between two homogeneous functions.

Lemma 8 ([64])**.** *Suppose* $V_1(x), \; V_2(x) : \; \mathscr{R}^n \longrightarrow \mathscr{R}$ *are continuous functions on* $\mathscr{R}^n$*, homogeneous with respect to the weights* $\nu = (\nu_1, \ldots, \nu_n)^T$ *of degrees* $d_1 > 0$ *and* $d_2 > 0$*, respectively, and function*

$V_1(x)$ is positive definite. Then, for every $x \in \mathscr{R}^n$, one finds

$$\left[\min_{\{z:V_1(z)=1\}} V_2(z)\right] V_1(x)^{\frac{d_2}{d_1}} \leq V_2(x) \leq \left[\max_{\{z:V_1(z)=1\}} V_2(z)\right] V_1(x)^{\frac{d_2}{d_1}}.$$

In this thesis, the goal is to study the global finite-time observers design problem for the following nonlinear systems whose solutions exist globally and are unique for all positive time:

$$\begin{cases} \dot{x}_1 &= x_2 + f_1(y,u), \\ \dot{x}_2 &= x_3 + f_2(y,x_2,u), \\ &\vdots \\ \dot{x}_n &= f_n(y,x_2,\ldots,x_n,u), \\ y &= x_1, \end{cases} \tag{2.4}$$

where $x \in \mathscr{R}^n$, $u \in \mathscr{R}^m$, $y \in \mathscr{R}$.

The problem will be tried to solve, when the nonlinear terms $f_i(\cdot)$ $(i = 2,\ldots,n)$ in systems (2.4) satisfy the following two conditions:

- with bounded varying rational powers in the nonlinearities

$$\begin{aligned} |f_i(y,x_2,\ldots,x_i,u) - f_i(y,\hat{x}_2,\ldots,\hat{x}_i,u)| &\leq \Gamma(u,y)(1+\sum_{j=2}^{n}|\hat{x}_j|^{\upsilon_j}) \\ &\times \sum_{j=2}^{i}|x_j-\hat{x}_j| + l\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\theta_{ij}}, \end{aligned} \tag{2.5}$$

  where θ_{ij} $(2 \leq j \leq i \leq n)$ are called the rational powers, which are varying and bounded and satisfy $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, x_j, $\hat{x}_j$ $(2 \leq j \leq i)$ are arbitrary, $\Gamma(\cdot)$ is a continuous function, $l \geq 0$, $\upsilon_j \in [0, \frac{1}{j-1})$ $(j = 2,\ldots,n)$.

- with bounded varying mixed rational powers in the nonlinearities

$$\begin{aligned} |f_i(y,x_2,\ldots,x_i,u) - f_i(y,\hat{x}_2,\ldots,\hat{x}_i,u)| &\leq \Gamma(u,y)(1+\sum_{j=2}^{n}|\hat{x}_j|^{\upsilon_j}) \\ &\times \sum_{j=2}^{i}|x_j-\hat{x}_j| + l_1\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\theta_{1,ij}} + l_2\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\theta_{2,ij}}, \end{aligned} \tag{2.6}$$

  where $\theta_{1,ij}$, $\theta_{2,ij}$ are called the mixed rational powers, which are varying and bounded and satisfy $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, x_j, $\hat{x}_j$ $(2 \leq j \leq i)$ are arbitrary, l_1, $l_2 > 0$ are two positive real numbers.

Note that research has been done on the observer design for nonlinear systems (2.4): in paper [62], semi-global finite-time observers are studied for nonlinear systems (2.4) with the nonlinear terms satisfying conditions (2.5) and (2.6) but with different rational and mixed rational powers.

2.3 SEMI-GLOBAL FINITE-TIME OBSERVERS

In this section, the semi-global finite-time observers designed in [62] for nonlinear systems (2.4) with conditions (2.5) and (2.6) (where $l_1 = l_2$ in [62]) will be reviewed, where the rational and mixed rational powers in the nonlinearities satisfy $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{ij} < \frac{i}{j-1}$ and $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ (where $\bar{q} > n$ is a positive real number and will be given in Lemma 10), respectively. The semi-global finite-time observers in [62] are shown in the following:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + \zeta a_1 \lceil e_1 \rfloor^{\alpha_1} + f_1(y,u), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + \zeta^2 a_2 \lceil e_1 \rfloor^{\alpha_2} + f_2(y,\hat{x}_2,u), \\ &\vdots \\ \dot{\hat{x}}_n &= \zeta^n a_n \lceil e_1 \rfloor^{\alpha_n} + f_n(y,\hat{x}_2,\ldots,\hat{x}_n,u), \end{cases} \tag{2.7}$$

with the high gain ζ being dynamically updated by

$$\dot{\zeta} = -\zeta[\mu_1(\zeta^{1-\sigma} - \mu_2) - \mu_3 \Psi(u,y,\hat{x})],\ \zeta(0) > \mu_2, \tag{2.8}$$

where μ_1, $\mu_2 > 1$, μ_3 are three positive real numbers, $0 < \sigma < 1$ is chosen such that $\theta_{ij} < \frac{i-\sigma}{j-1+\sigma}$, $\upsilon_j < \frac{1-2\sigma}{j-1+\sigma}$ holds, $\Psi(u,y,\hat{x}) = \Gamma(u,y)(1+\sum_{j=2}^{n}|\hat{x}_j|^{\upsilon_j})$, $\alpha_i = i\alpha' - (i-1)$, $(i = 0,1,\ldots,n)$, $\alpha' \in (1-\varepsilon,1)$, $\varepsilon \in (0,\frac{1}{n})$ and $a_i > 0$ $(i = 1,\ldots,n)$ are the coefficients of Hurwitz polynomial $s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$.

The observer gain $L(t)$ in (2.8) has the following property.

Lemma 9 ([62]). *For the observer gain $\zeta(t)$ defined in (2.8), there exists an $M > 0$ such that $\zeta(t) < M, t \in [0,T]$, $\forall T \in (0,\infty)$.*

For α_i $(0 \leq i \leq n)$ in the semi-global finite-time observer (2.7), one uses the following lemma:

Lemma 10 ([62]). *Let $\bar{r} = \Pi_{i=1}^{n-1}\alpha_i$, $\bar{\gamma} = \frac{\bar{r}}{2}(\frac{2}{\bar{r}} + \alpha' - 1)$. Then, there exists a $\bar{q} > n$ such that $\alpha' < \frac{\bar{q}-1}{\bar{q}}$. And if $\theta_{ij} > \frac{\bar{q}-i}{\bar{q}-j+1}$ $(2 \leq j \leq i \leq n)$, then $\min_{2\leq j\leq i\leq n}\{\frac{\alpha_{j-1}}{\alpha_{i-1}}(1+(\theta_{ij}-1)\alpha_{i-1}\bar{r})\} > 2\bar{\gamma} - 1$ holds.*

The main contribution of the thesis is reduce the $\bar{q}$ to $\bar{q} = n$ for the new class of nonlinear systems (2.4) with conditions (2.5) and (2.6). The following remark illustrates the differences between the systems studied in paper [62] and the nonlinear systems that will be researched in the thesis.

Remark 11. *In paper [62], the semi-global finite-time observers are designed for nonlinear systems (2.4) with the rational powers and mixed rational powers in conditions (2.5) and (2.6) satisfying* $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{ij} < \frac{i}{j-1}$ *and* $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$*, where* $\bar{q} > n$ *is a positive real number related to* $\alpha',\ \alpha' \in (1-\varepsilon, 1),\ \varepsilon \in (0, \frac{1}{n})$*. Then, in paper [63], global asymptotic stability and finite-time stability are studied for a class of homogeneous nonlinear systems where the best lower bound* $1-\frac{1}{n}$ *of* α' *is achieved and it is denoted by* α^* *in the thesis,* $\alpha^* \in (1-\frac{1}{n}, 1)$*. Motivated by this result, the thesis will relax the problem of finite-time observers design to a broader class of nonlinear systems (2.4) with lower bounds of the rational and mixed rational powers in the conditions (2.5) and (2.6):* $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ *and* $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$*.*

CHAPTER 3

GLOBAL FINITE-TIME OBSERVERS — WITH TWO HOMOGENEOUS TERMS

In this chapter, for a class of nonlinear systems with both bounded rational and mixed powers in the nonlinearities, global finite-time observers are designed.

3.1 INTRODUCTION

Recently, finite-time observer design has attracted a great deal of attention, resulting in progress in finite-time stability theory [42], [43], [44]. In particular, [44] presents a necessary and sufficient condition of finite-time stability for a class of nonlinear autonomous systems. Then, in [64], the link between finite-time stability and homogeneity is investigated: a homogeneous system is finite-time stable if and only if it is asymptotically stable and is homogeneous of a negative degree. Motivated by this result, [71] designed a finite-time observer based on a new kind of homogeneous Lyapunov function for nonlinear systems, which can be put into a linear canonical form by output injections. A homogeneous dual-observer is also employed for global output feedback stabilization of nonlinear systems with low-order and high-order nonlinearities in [59]. Then, based on the homogeneous Lyapunov function proposed by [71], many results on finite-time observers have been obtained in the past several years. Among others, [55], [72], [73] have made considerable progress in finite-time observer design for Lipschitz nonlinear systems with a triangular structure. [55] proposes a semi-global finite-time observer for single output nonlinear systems that are uniformly observable and globally Lipschitz. Then for the same class of nonlinear systems, [72] and [73] propose designs of global finite-time observers: [72] gives a global finite-time observer with an update gain which is an exponential function with arbitrary growth rate, and [73] presents a global finite-time observer based on a dedicated high-gain approach. Then, motivated by the proof method of [73]: global asymptotic stability and local finite-time stability mean global finite-time stability, [54] extends global finite-time observers to a class of Lipschitz nonlinear systems with a non-triangular structure where interconnections between all the states of the nonlinear terms are allowed.

However, in all these papers, the derivatives of the homogeneous Lyapunov function along the observation error systems are not continuous. Then [62] gives correct proof of the convergence of observation error. And semi-global finite-time observers are studied in [62] for nonlinear systems (2.4) under condition (2.5) and condition (2.6), with the varying incremental rational and mixed rational powers in the nonlinearities satisfying $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ and $\frac{\bar{q}-i}{\bar{q}-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ (where $\bar{q} > n$ is a positive number which is given in Lemma 10), respectively.

Subsequently, in [63] global asymptotic stability and finite-time stability were studied for a class of homogeneous nonlinear systems and the best possible lower bound $-\frac{1}{n}$ of the degree of the homogeneity was obtained. Motivated by the result in [63], for the rational and mixed rational powers with smaller lower bound satisfying $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in conditions (2.5) and (2.6) of nonlinear systems (2.4) for $n \geq 3$, there have still not been any related results on asymptotic and finite-time observer design. In this chapter, the aim is to solve the problem of designing global finite-time observers. The attention in the thesis is restricted to estimating the states only for those nonlinear systems (2.4) whose solutions exist globally and are unique for all positive time.

In order to solve the problem of designing a global finite-time observer, homogeneity properties [64] and the argument method in [73] will be employed together. Under exactly the same gain update law as that in semi-global finite-time results [62], the global finite-time observers that will be designed have two homogeneous terms, one of degree smaller than 1, the other of degree greater than 1. Moreover, the global finite-time convergence of the observation error system is derived based on two different homogeneous Lyapunov functions. The derivatives of the Lyapunov functions are calculated by splitting the whole space into three different sets to obtain the global asymptotic stability and local finite-time stability.

This chapter is organized as follows: The global finite-time observers for nonlinear system (2.4) for $n \geq 3$ with condition (2.5) with single rational powers $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ are presented in Section 3.3 and in Section 3.4 for nonlinear systems (2.4) for $n \geq 3$ under condition (2.6) with mixed rational powers satisfying $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, respectively. The chapter is concluded in Section 3.5.

3.2 A USEFUL LEMMA FOR A CLASS OF NONLINEAR HOMOGENOUS SYSTEMS

In order to solve the problem of global finite-time observers for nonlinear systems (2.4) with conditions (2.5) and (2.6), in this section, some properties will be investigated for the following class of nonlinear homogenous systems:

$$\begin{cases} \dot{\varepsilon}_1 &= \rho\varepsilon_2 - \rho^{(\lambda_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\lambda_1}, \\ \dot{\varepsilon}_2 &= \rho\varepsilon_3 - \rho^{(\lambda_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\lambda_2}, \\ &\vdots \\ \dot{\varepsilon}_n &= -\rho^{(\lambda_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\lambda_n}, \end{cases} \tag{3.1}$$

where $\rho > 0,\ \lceil\varepsilon_1\rfloor^{\lambda_i} = |\varepsilon_1|^{\lambda_i}\text{sign}(\varepsilon_1),\ \lambda_i = i\lambda - (i-1)\ (i = 0, 1, \ldots, n),\ \lambda > 1 - \frac{1}{n},\ 0 < \sigma < 1,\ a_i > 0\ (1 \leq i \leq n)$ are the coefficients of Hurwitz polynomial

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n. \tag{3.2}$$

From papers [67], [68] and [69], a_i $(1 \leq i \leq n)$ in (3.1) are appropriately chosen such that there exists a matrix $P \in \mathscr{R}^{n\times n}$, $P^T = P > 0$ satisfying

$$A^TP + PA \leq -I,\ h_1 I \leq D_1P + PD_1 \leq h_2 I, \tag{3.3}$$

where $A = \begin{bmatrix} -a_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \ldots & 1 \\ -a_n & 0 & \ldots & 0 \end{bmatrix}$, $D_1 = \text{diag}\{\sigma, 1+\sigma, \ldots, n-1+\sigma\},\ h_1,\ h_2 > 0$ are two real constants.

The following lemma summarizes some results for the nonlinear system (3.1). It can be seen that under a new homogeneous Lyapunov function, nonlinear system (3.1) is finite-time stable for $\lambda \in (1 - \frac{1}{n}, 1)$ and asymptotically stable for $\lambda \geq 1$.

Lemma 12. *Construct the following function as in [70]*

$$V(\varepsilon) = \begin{cases} \int_0^\infty \frac{1}{v^{q+1}}(\chi \circ \bar{V})(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \ldots, v^{\lambda_{n-1}}\varepsilon_n)dv, & \varepsilon \in \mathscr{R}^n \setminus \{0\}, \\ 0, & \varepsilon = 0, \end{cases}$$

where $\chi(s) = \begin{cases} 0, & s \in (-\infty, 1] \\ 2(s-1)^2, & s \in (1, \frac{3}{2}) \\ 1-2(s-2)^2, & s \in [\frac{3}{2}, 2) \\ 1, & s \in [2, \infty) \end{cases}$, $0 \le \chi'(s) \le 2$, $\bar{V}(\varepsilon) = \varepsilon^T P \varepsilon$, $\lambda > 1 - \frac{1}{n}$, *P satisfies condition (3.3),* $q > 0$ *is a positive integer. Let* $\gamma = \frac{q+\lambda-1}{q}$*. Then*

(i) $V(\varepsilon)$ *is a positive definite function homogeneous of degree q with respect to the weights* $\{\lambda_{i-1}\}_{1 \le i \le n}$*.* $V(\varepsilon)$ *is called a q h-Lyapunov function of* $\bar{V}(\varepsilon)$ *w.r.t.* $\chi, \rho, (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$*.*

(ii) If a_i $(1 \le i \le n)$ *are chosen to satisfy condition (3.3), then there exist* w_1, $w_2 > 0$ *such that*

$$w_1 V(\varepsilon) \le \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T D_1 \varepsilon \le w_2 V(\varepsilon). \tag{3.4}$$

(iii) For $1 - \frac{1}{n} < \lambda < 1$*, if* $q > 1 + \max\{\lambda_i\}_{0 \le i \le n}$*,* a_i $(1 \le i \le n)$ *and P satisfy condition (3.3),* $\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}$ *is* C^1 *on* $\mathscr{R}^n$*, then there exists a* $w_3 > 0$ *such that*

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} \le -w_3 \rho^{1-\sigma} V(\varepsilon)^{\gamma}. \tag{3.5}$$

(iv) For $\lambda \ge 1$*,* $n \ge 3$*, if* $q > 1 + \max\{\lambda_i\}_{0 \le i \le n}$*,* a_i $(1 \le i \le n)$ *and P satisfy condition (3.3),* $a_n P_{1n} > 0$ *(where* P_{1n} *is the element of P at the first line and nth column), then* $\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}$ *is* C^1 *on* $\mathscr{R}^n$*, and there exists a* $w_4 > 0$ *such that*

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} \le -w_4 \rho^{1-\sigma} V(\varepsilon)^{\gamma}. \tag{3.6}$$

The proofs of (i) and (ii) are quite easy and the proofs of (iii) and (iv) are very similar. The main idea of proofs (iii) and (iv) is to construct a compact set containing the origin on which the derivative of the constructed homogeneous Lyapunov function satisfies some key inequalities. Then inequalities (3.5) and (3.6) are derived by using the homogeneity properties of both the Lyapunov function and system (3.1). The detailed proof is given in the Appendix A.

3.3 GLOBAL FINITE-TIME OBSERVERS FOR NONLINEAR SYSTEM WITH SINGLE RATIONAL POWER IN THE NONLINEAR TERM

In this section, the global finite-time converging observers for a nonlinear system (2.4) for $n \ge 3$ with the condition (2.5) where the rational powers satisfy $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$ will be designed.

In what follows, it will be shown that under the same gain update law as that of semi-global finite-time observers in paper [62], the global finite-time observers for a nonlinear system (2.4) for $n \geq 3$ with the rational powers satisfying $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in the condition (2.5) can be constructed as:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1\lceil e_1 \rfloor^{\alpha_1} + L^{1-(\beta_1-1)(1-\eta)\sigma} a_1 \lceil e_1 \rfloor^{\beta_1} + f_1(y,u), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + L^2 a_2 \lceil e_1 \rfloor^{\alpha_2} + L^{2-(\beta_2-1)(1-\eta)\sigma} a_2 \lceil e_1 \rfloor^{\beta_2} + f_2(y,\hat{x}_2,u), \\ &\vdots \\ \dot{\hat{x}}_n &= L^n a_n \lceil e_1 \rfloor^{\alpha_n} + L^{n-(\beta_n-1)(1-\eta)\sigma} a_n \lceil e_1 \rfloor^{\beta_n} + f_n(y,\hat{x}_2,\ldots,\hat{x}_n,u), \end{cases} \tag{3.7}$$

with the high gain

$$\dot{L} = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3 \Psi(u,y,\hat{x})], \; L(0) > \varphi_2, \tag{3.8}$$

where $\varphi_1, \; \varphi_2 > 1, \; \varphi_3$ are three positive real numbers, $0 < \sigma < 1$ is chosen such that $\theta_{ij} < \frac{i-\sigma}{j-1+\sigma}, \; \upsilon_j < \frac{1-2\sigma}{j-1+\sigma}$ holds, $\Psi(u,y,\hat{x}) = \Gamma(u,y)(1 + \sum_{j=2}^{n} |\hat{x}_j|^{\upsilon_j}), \; \alpha_i = i\alpha^* - (i-1) \; (i = 0,1,\ldots,n), \; \alpha^* \in (1 - \frac{1}{n}, 1), \; \beta_i = i\beta^* - (i-1) \; (i = 0,1,\ldots,n), \; \beta^* > \frac{1+\sigma}{\sigma}, \; 0 < \sigma < 1, \; 0 < \eta < 1 - \alpha^* < 1$.

Note that in paper [59], two homogeneous observers with different degrees are constructed for the global output feedback stabilization problem of a class of nonlinear systems. The following remark summarizes the differences between the homogeneous observer (3.7) in the thesis and the homogeneous observers proposed in [59].

Remark 13. *Note that in [59], a dual observer is employed to solve the problem of global output feedback stabilization for a class of nonlinear systems whose nonlinearities are bounded by both low-order and high-order terms. Comparing the results in [59] with the global finite-time observer (3.7) in the thesis, results in the following statements.*

- *The dual observer [59] is comprised of two seperate homogeneous observers, one estimating the low-order part of unmeasurable states and the other estimating the high-order components. However, here, two homogeneous terms, one of degree less than 1 and the other greater than 1, are introduced in the design of the global finite-time observer simultaneously.*

- *In [59], either the low-order or the high-order observer can only estimate those states in a limited region either close to or far away from the origin, but not all the states in the space. However, the observer (3.7) in the thesis can estimate the states in the whole space.*

- *Both the low-order observer and high-order observer, as well as the coefficients in the observers are derived by a recursive method in [59]. Here, one will see that the global finite-time stability* *of the proposed observer in the thesis will be proved based on Lyapunov theory and all the coefficients in the observer are given explicitly.*

For α_i $(1 \le i \le n)$ and θ_{ij} $(2 \le j \le i \le n)$ in (2.5), they satisfy the following properties:

Lemma 14. *For θ_{ij} $(2 \le j \le i \le n)$ being given by (2.5), if $\frac{i}{j-1} > \theta_{ij} > \frac{n-i}{n-j+1}$, one has $-\alpha_{i-1} + \theta_{ij}\alpha_{j-1} - \alpha^* + 1 > 0$ $(2 \le j \le i \le n)$. Moreover, select $0 < \sigma < 1$ such that $\beta^* > \frac{1+\sigma}{\sigma}$, then one has $\theta_{ij}\beta_{j-1} - \beta_{i-1} < \beta^* - 1$ $(2 \le j \le i \le n)$.*

Proof. To prove $\alpha^* - 1 - \alpha_{j-1}\theta_{ij} + \alpha_{i-1} < 0$ is equivalent to proving $\theta_{ij} > \frac{\alpha_i}{\alpha_{j-1}}$ for $2 \le j \le i \le n$. For $1 - \frac{1}{n} < \alpha^* < 1$, one finds $\frac{\alpha_i}{\alpha_{j-1}} = \frac{i\alpha^* - (i-1)}{(j-1)\alpha^* - (j-2)}$ which is strictly increasing with respect to α^*. Because $\alpha^* < 1$, $\theta_{ij} > \frac{n-i}{n-j+1}$, there exists a $\varepsilon > 0$ such that $\alpha^* < \frac{n-1+\varepsilon}{n}$ and $\theta_{ij} > \frac{n-i+i\varepsilon}{n-j+1+j\varepsilon-\varepsilon}$. Then one finds $\frac{\alpha_i}{\alpha_{j-1}} < \frac{i\frac{n-1+\varepsilon}{n} - (i-1)}{(j-1)\frac{n-1+\varepsilon}{n} - (j-2)} = \frac{n-i+i\varepsilon}{n-j+1+j\varepsilon-\varepsilon} < \theta_{ij}$.

Similarly, for $\beta^* > \frac{1+\sigma}{\sigma}$, it is clear that $\frac{\beta_i}{\beta_{j-1}} = \frac{i\beta^* - (i-1)}{(j-1)\beta^* - (j-2)}$ is strictly increasing with respect to β^*. Then, $\frac{\beta_i}{\beta_{j-1}} > \frac{i\frac{1+\sigma}{\sigma} - (i-1)}{(j-1)\frac{1+\sigma}{\sigma} - (j-2)} = \frac{i+\sigma}{j-1+\sigma} > \theta_{ij}$ $(2 \le j \le i \le n)$. Then one gets $\theta_{ij}\theta_{j-1} - \beta_{i-1} < \beta^* - 1$ $(2 \le j \le i \le n)$.

Thus, the proof is completed. □

In the following, for $n \ge 3$, it will be proven that system (3.7) is a global finite-time observer for a nonlinear system (2.4) with condition (2.5) where the rational powers satisfy $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$. It is in two steps. First the change of the coordinates of the observation error system will be made. Then it will be shown that the observer (3.7) that is proposed can render the error system globally finite-time stable for system (2.4) with condition (2.5) where the rational powers satisfy $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$.

The dynamics of the observation error $e = x - \hat{x}$ between system (2.4) and the global finite-time observer (3.7) are given by

$$\left\{\begin{array}{rcl} \dot{e}_1 &=& e_2 - La_1\lceil e_1\rfloor^{\alpha_1} - L^{1-(\beta_1-1)(1-\eta)\sigma}a_1\lceil e_1\rfloor^{\beta_1}, \\ \dot{e}_2 &=& e_3 - L^2a_2\lceil e_1\rfloor^{\alpha_2} - L^{2-(\beta_2-1)(1-\eta)\sigma}a_2\lceil e_1\rfloor^{\beta_2} + \tilde{f}_2, \\ &\vdots& \\ \dot{e}_n &=& -L^na_n\lceil e_1\rfloor^{\alpha_n} - L^{n-(\beta_n-1)(1-\eta)\sigma}a_n\lceil e_1\rfloor^{\beta_n} + \tilde{f}_n, \end{array}\right. \tag{3.9}$$

where $\tilde{f}_i = f_i(y, x_2, \ldots, x_i, u) - f_i(y, \hat{x}_2, \ldots, \hat{x}_i, u)$ $(2 \le i \le n)$.

Consider the change of coordinates

$$\varepsilon_i = \frac{e_i}{L^{i-1+\sigma}}.$$

Then, the observation error system (3.9) can be expressed as

$$\left\{\begin{array}{rcl} \dot{\varepsilon}_1 & = & L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} - \frac{\dot{L}}{L}\sigma\varepsilon_1 - L^{(\beta_1-1)\eta\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\beta_1}, \\ \dot{\varepsilon}_2 & = & L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2} - \frac{\dot{L}}{L}(\sigma+1)\varepsilon_2 - L^{(\beta_2-1)\eta\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\beta_2} \\ & & +\frac{\tilde{f}_2}{L^{1+\sigma}}, \\ & \vdots & \\ \dot{\varepsilon}_n & = & -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} - \frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n - L^{(\beta_n-1)\eta\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\beta_n} \\ & & +\frac{\tilde{f}_n}{L^{n-1+\sigma}}. \end{array}\right. \tag{3.10}$$

By Lemma 12, for the following two systems with $n \geq 3$

$$\left\{\begin{array}{rcl} \dot{\varepsilon}_1 & = & L\varepsilon_2 - L^{(\beta_1-1)\eta\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\beta_1}, \\ \dot{\varepsilon}_2 & = & L\varepsilon_3 - L^{(\beta_2-1)\eta\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\beta_2}, \\ & \vdots & \\ \dot{\varepsilon}_n & = & -L^{(\beta_n-1)\eta\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\beta_n}, \end{array}\right. \tag{3.11}$$

and

$$\left\{\begin{array}{rcl} \dot{\varepsilon}_1 & = & L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1}, \\ \dot{\varepsilon}_2 & = & L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2}, \\ & \vdots & \\ \dot{\varepsilon}_1 & = & -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n}, \end{array}\right. \tag{3.12}$$

there exist $\underline{c}_1$, $\bar{c}_1$, $c_1 > 0$ and $\underline{c}_2$, $\bar{c}_2$, $c_2 > 0$ such that

$$\underline{c}_1 V_\beta(\varepsilon) \leq \frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon} D_1\varepsilon \leq \bar{c}_1 V_\beta(\varepsilon), \; \left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.11)} < -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1}, \tag{3.13}$$

and

$$\underline{c}_2 V_\alpha(\varepsilon) \leq \frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon} D_1\varepsilon \leq \bar{c}_2 V_\alpha(\varepsilon), \; \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.12)} \leq -c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2}, \tag{3.14}$$

hold, where $V_\beta(\varepsilon)$ is a q_1 h-Lyapunov function of $\bar{V}_\beta(\varepsilon)$ w.r.t. $\chi, L, (\beta_0, \beta_1, \ldots, \beta_{n-1})$, $V_\alpha(\varepsilon)$ is a q_2 h-Lyapunov function of $\bar{V}_\alpha(\varepsilon)$ w.r.t. $\chi, L, (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, q_1, $q_2 > 0$ are two positive real numbers, $\bar{V}_\beta(\varepsilon) = \bar{V}_\alpha(\varepsilon) = \varepsilon^T P\varepsilon$, P satisfies condition (3.3), $\gamma_1 = \frac{q_1+\beta^*-1}{q_1}$, $\gamma_2 = \frac{q_2+\alpha^*-1}{q_2}$.

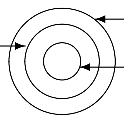
In the following, the global finite-time convergence of the error system (3.9) between the observer (3.7) in the thesis and the nonlinear system (2.4) for $n \geq 3$ with condition (2.5) (where the rational powers satisfy $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$) is proved.

Theorem 15. *If* $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$, *then for* $n \ge 3$, *any* $\alpha^* \in (1-\frac{1}{n}, 1)$, *there exist* $\varphi_i > 0$ $(i = 1,2,3)$, $0 < \sigma < 1$, $\beta^* > \frac{1+\sigma}{\sigma}$ *and* $0 < \eta < 1-\alpha^*$ *such that the system (3.7) with the observer gain (3.8) is a global finite-time observer for the nonlinear system (2.4) under the condition (2.5).*

Proof. From [65] and [73], it is evident that global asymptotic stability and local finite-time stability mean global finite-time stability. Here, this principle will be employed and the proof of the global finite-time convergence of the observation error system will be divided into global asymptotic stability and local finite-time stability.

First of all, for $n \ge 3$, by suitably choosing a_i $(1 \le i \le n)$ such that there exists $P^T = P > 0$ satisfying condition (3.3) and $a_n P_{1n} > 0$, which is always possible.

For $\delta > 0$, define $\mathscr{B}_{V_\alpha,\delta} \triangleq \{\varepsilon : V_\alpha(\varepsilon) < \delta\}$, $\mathscr{B}_{V_\beta,\delta} \triangleq \{\varepsilon : V_\beta(\varepsilon) < \delta\}$. As shown in the following figure, one has $\mathscr{B}_{V_\beta,\delta_3} \subset \mathscr{B}_{V_\beta,\delta_1} \subset \mathscr{B}_{V_\beta,1}$ by choosing $1 > \delta_1 > \delta_3 > 0$ (where δ_1, δ_3 will be given in the proof). The proof is in three parts. First, one uses $V_\beta(\varepsilon)$ to derive $\frac{dV_\beta(\varepsilon)}{dt} < 0$ for $\varepsilon \in \mathscr{R}^n \setminus \mathscr{B}_{V_\beta,1}$ and $\varepsilon \in \mathscr{B}_{V_\beta,1} \setminus \mathscr{B}_{V_\beta,\delta_1}$, respectively. When $\varepsilon \in \mathscr{B}_{V_\beta,\delta_3}$, $V_\alpha(\varepsilon)$ is employed to prove the finite-time stability of the system (3.10). Finally, when $\varepsilon \in \mathscr{B}_{V_\beta,\delta_1} \setminus \mathscr{B}_{V_\beta,\delta_3}$, for $\forall \varepsilon > 0$, there exist $\varphi_i > 0$ $(i = 1,2,3)$ such that $\delta_1 - \delta_3 < \varepsilon$, then by continuity of $\frac{dV_\alpha(\varepsilon)}{dt}$, we obtain $\frac{dV_\alpha(\varepsilon)}{dt} < 0$.



Part I:

When $\varepsilon \in \mathscr{P} = \mathscr{R}^n \setminus \mathscr{B}_{V_\beta,1}$, consider the q_1 h-Lyapunov function $V_\beta(\varepsilon)$. Based on (3.13), calculating the derivative of $V_\beta(\varepsilon)$ along the solution of the system (3.10), one has

$$\begin{aligned}\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} &= \left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.11)} + \varphi_1(L^{1-\sigma} - \varphi_2)\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T D_1\varepsilon - \varphi_3\Psi(u,y,\hat{x})\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \\ &\times D_1\varepsilon + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1 + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F} \le -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1\varphi_1(L^{1-\sigma} - \varphi_2)V_\beta(\varepsilon) \\ &- \underline{c}_1\varphi_3\Psi(u,y,\hat{x})V_\beta(\varepsilon) + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1 + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F},\end{aligned} \tag{3.15}$$

where $\tilde{G}_1 = \begin{pmatrix} -L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} \\ \vdots \\ -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} \end{pmatrix}$, $\tilde{F} = \begin{pmatrix} 0 \\ \frac{\tilde{f}_2}{L^{1+\sigma}} \\ \vdots \\ \frac{\tilde{f}_n}{L^{n-1+\sigma}} \end{pmatrix}$.

For $\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon}^T \tilde{G}_1$, by Lemma 8, one has

$$\begin{aligned} \frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon}^T \tilde{G}_1 &\le L^{1-(1-\alpha^*)\sigma}a^*\sum_{i=1}^{n}\left|\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_1|^{\alpha_i} \le L^{1-(1-\alpha^*)\sigma}a^*k_1\sum_{i=1}^{n}V_\beta(\varepsilon)^{\frac{q_1-\beta_{i-1}+\alpha_i}{q_1}} \\ &\le L^{1-(1-\alpha^*)\sigma}a^*k_1 n V_\beta(\varepsilon), \end{aligned} \tag{3.16}$$

where $k_1 = \max_{\{z:V_\beta(z)=1\}}\left|\frac{\partial V_\beta(z)}{\partial z_i}\right||z_1|^{\alpha_i}$, $a^* = \max_{1\le i\le n}\{a_i\}$.

For $\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon}^T \tilde{F}$, one has

$$\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon}^T \tilde{F} \le \Psi(u,y,\hat{x})\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon_i}\right|\frac{|e_j|}{L^{i-1+\sigma}} + l\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon_i}\right|\frac{|e_j|^{\theta_{ij}}}{L^{i-1+\sigma}}.$$

Note that under the condition $\theta_{ij} < \frac{i}{j-1}$, there exists a $\sigma_1 > 0$ such that $\theta_{ij} < \frac{i}{j-1+\sigma_1}$, $v_j < \frac{1-\sigma_1}{j-1+\sigma_1}$ $(2 \le j \le i \le n)$, and let $0 < \sigma < \sigma_1$. Because $L(t) > \varphi_2 > 1$, one has

$$L^{(j-1+\sigma)\theta_{ij}-(i-1+\sigma)} < L^{1-\sigma}.$$

Then, similarly by Lemma 8, one has

$$\begin{aligned} \frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon}^T \tilde{F} &\le \Psi(u,y,\hat{x})\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j| + lL^{1-\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V_\beta(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j|^{\theta_{ij}} \\ &\le k_2\Psi(u,y,\hat{x})\sum_{i=2}^{n}\sum_{j=2}^{i}V_\beta(\varepsilon)^{\frac{q_1-\beta_{i-1}+\beta_{j-1}}{q_1}} + lk_3L^{1-\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}V_\beta(\varepsilon)^{\frac{q_1-\beta_{i-1}+\theta_{ij}\beta_{j-1}}{q_1}} \\ &\le k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon) + lk_3n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{q_1+\bar{\beta}}{q_1}}, \end{aligned} \tag{3.17}$$

where $\bar{\beta} = \max_{2\le j\le i\le n}\{\theta_{ij}\beta_{j-1} - \beta_{i-1}\}$, $k_2 = \max_{\{z:V_\beta(z)=1\}}\left|\frac{\partial V_\beta(z)}{\partial z_i}\right||z_j|$, $k_3 = \max_{\{z:V_\beta(z)=1\}}\left|\frac{\partial V_\beta(z)}{\partial z_i}\right||z_j|^{\theta_{ij}}$.

Then, by substituting (3.16) and (3.17) into (3.15), one finds

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} \le -c_1L^{1-\eta\sigma}V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1\varphi_1L^{1-\sigma}V_\beta(\varepsilon) - \bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon) - \underline{c}_1\varphi_3\Psi(u,y,\hat{x})V_\beta(\varepsilon)$$

$$+L^{1-(1-\alpha^*)\sigma}a^*k_1nV_\beta(\varepsilon)+k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon)+lk_3n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{q_1+\bar{\beta}}{q_1}}. \tag{3.18}$$

From Lemma 10, it is known that $\gamma_1 > \frac{q_1+\bar{\beta}}{q_1}$. Then, for all $\varepsilon \in \mathscr{P}$, there exist $d_{11} > 0,\ d_{21} > 1,\ d_{31} > 0$ such that when $0 < \varphi_1 < d_{11},\ \varphi_2 > d_{21},\ \varphi_3 > d_{31}$, one has

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} \leq -\bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon),\ \varepsilon \in \mathscr{P}, \tag{3.19}$$

where $d_{11} = \frac{c_1}{3\bar{c}_1},\ d_{21} = \max\{(\frac{3a^*k_1n}{c_1})^{\frac{1}{(1-\alpha^*-\eta)\sigma}}, (\frac{3lk_3n^2}{c_1})^{\frac{1}{(1-\eta)\sigma}}\},\ d_{31} = \frac{k_2n^2}{\underline{c}_1}$.

When $\varepsilon \in \mathscr{B}_{V_\beta,1}$, one again uses the q_1 h-Lyapunov function $V_\beta(\varepsilon)$. First, one has

$$\begin{aligned}
\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T\tilde{G}_1 &\leq L^{1-(1-\alpha^*)\sigma}a^*k_1nV_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\alpha_n}{q_1}},\\
\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T\tilde{F} &\leq k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\beta^*}{q_1}}+lk_3n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{q_1+\underline{\beta}}{q_1}}.
\end{aligned} \tag{3.20}$$

Then from (3.15) and (3.20), one obtains

$$\begin{aligned}
\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} \leq\ & -c_1L^{1-\eta\sigma}V_\beta(\varepsilon)^{\gamma_1}+\bar{c}_1\varphi_1(L^{1-\sigma}-\varphi_2)V_\beta(\varepsilon)-\underline{c}_1\varphi_3\Psi(u,y,\hat{x})V_\beta(\varepsilon)\\
&+L^{1-(1-\alpha^*)\sigma}a^*k_1nV_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\alpha_n}{q_1}}+k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\beta^*}{q_1}}\\
&+lk_3n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{q_1+\underline{\beta}}{q_1}},
\end{aligned}$$

where $\underline{\beta} = \min_{2\leq j\leq i\leq n}\{\theta_{ij}\beta_{j-1}-\beta_{i-1}\}$.

There exists a $d_{22} > 1$ such that $0 < g_{11} < g_{13} < 1,\ 0 < g_{12},\ g_{14} < 1$ when $0 < \varphi_1 < d_{11},\ \varphi_2 > d_{22},\ \varphi_3 > d_{31}$. Then one has

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} \leq -\bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon),\ \varepsilon \in \mathscr{B}_{V_\beta,1}\setminus\mathscr{B}_{V_\beta,\delta_1}, \tag{3.21}$$

where $\delta_1 = \max\{g_{12},\ g_{13},\ g_{14}\},\ g_{11} = (\frac{3\bar{c}_1\varphi_1}{c_1})^{\frac{q_1}{\beta^*-1}}\varphi_2^{-\frac{(1-\eta)\sigma q_1}{\beta^*-1}},\ g_{12} = (\frac{3lk_3n^2}{c_1})^{\frac{q_1}{\beta^*-\underline{\beta}-1}}\varphi_2^{-\frac{(1-\eta)\sigma q_1}{\beta^*-\underline{\beta}-1}},\ g_{13} = (\frac{3a^*k_1n}{c_1})^{\frac{q_1}{\beta_n-\alpha_n}}\varphi_2^{-\frac{(1-\alpha^*-\eta)\sigma q_1}{\beta_n-\alpha_n}},\ g_{14} = (\frac{k_2n^2}{\underline{c}_1})^{\frac{q_1}{\beta_{n-1}-\beta^*}}\varphi_3^{-\frac{q_1}{\beta_{n-1}-\beta^*}}$.

Thus, from (3.19) and (3.21), one can derive

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.10)} \leq -\bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon),\ \varepsilon \in \mathscr{R}^n\setminus\mathscr{B}_{V_\beta,\delta_1}. \tag{3.22}$$

Part II:

In this part, the set $\varepsilon \in \mathscr{B}_{V_\beta,\delta_1}$ will be considered. Here, the q_2 h-Lyapunov function $V_\alpha(\varepsilon)$ is used.

Because $V_\beta(\varepsilon)$, $V_\alpha(\varepsilon)$ are homogeneous of degrees q_1 and q_2, respectively, one has $V_\alpha(\varepsilon) \leq k^* V_\beta(\varepsilon)^{\frac{q_2}{q_1}}$, where $k^* = \max_{\{z:V_\beta(z)=1\}} V_\alpha(z)$. Then there exist $d_{23} > 1,\ d_{32} > 0$ such that $k^* \delta_1^{\frac{q_2}{q_1}} \leq 1$, i.e., $V_\alpha(\varepsilon) \leq 1$ when $\varphi_2 > d_{23},\ \varphi_3 > d_{32}$.

Under the above condition, based on (3.14), calculating the derivative of $V_\alpha(\varepsilon)$ along the solution of system (3.10), using the same method as that in part I, one finds

$$\begin{aligned} \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)} &\leq -c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2} + \bar{c}_2 \varphi_1 (L^{1-\sigma} - \varphi_2) V_\alpha(\varepsilon) - \underline{c}_2 \varphi_3 \Psi(u,y,\hat{x}) V_\alpha(\varepsilon) \\ &\quad + \frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_2 + \frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{F}, \end{aligned} \tag{3.23}$$

where $\tilde{G}_2 = (-L^{(\beta_1-1)\eta\sigma+1} a_1 \lceil \varepsilon_1 \rfloor^{\beta_1}, \ldots, -L^{(\beta_n-1)\eta\sigma+1} a_n \lceil \varepsilon_1 \rfloor^{\beta_n})^T$.

For $\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_2$ and $\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{F}$, similarly, one has

$$\begin{aligned} \frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_2 &\leq L^{(\beta_n-1)\eta\sigma+1} a^* k_4 n V_\alpha(\varepsilon), \\ \frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{F} &\leq k_5 n^2 \Psi(u,y,\hat{x}) V_\alpha(\varepsilon) + l k_6 n^2 L^{1-\sigma} V_\alpha(\varepsilon)^{\frac{q_2+\underline{\alpha}}{q_2}}, \end{aligned} \tag{3.24}$$

where $\underline{\alpha} = \min_{2\leq j\leq i\leq n}\{\theta_{ij}\alpha_{j-1} - \alpha_{i-1}\}$, $k_4 = \max_{\{z:V_\alpha(z)=1\}} \left|\frac{\partial V_\alpha(z)}{\partial z_i}\right| |z_1|^{\beta_i}$, $k_5 = \max_{\{z:V_\alpha(z)=1\}} \left|\frac{\partial V_\alpha(z)}{\partial z_i}\right| |z_j|$, $k_6 = \max_{\{z:V_\alpha(z)=1\}} \left|\frac{\partial V_\alpha(z)}{\partial z_i}\right| |z_j|^{\theta_{ij}}$.

Then by substituting (3.24) into (3.23), one has

$$\begin{aligned} \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)} &\leq -c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2} + \bar{c}_2 \varphi_1 (L^{1-\sigma} - \varphi_2) V_\alpha(\varepsilon) - \underline{c}_2 \varphi_3 \Psi(u,y,\hat{x}) V_\alpha(\varepsilon) \\ &\quad + L^{(\beta_n-1)\eta\sigma+1} a^* k_4 n V_\alpha(\varepsilon) + k_5 n^2 \Psi(u,y,\hat{x}) V_\alpha(\varepsilon) + l k_6 n^2 L^{1-\sigma} V_\alpha(\varepsilon)^{\frac{q_2+\underline{\alpha}}{q_2}}. \end{aligned}$$

From Lemma 9 and Lemma 10, it is known $\gamma_2 < \frac{q_2+\underline{\alpha}}{q_2}$, $\varphi_2 < L(t) < M$. In addition, because $0 < \varphi_1 < d_{11}$, there exists a $d_{24} > 0$ such that $g_{22} < g_{21},\ g_{22} < g_{23}$ when $L(t) > \varphi_2 > d_{24}$. Moreover, there exists a $d_{33} > 0$ such that when $\varphi_3 > d_{33},\ \varphi_2 > d_{24}$, one has

$$\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)} \leq -\frac{1}{4} c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2},\ \varepsilon \in \mathscr{B}_{V_\alpha,\delta_2} \setminus \{0\},$$

where $\delta_2 = g_{22}$, $d_{33} = \frac{k_5 n^2}{\underline{c}_2}$, $g_{21} = (\frac{c_2}{4\bar{c}_2\varphi_1})^{\frac{q_2}{1-\alpha^*}}$, $g_{22} = (\frac{c_2}{4a^* k_4 n})^{\frac{q_2}{1-\alpha^*}} \varphi_2^{-\frac{\sigma(1+(\beta_n-1)\eta)q_2}{1-\alpha^*}}$, $g_{23} = (\frac{c_2}{4lk_6 n^2})^{\frac{q_2}{\underline{\alpha}-\alpha^*+1}}$.

Then, by Lemma 3, the system (3.10) is locally finite-time stable on $\mathscr{B}_{V_\alpha,\delta_2}$.

From $V_\alpha(\varepsilon) \leq k^* V_\beta(\varepsilon)^{\frac{q_2}{q_1}}$, one can obtain $\mathscr{B}_{V_\beta,\delta_3} \subset \mathscr{B}_{V_\alpha,\delta_2}$, where $\delta_3 = (\frac{g_{22}}{k^*})^{\frac{q_1}{q_2}} =$

$\varphi_2^{-\frac{\sigma(1+(\beta_n-1)\eta)q_1}{1-\alpha^*}}(\frac{1}{k^*})^{\frac{q_1}{q_2}}(\frac{c_2}{4a^*k_4n})^{\frac{q_1}{1-\alpha^*}}$. Then, $\mathscr{B}_{V_\beta,\delta_3}$ is a domain of observer attraction, i.e.,

$$\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)} \leq -\frac{1}{4}c_2L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2},\ \varepsilon \in \mathscr{B}_{V_\beta,\delta_3} \setminus \{0\}. \tag{3.25}$$

Part III:

For any $\varepsilon > 0$, there exist sufficiently large φ_2, φ_3 and $0 < \varphi_1 < d_{11}$, $\varphi_2 > d_{2i}$ $(1 \leq i \leq 4)$, $\varphi_3 > d_{3j}$ $(1 \leq j \leq 3)$ such that $0 < \delta_1 - \delta_3 < \varepsilon$. Because $\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)}$ is continuous on $\mathscr{R}^n$, one has

$$\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.10)} < 0,\ \varepsilon \in \mathscr{B}_{V_\beta,\delta_1} \setminus \mathscr{B}_{V_\beta,\delta_3}. \tag{3.26}$$

Thus, from (3.22), (3.25) and (3.26), by combining global asymptotic stability and local finite-time stability, one gets the result that the system (3.10) is globally finite-time stable, i.e., there exists a $T_1 > 0$ such that $\varepsilon_i(t) = 0$ when $t > T_1$.

From Lemma 9, there exists an $M^* > 0$ such that $L^{i-1+\sigma} \leq M^*$ $(i = 1,\ldots,n)$. Then, one has $\frac{e_i(t)}{M^*} \leq \frac{e_i(t)}{L^{i-1+\sigma}} = \varepsilon_i(t) = 0$ $(t > T_1)$, i.e., $e_i(t) = 0$ $(t > T_1)$ $(i = 1,\ldots,n)$, which means system (3.7) is a global finite-time observer for system (2.4) under the condition (2.5).

This completes the proof. □

3.4 GLOBAL FINITE-TIME OBSERVERS FOR NONLINEAR SYSTEM WITH MIXED RATIONAL POWERS IN THE NONLINEAR TERM

In this section, it will be proven that system (3.7) with high gain (3.8) is also a global finite-time observer for nonlinear system (2.4) under condition (2.6) with the mixed rational powers in the non-linearities satisfying $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$.

Theorem 16. *If* $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, *then for any* $\alpha^* \in (1-\frac{1}{n},1)$, *there exist* $0 < \sigma < 1$, $\beta^* > \frac{1+\sigma}{\sigma}$ *and* $0 < \eta < 1-\alpha^*$ *such that global finite-time observers in the form (3.7) with the observer gain (3.8) can be designed for the nonlinear systems (2.4) with condition (2.6).*

Similar to what is done in Section 3.3, through changing the coordinates $\varepsilon_i = \frac{e_i}{L^{i-1+\sigma}}$, $e_i = x_i - \hat{x}_i$, the observation error systems between the nonlinear system (2.4) and the designed global finite-time observers (3.7) are shown as follows:

$$\begin{cases} \dot{\varepsilon}_1 = L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} - \frac{\dot{L}}{L}\sigma\varepsilon_1 - L^{(\beta_1-1)\eta\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\beta_1}, \\ \dot{\varepsilon}_2 = L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2} - \frac{\dot{L}}{L}(\sigma+1)\varepsilon_2 - L^{(\beta_2-1)\eta\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\beta_2} \\ \quad +\frac{\tilde{f}_2}{L^{1+\sigma}}, \\ \vdots \\ \dot{\varepsilon}_n = -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} - \frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n - L^{(\beta_n-1)\eta\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\beta_n} \\ \quad +\frac{\tilde{f}_n}{L^{n-1+\sigma}}, \end{cases} \tag{3.27}$$

where $\tilde{f}_i = f_i(y,x_2,\ldots,x_i,u) - f_i(y,\hat{x}_2,\ldots,\hat{x}_i,u)$ $(2 \le i \le n)$.

Before the global finite-time stability of the proposed result is proved, similarly to Lemma 14, the following lemma is given to show some properties of α_i $(1 \le i \le n)$ and $\theta_{1,ij}$, $\theta_{2,ij}$ $(2 \le j \le i \le n)$ in (2.5).

Lemma 17. *For $\theta_{1,ij}$, $\theta_{2,ij}$ $(2 \le j \le i \le n)$ in (2.5), if $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$, one has $-\alpha_{i-1} + \theta_{1,ij}\alpha_{j-1} - \alpha^* + 1 > 0$, $-\alpha_{i-1} + \theta_{2,ij}\alpha_{j-1} - \alpha^* + 1 > 0$ $(2 \le j \le i \le n)$. Moreover, select $0 < \sigma < 1$ such that $\beta^* > \frac{1+\sigma}{\sigma}$, then one has $\theta_{2,ij}\beta_{j-1} - \beta_{i-1} < \beta^* - 1$ $(2 \le j \le i \le n)$.*

In the following, the detailed proof of the global finite-time stability of the observation error system (3.27) will be given.

Proof. The proof of Theorem 16 is similar to that of Theorem 15 and is also divided into three parts. First, choose a_i $(1 \le i \le n)$, P to satisfy (3.3) and $a_nP_{1n} > 0$.

Part I:

When $\varepsilon \in \mathscr{P}$ ($\mathscr{P}$ is the same as defined in Section 3.3), one uses the Lyapunov function $V_\beta(\varepsilon)$. Using the same method as that in the proof of Theorem 15, one has

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.27)} \le -c_1L^{1-\eta\sigma}V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1\varphi_1(L^{1-\sigma}-\varphi_2)V_\beta(\varepsilon) - \underline{c}_1\varphi_3\Psi(u,y,\hat{x})V_\beta(\varepsilon)$$
$$+L^{1-(1-\alpha^*)\sigma}a^*k_1nV_\beta(\varepsilon) + k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon) + l_1k_7n^2V_\beta(\varepsilon) + l_2k_8n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{r_1+\bar{\beta}_2}{r_1}},$$

where $\bar{\beta}_2 = \max_{2\le j\le i\le n}\{\theta_{2,ij}\beta_{j-1} - \beta_{i-1}\}$, $k_7 = \max_{\{z:V_\beta(z)=1\}}|\frac{\partial V_\beta(z)}{\partial z_i}||z_j|^{\theta_{1,ij}}$, $k_8 = \max_{\{z:V_\beta(z)=1\}}|\frac{\partial V_\beta(z)}{\partial z_i}||z_j|^{\theta_{2,ij}}$.

Then, for all $\varepsilon \in \mathscr{P}$, there exist $s_{11} > 0$, $s_{21} > 1$, $s_{31} > 0$ such that when $0 < \varphi_1 < s_{11}$, $\varphi_2 > s_{21}$, $\varphi_3 >$

s_{31}, one has

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.27)} \leq -\bar{c}_1\varphi_1\varphi_2 V_\beta(\varepsilon),$$

where $s_{11} = \frac{c_1}{4\bar{c}_1}$, $s_{21} = \max\{(\frac{4a^*k_1n}{c_1})^{\frac{1}{(1-\alpha^*-\eta)\sigma}}, (\frac{4l_1k_7n^2}{c_1})^{\frac{1}{1-\eta\sigma}}, (\frac{4l_2k_8n^2}{c_1})^{\frac{1}{(1-\eta)\sigma}}\}$, $s_{31} = d_{31} = \frac{k_2n^2}{\underline{c}_1}$.

When $\varepsilon \in \mathscr{B}_{V_\beta,1}$, one has

$$\begin{aligned}\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.27)} \leq\; & -c_1L^{1-\eta\sigma}V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1\varphi_1(L^{1-\sigma}-\varphi_2)V_\beta(\varepsilon) - \underline{c}_1\varphi_3\Psi(u,y,\hat{x})V_\beta(\varepsilon) \\ & + L^{1-(1-\alpha^*)\sigma}a^*k_1nV_\beta(\varepsilon)^{\frac{r_1-\beta_{n-1}+\alpha_n}{r_1}} + k_2n^2\Psi(u,y,\hat{x})V_\beta(\varepsilon)^{\frac{r_1-\beta_{n-1}+\beta^*}{r_1}} \\ & + l_1k_7n^2V_\beta(\varepsilon)^{\frac{r_1+\underline{\beta}_1}{r_1}} + l_2k_8n^2L^{1-\sigma}V_\beta(\varepsilon)^{\frac{r_1+\underline{\beta}_2}{r_1}},\end{aligned}$$

where $\underline{\beta}_1 = \min_{2\leq j\leq i\leq n}\{\theta_{1,ij}\beta_{j-1}-\beta_{i-1}\}$, $\underline{\beta}_2 = \min_{2\leq j\leq i\leq n}\{\theta_{2,ij}\beta_{j-1}-\beta_{i-1}\}$.

There exist $s_{22} > 1$, $s_{32} > 0$ such that $g_{42} > g_{41}$, $g_{43} > g_{44}$ when $\varphi_2 > s_{22}$, $\varphi_3 > s_{32}$, then one can get

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.27)} \leq -\bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon),\ \varepsilon \in \mathscr{B}_{V_\beta,1} \setminus \mathscr{B}_{V_\beta,\delta_4},$$

where $\delta_4 = \max\{g_{42},\ g_{43},\ g_{45}\}$, $g_{41} = (\frac{4\bar{c}_1\varphi_1}{c_1})^{\frac{r_1}{\beta^*-1}}\varphi_2^{-\frac{(1-\eta)\sigma r_1}{\beta^*-1}}$, $g_{42} = (\frac{4a^*k_1n}{c_1})^{\frac{r_1}{\beta_n-\alpha_n}}\varphi_2^{-\frac{(1-\alpha^*-\eta)\sigma r_1}{\beta_n-\alpha_n}}$, $g_{43} = (\frac{4l_1k_7n^2}{c_1})^{\frac{r_1}{\beta^*-1-\underline{\beta}_1}}\varphi_2^{-\frac{(1-\eta\sigma)r_1}{\beta^*-1-\underline{\beta}_1}}$, $g_{44} = (\frac{4l_2k_8n^2}{c_1})^{\frac{r_1}{\beta^*-1-\underline{\beta}_2}}\varphi_2^{-\frac{(1-\eta)\sigma r_1}{\beta^*-1-\underline{\beta}_2}}$, $g_{45} = (\frac{k_2n^2}{\underline{c}_1})^{\frac{r_1}{\beta_{n-1}-\beta^*}}\varphi_3^{-\frac{r_1}{\beta_{n-1}-\beta^*}}$.

Then, one has

$$\left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(3.27)} \leq -\bar{c}_1\varphi_1\varphi_2V_\beta(\varepsilon),\ \varepsilon \in \mathscr{R}^n \setminus \mathscr{B}_{V_\beta,\delta_4}, \tag{3.28}$$

Part II:

For $\varepsilon \in \mathscr{B}_{V_\beta,\delta_4}$, the Lyapunov function $V_\alpha(\varepsilon)$ is considered.

From the proof of Theorem 15, it is known that there exist $s_{23} > 1$, $s_{33} > 0$ such that $V_\alpha(\varepsilon) \leq 1$ when $\varphi_2 > s_{23}$, $\varphi_3 > s_{33}$, where $s_{23} = d_{23}$, $s_{33} = d_{32}$. Under this condition, calculating the derivative along the system (3.27), one finds

$$\begin{aligned}\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(3.27)} \leq\; & -c_2L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2} + \bar{c}_2\varphi_1(L^{1-\sigma}-\varphi_2)V_\alpha(\varepsilon) - \underline{c}_2\varphi_3\Psi(u,y,\hat{x})V_\alpha(\varepsilon) \\ & + L^{(\beta_n-1)\eta\sigma+1}a^*k_4nV_\alpha(\varepsilon) + k_5n^2\Psi(u,y,\hat{x})V_\alpha(\varepsilon) + l_1k_9n^2V_\alpha(\varepsilon)^{\frac{r_2+\alpha_1}{r_2}} \\ & + l_2k_{10}n^2L^{1-\sigma}V_\alpha(\varepsilon)^{\frac{r_2+\alpha_2}{r_2}},\end{aligned}$$

where $\underline{\alpha}_1 = \min_{2\le j\le i\le n}\{\theta_{1,ij}\alpha_{j-1} - \alpha_{i-1}\}$, $\underline{\alpha}_2 = \min_{2\le j\le i\le n}\{\theta_{2,ij}\alpha_{j-1} - \alpha_{i-1}\}$, $k_9 = \max_{\{z:V_\alpha(z)=1\}} |\frac{\partial V_\alpha(z)}{\partial z_i}||z_j|^{\theta_{1,ij}}$, $k_{10} = \max_{\{z:V_\alpha(z)=1\}} |\frac{\partial V_\alpha(z)}{\partial z_i}||z_j|^{\theta_{2,ij}}$.

Because $0 < \varphi_1 < s_{11}$, there exists an $s_{24} > 0$ such that $g_{51} < g_{5i}$ $(2 \le i \le 4)$ when $\varphi_2 > s_{24}$. Moreover, there exists an $s_{34} > 0$ such that when $\varphi_3 > s_{34}$, $\varphi_2 > s_{24}$, one has

$$\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(3.27)} \le -\frac{1}{4}c_2L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2} < 0,\ \varepsilon \in \mathscr{B}_{V_\alpha,\delta_5} \setminus \{0\},$$

where $\delta_5 = g_{51}$, $s_{34} = d_{33} = \frac{k_5 n^2}{c_2}$, $g_{51} = (\frac{c_2}{5a^* k_4 n})^{\frac{r_2}{1-\alpha^*}} M^{-\frac{\sigma(1+(\beta_n-1)\eta)r_2}{1-\alpha^*}}$, $g_{52} = (\frac{c_2}{5\bar{c}_2\varphi_1})^{\frac{r_2}{1-\alpha^*}}$, $g_{53} = (\frac{c_2}{5l_1 k_9 n^2})^{\frac{r_2}{\underline{\alpha}_1+1-\alpha^*}}$, $g_{54} = (\frac{c_2}{5l_2 k_{10} n^2})^{\frac{r_2}{\underline{\alpha}_2+1-\alpha^*}}$.

Let $\delta_6 = (\frac{g_{51}}{k^*})^{\frac{r_1}{r_2}}$, then $\mathscr{B}_{V_\beta,\delta_6} \subset \mathscr{B}_{V_\alpha,\delta_5}$. Then,

$$\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(3.27)} \le -\frac{1}{5}c_2L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2},\ \varepsilon \in \mathscr{B}_{V_\beta,\delta_6} \setminus \{0\}, \tag{3.29}$$

holds, i.e., $\mathscr{B}_{V_\beta,\delta_6}$ is a domain of observer attraction.

Part III:

For any $\varepsilon^* > 0$, choose φ_2, φ_3 sufficiently large and $0 < \varphi_1 < s_{11}$, $\varphi_2 > s_{2i}$, $\varphi_3 > s_{3i}$ $(1 \le i \le 4)$ such that $0 < \delta_4 - \delta_6 < \varepsilon^*$. By continuity of $\frac{dV_\alpha(\varepsilon)}{dt}\Big|_{(3.27)}$ on $\mathscr{R}^n$, one has

$$\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(3.27)} < 0,\ \varepsilon \in \mathscr{B}_{V_\beta,\delta_4} \setminus \mathscr{B}_{V_\beta,\delta_6}. \tag{3.30}$$

Thus, from (3.28), (3.29) and (3.30), by combining global asymptotic stability and local finite-time stability, the result is that the observation error system (3.27) is globally finite-time stable, i.e., the system (3.7) is also a global finite-time observer for the system (2.4) with condition (2.6) where the mixed rational powers in the nonlinearities satisfying $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1$, $1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \le j \le i \le n)$.

The proof is completed. □

From what is stated in Chapter 2, it is known that the main results of the thesis is to extend the semi-global finite-time observers designed in paper [62] to global finite-time observers for a broader class of nonlinear systems (2.4) with lower bounds of θ_{ij}, $\theta_{1,ij}$, $\theta_{2,ij}$ $(2 \le j \le i \le n)$ in conditions (2.5) and (2.6). The following remark will show that compared with the result in paper [62], how the global finite-time observers (3.7) proposed in this chapter are obtained.

Remark 18. *Compared with the result in this chapter with that in paper [62], in both of the results, the same dynamic high gain is used and it is the term* $\lceil e_1 \rfloor^{\alpha_1}, \lceil e_1 \rfloor^{\alpha_2}, \ldots, \lceil e_1 \rfloor^{\alpha_n}$ *that makes the observation error systems locally finite-time stable. In this chapter, based on the result from Lemma 12, the result in paper [62] is able to be extended to nonlinear systems (2.4) with lower-bound rational and mixed rational powers satisfying* $\frac{n-i}{n-j+1} \leq \theta_{ij} \leq \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ *and* $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ *in condition (2.5) and condition (2.6). Moreover, the term* $\lceil e_1 \rfloor^{\beta_1}, \lceil e_1 \rfloor^{\beta_2}, \ldots, \lceil e_1 \rfloor^{\beta_n}$ *in the observers proposed in this chapter ensures that the observation error systems converge into a spherical area of the origin and the proof is based on two new different homogeneous Lyapunov functions. That is how the global finite-time observers are achieved in this chapter.*

3.5 SUMMARY

In this chapter, global finite-time observers are designed for a class of nonlinear systems with bounded rational and mixed rational powers imposed on the nonlinear terms. The proposed global finite-time observers have a dynamic high gain and two homogeneous terms, one ensuring the observation error systems asymptotically converge to a spherical area around the origin, the other ensuring local finite-time stability. Global finite-time stability is obtained through the combination of the global asymptotic stability and local finite-time stability.

CHAPTER 4

GLOBAL FINITE-TIME OBSERVERS — WITH A NEW GAIN UPDATE LAW

In this chapter, a new kind of global finite-time observer is designed for the same class of nonlinear systems as those in Chapter 3, which have bounded rational and mixed rational powers imposed on the incremental nonlinearities. Compared with the results in Chapter 3, the newly designed observers have a new high gain and can be applied to nonlinear systems with any dimension. Moreover, the proposed observers in this chapter do not impose any condition for $a_n P_{1n}$ (where a_n, P_{1n} are given in Chapter 3).

4.1 INTRODUCTION

Nonlinear observer design is one of the most important problems in the field of nonlinear control. Over the years, a great deal of work has been done and various nonlinear observer design methods [13], [12], [14], [16] have been developed, where the high gain method [73], [72], [17], [28] plays a very important role. Asymptotic and finite-time observers exist for Lipschitz nonlinear systems based on dynamic high gain [72] and constant high gains [13], [55], [73], [54]. High-gain observers in the presence of measurement noise are employed to the output feedback control problem for a class of nonlinear systems through a switched-gain approach in [17]. Then, for a class of nonlinear systems with the nonlinear terms admitting an incremental rate of the measured output, asymptotic high-gain observers are employed for a global output feedback design in [23] and [28] through different methods. Motivated by the result in [28], for the same kind of nonlinear systems with nonlinear terms admitting an incremental rate depending only on the output, [60] designed a global high-gain finite-time observer.

Then [61] and [62] made further extensions to a high-gain asymptotic and semi-global finite-time observer design to two broader classes of nonlinear systems with bounded rational and mixed rational powers imposed on the nonlinear incremental rate. Based on the result in [63] where asymptotic stability and finite-time stability are studied for a class of nonlinear homogeneous systems and the best possible lower bound of homogeneity of degree is obtained, [74] constructs global finite-time observers for nonlinear systems (2.4) under conditions (2.5) and (2.6) with smaller lower bounds of the rational and mixed rational powers than those in either [61] or [62].

The purpose of this chapter is to make an attempt to design global finite-time observers with a new gain update law for the nonlinear system (2.4) with rational and mixed rational powers satisfying $\frac{n-i}{n-j+1} \leq \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in condition (2.5) and $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in condition (2.6), respectively.

This chapter is organized as follows: In Section 4.2, global finite-time observers with a new gain update law are designed for nonlinear systems (2.4) under condition (2.5) with rational powers in the nonlinearities. Then, in Section 4.3, it will be shown that the designed observers in Section 4.2 can also be applied to nonlinear systems (2.4) under condition (2.5) with mixed rational powers in the nonlinearities. A summary is given in Section 4.4.

4.2 GLOBAL FINITE-TIME OBSERVERS FOR NONLINEAR SYSTEM WITH SINGLE RATIONAL POWER IN THE NONLINEAR TERM

Before the result presented, a useful result is reviewed first.

For the rational powers $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in condition (2.5) and $1 - \frac{1}{n} < \alpha^* < 1$, from Lemma 14 in Chapter 3, the result is: $\alpha^* - 1 - \alpha_{j-1}\theta_{ij} + \alpha_{i-1} < 0$.

In the following, it will be proven that the observer of the following form

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1\lceil e_1 \rfloor^{\alpha_1} + f_1(y,u), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + L^2 a_2\lceil e_1 \rfloor^{\alpha_2} + f_2(y,\hat{x}_2,u), \\ &\vdots \\ \dot{\hat{x}}_n &= L^n a_n\lceil e_1 \rfloor^{\alpha_n} + f_n(y,\hat{x}_2,\ldots,\hat{x}_n,u), \end{cases} \tag{4.1}$$

with the following dynamic gain

$$\begin{aligned}\dot{L} &= -L[\varphi_1(L^{1-\sigma}-\varphi_2)-\varphi_3\Psi(u,y,\hat{x})-\varphi_4 L^{1-2\sigma}|y-\hat{x}_1|^m \\ &\quad -\varphi_5\Psi(u,y,\hat{x})|y-\hat{x}_1|^m],\ L(0)>\varphi_2, \end{aligned} \tag{4.2}$$

is a global finite-time observer for a nonlinear system (2.4) with condition (2.5), where $\varphi_1,\ \varphi_2 > 1,\ \varphi_3,\ \varphi_4,\ \varphi_5$ are five positive numbers, m is a positive number satisfying

$$m \geq \max\{\alpha_{j-1}\theta_{ij}-\alpha_{i-1},\ 1\},\ 2\leq j\leq i\leq n, \tag{4.3}$$

where $\Psi(u,y,\hat{x}) = \Gamma(u,y)(1+\sum_{j=2}^{n}|\hat{x}_j|^{\nu_j})$.

For the gain update law $L(t)$ in (4.2), one has the following result:

Lemma 19. *For the observer gain $L(t)$ in (4.2), there exists $M>0$ such that $L(t)<M, t\in[0,T],\ \forall T\in(0,\infty)$.*

Proof. The proof is simple and similar to the proof of the boundedness of the dynamic observer high gain in [62], thus omitted here. □

The dynamics of the observation error $e = x-\hat{x}$ are given by

$$\begin{cases}\dot{e}_1 = e_2 - La_1\lceil e_1\rfloor^{\alpha_1}, \\ \dot{e}_2 = e_3 - L^2a_2\lceil e_1\rfloor^{\alpha_2}+\bar{f}_2, \\ \quad\vdots \\ \dot{e}_n = -L^na_n\lceil e_1\rfloor^{\alpha_n}+\bar{f}_n, \end{cases} \tag{4.4}$$

where $\bar{f}_2 = f_2(y,x_2,u)-f_2(y,\hat{x}_2,u),\ldots,\bar{f}_n = f_n(y,x_2,\ldots,x_n,u)-f_n(y,\hat{x}_2,\ldots,\hat{x}_n,u)$. Consider the change of coordinates

$$\varepsilon_i = \frac{e_i}{L^{i-1+\sigma}},$$

where $0<\sigma<1$ will be given later. Then (4.4) can be expressed as

$$\begin{cases}\dot{\varepsilon}_1 = L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1}-\frac{\dot{L}}{L}\sigma\varepsilon_1, \\ \dot{\varepsilon}_2 = L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2}-\frac{\dot{L}}{L}(\sigma+1)\varepsilon_2+\frac{\bar{f}_2}{L^{1+\sigma}}, \\ \quad\vdots \\ \dot{\varepsilon}_n = -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n}-\frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n+\frac{\bar{f}_n}{L^{n-1+\sigma}}. \end{cases} \tag{4.5}$$

Before the global finite-time stability of the error system (4.5) is proven, some properties of the following homogeneous nonlinear system are investigated:

$$\begin{cases} \dot{\varepsilon}_1 &= L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1}, \\ \dot{\varepsilon}_2 &= L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2}, \\ &\vdots \\ \dot{\varepsilon}_n &= -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n}. \end{cases} \tag{4.6}$$

By Lemma 12, for system (4.6), suitably choose a_i $(1 \le i \le n)$ such that there exists $P^T = P > 0$ satisfying

$$A^T P + PA \le -I,\ h_1 I \le D_1 P + PD_1 \le h_2 I, \tag{4.7}$$

where $A = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix}$, $D_1 = \text{diag}\{\sigma, 1+\sigma, \dots, n-1+\sigma\}$, $h_1, h_2 > 0$ are real constants.

Then, there exists a homogeneous Lyapunov function

$$V(\varepsilon) = \begin{cases} \int_0^\infty \frac{1}{v^{q+1}}(\chi \circ \bar{V})(v\varepsilon_1, v^{\alpha_1}\varepsilon_2, \dots, v^{\alpha_{n-1}}\varepsilon_n)dv, & \varepsilon \in \mathscr{R}^n \setminus \{0\}, \\ 0, & \varepsilon = 0, \end{cases} \tag{4.8}$$

where $\bar{V}(\varepsilon) = \varepsilon^T P \varepsilon$, $q > 0$ is an integer, $\chi(s) = \begin{cases} 0, & s \in (-\infty, 1] \\ 2(s-1)^2, & s \in (1, \frac{3}{2}) \\ 1 - 2(s-2)^2, & s \in [\frac{3}{2}, 2) \\ 1, & s \in [2, \infty) \end{cases}$, $\chi(s) \in C'(\mathscr{R}, \mathscr{R})$, under which the nonlinear homogeneous system (4.6) is finite-time stable.

Moreover, there exist $c_1,\ c_2 > 0$ such that

$$c_1 V(\varepsilon) \le \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T D_1 \varepsilon \le c_2 V(\varepsilon). \tag{4.9}$$

If $q > \max\{\alpha_i\}_{0 \le i \le n-1} + 1$, $\frac{dV(\varepsilon)}{dt}\Big|_{(4.6)}$ is C^1 on $\mathscr{R}^n$, then there exists a $c_3 > 0$ such that

$$\frac{dV(\varepsilon)}{dt}\Big|_{(4.6)} \le -c_3 L^{1-\sigma} V(\varepsilon)^\gamma, \tag{4.10}$$

where $\gamma = \frac{q+\alpha^*-1}{q}$.

Based on Lemma 19 and the above preliminaries, the main result with explicit proof is given in the following:

Theorem 20. *If* $\frac{n-i}{n-j+1} \leq \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$*, then for any* $1-\frac{1}{n} < \alpha^* < 1$*, there exist* $\varphi_i > 0$ $(1 \leq i \leq 5)$ *and* $0 < \sigma < 1$ *such that the system (4.1) with dynamic high gain (4.2) is a global finite-time observer for nonlinear system (2.4) with condition (2.5).*

Proof. Under the condition that $1-\frac{1}{n} < \alpha^* < 1$, a_i $(1 \leq i \leq n)$ satisfying (4.7), $0 < \sigma < 1$ (which will be given later), the homogeneous Lyapunov function $V(\varepsilon)$ defined in (4.8) will be used to derive the global finite-time stability.

For all $\varepsilon \in \mathscr{R}^n$, calculating the derivative of the Lyapunov function $V(\varepsilon)$ defined in (4.8) along the solution of system (4.5), from inequalities (4.9) and (4.10), one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} \leq -c_3 L^{1-\sigma} V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma}-\varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)$$
$$-c_1\varphi_4 L^{1+(m-2)\sigma}|\varepsilon_1|^m V(\varepsilon) - c_1\varphi_5 L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^m V(\varepsilon) + \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}, \tag{4.11}$$

where $\bar{F} = \left(0, \frac{\bar{f}_2}{L^{1+\sigma}}, \ldots, \frac{\bar{f}_n}{L^{n-1+\sigma}}\right)^T$.

For $\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}$, one has

$$\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}\right| = \left|\sum_{i=2}^{n} \frac{\partial V(\varepsilon)}{\partial \varepsilon_i} \frac{\bar{f}_i}{L^{i-1+\sigma}}\right| \leq \sum_{i=2}^{n}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right| \frac{1}{L^{i-1+\sigma}}(\Psi(u,y,\hat{x})\sum_{j=2}^{i}|x_j-\hat{x}_j|$$
$$+l\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\theta_{ij}}) \leq \sum_{i=2}^{n}\sum_{j=2}^{i}\Psi(u,y,\hat{x})\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j| + l\sum_{i=2}^{n}\sum_{j=2}^{i}$$
$$\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j|^{\theta_{ij}} L^{(j-1+\sigma)\theta_{ij}-(i-1+\sigma)}.$$

If $\theta_{ij} < \frac{i}{j-1}$, there exist a $\sigma_1 > 0$ such that $\theta_{ij} < \frac{i-\sigma_1}{j-1+\sigma_1}$, $v_j < \frac{1-\sigma_1}{j-1+\sigma_1}$, $(2 \leq j \leq i \leq n)$. Choose $0 < \sigma < \sigma_1$, then one gets

$$L^{(j-1+\sigma)\theta_{ij}-(i-1+\sigma)} < L^{1-2\sigma}.$$

Then, by Lemma 8, one has

$$\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}\right| \leq \sum_{i=2}^{n}\sum_{j=2}^{i}\Psi(u,y,\hat{x})\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j| + lL^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j|^{\theta_{ij}}$$

$$\leq k_1\Psi(u,y,\hat{x})\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}}{q}} + k_2 lL^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}\theta_{ij}}{q}}, \tag{4.12}$$

where $k_1 = \max_{\{z:V(z)=1\}}\left|\frac{\partial V(z)}{\partial z_i}\right||z_j|$, $k_2 = \max_{\{z:V(z)=1\}}\left|\frac{\partial V(z)}{\partial z_i}\right||z_j|^{\theta_{ij}}$.

Then, for $\delta > 0$, define $\overline{\mathscr{B}}_\delta \triangleq \{\varepsilon : V(\varepsilon) \leq \delta\}$, $\mathscr{P}_\delta = \{\varepsilon : |\varepsilon_1| < \delta\}$. Let $\Omega = \{\varepsilon : (0,\varepsilon_2,\ldots,\varepsilon_n) \in \mathscr{R}^n\}$.

The proof is divided into two parts: $\varepsilon \in \mathscr{R}^n \setminus \Omega$ and $\varepsilon \in \Omega$, where part I consists of two small parts $\varepsilon \in \overline{\mathscr{B}}_1 \setminus \Omega$ and $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, respectively. When $\varepsilon \in \overline{\mathscr{B}}_1 \setminus \Omega$, one can get $\frac{dV(\varepsilon)}{dt}\Big|_{(4.5)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^\gamma$. Then one has $\frac{dV(\varepsilon)}{dt}\Big|_{(4.5)} \leq -c_3L^{1-\sigma}V(\varepsilon)^\gamma$ for $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$. Thus, one obtains $\frac{dV(\varepsilon)}{dt}\Big|_{(4.5)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^\gamma$ for all $\varepsilon \in \mathscr{R}^n \setminus \Omega$. Then when $\varepsilon \in \Omega$, it can be verified that the non-trivial solution of system (4.5) can only pass through Ω finite times. Thus, from the combination of these two parts, one obtains the global finite-time stability of the error system (4.5).

Part I:

(1). When $\varepsilon \in \overline{\mathscr{B}}_1 \setminus \Omega$, from (4.11) and (4.12), one has

$$\begin{aligned}\frac{dV(\varepsilon)}{dt}\Big|_{(4.5)} \leq\; & -c_3L^{1-\sigma}V(\varepsilon)^\gamma + c_2\varphi_1(L^{1-\sigma}-\varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \\ & -c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^m V(\varepsilon) - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^m V(\varepsilon) \\ & +k_1n^2\Psi(u,y,\hat{x})V(\varepsilon) + k_2n^2 lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\beta}{q}},\end{aligned} \tag{4.13}$$

where $\underline{\beta} = \min_{2\leq j\leq i\leq n}\{\alpha_{j-1}\theta_{ij} - \alpha_{i-1}\}$. From Lemma 14, one can derive $\gamma < \frac{q+\beta}{q}$, then, there exist $d_{11},\ d_{21},\ d_{31} > 0$ such that when $\varphi_1 < d_{11},\ \varphi_2 > d_{21},\ \varphi_3 > d_{31}$ one has

$$\begin{aligned}\frac{dV(\varepsilon)}{dt}\Big|_{(4.5)} \leq\; & -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^\gamma - c_2\varphi_1\varphi_2V(\varepsilon) - c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^m V(\varepsilon) \\ & -c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^m V(\varepsilon) \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^\gamma,\end{aligned} \tag{4.14}$$

where $d_{11} = \frac{c_3}{3c_2}$, $d_{21} = \left(\frac{3k_2n^2l}{c_3}\right)^{\frac{1}{\sigma}}$, $d_{31} = \frac{k_1n^2}{c_1}$.

(2). When $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, from (4.11) and (4.12), one can derive

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} \leq\; & -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \\ & -c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon) - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon) \\ & +k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}} + k_2n^2lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}}{q}}, \end{aligned} \tag{4.15}$$

where $\bar{\beta} = \max_{2\leq j\leq i\leq n}\{\alpha_{j-1}\theta_{ij} - \alpha_{i-1}\}$.

Let $\mathscr{G} = \{z : V(z) = 1\}$. For any $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, there exist $\delta > 0$ and λ such that $\varepsilon = (\lambda\varepsilon_1^{\delta}, \lambda^{\alpha_1}\varepsilon_2^{\delta}, \ldots, \lambda^{\alpha_{n-1}}\varepsilon_n^{\delta})^T = \mathrm{diag}\{\lambda, \lambda^{\alpha_1}, \ldots, \lambda^{\alpha_{n-1}}\}\varepsilon^{\delta}$, $\varepsilon^{\delta} = (\varepsilon_1^{\delta}, \ldots, \varepsilon_n^{\delta})^T \in \mathscr{G} \setminus \mathscr{P}_{\delta}$. Then one has

$$|\varepsilon_1|^mV(\varepsilon) = \lambda^{m+q}|\varepsilon_1^{\delta}|^mV(\varepsilon^{\delta}) = \lambda^{m+q}|\varepsilon_1^{\delta}|^m = V(\varepsilon)^{\frac{m+q}{q}}|\varepsilon_1^{\delta}|^m,$$

Because $|\varepsilon_1^{\delta}|^m \geq \min_{\varepsilon\in\mathscr{G}\setminus\mathscr{P}_{\delta}}|\varepsilon_1|^m = \delta^m$, then one can get the following inequality

$$|\varepsilon_1|^mV(\varepsilon) \geq \delta^mV(\varepsilon)^{\frac{m+q}{q}},\ \varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega. \tag{4.16}$$

Thus, from (4.15) and (4.16), one obtains

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} \leq\; & -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \\ & -c_1\varphi_4L^{1+(m-2)\sigma}\delta^mV(\varepsilon)^{\frac{m+q}{q}} - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})\delta^mV(\varepsilon)^{\frac{m+q}{q}} \\ & +k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}} + k_2n^2lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}}{q}}. \end{aligned} \tag{4.17}$$

Because $m \geq \max\{\alpha_{j-1}\theta_{ij} - \alpha_{i-1},\ 1\}$ $(2 \leq j \leq i \leq n)$, one can get $L^{1+(m-2)\sigma} \geq L^{1-\sigma}$. Then, there exist $d_{41},\ d_{51} > 0$ such that $\varphi_4 > \frac{2c_2}{c_1\delta^m}\varphi_1$ holds when $\varphi_4 > d_{41},\ \varphi_5 > d_{51}$.

Thus, for $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, one has

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} & \leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} - c_2\varphi_1\varphi_2V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \\ & \leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \end{aligned} \tag{4.18}$$

where $d_{41} = \max\{\frac{2k_2n^2l}{c_1\delta^m},\ \frac{2c_3}{3c_1\delta^m}\}$, $d_{51} = \frac{k_1n^2}{c_1\delta^m}$.

Finally, from (4.14) and (4.18), by combining part **(1)** and **(2)**, one can get that the following inequality

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \tag{4.19}$$

holds for $\varepsilon \in \mathscr{R}^n \setminus \Omega$.

Part II:

When $\varepsilon \in \Omega$, let $\varepsilon(t, t_0, \varepsilon_0)$ denote a non-trivial solution of system (4.5).

In the following, it will be verified that there does not exist such $t_2 > t_1 \geq t_0$ that $\varepsilon(t, t_0, \varepsilon_0)$ stays on Ω in the interval (t_1, t_2). It will be proven by using a contradiction argument. Suppose there exists such interval that $\varepsilon(t, t_0, \varepsilon_0)$ can stay on Ω. From the first equation of system (4.5), one can derive $\varepsilon_2 = 0$ on (t_1, t_2). Then, from the second equation, one can obtain $\varepsilon_3 = 0$ on (t_1, t_2). Then following the same steps, one has $\varepsilon_i = 0$ $(2 \leq i \leq n)$ on (t_1, t_2), which is a contradiction. Thus, $\varepsilon(t, t_0, \varepsilon_0)$ can only pass through Ω.

Let t_k denote the time when $\varepsilon(t, t_0, \varepsilon_0)$ passes through Ω. From (4.19), one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.5)} V(\varepsilon)^{-\gamma} \leq -\frac{1}{3} c_3 L^{1-\sigma} \leq -\frac{1}{3} c_3 \varphi_2^{1-\sigma}. \tag{4.20}$$

Integrate both sides of (4.20), one has

$$\sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} V(\varepsilon)^{-\gamma} dV(\varepsilon) \leq -\frac{1}{3} c_3 \varphi_2^{1-\sigma} \int_{t_k}^{t_{k+1}} dt,$$

i.e.,

$$\frac{1}{1-\gamma} V(\varepsilon(t_{n+1}))^{1-\gamma} \leq \frac{1}{1-\gamma} V(\varepsilon(t_1))^{1-\gamma} - \frac{1}{3} c_3 \varphi_2^{1-\sigma} (t_{n+1} - t_1). \tag{4.21}$$

Here, the contradiction argument will be still used to prove that $\{t_k\}$ is a finite sequence. If $\{t_k\}$ is not a finite sequence, then one has $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$. And it could happen that the left side of (4.21) approaches zero while the right side of (4.21) approaches $-\infty$, which is a contradiction. Thus, $\{t_k\}$ is a finite sequence.

Therefore, there exists a T_1 such that (4.19) holds for all $\varepsilon \in \mathscr{R}^n$ $(t > T_1)$.

Thus, from Lemma 3 and by combining part I and part II, one gets the global finite-time convergence of the observation error ε_i $(i = 1, \ldots, n)$. The settling time $T(\varepsilon^0)$ is $T(\varepsilon^0) \leq \frac{3}{c_3 {\varphi_2}^{1-\sigma}(1-\gamma)} V(\varepsilon^0)^{1-\gamma} + T_1$, where t_0 is the initial time, $\varepsilon^0 = (e_1^0, \frac{e_2^0}{{\varphi_2}^{\sigma}}, \ldots, \frac{e_n^0}{{\varphi_2}^{n-1+\sigma}})^T$ is the initial state.

Then from Lemma 19, one gets $\frac{e_i}{M^{i-1+\sigma}} < \frac{e_i}{L^{i-1+\sigma}} = \varepsilon_i = 0$ when $t > T(\varepsilon^0) + T_1$ $(1 \leq i \leq n)$, i.e., the system (4.1) with update gain (4.2) is a global finite-time observer for system (2.4) with condition (2.5).

This completes the proof. □

4.3 GLOBAL FINITE-TIME OBSERVERS FOR NONLINEAR SYSTEM WITH MIXED RATIONAL POWERS IN THE NONLINEAR TERM

In this section, it will be shown that the system (4.1) with update gain (4.2) (where $m \geq \max\{\alpha_{j-1}\theta_{2,ij} - \alpha_{i-1},\ 1\}$ $(2 \leq j \leq i \leq n)$) is also global finite-time observers for the nonlinear system (2.4) with condition (2.6) where the mixed rational powers in the nonlinearities satisfying $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$.

For system (2.4) with condition (2.6) where the mixed rational powers in the nonlinearities satisfy $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, one has the following result:

Theorem 21. *If $\frac{n-i}{n-j+1} \leq \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, then for any $1 - \frac{1}{n} < \alpha^* < 1$, there exist $\varphi_i > 0$ $(1 \leq i \leq 5)$ and $0 < \sigma < 1$ such that the system (4.1) with dynamic high gain (4.2) is a global finite-time observer for nonlinear system (2.4) with condition (2.5).*

Similar to what is done in Section 4.2, through changing coordinates, the observation error system between system (2.4) with condition (2.6) and the global finite-time observers (4.1) is shown as follows:

$$\begin{cases} \dot{\varepsilon}_1 &= L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} - \frac{\dot{L}}{L}\sigma\varepsilon_1, \\ \dot{\varepsilon}_2 &= L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2} - \frac{\dot{L}}{L}(\sigma+1)\varepsilon_2 + \frac{\bar{f}_2}{L^{1+\sigma}}, \\ &\vdots \\ \dot{\varepsilon}_n &= -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} - \frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n + \frac{\bar{f}_n}{L^{n-1+\sigma}}, \end{cases} \tag{4.22}$$

where $\bar{f}_i$ $(1 \leq i \leq n)$ are the same as that in Section 4.2.

The detailed proof of the global finite-time stability of the observation error system (4.22) is given in the following:

Proof. Here, the homogeneous Lyapunov function $V(\varepsilon)$ defined in (4.8) will be used.

For all $\varepsilon \in \mathscr{R}^n$, calculating the derivative of the Lyapunov function $V(\varepsilon)$ defined in (4.8) along the solution of the observation error system (4.22), one has

$$\begin{aligned} \left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} &\leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \\ &\quad -c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon) - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon) + \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T\bar{F}, \end{aligned}$$

where $\bar{F} = \left(0, \frac{\bar{f}_2}{L^{1+\sigma}}, \ldots, \frac{\bar{f}_n}{L^{n-1+\sigma}}\right)^T$.

For $\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}$, one has

$$\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}\right| \leq \sum_{i=2}^{n}\sum_{j=2}^{i}\Psi(u,y,\hat{x})\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j| + l_1\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j|^{\theta_{1,ij}}$$
$$\times L^{(j-1+\sigma)\theta_{1,ij}-(i-1+\sigma)} + l_2\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon_i}\right||\varepsilon_j|^{\theta_{2,ij}}L^{(j-1+\sigma)\theta_{2,ij}-(i-1+\sigma)}.$$

If $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$, there exist a $\sigma_1 > 0$ such that $\theta_{2,ij} < \frac{i-\sigma_1}{j-1+\sigma_1}$, $v_j < \frac{1-\sigma_1}{j-1+\sigma_1}$, $(2 \leq j \leq i \leq n)$. Choose $0 < \sigma < \sigma_1$, then one gets

$$L^{(j-1+\sigma)\theta_{1,ij}-(i-1+\sigma)} < 1,\ L^{(j-1+\sigma)\theta_{2,ij}-(i-1+\sigma)} < L^{1-2\sigma}.$$

Then, by Lemma 8, one has

$$\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T \bar{F}\right| \leq k_1\Psi(u,y,\hat{x})\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}}{q}} + k_3 l_1\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}\theta_{1,ij}}{q}}$$
$$+k_4 l_2 L^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}\theta_{2,ij}}{q}},$$

where $k_3 = \max_{\{z:V(z)=1\}}\left|\frac{\partial V(z)}{\partial z_i}\right||z_j|^{\theta_{1,ij}}$, $k_4 = \max_{\{z:V(z)=1\}}\left|\frac{\partial V(z)}{\partial z_i}\right||z_j|^{\theta_{2,ij}}$, k_1 is given in Section 4.2.

The proof is also divided into two parts: $\varepsilon \in \mathscr{R}^n \setminus \Omega$ and $\varepsilon \in \Omega$, where part I consists of two small parts $\varepsilon \in \overline{\mathscr{B}}_1 \setminus \Omega$ and $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, respectively (where $\overline{\mathscr{B}}_1$, Ω are the same as those defined in Section 4.2).

In part I, one can obtain $\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}$ for all $\varepsilon \in \mathscr{R}^n \setminus \Omega$. Then when $\varepsilon \in \Omega$, it will be proven that the non-trivial solution of system (4.22) can only pass through Ω finite times. Then, from these two parts, the global finite-time stability of the observation error system (4.22) is derived.

Part I:

(1). When $\varepsilon \in \overline{\mathscr{B}}_1 \setminus \Omega$, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} \leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)$$
$$-c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon) - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon)$$
$$+k_1n^2\Psi(u,y,\hat{x})V(\varepsilon) + k_3n^2l_1V(\varepsilon)^{\frac{q+\beta_1}{q}} + k_4n^2l_2L^{1-2\sigma}V(\varepsilon)^{\frac{q+\beta_2}{q}},$$

where $\underline{\beta}_1 = \min_{2\leq j\leq i\leq n}\{\alpha_{j-1}\theta_{1,ij} - \alpha_{i-1}\}$, $\underline{\beta}_2 = \min_{2\leq j\leq i\leq n}\{\alpha_{j-1}\theta_{2,ij} - \alpha_{i-1}\}$.

From Lemma 14, one can have $\gamma < \frac{q+\beta_1}{q}$, then, there exist $d_{11},\ d_{21}^*,\ d_{22}^*,\ d_{31} > 0$ such that when $\varphi_1 < d_{11},\ \varphi_2 > \max\{d_{21}^*,\ d_{22}^*\},\ \varphi_3 > d_{31}$ one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma},$$

where $d_{21}^* = \frac{6k_3n^2l_1}{c_3},\ d_{22}^* = \left(\frac{6k_4n^2l_2}{c_3}\right)^{\frac{1}{\sigma}}$, $d_{11},\ d_{31}$ are the same as that in Section 4.2.

(2). When $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, one can derive

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} &\leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)\\ &-c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon) - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon)\\ &+k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}} + k_3n^2l_1V(\varepsilon) + k_4n^2l_2L^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}_2}{q}},\end{aligned}$$

where $\bar{\beta}_2 = \max_{2\leq j\leq i\leq n}\{\alpha_{j-1}\theta_{2,ij} - \alpha_{i-1}\}$.

Similarly, for any $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, there exist $\delta > 0$ and λ such that $\varepsilon = (\lambda\varepsilon_1^{\delta}, \lambda^{\alpha_1}\varepsilon_2^{\delta}, \ldots, \lambda^{\alpha_{n-1}}\varepsilon_n^{\delta})^T = \mathrm{diag}\{\lambda, \lambda^{\alpha_1}, \ldots, \lambda^{\alpha_{n-1}}\}\varepsilon^{\delta}$, $\varepsilon^{\delta} = (\varepsilon_1^{\delta}, \ldots, \varepsilon_n^{\delta})^T \in \mathscr{G} \setminus \mathscr{P}_{\delta}$ (where $\mathscr{G},\ \mathscr{P}_{\delta}$ are the same as that in Section 4.2).

Similar to what is done in Section 4.2, one can get

$$|\varepsilon_1|^mV(\varepsilon) \geq \delta^mV(\varepsilon)^{\frac{m+q}{q}},\ \varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega.$$

Thus, from the above steps, one can obtain

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} &\leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon) - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)\\ &-c_1\varphi_4L^{1+(m-2)\sigma}\delta^mV(\varepsilon)^{\frac{m+q}{q}} - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})\delta^mV(\varepsilon)^{\frac{m+q}{q}}\\ &+k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}} + k_3n^2l_1V(\varepsilon) + k_4n^2l_2L^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}_2}{q}}.\end{aligned}$$

Because $m \geq \max\{\alpha_{j-1}\theta_{2,ij} - \alpha_{i-1},\ 1\}$ $(2 \leq j \leq i \leq n)$, it is clear $L^{1+(m-2)\sigma} \geq L^{1-\sigma}$. Then, there exist $d_{41}^*,\ d_{51} > 0$ such that $\varphi_4 > \frac{2c_2}{c_1\delta^m}\varphi_1$ holds when $\varphi_4 > d_{41}^*,\ \varphi_5 > d_{51}$.

Thus, for $\varepsilon \in (\mathscr{R}^n \setminus \overline{\mathscr{B}}_1) \setminus \Omega$, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} \leq -c_3L^{1-\sigma}V(\varepsilon)^{\gamma},$$

where $d_{41}^* = \max\{\frac{4k_3n^2l_1}{c_1\delta^m},\ \frac{4k_4n^2l_2}{c_1\delta^m},\ \frac{2c_3}{3c_1\delta^m}\}$, d_{51} is the same as that in Section 4.2.

Then, from part **(1)** and part **(2)**, one finds that the following inequality

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(4.22)} \leq -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \tag{4.23}$$

holds for $\varepsilon \in \mathscr{R}^n \setminus \Omega$.

Part II:

When $\varepsilon \in \Omega$, similarly, let $\varepsilon(t, t_0, \varepsilon_0)$ denote a non-trivial solution of system (4.22).

Similar to what is done in Section 4.2, it can be proved that $\varepsilon(t, t_0, \varepsilon_0)$ can only pass through Ω. Let t_k denote the time when $\varepsilon(t, t_0, \varepsilon_0)$ passes through Ω. From (4.19), through integration, one can get

$$\sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} V(\varepsilon)^{-\gamma} dV(\varepsilon) \leq -\frac{1}{3}c_3\varphi_2^{1-\sigma}\int_{t_k}^{t_{k+1}} dt,$$

i.e.,

$$\frac{1}{1-\gamma}V(\varepsilon(t_{n+1}))^{1-\gamma} \leq \frac{1}{1-\gamma}V(\varepsilon(t_1))^{1-\gamma} - \frac{1}{3}c_3\varphi_2^{1-\sigma}(t_{n+1}-t_1).$$

By using the contradiction argument, it can easily be proven that $\{t_k\}$ is a finite sequence.

Then, from part I and part II, the global finite-time convergence of the observation error system (4.22) can be derived, i.e., the system (4.1) with the update gain (4.2) is also global finite-time observers for system (2.4) with condition (2.5) where the mixed rational powers in the nonlinearities satisfy $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}\ (2 \leq j \leq i \leq n)$.

This completes the proof. □

Similarly to what is done in Chapter 3, in the following, a remark will be provided to show that how the global finite-time observers are obtained in this chapter compared with the semi-global finite-time observers in paper [62].

Remark 22. *Based on the result in Lemma 12, the thesis successfully extends the problem of observer design to a broader class of nonlinear systems (2.4) compared with paper [62]. In the thesis, the nonlinear systems (2.4) are with lower bounds of the rational and mixed rational powers in conditions (2.5) and (2.6):* $\frac{n-i}{n-j+1} \leq \theta_{ij} \leq \frac{i}{j-1}\ (2 \leq j \leq i \leq n)$ *and* $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}\ (2 \leq j \leq i \leq n)$. *In both of the global finite-time observers designed in this chapter and the semi-global finite-time observers in paper [62], the term* $\lceil e_1 \rfloor^{\alpha_1}, \lceil e_1 \rfloor^{\alpha_2}, \ldots, \lceil e_1 \rfloor^{\alpha_n}$ *ensures the local finite-time stability. The global finite-time stability of the error systems in this chapter is obtained based on the new high gain, where two items are introduced compared with that in paper [62].*

4.4 SUMMARY

In this chapter, a new kind of global finite-time observer is designed for the same class of nonlinear systems as in Chapter 3, both with bounded rational and mixed rational powers imposed on the nonlinearities. Like those in Chapter 3, the proposed observers in this chapter are also constructed by employment of the high-gain technique. Compared with the results in Chapter 3, the newly proposed observers have one homogeneous term and a new dynamic high gain where two homogeneous terms are introduced.The global finite-time stability is derived in one step, which is based on the finite-time theory in [44]. Moreover, the designed global finite-time observers in this chapter do not place any limitations on either the dimension of the nonlinear systems or $a_n P_{1n}$.

CHAPTER 5

EXAMPLES

In this chapter, some simulation results are provided to illustrate the performance of the designed two kinds of nonlinear global finite-time observers as proposed in Chapter 3 and Chapter 4 for nonlinear systems (2.4) with conditions (2.5) and (2.6) where the nonlinear terms are with certain bounded rational and mixed rational powers, respectively. Moreover, through several examples, some comparisons are made between the two kinds of observers, which show the effectiveness of the proposed results more clearly.

5.1 EXAMPLES OF GLOBAL FINITE-TIME OBSERVERS WITH TWO HOMOGENEOUS TERMS DESIGNED IN CHAPTER 3

In the following, two simulation results under different conditions are given to show the validity of the designed global finite-time observers for nonlinear systems (2.4) with conditions (2.5) and (2.6) where the nonlinear terms are with bounded rational and mixed rational powers, respectively.

5.1.1 Example of nonlinear system with single rational power in the nonlinear term

In this subsection, an example is given to illustrate the effectiveness of the designed observers for nonlinear systems (2.4) with condition (2.5) where the rational powers in the nonlinearities satisfy $\frac{n-i}{n-j+1} \leq \theta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in Section 3.3, Chapter 3.

Example 23. *Consider the following nonlinear system*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_3^{\frac{3}{2}} - x_3, \\ y &= x_1. \end{cases} \tag{5.1}$$

It can be verified that the nonlinear condition holds: $|(x_3^{\frac{3}{2}} - x_3) - (\hat{x}_3^{\frac{3}{2}} - \hat{x}_3)| \leq (1 + \frac{3}{2}|\hat{x}_3|^{\frac{1}{2}})|x_3 - \hat{x}_3| + |x_3 - \hat{x}_3|^{\frac{3}{2}}$.

Following the result in Section 3.3, Chapter 3, an observer can be designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y - \hat{x}_1 \rfloor^{\alpha^*} + 3L^{1-(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{\beta^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha^*-1} + 3L^{2-2(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{2\beta^*-1}, \\ \dot{\hat{x}}_3 &= \hat{x}_3^{\frac{3}{2}} - \hat{x}_3 + L^3\lceil y - \hat{x}_1 \rfloor^{3\alpha^*-2} + L^{3-3(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{3\beta^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1 + \frac{3}{2}|\hat{x}_2|^{\frac{1}{2}})]. \end{cases}$$

Here, one chooses $P = \begin{pmatrix} 5 & -3 & 0.2 \\ -3 & 4 & -3 \\ 0.2 & -3 & 7 \end{pmatrix} > 0$ *and* $a_1 = a_2 = 3,\ a_3 = 1$, *i.e.,* $A = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$.
It can be verified that A *and* P *satisfy* $A^T P + PA \leq -I$ *and* $a_3 P_{13} = 0.2 > 0$.

In order to highlight the performance of the proposed result, it will be illustrated under the following four different conditions, which will show the finite-time convergence of the observation error system under different parameters and different initial values, as well as the simulations under noise injection in each case.

Condition I

Parameters: $\alpha^* = 0.95,\ \beta^* = 10^5,\ \sigma = 0.01,\ \eta = 0.01,\ \varphi_1 = 0.1,\ \varphi_2 = 1.2,\ \varphi_3 = \frac{1}{3}$. *The initial values:* $x_1(0) = 1.2,\ x_2(0) = 0.1,\ x_3(0) = 0.2,\ \hat{x}_1(0) = 0.2,\ \hat{x}_2(0) = 0.4,\ \hat{x}_3(0) = 0.1,\ L(0) = 1.5$.

Condition II

Parameters: $\alpha^* = 0.95,\ \beta^* = 10^5,\ \sigma = 0.01,\ \eta = 0.01,\ \varphi_1 = 0.1,\ \varphi_2 = 1.2,\ \varphi_3 = \frac{1}{3}$. *The initial values:* $x_1(0) = 0.6,\ x_2(0) = 0.1,\ x_3(0) = 0.2,\ \hat{x}_1(0) = 0.2,\ \hat{x}_2(0) = 0.4,\ \hat{x}_3(0) = 0.1,\ L(0) = 15$.

Condition III

Parameters: $\alpha^* = 0.85$, $\beta^* = 10^4$, $\sigma = 0.001$, $\eta = 0.002$, $\varphi_1 = 0.3$, $\varphi_2 = 1$, $\varphi_3 = \frac{5}{6}$. *The initial values:* $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.1$, $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = 0.1$, $\hat{x}_3(0) = 0.4$, $L(0) = 1.5$.

Condition IV

Parameters: $\alpha^* = 0.85$, $\beta^* = 10^4$, $\sigma = 0.001$, $\eta = 0.002$, $\varphi_1 = 0.3$, $\varphi_2 = 1$, $\varphi_3 = \frac{5}{6}$. *The initial values:* $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.1$, $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = 0.1$, $\hat{x}_3(0) = 0.4$, $L(0) = 15$.

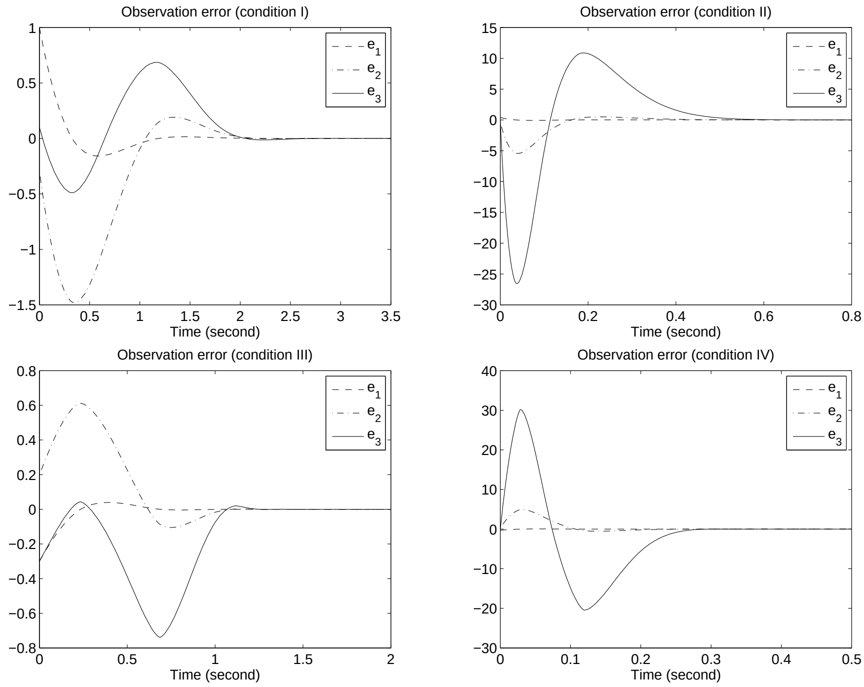


Figure 5.1: Trajectories of the observation error of system (5.1) under conditions I, II, III and IV without noise

From the simulation results as shown in Figures 5.1 and 5.2, one can see that the proposed result can make the observation error system converge in finite time. Different parameters and initial values do have some impact on the convergent time where the high gain plays a much more important role, the bigger the high gain, the more quickly the observation error system converges. Figure 5.2 shows the trajectories of the observation error of system (5.1) under the four conditions with uniform random number noise (where the magnitude of the noise is 1 in conditions (I) and (III), 0.1 in conditions (II) and (IV)) imposed on the observer as well as the high gain. It can be seen that although the observation error system converges faster with a bigger high gain, it is slightly more sensitive to the noise.

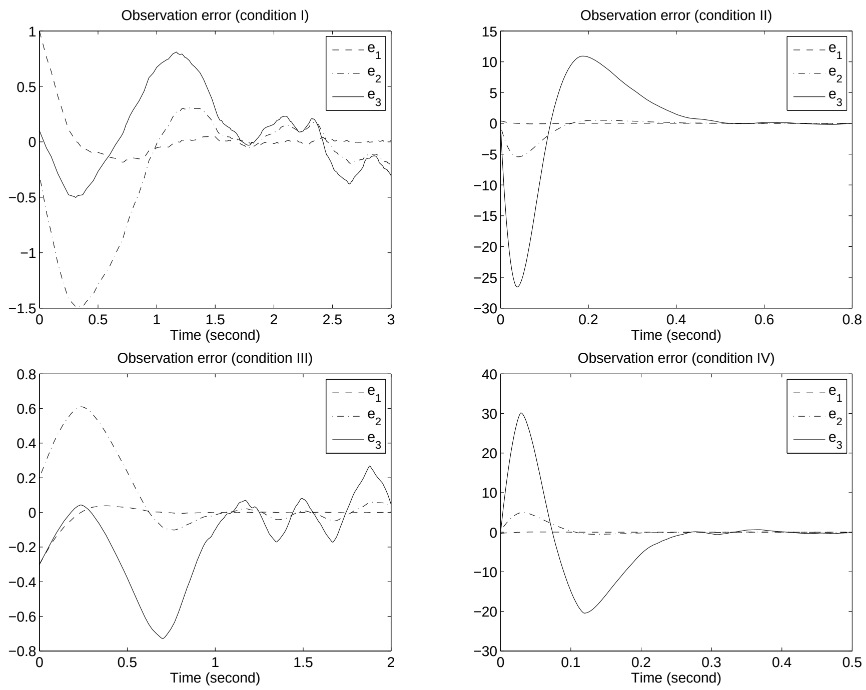


Figure 5.2: Trajectories of the observation error of system (5.1) under conditions I, II, III and IV with noise added on the observer and the high gain

5.1.2 Example of nonlinear system with mixed rational powers in the nonlinear term

In the following, the proposed global finite-time observers in Section 3.4, Chapter 3 are illustrated through an example.

Example 24. *For the following nonlinear system*

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & x_3, \\ \dot{x}_3 & = & -x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}, \\ y & = & x_1, \end{array}\right. \tag{5.2}$$

from Lemma A.4 in [59], one can find that nonlinear condition with mixed rational powers $|(-x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}) - (-\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}})| \leq (|x_3^{\frac{3}{5}} - \hat{x}_3^{\frac{3}{5}}| + |x_3^{\frac{5}{3}} - \hat{x}_3^{\frac{5}{3}}|) \leq \frac{5}{3}|\hat{x}_3|^{\frac{2}{3}}|x_3 - \hat{x}_3| + 2^{\frac{2}{5}}|x_3 - \hat{x}_3|^{\frac{3}{5}} + |x_3 - \hat{x}_3|^{\frac{5}{3}}$ *holds.*

From the result in Section 3.4, Chapter 3, the observer dynamics are designed as follows

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y-\hat{x}_1 \rfloor^{\alpha^*} + 3L^{1-(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1 \rfloor^{\beta^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y-\hat{x}_1 \rfloor^{2\alpha^*-1} + 3L^{2-2(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1 \rfloor^{2\beta^*-1}, \\ \dot{\hat{x}}_3 &= -\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}} + L^3\lceil y-\hat{x}_1 \rfloor^{3\alpha^*-2} + L^{3-3(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1 \rfloor^{3\beta^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma}-\varphi_2) - \frac{5}{3}\varphi_3|\hat{x}_3|^{\frac{2}{3}}]. \end{cases}$$

In this example, it is also chosen $P = \begin{pmatrix} 5 & -3 & 0.2 \\ -3 & 4 & -3 \\ 0.2 & -3 & 7 \end{pmatrix} > 0$ *and* $a_1 = a_2 = 3,\ a_3 = 1$, *i.e.,* $A = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$. *The conditions A and P satisfy* $A^T P + PA \leq -I$ *and* $a_3 P_{13} = 0.2 > 0$ *are satisfied.*

Similarly, the effectiveness of the proposed global finite-time observer will be illustrated from the following four different conditions.

Condition I

Parameters: $\alpha^* = 0.9,\ \sigma = 0.1,\ \eta = 0.01,\ \beta^* = 10^4,\ \varphi_1 = 0.2,\ \varphi_2 = 1.5,\ \varphi_3 = 2$. *The initial values:* $x_1(0) = 1,\ x_2(0) = 0.1,\ x_3(0) = 0.2,\ \hat{x}_1(0) = 0.5,\ \hat{x}_2(0) = 0.2,\ \hat{x}_3(0) = 0.1,\ L(0) = 2.$

Condition II

Parameters: $\alpha^* = 0.9,\ \sigma = 0.1,\ \eta = 0.01,\ \beta^* = 10^4,\ \varphi_1 = 0.2,\ \varphi_2 = 1.5,\ \varphi_3 = 2$. *The initial values:* $x_1(0) = 1,\ x_2(0) = 0.1,\ x_3(0) = 0.2,\ \hat{x}_1(0) = 0.5,\ \hat{x}_2(0) = 0.2,\ \hat{x}_3(0) = 0.1,\ L(0) = 20.$

Condition III

Parameters: $\alpha^* = 0.8,\ \sigma = 0.2,\ \eta = 0.1,\ \beta^* = 10^3,\ \varphi_1 = 0.1,\ \varphi_2 = 1,\ \varphi_3 = 4$. *The initial values:* $x_1(0) = 0.5,\ x_2(0) = 0.4,\ x_3(0) = 0.3,\ \hat{x}_1(0) = 0.6,\ \hat{x}_2(0) = 0.1,\ \hat{x}_3(0) = 0.5,\ L(0) = 2.$

Condition IV

Parameters: $\alpha^* = 0.8,\ \sigma = 0.2,\ \eta = 0.1,\ \beta^* = 10^3,\ \varphi_1 = 0.1,\ \varphi_2 = 1,\ \varphi_3 = 4$. *The initial values:* $x_1(0) = 0.5,\ x_2(0) = 0.4,\ x_3(0) = 0.3,\ \hat{x}_1(0) = 0.6,\ \hat{x}_2(0) = 0.1,\ \hat{x}_3(0) = 0.5,\ L(0) = 20.$

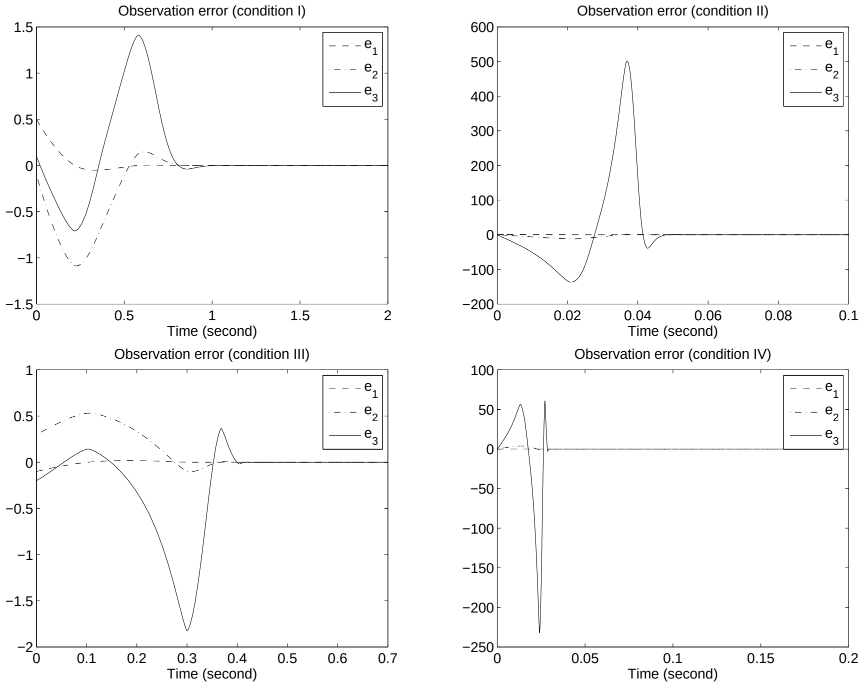


Figure 5.3: Trajectories of the observation error of system (5.2) under conditions I, II, III and IV without noise

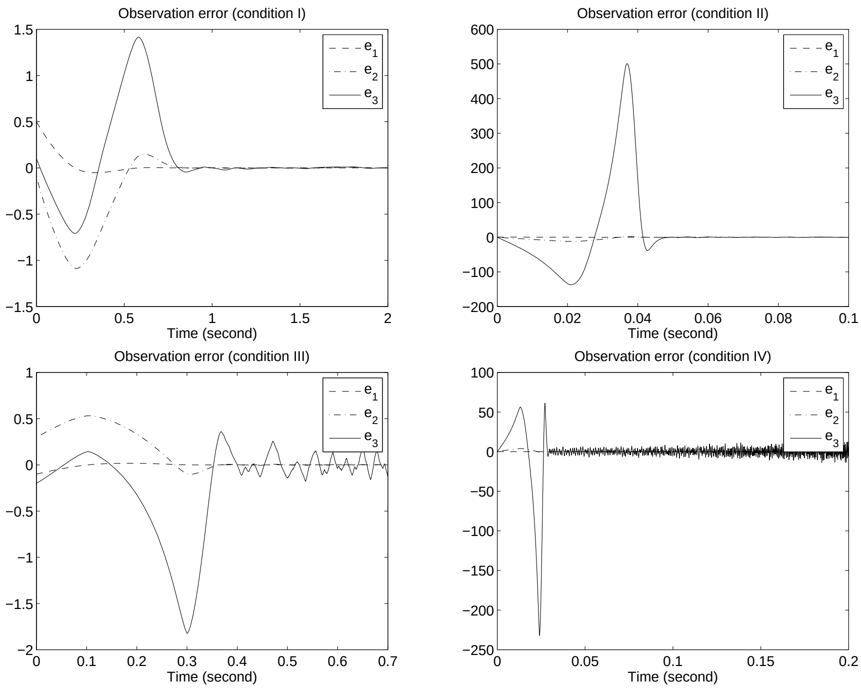


Figure 5.4: Trajectories of the observation error of system (5.2) under conditions I, II, III and IV with noise added on the observer and the high gain

The simulations without noise in Figure 5.3 and with uniform random number noise (where the magnitude of the noise is 1 in conditions (I) and (III), 0.1 in conditions (II) and (IV)) imposed on the observer and the high gain in Figure 5.4 show the dynamics of the observation errors, respectively, which can make the error systems converge in finite time. Similarly, compared with other parameters, the high gain has much more impact on the convergence time. It can be seen that the observation errors converge much faster with a bigger high gain, but they are slightly more noise-sensitive.

5.2 EXAMPLES OF GLOBAL FINITE-TIME OBSERVERS WITH A NEW GAIN UPDATE LAW DESIGNED IN CHAPTER 4

In this section, the performance of the proposed observers in Chapter 4 will be verified by two examples for a nonlinear system (2.4) with rational powers and mixed rational powers in conditions (2.5) and (2.6) satisfying $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, respectively.

5.2.1 Example of nonlinear system with single rational power in the nonlinear term

In the following, the effectiveness of the proposed results will be illustrated through an example that shows the finite-time stability of the proposed observers.

Example 25. *Consider the following nonlinear system*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -1.5x_2 - x_2^{1.4}, \\ y &= x_1, \end{cases}$$

where the following nonlinear condition holds: $|(-1.5x_2 - x_2^{1.4}) - (-1.5\hat{x}_2 - \hat{x}_2^{1.4})| \leq (1.5 + 1.4|\hat{x}_2|^{0.4})|x_2 - \hat{x}_2| + |x_2 - \hat{x}_2|^{1.4}$. *Here m is given as* $m = 2 \geq \max\{\alpha^*\theta_{22} - \alpha^*,\ 1\} = 1,\ \theta_{22} = 1.4$.

Following the result in Section 4.2, Chapter 4, the global finite-time observer is designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 4L\lceil y - \hat{x}_1 \rfloor^{\alpha^*}, \\ \dot{\hat{x}}_2 &= 3L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha^*-1} - 1.5\hat{x}_2 - \hat{x}_2^{1.4}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1.5 + 1.4|\hat{x}_2|^{0.4}) - \varphi_4 L^{1-2\sigma}|x_1 - \hat{x}_1|^2 \\ & \quad -\varphi_5(1.5 + 1.4|\hat{x}_2|^{0.4})|x_1 - \hat{x}_1|^2]. \end{cases} \tag{5.3}$$

In order to illustrate the proposed result more clearly, three simulation results are given under different initial conditions and parameters.

Condition I

Parameters: $\alpha^* = 0.95$, $\beta^* = 10^5$, $\sigma = 0.01$, $\eta = 0.01$, $\varphi_1 = 0.1$, $\varphi_2 = 1.2$, $\varphi_3 = 0.2$, $\varphi_4 = 500$, $\varphi_5 = 400$. *The initial values:* $x_1(0) = 0.2$, $x_2(0) = 0.3$, $\hat{x}_1(0) = 0.1$, $\hat{x}_2(0) = 0.4$, $L(0) = 1.5$.

Condition II

Parameters: $\alpha^* = 0.8$, $\beta^* = 10^4$, $\sigma = 0.001$, $\eta = 0.1$, $\varphi_1 = 0.01$, $\varphi_2 = 1$, $\varphi_3 = 1$, $\varphi_4 = 20$, $\varphi_5 = 30$. *The initial values:* $x_1(0) = 2$, $x_2(0) = 5$, $\hat{x}_1(0) = 3$, $\hat{x}_2(0) = 1$, $L(0) = 10$.

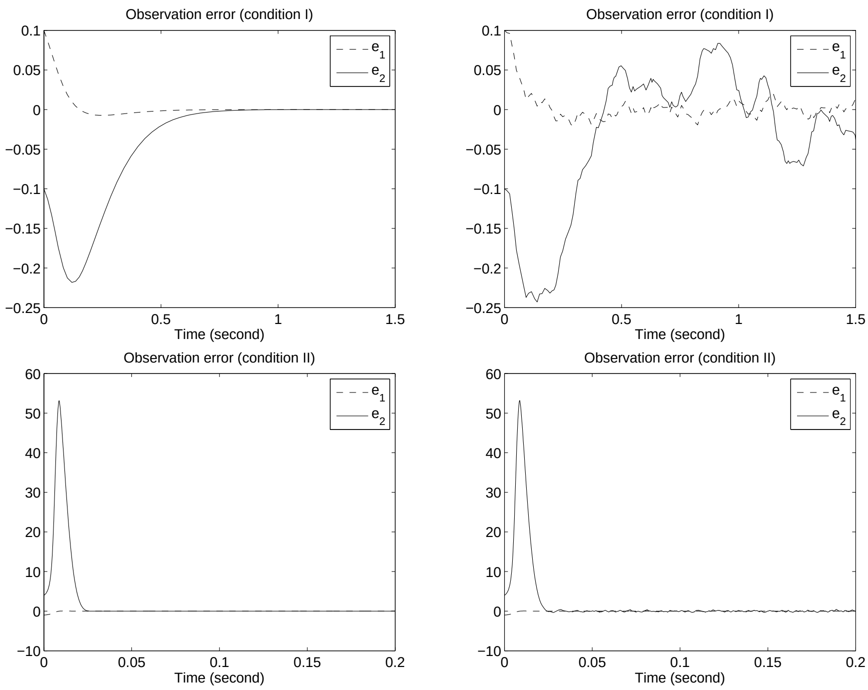


Figure 5.5: Observation errors of system (5.3) under conditions I and II, without noise and with uniform random number noise

From the simulations in Figure 5.5 (where the magnitude of the noise is 1 in condition (I), 0.1 in condition (II)), the finite-time stability of the proposed observers is shown. One can see that the faster the observation error system converges, the more noise-sensitive it is. The change of parameters, the initial values of the states and the high gain L do have some effect on the convergence time of the error system, where the high gain plays a much more important role.

5.2.2 Example of nonlinear system with mixed rational powers in the nonlinear term

Below, through an example, the effectiveness of the proposed global finite-time observers in Section 4.3, Chapter 4 will be shown.

Example 26. *For the following nonlinear system*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_2^{\frac{7}{5}} + x_2^{\frac{5}{7}}, \\ y &= x_1, \end{cases} \tag{5.4}$$

from Lemma 4, one has the nonlinear condition with mixed rational powers $|(x_2^{\frac{7}{5}} + x_2^{\frac{5}{7}}) - (\hat{x}_2^{\frac{7}{5}} + \hat{x}_2^{\frac{5}{7}})| \leq (|x_2^{\frac{7}{5}} - \hat{x}_2^{\frac{7}{5}}| + |x_2^{\frac{5}{7}} - \hat{x}_2^{\frac{5}{7}}|) \leq \frac{7}{5}|\hat{x}_2|^{\frac{2}{5}}|x_2 - \hat{x}_2| + 2^{\frac{2}{7}}|x_2 - \hat{x}_2|^{\frac{5}{7}} + |x_2 - \hat{x}_2|^{\frac{7}{5}}$.

From the result in Section 4.3, Chapter 4, the global finite-time observer is designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 2L\lceil y - \hat{x}_1 \rfloor^{\alpha^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_2^{\frac{7}{5}} + \hat{x}_2^{\frac{5}{7}} + L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha^*-1}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \frac{7}{5}\varphi_3|\hat{x}_2|^{\frac{2}{5}} - \varphi_4 L^{1-2\sigma}|y - \hat{x}_1|^2 - \frac{7}{5}\varphi_5|\hat{x}_2|^{\frac{2}{5}}|y - \hat{x}_1|^2], \end{cases}$$

where $m = 2 \geq \max\{\alpha^*\theta_{2,22} - \alpha^*,\ 1\} = 1,\ \theta_{2,22} = \frac{7}{5}$ *is chosen.*

The simulations are made under the following two different conditions.

Condition I

Parameters: $\alpha^* = 0.9,\ \sigma = 0.1,\ \eta = 0.01,\ \beta^* = 10^4,\ \varphi_1 = 0.2,\ \varphi_2 = 1.5,\ \varphi_3 = 2,\ \varphi_4 = 12,\ \varphi_5 = 15.$ *The initial values:* $x_1(0) = 1,\ x_2(0) = 0.1,\ \hat{x}_1(0) = 2,\ \hat{x}_2(0) = 0.3,\ L(0) = 2.$

Condition II

Parameters: $\alpha^* = 0.8,\ \sigma = 0.2,\ \eta = 0.1,\ \beta^* = 10^3,\ \varphi_1 = 1,\ \varphi_2 = 2,\ \varphi_3 = 3,\ \varphi_4 = 5,\ \varphi_5 = 27.$ *The initial values:* $x_1(0) = 0.5,\ x_2(0) = 2,\ \hat{x}_1(0) = 0.3,\ \hat{x}_2(0) = 0.6,\ L(0) = 20.$

The trajectories (without noise and with uniform random number noise, where the magnitude of the noise is 1 in condition (I), 0.1 in condition (II)) in Figure 5.6 show the effectiveness of the proposed observers. It can be seen that the bigger the initial value of the high gain, the faster the observation error system converges to the origin.

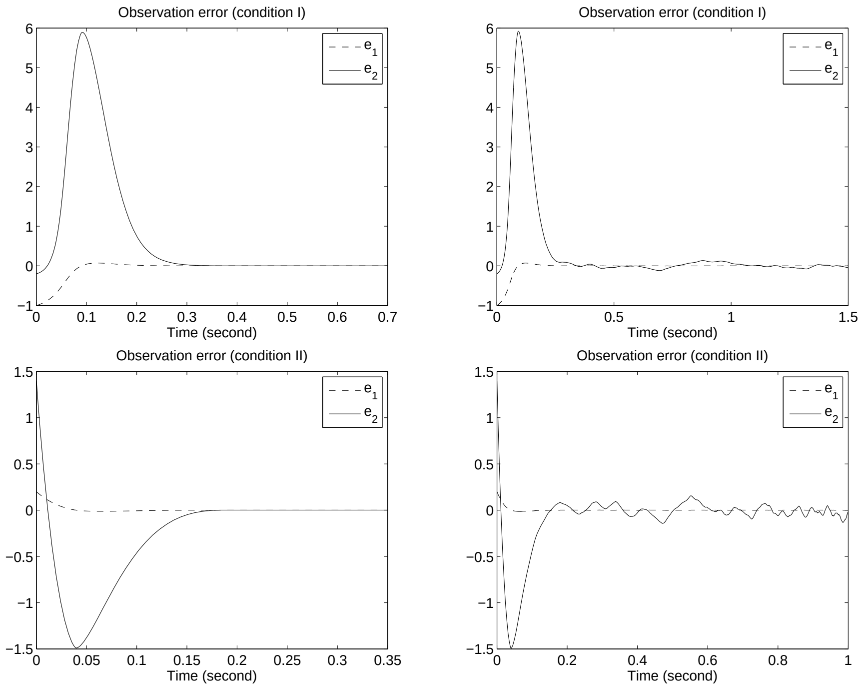


Figure 5.6: Observation errors of system (5.4) under conditions I and II (without noise and with uniform random number noise)

5.3 COMPARISONS BETWEEN THE TWO KINDS OF GLOBAL FINITE-TIME OBSERVERS PROPOSED IN CHAPTER 3 AND CHAPTER 4

In this section, two examples will be given to show some comparisons between the two global finite-time observers designed for nonlinear system (2.4) in Chapter 3 and Chapter 4 in the two cases: with single rational powers and mixed rational powers in conditions (2.5) and (2.6) satisfying $\frac{n-i}{n-j+1} < \theta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \theta_{1,ij} < 1,\ 1 < \theta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, respectively.

5.3.1 Example of nonlinear system with single rational power in the nonlinear term

Compared with the global finite-time observers designed in Section 3.3, Chapter 3, the proposed result in Section 4.2, Chapter 4 neither has any limitation on the dimension of the nonlinear system nor imposes any requirements on $a_n P_{1n}$. In order to illustrate the comparisons more clearly, in the following, an example is given.

Example 27. *Consider the same nonlinear system as that in example 23:*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_3^{\frac{3}{2}} - x_3, \\ y &= x_1, \end{cases}$$

with nonlinear condition $|(x_3^{\frac{3}{2}} - x_3) - (\hat{x}_3^{\frac{3}{2}} - \hat{x}_3)| \leq (1 + \frac{3}{2}|\hat{x}_3|^{\frac{1}{2}})|x_3 - \hat{x}_3| + |x_3 - \hat{x}_3|^{\frac{3}{2}}$ *holds.*

Following what is done in example 23, the global finite-time observer is designed:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y - \hat{x}_1 \rfloor^{\alpha^*} + 3L^{1-(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{\beta^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha^*-1} + 3L^{2-2(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{2\beta^*-1}, \\ \dot{\hat{x}}_3 &= \hat{x}_3^{\frac{3}{2}} - \hat{x}_3 + L^3\lceil y - \hat{x}_1 \rfloor^{3\alpha^*-2} + L^{3-3(\beta^*-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{3\beta^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1 + \frac{3}{2}|\hat{x}_2|^{\frac{1}{2}})]. \end{cases} \tag{5.5}$$

Here, the choice is $m = 2 \geq \max\{\alpha_2\theta_{33} - \alpha_2,\ 1\} = 1,\ \theta_{33} = \frac{3}{2}$.

From the result in Section 4.2, Chapter 4, the global finite-time observer with a new high gain is as follows:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y - \hat{x}_1 \rfloor^{\alpha^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha^*-1}, \\ \dot{\hat{x}}_3 &= \hat{x}_3^{\frac{3}{2}} - \hat{x}_3 + L^3\lceil y - \hat{x}_1 \rfloor^{3\alpha^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1 + \frac{3}{2}|\hat{x}_2|^{\frac{1}{2}}) - \varphi_4 L^{1-2\sigma}|x_1 - \hat{x}_1|^2 \\ & \quad -\varphi_5(1 + \frac{3}{2}|\hat{x}_2|^{\frac{1}{2}})|x_1 - \hat{x}_1|^2]. \end{cases} \tag{5.6}$$

As in example 23, P and A are given as $P = \begin{pmatrix} 5 & -3 & 0.2 \\ -3 & 4 & -3 \\ 0.2 & -3 & 7 \end{pmatrix} > 0$ *and* $a_1 = a_2 = 3,\ a_3 = 1$*, i.e.,* $A = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$.

The comparisons between observers (5.5) and (5.6) will be made under the following two different conditions:

Condition I

Parameters: $\alpha^* = 0.9$, $\beta^* = 10^4$, $\sigma = 0.002$, $\eta = 0.001$, $\varphi_1 = 0.2$, $\varphi_2 = 1$, $\varphi_3 = 0.7$, $\varphi_4 = 30$, $\varphi_5 = 50$. *The initial values:* $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.1$, $\hat{x}_1(0) = 0.7$, $\hat{x}_2(0) = 0.1$, $\hat{x}_3(0) = 0.3$, $L(0) = 2$.

Condition II

Parameters: $\alpha^* = 0.8$, $\beta^* = 10^3$, $\sigma = 0.001$, $\eta = 0.002$, $\varphi_1 = 0.5$, $\varphi_2 = 2$, $\varphi_3 = 1$, $\varphi_4 = 10$, $\varphi_5 = 20$. *The initial values:* $x_1(0) = 1$, $x_2(0) = 0.5$, $x_3(0) = 0.1$, $\hat{x}_1(0) = 0.2$, $\hat{x}_2(0) = 0.3$, $\hat{x}_3(0) = 0.6$, $L(0) = 10$.

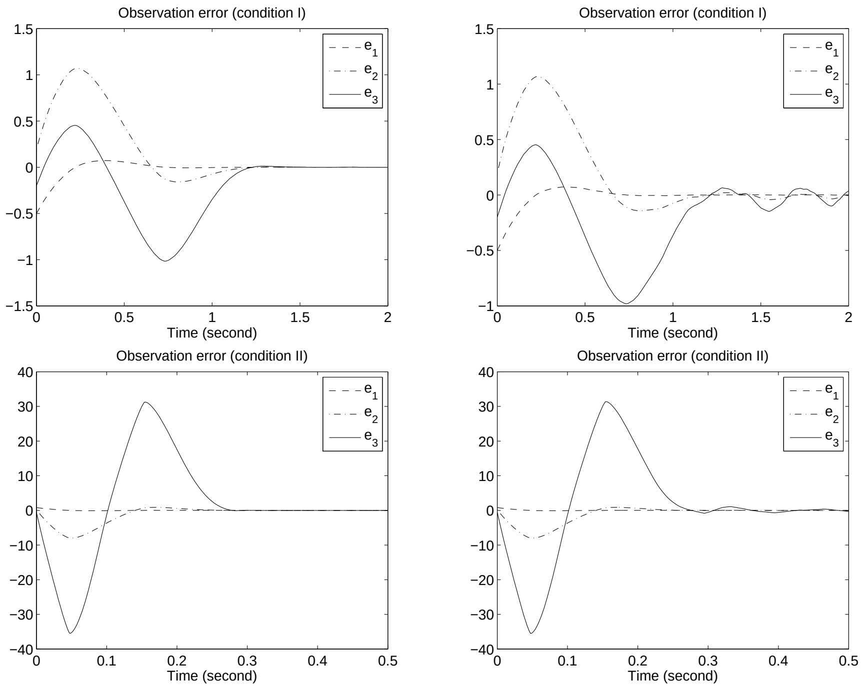


Figure 5.7: Observation errors of system (5.5) under conditions I and II (without noise and with uniform random number noise)

From the simulations (without noise and with uniform random number noise added to the observers, where the magnitude of the noise is 1 in condition (I), 0.1 in condition (II)) as shown in Figure 5.7 and Figure 5.8, it is very clear that no matter under which condition, although it is slightly more noise-sensitive, the observer (5.6) proposed in Section 4.2, Chapter 4 can make the error systems converge more quickly than the observer (5.5) obtained from Section 3.3, Chapter 3.

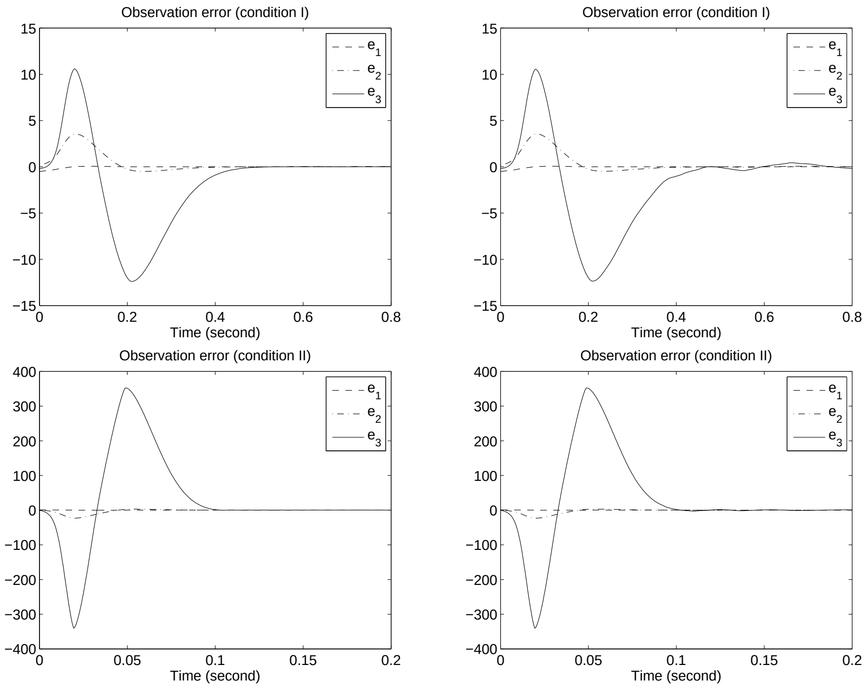


Figure 5.8: Observation errors of system (5.6) under conditions I and II (without noise and with uniform random number noise)

5.3.2 Example of nonlinear system with mixed rational powers in the nonlinear term

Similar to what is done above, compared with the global finite-time observers designed in Section 3.4, Chapter 3, the proposed result in Section 4.3, Chapter 4 also do not impose any condition on the system dimension and $a_n P_{1n}$. The detailed differences will be shown through an example in the following:

Example 28. *Consider the same nonlinear system as that in example 24:*

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}, \\ y &= x_1, \end{cases}$$

with mixed rational powers in the nonlinear terms $|(-x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}) - (-\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}})| \leq (|x_3^{\frac{3}{5}} - \hat{x}_3^{\frac{3}{5}}| + |x_3^{\frac{5}{3}} - \hat{x}_3^{\frac{5}{3}}|) \leq \frac{5}{3}|\hat{x}_3|^{\frac{2}{3}}|x_3 - \hat{x}_3| + 2^{\frac{2}{5}}|x_3 - \hat{x}_3|^{\frac{3}{5}} + |x_3 - \hat{x}_3|^{\frac{5}{3}}$.

The global finite-time observer designed in example 24 is as follows

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y-\hat{x}_1\rfloor^{\alpha^*} + 3L^{1-(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1\rfloor^{\beta^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y-\hat{x}_1\rfloor^{2\alpha^*-1} + 3L^{2-2(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1\rfloor^{2\beta^*-1}, \\ \dot{\hat{x}}_3 &= -\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}} + L^3\lceil y-\hat{x}_1\rfloor^{3\alpha^*-2} + L^{3-3(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1\rfloor^{3\beta^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma}-\varphi_2) - \frac{5}{3}\varphi_3|\hat{x}_3|^{\frac{2}{3}}]. \end{cases} \tag{5.7}$$

On the basis of the result in Section 4.3, Chapter 4, the global finite-time observer with a new high gain is given as:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y-\hat{x}_1\rfloor^{\alpha^*} + 3L^{1-(\beta^*-1)(1-\eta)\sigma}\lceil y-\hat{x}_1\rfloor^{\beta^*}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y-\hat{x}_1\rfloor^{2\alpha^*-1}, \\ \dot{\hat{x}}_3 &= -\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}} + L^3\lceil y-\hat{x}_1\rfloor^{3\alpha^*-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma}-\varphi_2) - \frac{5}{3}\varphi_3|\hat{x}_3|^{\frac{2}{3}} - \varphi_4 L^{1-2\sigma}|y-\hat{x}_1|^2 \\ & \quad -\frac{5}{3}\varphi_5|\hat{x}_3|^{\frac{2}{3}}|y-\hat{x}_1|^2], \end{cases} \tag{5.8}$$

where $m = 2 \geq \max\{\alpha^*\theta_{2,22} - \alpha^*,\ 1\} = 1,\ \theta_{2,22} = \frac{5}{3}$. $P,\ A,\ a_i\ (1 \leq i \leq 3)$ *are the same as those in example 24.*

In what follows, the comparisons of the two kinds of global finite-time observers (5.7) and (5.8) will be illustrated under two different conditions without noise and with uniform random number noise imposed on both the observer and the high gain.

Condition I

Parameters: $\alpha^* = 0.95,\ \sigma = 0.01,\ \eta = 0.001,\ \beta^* = 10^3,\ \varphi_1 = 1.2,\ \varphi_2 = 1,\ \varphi_3 = 2.5,\ \varphi_4 = 15,\ \varphi_5 = 20$. *The initial values:* $x_1(0) = 1.2,\ x_2(0) = 0.3,\ x_3(0) = 0.1,\ \hat{x}_1(0) = 0.4,\ \hat{x}_2(0) = 0.5,\ \hat{x}_3(0) = 0.6,\ L(0) = 2$.

Condition II

Parameters: $\alpha^* = 0.83,\ \sigma = 0.002,\ \eta = 0.01,\ \beta^* = 10^4,\ \varphi_1 = 0.8,\ \varphi_2 = 1.2,\ \varphi_3 = 4,\ \varphi_4 = 40,\ \varphi_5 = 50$. *The initial values:* $x_1(0) = 0.4,\ x_2(0) = 0.6,\ x_3(0) = 0.1,\ \hat{x}_1(0) = 0.8,\ \hat{x}_2(0) = 0.2,\ \hat{x}_3(0) = 0.3,\ L(0) = 11$.

According to the simulation results in Figure 5.9 and Figure 5.10, (where the magnitude of the noise is 1 in condition (I), 0.1 in condition (II)) in both cases, compared with the observer (5.7) designed in

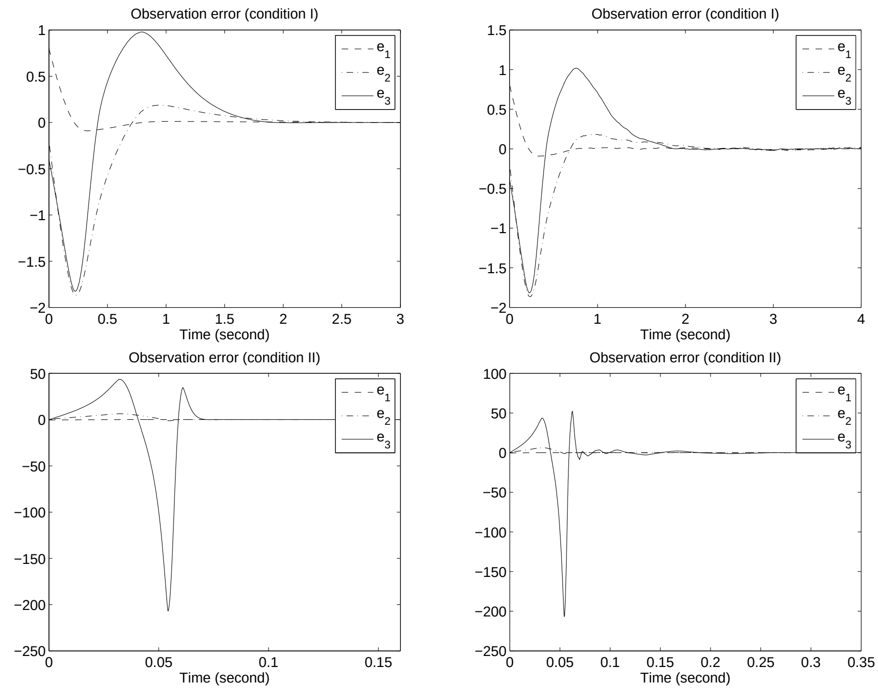


Figure 5.9: Trajectories of the observation error of system (5.7) under conditions I, II without noise and with uniform random number noise

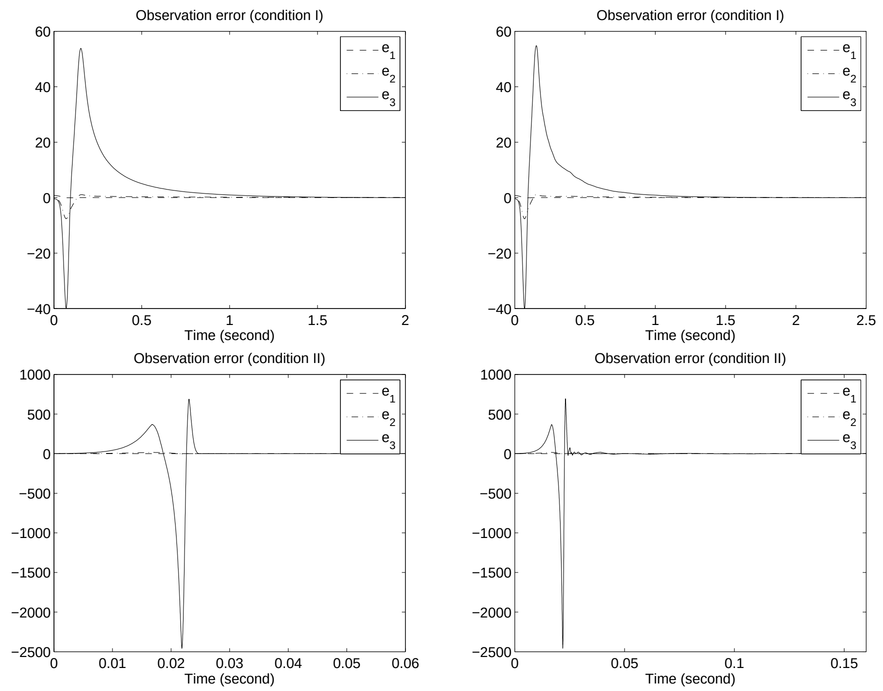


Figure 5.10: Trajectories of the observation error of system (5.8) under conditions I, II without noise and with uniform random number noise

Section 3.4, Chapter 3, although it is more noise-sensitive, the observer (5.8) proposed in Section 4.3, Chapter 4 can really reduce the convergence time of the observation error systems.

5.4 SUMMARY

In this chapter, the validity of the proposed two kinds of global finite-time observers in Chapter 3 and Chapter 4 are illustrated through several examples, which show that the observers designed in Chapter 3 and Chapter 4 can make the observation error systems converge in finite time. Moreover, some comparisons are made between the two kinds of observers, through which it can be seen that although the observation error system is slightly more sensitive to the noise, under the observers with a new gain update law proposed in Chapter 4, the observation error system can converge much more quickly than the observers with two homogeneous terms designed in Chapter 3.

CHAPTER 6

CONCLUSIONS

This thesis contributes to the area of global finite-time observer design for a class of nonlinear systems with certain bounded rational and mixed rational powers in the incremental rate of the nonlinear terms. This chapter summarizes the main results of the thesis and indicates some directions for future research.

6.1 MAIN RESULTS OF THE THESIS

In the thesis, for the same kind of nonlinear systems with both bounded rational and mixed rational powers in the nonlinearities, by employment of the finite-time theory and homogeneity properties that are reviewed in Chapter 2, two kinds of global finite-time observers are proposed with different methods. Both of the observers are derived based on Lyapnov theory and the high-gain technique is employed in the design of both the global finite-time observers.

The designed global finite-time observers in Chapter 3 have a high gain and two homogeneous terms, one of degree greater than 1, and the other of degree less than 1. Besides some conditions regarding the coefficients of the global finite-time observers, the designed observers in Chapter 3 can only be designed to those nonlinear systems with a dimension greater than 3. The global finite-time stability of the designed observers is derived by a combination of global asymptotic stability and local finite-time stability, which is proved by employment of two different homogeneous Lyapunov functions in accordance with the two homogeneous terms.

In Chapter 4, a new kind of global finite-time observer is proposed for the same kind of nonlinear systems under the same two cases: with bounded rational and mixed rational powers in the nonlinearities. The observers have one homogeneous term and a new dynamic high gain where two new homogeneous terms are introduced. Compared with the observers designed in Chapter 3, in Chapter 4, the global finite-time stability is obtained in only one step. The proposed observers do not have any limitation on either the dimension of the nonlinear systems or $a_n P_{1n}$.

The performance of the proposed two kinds of nonlinear global finite-time observers is illustrated through several examples in Chapter 5. For the observers designed in Chapter 3 and Chapter 4, two examples are given for each chapter to show the effectiveness of the proposed results in two cases: for nonlinear systems with bounded rational and mixed rational powers in the nonlinearities, respectively. Moreover, examples are also provided for the comparisons between the results in Chapter 3 and Chapter 4, through which it can be seen that the proposed observers in Chapter 4 can make the observation error system converge more quickly to the origin although they are a little more noise-sensitive.

6.2 FUTURE RESEARCH

With respect to future research, the following problems appear to be interesting:

- As shown from the simulation results in Chapter 3 and Chapter 4, both of the proposed global finite-time observers are noise-sensitive. Thus, robust and adaptive observers can be interesting topics.

- In the thesis, the high-gain method and homogeneity theory are employed to obtain the global finite-time observers. The method can be extended to observer design of other kinds of nonlinear systems with different conditions in the nonlinearities.

- Compared with finite-time nonlinear observer design, finite-time nonlinear controller design can be more challenging and interesting. This is a largely open problem and research would be worthwhile.

- According to the pseudo-global observers proposed in [18], the problem of observer design for nonlinear systems can be possibly relaxed and studied to larger family of nonlinear systems.

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APPENDIX A

PROOF OF LEMMA 12

In this section, the detailed proof of Lemma 12 is included. Before giving the explicit proof of Lemma 12, a useful result is introduced first.

Lemma 29 ([63]). *If a_i $(1 \leq i \leq n)$ in (3.2) are chosen such that condition (3.3) holds, then, for any $x = (0, x_2, \ldots, x_n)^T$, $y = (x_2, \ldots, x_n, 0)^T \in \mathscr{R}^n$, one has $x^T P y + y^T P x \leq -\sum_{i=2}^{n} x_i^2$.*

The following is the detailed proof of Lemma 12.

Proof. First, some definitions are introduced. For $\pi > 0$, $0 < \sigma < 1$, define $\mathscr{F}_\pi \triangleq \{\varepsilon : |\varepsilon_1| = \pi\}$, $\overline{\mathscr{B}}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T \varepsilon \leq \pi\}$, $\mathscr{B}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T \varepsilon < \pi\}$, $\overline{\mathscr{B}}_{2,\pi} \triangleq \{(\varepsilon_1, \rho^{-(\lambda_n - 1)\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-(\lambda_n - 1)\lambda_{n-1} \sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2\}$, $\overline{\mathscr{B}}_{3,\pi} \triangleq \{(\varepsilon_1, \rho^{-\lambda_n \lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-\lambda_n \lambda_{n-1} \sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2\}$, $\mathscr{B}_{3,\pi} \triangleq \{(\varepsilon_1,\ \rho^{-\lambda_n \lambda_1 \sigma} \varepsilon_2, \ldots,\ \rho^{-\lambda_n \lambda_{n-1} \sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 < \pi^2\}$, $\overline{\mathscr{P}}_\pi \triangleq \{\varepsilon : |\varepsilon_1| \leq \pi\}$, $\overline{\mathscr{B}}_{4,\pi} \triangleq \{(\varepsilon_1, \rho^{-2\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-2\lambda_{n-1} \sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2\}$, $\mathscr{B}_{4,\pi} \triangleq \{(\varepsilon_1, \rho^{-2\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-2\lambda_{n-1} \sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 < \pi^2\}$, $\mathscr{P}_\pi \triangleq \{\varepsilon : |\varepsilon_1| < \pi\}$ and $\mathscr{S}_\pi \triangleq \{\varepsilon : \varepsilon^T \varepsilon = \pi\}$.

It is not difficult to determine that $V(\varepsilon)$ is differentiable for any $\varepsilon \in \mathscr{R}^n \setminus \{0\}$. And $\frac{\partial V(\varepsilon)}{\partial \varepsilon_i} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, i.e., $V(\varepsilon)$ is C^1 at the origin.

The proofs of (i) and (ii) are quite easy. For (i), by change of integration, it is very easy to verify that $V(\varepsilon)$ is homogeneous of degree q with respect to the weights $\{\lambda_i\}_{0 \leq i \leq n-1}$. From condition (3.3), it is also not difficult to derive the inequality (3.4) in (ii).

The proofs of (iii) and (iv) are a bit complicated, but the main ideas are the same. Thus, in the following, only the proof of (iv) is given, but the main difference between the proofs of (iii) and (iv) will also be stated.

First, it is not difficult to verify that for $n=2$, there does not exist such $a_1,\ a_2>0$ and $P>0$ which satisfy the condition (3.3) and $a_2P_{12}>0$. For $n\geq 3$, it is always possible to find $a_i>0$ $(1\leq i\leq n)$ such that there exists $P^T=P>0$ satisfying the condition (3.3) and $a_nP_{1n}>0$.

The proof is divided into two parts. The first part is to construct a compact set $\overline{\mathscr{A}}$ (where $\overline{\mathscr{A}}$ will be given later) encircling the origin where some inequalities are obtained. Actually, the compact set is constructed in four parts. In each part, $\frac{dV(\varepsilon)}{dt}\Big|_{(3.1)}$ and $V(\varepsilon)$ satisfy some inequalities on a certain set. Then, the compact set $\overline{\mathscr{A}}$ is derived by combining the four sets. In the second part, for any $\varepsilon\in\mathscr{R}^n\setminus\{0\}$, the relationship between $\frac{dV(\varepsilon)}{dt}\Big|_{(3.1)}$ and $\frac{dV(\varepsilon_0)}{dt}\Big|_{(3.1)}$, $\varepsilon_0\in\overline{\mathscr{A}}$ is established by use of the homogeneity theory. Then, one gets the inequality (3.6) in (iv).

Part I:

This part is divided into six parts. In the first four parts, the researcher will show that $\frac{dV(\varepsilon)}{dt}\Big|_{(3.1)}$ satisfies some inequalities on the following sets $\mathscr{S}_1\cap\overline{\mathscr{P}}_{\rho^{-\sigma}}$, $(\overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}}\setminus\mathscr{P}_{(1-\pi_1)\rho^{-\sigma}})\cap\overline{\mathscr{B}}_{3,\pi_1}$, $\mathscr{F}_{\rho^{-h\sigma}}\cap(\overline{\mathscr{B}}_{1,1}\setminus\mathscr{B}_{3,\pi_1})$ and $(\overline{\mathscr{P}}_{\rho^{-\sigma}}\setminus\mathscr{P}_{\rho^{-h\sigma}})\cap(\overline{\mathscr{B}}_{3,\pi_1}\setminus\mathscr{B}_{3,\pi_1})$, separately, where $\pi_1>0$, $h>\{\bar h_1,\bar h_2\}$, $\rho>\{\rho_1,\rho_2\}$ will be given later. Then in the fifth part, $V(\varepsilon)$ admits some inequalities for ε belonging to each of these four sets. Finally, in the sixth part, by combining these four sets, one derives the compact set $\overline{\mathscr{A}}$.

(1) Let l_1 be the largest $l>0$ such that

$$\max_{\{v\leq l\}}\max_{\{\varepsilon\in\overline{\mathscr{B}}_{1,2}\setminus\mathscr{B}_{1,\frac{1}{2}}\}}\bar V(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n)\leq 1.$$

Let l_2 be the smallest $l>0$ such that

$$\min_{\{v\geq l\}}\min_{\{\varepsilon\in\overline{\mathscr{B}}_{1,2}\setminus\mathscr{B}_{1,\frac{1}{2}}\}}\bar V(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n)\geq 2.$$

Then one has

$$V(\varepsilon)=\int_{l_1}^{l_2}\frac{1}{v^{q+1}}(\chi\circ\bar V(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n))dv+\frac{1}{ql_2^q},\ \varepsilon\in\overline{\mathscr{B}}_{1,2}\setminus\mathscr{B}_{1,\frac{1}{2}}.$$

And for $\varepsilon\in\overline{\mathscr{B}}_{1,2}\setminus\mathscr{B}_{1,\frac{1}{2}}$, one can get

$$\frac{dV(\varepsilon)}{dt}\Big|_{(3.1)}=2\rho\int_{l_1}^{l_2}\frac{\chi'(\bar V(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n))}{v^{q+\lambda}}K(v,\varepsilon_1,\ldots,\varepsilon_n)dv, \tag{B.1}$$

where

$$K(v,\varepsilon_1,\ldots,\varepsilon_n)=\begin{bmatrix} v\varepsilon_1\\ 0\\ \vdots\\ 0\end{bmatrix}^T P\begin{bmatrix} v^{\lambda_1}\varepsilon_2\\ \vdots\\ v^{\lambda_{n-1}}\varepsilon_n\\ 0\end{bmatrix}+\begin{bmatrix} v\varepsilon_1\\ 0\\ \vdots\\ 0\end{bmatrix}^T P\begin{bmatrix} -a_1\rho^{(\lambda_1-1)\sigma}\lceil v\varepsilon_1\rfloor^{\lambda_1}\\ \vdots\\ -a_n\rho^{(\lambda_n-1)\sigma}\lceil v\varepsilon_1\rfloor^{\lambda_n}\end{bmatrix}$$

$$+\begin{bmatrix} 0\\ v^{\lambda_1}\varepsilon_2\\ \vdots\\ v^{\lambda_{n-1}}\varepsilon_n\end{bmatrix}^T P\begin{bmatrix} v^{\lambda_1}\varepsilon_2\\ \vdots\\ v^{\lambda_{n-1}}\varepsilon_n\\ 0\end{bmatrix}+\begin{bmatrix} 0\\ v^{\lambda_1}\varepsilon_2\\ \vdots\\ v^{\lambda_{n-1}}\varepsilon_n\end{bmatrix}^T P\begin{bmatrix} -a_1\rho^{(\lambda_1-1)\sigma}\lceil v\varepsilon_1\rfloor^{\lambda_1}\\ \vdots\\ -a_n\rho^{(\lambda_n-1)\sigma}\lceil v\varepsilon_1\rfloor^{\lambda_n}\end{bmatrix}. \tag{B.2}$$

When $\varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_{\rho^{-\sigma}}$, from Lemma 29, equations (B.1) and (B.2), there exists $\rho_1>2$ such that when $\rho>\rho_1$, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}<-\frac{\rho}{2}\int_{l_1}^{l_2}\frac{1}{v^{q+\lambda}}\sum_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2\chi'(\bar{V}(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n))dv,\ \varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_{\rho^{-\sigma}},$$

where $a^*=\max_{\{1\le i\le n\}}a_i,\ \bar{p}=\max_{\{1\le i,\ j\le n\}}|P_{ij}|$.

And clearly, one has $(\mathscr{S}_1\cap\overline{\mathscr{P}}_0)\subset(\mathscr{S}_1\cap\overline{\mathscr{P}}_{\rho^{-\sigma}})\subset(\mathscr{S}_1\cap\overline{\mathscr{P}}_{2^{-\sigma}})$. Let l_3 be the largest $l>0$ such that

$$\max_{\{v\le l\}}\max_{\{\varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_0\}}\bar{V}(v\varepsilon,\ldots,v^{\lambda_{n-1}}\varepsilon_n)\le 1.$$

Let l_4 be the smallest $l>0$ such that

$$\min_{\{v\ge l\}}\min_{\{\varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_0\}}\bar{V}(v\varepsilon,\ldots,v^{\lambda_{n-1}}\varepsilon_n)\ge 2.$$

It is not difficult to get $l_3\ge l_1,\ l_4\le l_2$. Then one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}<-\rho d_1,\ \varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_{\rho^{-\sigma}}, \tag{B.3}$$

where $d_1=\frac{1}{2}\min_{\{\varepsilon\in\mathscr{S}_1\cap\overline{\mathscr{P}}_{2^{-\sigma}}\}}\int_{l_3}^{l_4}\frac{1}{v^{q+\lambda}}\sum_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2\chi'(\bar{V}(v\varepsilon_1,\ldots,v^{\lambda_{n-1}}\varepsilon_n))dv$.

(2) For $\varepsilon=(\pm1,0,\ldots,0)^T$, from Lemma 29, (B.1) and (B.2), one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}=-2\rho\int_{l_1}^{l_2}\frac{\chi'(\bar{V}(\pm v,\ldots,0))}{v^{q+\lambda}}\sum_{i=1}^{n}a_iP_{1i}\rho^{(\lambda_i-1)\sigma}|v|^{1+\lambda_i}dv.$$

Because $a_1P_{11}>0,\ a_nP_{1n}>0,\ \lambda_n>\lambda_i\ (1\le i\le n)$ when $\lambda>1$, there exist $\pi_1\in(0,1)$ and $\rho_2>1$ such that when $\rho>\rho_2$, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}<-\rho^{1-\sigma}\int_{l_1}^{l_2}\frac{a_nP_{1n}|v|^{1+\lambda_n}}{v^{q+\lambda}}\chi'(\bar{V}(\pm v,0,\ldots,0))dv,$$

for $\varepsilon \in (\overline{\mathscr{P}}_{1+\pi_1} \setminus \mathscr{P}_{1-\pi_1}) \cap \overline{\mathscr{B}}_{2,\pi_1}$.

Because $\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}$ is homogeneous of degree $q+\lambda-1$ with respect to the weights $\{\lambda_i\}_{0\le i\le n-1}$, one gets

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} < -d_2\rho^{1-(q+\lambda)\sigma},\ \varepsilon \in (\overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \setminus \mathscr{P}_{(1-\pi_1)\rho^{-\sigma}}) \cap \overline{\mathscr{B}}_{3,\pi_1}, \tag{B.4}$$

where $d_2 = \int_{l_1}^{l_2} \frac{a_n P_{1n} v^{1+\lambda_n}}{v^{q+\lambda}} \chi'(\bar{V}(\pm v, 0, \ldots, 0))dv$.

(3) Let l_5 be the largest $l > 0$ such that

$$\max_{\{v\le l\}} \max_{\{\varepsilon \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \le 1.$$

And let l_6 be the smallest $l > 0$ such that

$$\min_{\{v\ge l\}} \min_{\{\varepsilon \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \ge 2.$$

Then for $\varepsilon \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, one has

$$V(\varepsilon) = \int_{l_5}^{l_6} \frac{1}{v^{q+\lambda}} (\chi \circ \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))dv + \frac{1}{q l_6^q}$$

and

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} = 2\rho \int_{l_5}^{l_6} \frac{1}{v^{q+\lambda}} \chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))K(v, \varepsilon_1, \ldots, \varepsilon_n)dv.$$

And for any $\varepsilon \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, there exists $\tilde{\rho} \ge 1$ such that $\varepsilon = (\tilde{\rho}^{\sigma}(\tilde{\rho}^{-\sigma}\rho^{-\sigma}\varepsilon_1), \tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T$, $|\varepsilon_1| \le 1+\pi_1$, $\sum_{i=2}^n \varepsilon_i^2 = \pi_1^2$. By using the boundedness of the compact set $\varepsilon \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, one can get that $\tilde{\rho}$ is upper bounded with respect to ρ.

For any $\varepsilon \in \mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, there exists $\bar{h}_1 > \lambda_n\lambda_{n-1}$ such that when $h \ge \bar{h}_1$, one has

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} \quad < \quad & -\frac{\rho}{2}\int_{l_5}^{l_6} \frac{\chi'(\bar{V}(v\rho^{-h\sigma}, \ldots, v^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n))}{v^{q+\lambda}} \\ & \times \sum_{i=2}^n \tilde{\rho}^{2\lambda_{i-1}\sigma}\rho^{-2\lambda_n\lambda_{i-1}\sigma}v^{2\lambda_{i-1}}\varepsilon_i^2 dv.\end{aligned}$$

And for any $\varepsilon \in \mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, let $l_7(\varepsilon)$ and $l_8(\varepsilon)$ be such that $\frac{5}{4} \le \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \le \frac{7}{4}$ when $l_7(\varepsilon) \le l \le l_8(\varepsilon)$ (without loss of generality, it is assumed that $0 \le l_7(\varepsilon) \le l_8(\varepsilon)$).

Note that from the definition of $\chi(s)$, $1 \leq \chi'(s) \leq 2$ for $\frac{5}{4} \leq s \leq \frac{7}{4}$. Then, there exists $\bar{h}_2 > \lambda_n\lambda_{n-1}$ such that when $h > \bar{h}_2$ one can have

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} < -\frac{\rho}{2}\int_{l_7(\varepsilon)}^{l_8(\varepsilon)} \frac{\sum_{i=2}^{n} \tilde{\rho}^{2\lambda_{i-1}\sigma}\rho^{-2\lambda_n\lambda_{i-1}\sigma}\nu^{2\lambda_{i-1}}\varepsilon_i^2}{\nu^{q+\lambda}}d\nu$$
$$< -\frac{5\rho}{16\bar{\lambda}(q+\lambda-1)}\frac{l_8(\varepsilon)^{q+\lambda-1}-l_7(\varepsilon)^{q+\lambda-1}}{l_7(\varepsilon)^{q+\lambda-1}l_8(\varepsilon)^{q+\lambda-1}},$$

where $\bar{\lambda} = \lambda_{\max}(P)$.

It is clear that $\{z : z^TPz = \frac{5}{4}\} \cap \{z : z^TPz = \frac{7}{4}\} = \emptyset$, thus, one can derive the following inequality

$$M_1 < \sum_{i=1}^{n}(z_i^{1\,\frac{q+\lambda-1}{\lambda_{i-1}}} - z_i^{2\,\frac{q+\lambda-1}{\lambda_{i-1}}})^2,$$

where $M_1 > 0$ is a positive real number, $z^1 = (z_1^1, \ldots, z_n^1)^T \in \{z : z^TPz = \frac{7}{4}\}$ and $z^2 = (z_1^2, \ldots, z_n^2)^T \in \{z : z^TPz = \frac{5}{4}\}$.

Because $(l_8(\varepsilon)\tilde{\rho}^{\sigma}(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1), l_8(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, l_8(\varepsilon)^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T \in \{z : z^TPz = \frac{7}{4}\}$, and $(l_7(\varepsilon)\tilde{\rho}^{\sigma}(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1), l_7(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, l_7(\varepsilon)^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T \in \{z : z^TPz = \frac{5}{4}\}$, we can get

$$M_1 \leq \tilde{\rho}^{2(q+\lambda-1)\sigma}\rho^{-2\lambda_n(q+\lambda-1)\sigma}(l_8(\varepsilon)^{q+\lambda-1} - l_7(\varepsilon)^{q+\lambda-1})^2(1 + \sum_{i=2}^{n}\varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}),\ \sum_{i=2}^{n}\varepsilon_i^2 = \pi_1^2.$$

Note that $\{z : 1 \leq z^TPz \leq 2\}$ is a bounded compact set. Then, there exist $M_2,\ M_3 > 0$ such that

$$M_2 < \sum_{i=2}^{n} z_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}} < M_3,\ z \in \{z : 1 \leq z^TPz \leq 2\}.$$

It is clear to get that there exist $\varepsilon^j \in \overline{\mathscr{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$ such that $(l_j(\varepsilon)\tilde{\rho}^{\sigma}(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1^j), l_j(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2^j, \ldots, l_j(\varepsilon)^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n^j)^T \in \{z : 1 \leq z^TPz \leq 2\},\ j = 7,\ 8$, and

$$M_3 > \tilde{\rho}^{2(q+\lambda-1)\sigma}\ \rho^{-2\lambda_n(q+\lambda-1)\sigma}l_j(\varepsilon)^{2(q+\lambda-1)}\sum_{i=2}^{n}\varepsilon_i^{j\,\frac{2(q+\lambda-1)}{\lambda_{i-1}}},\ j = 7, 8,\ \sum_{i=2}^{n}\varepsilon_i^{j\,2} = \pi_1^2.$$

Thus, one gets

$$l_8(\varepsilon)^{q+\lambda-1} - l_7(\varepsilon)^{q+\lambda-1} > \min_{\{\varepsilon:\sum_{i=2}^{n}\varepsilon_i^2=\pi_1^2\}}\sqrt{\frac{\rho^{2\lambda_n(q+\lambda-1)\sigma}M_1}{\tilde{\rho}^{2(q+\lambda-1)\sigma}(\sum_{i=2}^{n}\varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}+1)}}$$

and

$$\frac{1}{l_j(\varepsilon)^{q+\lambda-1}} > \min_{\{\varepsilon:\sum_{i=2}^{n}\varepsilon_i^2=\pi_1^2\}}\sqrt{\frac{\tilde{\rho}^{2(q+\lambda-1)\sigma}\sum_{i=2}^{n}\varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}}{\rho^{2\lambda_n(q+\lambda-1)\sigma}M_3}},\ j = 7, 8.$$

Therefore, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} < -\rho^{1-\lambda_n(q+\lambda-1)\sigma}\tilde{\rho}^{(q+\lambda-1)\sigma}d_3,\ \varepsilon \in \mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1}), \tag{B.5}$$

where $d_3 = \min_{\{\varepsilon:\sum_{i=2}^n \varepsilon_i^2=\pi_1^2\}} \dfrac{5\sqrt{M_1}\sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}}{16\bar{\lambda}(q+\lambda-1)M_3\sqrt{\sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}+1}}$.

(4) Fourthly, when $\varepsilon \in (\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1})$, because for any $\varepsilon^1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_n^1)^T \in (\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1})$ and any $\varepsilon^2 = (\pm\rho^{-\sigma}, \varepsilon_2^1, \ldots, \varepsilon_n^1)^T \in \mathscr{F}_{\rho^{-\sigma}} \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1})$, one has $\|\varepsilon^1 - \varepsilon^2\|_2^2 \leq 4\rho^{-2\sigma}$. Because of the continuity of $\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}$ on $\varepsilon \in \mathscr{R}^n$, one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} < -\frac{d_2}{2}\rho^{1-(q+\lambda)\sigma} < 0,\ \varepsilon \in (\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1}). \tag{B.6}$$

(5) From (B.3), one can select $\rho > \max_{\{1\leq i\leq 2\}}\{2, \rho_i\}$ such that

$$V(\varepsilon)^{-\gamma} \geq d_4^{-\gamma},\ \varepsilon \in \mathscr{S}_1 \cap \mathscr{P}_{\rho^{-\sigma}}, \tag{B.7}$$

where $d_4 = \max_{\sum_{i=2}^n \varepsilon_i^2=1} V(\varepsilon)$.

When $\varepsilon \in \mathscr{F}_{\rho^{-\sigma}} \cap \overline{\mathscr{B}}_{3,\pi_1}$, one can have

$$\begin{aligned} V(\pm\rho^{-\sigma}, \rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n) &= \rho^{-q\sigma}V(\pm 1, \rho^{-(\lambda_n-1)\lambda_1\sigma}\varepsilon_2, \ldots, \rho^{-(\lambda_n-1)\lambda_{n-1}\sigma} \\ &\quad \varepsilon_n) \leq d_5\rho^{-q\sigma}, \end{aligned}$$

where $d_5 = \max_{\sum_{i=2}^n \varepsilon_i^2 \leq \pi_1^2} V(\pm 1, \varepsilon_2, \ldots, \varepsilon_n)$. Then, one has

$$V(\varepsilon)^{-\gamma} > d_5^{-\gamma}\rho^{\sigma(q+\lambda-1)},\ \varepsilon \in \mathscr{F}_{\rho^{-\sigma}} \cap \overline{\mathscr{B}}_{3,\pi_1}. \tag{B.8}$$

When $\varepsilon \in \mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})$, one has

$$\begin{aligned} &V(\pm\tilde{\rho}^{\sigma}\tilde{\rho}^{-\sigma}\rho^{-h\sigma}, \tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n) = \tilde{\rho}^{q\sigma}\rho^{-\lambda_n q\sigma} \\ &\times V(\pm\tilde{\rho}^{-\sigma}\rho^{-(h-\lambda_n)\sigma}, \varepsilon_2, \ldots, \varepsilon_n) \leq d_6\tilde{\rho}^{q\sigma}\rho^{-\lambda_n q\sigma}, \end{aligned}$$

where $d_6 = \max_{|\varepsilon_1|\leq 1, \sum_{i=2}^n \varepsilon_i^2 \leq \pi_1^2} V(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

Then the following inequality holds:

$$V(\varepsilon)^{-\gamma} > d_6^{-\gamma}\rho^{\lambda_n(q+\lambda-1)\sigma}\tilde{\rho}^{-(q+\lambda-1)\sigma},\ \varepsilon \in \mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1}). \tag{B.9}$$

When $\varepsilon \in (\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1})$, it can be obtained that

$$V(\pm\rho^{-(1+(h-1)s)\sigma}, \rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \dots, \rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n) = \rho^{-q\sigma}V(\pm\rho^{-(h-1)s\sigma}, \rho^{-(\lambda_n-1)\lambda_1\sigma}\varepsilon_2,$$

$$\dots, \rho^{-(\lambda_n-1)\lambda_{n-1}\sigma}\varepsilon_n) \le d_6\rho^{-q\sigma},$$

where $0 < s < 1$.

Therefore, one has

$$V(\varepsilon)^{-\gamma} > d_6^{-\gamma}\rho^{(q+\lambda-1)\sigma}, \ \varepsilon \in (\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1}). \tag{B.10}$$

(6) Thus, selecting $h > \{\bar{h}_1, \bar{h}_2\}$ and $\rho > \{\rho_1, \rho_2\}$, from the above inequalities (B.3), (B.7); (B.4), (B.8); (B.5), (B.9) and (B.6), (B.10), one can obtain a compact set which encircles the origin

$$\overline{\mathscr{A}} \triangleq (\mathscr{S}_1 \cap \overline{\mathscr{P}}_{\rho^{-h\sigma}}) \cup (\mathscr{F}_{\rho^{-\sigma}} \cap \overline{\mathscr{B}}_{3,\pi_1}) \cup (\mathscr{F}_{\rho^{-h\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{3,\pi_1})) \cup ((\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathscr{B}}_{3,\pi_1} \setminus \mathscr{B}_{3,\pi_1})).$$

And

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} V(\varepsilon)^{-\gamma} \le -w_4\rho^{1-\sigma}, \ \varepsilon \in \mathscr{A}, \tag{B.11}$$

where $w_4 = \min\{d_1d_4^{-\gamma}, d_2d_5^{-\gamma}, d_3d_6^{-\gamma}, \frac{d_2d_6^{-\gamma}}{2}\} > 0$.

Part II:

It is clear that $V(\varepsilon)$ and $\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)}$ are homogeneous of degrees q and $q+\lambda-1$ with respect to the weights $\{\lambda_i\}_{0\le i\le n-1}$. For any $\varepsilon \in \mathscr{R}^n \setminus \{0\}$, there exist $v_0 > 0$ and $\varepsilon^0 \in \mathscr{A}$ such that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T = (v_0\varepsilon_1^0, \dots, v_0^{\lambda_{n-1}}\varepsilon_n^0)^T$.

Then one has

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} = v_0^{q+\lambda-1}\left.\frac{dV(\varepsilon^0)}{dt}\right|_{(3.1)}$$

and

$$V(\varepsilon) = v_0^q V(\varepsilon^0).$$

Finally, from (B.11), for $\varepsilon \in \mathscr{R}^n \setminus \{0\}$, one can obtain

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(3.1)} = V(\varepsilon)^{-\gamma}\left.\frac{dV(\varepsilon^0)}{dt}\right|_{(3.1)} V(\varepsilon^0)^{-\gamma} \le -w_4\rho^{1-\sigma}V(\varepsilon)^{-\gamma}. \tag{B.12}$$

As for the proof of (iii), it follows the same procedure as the proof of (iv). The main difference compared with the proof (iv) is that in the proof of (iii), the compact set is constructed from the following four parts:

$\mathscr{S}_1 \cap \overline{\mathscr{P}}_{\rho^{-\sigma}}$, $(\overline{\mathscr{P}}_{(1+\pi_2)\rho^{-\sigma}} \setminus \mathscr{P}_{(1-\pi_2)\rho^{-\sigma}}) \cap \overline{\mathscr{B}}_{4,\pi_2}$, $\mathscr{F}_{\rho^{-h^*\sigma}} \cap (\overline{\mathscr{B}}_{1,1} \setminus \mathscr{B}_{4,\pi_2})$ and $(\overline{\mathscr{P}}_{\rho^{-\sigma}} \setminus \mathscr{P}_{\rho^{-h^*\sigma}}) \cap (\overline{\mathscr{B}}_{4,\pi_2} \setminus \mathscr{B}_{4,\pi_2})$, where $\pi_2 > 0$, $h^* > 2$ are two positive numbers.

This completes the proof.

□