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POISEUILLE FLOW OF POWER-LAW FLUIDS IN CONCENTRIC ANNULI - LIMITING CASES

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ABSTRACT

The flow of non-Newtonian fluids through an annulus is often encountered in various industrial processes such as transportation of drilling fluids in petroleum industry and extrusion of polymers (in a mandrel region).

Roughly speaking there are two approaches how to cope with the description of these flow situations. The numerical approach aims at a calculation of the quantities (e.g. velocity components, flow rate) describing the concrete problem, and with an arbitrary change of the entry parameters (geometry, kinematics, rheological characteristics) it is necessary to repeat the whole procedure from the beginning.

The other approach lays emphasis on the functional participation of the individual entry parameters in the whole solution. This method enables to decide which parameters should be altered (and in which way) to obtain more favourable results e.g. from the viewpoint of production rate. In this case the optimum approach is represented by an explicit solution. However in more complicated problems the chance to obtain an explicit solution is rather limited.

A number of papers have aimed at an analytical solution of an axial annular flow of power-law fluids, especially a relation: volumetric flow rate vs. pressure gradient. No complete analytical solution has been yet achieved. The only analytical solutions - that have been hitherto derived - concern the limiting cases of the geometrical parameter κ (inner-to-outer diameters ratio) or flow behaviour index n.

The present contribution discusses an applicability of these limiting solutions for a broader region of entry parameters and proves that in many cases usage of these relations is fully acceptable (and comparable with an inaccuracy in experimental determination of flow behaviour index n and consistency parameter k of the power-law model).

INTRODUCTION

Starting with a paper by Fredrickson and Bird [1] published in 1958, a derivation of the analytical relation volumetric flow rate vs. pressure gradient for steady laminar isothermal flow of incompressible axial annular flow of power-law fluids (see Fig.1) with no-slip at the boundaries has become an intensively studied topic up to now (for references see e.g. Escudier et al. [2], Filip and David [3]). Unlike laminar Newtonian flow, where complexity is caused almost exclusively by the geometric conditions of the given problem, for laminar non-Newtonian flow this complexity is intensified by nonlinear dependence between shear stress and shear rate.

In this case a power-law model is governed by the relation

$$\tau = -k\left|\frac{dv_z}{dr}\right|^n \operatorname{sgn}\left(\frac{dv_z}{dr}\right) \quad , \quad \dot{\gamma} = \left|\frac{dv_z}{dr}\right| \qquad (1)$$

where n represents the flow behaviour index, k the consistency parameter, v_z the axial velocity component (see Fig.1), and $\dot{\gamma}$ is the shear rate.

Each hitherto published semi-analytical solution encounters the problem how to determine a parameter λ, where λR represents the radial location of maximum of axial velocity component and simultaneously the point where shear stress nullifies.

Hanks and Larson [4], and Prasanth and Shenoy [5] independently (using different approaches) derived the relation

$$Q_{ax} = \frac{\pi n R^3}{1+3n}\left[\left(1-\lambda^2\right)^{1+\frac{1}{n}} - \kappa^{1-\frac{1}{n}}\left(\lambda^2-\kappa^2\right)^{1+\frac{1}{n}}\right]\left(\frac{PR}{2k}\right)^{\frac{1}{n}} \qquad (2)$$

where Q_{ax} stands for the volumetric flow rate, P is the pressure drop defined as a change of pressure per unit of length ($\Delta P/L$). The parameter λ is necessary to determine numerically from an integral equation introduced already in Fredrickson and Bird [1]

$$\int_{\kappa}^{\lambda} \left(\frac{\lambda^2}{\xi} - \xi\right)^{\frac{1}{n}} d\xi = \int_{\lambda}^{1} \left(\xi - \frac{\lambda^2}{\xi}\right)^{\frac{1}{n}} d\xi \qquad (3)$$

The term in the square brackets (rel.(2)) represents a weight function reducing an axial annular flow rate from that through a pipe given by two remaining terms in rel.(2).

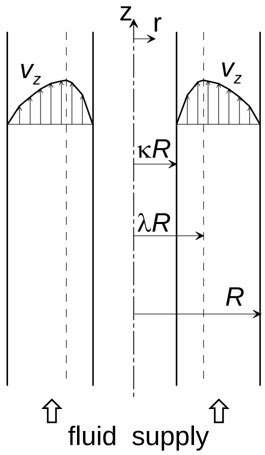

Fig.1. A definition sketch.

There are approximately three possibilities how to eliminate a necessity to solve numerically the integral equation (3)

1) to express an approximate relation for the parameter λ;
2) to propose a fully analytical (algebraic) form Q_{ax} vs. P eliminating λ;
3) determination of quasisimilarity transformations providing almost exact relation Q_{ax} vs. P in a broad region of entry parameters.

ad 1)

Substituting the approximate relations for pseudoplastic fluids

$$\lambda(n,\kappa) = \sqrt{\kappa} + n^{0.37+0.28(1-n)^2+\frac{1}{100\kappa}} \cdot \left(\sqrt{\frac{1-\kappa^2}{\ln(1/\kappa^2)}} - \sqrt{\kappa}\right) \qquad (4)$$

and dilatant fluids

$$\lambda(n,\kappa) = \sqrt{\frac{1-\kappa^2}{\ln(1/\kappa^2)}} + \left(1-\left(\frac{1}{n}\right)^{0.66+0.2\left(1-\frac{1}{n}\right)^{1.5}+\frac{1}{200n^{0.8}}}\right) \cdot \left(\frac{1+\kappa}{2} - \sqrt{\frac{1-\kappa^2}{\ln(1/\kappa^2)}}\right) \qquad (5)$$

proposed by David et al. [6] into rel.(2) the deviations of Q_{ax} from the exact values do not exceed 4% for a pseudoplastic case ($\kappa \geq 0.5$, $0.1 \leq n \leq 1$) and 0.15% for a dilatant case ($0 < \kappa < 1$, $n \geq 1$).

ad 2)

David and Filip [7] proposed an explicit approximate algebraic expression relating volumetric flow rate with pressure gradient in the form

$$Q_{ann,appr} = \frac{\pi R^3 n}{1+3n}(1-\kappa)^{2+\frac{1}{n}}(1+\kappa)\cdot Q_{norm}(\kappa,n)\cdot\left(\frac{PR}{2k}\right)^{\frac{1}{n}} \qquad (6)$$

where for pseudoplastic fluids

$$Q_{norm}(\kappa,n) = \frac{1}{1+2n}\left[(1-n)Q_{norm}(\kappa,0) + 3nQ_{norm}(\kappa,1)\right] , \qquad (7)$$

and for dilatant fluids

$$Q_{norm}(\kappa,n) = \frac{3}{1+2n}\left(Q_{norm}(\kappa,1) + \frac{n-1}{2}\right) , \qquad (8)$$

where

$$Q_{norm}(\kappa,1) = \frac{1+\kappa}{1-\kappa}\left(\frac{1+\kappa^2}{1-\kappa^2} + \frac{1}{\ln\kappa}\right) , \qquad (9)$$

$$Q_{norm}(\kappa,0) = \frac{1}{2}\left[1 + (3Q_{norm}(\kappa,1) - 2)\right] . \qquad (10)$$

The deviation of these expressions from the exact values for $0.025 < \kappa < 1$ in the whole pseudoplastic region $0 < n < 1$ does not exceed 2.15%. For $0.5 < \kappa < 1$ the deviation is even less than 0.4%; for $0.6 < \kappa < 1$ less than 0.16%. In the case of dilatant fluids the situation is even better, the deviation does not exceed 1.5% for $0.025 < \kappa < 1$ and 0.1% for $0.4 < \kappa < 1$.

ad 3)

The given problem is also possible to treat from the viewpoint of similarity behaviour. It was shown (David and Filip [8]) that the relation Q_{ax} vs. P exhibits various features of similarity behaviour – not in an exact form but only approximately (it implies the term 'quasisimilarity'). Nevertheless, even this 'weak' similarity enables one to derive a 'universal' solution which is possible to rewrite to a concrete form for given entry parameters by means of certain derived transformations. This fully eliminates the role of the parameter λ; however, quasisimilarity is not valid in the whole range of entry parameters κ, n (it was shown in the region $\kappa \geq 0.4$ and $(1-\kappa)^{1.8}/n \leq 2.44$). In this connection it is still necessary to have in mind that the notion 'exact solution' is only hypothetical with respect to the approximate determination of the entry rheological parameters k and n.

LIMITING CASES

Parallely to the papers referred to in ad 1), 2), 3), there is a group of the papers using for a determination of the relation Q_{ax} vs. P the limiting values of the parameter $\lambda(\kappa,n)$ both for A) a flow behaviour index n and B) an annular aspect ratio κ.

ad A)

In the case of a flow behaviour index n Vaughn [9] proved that for all aspect ratios κ the following relations are exact

$$\lim_{n\to 0} \lambda(\kappa,n) = \sqrt{\kappa} , \qquad (11)$$

$$\lim_{n\to 1} \lambda(\kappa,n) = \sqrt{(1-\kappa^2)/\ln(1/\kappa^2)} , \qquad (12)$$

$$\lim_{n\to \infty} \lambda(\kappa,n) = \frac{1+\kappa}{2} . \qquad (13)$$

It implies that for power-law fluids whose behaviour approaches that of solid-like materials (see e.g. Sitzer and Durban [10]) it is possible to use rel.(11) as a first approximation.

Rel.(12) valid for Newtonian liquids was applied by Luo and Peden [11] as the approximation for power-law fluids because no exact solution of rel.(3) is known for n≠1. In this case the deviation of the approximate value of λ (rel.(12)) from the exact one does not exceed approximately 3% in the region $0.3 < \kappa < 1$ and $0.5 < n < 1$ as illustrated by Luo and Peden [11, Fig.1].

Rel.(13) indicates that for strongly dilatant fluids a location of the parameter λ for any κ roughly corresponds to its location for the case of parallel-plate geometry.

ad B)

In the case of an annular aspect ratio κ there are two limiting cases, either $\kappa \to 0$ (pipe flow) or $\kappa \to 1$ (flow between parallel plates) for which

$$\lim_{\kappa \to 1} \lambda(\kappa, n) = \frac{1+\kappa}{2} \quad . \tag{14}$$

A combination of the flow situations for which there are valid rels.(13,14) elucidates why for a description of dilatant fluids in a narrow annular gap an application of the relation $\lambda(\kappa,n)=(1+\kappa)/2$ is fully justified and provides almost exact results.

SOLUTION FOR A PARALLEL-PLATE GEOMETRY AS A STARTING POINT

In this case there is no problem with a determination of the parameter λ, moreover this relation does not depend on a flow behaviour index n

$$\lambda_{par\,pl}(\kappa, n) = \frac{1+\kappa}{2} \quad . \tag{15}$$

There are approximately four papers trying to use a solution for the parallel-plate geometry for that describing flow through an annular passage.

1) Worth [12] studied the deviations of the exact solutions Q vs. P from those for an equivalent parallel-plate geometry (i.e. a width between the parallel plates corresponds to a clearance between the cylinders) for the following four cases of concentric annular flows: tangential drag flow, tangential pressure flow, axial drag flow, and axial pressure flow. In his analysis of axial pressure flow he concentrated to a region $0.5 \le \kappa \le 1$, and n = 1/5, 1/4, 1/3, 1/2, and 1. His choice of the individual n's as the reciprocal values of the natural numbers reflects the results of Fredrickson and Bird [1] as rel.(2) was not yet known. Worth [12, Fig.8] compared graphically the flow rate $Q_{ax}(\kappa,k,n,\lambda,P)$ for a given annular geometry, pseudoplastic power-law fluids and pressure drop with the corresponding flow rate $Q_{par\,pl}(W,k,n,P)$ for the parallel plate geometry (with a width $W=(1-\kappa)R$, $\lambda=(1+\kappa)/2$). He showed that the ratio $Q_{ax}/Q_{par\,pl}$ monotonously decreases (from the value 1) with decreasing annular aspect ratio κ and flow behavior index n, but for greater κ and n this ratio is very close to one.

2) Bird et al.[13] succeeded in eliminating the parameter λ from the relation Q_{ax} against P using a variational method supposing one-parametrical velocity profile. However, their relation is only approximate. It seems that there is no possibility to improve their result using two- or multi-parametrical velocity profile as the resulting algebraic equations for determination of individual variational parameters are more complex than the original integral equation for a determination of λ (rel.(3)). In fact, their resulting relation (see rel.(4.3-37), p.203)

$$\mathrm{Q}_{\mathrm{par\,pl}} = \frac{\pi \mathrm{n R}}{2(2\mathrm{n}+1)} \left(1+\kappa\right) \left(1-\kappa\right)^{2+\frac{1}{\mathrm{n}}} \left(\frac{\mathrm{PR}}{2\mathrm{k}}\right)^{\frac{1}{\mathrm{n}}} \tag{16}$$

coincides with the relation for volumetric flow rate between parallel plates (rel.(3-101), p.102 in McKelvey [14]). As stated in Bird et al. [14] the inaccuracy of rel.(16) related to the exact rel.(2) is less than 2% for $\kappa \ge 0.5$, $n \ge 0.5$; this deviation corresponds to Fig.8 in Worth [12].

3) Tuoc and McGiven [15] proposed a generalised Mooney-Rabinowitsch equation (independent on a specific non-Newtonian constitutive model) respecting the limiting cases of flow in cylindrical pipes and between parallel plates. This equation was tested applying a power-law model and examined using the experimental data in an annular flow.

4) Based on the quasisimilarity behaviour of axial annular flow (David and Filip [8]), i.e. the continuous convergence (for $\kappa \to 1$) of flow to the parallel-plate flow, in other words

$$\lim_{\kappa \to 1} Q_{ax} = Q_{par\,pl} \quad , \tag{17}$$

it is possible to propose the approximate relation

$$Q_{ax\,appr} = Q_{par\,pl} \left[1 - \frac{1}{93} n^{-5/9} \left(\frac{1}{\kappa} - 1 \right)^{9/10} \right]^{-1} \tag{18}$$

for volumetric flow rate of axial annular flow, see Filip and David [16]. This algebraic relation does not explicitly depend on the relative radial location λ of the maximum velocity, and therefore eliminates the necessity of computation of the integral equation (3). The relative deviations do not exceed 3.5% in the region $\kappa \ge 0.1$, $n \ge 0.1$; for $\kappa \ge 0.1$, $n \ge 0.6$ or $\kappa \ge 0.4$, $n \ge 0.1$ the relative deviations are less than 1%.

RESULTS AND DISCUSSION

The above analysis proves that not always it is indispensable to apply numerical procedures for calculating a set of integro-differential equations describing the balance equations of the chosen problem, as e.g. a flow through a concentric annulus. Sometimes it is more efficient to compare a deviation of the limiting case (parallel-plate geometry) from the exact values and to 'suppress' this discrepancy through a weight function, see rel.(18). This approach gives the possibility to determine how the individual entry parameters influence the resulting relation volumetric flow rate vs. pressure drop, and thus how to simply encounter the demands from practice. If in rel.(18) we compare the relative deviations (less than 3.5%) in the region $\kappa \ge 0.1$, $n \ge 0.1$ with the experimental errors in determining flow behaviour index n and consistency parameter *k*, we can conclude that the proposed relation (18) is from the practical point of view fully acceptable.

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