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Ergodic properties of noncommutative dynamical systems

by

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DECLARATION

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract. In this dissertation we develop aspects of ergodic theory for C^*-dynamical systems for which the C^*-algebras are allowed to be noncommutative. We define four ergodic properties, with analogues in classic ergodic theory, and study C^*-dynamical systems possessing these properties. Our analysis will show that, as in the classical case, only certain combinations of these properties are permissable on C^*-dynamical systems. In the second half of this work, we construct concrete noncommutative C^*-dynamical systems having various permissable combinations of the ergodic properties. This shows that, as in classical ergodic theory, these ergodic properties continue to be meaningful in the noncommutative case, and can be useful to classify and analyse C^*-dynamical systems.

Contents

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Introduction

Ergodic theory, in the most general sense, is the study of the asymptotic behaviour of some dynamical system, where the latter term is used very loosely. This description, by no accident, also describes statistical mechanics. A little less generally, ergodic theory is the study of the conditions under which a dynamical system's asymptotic behaviour has certain preferred/interesting properties. These are the ergodic properties in the sense that, if a system has ergodic property A, then that tells us something about how the system evolves in the long term, i.e. asymptotically. In ergodic theory there are

ergodic systems
weakly mixing systems
strongly mixing systems
compact systems

which, note, do not form a partition. There is a significant body of knowledge around these properties in the classical case, i.e. commutative case, the most standard of which is a measure space X with a measure preserving transformation $T : X \to X$, or a whole group of measure preserving transformations $\{T_g : g \in \mathbb{R}\}$. Usually the measure space is finite and is then normalized to a probability space. In this setting, it is known that there are several different, but equivalent, formulations, and interesting relationships between the various properties. There is also no manner of shortage of concrete dynamical systems having various combinations of these ergodic properties.

Our main objective is to define analogues of these properties on noncommutative dynamical systems, and to determine whether these properties can then be similarly (to the classical case) analysed and identified on concrete noncommutative dynamical systems. By a noncommutative dynamical system we mean for example a quantum dynamical system which, in the C^*-algebraic formulation of quantum dynamics, has an observable algebra given by a noncommutative C^*-algebra. For example, the observable algebra of the quantum harmonic oscillator is the operator algebra $\mathscr{L}(L^2(\mathbb{R}))$ which contains the position measurement projection X_V and momentum measurement projection P_V, for any Borel set $V \subseteq \mathbb{R}$, with $X_V P_V \neq P_V X_V$ in general. C^*-dynamical systems provide a fertile ground for such noncommutative dynamical systems and so will form the basis of our study. The study of C^*-dynamical systems play a prominent role in quantum mechanics, and [**4**, p. 3-p. 15] provides historical context and an outline of how the importance of these systems were gradually understood.

It would be a great pity to dive headlong into the C^*-dynamical formulations of these ergodic properties without first giving a brief account of the elegant theoretical considerations that naturally gave rise to them, and in so doing develop a deeper understanding of what these properties represent, at least classically. We will then also be afforded the opportunity to understand how these properties can be translated to a C^*-algebraic setting. As a bonus we will also learn why the ergodic property, ergodicity, is the namesake of the entire field of ergodic theory. An excellent, and far more detailed, account of the development of ergodic theory is given in [**15**], which will also act as the inspiration for the discussion to follow.

A general starting point is a measure space (X, μ), with σ-algebra Σ, and a one-parameter group of measure preserving transformations $\{T_t : X \to X : t \in \mathbb{R}\}$. The motivation for these mathematical structures comes from the evolution of a system governed by Hamiltonian equations and a fundamental theorem from statistical mechanics. The state space of a system of N particles is $\mathbb{R}^{6N}$ and the Hamiltonian equations determine the Hamiltonian flow $\{T_t : \mathbb{R}^{6N} \to \mathbb{R}^{6N} : -\infty < t < \infty\}$ of the system. That is, if the system is initially in state $x_0 \in \mathbb{R}^{6N}$, then after time t it is in state $x_t = T_t x_0$. According to Liouville's Theorem, the Hamiltonian flow preserves the Lebesgue measure on $\mathbb{R}^{2n}$ for any $n \in \mathbb{N}$. The Hamiltonian flow is therefore a one-parameter group of Lebesgue measure preserving transformations on $X = \mathbb{R}^{6N}$. It is within the context of statistical mechanics, i.e. when N is very large, that Ludwig Boltzmann started to think in terms of asymptotic behaviour to get past physical problems he encountered. We will return to this a little later, and will first look at some of the questions raised by Poincaré which, even though it came later, is simpler and more natural from a purely mathematical perspective. Nonetheless, Poincaré himself was also motivated by physical problems.

Returning to (X, μ) and $\{T_t : t \in \mathbb{R}\}$, we would like to make two simplifications for the sake of this introductory discussion. Firstly, let us suppose measurements occur only at discrete time intervals in $\mathbb{R}$, so that our time group in effect becomes $\mathbb{Z}$, i.e. we consider $T_{nt_0} = T_{t_0}^n$ for all $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}\backslash\{0\}$. So, instead of the one-parameter group $\{T_\alpha\}$, we only work with a single T representing time evolution over a fixed time interval. We will cover more general "time" groups, including $\mathbb{R}$ as a special case, later on in the dissertation. Secondly, let us assume that X is a finite measure space. Even though $\mathbb{R}^{6n}$ is not a finite measure space, in many cases the system can be restricted to a subset of $\mathbb{R}^{6n}$ with finite measure. That is, there is a set with finite measure in $\mathbb{R}^{6n}$ that is invariant under the Hamiltonian flow. For example, this would be the case for a classical gas in a closed container.

Some of the earlier asymptotic questions were raised by Poincaré. If $x \in E \subseteq X$ then x is called recurrent, with respect to E, if $T^n x \in E$ for some positive integer n. This lead to the recurrence theorem:

THEOREM 0.1. *(The Recurrence Theorem) If $T : X \to X$ is a measure preserving transformation on a space X of finite measure, and*

$E \subseteq X$ is a measurable set, then almost every point of E is recurrent. Moreover, for almost every point $x \in E$ there is a sequence $(n_i) \in \mathbb{N}$ such that $T^{n_i}x \in E$ for all $i \in \mathbb{N}$

The Recurrence Theorem states that, not only does almost every point of E return to E eventually, it does so infinitely often. Naturally one can now ask, if a point $x \in E$ returns to E infinitely often, how long does it spend in E on average? If we consider the orbit of x, $\{T^n x : n \in \mathbb{N}\}$, then the ratio of the number of these points in E to the total number of points, after n "steps" is given by

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \tag{1}$$

where f is the characteristic function of E, i.e. χ_E. How long the orbit of x spends in E on average, is then given by the limit of (1) as n tends to infinity. An ingenious generalization comes from relaxing the restriction that f is a characteristic function. Of course the immediate question then becomes: when does $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ exist, i.e. when does the sequence $f(T^n x)$ converge in the sense of Cesàro? If we define $U : f \mapsto f \circ T$, then U defines an operator between function spaces. We say that T induces the operator U. The investigation of U yielded some undoubtedly surprising results. Restricted to $L^2(X)$, the induced operator $U : L^2(X) \to L^2(X)$ is a linear isometry, and if $T : X \to X$ is an *invertible* measure preserving transformation, then the invertibility carries over to U. $L^2(X)$ is of course a Hilbert space, and any linear invertible isometry on a Hilbert space is a unitary operator. The problem of Cesàro convergence thus reduces to the study of the limiting behaviour of

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k x \tag{2}$$

where U is a unitary operator on a Hilbert space H and $x \in H$. That mean convergence, i.e. the limits of these Cesàro averages, always takes place in the strong topology, is considered the starting point of modern ergodic theory. It is addressed in two important theorems, the mean ergodic Theorem of von Neumann and the pointwise ergodic theorem of G.D. Birkhoff. The mean ergodic theorem establishes the convergence of (2) for unitary operators U, and the individual ergodic theorem directly establishes the convergence of (1). In this dissertation, we will rely on the Hilbert space approach, instead of the pointwise approach, and so we focus on the former case. [**15**] focuses more on the pointwise approach. A precise formulation of the mean ergodic theorem, in the case of discrete time, is contained in Theorem 0.2.

THEOREM 0.2. *(Mean Ergodic Theorem) If U is a contraction on a complex Hilbert space H and if P is the projection onto the fixed point* *space of* U, $\{x \in H : Ux = x\}$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k x = Px \quad \textit{for all } x \in H.$$

This theorem naturally leads to the notion of ergodicity which we now explain. Ergodicity is one of the formulations of the natural, and preferred, asymptotic behaviour wherein a transformation does a good job of stirring up the points of the space it acts upon. A precise, and quite straight-forward, formulation is as follows: T is ergodic if and only if

$$T^{-1}(E) = E \;\Rightarrow\; \mu(E) = 0 \text{ or } \mu(X\backslash E) = 0$$

for all measurable sets $E \subseteq X$. The connection between this formulation and a transformation that mixes up the space, is quite clear. However it is not especially convenient, so an immediate incentive is to find more useful formulations of ergodic transformations. A more useful, and equivalent, formulation is: T is ergodic if and only if every measurable invariant function is constant up to a set of measure zero. An L^2 function f is invariant if $f \circ T = f$. This formulation can now show the connection of ergodicity with its historic anchorage. If the measure space X is finite, $\mu(X) < \infty$, and T is ergodic then, by Theorem 0.2

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = Pf = \left(\frac{1}{\mu(X)} \int_X f \; d\mu \right) \Omega$$

where $\Omega = 1 \in L^2(X)$. Here P denotes the projection $P = \frac{1}{\mu(X)}\Omega \otimes \Omega$ in L^2, i.e. $Pf = \frac{1}{\mu(X)}\Omega\langle\Omega, f\rangle_{L^2}$. What we are looking at here is an L^2 version of the highly sought after conclusion of Boltzmann's ergodic hypothesis, that the phase space mean of a physical variable f (right) should equal the, generally harder to obtain, time mean of the variable (left). All that was wrong with the hypothesis was the premise. Boltzman speculated that if the orbit of a point reached all corners of the space X then this conclusion should follow. So, the ergodic property as we formulated it, yields the conclusion of the ergodic hypothesis and the subsequent further development of these ideas became known as *ergodic* theory.

For convenience, we will now assume that $\mu(X) = 1$, if $\mu(X) < \infty$. This is mostly the standard framework for work in ergodic theory, and it is what we will generalize to C^*-dynamical systems.

So how are the other ergodic properties formulated? From our initial definition of ergodic transformations it is not exactly clear how a variation in the definition might suggest other interesting asymptotic behaviours. This is one of the advantages in alternate equivalent formulations, as we shall shortly observe. Not only can an equivalent formulation prove to be more useful analytically, it can also suggest other interesting properties worth investigating. Let us humour two additional formulations of ergodic transformations, one because it will lead us to the weak mixing and strong mixing property, and the other since it connects to spectral properties of U, which will prove to be very useful.

First the latter, which follows from the discussion above regarding invariant functions. The ergodic property of a transformation T has a surprising influence on the spectral properties of the unitary operator induced by T, as discussed earlier.

Theorem 0.3. *A measure preserving transformation $T : X \to X$ on a probability space X is ergodic if and only if 1 is a simple eigenvalue of U, or equivalently, the fixed point space of U is one-dimensional, namely $\mathbb{C}\Omega$.*

In Chapter 2 we will prove this for the C^*-dynamical case using the mean ergodic theorem, albeit a generalized version.

For the former, again consider an ergodic transformation T on a probability space X. Consider any measurable $F, G \subseteq X$ and let $f = \chi_F, g = \chi_G$. Using the mean ergodic theorem we can show that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(F \cap T^{-k}(G)) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \langle \chi_F, U^k \chi_G\rangle \\
&= \langle \chi_F, P\chi_G\rangle \\
&= \langle \chi_F, \Omega\langle \Omega, \chi_G\rangle\rangle \\
&= \mu(F)\mu(G) \qquad (3)
\end{aligned}$$

In fact, conversely it can easily be shown that if (3) holds for all measurable $F, G \subseteq X$, then T is ergodic. The formulation of ergodicity as in (3) readily allows us to formulate other possible properties that a system may have:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \left|\mu(F \cap T^{-k}G) - \mu(F)\mu(G)\right| = 0 \quad \forall\ F, G \in \Sigma \qquad (4)$$

$$\lim_{n\to\infty} \mu(F \cap T^{-n}G) = \mu(F)\mu(G) \quad \forall\ F, G \in \Sigma \qquad (5)$$

It is easy to see that (5)$\Rightarrow$(4)$\Rightarrow$(3). If a transformation T satisfies (5), we call it *strongly mixing*, and if T satisfies (4) we call it *weakly mixing*. It turns out that these properties are very useful in classifying dynamical systems, and indeed there are systems that are ergodic but not weakly mixing, weakly mixing but not strongly mixing, and systems which are strongly mixing.

Weak mixing in particular is of particular structural importance, In a sense it is one type of "extreme" behaviour a system can have, with the "opposite extreme" being *compactness*. This will be discussed further below and in Chapter 2, by building on the theory developed in Chapter 1.

The simplest way to transfer these ergodic properties to a C^*-algebraic setting is to consider a C^*-algebraic formulation of the classical setting described above, and to literally translate (3),(4) and (5) in terms of C^*-algebraic related structures.

All of the physical information of the system (X, μ, T) is given by real, or complex, valued measurable functions on X. Mathematically, we are thus looking at a C^*-algebra $\mathfrak{A}$ of functions, e.g. the space $B_\infty(X)$ of all bounded complex valued measurable functions on X. There are other spaces as well, such as $L^\infty(X)$, or $C(X)$ if X is topological. Suppose that $\mathfrak{A} = B_\infty(X)$. We can recover all of the probabilities from the functions in $B_\infty(X)$ since for any measurable $W \subseteq X$, $\mu(W) = \int_X \chi_W \, d\mu$, and $\chi_W \in B_\infty(X)$. For instance, if

$$f : X \to \mathbb{R}$$

represents some physical quantity of the system, defined at each point of the phase space X, then the quantity after one step is given by $f \circ T$ and the probability that, at some initial time, the quantity will have a value in $V \subseteq \mathbb{R}$ is

$$\int_X \chi_{f^{-1}(V)} \, d\mu = \mu(f^{-1}(V))$$

So, the probabilities are given by the mapping

$$\omega : \mathfrak{A} \to \mathbb{C} : g \mapsto \int_X g \, d\mu$$

and the time evolution by the mapping

$$\tau : \mathfrak{A} \to \mathfrak{A} : f \mapsto f \circ T$$

ω is a state on $\mathfrak{A}$ and τ is a $*$-automorphism of $\mathfrak{A}$. We delay precise definitions until Chapter 2.

So let us see what (3) translates into, in terms of the *C^*-algebra* $\mathfrak{A}$, *state* ω and *$*$-automorphism* τ. Let $f = \chi_F$, and $g = \chi_G$ for measurable F, G. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(F \cap T^{-k}G) = \mu(F)\mu(G)$$

becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega(f\tau^k(g)) = \omega(f)\omega(g) \tag{6}$$

(6) is precisely how ergodicity is formulated in a C^*-algebraic system, for a general, possibly noncommutative, C^*-algebra $\mathfrak{A}$, $*$-automorphism τ and τ-invariant state ω. Of course the weak mixing and strong mixing properties' formulation may similarly be obtained. The triple $(\mathfrak{A}, \omega, \tau)$ is called a C^*-dynamical system. We will however opt for a more general approach not confined to discrete time. That is, in the preceding discussion, time evolution operated in a discrete manner through iterations of the measure preserving transformation T, or translated, iterations of the $*$-automorphism τ. We say the time evolution of the system is a $\mathbb{Z}$-action. Instead we will allow for more general groups G that determine a group of $*$-automorphisms $\{\tau_g : g \in G\}$ such that $\tau_g \circ \tau_h = \tau_{gh}$ for all $g, h \in G$.

To allow for the successful characterization of the ergodic properties on a dynamical system that evolves under the action of a general group G, certain structural features will have to be assumed on G. These assumptions, the definition of a C^*-dynamical system and the characterization of the ergodic properties on C^*-dynamical systems will be the topic of the first two sections of Chapter 2. The rest of Chapter 2 will be devoted to analysing the defined properties to determine what combinations of the properties are permissable on the same C^*-dynamical system. This will be done by deriving different, but equivalent, characterizations of the ergodic properties. Some important tools and results used to this end form the entirety of Chapter 1. We will show in Chapter 2 how the group G may always be represented by a group of unitary operators on a Hilbert space, known as the GNS representation, and how the ergodic properties can be characterized in terms of this representation. We will show that ergodicity relates to the fixed point space of the representation, and the weak mixing and compactness properties to the eigenspace of the representation. We will prove that a C^*-dynamical system is weak mixing if and only if the eigenspace of the GNS representation is one dimensional. The other end of this extreme, is when the eigenspace spans the entire Hilbert space in which case the C^*-dynamical system will be called compact. Why the property is coined "compact" is explained in Chapter 2.

By the end of Chapter 2 we will know which combinations of the ergodic properties on a C^*-dynamical system are ruled out by general theory. It will then become our final objective to obtain/construct concrete examples of C^*-dynamical systems exhibiting the remaining combinations. We will consider C^*-dynamical systems on three different noncommutative C^*-algebras:

(i) Quantum mechanical systems with discrete energy spectrums on $\mathscr{L}(H)$, with H separable (Chapter 3),
(ii) Systems of algebraic origin on reduced group C^*-algebras (Chapter 4),
(iii) Two types of systems on quantum tori (Chapter 5).

These examples are not arbitrary and do carry significance, which we will elaborate on in their respective Chapters. Here we only note that these examples are either basic in quantum physics (namely the quantum harmonic oscillator as a special case of a system with discrete energy spectrum), or are built on C^*-algebras, which are some of the most important examples of noncommutative C^*-algebras other than $\mathscr{L}(H)$ (namely the reduced group C^*-algebras and quantum tori). These examples are therefore very natural in the noncommutative context. This is a relevant point, since in principle we could simply take classical systems with various combinations of properties and extend them to $\mathscr{L}(H)$ using the GNS representation U of T, to obtain noncommutative systems with the same combinations of properties. However, unlike the examples mentioned above, this would not be very natural or interesting from the point of view of quantum physics or even C^*-algebras, and would therefore not have any true significance.

The ultimate goal of this dissertation is to show that the various ergodic properties discussed above, remain meaningful in the noncommutative case. In particular one would like to know that there are noncommutative systems having the various properties, and that the properties are indeed distinct, as they are in the classical case. Our study of these systems will yield all of the combinations, except for two, and the possible resolution of this shortcoming will form part of the discussion in the section "Further research". To connect the beginning with the end, we end by providing a summary of what our efforts will yield:

The last column on the following page list the examples we obtained having the properties indicated in the various rows, and indicates the combinations of properties which are impossible to have in the same system due to general theory. It is important to note that aside from the combinations marked "?", this is in exact correspondence to the classical case, where the same combinations are impossible, and there are classical examples satisfying the remaining combinations.

	ergodic	weakly mixing	strongly mixing	compact	examples
(a)	$\times$	$\times$	$\times$	$\times$	ReFG3 QTA3(θ)
(b)	$\times$	$\times$	$\times$	$\checkmark$	DESS2 ReFG2 QTA2(θ)
(c)	$\times$	$\times$	$\checkmark$	$\times$	impossible
(d)	$\times$	$\times$	$\checkmark$	$\checkmark$	impossible
(e)	$\times$	$\checkmark$	$\times$	$\times$	impossible
(f)	$\times$	$\checkmark$	$\times$	$\checkmark$	impossible
(g)	$\times$	$\checkmark$	$\checkmark$	$\times$	impossible
(h)	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	impossible
(i)	$\checkmark$	$\times$	$\times$	$\times$	?
(j)	$\checkmark$	$\times$	$\times$	$\checkmark$	DESS1 QTT(θ)
(k)	$\checkmark$	$\times$	$\checkmark$	$\times$	impossible
(l)	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	impossible
(m)	$\checkmark$	$\checkmark$	$\times$	$\times$	?
(n)	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	impossible
(o)	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	ReFG1 QTA1(θ)
(p)	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	impossible

$\checkmark$: property present, $\times$: property absent

For more details, consult:

(a) ReFG3 : Def. 3.3, Proposition 3.6, Theorems 5.1 and 5.2
QTA3(θ) : Def. 6.4, Proposition 6.7, Theorems 5.1 and 5.2

(b) DESS2 : Def. 2.4, Proposition 2.6 and Theorem 5.1
ReFG2 : Def. 3.3, Proposition 3.5, Theorems 5.1 and 5.2
QTA2(θ) : Def. 6.4, Proposition 6.6, Theorems 5.1 and 5.2

(j) DESS1 : Def. 2.4, Proposition 2.5 and Theorem 5.3
QTT(θ) : Def. 3.2, Propositions 4.4, 4.5 and Theorem 5.3

(o) ReFG1 : Def. 3.3, Proposition 3.4, Theorems 5.1, 5.2 and 5.3
QTA1(θ) : Def. 6.4, Proposition 6.5, Theorems 5.1 - 5.3

(c),(d),(k),(l) : Theorem 5.2
(f),(g),(h) : Theorem 5.1
(n),(p) : Theorem 5.3

(i),(m) : refer to the section "Further research"

Index of Symbols and Conventions

$\mathbb{N}$	natural numbers, $\{1, 2, 3, ...\}$	$\mathbb{Z}$	set of Integers
$\mathbb{R}$	real field	$\mathbb{C}$	complex field
$\mathbb{S}^1$	unit circle in $\mathbb{C}$	$\mathbb{T}^2$	torus, $\mathbb{T} \times \mathbb{T} \equiv \mathbb{R}^2/2\pi\mathbb{Z}^2$
$\mathbb{Q}$	rational numbers		
X	Banach space	H	Hilbert space
$\mathfrak{A}$	C^*-algebra	$\mathscr{B}$	Borel σ-algebra

$\mathscr{L}(X)$	space of bounded linear operators $X \to X$
$\mathfrak{U}(X)$	collection of unitary operators in $\mathscr{L}(X)$
$C(K)$	space of continuous functions on the compact set K
X^*	continuous dual of X
κ	canonical mapping $X \to X^*$, with $\kappa(x)(f) = f(x)$ for all $x \in X$
$\langle h, x\rangle$	alternative notation for $h(x)$, with $h \in X^*$ and $x \in X$, used in Chapter 1
T^*	adjoint operator in $\mathscr{L}(X^*)$ of $T \in \mathscr{L}(X)$, defined $\langle T^*h, x\rangle = \langle h, Tx\rangle$ for all $x \in X$
A^*	adjoint operator in $\mathscr{L}(H)$ of $A \in \mathscr{L}(H)$
$\Omega \otimes \Omega$	projection operator in $\mathscr{L}(H)$ defined $x \mapsto \Omega\langle \Omega, x\rangle$
id	identity mapping of a space onto itself
$E_{m,n}$	the function $\mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto e^{imx}e^{iny}$
1	either $1 \in \mathbb{R}$, unit element of a group, unit element of a C^*-algebra or a constant valued function with value $1 \in \mathbb{R}$
χ_V	characteristic/step function of V
Aut $(\mathfrak{A})$	collection of all $*$-automorphisms of $\mathfrak{A}$

wo-	weak operator topology prefix	w-	weak topology prefix
so-	strong operator topology prefix	w*-	weak* topology prefix

$\overline{z}$	complex conjugate of $z \in \mathbb{C}$
$\mathrm{Re}(f)$	real part of f, $\mathrm{Re}(f)(x) := \frac{1}{2}\left(f(x) + \overline{f(x)}\right)$
$\mathrm{Im}(f)$	imaginary part of f, $\mathrm{Im}(f)(x) := \frac{i}{2}\left(\overline{f(x)} - f(x)\right)$
f^+	positive part of f, $f(x) := \max\{0, f(x)\}$
f^-	negative part of f, $f(x) := \max\{0, -f(x)\}$
$x_n \longrightarrow x$	convergence in associated metric/norm topology
$\succeq$	partial order on a directed set
cl V	closure of V in associated topology
co (V)	convex hull of V
$\overline{\text{co}}$ (V)	norm closure of co (V)
span V	vector/algebra span of V depending on whether V is a subset of a vector space or algebra

(i) Unless explicitly stated otherwise, any vector space or algebra will be over the field of complex numbers.
(ii) A locally compact space will always be assumed to be Hausdorff.
(iii) An inner product in a Hilbert space will always be taken to be linear in the second argument and conjugate linear in the first argument.
(iv) Concerning the *function* space L^2, we will adopt the customary approach as pointed out in [**26**]: L^2 is a not a function space but a space whose elements are equivalence classes of functions, however, for the sake of simplicity of language we relegate this distinction to the status of a tacit understanding and continue to speak of L^2 as a function space.

CHAPTER 1

Splitting Theorem of Jacobs-Deleeuw-Glicksberg

In this chapter we review the splitting Theorem of Jacobs-Deleeuw-Glicksberg, closely following [**19**, 2.4 p. 103 -p. 109], but expanding and expounding the arguments and proofs given there. The results that we shall derive are for semigroups of continuous linear operators in Banach Spaces. In Chapter 2 we will restrict these results to the case of Hilbert spaces, and use them to derive the spectral formulations of a weakly mixing C^*-dynamical system and a compact C^*-dynamical system, in terms of the GNS representation. The advantage in deriving these spectral formulations will be twofold. First, it will illuminate more clearly the relationship between the two properties. Secondly, for the majority of systems that we will consider, it will be a far simpler matter to determine whether a system is weak-mixing or compact using these spectral formulations. The reason, as we shall see, will be due to the relative ease in which the GNS representation can be obtained in most cases, and the remarkably concise formulation of the two properties in terms of the GNS representation.

The reason that the results are proved for Banach spaces and not for Hilbert spaces, which is all that we need, is that it is serves as a great example of the richness of Banach spaces even when compared to Hilbert spaces. Several of the results to follow are far easier to obtain when the space under consideration is not only a Banach space but also a Hilbert space. Yet, perhaps surprisingly, the results hold even when all that we know is that the space under consideration is a Banach space. In addition, adopting such a general framework also opens the door for possible further work on ergodic properties within the context of Banach spaces.

The main results in this chapter are Theorems 2.8, 2.13 and 2.18. Theorem 2.8 is the Splitting Theorem, and it is states how a complex Banach space can be “split” into a direct sum derived from a semigroup of bounded linear operator with certain properties. Theorems 2.13 and 2.18 then derive alternate characterizations of the the closed subspaces in the direct sum of Theorem 2.8. Theorem 2.18 is a modified version of [**19**, Theorem 4.7].

Take note that in this section X will always refer to a complex Banach space and that the value of a functional $h \in X^*$ at $x \in X$ will be denoted [1] by $\langle h, x \rangle$.

1. Locally convex topologies

Apart from the norm topology on X, we will consider three different topologies in this chapter, on X and on $\mathscr{L}(X) = \{T : X \to X : T \text{ is bounded linear}\}$. $\mathscr{L}(X)$ is equipped with the operator norm of its bounded linear operators, and under this norm it is itself a Banach space. There is the weak topology (w-topology) on X, the weak* topology (w*-topology) on X^* and the weak operator topology (wo-topology) on $\mathscr{L}(X)$.

(i) In the w-topology on X each $x \in X$ has a base of neighbourhoods consisting of all sets of the form :
$V_{x,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_j, x\rangle - \langle h_j, z\rangle| < \epsilon \ \forall\ j = 1, ..., n\}$
where $\epsilon > 0$ and $h_1, ..., h_n \in X^*$.

(ii) In the w*-topology on X^* each $h \in X^*$ has a base of neighbourhoods consisting of all sets of the form :
$V_{h,x_1,...,x_n,\epsilon} = \{g \in X^* : |\langle h, x_j\rangle - \langle g, x_j\rangle| < \epsilon \ \forall\ j = 1, ..., n\}$
where $\epsilon > 0$ and $x_1, ..., x_n \in X$.

(iii) In the wo-topology on $\mathscr{L}(X)$ each $T \in \mathscr{L}(X)$ has a base of neighbourhoods consisting of all sets of the form :
$V_{T,x_1,...,x_n,h_1...h_n,\epsilon} =$
$\{R \in \mathscr{L}(X) : |\langle h_j, Tx_j\rangle - \langle h_j, Rx_j\rangle| < \epsilon \ \forall\ j = 1, ..., n\}$
where $\epsilon > 0$,$h_1, ..., h_n \in X^*$ and $x_1, ..., x_n \in X$.

It it is easy to see that the w-topology on X is the smallest topology such that all elements in X^* are continuous and is therefore coarser than the norm-topology of X. Similarly the w*-topology is the smallest topology such that the mappings $\hat{x} : X^* \to \mathbb{C} : h \mapsto \langle h, x\rangle$, where $x \in X$, are continuous. The wo-topology is the smallest topology such that the mappings $\mathscr{L}(X) \to \mathbb{C} : T \mapsto \langle h, Tx\rangle$, where $h \in X^*$ and $x \in X$, are continuous. Hence it can be seen that the wo-topology is coarser than the operator-norm topology of $\mathscr{L}(X)$. The w-topology, w*-topology and wo-topology are typically defined as the smallest topologies for which the above mappings are, respectively, continuous. That their bases are of the above forms is then easily derived.

[1] This notation is akin to attaching a Ferrari badge to a Porsche Boxster and painting it red. It might look a bit like a Ferrari but it wont get you around the Nurburgring quite as fast...

Two tools, in particular, from functional analysis are of indispensable use when working with these topologies. The first, unsurprisingly, is the Hahn-Banach theorem [**20**, Theorem 4.3-2] and the second Banach-Alaoglu [**8**, V.4.2]. The latter states that the closed unit ball in X^* is w*-compact. As a first application of the Hahn-Banach theorem, we show that a w-compact set of X is w-closed and bounded in the norm of X.

THEOREM 1.1. *Consider a subset $V \subseteq X$. If $h(V)$ is bounded in $\mathbb{C}$, for all $h \in X^*$, then V is bounded in the norm of X.*

PROOF. If $h(V)$ is bounded in $\mathbb{C}$ for all $h \in X^*$, then it follows that for any $h \in X^*$ there is some $M_h > 0$ such that $|\langle h, x\rangle| < M_h$ for all $x \in X$. Therefore, for any $\epsilon > 0$ and $h \in X^*$, it follows that

$$\exists\ \delta > 0 : x \in V, \|x\| > \delta \Rightarrow |\langle h, x\rangle| < \epsilon\|x\|$$

which can be seen to follow from the fact that for any $x \in V$ and $\delta > 0$

$$\|x\| \geq \delta \Rightarrow \frac{\|x\|}{\delta} \geq 1 \Rightarrow |\langle h, x\rangle| < \frac{M_h}{\delta}\|x\|.$$

Suppose that V is unbounded, so for all $N \in \mathbb{N}$ there is some $x \in V$ for which $\|x\| > N$. Since for any $x \in X$ there exists, by the Hahn-Banach Theorem, an $h \in X^*$ such that $\langle h, x\rangle = \|x\|$ and $\|h\| = 1$, we can define a sequence (h_n) in X^* in the following way:

h_1 : Pick any $x_1 \in V$ such that $\|x_1\| \neq 0$. Choose an $h_1 \in X^*$ such that $\langle h_1, x_1\rangle = \|x_1\|$ and $\|h_1\| = 1$

h_n $(n > 1)$: Pick any x_n, large enough, such that $|\langle h_1, x_n\rangle|, ..., |\langle h_{n-1}, x_n\rangle| \leq \frac{1}{3^{n+1}}\|x_n\|$ and $\|x_n\| \geq 3^n n$. Choose an $h_n \in X^*$ such that $\langle h_n, x_n\rangle = \|x_n\|$ and $\|h_n\| = 1$

As $\|h_n\| = 1$ for all $n \in \mathbb{N}$, the sequence

$$\left\{\sum_{n=1}^{m} \frac{1}{3^n} h_n\right\}_{m\in\mathbb{N}}$$

is clearly Cauchy and hence convergent in X^*. Define

$$h' := \sum_{n=1}^{\infty} \frac{1}{3^n} h_n$$

Thus $h' \in X^*$. However it can be shown that $h'(V)$ is unbounded as for any $m \in \mathbb{N}$, with $m \geq 2$, it follows that

$$\begin{aligned}
|h'(x_m)| &= |\sum_{n=1}^{\infty} \frac{1}{3^n} h_n(x_m)| \\
&= \left| \frac{1}{3^m} h_m(x_m) + \sum_{n=1}^{m-1} \frac{1}{3^n} h_n(x_m) + \sum_{n=m+1}^{\infty} \frac{1}{3^n} h_n(x_m) \right| \\
&\geq \frac{1}{3^m} |h_m(x_m)| - \left| \sum_{n=1}^{m-1} \frac{1}{3^n} h_n(x_m) + \sum_{n=m+1}^{\infty} \frac{1}{3^n} h_n(x_m) \right| \\
&\geq \frac{1}{3^m} |h_m(x_m)| - \sum_{n=1}^{m-1} \frac{1}{3^n} |h_n(x_m)| - \sum_{n=m+1}^{\infty} \frac{1}{3^n} |h_n(x_m)| \\
&\geq \frac{1}{3^m} \|x_m\| - \sum_{n=1}^{m-1} \frac{1}{3^{n+m+1}} \|x_m\| - \sum_{n=m+1}^{\infty} \frac{1}{3^n} \|x_m\| \\
&= \frac{1}{3^m} \|x_m\| \left(1 - \sum_{n=1}^{m-1} \frac{1}{3^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{3^n} \right) \\
&\geq \frac{1}{3^m} \|x_m\| \left(1 - \frac{1}{3} \cdot \frac{1}{2} - \frac{1}{2} \right) \\
&= \frac{1}{3^m} \|x_m\| \frac{1}{3} \\
&\geq \frac{m}{3}
\end{aligned}$$

Thus h' has arbitrarily large values on V which, since $h' \in X^*$, contradicts the premise that $h(V)$ is bounded for *all* $h \in X^*$. $\square$

COROLLARY 1.2. *If V is a w-compact subset of X, then it is w-closed and bounded in the norm of X.*

PROOF. If V is w-compact, then $h(V)$ is compact for all $h \in X^*$, by the w-continuity of the elements of X^*. Thus $h(V) \subseteq \mathbb{C}$ is bounded for all $h \in X^*$ from which it follows that V is norm bounded by Theorem 1.1.

Since the w-topology is Hausdorff, all w-compact sets are w-closed [**21**, Theorem 26.3]. $\square$

The bases generating these weak topologies suggest that communication between them ought to be fairly easy. This is indeed the case with several easily verifiable relationships between the topologies. For the sake of completeness and easy reference, we include in the following theorems those that will most aid our analysis in the subsequent section.

For any $T \in \mathscr{L}(X)$, the adjoint operator of T is the operator

$$T^* : X^* \to X^*$$

with $\langle T^* h, x\rangle = \langle h, Tx\rangle$ for all $x \in X, h \in X^*$. $T^* \in \mathscr{L}(X^*)$ and

$$\begin{aligned}
\|T^*\| &= \sup_{\|h\|=1} \|T^* h\| \\
&= \sup_{\|h\|=1} \sup_{\|x\|=1} |\langle T^* h, x\rangle| \\
&= \sup_{\|h\|=1} \sup_{\|x\|=1} |\langle h, Tx\rangle| \\
&\leq \|T\| \sup_{\|h\|=1} \sup_{\|x\|=1} \|h\|\|x\| \\
&= \|T\|
\end{aligned}$$

THEOREM 1.3. *Let $(T_\alpha)_{\alpha\in\Lambda}$ be a net in $\mathscr{L}(X)$, and T an operator in $\mathscr{L}(X)$. Then the following are equivalent:*

(1) $\lim\langle h, T_\alpha x\rangle = \langle h, Tx\rangle \; \forall \; x \in X, h \in X^*$
(2) wo-lim $T_\alpha = T$
(3) w-lim $T_\alpha x = Tx \; \forall \; x \in X$
(4) w*-lim $T_\alpha^* h = T^* h \; \forall \; h \in X^*$

PROOF. (1)⇒(2) Let W be a wo-neighbourhood of T. It follows that there are $x_1, ..., x_n \in X$, $h_1, ..., h_n \in X^*$ and an $\epsilon > 0$ such that

$$T \in V = V_{T,x_1,...,x_n,h_1,...,h_n,\epsilon} \subseteq W$$

where

$$\begin{aligned}
&V_{T,x_1,...,x_n,h_1,...,h_n,\epsilon} \\
&= \{R \in \mathscr{L}(X) : |\langle h_i, Rx_i\rangle - \langle h_i, Tx_i\rangle| < \epsilon \; \forall \; i = 1, ..., n\}.
\end{aligned}$$

Clearly, $V = \bigcap_{i=1}^n V_{T,h_i,x_i,\epsilon}$. For each $i = 1, ..., n$ it follows that, as $\langle h_i, T_\alpha x_i\rangle \longrightarrow \langle h_i, Tx_i\rangle$, there is a $\beta_i \in \Lambda$ such that $|\langle h_i, T_\alpha x_i\rangle - \langle h_i, Tx_i\rangle| < \epsilon$ for all $\alpha \succeq \beta_i$, i.e. $T_\alpha \in V_{T,h_i,x_i,\epsilon}$ for all $\alpha \succeq \beta_i$. As there is a $\beta \in \Lambda$ with $\beta \succeq \beta_i$ for all $i = 1, ..., n$, it thus follows that $T_\alpha \in \bigcap_{i=1}^n V_{T,h_i,x_i,\epsilon} = V \subseteq W$ for all $\alpha \succeq \beta$, and as W was arbitrary, (2) follows.

(2)⇒(3) Let $x \in X$ be arbitrary and let W be a w-neighbourhood of Tx. It follows that there are $h_1, ..., h_n \in X^*$ and an $\epsilon > 0$ such that

$$Tx \in V := V_{Tx,h_1,...,h_n,\epsilon} \subseteq W$$

where

$$V_{Tx,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_i, z\rangle - \langle h_i, Tx\rangle| < \epsilon \; \forall \; i = 1, ..., n\}.$$

The wo-base element

$$U := U_{T,x,h_1...h_n,\epsilon} = \{R \in \mathscr{L}(X) : |\langle h_i, Rx\rangle - \langle h_i, Tx\rangle| < \epsilon \, \forall \, i = 1, ..., n\}$$

is a wo-neighbourhood of T so that, since wo-lim $T_\alpha = T$, there is a $\beta \in \Lambda$ such that $T_\alpha \in U$ for all $\alpha \succeq \beta$. In other words, $|\langle h_i, T_\alpha x\rangle - \langle h_i, Tx\rangle| < \epsilon$, i.e. $T_\alpha x \in V \subseteq W$, for all $\alpha \succeq \beta$. As W was arbitrary, (3) follow for arbitrary $x \in X$.

(3)⇒(4) Let $h \in X^*$ be arbitrary and let W be a w*-neighbourhood of T^*h. It follows that there are $x_1, ..., x_n \in X$ and an $\epsilon > 0$ such that

$$T^*h \in V := V_{T^*x,h_1,...,h_n,\epsilon>0} \subseteq W$$

where

$$V_{T^*h,x_1...x_n,\epsilon} = \{g \in X^* : |\langle g, x_i\rangle - \langle T^*h, x_i\rangle| < \epsilon \ \forall \ i = 1, ..., n\}.$$

Clearly $V = \bigcap_{i=1}^n V_{T^*h,x_i,\epsilon}$. For each $i = 1, ..., n$, the w-base element

$$U_i := U_{Tx_i,h,\epsilon} = \{z \in X : |\langle h, z\rangle - \langle h, Tx_i\rangle| < \epsilon \ \forall \ i = 1, ..., n\}$$

is a w-neighbourhood of Tx_i so that, since w-lim $T_\alpha x_i = Tx$, there is a $\beta_i \in \Lambda$ such that $T_\alpha x_i \in U_i$ for all $\alpha \succeq \beta_i$. In other words, for each $i = 1, ..., n$

$$|\langle h, T_\alpha x_i\rangle - \langle h, Tx_i\rangle| = |\langle T_\alpha^* h, x_i\rangle - \langle T^*h, x_i\rangle| < \epsilon$$

i.e. $T_\alpha^* h \in V_{T^*h,x_i,\epsilon}$, for all $\alpha \succeq \beta_i$. As there is a $\beta \in \Lambda$ such that $\beta \succeq \beta_i$, for each $i = 1, ..., n$, it thus follows that $T_\alpha^* h \in V = \bigcap_{i=1}^n V_{T^*h,x_i,\epsilon} \subseteq W$ for all $\alpha \succeq \beta$. As W was arbitrary, (4) follows for arbitrary $h \in X^*$.

(4)⇒(1) Let $x \in X, h \in X^*$ and $\epsilon > 0$ be arbitrary. The w*-base element

$$V = V_{T^*h,x,\epsilon} = \{g \in X^* : |\langle g, x\rangle - \langle T^*h, x\rangle| < \epsilon\}$$

is a w*-neighbourhood of T^*h so that, since w*-lim $T_\alpha^* = T^*h$, there is a $\beta \in \Lambda$ such that $T_\alpha^* \in V$ for all $\alpha \succeq \beta$. In other words, $|\langle h, T_\alpha x\rangle - \langle h, Tx\rangle| = |\langle T_\alpha^* h, x\rangle - \langle T^*h, x\rangle| < \epsilon$ for all $\alpha \succeq \beta$, and as $\epsilon > 0$ was arbitrary, (1) follows. □

As a bounded linear operator between normed spaces, each $T \in \mathscr{L}(X)$ is continuous with respect to the norm topology of X. That T is continuous with respect to the weaker w-topology on X, is easy to show.

Theorem 1.4. *If* w-lim $x_\alpha = x$, *for some net* $(x_\alpha)_{\alpha\in\Lambda}$ *in* X, *then* w-lim $Tx_\alpha = Tx$ *for all* $T \in \mathscr{L}(X)$.

Proof. Let $T \in \mathscr{L}(X)$ be arbitrary and let W be a w-neighbourhood of Tx. It follows that there are $h_1, ..., h_n \in X^*$ and an $\epsilon > 0$ such that

$$Tx \in V := V_{Tx,h_1,...,h_n,\epsilon} \subseteq W$$

where

$$V_{Tx,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_i, z\rangle - \langle h_i, Tx\rangle| < \epsilon \ \forall \ i = 1, ..., n\}.$$

Clearly $V = \bigcap_{i=1}^{n} V_{Tx,h_i,\epsilon}$. For each $i = 1, ..., n$, the w-base element

$$U_i := U_{x,T^*h_i,\epsilon} = \{z \in X : |\langle T^* h_i, z\rangle - \langle T^* h_i, x\rangle| < \epsilon \ \forall\ i = 1, ..., n\}$$

is a w-neighbourhood of x so that, since w-lim $x_\alpha = x$, there is a $\beta_i \in \Lambda$ such that $x_\alpha \in U_i$ for all $\alpha \succeq \beta_i$. In other words, $|\langle h_i, Tx_\alpha\rangle - \langle h_i, Tx\rangle| = |\langle T^* h_i, x_\alpha\rangle - \langle T^* h_i, x\rangle| < \epsilon$, i.e. $Tx_\alpha \in V_{Tx,h_i,\epsilon}$, for all $\alpha \succeq \beta_i$. Since there is a $\beta \in \Lambda$ such that $\beta \succeq \beta_i$ for each $i = 1, ..., n$, it thus follows that $Tx_\alpha \in V = \bigcap_{i=1}^{n} V_{Tx,h_i,\epsilon} \subseteq W$ for all $\alpha \succeq \beta$, and as W was arbitrary the result follows. □

If (x_α) and (c_α) are nets in X and $\mathbb{C}$ respectively then, if $x_\alpha \longrightarrow x$, i.e. in the norm topology, and $c_\alpha \longrightarrow c$, for some $x \in X$ and $c \in \mathbb{C}$, then it easy to show that $c_\alpha x_\alpha \longrightarrow cx$. Of greater significance to us is that the same result holds in the w-topology.

THEOREM 1.5. *If* w-lim $x_\alpha = x$ *and* $\lim c_\alpha = c$ *for nets* $(x_\alpha)_{\alpha\in\Lambda} \subseteq X$ *and* $(c_\alpha)_{\alpha\in\Lambda} \subseteq \mathbb{C}$, *then*

$$\text{w-lim } c_\alpha x_\alpha = cx$$

PROOF. Let W be a w-neighbourhood of cx. It follows that there are $h_1, ..., h_n \in X^*$ and an $\epsilon > 0$ such that

$$cx \in V := V_{cx,h_1,...,h_n,\epsilon} \subseteq W$$

where

$$V_{cx,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_i, z\rangle - \langle h_i, cx\rangle| < \epsilon \ \forall\ i = 1, ..., n\}.$$

Clearly $V = \bigcap_{i=1}^{n} V_{cx,h_i,\epsilon}$. For each $i = 1, ..., n$ it follows that

$$\begin{aligned} |\langle h_i, c_\alpha x_\alpha\rangle - \langle h_i, cx\rangle| &\le |\langle h_i, c_\alpha x_\alpha\rangle - \langle h_i, cx_\alpha\rangle| + |\langle h_i, cx_\alpha\rangle - \langle h_i, cx\rangle| \\ &= |c_\alpha - c||\langle h_i, x_\alpha\rangle| + |\langle ch_i, x_\alpha\rangle - \langle ch_i, x\rangle|. \end{aligned}$$

Since h_i is w-continuous and w-lim $x_\alpha = x$, $\langle h_i, x_\alpha\rangle \longrightarrow \langle h_i, x\rangle$. Thus $|\langle h_i, x_\alpha\rangle| \longrightarrow |\langle h_i, x\rangle|$ so that if we fix some $M > |\langle h_i, x\rangle|$ then there is a $\rho_1 \in \Lambda$ such that $|\langle h_i, x_\alpha\rangle| < M$ for all $\alpha \succeq \rho_1$. Since $c_\alpha \longrightarrow c$ there is a $\rho_2 \in \Lambda$ such that $|c_\alpha - c| < \frac{\epsilon}{2M}$ for all $\alpha \succeq \rho_2$. Since $ch_i \in X^*$, and is therefore w-continuous, we likewise have that $\langle ch_i, x_\alpha\rangle \longrightarrow \langle ch_i, x\rangle$. Thus there is a $\rho_3 \in \Lambda$ such that $|\langle ch_i, x_\alpha\rangle - \langle ch_i, x\rangle| < \frac{\epsilon}{2}$ for all $\alpha \succeq \rho_3$. As there is a $\beta \in \Lambda$ such that $\beta_i \succeq \rho_j$ for $j = 1, 2$ and 3,

$$|c_\alpha - c||\langle h_i, x_\alpha\rangle| + |\langle ch_i, x_\alpha\rangle - \langle ch_i, x\rangle| < M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon$$

for all $\alpha \succeq \beta_i$. In other words, $c_\alpha x_\alpha \in V_{cx,h_i,\epsilon}$ for all $\alpha \succeq \beta_i$, so that as there is a $\beta \in \Lambda$ such that $\beta \succeq \beta_i$ for all $i = 1, ..., n$, $c_\alpha x_\alpha \in V = \bigcap_{i=1}^{n} V_{cx,h_i,\epsilon} \subseteq W$ for all $\alpha \succeq \beta$. As W was arbitrary, the result follows. □

2. The splitting theorem

We will now begin our study of a more abstract version of the $\mathbb{Z}$ actions briefly considered in the introduction. Our aim is simply to be able to consider more general group actions than the $\mathbb{Z}$ case. We also mentioned, that if a dynamical system has a G action, where G is a general group, then G has a representation as a group of unitary operators on a Hilbert space in terms of which we can describe the time evolution of the system. This will be described in detail in Chapter 2. To consider this in a more abstract setting we will focus on semigroups on which we assume, but do not explicitly mention, the presence of a unit element. In particular we will consider semigroups of operators in $\mathscr{L}(X)$, containing the identity operator.

For an abstract semigroup we will denote by, and refer to, the semigroup operation as multiplication.

If S is an abstract semigroup equipped with some topology then multiplication in S is said to be *separately* continuous if for each $s \in S$ the mappings $S \to S : t \mapsto st$ and $S \to S : t \mapsto ts$ are continuous. The multiplication is said to be *jointly continuous* if the mapping $S \times S :\to S : (s,t) \mapsto st$ is continuous.

Definition 2.1. A *semitopological* semigroup is a semigroup which is a Hausdorff space in which multiplication is separately continuous, and a *topological* semigroup is a semigroup which is a Hausdorff space in which multiplication is jointly continuous.

Theorem 2.2. *Any compact Abelian semitopological semigroup S contains a unique minimal ideal K, the kernel of S. K is contained in any ideal, and*

$$K = \bigcap_{t \in S} tS \tag{7}$$

Furthermore, K is a group, and if q denotes the unit of this group, then $K = qS$.

Proof. If $J_1, ..., J_k$ are ideals in S, their product $J_1 \cdot ... \cdot J_k$ is nonempty, and is contained in their intersection $J_1 \cap ... \cap J_k$. Therefore the collection of all closed ideals of S has the finite intersection property. As S is compact it thus follows that the intersection of all closed ideals is nonempty [**21**, Theorem 26.9], and is itself an ideal which we take to be K.

If J is an ideal in S and $t \in J$ then $tS \subseteq J$ and since clearly $ts_1s_2, s_2ts_1 \in tS$ for all $s_1, s_2 \in S$, tS is itself an ideal. As the continuous image of a compact set in a Hausdorff space, tS is compact and therefore a *closed* ideal contained in J. As each ideal contains a closed ideal of the form tS, it follows that K is the intersection of *all* ideals in S. Thus K is minimal and necessarily unique. As each ideal contains a closed ideal of the form tS, *and* each tS is a closed ideal, (7) follows.

For any $s \in K$, sK is an ideal contained in K which, by the minimality of K, is equal to K. Thus since $sK = K$ there exists some $q \in K$ such that $sq = s$. If $t \in K$ is arbitrary, then again since $sK = K$, there is some $r \in K$ such that $sr = t$. It follows that

$$qt = qsr = sr = t$$

which reveals that q is the unit element in K. Similarly, since $sK = K$, there is some $s' \in K$ such that $ss' = q$. Therefore every element in K can be seen to have an inverse and we have that K is a group.

The last assertion follows from the fact that $qS \subseteq K$, since q is an element of the ideal K, and the fact that $K = qK \subseteq qS$. $\square$

We now move towards semigroups of operators in $\mathscr{L}(X)$ and start by proving some preliminary results.

Proposition 2.3. *Multiplication in $\mathscr{L}(X)$ is separately continuous with respect to the wo-topology.*

Proof. Let $R \in \mathscr{L}(X)$ be arbitrary, and consider any wo-convergent net $(T_\lambda)_{\lambda\in\Lambda}$ in $\mathscr{L}(X)$ with, say, wo-lim $T_\lambda = T$. To establish the separate continuity we wish to show that wo-lim $T_\lambda R = TR$ and wo-lim $RT_\lambda = RT$. By Theorem 1.3

$$\text{wo-lim } T_\lambda = T \;\Leftrightarrow\; \langle h, T_\lambda x\rangle \longrightarrow \langle h, Tx\rangle \;\forall\; x \in X, h \in X^* \tag{8}$$

Therefore, in particular, we have that

$$\langle h, T_\lambda Rx\rangle \longrightarrow \langle h, TRx\rangle \text{ , and}$$
$$\langle h, RT_\lambda x\rangle = \langle R^* h, T_\lambda x\rangle \longrightarrow \langle R^* h, Tx\rangle = \langle h, RTx\rangle$$

for all $x \in X, h \in X^*$ and $R \in \mathscr{L}(X)$. Hence, by (8), wo-lim $T_\lambda R = TR$ and wo-lim $RT_\lambda = RT$ as required. $\square$

Proposition 2.3 allows us to easily show that any semigroup in $\mathscr{L}(X)$ becomes a semitopological semigroup under its wo-closure.

Proposition 2.4. *If $\mathscr{S}$ is a semigroup in $\mathscr{L}(X)$, then $\overline{\mathscr{S}} :=$ wo-cl $\mathscr{S}$ is a semigroup, and in particular a semitopological semigroup. If $\mathscr{S}$ is Abelian, then so is $\overline{\mathscr{S}}$.*

Proof. We will chiefly rely on the separate continuity of $\mathscr{L}(X)$, in terms of the wo-topology, established in Proposition 2.3.

Since $\mathscr{S}$ is a semigroup, $\mathscr{S}\mathscr{S} \subseteq \mathscr{S}$ and therefore, by the separate continuity of $\mathscr{L}(X)$, $\mathscr{S}\overline{\mathscr{S}} \subseteq \overline{\mathscr{S}}$. Applying separate continuity once more it follows that $\overline{\mathscr{S}}\,\overline{\mathscr{S}} \subseteq \overline{\mathscr{S}}$. To see why, consider any $R, T \in \overline{\mathscr{S}}$. Hence there are two wo-convergent nets, say $(R_\gamma)_{\gamma\in\Gamma}$ and $(T_\lambda)_{\lambda\in\Lambda}$, in $\mathscr{S}$ converging to R and T respectively. Since $R_\gamma T_\lambda \in \mathscr{S}$, for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$, $R_\gamma T = \text{wo-lim}_\lambda\, R_\gamma T_\lambda \in \overline{\mathscr{S}}$ for all $\gamma \in \Gamma$, and thus $RT = \text{wo-lim}_\gamma\, R_\gamma T \in \overline{\mathscr{S}}$. Thus $\overline{\mathscr{S}}$ is a semigroup, and in particular a semitopological semigroup since it is a wo-closed subspace of $\mathscr{L}(X)$ on which we have separate continuity.

To show that $\overline{\mathscr{S}}$ is Abelian if $\mathscr{S}$ is Abelian, let us again consider the arbitrary operators $R, T \in \overline{\mathscr{S}}$. It similarly follows that $TR_\gamma =$ wo-$\lim_\lambda T_\lambda R_\gamma \in \overline{\mathscr{S}}$ for all $\gamma \in \Gamma$, and $TR \in \overline{\mathscr{S}} =$ wo-$\lim_\gamma TR_\gamma$. Since, now, $T_\lambda R_\gamma = R_\gamma T_\lambda$ for all $\lambda \in \Lambda$ and $\gamma \in \Gamma$, $RT = TR$ follows from the uniqueness of limits in the Hausdorff space $\mathscr{L}(X)$. □

Lemma 2.5. *A subset $\mathscr{S} \subseteq \mathscr{L}(X)$ is conditionally wo-compact if and only if $\mathscr{S}x$ is conditionally w-compact in X for each $x \in X$. In this case $\overline{\mathscr{S}}$ is norm bounded in $\mathscr{L}(X)$, and $\overline{\mathscr{S}}x =$ w-cl $(\mathscr{S}x)$ holds for all $x \in X$.*

Proof. Consider the mapping

$$p_x : \mathscr{L}(X) \to X : T \mapsto Tx$$

for an arbitrary $x \in X$ and, to communicate compactness between X and $\mathscr{L}(X)$, let us proceed by showing that p_x is wo-w-continuous. If $T \in \mathscr{L}(X)$ is arbitrary and U is some w-neighbourhood of $p_x(T) = Tx$, then there is some w-open set of the form

$$V = V_{Tx,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_j, Tx\rangle - \langle h_j, z\rangle| \ \forall\ j = 1, ..., n\}$$

such that $V \subseteq U$. Therefore, if

$$W = W_{T,x,h_1,...,h_n,\epsilon} = \{R \in X : |\langle h_j, Tx\rangle - \langle h_j, Rx\rangle| \ \forall\ j = 1, ..., n\}$$

then W is a wo-neighbourhood of T and it clear that $p_x(W) \subseteq V \subseteq U$ which, as T and U were arbitrary, establishes the continuity of p_x.

Now, if $\mathscr{S}$ is conditionally wo-compact, then $\overline{\mathscr{S}}$ is wo-compact, by definition, and $\overline{\mathscr{S}}x$ is w-compact as the image of a wo-compact set under the continuous mapping p_x. Since the w-topology is Hausdorff, $\overline{\mathscr{S}}x$ is w-closed and hence $\overline{\mathscr{S}}x \supseteq$ w-cl $(\mathscr{S}x)$. To show the forward inclusion let $Tx \in \overline{\mathscr{S}}x$ be arbitrary, with $T \in \overline{\mathscr{S}}$. Thus $T =$ wo-lim T_λ for some net (T_λ) in $\mathscr{S}$. It follows by the continuity of p_x that

$$Tx = p_x(T) = p_x(\text{wo-lim } T_\lambda) = \text{w-lim } p_x(T_\lambda) = \text{w-lim } T_\lambda x$$

which, as $T_\lambda x \in \mathscr{S}x$ for all λ, shows that $Tx =$ w-lim $T_\lambda x \in$ w-cl $(\mathscr{S}x)$. Thus, $\overline{\mathscr{S}}x =$ w-cl $(\mathscr{S}x)$ for all $x \in X$, and we already know the former to be w-compact. Thus $\mathscr{S}x$ is conditionally w-compact for all $x \in X$.

Note that the inclusion $\overline{\mathscr{S}}x \subseteq w\text{-}cl(\mathscr{S}x)$ holds in general, as to show it, we did not require the conditional wo-compactness of $\mathscr{S}$.

Conversely, assume that $\mathscr{S}x$ is conditionally w-compact for all $x \in X$. Then by Corollary 1.2 w-cl $(\mathscr{S}x)$ is norm bounded for all $x \in X$. Since $\overline{\mathscr{S}}x \subseteq w\text{-}cl(\mathscr{S}x)$, as established in the first part, it follows by the uniform boundedness principle [**8**, II.3.21], that $\overline{\mathscr{S}}$ is norm bounded by, say $M > 0$.

For each $x \in X$, let $E_x = w\text{-}cl(\mathscr{S}x)$ and $E = \prod_{x \in X} E_x$ be the Cartesian product of the spaces E_x with the product topology. Since each E_x is w-compact, E is compact by Tychonoff's Theorem [**21**, Theorem 37.3]. Now let $(T_\lambda)_{\lambda \in \Lambda}$ be an arbitrary net in $\overline{\mathscr{S}}$. The wo-compactness of $\overline{\mathscr{S}}$ will follow if we can identify a wo-convergent subnet of $(T_\lambda)_{\lambda \in \Lambda}$. Since $\overline{\mathscr{S}}x \subseteq$ w-cl $(\mathscr{S}x)$ for each $x \in X$, $((T_\lambda x)_{x \in X})_{\lambda \in \Lambda}$ is a net in E, which, by *its* compactness, contains a subnet $((T_{\lambda_\beta} x)_{x \in X})_{\beta \in \Lambda'}$ converging to some element in E, say $(\theta_x)_{x \in X}$. We will now show that w-lim $T_{\lambda_\beta} x = \theta_x$, for all $x \in X$, and that the operator

$$T : X \to X : x \mapsto \theta_x \tag{9}$$

is the wo-limit of $(T_{\lambda_\beta})_{\beta \in \Lambda'}$, a subnet of $(T_\lambda)_{\lambda \in \Lambda}$. To show that w-lim $T_{\lambda_\beta} x = \theta_x$ in X, for all $x \in X$, consider any $y \in X$ and w-neighbourhood V of θ_y in X. The set

$$W = \prod_x \{A_x : A_x = E_x \text{ if } x \neq y \text{ and } A_y = E_y \cap V\}$$

is a neighbourhood of (θ_x) in the product topology of E. Hence, as (θ_x) is the limit of the subnet $((T_{\lambda_\beta} x)_{x \in X})_{\beta \in \Lambda'}$, and $(T_{\lambda_\beta} x)_{x \in X} \in W$ if and only if $T_\beta y \in V$, we have that

$$\forall\, \epsilon > 0\ \exists\, \varphi \in \Lambda' \ :\ \beta \succeq \varphi \Rightarrow T_\beta y \in V.$$

In other words, w-lim $T_\beta y = \theta_y$.

Now we wish to show that (9) defines an operator in $\mathscr{L}(X)$. Since

$$T : x \mapsto \text{w-lim } T_{\lambda_\beta} x \tag{10}$$

it is easy to see that T is linear. To see why T is bounded linear, consider any $x \in X$ and let $g \in X^*$ be a functional such that $\langle g, Tx\rangle = \|Tx\|$ and $\|g\| = 1$. That there is such a $g \in X^*$, follows by the Hahn-Banach Theorem. By (10), $\langle h, Tx\rangle = \lim_\beta \langle h, T_\beta x\rangle$ for all $h \in X^*$. Hence, in particular, for any $\epsilon > 0$, there is a ν such that for any $\beta \succeq \nu$

$$\|Tx\| = |\langle g, Tx\rangle| \leq |\langle g, T_{\lambda_\beta} x\rangle| + \epsilon \leq Mx + \epsilon.$$

Thus $\|Tx\| \leq M\|x\|$ for all $x \in X$ and we have that T is a bounded linear operator in $\mathscr{L}(X)$. To complete the proof we need only recall from Theorem 1.3 that, for an operator $T \in \mathscr{L}(X)$, and net $\{T_\alpha\} \subseteq \mathscr{L}(X)$

$$\text{wo-lim } T_\alpha = T \Leftrightarrow \text{w-lim } T_\alpha x = Tx\ \forall\ x \in X.$$

Thus, by (10), $T =$ wo-lim T_β which completes the proof. □

DEFINITION 2.6. Let $\mathscr{S} \subseteq \mathscr{L}(X)$ be a semigroup. A vector $x \in X$ is called *reversible* if for any $T \in \overline{\mathscr{S}}$ there exists an $R \in \overline{\mathscr{S}}$ with $RTx = x$. A vector $x \in X$ is called a *flight vector* if there exists an $S \in \overline{\mathscr{S}}$ with $Sx = 0$. The set of reversible vectors is denoted by X_{rev} and the set of flight vectors by X_{fl}.

For our intended purposes, the real interesting and useful properties of reversible and flight vectors are obtained, when instead of considering general semigroups in $\mathscr{L}(X)$, we consider only those semigroups under which the elements of X have "almost" compact orbits:

DEFINITION 2.7. A semigroup $\mathscr{S} \subseteq \mathscr{L}(X)$ of continuous linear operators in a Banach space X is called *weakly almost periodic* if for any $x \in X$ the orbit $\mathscr{S}x = \{Tx : T \in \mathscr{S}\}$ is conditionally w-compact, i.e. if $w\text{-}cl(\mathscr{S}x)$ is w-compact.

Clearly, $X_{rev} \cap X_{fl} = \{0\}$. We shall only consider the case in which $\mathscr{S}$ is weakly almost periodic, i.e. when w-cl $(\mathscr{S}x)$ is w-compact for all $x \in X$. By Lemma 2.5 this means that $\overline{\mathscr{S}}$ wo-compact, and the identity $\overline{\mathscr{S}}x = \text{w-cl } (\mathscr{S}x)$ then yields

$$X_{rev} = \{x \in X : y \in \text{w-cl } (\mathscr{S}x) \Rightarrow x \in \text{w-cl } (\mathscr{S}y)\}$$
$$X_{fl} = \{x \in X : 0 \in \text{w-cl } (\mathscr{S}x)\}$$

If $\mathscr{S}$ is an Abelian weakly almost periodic semigroup in $\mathscr{L}(X)$ then, by Proposition 2.4, $\overline{\mathscr{S}}$ is a semitopological Abelian semigroup. By Lemma 2.5, $\overline{\mathscr{S}}$ is wo-compact and we can therefore apply Theorem 2.2 to $\overline{\mathscr{S}}$.

THEOREM 2.8. *(Jacobs-Deleeuw-Glicksberg). Let $\mathscr{S}$ be an Abelian weakly almost periodic semigroup in $\mathscr{L}(X)$ and Q the unit in the kernel K of $\overline{\mathscr{S}}$. Then $X_{rev} = QX$ and $X_{fl} = Q^{-1}(0) = (1-Q)X$ where 1 is the unit operator in $\mathscr{L}(X)$. In particular, X is the direct sum of the w-closed subspaces X_{rev} and X_{fl}, which are both invariant under $\overline{\mathscr{S}}$. The restriction of $\overline{\mathscr{S}}$ to X_{rev} is a group.*

PROOF. If $x \in QX$, then $x = Qy$ for some $y \in X$. By Theorem 2.2, for any $T \in \overline{\mathscr{S}}$, $TQ = QT \in K$ and therefore there is an $R \in K$ such that $RTQ = Q$. Thus $RTx = RTQy = Qy = x$, which shows that $x \in X_{rev}$. Conversely if $x \in X_{rev}$, then there is an $R \in \overline{\mathscr{S}}$ such that $x = RQx = QRx$, which reveals that $x \in QX$. Hence

$$X_{rev} = QX. \tag{11}$$

It is clear that $Q^{-1}(0) \subseteq X_{fl}$. Conversely, if $x \in X_{fl}$ then $Tx = 0$ for some $T \in \overline{\mathscr{S}}$. Thus $QTx = 0$ and since, by Theorem 2.2, $QT \in K$ there is an $R \in K$ such that $RQT = Q$. Therefore $Qx = RQTx = 0$ revealing that $x \in Q^{-1}(0)$. Hence $X_{fl} = Q^{-1}(0)$ follows.

By the projection property $Q^2 = Q$ it is clear that $(1-Q)X \subseteq Q^{-1}(0)$. Conversely, if $Qx = 0$ for some $x \in X$ then $Qx = x - x$ or, rearranged, $x = x - Qx = (1-Q)x$. Thus $Q^{-1}(0) \subseteq (1-Q)X$ and we have that

$$X_{fl} = (1-Q)X. \tag{12}$$

(11) and (12) clearly show that X_{rev} and X_{fl} are subspaces invariant under $\overline{\mathscr{S}}$, since $\overline{\mathscr{S}}$ is Abelian. By Theorem 1.4, Q is w-continuous and hence $X_{fl} = Q^{-1}(0)$ is w-closed. If (Qx_α) is any w-convergent net in $QX = X_{rev}$ with, say, w-lim $Qx_\alpha = x$, then w-lim $Qx_\alpha =$ w-lim $Q^2 x_\alpha = Qx$ by Theorem 1.4. Since the w-topology is Hausdorff, limits are unique, and thus $x = Qx \in QX = X_{rev}$. That is, X_{rev} is w-closed.

Any $x \in X$ can be written in the form $x = x_1 + x_2$ with $x_1 \in QX, x_2 \in (1-Q)X$ by setting

$$x_1 = Qx \text{ and } x_2 = (1-Q)x. \tag{13}$$

Conversely, if $x = x_1 + x_2$ with $x_1 \in QX, x_2 \in (1-Q)X$, then since $Qx_1 = x_1$ and $Qx_2 = 0$, we have that $Qx = Qx_1 + Qx_2 = x_1$ and consequently $x = Qx + x_2$, or $x_2 = x - Qx$. Hence, the decomposition (13) is necessarily unique and we conclude that X is the direct sum of X_{rev} and X_{fl}.

The group property of $\overline{\mathscr{S}}$ restricted to X_{rev}, $\overline{\mathscr{S}}|_{X_{rev}} = \{T|_{X_{rev}} : T \in \overline{\mathscr{S}}\}$, follows from

$$\overline{\mathscr{S}}|_{X_{rev}} = \overline{\mathscr{S}}|_{QX} = \overline{\mathscr{S}}Q|_{QX} = Q\overline{\mathscr{S}}|_{QX}$$

since $Q^2 = Q$ and $\overline{\mathscr{S}}$ is Abelian. That is, $\overline{\mathscr{S}}|_{X_{rev}} = Q\overline{\mathscr{S}}|_{QX}$ is a group by Theorem 2.2.

The group property of $\overline{\mathscr{S}}$ in X_{rev} follows from the group property of K. □

Definition 2.9. A nonzero vector $x \in X$ is called an *eigenvector* of $\mathscr{S} \subseteq \mathscr{L}(X)$ if there is a map $\lambda : \mathscr{S} \to \mathbb{C}$ such that $Tx = \lambda(T)x$ for all $T \in \mathscr{S}$. We say that the *eigenvalues* of x is given by λ. If $|\lambda(T)| = 1$ for all $T \in \mathscr{S}$, then x is called an *eigenvector with unimodular eigenvalues*. The norm closure of the subspace of X spanned by all eigenvectors of $\mathscr{S}$ with unimodular eigenvalues is denoted by X_{uds}.

We will show that, in the case that $\mathscr{S}$ is an Abelian weakly almost periodic semigroup, any reversible vector can be approximated to an arbitrary accuracy by a linear combination of eigenvectors with unimodular eigenvalues, which motivates the definition of X_{uds}. This will be our next main result. However, we first prove some useful properties of the eigenvectors in definition 2.9.

Proposition 2.10. *For any $\mathscr{S} \subseteq \mathscr{L}(X)$, if x is an eigenvector with unimodular eigenvalues, of $\overline{\mathscr{S}}$, and the eigenvalues of x given by $\lambda : \overline{\mathscr{S}} \to \mathbb{C}$, then λ is wo-continuous.*

Proof. Let $T \in \overline{\mathscr{S}}, \epsilon > 0$ be arbitrary and let $h \in X^*$ be a functional, obtained with the Hahn-Banach Theorem, such that $\langle h, x\rangle =$

$\|x\|$. The set

$$\begin{aligned} V := V_{T,x,h,\epsilon\|x\|} &= \left\{R \in \overline{\mathscr{S}} : |\langle h, Rx\rangle - \langle h, Tx\rangle| < \epsilon\|x\|\right\} \\ &= \left\{R \in \overline{\mathscr{S}} : |\lambda(R)\|x\| - \lambda(T)\|x\|| < \epsilon\|x\|\right\} \end{aligned}$$

is a wo-neighbourhood of T in $\overline{\mathscr{S}}$ and it directly shows that if $R \in V$ then $|\lambda(R) - \lambda(T)| < \epsilon$. As $\epsilon > 0$ and $T \in \overline{\mathscr{S}}$ were arbitrary, the wo-continuity of $\lambda : \overline{\mathscr{S}} \to \mathbb{C}$ follows. □

PROPOSITION 2.11. *For any $\mathscr{S} \subseteq \mathscr{L}(X)$, if $x \in X$ is an eigenvector with unimodular eigenvalues, of $\mathscr{S}$, then x is also an eigenvector with unimodular eigenvalues, of $\overline{\mathscr{S}}$.*

PROOF. Suppose that $x \in X$ is an eigenvector with unimodular eigenvalues of $\mathscr{S} \subseteq \mathscr{L}(X)$, given by $\lambda : \mathscr{S} \to \mathbb{C}$, and consider any $T \in \overline{\mathscr{S}}$. If we let (T_α) be any net in $\mathscr{S}$ such that wo-lim $T_\alpha = T$, then the numbers $\lambda(T_\alpha)$ necessarily converge to a uniquely determined $\omega_T \in \mathbb{C}$, and if $T \in \mathscr{S}$ then $\omega_T = \lambda(T)$. To see why this is the case, recall from Theorem 1.3 that

$$\text{wo-lim } T_\alpha = T \Leftrightarrow \langle h, \lambda(T_\alpha)y\rangle = \langle h, T_\alpha y\rangle \longrightarrow \langle h, Ty\rangle \;\forall\, y \in X, h \in X^*.$$

In particular, using the Hahn-Banach theorem, for $y = x$ and an $h \in X^*$ such that $\langle h, x\rangle = \|x\|$ we have that $\lambda(T_\alpha)\|x\| \longrightarrow \langle h, Tx\rangle$ and therefore that $\lambda(T_\alpha) \longrightarrow \langle h, Tx\rangle\|x\|^{-1}$. If we let $\omega_T = \langle h, Tx\rangle\|x\|^{-1}$, then it promptly follows that $\langle h, \lambda(T_\alpha)x\rangle \longrightarrow \langle h, \omega_T x\rangle$ for all $h \in X^*$ and therefore that w-lim $\lambda(T_\alpha)x = \omega_T x$. As wo-lim $T_\alpha = T$ implies that w-lim $\lambda(T_\alpha)x =$ w-lim $T_\alpha x = Tx$, by Theorem 1.3, it follows by the uniqueness of the weak-limit that $Tx = \omega_T x$. If $T \in \mathscr{S}$, then $Tx = \lambda(T)x$ which shows why $\omega_T = \lambda(T)$. If the limit $T \notin \mathscr{S}$, i.e. if $T \in \overline{\mathscr{S}}\backslash\mathscr{S}$, then this reveals that x is also an eigenvector of $\overline{\mathscr{S}}$ and that λ can be extended to all of $\overline{\mathscr{S}}$, by simply setting $\lambda(T) = \omega_T$. To show that $|\lambda(T)| = 1$ consider again the wo-convergent net (T_α) in $\mathscr{S}$. By Proposition 2.10, λ is wo-continuous so that $\lambda(T) = \lim_\alpha \lambda(T_\alpha)$ from which it follows that

$$|\lambda(T)| = \lim_\alpha |\lambda(T_\alpha)| = \lim_\alpha 1 = 1.$$

□

There is a convenient feature in how the space X_{uds} is defined and to see that, let us denote, temporarily, the w-closure of the subspace of X spanned by the eigenvectors with unimodular eigenvalues, of $\mathscr{S} \subseteq \mathscr{L}(X)$, by X^w_{uds}. Since X_{uds} is convex, as the norm-closure of convex set, X_{uds} is w-closed by [**8**, V3.13]. Hence, $X^w_{uds} \subseteq X_{uds}$. However, as the w-topology is coarser than the norm topology, $X_{uds} \subseteq X^w_{uds}$. Thus $X_{uds} = X^w_{uds}$, or in other words, X_{uds} is also the w-closure of the subspace of X spanned by all eigenvectors of $\mathscr{S}$ with unimodular eigenvalues.

In the theorem to follow we will investigate the relationship between X_{uds},X_{rev} and find that in our case, i.e. when $\mathscr{S}$ is an Abelian weakly almost periodic semigroup, the two spaces are in fact the same. However, a simple question of geometric nature will be encountered that we wish to consider beforehand: If $a \in \mathbb{S}^1$, then is there a sequence $(n_i) \subseteq \mathbb{N}$ such that $a^{n_i} \longrightarrow 1$? The question is geometric in the sense that, if $a = e^{i\theta}$, then the powers of a correspond to rotations on $\mathbb{S}^1 \subseteq \mathbb{C}$ by multiples of θ. In the event that θ is of the form $\frac{m}{n}\pi$, for some $m, n \in \mathbb{Z}$, then the answer to the question is obviously affirmative. The more interesting case is when the rotations are instead irrational. For irrational rotations the answer is still affirmative and this can be proven directly, barring some unpleasant algebraic wizardry[2]. Instead, we will find the solution in classic ergodic theory.

LEMMA 2.12. *For any $a \in \mathbb{S}^1$, there is a sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that $a^{n_i} \longrightarrow 1$.*

PROOF. The transformation

$$T_a : \mathbb{S}^1 \to \mathbb{S}^1 : z \mapsto az$$

for any $a \in \mathbb{S}^1$, is an example of a measure preserving transformation on $\mathbb{S}^1$. Suppose first that a is not a root of unity, i.e. a is not in the form $e^{i\frac{m}{n}\pi}$ with $m, n \in \mathbb{N}$. By [**28**, Theorem 1.8] T_a is ergodic if and only if a is not a root of unity. Thus T_a is ergodic, so that by [**28**, Theorem 1.9], $\{a^n : n \in \mathbb{Z}\}$ is dense in the compact group $\mathbb{S}^1$. To see that $\{a^n : n \in \mathbb{N}\}$ is therefore also dense in $\mathbb{S}^1$, let $z \in \mathbb{S}^1$ and $d > 0$ be arbitrary. Since $\{a^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}^1$, we can find $n_1 < n_2 \in \mathbb{Z}$ such that $a^{n_1}, a^{n_2} \in B_{z,d} = \{x \in \mathbb{S}^1 : |x - z| < d\}$. Let $n = n_2 - n_1$. Then $n \in \mathbb{N}$ and it is easy to see that $a^{mn} \in B_{z,d}$ for some $m \in \mathbb{N}$. Hence, as $z \in \mathbb{S}^1$ and $d > 0$ was arbitrary, it follows that $\{a^n : n \in \mathbb{N}\}$ is dense in $\mathbb{S}^1$ as well, from which a sequence $(n_i)_{i\in\mathbb{N}}$ can readily be found such that $a^{n_i} \longrightarrow 1$.

If $a = e^{i\frac{m}{n}\pi}$, for some $m, n \in \mathbb{N}$, then $a^{2n} = 1$ so that the lemma follows trivially. □

THEOREM 2.13. *(Jacobs-Deleeuw-Glicksberg). If $\mathscr{S}$ is an Abelian weakly almost periodic semigroup in $\mathscr{L}(X)$, then $X_{rev} = X_{uds}$.*

PROOF. To show that $X_{rev} \supseteq X_{uds}$, consider any eigenvector x of $\mathscr{S}$, with unimodular eigenvalues, and any $T \in \overline{\mathscr{S}}$. It follows from the identity $T^n x = \lambda(T)^n x$, and Lemma 2.12, that a sequence $(n_i) \subseteq \mathbb{N}$ may be found such that $T^{n_i} x \longrightarrow x$. As a consequence, for any $h \in X^*$ and $\epsilon > 0$, the set

$$W_{h,\epsilon} = \{R \in \overline{\mathscr{S}} : |\langle h, RTx - x\rangle| \leq \epsilon\}$$

[2]The author worked out a proof, misplaced it, and made no effort to find it again.

is non-empty and any finite intersection of such sets is non-empty. This simply follows from the fact that $|\langle h, RTx - x\rangle| \leq \|h\|\|RTx - x\|$ can be made as small as needed by setting $R = T^{n_i - 1}$ and choosing i large enough. Moreover, $W_{h,\epsilon}$ is closed, which can be seen by letting $(R_\alpha)_{\alpha\in\Lambda}$ be any net in $W_{h,\epsilon}$ with, say, wo-lim $R_\alpha = R$ and establishing that $R \in W_{h,\epsilon}$. As the limit of a net in $\overline{\mathscr{S}}$, $R \in \overline{\mathscr{S}}$. Since $\langle h, R_\alpha y\rangle \longrightarrow \langle h, Ry\rangle$ for all $y \in X$ implies that $\langle h, R_\alpha Ty\rangle = \langle R_\alpha^* h, Ty\rangle \longrightarrow \langle R^* h, Ty\rangle = \langle h, RTy\rangle$ for all $y \in Y$, it follows by Theorem 1.3 that wo-lim $R_\alpha T = RT$ and furthermore that, for any $\delta > 0$, there is a $\beta \in \Lambda$ such that $|\langle h, R_\alpha Tx\rangle - \langle h, RTx\rangle| \leq \delta$ for all $\alpha \succeq \beta$. Thus

$$\begin{aligned}|\langle h, RTx\rangle - \langle h, x\rangle| &\leq |\langle h, RTx\rangle - \langle h, R_\alpha Tx\rangle| + |\langle h, R_\alpha Tx\rangle - \langle h, x\rangle| \\ &\leq \delta + \epsilon\end{aligned}$$

which, since $\delta > 0$ is arbitrary, shows that $|\langle h, RTx - x\rangle| = |\langle h, RTx\rangle - \langle h, x\rangle| \leq \epsilon$. That is, $R \in W_{h,\epsilon}$ as required. By the wo-compactness of $\overline{\mathscr{S}}$ it now follows that the intersection over all sets of the form $W_{h,\epsilon}$ is non-empty. Consequently, there is an $S \in \overline{\mathscr{S}}$ such that $|\langle h, STx - x\rangle| \leq \epsilon$ for all $\epsilon > 0$ and $h \in X^*$, in particular for $h \in X^*$ such that $\langle h, STx - x\rangle = \|STx - x\|$. But then $STx = x$ and since $T \in \overline{\mathscr{S}}$ was arbitrary, this shows that $x \in X_{rev}$. Thus X_{rev} contains all eigenvectors with unimodular eigenvalues. By Theorem 2.8 we know that X_{rev} is a w-closed, and hence norm closed, subspace of X. Thus, as X_{uds} is the norm closure of the span of all eigenvectors of $\mathscr{S}$ with unimodular eigenvalues, it follows that $X_{uds} \subseteq X_{rev}$.

The proof of the converse will be lengthy and technical and so it is prudent to first summarize the argument that we will follow:

(i) Let Γ be the dual group of K where K is the kernel, in the sense of Theorem 2.2, of $\overline{\mathscr{S}}$. Establish that K is compact, equip it with the normalized Haar measure ϱ and for each $\gamma \in \Gamma, x \in X$ cleverly define a functional $\Phi_{\gamma,x} \in X^{**}$ using ϱ.
(ii) Show that for each $\gamma \in \Gamma$ and $x \in X$, $\Phi_{\gamma,x}$ corresponds to $\kappa(z_{\gamma,x})$, for some $z_{\gamma,x}$ in a certain w-compact set $A \subseteq X$, where κ is the natural/canonical mapping of X into X^{**}
(iii) For each $\gamma \in \Gamma$, use the correspondence $x \mapsto z_{\gamma,x}$ to define a operator $T_\gamma : X \to X$.
(iv) Show that the range of these operators T_γ consist of eigenvectors with unimodular eigenvalues.
(vi) Define U to be the norm-closure of the span of the ranges of these operators $\{T_\gamma\}$ so that $U \subseteq X_{uds}$.
(vii) By using the result from harmonic analysis that Γ is a total orthonormal set in $L^2(K, \varrho)$, show that for any $Qx \in X_{rev}$ (see Theorem 2.8), Qx is necessarily an element of U.

Our proof of the converse starts with the kernel K of $\overline{\mathscr{S}}$ obtained by yet again applying Theorem 2.2 to $\overline{\mathscr{S}}$. By Theorem 2.2

$$K = \bigcap_{T \in \overline{\mathscr{S}}} T\overline{\mathscr{S}} \tag{14}$$

and is a group. We know that $\overline{\mathscr{S}}$ is wo-compact and that multiplication in $\mathscr{L}(X)$ is separately continuous. Thus $T\overline{\mathscr{S}}$ is wo-compact and hence wo-closed for all $T \in \overline{\mathscr{S}}$. Therefore (14) ensures that K is wo-closed as well, and as $K \subseteq \overline{\mathscr{S}}$, K is wo-compact. It now follows, by Proposition 2.4, that K is a semitopological group. That K is in fact a *topological* group follows from [**13**, Theorem 2] where Ellis shows that any locally compact Hausdorff semitopological group is necessarily a topological group.

Let Γ be the dual group of K as in [**14**, p. 88], that is, the group of continuous homomorphisms $K \to \mathbb{S}^1$ also known as characters. Let ϱ be the normalized Haar measure on K. For any $\gamma \in \Gamma$, $\overline{\gamma}$ will denote the complex conjugate of γ, which is itself a character. For each $\gamma \in \Gamma$ and $x \in X$, define

$$\Phi_{\gamma,x} : X^* \to \mathbb{C} : h \mapsto \int_K \langle h, \overline{\gamma}(T)Tx \rangle d\varrho(T). \tag{15}$$

To show that $\Phi_{\gamma,x}$ is well defined we consider the mapping

$$K \to \mathbb{C} : T \mapsto \langle h, \overline{\gamma}(T)Tx \rangle \tag{16}$$

which we need to show is integrable. (16) is the composition of $h : X \to \mathbb{C}$, which is w-continuous, and $\vartheta : K \to X : T \mapsto \overline{\gamma}(T)Tx$, which we can easily show to be wo-w-continuous. Let (T_α) be a wo-convergent net in K with wo-lim $T_\alpha = T$ for some $T \in K$. Then w-lim $T_\alpha x = Tx$ by Theorem 1.3, and $\overline{\gamma}(T_\alpha) \longrightarrow \overline{\gamma}(T)$ since $\overline{\gamma}$ is a character, from which it follows that w-lim $\overline{\gamma}(T_\alpha)T_\alpha x = \overline{\gamma}(T)Tx$, by Theorem 1.5. Thus (16) is wo-continuous and hence measurable with respect to the Haar measure ϱ. Furthermore, $\overline{\mathscr{S}}$ and in particular K are bounded in norm, by Lemma 2.5, and so it is clear that (16) is a bounded mapping. Since K is compact, $\varrho(K) < \infty$, and we thus have that (16) is integrable.

It is clear in addition that $\Phi_{\gamma,x}$ is bounded linear and thus defines a functional in X^{**}.

As the image of a compact set under a continuous mapping, the set $\vartheta(K) = \{\overline{\gamma}(T)Tx : T \in K\}$ is w-compact. By the Krein-Shmulian theorem [**8**, V.6.4], the norm closure of its convex hull, $\overline{\text{co}}\,\{\overline{\gamma}(T)Tx : T \in K\} =: A$ is w-compact. We will now show that $\Phi_{\gamma,x}$ is an element of the canonical embedding of A in X^{**}, denoted κA. Since the canonical embedding $\kappa : X \to X^{**}$ is continuous with respect to the w-topology on X and w*-topology on X^{**}, which is easy to confirm, it follows that κA is w*-compact and thus w*-closed. Therefore to show that $\Phi_{\gamma,x}$

is an element of κA, it suffices to show that any w*-neighbourhood of $\Phi_{\gamma,x}$ contains an element of κA.

Let W be a w*-neighbourhood of $\Phi_{\gamma,x}$. It follows that there are $h_1, ..., h_m \in X^*$ and an $\epsilon > 0$ such that

$$\Phi_{\gamma,x} \in V := V_{\Phi_{\gamma,x},h_1,...,h_m,\epsilon} \subseteq W$$

where

$$\begin{aligned} &V_{\Phi_{\gamma,x},h_1,...,h_m,\epsilon} \\ &= \{\xi \in X^{**} : |\xi(h_i) - \Phi_{\gamma,x}(h_i)| < \epsilon \ , i = 1, ..., m \ \} \\ &= \{\xi \in X^{**} : |\xi(h_i) - \int_K \langle h_i, \overline{\gamma}(T)Tx\rangle d\varrho(T)| < \epsilon \ , i = 1, ..., m \ \} \end{aligned}$$

We will now proceed to define, though a measure theoretic construction, a vector $z \in A$ such that $\hat{z} \equiv \kappa(z) : X^* \to \mathbb{C} : h \mapsto \langle h, z\rangle$ is an element in V.

Let us use the notation $\Theta_i(T) = \langle h_i, \overline{\gamma}(T)Tx\rangle$. So, as shown above, for all $i = 1, .., m$, $\Theta_i : K \to \mathbb{C}$ is w-continuous on K and bounded with, say, $|\Theta_i(K)| < M_i < \infty$. Let $M = \max_i M_i$. Define

$$W_{n,i,p,q} = \Theta_i^{-1}\left(\left(\frac{M}{n}p, \frac{M}{n}(p+1)\right] \times \left(\frac{M}{n}q, \frac{M}{n}(q+1)\right]\right)$$

It is easy to see that, for all $i = 1, ..., m$ and $n \in \mathbb{N}$, $W_{n,i,p,q}$ for $p, q = -n, ..., n-1$ defines a measurable partition of K. The partition $W_{n,i,p,q}$ for $p, q = -n, ..., n-1$ is relative to h_i only. However, for any $n \in \mathbb{N}$, we would like a common partition relative to all h_i. Therefore we consider, for each $n \in \mathbb{N}$, the common refinement of these partitions. So, for a given n we consider all intersections of the form

$$\bigcap_{i=1}^{m} W_{n,i,p_i,q_i}$$

where $p_1, ..., p_m, q_1, ..., q_m \in \{-n, ..., n-1\}$. The collection of all *nonempty* intersections of this form now forms a new partition of K which we denote by

$$V_{n,1}, ..., V_{n,r_n}$$

for some $r_n \in \mathbb{N}$, the number of nonempty intersections. Using this partition, we define a simple function by first taking, for all $n \in \mathbb{N}$ and $j \in \{1, ..., r_n\}$, $T_{n,j}$ to be any element of $V_{n,j}$. Then, for each $i = 1, ..., m$ and $n \in \mathbb{N}$ we define a simple function $K \to \mathbb{C}$ as follows:

$$\Theta_{i,n} = \sum_{j=1}^{r_n} \chi_{V_{n,j}} \Theta_i(T_{n,j})$$

It can now be shown that, for each $i = 1, ..., m$, $\Theta_{i,n}(T) \longrightarrow \Theta_i(T)$ for all $T \in K$. To see why, consider any $R \in K, i \in \{1, ..., m\}$ and

$\epsilon > 0$. Let $k \in \mathbb{N}$ be a number such that $\frac{M}{k} < \frac{\epsilon}{\sqrt{2}}$. Now, consider any $n \geq k$ and the partition $V_{n,1}, ..., V_{n,r_n}$ of K. Hence $R \in V_{n,j}$ for some $j \in \{1, ..., r_n\}$, so that by the definition of $V_{n,j}$ it follows, for any $T \in V_{n,j}$, that

$$|\Theta_i(T) - \Theta_i(R)| \leq \sqrt{\left(\frac{M}{n}\right)^2 + \left(\frac{M}{n}\right)^2} = \sqrt{2}\frac{M}{n} < \epsilon$$

In particular, for $T_{n,j} \in V_{n,j}$,

$$|\Theta_{i,n}(R) - \Theta_i(R)| = |\Theta_i(T_{n,j}) - \Theta_i(R)| < \epsilon$$

holds for our arbitrary choice in $n \geq k$. Thus $\lim_n \Theta_{i,n}(R) = \Theta_i(R) = \langle h_j, \overline{\gamma}(R)Rx\rangle$.

For all $i \in \{1, ..., m\}$, the fact that $|\Theta_i(K)| < M < \infty$ clearly implies that $\Theta_{i,n}$ is "dominated" by the measurable $L^1(K)$ function with constant value M over K. Consequently, by Lebesgue's dominated convergence theorem for complex measurable functions [**26**, 1.34]

$$\int_K \Theta_{i,n}(T)d\varrho(T) \xrightarrow{n} \int_K \langle h_i, \overline{\gamma}(T)Tx\rangle d\varrho(T) \tag{17}$$

However, by the definition of $\Theta_{i,n}$,

$$\begin{aligned}\int_K \Theta_{i,n}(T)d\varrho(T) &= \sum_{j=1}^{r_n} \varrho(V_{n,j})\Theta_i(T_{n,j}) \\ &= \sum_{j=1}^{r_n} \varrho(V_{n,j})\langle h_i, \overline{\gamma}(T_{n,j})T_{n,j}x\rangle \\ &= \langle h_i, \mathcal{S}_n\rangle\end{aligned}$$

where $\mathcal{S}_n = \sum_{j=1}^{r_n} \varrho(V_{n,j})\overline{\gamma}(T_{n,j})T_{n,j}x$. Importantly, $\mathcal{S}_n \in A$ for all $n \in \mathbb{N}$ since $\varrho(K) = 1$ and the sets $V_{n,j}$ form a partition of K.

Let us now again consider the w*-neighbourhood V. By (17), for every $i \in \{1, ..., m\}$, there is an $n_i \in \mathbb{N}$ such that $|\int_K \Theta_{i,n}(T)\varrho(dT) - \int_K \langle h_i, \overline{\gamma}(T)Tx\rangle\varrho(dT)| < \epsilon$ for all $n \geq n_i$, or in other words, such that

$$|\langle h_i, \mathcal{S}_{n_i}\rangle - \Phi_{\gamma,x}(h_i)| < \epsilon$$

That is, in terms of the canonical mapping $\kappa : K \to X^{**}$,

$$|\kappa(\mathcal{S}_{n_i})(h_i) - \Phi_{\gamma,x}(h_i))| < \epsilon$$

Thus, if $n' = \max_i n_i$ then it is clear that $\kappa(\mathcal{S}_{n'}) \in V \subseteq W$. As W was arbitrary and $\kappa(\mathcal{S}_{n'}) \in \kappa A$ and since w*-closed we thus have that $\Phi_{\gamma,x} \in \kappa A$ for our arbitrary choice in $\gamma \in \Gamma$ and $x \in X$.

Since $\Phi_{\gamma,x} \in \kappa A$ there is, for each $x \in X$, a necessarily unique element $z_{\gamma,x} \in A$ such that $\Phi_{\gamma,x}(h) = \langle h, z_{\gamma,x}\rangle$ for all $h \in X^*$. We can now, for all $\gamma \in \Gamma$ define the operator

$$T_\gamma : X \to X : x \mapsto z_{\gamma,x}$$

Thus, for any $h \in X^*$

$$\langle h, T_\gamma x\rangle = \int_K \langle h, \overline{\gamma}(T)Tx\rangle d\varrho(T).$$

We will now show that the range of this operator consists only of eigenvectors with unimodular eigenvalues. Consider any $R \in K$. Then, for any $x \in X, h \in X^*$ it follows by the invariance of ϱ that

$$\begin{aligned}
\langle h, RT_\gamma x\rangle &= \langle R^*h, T_\gamma x\rangle \\
&= \int_K \langle R^*h, \overline{\gamma}(T)Tx\rangle d\varrho(T) \\
&= \int_K \langle h, \gamma(R)\overline{\gamma}(R)\overline{\gamma}(T)RTx\rangle d\varrho(T) \\
&= \gamma(R)\int_K \langle h, \overline{\gamma}(RT)RTx\rangle d\varrho(T) \\
&= \gamma(R)\int_K \langle h, \overline{\gamma}(T)Tx\rangle d\varrho(T) \\
&= \gamma(R)\langle h, T_\gamma x\rangle \\
&= \langle h, \gamma(R)T_\gamma x\rangle.
\end{aligned}$$

Thus, for every $x \in X$, $RT_\gamma x = \gamma(R)T_\gamma x$, or simply,

$$RT_\gamma = \gamma(R)T_\gamma \ \forall\ R \in K. \tag{18}$$

In particular, for the unit $Q \in K$, $\gamma(Q) = 1$ and therefore $QT_\gamma = T_\gamma$. Now consider any $S \in \overline{\mathscr{S}}$. Since $SQ \in K$ we may apply (18) to find that

$$ST_\gamma = S(QT_\gamma) = (SQ)T_\gamma = \gamma(SQ)T_\gamma \ \ \forall\ S \in \overline{\mathscr{S}}. \tag{19}$$

Therefore $S(T_\gamma x) = \gamma(SQ)T_\gamma x$ for all $x \in X$, which thus reveals that $T_\gamma X$ consists of eigenvectors of $\overline{\mathscr{S}}$ with unimodular eigenvalues.

To conclude the proof we will now show that X_{rev} is contained in the space U, the norm-closure, which is contained in the w-closure, of the span of the spaces $T_\gamma X$,$\gamma \in \Gamma$. Suppose we consider some nonzero $z \in X_{rev}$. If $\langle h, z\rangle = 0$ for all those $h \in X^*$ that vanish on U then we must have that $z \in U$. This is a consequence of the Hahn-Banach theorem [**16**, Corollary 1.2.13] for if $z \notin U$ then there has to be a $h' \in X^*$ that vanishes on U but is nonzero at z, which is a contradiction.

Consider any $h \in X^*$ that vanishes on U and any element $Qx \in QX = X_{rev}$. We wish to show that $\langle h, Qx\rangle = 0$ so let us assume that $\langle h, Qx\rangle \neq 0$ and attempt to obtain a contradiction. We know that, since h vanishes on U,

$$\int_K \langle h, \overline{\gamma}(T)Tx\rangle d\varrho(T) = \langle h, T_\gamma x\rangle = 0\ \forall\ \gamma \in \Gamma. \tag{20}$$

We will now show that $\langle h, Qx\rangle \neq 0$ necessitates the existence of a character $\gamma \in \Gamma$ for which the integral in (20) is nonzero. As $\mathscr{Z}$:

$K \to \mathbb{C} : T \mapsto \langle h, Tx\rangle$ is wo-continuous and $\langle h, Qx\rangle \neq 0$, there is a wo-neighbourhood F of Q such that either $\langle h, Tx\rangle > 0$ or $\langle h, Tx\rangle < 0$ for all $T \in F$. Since the collection of all translations RF,$R \in K$ forms a wo-open covering of K which is wo-compact, there is a finite subcollection of translations $R_1F, ..., R_nF$ that covers K. Hence, as $\varrho(K) = 1$ and $\varrho(R_iF) = \varrho(F)$ for all i, $\varrho(F) > 0$. Therefore $\int_K \chi_F \langle h, Tx\rangle d\varrho(T) \neq 0$. However, as the characters $\gamma \in \Gamma$ of K form an orthonormal basis for $L^2(K, \varrho)$ (see [**9**, XI.1.6] and [**14**, Prop 4.3]), there is a sequence $S_n = \sum_{i=1}^{m_n} a_{n,i}\gamma_{n,i}$, with $a_{n,i} \in \mathbb{C}$, in span Γ such that $S_n \longrightarrow \chi_F$ in $L^2(K, \varrho)$. Therefore

$$\langle S_n, \mathscr{X}\rangle_{L^2} \longrightarrow \langle \chi_F, \mathscr{X}\rangle_{L^2} \ ,$$

in other words

$$\sum_{i=1}^{n} \overline{a_i} \int_K \langle h, \overline{\gamma_i}(T)Tx\rangle d\varrho(T) \longrightarrow \int_K \chi_F \langle h, Tx\rangle d\varrho(T) \neq 0 \ ,$$

from which we deduce the existence of a character γ such that

$$\langle h, T_\gamma x\rangle = \int_K \langle h, \overline{\gamma_j}(T)Tx\rangle d\varrho(T) \neq 0 \ ,$$

which contradicts (20). Thus $\langle h, Qx\rangle = 0$ and as $h \in X^*$ was an arbitrary functional vanishing on U, $Qx \in U$. The result therefore follows since $Qx \in X_{rev}$ was arbitrary, and U is clearly contained in X_{uds}. $\square$

Remarks 2.14. There are a couple of important remarks to be made regarding this proof, all relevant to the proposition to follow. To that end we continue to use the symbols and definitions used in the proof.

Although it was not needed in the above proof, for the proposition to follow it will be required to know that $T_\gamma \in \mathscr{L}(X)$, for all $\gamma \in \Gamma$. This can easily be shown. Since $\langle h, T_\gamma x\rangle = \int_K \langle h, \overline{\gamma}(T)Tx\rangle \varrho(dT)$, it is clear that, for any $c_1, c_2 \in \mathbb{C}$ and $x_1, x_2 \in X$

$$\langle h, T_\gamma(c_1x_1 + c_2x_2)\rangle = \langle h, c_1T_\gamma x_1 + c_2T_\gamma x_2\rangle$$

for all $h \in X^*$. Thus $T_\gamma(c_1x_1 + c_2x_2) = c_1T_\gamma x_1 + c_2T_\gamma x_2$ and we have that T_γ is linear for all $\gamma \in \Gamma$.

Let $x \in X$ be arbitrary and let $h \in X^*$ be a functional, obtained with the Hahn-Banach theorem, such that $\langle h, T_\gamma x\rangle = \|T_\gamma x\|$ and $\|h\| = 1$. It follows that

$$\begin{aligned}\|T_\gamma x\| = |\langle h, T_\gamma x\rangle| &= \left|\int_K \langle h, \overline{\gamma}(T)Tx\rangle \varrho(dT)\right| \\ &\leq M\|x\|\end{aligned}$$

where $M = \sup\{\|T\| : T \in K\} < \infty$ since $|\langle h, \overline{\gamma}(T)Tx\rangle| \leq M\|x\|$. Thus, for all $\gamma \in \Gamma$, $T_\gamma \in \mathscr{L}(X)$.

Consider any $R \in K$. Similar to how (18) was derived, it follows for any $x \in X, h \in X^*$ that

$$\begin{aligned}\langle h, T_\gamma Rx\rangle &= \gamma(R)\int_K \langle h, \overline{\gamma}(TR)TRx\rangle d\varrho(T)\\ &= \gamma(R)\int_K \langle h, \overline{\gamma}(T)Tx\rangle d\varrho(T)\\ &= \langle h, \gamma(R)T_\gamma x\rangle.\end{aligned}$$

Thus, since $x \in X, h \in X^*$ and $R \in K$ was arbitrary, it follows that

$$T_\gamma R = \gamma(R)T_\gamma. \tag{21}$$

In Theorem 2.13 we established an equivalent characterization of X_{rev}, namely X_{uds}. In the following proposition we consider X_{fl} and discover that its elements are exactly those that are mapped to zero by the eigenvectors of $\mathscr{S}^*$ in X^* with unimodular eigenvalues. Take note that, in the proof, we will continue working with the operators T_γ and character group Γ introduced in the proof of Theorem 2.13 and in remarks 2.14.

PROPOSITION 2.15. *If $\mathscr{S}$ is an Abelian weakly almost periodic semigroup in $\mathscr{L}(X)$, then $x \in X$ is a flight vector if and only if $\langle h, x\rangle = 0$ for all eigenvectors $h \in X^*$ of $\mathscr{S}^*$ having unimodular eigenvalues.*

PROOF. Let $x \in X_{fl}$ be arbitrary and consider any eigenvector $h \in X^*$ of $\mathscr{S}^*$ with unimodular eigenvalues $\lambda : \mathscr{S}^* \to \mathbb{S}^1$. By definition there is a $T \in \overline{\mathscr{S}}$ such that $Tx = 0$. Let $(T_\alpha)_{\alpha\in\Lambda}$ be a net in $\mathscr{S}$ such that wo-lim $T_\alpha = T$. By Theorem 1.3,

$$\langle h, T_\alpha x\rangle \longrightarrow \langle h, Tx\rangle = 0$$

and as h is an eigenvector of $\mathscr{S}^*$, $\langle h, T_\alpha x\rangle = \langle T_\alpha^* h, x\rangle = \lambda(T_\alpha^*)\langle h, x\rangle$ so that we have

$$\lambda(T_\alpha^*)\langle h, x\rangle \longrightarrow 0.$$

Thus, as $|\lambda(T_\alpha)| = 1$ for all $\alpha \in \Lambda$, $\langle h, x\rangle = 0$ follows.

Conversely, consider any $x \in X$ such that $\langle h, x\rangle = 0$ for all eigenvectors $h \in X^*$ of $\mathscr{S}^*$ with unimodular eigenvalues. By Theorem 2.8, $x = x_{rev} + x_{fl}$, so to show that $x \in X_{fl}$ it will suffice to show that $x_{rev} = 0$. By the same Theorem $x_{fl} = z - Qz$ for some $z \in X$. Thus, for any $\gamma \in \Gamma$, $T_\gamma x_{fl} = T_\gamma z - T_\gamma Qz = T_\gamma z - T_\gamma z = 0$ by (21), since $\gamma(Q) = 1$, so we have that

$$T_\gamma x = T_\gamma x_{rev} \ \forall\, \gamma \in \Gamma. \tag{22}$$

By Theorem 2.8, $X_{rev} = QX$, and therefore $Qx_{rev} = x_{rev}$ since $Q^2 = Q$. We saw in the final part of the proof of Theorem 2.13 that if

$$\langle h, x_{rev}\rangle = \langle h, Qx_{rev}\rangle \neq 0 \ \text{ for some functional } \ h \in X^* \tag{23}$$

then

$$(24) \qquad \langle h, T_\gamma x\rangle = \langle h, T_\gamma x_{rev}\rangle \neq 0 \text{ for some character . } \gamma \in \Gamma$$

Note that the negation of (24), implies the negation of (23) which is that $x_{rev} = 0$. To establish the negation of (24) we first consider any $\gamma \in \Gamma, x \in X$ and $h \in X^*$. Then it follows from (21), that for any $R \in \mathscr{S}$,

$$\begin{aligned}\langle R^* T_\gamma^* h, x\rangle &= \langle h, T_\gamma R x\rangle \\ &= \langle h, \gamma(R) T_\gamma x\rangle \\ &= \langle \gamma(R) T_\gamma^* h, x\rangle.\end{aligned}$$

Thus $R^* T_\gamma^* h = \gamma(R) T_\gamma^* h$, or in other words, $T_\gamma^* h$ is either zero or an eigenvector of $\mathscr{S}^* = \{T^* : T \in \mathscr{S}\}$ with unimodular eigenvalues. Therefore $\langle h, T_\lambda x\rangle = \langle T_\lambda^* h, x\rangle = 0$ as required. □

We can now prove the final result in this chapter which, as mentioned at the beginning of the chapter, will be a modified version of [**19**, Theorem 4.7]. To that end we first define what we mean by an invariant mean and by a function with zero average. An invariant mean is defined prior to [**19**, Theorem 4.7] but we will be using a different definition. [**19**] defines it as a kind of linear functional on the space of *all* bounded complex valued functions g on $\mathscr{S}$. However in Chapter 2, we will have an interest in defining an invariant mean in terms of an integral, and so it will not be defined for *all* complex functions. In other words, [**19**, Theorem 4.7] is too general for our intended purpose.

Definition 2.16. For any $\mathscr{S} \subseteq \mathscr{L}(X)$ and function $g : \mathscr{S} \to \mathbb{C}$, we say that g has *zero average* if, for any $\epsilon > 0$ there are real values $a_1, ..., a_n$ with $\sum_{i=1}^n a_i = 1$ and operators $J_1, ..., J_n \in \mathscr{S}$ such that

$$\sum_{i=1}^n a_i g(TJ_i) < \epsilon \ \forall\ T \in \mathscr{S}$$

As a matter of convenience in the definition to follow, for any $R \in \mathscr{L}(X)$, we denote by $\overline{R}$ the composite mapping

$$\overline{R} : \mathscr{L}(X) \to \mathscr{L}(X) : T \mapsto T \circ R \equiv TR.$$

Definition 2.17. If $\mathscr{S} \subseteq \mathscr{L}(X)$, then an *invariant mean* on $\mathscr{G}$, a space of bounded complex valued functions $\mathscr{S} \to \mathbb{C}$ containing the constant function $1 : \mathscr{G} \to \mathbb{C} : g \mapsto 1$, is a linear functional $L : \mathscr{G} \to \mathbb{C}$ such that

$$\begin{aligned}&g \in \mathscr{G}, g \geq 0 \Rightarrow L(g) \geq 0\ , \\ &L(1) = 1\ , \\ &L(g \circ \overline{U}) = L(g)\ \forall\ g \in \mathscr{G}, U \in \mathscr{S}.\end{aligned}$$

THEOREM 2.18. *Let $\mathscr{S}$ be an Abelian weakly almost periodic semigroup in $\mathscr{L}(X)$, $x \in X$ and L an invariant mean on a subspace $\mathscr{G}'$ of*

$$\mathscr{G} = \text{span}\,\{g_{x,h}, 1 : x \in X, h \in X^*\}$$

where $g_{x,h} : \mathscr{S} \to \mathbb{C} : T \mapsto |\langle h, Tx\rangle|$. Assume that $\mathscr{G}'$ contains all $g \in \mathscr{G}$ with zero average, and that the Banach space $X(x)$ generated by $\mathscr{S}x$ is norm-separable. Then the following are equivalent, for any $x \in X$:

(a) *$x \in X_{fl}$*

(b) *there is a sequence (T_n) in $\mathscr{S}$ such that* w-lim $T_n x = 0$

(c) *for every $h \in X^*$ we have $g_{x,h} \in \mathscr{G}'$ and $L(g_{x,h}) = 0$*

PROOF. (a) $\Rightarrow$ (b) Consider any nonzero $x \in X_{fl}$. Hence there is some $T \in \overline{\mathscr{S}}$ such that $Tx = 0$. In the event that $Tx \in \mathscr{S}x$, (b) follows trivially. So suppose that $Tx \notin \mathscr{S}x$. To find a sequence (T_n) in $\mathscr{S}$ such that $T_n x$ converges weakly to zero, we will prove that 0 is a w-limit point of $\mathscr{S}x$ in the w-topology and that the w-topology on $\overline{\text{co}}(\mathscr{S}x) \supseteq \mathscr{S}x$ is given by a metric. Once we have the metric it will be a simple matter to find the sequence.
To show that $0 \in X$ is a w-limit point of $\mathscr{S}x$, let W be a w-neighbourhood of 0. It follows that there are $h_1, ..., h_n \in X^*$ and an $\epsilon > 0$ such that

$$0 \in V := V_{0,h_1,...,h_n,\epsilon} = \{z \in X : |\langle h_i, z\rangle| < \epsilon \;\forall\, i = 1, ..., n\} \subseteq W.$$

The wo-base element

$$U := U_{T,x,h_1,...,h_n,\epsilon} = \{R \in \mathscr{L}(X) : |\langle h_i, Rx\rangle - \langle h_i, Tx\rangle| < \epsilon \;\forall\, i = 1, ..., n\}$$

is a wo-neighbourhood of T. Thus, since $T \in \overline{\mathscr{S}} :=$ wo-cl $(\mathscr{S})$, there is an $S \in \mathscr{S}$ such that $S \in U$, i.e. such that $|\langle h_i, Sx\rangle - \langle h_i, Tx\rangle| < \epsilon$ for all $i = 1, ..., n$. In other words $Sx \in V \subseteq W$ since $Tx = 0$. If $Sx = 0$ then (b) follows trivially. If $Sx \neq 0$ then, since W was arbitrary, we have that 0 is a w-limit point of $\mathscr{S}x$.

co $(\mathscr{S}x)$ is convex and therefore so is its norm closure $\overline{\text{co}}\,(\mathscr{S}x)$. By [**8**, V.3.13], $\overline{\text{co}}\,(\mathscr{S}x)$ is w-closed. Since $\overline{\mathscr{S}}x$ is w-compact, $\overline{\text{co}}\,(\overline{\mathscr{S}}x)$ is w-compact by the Krein-Smulian Theorem [**8**, V.6.4]. Thus, as $\overline{\text{co}}\,(\mathscr{S}x) \subseteq \overline{\text{co}}\,(\overline{\mathscr{S}}x)$, it follows that $\overline{\text{co}}\,(\mathscr{S}x)$ is w-compact. At this point it is important to understand that the w-topology of the Banach space $X(x)$ *is* the subspace topology of the w-topology of X on $X(x)$. This is a consequence of the Hahn Banach theorem in that $g \in X(x)^*$ if and only if $g = h|_{X(x)}$ for some $h \in X^*$. Hence, for any $y \in X(x)$

$$\{z \in X(x) : |\langle g, z\rangle - \langle g, y\rangle| < \epsilon\} = X(x) \cap \{z \in X : |\langle h, z\rangle - \langle h, y\rangle| < \epsilon\}$$

which shows us how the weak topology of $X(x)$ and the weak topology of X on $X(x)$ share a common basis. Thus, since $\overline{\text{co}}\,(\mathscr{S}x) \subseteq X(x)$, $\overline{\text{co}}\,(\mathscr{S}x)$ is also compact in the w-topology of $X(x)$. It now follows that the subspace topology of the w-topology of $X(x)$ on $\overline{\text{co}}\ (\mathscr{S}x)$ is metric by [**8**, V6.3], since $X(x)$ is separable.

Now, since 0 is a limit point of $\mathscr{S}x$ in the w-topology of X, and $\mathscr{S}x \subseteq \overline{\text{co}}\ (\mathscr{S}x)$, 0 is a limit point of $\mathscr{S}x$ in the subspace topology of the w-topology of X on $\overline{\text{co}}\ (\mathscr{S}x)$. Hence, by the preceding observation, 0 is therefore a limit point of $\mathscr{S}x$ in the subspace topology of the w-topology of $X(x)$ on $\overline{\text{co}}\ (\mathscr{S}x)$. Since the latter topology, say $\mathfrak{T}$, is metric, there is a sequence (T_nx) in $\mathscr{S}x$ that converges to 0 in $\mathfrak{T}$. Since $\mathfrak{T}$ is, again by the above observation, the same as the subspace topology of the w-topology of X on $\overline{\text{co}}\ (\mathscr{S}x)$, it is easy to see that w-lim $T_nx = 0$, i.e. that (T_nx) converges to 0 in the w-topology of X.

(b) $\Rightarrow$ (c) By the Banach-Alaoglu Theorem, the closed unit ball $B \subseteq X^*$ is w*-compact. For each $n \in \mathbb{N}$ define a function $f_n \in C(B)$ as follows

$$f_n : B \to \mathbb{C} : h \mapsto |\langle h, T_nx\rangle|.$$

As the composition of the continuous mapping $\mathbb{C} \to \mathbb{C} : a \mapsto |a|$, and the clearly w*-continuous mapping $X^* \to \mathbb{C} : h \mapsto \langle h, T_nx\rangle$ restricted to B, $f_n \in C(B)$ is well defined. By recalling from Lemma 2.5 that $\mathscr{S}$ is norm bounded, it follows that for each $n \in \mathbb{N}$, f_n is bounded by $M\|x\|$, where $M = \sup\{\|T\| : T \in \mathscr{S}\}$. It also follows, since w-lim $T_nx = 0$ and each $h \in X^*$ is w-continuous, that $f_n(h) = \langle h, T_nx\rangle \longrightarrow 0$ for all $h \in B$. That is, the sequence (f_n) converges pointwise to $0 \in C(B)$. We will show that these results are sufficient to show that f_n converges weakly to $0 \in C(B)$.

Consider any positive functional $\Lambda \in C(B)^*$. Since the w*-topology on X^* is Hausdorff and B is w*-compact, we may apply The Riesz Representation Theorem [**26**, 2.14 Theorem] to Λ. Thus, for all $g \in C(B)$, $\Lambda(g) = \int_B g\, d\mu$ for some finite positive measure μ on B. Hence, $M\|x\|1 \in L^1(\mu)$ with $1 \in C(B)$ the constant function with value 1. As the sequence (f_n) is uniformly bounded by $\|x\|M1$ and converges pointwise to $0 \in C(B)$, it therefore follows by Lebesgue's Dominated Convergence Theorem for complex valued functions [**26**, 1.34], that

$$\lim_{n\to\infty} \Lambda(f_n) = \lim_{n\to\infty} \int_B f_n\, d\mu = \int_B 0\, d\mu = 0.$$

Thus, for an arbitrary $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|\Lambda(f_n)| < \epsilon$ for all $n \geq N$.

More generally, if $\Lambda \in C(B)^*$ is such that $\Lambda = \sum_{k=1}^m b_k\Lambda_k$ where each $\Lambda_k \in C(B)^*$ is positive, then for each $k = 1, .., m$ there is an N_k such that $|\Lambda_k(f_n)| < \frac{\epsilon}{|b_k|m}$ so that

$$|\Lambda(f_n)| = |\sum_{k=1}^m b_k\Lambda_k(f_n)| \leq \sum_{k=1}^m |b_k||\Lambda_k(f_n)| < \epsilon.$$

Thus in this case $\Lambda(f_n) \longrightarrow 0$ as well. If $\Lambda \in C(B)^*$ is completely arbitrary, then Λ is the linear combination of at most four positive functionals in $C(B)^*$, by [**16**, Corollary 4.2.4], so it still follows that $\Lambda(f_n) \longrightarrow 0$. We conclude therefore, that f_n converges to 0 in the weak topology of $C(B)$. This will be sufficient to show that $g_{x,h} \in \mathscr{G}'$ for all $h \in X^*$.

Fix an arbitrary nonzero $h \in X^*$. It now follows by Mazur's Theorem [**29**, Theorem 2 p. 120] that, for any $\epsilon > 0$, there is a convex combination $\sum_{i=1}^m a_i f_{n_i}$ such that

$$\left\|\sum_{i=1}^m a_i f_{n_i}\right\|_{C(B)} = \left\|\sum_{i=1}^m a_i f_{n_i}\right\|_\infty < \frac{\epsilon}{\|h\|}$$

Therefore $\sum_{i=1}^m a_i|\langle h', T_{n_i}x\rangle| < \frac{\epsilon}{\|h\|}$ for all $h' \in B$, hence, for any $T \in \mathscr{S}$

$$\begin{aligned}
(\sum_{i=1}^m a_i g_{x,h} \circ \overline{T}_{n_i})(T) &= \sum_{i=1}^m a_i g_{x,h}(TT_{n_i}) \\
&= \sum_{i=1}^m a_i |\langle h, TT_{n_i}x\rangle| \\
&= \sum_{i=1}^m a_i(\|T^*h\|+1)\left|\left\langle \frac{T^*h}{(\|T^*h\|+1)}, T_{n_i}x\right\rangle\right| \\
&= (\|T^*h\|+1)\sum_{i=1}^m a_i\left|\left\langle \frac{T^*h}{(\|T^*h\|+1)}, T_{n_i}x\right\rangle\right| \\
&< (\|T^*h\|+1)\frac{\epsilon}{\|h\|} \quad (25)\\
&\leq \epsilon M + \frac{\epsilon}{\|h\|} \quad (26)
\end{aligned}$$

where (25) follows from the fact that $\frac{T^*h}{(\|T^*h\|+1)} \in B$, and (26) follows from the fact that if $\|T\| \leq M$, then $\|T^*\| \leq M$. As $\epsilon > 0$ was arbitrary this shows that $g_{x,h}$ has zero average and is therefore an element of $\mathscr{G}'$. Since L is now known to be defined at $g_{x,h}$ it follows from the invariance of L under translations by T_{n_i} that

$$\begin{aligned}
L(g_{x,h}) &= \sum a_i L(g_{x,h}) \\
&= \sum a_i L(g_{x,h} \circ \overline{T}_{n_i}) \\
&= L(\sum a_i g_{x,h} \circ \overline{T}_{n_i}) \\
&\leq \epsilon M \quad (27)
\end{aligned}$$

The last inequality follows from the observation that

$$\epsilon M \geq (\sum \alpha_i g_{x,h} \circ \overline{T}_{n_i})(T) =: h'(T)$$

which shows that $h' \leq \epsilon M1$ and therefore that $\epsilon M1 - h' \geq 0$. Thus, $L(\epsilon M1 - h') = \epsilon M - L(h') \geq 0$, which yields (27). As $\epsilon > 0$ was arbitrary, (c) follows for our arbitrary choice of a nonzero $h \in X^*$. If $h = 0$ then $g_{x,h} = 0$ and (c) follows trivially since L is linear.

(c) $\Rightarrow$ (a) If $h \in X^*$ is an eigenvector of $\mathscr{S}^*$ with unimodular eigenvalues then $g_{x,h}(T) = |\langle h, Tx\rangle| = |\langle T^* h, x\rangle| = \langle h, x\rangle$ for all $T \in \mathscr{S}$. In other words $g_{x,h} = \langle h, x\rangle 1$ and therefore, by (c) and the definition of an invariant mean, $\langle h, x\rangle = L(g_{x,h}) = 0$. Hence (a) follows from Proposition 2.15. $\square$

CHAPTER 2

C*-dynamical systems

The ergodic properties of noncommutative C^*-dynamical systems, and in particular W^*-dynamical systems in terms of von Neumann algebras, remain a relevant and active research field. For some of the more recent advances in this field, the reader is referred to [**3**],[**24**] and [**1**].

In this chapter C^*-dynamical systems are defined as well as several ergodic properties that such systems can possess. The second half of the chapter consists of the derivation of equivalent Hilbert space and spectral characterizations of the ergodic properties. It is in some of these spectral characterizations that the theory from Chapter 1 will be used. The chapter concludes with an investigation into the ergodic properties' interrelationships. In particular, we will find which combinations of properties are impossible.

1. Basic definitions and concepts

The following definition shows how a dynamical system can be defined on a C^*-algebra, given a state on the C^*-algebra, and a group of $*$-automorphisms under which the state is invariant. Beyond the definition we will show how such systems can be represented on Hilbert space level by a group of unitary operators on a Hilbert space H, called the GNS representation. The systems defined in the following definition are quite abstract, however this definition provides a convenient setting for the GNS representation. Once the GNS representation is in place, we will identify a subset of such systems on which to develop our ergodic ideas.

Definition 1.1. Let ω be a state on a unital C^*-algebra $\mathfrak{A}$, i.e. $\omega : \mathfrak{A} \to \mathbb{C}$ is linear and

$$\omega(A) \geq 0 \text{ for all } A \geq 0$$
$$\omega(1) = 1$$

Let G be a group and let $\tau : G \to \operatorname{Aut}(\mathfrak{A}) : g \mapsto \tau_g$ be a group homomorphism, i.e. a mapping into the group of $*$-automorphisms of $\mathfrak{A}$ that satisfies

$$\tau_g \circ \tau_h = \tau_{gh} \ , \quad \forall\, g, h \in G$$

If ω is τ-invariant, i.e. if

$$\omega \circ \tau_g = \omega \quad \forall\, g \in G$$

then we call $(\mathfrak{A}, \omega, \tau, G)$ an *abstract dynamical system.*

Note that, in the above definition, it follows that $\tau_1 = \mathrm{id}$ where $1 \in G$ is the unit element of G.

We now proceed towards the definition of a GNS representation of an abstract dynamical system, which allows us to relegate much of our study of an abstract dynamical system to a C^*-algebra of bounded linear operators on a Hilbert space H. The first step in finding a GNS representation, is to identify an appropriate cyclic representation of $(\mathfrak{A}, \omega)$, which we describe next.

Definition 1.2. A *representation* of a C^*-algebra $\mathfrak{A}$ is a pair (H, π), where H is a complex Hilbert space and $\pi : \mathfrak{A} \to \mathscr{L}(H)$ is a $*$-homomorphism. A *cyclic representation* of a C^*-algebra $\mathfrak{A}$ is a triple (H, π, Ω), where (H, π) is a representation of $\mathfrak{A}$ and $\Omega \in H$ is a cyclic vector, i.e. $\overline{\pi(\mathfrak{A})\Omega} = \overline{\{\pi(A)\Omega : A \in \mathfrak{A}\}} = H$.

If we were to consider some cyclic representation (H, π, Ω) of a C^*-algebra $\mathfrak{A}$ and define

$$\omega_\Omega : \mathfrak{A} \to \mathbb{C} : A \mapsto \langle \Omega, \pi(A)\Omega \rangle$$

then it is easy to see that ω_Ω is a state over $\mathfrak{A}$, provided Ω has unit norm. This is referred to as a vector state of the representation, in Ω. What the Gelfand-Naimark-Segal construction (GNS construction) shows is that every state ω over a C^*-algebra $\mathfrak{A}$ is the vector state of some cyclic representation (H, π, Ω) of $\mathfrak{A}$, in Ω. Moreover, and importantly, such a cyclic representation is unique up to unitary equivalence [**4**, Theorem 2.3.16]. That is if (H', π', Ω') is a another cyclic representation such that ω is the vector state of (H', π', Ω') in Ω' then there is a unitary operator $U \in \mathscr{L}(H, H')$ satisfying

$$U^{-1}\pi'(A)U = \pi(A) \ \forall\ A \in \mathfrak{A}$$
$$U\Omega = \Omega'.$$

Thus, combined, these two properties give

$$U\pi(A)\Omega = \pi'(A)\Omega' \ \forall\ A \in \mathfrak{A}$$

revealing that $U \in \mathscr{L}(H, H')$ is necessarily uniquely determined since $\overline{\pi(\mathfrak{A})\Omega} = H$. Hence we can call (H, π, Ω) *the* cyclic representation of $(\mathfrak{A}, \omega)$. That is, (H, π, Ω) is a cyclic representation of $\mathfrak{A}$ such that ω is the vector state of (H, π, Ω) in Ω. If we consider an abstract dynamical system $(\mathfrak{A}, \omega, \tau, G)$ and a cyclic representation (H, π, Ω) of $(\mathfrak{A}, \omega)$, then it can easily be seen that, for all $g \in G$, $(H, \pi \circ \tau_g, \Omega)$ is also a cyclic representation of $(\mathfrak{A}, \omega)$, since $\omega \circ \tau_g = \omega$. Therefore there exists a unique unitary operator $U_g \in \mathscr{L}(H)$ such that $U_g\pi(A)\Omega = \pi \circ \tau_g(A)\Omega$ for all $A \in \mathfrak{A}$ [**4**, Corollary 2.3.17]. This is how we will define the GNS representation of a C^*-dynamical system.

DEFINITION 1.3. Let $(\mathfrak{A}, \omega, \tau, G)$ be an abstract dynamical system and let (H, π, Ω) be the cyclic representation of $(\mathfrak{A}, \omega)$. Define $U : G \to \mathscr{L}(H) : g \mapsto U_g$ by

$$U_g : \pi(A)\Omega \mapsto \pi(\tau_g(A))\Omega \quad \forall\ A \in \mathfrak{A},$$

(H, π, Ω, U) is called the *GNS representation* of the abstract dynamical system $(\mathfrak{A}, \omega, \tau, G)$, and U the *GNS representation* of τ.

Several important properties of the GNS representation in Definition 1.3 can immediately be derived, which constitutes the following proposition. The GNS representation is a representation of the group G, the $*$-automorphisms τ_g can be expressed in terms of the GNS representation, and it is an important fact that the GNS representation always has at least one nonzero fixed vector:

PROPOSITION 1.4. *Let (H, π, Ω, U) be the GNS representation of an abstract dynamical system $(\mathfrak{A}, \omega, \tau, G)$. Then*

$$\begin{aligned} U_g U_h &= U_{gh}, \\ U_g \pi(A) U_g^{-1} &= \pi(\tau_g(A)) \quad \forall\ A \in \mathfrak{A}, \text{ and} \\ U_g \Omega &= \Omega \end{aligned}$$

for all $g, h \in G$.

PROOF. Let $g, h \in G$ and $A \in \mathfrak{A}$ be arbitrary. It follows that

$$\begin{aligned} U_g U_h \pi(A)\Omega &= U_g \pi(\tau_h(A))\Omega \\ &= \pi(\tau_g \circ \tau_h(A))\Omega \\ &= \pi(\tau_{gh}(A))\Omega \\ &= U_{gh}\pi(A)\Omega. \end{aligned}$$

Since $A \in \mathfrak{A}$ was arbitrary, and $\overline{\pi(\mathfrak{A})} = H$, we have that $U_g U_h = U_{gh}$.

Let $g \in G$ be arbitrary and consider any $A, B \in \mathfrak{A}$. Then, $\tau_g(D) = B$ for some $D \in \mathfrak{A}$. It follows that

$$\begin{aligned} U_g \pi(A) U_g^{-1} \pi(B)\Omega &= U_g \pi(A) U_g^{-1} \pi(\tau_g(D))\Omega \\ &= U_g \pi(A)\pi(D)\Omega \\ &= U_g \pi(AD)\Omega \\ &= \pi(\tau_g(A))\pi(B)\Omega. \end{aligned}$$

Thus $U_g \pi(A) U_g^{-1} = \pi(\tau_g(A))$ on $\pi(\mathfrak{A})\Omega$, and therefore on $\overline{\pi(\mathfrak{A})\Omega} = H$.

For the third assertion, we have

$$U_g\Omega = U_g\pi(1)\Omega = \pi(\tau_g(1))\Omega = \pi(1)\Omega = \Omega$$

for all $g \in G$. □

We wish to define and study various ergodic properties on noncommutative dynamical systems, such as abstract dynamical systems. However, as [**4**, p. 400] highlights, apart from a few general results, it would be difficult to develop and elaborate on the theory without further assumptions on the structure of the group G and the continuity of its action $\tau : G \mapsto \tau_g$. In Definition 1.6, we state these further assumptions, and define abstract dynamical systems for which these assumptions hold, as *C^*-dynamical systems*. These systems will be the object of our subsequent study. However, we first include a preliminary definition

Definition 1.5. A locally compact group G is called *amenable* if it possesses a Følner sequence (Λ_n), i.e. a sequence of compact sets $(\Lambda_n) \subseteq G$, with $\mu(\Lambda_n) > 0$, such that

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \Delta(\Lambda_n g))}{\mu(\Lambda_n)} = 0 \ \ \forall\, g \in G$$

where μ is a right Haar measure on G and Δ denotes the symmetric difference.

What we are stating here as a definition of amenability is actually a derived result from amenable group theory, for locally compact groups. Amenability for a locally compact group is defined in terms of the existence of a certain invariant mean, hence the term a*mean*able (see [**25**, p. 1]). It can then be proven that amenability on locally compact groups, under certain conditions, is equivalent to the existence of such Følner sequences. However, we wish to avoid delving into amenability, and the derivation of the Følner conditions, as our only interest at this stage is in groups that possess such sequences. We opt to use the term *amenable* for such groups, as in Definition 1.5, simply because it is more convenient than to say “group with a Følner sequence”.

For the locally compact groups that we will encounter in our examples, which will either be $\mathbb{Z}, \mathbb{R}$ or $\mathbb{R}^2$, it will not be necessary to go to any great lengths to prove amenability. Some familiarity with these elementary groups will be sufficient to identify a Følner sequence, and so when encountered for the first time we will deem it sufficient to supply an example of a Følner sequence without any additional motivation.

Definition 1.6. A *C^*-dynamical system* is an abstract dynamical system $(\mathfrak{A}, \omega, \tau, G)$ where

(i) G is a separable Abelian amenable group.
(ii) The GNS representation $U : G \mapsto \mathfrak{U}(H) : g \mapsto U_g$ of τ is continuous in the strong operator topology, where $\mathfrak{U}(H)$ is the group of all unitary operators in $\mathscr{L}(H)$.

Recall that the strong operator topology (so-topology) on $\mathscr{L}(H)$ is the locally convex topology generated by the family of seminorms

$$p_x : \mathscr{L}(H) \to \mathbb{R}^+ : T \mapsto \|Tx\|$$

for all $x \in H$. So so-continuity of U is equivalent to $g \mapsto U_g x$ being continuous in the norm of H, for all $x \in H$. We can immediately establish two important properties of C^*-dynamical systems. The first will be needed when we apply Theorem 2.18 in section 2.4.

PROPOSITION 1.7. *Let $(\mathfrak{A}, \omega, \tau, G)$ be a C^*-dynamical system with GNS representation (H, π, Ω, U). Then the Hilbert space spanned by the orbit $\{U_g x : g \in G\}$ is separable in H, for all $x \in X$.*

PROOF. Consider any $x \in X$ and denote by $H(X)$ the Hilbert space spanned by the orbits $\{U_g x : g \in G\}$. Since G is separable it has a countable dense subset $\Gamma \subseteq G$. Let $g \in G$ and $\epsilon > 0$ be arbitrary. Suppose now that $z \in \operatorname{span}\{U_g x : g \in G\}$ is arbitrary, say $z = \sum_{k=1}^{n} a_k U_{g_k} x$, with $a_k \neq 0$. By the strong continuity of U, for every $k = 1, ..., n$ there is a $h_k \in \Gamma$ such that $\|U_{h_k} x - U_{g_k} x\| < \frac{\epsilon}{n|a_k|}$. Let $y = \sum_{k=1}^{n} a_k U_{h_k} x$. It follows by the triangle inequality that

$$\|y - z\| < \epsilon$$

Let $\mathfrak{F} = \{\sum_{j=1}^{n} b_j U_{h_j} x : h_j \in \Gamma,\ b_j \in \mathbb{Q} + i\mathbb{Q}, n \in \mathbb{N}\} \subseteq \operatorname{span}\{U_h x : h \in \Gamma\}$. Then $\mathfrak{F}$ is countable, and it is easy to see that $\mathfrak{F}$ is dense in $\operatorname{span}\{U_h x : h \in \Gamma\}$. Now, let $z \in H(x)$ and $\epsilon > 0$ be arbitrary. It follows by the definition of $H(x)$ and the preceding arguments that there is a $z' \in \operatorname{span}\{U_g x : g \in G\}$ such that $\|z - z'\| < \frac{\epsilon}{3}$, a $y' \in \operatorname{span}\{U_h x : h \in \Gamma\}$ such that $\|z' - y'\| < \frac{\epsilon}{3}$ and a $y \in \mathfrak{F}$ such that $\|y' - y\| < \frac{\epsilon}{3}$. We thus have that

$$\|z - y\| \leq \|z - z'\| + \|z' - y'\| + \|y' - y\| < \epsilon$$

so that since $z \in \mathfrak{H}$ and $\epsilon > 0$ was arbitrary, $\mathfrak{F}$ is dense in $\mathfrak{H}$. As $\mathfrak{F}$ is countable, the separability of $\mathfrak{H}$ follows. □

The second property is relevant for the definition of ergodic properties in the next section.

PROPOSITION 1.8. *Let $(\mathfrak{A}, \omega, \tau, G)$ be a C^*-dynamical system. Then the mapping*

$$G \mapsto \mathbb{C} : g \mapsto \omega(A\tau_g(B))$$

is continuous for all $A, B \in \mathfrak{A}$.

PROOF. Let (H, π, Ω, U) be the GNS representation of $(\mathfrak{A}, \omega, \tau, G)$. Let (g_α) be an arbitrary convergent net in G, with $\lim_\alpha g_\alpha = g$ for some $g \in G$. Since $G \to \mathscr{L}(H) : g \mapsto U_g$ is so-continuous, $\lim_\alpha U_{g_\alpha} x = U_g x$ for all $x \in H$, and hence

$$\lim_\alpha \langle \pi(A^*)\Omega, U_{g_\alpha} \pi(B)\Omega \rangle = \langle \pi(A^*)\Omega, U_g \pi(B)\Omega \rangle \tag{28}$$

for any $A, B \in \mathfrak{A}$, by the continuity of the inner product. Since $\langle \pi(A^*)\Omega, U_h \pi(B)\Omega \rangle = \langle \Omega, \pi(A)\pi(\tau_h(B))\Omega \rangle = \omega(A\tau_h(B))$, (28) becomes

$$\lim_\alpha \omega(A\tau_{g_\alpha}(B)) = \omega(A\tau_g(B)).$$

Thus $G \mapsto \mathbb{C} : g \mapsto \omega(A\tau_g(B))$ is continuous. □

REMARKS 1.9. As a final note before we proceed with the definition of the ergodic properties, it should be noted that the properties in Definition 1.6 are trivially satisfied when $G = \mathbb{Z}$. In such a case, i.e. when we have a $\mathbb{Z}$ action, we will opt to abuse the notation somewhat: The τ in $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ will *not* denote a homomorphism $\mathbb{Z} \to \mathrm{Aut}\,(\mathfrak{A})$, but an element of $\mathrm{Aut}\,(\mathfrak{A})$ itself, with the homomorphism then given by $n \to \tau^n$. Similarly, if (H, π, Ω, U) is the GNS representation of $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$, then U will denote the unitary operator $H \to H : \pi(A)\Omega \mapsto \pi(\tau(A))\Omega$, instead of a homomorphism of $\mathbb{Z}$ into the collection of unitary operators in $\mathscr{L}(H)$. The homomorphism is then given by $n \mapsto U^n$.

2. Ergodic properties

In this section we will define what is meant by an ergodic C^*-dynamical system, a weak-mixing C^*-dynamical system, a strong-mixing C^*-dynamical system and a compact C^*-dynamical system. These are the ergodic properties that will occupy us for the remainder of the dissertation. For the remainder of the chapter, μ will be used to denote a Haar measure on the Abelian locally compact group G in question.

DEFINITION 2.1. A C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ is said to be *ergodic* if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g) = \omega(A)\omega(B)$$

for all $A, B \in \mathfrak{A}$ and some Følner-sequence (Λ_n) in G.

DEFINITION 2.2. A C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ is said to be *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(A\tau_g(B)) - \omega(A)\omega(B)|\; d\mu(g) = 0$$

for all $A, B \in \mathfrak{A}$ and some Følner-sequence (Λ_n) in G.

Note that the integrals in the definitions above are well defined by Proposition 1.8 and the fact that Λ_n is compact. It will be seen in section 2.4 that the above definitions are in fact independent of the Følner sequences used.

The following ergodic property will only be defined for actions on the group $\mathbb{Z}$, since this will be sufficient for our later purposes.

DEFINITION 2.3. If $G = \mathbb{Z}$ then a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ is said to be *strongly mixing* if

$$\lim_{n\to\infty} \omega(A\tau^n(B)) = \omega(A)\omega(B)$$

for all $A, B \in \mathfrak{A}$.

The final ergodic property that we will consider is defined in terms of the concept of total boundedness. A set B in a pseudo metric space (X, d) is said to be totally bounded if, for all $\epsilon > 0$, there exists a finite set $M_\epsilon \subseteq X$ such that

$$\forall\ x \in B\ \exists\ y \in M_\epsilon : d(x, y) < \epsilon.$$

DEFINITION 2.4. A C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ is said to be *compact* if, for all $A \in \mathfrak{A}$, the set $\{\tau_g(A) : g \in G\}$ is totally bounded in the semi normed space $(\mathfrak{A}, \|\cdot\|_\omega)$ where, for all $A \in \mathfrak{A}$, $\|A\|_\omega := \sqrt{\omega(A^*A)}$.

It is our intention to study the interrelationship between these properties, however, we will not do this for C^*-dynamical systems in general. The reason is that C^*-dynamical systems as defined, allow for certain systems that are entirely uninteresting from an ergodic point of view, in that all four the ergodic properties are trivially present on such systems. We wish to identify and exclude these kind of systems from consideration. In the following proposition we show how such triviality can come about.

PROPOSITION 2.5. *If a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ has a GNS representation (H, π, Ω, U) in which H is one dimensional, then the C^*-dynamical system is ergodic, weakly mixing and compact. If $G = \mathbb{Z}$ then it is also strongly mixing.*

PROOF. Since H is one dimensional, $H = \mathbb{C}\Omega$ so that since π maps into $\mathscr{L}(H)$ it follows, for any $A \in \mathfrak{A}$, that

$$\pi(A)\Omega = c_A\Omega$$

for some uniquely determined $c_A \in \mathbb{C}$. Thus, for any $A \in \mathfrak{A}$, we have that

$$\omega(A) = \langle\Omega, \pi(A)\Omega\rangle = c_A\langle\Omega, \Omega\rangle = c_A$$

and therefore for any $A, B \in \mathfrak{A}$ that

$$\begin{aligned} \omega(AB) &= \langle\Omega, \pi(AB)\Omega\rangle \\ &= \langle\Omega, \pi(A)\pi(B)\Omega\rangle \\ &= c_A c_B \\ &= \omega(A)\omega(B). \end{aligned} \tag{29}$$

In particular then,

$$\omega(A\tau_g(B)) = \omega(A)\omega(\tau_g(B)) = \omega(A)\omega(B)\ \forall\ g \in G \tag{30}$$

since $\omega \circ \tau_g = \omega$ for all $g \in G$. From (30), $(\mathfrak{A}, \omega, \tau, G)$ can easily be seen to be ergodic and immediately seen to be weakly mixing. If $G = \mathbb{Z}$, then (30) also immediately reveals the system to be strongly mixing.

Finally, consider the orbit of an arbitrary $A \in \mathfrak{A}$, $\{\tau_g(A) : g \in G\}$. As per the definition of total boundedness, let $\epsilon > 0$ be arbitrary and let $M = \{A\}$. Then, for all $g \in G$,

$$\begin{aligned}\|A - \tau_g(A)\|_\omega^2 &= \omega((A - \tau_g(A))^*(A - \tau_g(A))) \\ &= \omega(A^*A) - \omega(\tau_g(A)^*A) - \omega(A^*\tau_g(A)) + \omega(\tau_g(A)^*\tau_g(A)) \\ &= \omega(A^*A) - \omega(\tau_g(A^*)A) - \omega(A^*\tau_g(A)) + \omega(\tau_g(A^*)\tau_g(A)) \\ &= 0\end{aligned}$$

by (29) and (30). Thus it follows immediately and trivially from the definition of compactness that $(\mathfrak{A}, \omega, \tau, G)$ is compact. □

The existence of a cyclic representation of a C^*-dynamical system allows all of the ergodic properties to be expressed in terms of the GNS representation. This of course, is due to the fact that $\omega(A)$ can be replaced with $\langle \Omega, \pi(A)\Omega \rangle$. Proposition 2.5 shows that, because of this, if the representation is trivial in that its Hilbert space H is one dimensional, then the system is trivially ergodic, weak-mixing, strong-mixing and compact. However, we do not have to look at a cyclic representation to determine whether it's Hilbert space is one dimensional. Whenever the state of a C^*-dynamical system preserves algebraic structure this one dimensionality manifests, and conversely:

PROPOSITION 2.6. *Let (H, π, Ω) be a cyclic representation of a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$. Then H is one dimensional if and only if $\omega : \mathfrak{A} \to \mathbb{C}$ is a homomorphism, i.e. if and only if $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathfrak{A}$.*

PROOF. That $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathfrak{A}$ if H is one-dimensional, was proved in Proposition 2.5.

Conversely, suppose that $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathfrak{A}$. Define $\omega' : \mathfrak{A} \to \mathscr{L}(\mathbb{C}) : A \mapsto \omega'(A)$ by

$$\omega'(A) : \mathbb{C} \to \mathbb{C} : c \mapsto \omega(A)c.$$

ω' is linear, and since $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathfrak{A}$, ω' is a homomorphism. In addition, since $\omega(A^*) = \overline{\omega(A)}$ for all $A \in \mathfrak{A}$ by [**4**, Lemma 2.3.10], it follows that ω' is a $*$-homomorphism. It is now easy to see that $(\mathbb{C}, \omega', 1)$ is a cyclic representation of $(\mathfrak{A}, \omega)$. Hence, by the uniqueness of a cyclic representation, up to unitary equivalence, H is necessarily one dimensional since $\mathbb{C}$ is one dimensional. □

We may therefore now define what we mean by a trivial C^*-dynamical system.

DEFINITION 2.7. A C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ is said to be *trivial* if ω is a homomorphism.

It is important to note that, by this definition, a C^*-dynamical system is trivial only with respect to its ergodic properties, which we do not state in the definition precisely because our main objective is the ergodic analysis of C^*-dynamical systems. That is, these systems are trivial from our point of view.

3. Hilbert space characterization of ergodic properties

For any C^*-dynamical system, the existence of the GNS representation (H, π, Ω, U), allows us to completely characterize the ergodic properties of the preceding section in terms of (H, Ω, U), i.e. purely in Hilbert space terms.

PROPOSITION 3.1. *Consider a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$ with GNS representation (H, π, Ω, U).*

(a) For any Følner sequence (Λ_n) in G

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g) = \omega(A)\omega(B) \;\forall\; A, B \in \mathfrak{A} \tag{31}$$

if and only if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle x, U_g y, \rangle\, d\mu(g) = \langle x, \Omega\rangle\langle \Omega, y\rangle \;\forall\; x, y \in H. \tag{32}$$

(b) For any Følner sequence (Λ_n) in G

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(A\tau_g(B)) - \omega(A)\omega(B)|\; d\mu(g) \;\forall\; A, B \in \mathfrak{A} \tag{33}$$

if and only if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle|\; d\mu(g) \;\forall\; x, y \in H. \tag{34}$$

(c) If $G = \mathbb{Z}$, then

$$\lim_{n\to\infty} \omega(A\tau_n(B)) = \omega(A)\omega(B) \quad \forall\; A, B \in \mathfrak{A} \tag{35}$$

if and only if

$$\lim_{n\to\infty} \langle x, U_n y\rangle = \langle x, \Omega\rangle\langle \Omega, y\rangle \quad \forall\; x, y \in H. \tag{36}$$

PROOF. (a) Suppose (31) holds and consider any $x, y \in H$. Let $\epsilon > 0$ be arbitrary. Since $\{\pi(A)\Omega : A \in \mathfrak{A}\}$ is dense in H there are sequences $(A_k), (B_k) \subseteq \mathfrak{A}$ such that $x_k := \pi(A_k)\Omega \longrightarrow x$, $y_k := \pi(B_k)\Omega \longrightarrow y$. Therefore by the continuity of the inner product, and since $\|U_g\| = 1$ there is a $K_1 \in \mathbb{N}$ such that $|\langle x, U_g y\rangle - \langle x_k, U_g y_k\rangle| < \frac{\epsilon}{3}$ for all $k \geq K_1$

and all $g \in G$. Therefore

$$\begin{aligned}&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle x, U_g y\rangle\ d\mu(g) - \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle x_k, U_g y_k\rangle\ d\mu(g)\right|\\ <&\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\frac{\epsilon}{3}\ d\mu(g)\\ =&\frac{\epsilon}{3}\end{aligned}$$

for all $n \in \mathbb{N}$ and $k \geq K_1$. Also, as $\langle x_k, \Omega\rangle \longrightarrow \langle x, \Omega\rangle$ and $\langle \Omega, y_k\rangle \longrightarrow \langle \Omega, y\rangle$, there is a $K_2 \in \mathbb{N}$ such that

$$|\langle x_k, \Omega\rangle\langle \Omega, y_k\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle| < \frac{\epsilon}{3}$$

for all $k \geq K_2$. Let $K = \max\{K_1, K_2\}$. By (31) there is an $N \in \mathbb{N}$ such that

$$\begin{aligned}&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle x_K, U_g y_K\rangle\ d\mu(g) - \langle x_K, \Omega\rangle\langle \Omega, y_K\rangle\right|\\ =&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle \pi(A_K)\Omega, U_g\pi(B_K)\Omega\rangle\ d\mu(g) - \langle \pi(A_K)\Omega, \Omega\rangle\langle \Omega, \pi(B_K)\Omega\rangle\right|\\ =&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle \Omega, \pi(A_K)^*\pi(\tau_g(B_K))\Omega\rangle\ d\mu(g) - \langle \Omega, \pi(A_K)^*\Omega\rangle\langle \Omega, \pi(B_K)\Omega\rangle\right|\\ =&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle \Omega, \pi(A_K^*\tau_g(B_K))\Omega\rangle\ d\mu(g) - \langle \Omega, \pi(A_K^*)\Omega\rangle\langle \Omega, \pi(B_K)\Omega\rangle\right|\\ =&\left|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega(A_K^*\tau_g(B_K))\ d\mu(g) - \omega(A_K^*)\omega(B_K)\right|\\ <&\ \frac{\epsilon}{3}\end{aligned}$$

for all $n \geq N$. It now follows that

$$\begin{aligned}&\left|\frac{1}{\Lambda_n}\int_{\Lambda_n}\langle x, U_g y\rangle\ d\mu(g) - \langle x, \Omega\rangle\langle \Omega, y\rangle\right|\\ \leq&\ \left|\frac{1}{\Lambda_n}\int_{\Lambda_n}\langle x, U_g y\rangle\ d\mu(g) - \frac{1}{\Lambda_n}\int_{\Lambda_n}\langle x_K, U_g y_K\rangle\ d\mu(g)\right|\\ +&\ \left|\frac{1}{\Lambda_n}\int_{\Lambda_n}\langle x_K, U_g y_K\rangle\ d\mu(g) - \langle x_K, \Omega\rangle\langle \Omega, y_K\rangle\right|\\ +&\ |\langle x_K, \Omega\rangle\langle \Omega, y_K\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle|\\ \leq&\ \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\\ =&\ \epsilon\end{aligned}$$

for all $n \geq N$. Hence, as $\epsilon > 0$ was arbitrary,

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle x, U_g y, \rangle\ d\mu(g) = \langle x, \Omega\rangle\langle \Omega, y\rangle$$

and as $x, y \in H$ was arbitrary (32) follows.

Conversely, suppose that (32) holds and consider any $A, B \in \mathfrak{A}$. Set $x = \pi(A^*)\Omega$ and $y = \pi(B)\Omega$. Then, by (32)

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle x, U_g y, \rangle \; d\mu(g) = \langle x, \Omega\rangle\langle \Omega, y\rangle$$

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle \Omega, \pi(A)U_g\pi(B)\Omega, \rangle \; d\mu(g) = \langle \Omega, \pi(A)\Omega\rangle\langle \Omega, \pi(B)\Omega\rangle$$

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B_K)) \; d\mu(g) = \omega(A)\omega(B).$$

As $A, B \in \mathfrak{A}$ was arbitrary, (31) follows.

(b) The proof of (33) $\Rightarrow$ (34) is as for (31) $\Rightarrow$ (32), except we use the following instead:

$$\begin{aligned}
&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle| \; d\mu(g) \\
\leq\; &\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x_k, U_g y_k\rangle| d\mu(g) \\
&+ \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x_k, U_g y_k\rangle - \langle x_k, \Omega\rangle\langle \Omega, y_k\rangle| \; d\mu(g) \\
&+ \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x_k, \Omega\rangle\langle \Omega, y_k\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle| \; d\mu(g) \\
\leq\; &\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

The proof of (34) $\Rightarrow$ (33) is overly similar to that of (32) $\Rightarrow$ (31).

(c) We do not include any technical details as, again, (c) follows by arguments wholly similar to part (a), modified for the simpler case of $G = \mathbb{Z}$ and the absence of integrals. □

To characterize compactness on the Hilbert space H of the GNS representation we first have to understand how the GNS representation connects totally bounded orbits in the C^*-algebra, with totally bounded orbits in H, from which it will then be a simple task to derive the characterization of compactness on H.

PROPOSITION 3.2. *Let $\mathfrak{A}$ be a C^*-algebra with state ω, and let (H, π, Ω) be a cyclic representation of $(\mathfrak{A}, \omega)$. Then for any $V \subseteq \mathfrak{A}$, V is totally bounded in $(\mathfrak{A}, \|\cdot\|_\omega)$, if and only if $\pi(V)\Omega$ is totally bounded in H, if and only if $\overline{\pi(V)\Omega}$ is compact in H.*

PROOF. Consider any $V \subseteq \mathfrak{A}$ and suppose that V is totally bounded in $(\mathfrak{A}, \omega)$. Then, for any $\epsilon > 0$ there is a finite set $M_\epsilon \subseteq \mathfrak{A}$ such that, for any $A \in V$, there is a $B \in M_\epsilon$ such that

$$\begin{aligned} \epsilon > \|A - B\|_\omega &= \sqrt{\omega((A-B)^*(A-B))} \\ &= \sqrt{\langle \Omega, \pi((A-B)^*(A-B))\Omega \rangle} \\ &= \|\pi(A)\Omega - \pi(B)\Omega\|. \end{aligned} \tag{37}$$

Hence, for any $x \in \pi(V)\Omega$, there is a y in the finite set $\pi(M_\epsilon)\Omega$ such that $\|x - y\| < \epsilon$. As $\epsilon > 0$ was arbitrary, it follows that $\pi(V)\Omega$ is totally bounded in H.

Conversely, consider any $V \subseteq \mathfrak{A}$ and suppose that $\pi(V)\Omega$ is totally bounded in H. Then, for any $\epsilon > 0$ there is a finite set $M_H = \{x_1, ..., x_n\} \subseteq H$ such that, for any $A \in V$, there is a $y \in M$ such that

$$\|\pi(A)\Omega - y\| < \frac{\epsilon}{2}. \tag{38}$$

We know that $\pi(\mathfrak{A})\Omega$ is dense in H so we can find a set $M_{\mathfrak{A}} = \{B_1, ..., B_n\}$ such that $\|x_i - \pi(B_i)\Omega\| < \frac{\epsilon}{2}$. Hence, if $y = x_j$ in (38) for some $j = 1, ..., n$, then

$$\begin{aligned} \|\pi(A)\Omega - \pi(B_j)\Omega\| &\leq \|\pi(A)\Omega - x_j\| + \|x_j - \pi(B_j)\Omega\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

or, by (37), $\|A - B_j\|_\omega < \epsilon$. As $\epsilon > 0$ was arbitrary, it follows that V is totally bounded in $(\mathfrak{A}, \|\cdot\|_\omega)$.

The final "if and only if" follows from the fact that any set in a complete metric space is totally bounded if and only if the set is conditionally compact, by [**20**, Lemma 8.2-2]. □

Comparing Proposition 3.2 to Definition 2.4 we immediately obtain the following corollary:

COROLLARY 3.3. *Let $(\mathfrak{A}, \omega, \tau, G)$ be a C^*-dynamical system with GNS representation (H, π, Ω, U). Then $(\mathfrak{A}, \omega, \tau, G)$ is compact if and only if, for all $A \in \mathfrak{A}$, the orbits $\{U_g \pi(A)\Omega : g \in G\}$ are totally bounded in H, which in turn holds if and only if said orbits have compact closures in H.*

As promised in the introduction, Corollary 3.3 indicates why *compact* C^*-dynamical systems are referred to as such.

4. Spectral characterization of ergodic properties

In this section we will see how to characterize ergodic, weak mixing and compact C^*-dynamical systems in terms of the fixed point space and eigenspace of the GNS representation. These characterizations, together with the characterizations in Section 2.3, will become our main tools in identifying these ergodic properties on C^*-dynamical systems. The results of Chapter 1 will be used in the case of weak mixing and compactness, while ergodicity is more elementary to handle, given the mean ergodic theorem.

Whether or not a C^*-dynamical systems is ergodic, is a question that can be translated in terms of the dimension of the fixed point space of the GNS representation, i.e. $\{x \in H : U_g x = x \ \forall\ g \in G\}$. The mean ergodic theorem plays the role of the translator. [1] The form of the the mean ergodic theorem that will be of greatest use is stated in terms of a Gelfand integral, which provides a way of integrating a function with values in a Hilbert space. Consider a function $f : G \to H$ where G is a locally compact group with right Haar measure μ, and H a Hilbert space, such that $G \to \mathbb{C} : g \mapsto \langle f(g), x\rangle$ is Borel measurable for every $x \in H$. If Λ is a Borel set with $\mu(\Lambda) < \infty$ and f is bounded then

$$H \to \mathbb{C} : x \mapsto \int_\Lambda \langle f(g), x\rangle \ d\mu(g)$$

is a bounded linear functional on H. Using the Riesz-representation theorem for bounded linear functionals on Hilbert spaces, we then define $\int_\Lambda f \ d\mu \in H$ by

$$\left\langle \int_\Lambda f \ d\mu, x \right\rangle = \int_\Lambda \langle f(g), x\rangle \ d\mu(g) \ \ \forall\ x \in H. \tag{39}$$

$\int_\Lambda f \ d\mu$ is referred to as a Gelfand integral (see [**7**, p. 53,p. 58]). Note that (39) is equivalent to

$$\left\langle x, \int_\Lambda f \ d\mu \right\rangle = \int_\Lambda \langle x, f(g)\rangle \ d\mu(g) \ \ \forall\ x \in H \tag{40}$$

THEOREM 4.1. *(The mean ergodic theorem) Let G be a locally compact group with right Haar measure μ, and H a Hilbert space. Let $U : G \to \mathscr{L}(H) : g \mapsto U_g$ be such that $\|U_g\| \leq 1$, $U_g U_h = U_{gh}$ for all $g, h \in G$, and $G \to \mathbb{C} : g \mapsto \langle U_g x, y\rangle$ is Borel measurable for all $x, y \in H$. Take P to be the projection of H onto $V = \{x \in H : U_g x = x$ for all $g \in G\}$. For any Følner sequence (Λ_n) in G we then have*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} U_g x \ d\mu(g) = Px$$

for all $x \in H$.

A detailed proof of this particular version of the mean ergodic theorem can be found in [**10**, The mean ergodic theorem, p. 68] where it is proved in a slightly more general form.

[1]The mean ergodic Theorem is akin to a horse. Horses are found all over the world but their appearance always differ slightly, typically due to the different roles in society that they have been bred for.

THEOREM 4.2. *Let* $(\mathfrak{A}, \omega, \tau, G)$ *be a* C^**-dynamical system with GNS representation* (H, π, Ω, U). *If* P *is the projection of* H *onto the closed subspace* $V = \{x \in H : U_g x = x \text{ for all } g \in G\} \subseteq H$ *then* $(\mathfrak{A}, \omega, \tau, G)$ *is ergodic if and only if* $P = \Omega \otimes \Omega$, *where* $(\Omega \otimes \Omega)x := \Omega\langle \Omega, x\rangle$ *for all* $x \in H$.

PROOF. The mean ergodic Theorem is applicable to U. We know that $U_g U_h = U_{gh}$ and since the operators U_g are unitary, $\|U_g\| = 1$ for all g in the Abelian locally compact amenable group G. The mapping $G \to \mathbb{C} : g \mapsto \langle U_g x, y\rangle$ is continuous for all $x, y \in H$ by definition 1.6.

Suppose that $\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g)$ exists. From the mean ergodic theorem it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g) \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle \pi(A^*)\Omega, U_g \pi(B)\Omega\rangle\, d\mu(g) \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \left\langle \pi(A^*)\Omega, \int_{\Lambda_n} U_g \pi(B)\Omega\, d\mu(g) \right\rangle \qquad (41) \\
&= \langle \pi(A^*)\Omega, P\pi(B)\Omega\rangle\ ,
\end{aligned}$$

where (41) follows from (40), so in particular the limit exists.

Now, assume that $(\mathfrak{A}, \omega, \tau, G)$ is ergodic. Then, from Definition 2.1 it follows that, for all $A, B \in \mathfrak{A}$

$$\begin{aligned}
\langle \pi(A^*)\Omega, P(\pi(B)\Omega)\rangle &= \omega(A)\omega(B) \\
&= \langle \Omega, \pi(A)\Omega\rangle \langle \Omega, \pi(B)\Omega\rangle \\
&= \langle \pi(A^*)\Omega, \langle \Omega, \pi(B)\Omega\rangle \Omega\rangle \\
&= \langle \pi(A^*)\Omega, \Omega \otimes \Omega(\pi(B)\Omega)\rangle
\end{aligned}$$

Therefore, since $\overline{\{\pi(A)\Omega : A \in \mathfrak{A}\}} = H$, it follows that $P = \Omega \otimes \Omega$.

Conversely, suppose that $P = \Omega \otimes \Omega$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g) &= \langle \pi(A^*)\Omega, \Omega \otimes \Omega(\pi(B)\Omega)\rangle \\
&= \langle \pi(A^*)\Omega, \langle \Omega, \pi(B)\Omega\rangle \Omega\rangle \\
&= \langle \Omega, \pi(A)\Omega\rangle \langle \Omega, \pi(B)\Omega\rangle \\
&= \omega(A)\omega(B)
\end{aligned}$$

Therefore $(\mathfrak{A}, \omega, \tau, G)$ is ergodic. □

Theorem 4.2 states that, if the C^*-dynamical system is ergodic, the fixed point space of its GNS representation is the one-dimensional space $\mathbb{C}\Omega$. Conversely, if the fixed point space V of the GNS representation of a C^*-dynamical system is one-dimensional, then $V = \mathbb{C}\Omega$ since, necessarily, $\Omega \in V$ by Proposition 1.4. And if $V = \mathbb{C}\Omega$ then the projection operator is clearly $\Omega \otimes \Omega$. We state this as a corollary to Theorem 4.2.

Corollary 4.3. *Let $(\mathfrak{A}, \omega, \tau, G)$ be a C^*-dynamical system with GNS representation (H, π, Ω, U). Then $(\mathfrak{A}, \omega, \tau, G)$ is ergodic if and only if the fixed point space of U is one-dimensional*

The above characterization of ergodicity is remarkable for at least two reasons. First, it is significantly simpler than our original definition, and secondly, it answers a question that we have been deliberately avoiding up to this point. Our original definition states that a C^*-dynamical system is ergodic if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(A\tau_g(B))\, d\mu(g) = \omega(A)\omega(B) \tag{42}$$

for all $A, B \in \mathfrak{A}$ and some Følner sequence (Λ_n). The question we avoided was: If (42) holds/fails for some Følner sequence, does it also hold/fail for any other Følner sequence? The answer, of course, determines whether or not we should include the term "ergodic *with respect to the Følner sequence* (Λ_n)" in the definition of ergodicity. Corollary 4.3 informs us that this is not necessary, as ergodicity can be completely characterized by the GNS representation which has no connection whatever to Følner sequences. In other words, we can conclude that if (42) holds(fails) for some Følner sequence, it also holds(fails) for any other Følner sequence.

As was the case with ergodicity, whether or not a C^*-dynamical system is weak mixing, is a question that can be translated in terms of its GNS representation, not in terms of its fixed point space, but in terms of its eigenspace, which we define shortly. Theorem 2.18 in Chapter 1 will play the role of the translator this time round. Since Theorem 2.18 pertains to weakly almost periodic semigroups in $\mathscr{L}(X)$, with X a Banach space, we start by identifying such semigroups in our C^*-dynamical system.

Proposition 4.4. *Let $\mathscr{S}$ be an Abelian semigroup of contractions on a Hilbert space H. Then $\mathscr{S}$ is an Abelian weakly almost periodic semigroup in $\mathscr{L}(H)$.*

Proof. We have to show that

$$\text{w-cl }(\mathscr{S}x) = \text{w-cl }\{Ux : U \in \mathscr{S}\} \text{ is w-compact in } H$$

for all $x \in H$, so let $x \in H$ be arbitrary. Since each $U \in \mathscr{S}$ is a contraction, $\mathscr{S}x \subseteq \{z \in H : \|z\| \leq \|x\|\}$, that is, $\mathscr{S}x$ is contained in the closed $\|x\|$-ball $B_{\|x\|}$ in H. It therefore suffices to show that $B_{\|x\|}$ is w-compact and w-closed since then w-cl $(\mathscr{S}x) \subseteq$ w-cl $(B_{\|x\|}) = B_{\|x\|}$, from which it would follow that w-cl $(\mathscr{S}x)$ is w-compact. Since a Hilbert space is reflexive we have that B_1, the closed unit ball in H, is w-compact by Banach-Alaoglu [**8**, V.4.7]. As $B_{\|x\|}$ is the image of B_1 under the w-continuous mapping $H \to H : z \mapsto \|x\|z$, it follows that $B_{\|x\|}$ is w-compact. Finally, $B_{\|x\|}$ is w-closed since the w-topology is Hausdorff and compact sets in a Hausdorff topology are closed. Thus the result follows. □

Proposition 4.4 reveals that, for a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$, if U is the GNS representation of τ, then $\{U_g : g \in G\}$ is an Abelian weakly almost periodic semigroup. Of course in this case $\{U_g : g \in G\}$ is in fact a group. We now define, as per Definition 2.6, H_{fl} to be the collection of all flight vectors, and H_{rev} to be the collection of all reversible vectors, relative to $\mathscr{S} = \{U_g : g \in G\}$. As per Definition 2.9, we define H_{uds} to be the norm closure of the span of all eigenvectors of $\mathscr{S} = \{U_g : g \in G\}$. We will refer to H_{uds} as the eigenspace of the GNS representation.

PROPOSITION 4.5. *For any Abelian weakly almost periodic semigroup $\mathscr{S}$ in $\mathscr{L}(H)$ that is closed under the taking of adjoints, $H_{fl}^{\perp} = H_{rev}$.*

PROOF. Consider any $x \in H_{fl}$ and any eigenvector y of $\mathscr{S} = \{U_g : g \in G\}$ with unimodular eigenvalues. Thus $Ax = 0$ for some $A \in \overline{\mathscr{S}}$ and since $\mathscr{S}^* = \mathscr{S}$, for any net (A_α) such that wo-lim $A_\alpha = A$, it is clear from Theorem 1.3 (a) that wo-lim $A_\alpha^* = A^* \in \overline{\mathscr{S}}$. It now follows by Proposition 2.11 that $A^* y = cy$ for some $c \in \mathbb{S}^1$ so that

$$\langle x, y \rangle = \frac{1}{c}\langle x, A^* y \rangle = \frac{1}{c}\langle Ax, y \rangle = 0.$$

Thus by the linearity of the inner product, x is perpendicular to any element in the span of all eigenvectors with unimodular eigenvalues, so that by the continuity of the inner product, $x \perp y$ for all $y \in H_{uds}$. By Theorem 2.13 we thus have that $x \perp y$ for all $y \in H_{rev}$ so that since $H = H_{fl} \oplus H_{rev}$, we must have that $H_{fl}^{\perp} = H_{rev}$. □

Now that we know where Abelian weakly almost periodic groups are to be found in our C^*-dynamical context, we restrict the final result, Theorem 2.18, to Hilbert spaces and to a specific invariant mean. This is done in the proof of the next result.

THEOREM 4.6. *Consider a representation $U : G \to \mathscr{L}(H) : g \mapsto U_g$, of an Abelian locally compact group G, as contractions on any Hilbert space H. Assume that $G \to \mathbb{C} : g \mapsto \langle x, U_g y \rangle$ is Borel measurable, that G possesses a Følner-sequence (Λ_n) and that the Hilbert space spanned by $\{U_g x : g \in G\}$ is separable for every $x \in H$. Denote a Haar measure on G by μ. Then, for any $y \in H$ we have that: $y \in H_{fl}$, i.e. y is a flight vector, if and only if*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y \rangle| \, d\mu(g) = 0 \;\forall\; x \in H$$

PROOF. By Proposition 4.4 $\mathscr{S} = \{U_g : g \in G\}$ is an Abelian weakly almost periodic semigroup in $\mathscr{L}(H)$, and by assumption the Hilbert space spanned by $\mathscr{S} = \{U_g x : g \in G\}$ is separable for all $x \in H$. Therefore the proposition follows as a special case of Theorem 2.18 provided we prove that

(a) $L : \mathscr{G}' \to \mathbb{C} : f \mapsto \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g)$ is an invariant mean on $\mathscr{G}' \subseteq \mathscr{G} = \operatorname{span}(\{h_{x,y} : x, y \in H\} \cup \{1\})$ where $h_{x,y} : \mathscr{S} \to \mathbb{C} : U_g \mapsto |\langle x, U_g y\rangle|$, $1 : \mathscr{S} \to \mathbb{C} : U_g \mapsto 1$ and $\mathscr{G}'$ is simply the subspace of all $f \in \mathscr{G}$ for which the limit $L(f)$ exists.

(b) $\mathscr{G}'$ contains each element of $\mathscr{G}$ which has zero average.

(a) It is clear that $\mathscr{G}$, and therefore $\mathscr{G}'$, consist of bounded complex valued functions on $\mathscr{S}$, and that $1 \in \mathscr{G}'$ since

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} 1(U_g)\ d\mu(g) = \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} 1\ d\mu(g) = 1. \tag{43}$$

It is clear that $L(f) \geq 0$ for any $f \in \mathscr{G}'$ with $f \geq 0$ and by (43) that $L(1) = 1$. For $L : \mathscr{G}' \to \mathbb{C}$ to be an invariant mean, it remains to show that, for any $U_h \in \mathscr{S}$ and $f \in \mathscr{G}'$, $L(f) = L(f \circ \overline{U}_h)$. This can be shown from the invariance of μ and that (Λ_n) is a Følner sequence. For any $T = U_h \in \mathscr{S}$ and $f \in \mathscr{G}'$, it follows that

$$\begin{aligned}
&\left|\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \circ \overline{T}(U_g)\ d\mu(g)\right| \\
=&\left|\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_{gh})\ d\mu(g)\right| \\
=&\left|\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h} f(U_g)\ d\mu(g)\right| \\
=&\left|\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \setminus (\Lambda_n \cap \Lambda_n h)} f(U_g)\ d\mu(g) - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h \setminus (\Lambda_n \cap \Lambda_n h)} f(U_g)\ d\mu(g)\right| \\
\leq&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \setminus (\Lambda_n \cap \Lambda_n h)} |f(U_g)|\ d\mu(g) + \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h \setminus (\Lambda_n \cap \Lambda_n h)} |f(U_g)|\ d\mu(g) \\
=&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \triangle \Lambda_n h} |f(U_g)|\ d\mu(g) \\
\leq&\frac{\mu(\Lambda_n \triangle \Lambda_n h)}{\mu(\Lambda_n)} \sup_{U_g \in \mathscr{S}} |f(U_g)|
\end{aligned}$$

which, since (Λ_n) is a Følner sequence, is arbitrarily small for large enough n. Therefore, we can conclude that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \circ \overline{T}(U_g)\ d\mu(g) = \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g)$$

and in particular that this limit exists. In other words, $f \circ \overline{T} \in \mathscr{G}'$ and $L(f \circ \overline{T}) = L(f)$. Therefore, as $T \in \mathscr{S}$ and $f \in \mathscr{G}'$ was arbitrary, the final condition for L to be an invariant mean has been met.

(b) Similar to (a), to prove (b), we use the facts that that (Λ_n) is a Følner sequence and that μ is a Haar measure. Let $U_h \in \mathscr{S}, f \in \mathscr{G}, \epsilon > 0$ be arbitrary and set $M = \sup_{g \in G} |f(U_g)|$. It follows from the inequality above that there is an $N \in \mathbb{C}$ such that

$$\left| \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h} f(U_g)\ d\mu(g) \right| < \frac{\epsilon}{2} \tag{44}$$

for all $n \geq N$.

Now assume that f has zero average. If we let $\epsilon > 0$ be arbitrary then there are $g_1, ..., g_m \in G$ and $\alpha_1, .., \alpha_m > 0$, with $\sum_{i=1}^m \alpha_i = 1$, such that

$$\left| \sum_{i=1}^m \alpha_i f(U_g U_{g_i}) \right| < \frac{\epsilon}{2} \quad \forall\ g \in G. \tag{45}$$

We have from (44) that there are $z_1, ..., z_m \in \mathbb{C}$ with $|z_i| < \frac{\epsilon}{2}$ such that

$$\begin{aligned}
&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) \\
=&\sum_{i=1}^m \alpha_i \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) \\
=&\sum_{i=1}^m \alpha_i \left(\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n g_i} f(U_g)\ d\mu(g) + z_i \right) \\
=&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \sum_{i=1}^m \alpha_i f(U_g U_{g_i})\ d\mu(g) + \sum_{i=1}^m \alpha_i z_i
\end{aligned}$$

for $n \geq N$, for some $N \in \mathbb{N}$. Note that

$$\left| \sum_{i=1}^m \alpha_i z_i \right| \leq \sum_{i=1}^m \alpha_i |z_i| < \sum_{i=1}^m \alpha_i \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

Hence, since by (45)

$$\left| \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \sum_{i=1}^m \alpha_i f(U_g U_{g_i})\ d\mu(g) \right| < \frac{\mu(\Lambda_n)}{\mu(\Lambda_n)} \frac{\epsilon}{2}$$

we have that $\left| \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $n \geq N$. As $\epsilon > 0$ was arbitrary, $\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(U_g)\ d\mu(g) = 0$, which, since the limit exists, shows that $f \in \mathscr{G}'$. □

As a fairly straightforward consequence of Theorem 4.6 and Proposition 4.5 we have the following:

PROPOSITION 4.7. *Let* $(\mathfrak{A}, \omega, \tau, G)$ *be a* C^**-dynamical system with GNS representation* (H, π, Ω, U)*. Then* $(\mathfrak{A}, \omega, \tau, G)$ *is weakly mixing if and only if* $\dim H_{rev} = 1$*, i.e. if and only if* $\dim H_{uds} = 1$.

PROOF. Since $U_g\Omega = \Omega$ for all $g \in G$, it follows from Proposition 1.4 that

$$\mathbb{C}\Omega \subseteq H_{uds}$$

and therefore

$$H_{uds}^{\perp} \subseteq \mathbb{C}\Omega^{\perp}. \tag{46}$$

Assume first that $(\mathfrak{A}, \omega, \tau, G)$ is weakly mixing and consider any $y \in H$ perpendicular to Ω, i.e. such that $\langle \Omega, y \rangle = 0$. Then it follows by Proposition 3.1 that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle| \, d\mu(g) \\ &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle| \, d\mu(g) \end{aligned}$$

for all $x \in H$. By Proposition 1.7 we can apply Theorem 4.6, from which it thus follows that $y \in H_{fl}$. Thus we have that

$$H_{uds}^{\perp} \subseteq \mathbb{C}\Omega^{\perp} \subseteq H_{fl}$$

and since $H_{fl}^{\perp} = H_{rev} = H_{uds}$ by Proposition 4.5 and Theorem 2.13, we have that $\mathbb{C}\Omega = H_{uds}$.

Now assume that H_{uds} is one dimensional. Since $\mathbb{C}\Omega \subseteq H_{uds}$, we thus have that $H_{uds} = \mathbb{C}\Omega$. Let $x, y \in H$ be arbitrary. Then, by Theorems 2.8 and 2.13, $x = x_0 + x_1$ and $y = y_0 + y_1$ for some $x_0, y_0 \in H_{uds}$ and $x_1, y_1 \in H_{fl}$. Since $x_0, y_0 \in \mathbb{C}\Omega$, $U_g x_0 = x_0$ and $U_g y_0 = y_0$ for all $g \in G$, by Proposition 1.4. Since $H_{fl} \perp H_{uds}$ and $\Omega \in H_{uds}$, we also have that $\langle x_1, \Omega\rangle = \langle \Omega, y_1\rangle = 0$. It now follows by 4.6 that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle \Omega, y\rangle| \, d\mu(g) \\ &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y_1\rangle + \langle x, U_g y_0\rangle - \langle x_0, \Omega\rangle\langle \Omega, y_0\rangle| \, d\mu(g) \\ &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y_1\rangle + \langle x_0 + x_1, y_0\rangle - \langle x_0, \langle \Omega, y_0\rangle\Omega\rangle| \, d\mu(g) \\ &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y_1\rangle + \langle x_0, y_0\rangle - \langle x_0, y_0\rangle| \, d\mu(g) \\ &= 0. \end{aligned}$$

Thus $(\mathfrak{A}, \omega, \tau, G)$ is weakly mixing by Proposition 3.1 (b). □

Note how Proposition 4.7 implies that if (33) holds for one Følner sequence it also holds for any other as H_{uds} has no connection with any Følner sequence. Thus, similar to ergodicity, we do not have to include the term "weakly mixing *with respect to a Følner sequence* (Λ_n)" in the definition of a weak-mixing system.

We know that $U_g\Omega = \Omega$ always, so if a C^*-dynamical system is weak mixing, the one dimensional eigenspace of its GNS representation is given by $\mathbb{C}\Omega$.

Again applying the results of Chapter 1, we will now see how a compact C^*-dynamical system can also be characterized in terms of the eigenspace of its GNS representation. We start by taking a closer look at the notion of total boundedness and find that it has two easily verifiable properties.

LEMMA 4.8. *If A, B are two totally bounded sets in a semi normed space, then for any $\alpha, \beta \in \mathbb{C}$, $\alpha A + \beta B$ is also totally bounded.*

The proof of Lemma 4.8 is extremely elementary and is therefore not included.

PROPOSITION 4.9. *Consider a group $\mathscr{S}$ of unitary operators on a Hilbert space H. Then H_{uds}, for $\mathscr{S}$, is precisely the set of all elements $x \in H$ whose orbits $\mathscr{S}x$ are totally bounded.*

PROOF. Define the set

$$H_{tb} = \{x \in H : \mathscr{S}x \text{ is totally bounded}\}.$$

By lemma 4.8, H_{tb} is a vector space. Let E be the set of eigenvectors of $\mathscr{S}$ with unimodular eigenvalues, or simply the set of all eigenvectors of $\mathscr{S}$ since all eigenvectors of a unitary operator necessarily have unimodular eigenvalues. By definition, H_{uds} is the norm closure of span E, so we therefore have to show that $\overline{\text{span}\, E} = H_{tb}$. We can easily show that span $E \subseteq H_{tb}$:

Consider any $x \in E$. Since, for all $U_g \in \mathscr{S}$, $U_g x = c_g x$ for some $c_g \in \mathbb{S}^1$ we have that $Sx \subseteq \{cx : c \in \mathbb{S}^1\}$ which is a totally bounded set in H since it is compact. Thus $E \subseteq H_{tb}$ and, by Lemma 4.8, span $E \subseteq H_{tb}$.

Consider any sequence (x_n) in H_{tb} with $x_n \longrightarrow x$ for some $x \in H$. Let $\epsilon > 0$ be arbitrary. There is an $N \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\epsilon}{2}$ for all $n \geq N$. Since $\mathscr{S}x_N$ is totally bounded there is a finite set $M \subseteq H$ such that for any $U \in \mathscr{S}$

$$\|U_g x_N - z\| < \frac{\epsilon}{2}$$

for some $z \in M$, and therefore

$$\begin{aligned}\|U_g x - z\| &\leq \|U_g x - U_g x_N\| + \|U_g x_N - z\| \\ &= \|U_g(x - x_N)\| + \|U_g x_N - z\| \\ &= \|x - x_N\| + \|U_g x_N - z\| \\ &< \epsilon.\end{aligned}$$

Thus $x \in H_{tb}$ and hence H_{tb} is closed, from which we have that $H_{uds} = \overline{\text{span } E} \subseteq H_{tb}$

To prove that $H_{tb} \subseteq H_{uds}$, let $x \in H_{tb}$ be arbitrary. By Proposition 4.4 $\mathscr{S}$ is a weakly almost periodic semigroup in $\mathscr{L}(H)$, so that by Theorems 2.8 and 2.13, $x = x_0 + x_{fl}$ for some $x_0 \in H_{uds}$ and $x_{fl} \in H_{fl}$. Since $x \in H_{tb}$ and $x_0 \in H_{tb}$, from the first part, $x_{fl} = x - x_0 \in H_{tb}$ follows from Lemma 4.8. Therefore, for all $1 > \epsilon > 0$ there is a finite set $M_\epsilon \subseteq H$ such that, for any $R \in \mathscr{S}$, $\|Rx_{fl} - z\| < \epsilon$ for some $z \in M_\epsilon$

Since $x_{fl} \in H_{fl}$, $Tx_{fl} = 0$ for some $T \in$ wo-cl $\mathscr{S}$ by Definition 2.6. Thus wo-lim $T_\alpha = T$ for some net $(T_\alpha)_{\alpha \in \Lambda}$ in $\mathscr{S}$. By Theorem 1.3, and Riesz's representation Theorem for bounded linear functionals on a Hilbert space,

$$\langle z, T_\alpha x_{fl}\rangle \longrightarrow \langle z, Tx_{fl}\rangle = 0$$

for all $z \in H$. In particular, for all z in the *finite* set M_ϵ, we obtain

$$|\langle z, T_\alpha x_{fl}\rangle| < \epsilon \tag{47}$$

for all $z \in M_\epsilon$ and all $\alpha \succeq \beta$, for some $\beta \in \Lambda$. As mentioned above, $\|T_\beta x_{fl} - z'\| < \epsilon$ for some $z' \in M_\epsilon$. It now follows that

$$\begin{aligned}|\langle T_\beta x_{fl}, T_\beta x_{fl}\rangle| - |\langle T_\beta x_{fl}, z'\rangle| &\leq |\langle T_\beta x_{fl}, T_\beta x_{fl} - z'\rangle| \\ &\leq \|T_\beta x_{fl}\|\|T_\beta x_{fl} - z'\| \\ &\leq \|x_{fl}\|\epsilon\end{aligned}$$

which gives

$$\|x_{fl}\|^2 = \|T_\beta x_{fl}\|^2 \leq |\langle T_\beta x_{fl}, z'\rangle| + \epsilon\|x_{fl}\| < \epsilon + \epsilon\|x_{fl}\|$$

by (47). As $\epsilon > 0$ was arbitrary we must have that $\|x_{fl}\| = 0$. Thus $x = x_0 + x_{fl} = x_0 \in H_{uds}$ which completes the proof. □

THEOREM 4.10. *Let $(\mathfrak{A}, \omega, \tau, G)$ be a C^*-dynamical system with GNS representation (H, π, Ω, U). Then $(\mathfrak{A}, \omega, \tau, G)$ is compact if and only if $H_{uds} = H$.*

PROOF. Suppose that $(\mathfrak{A}, \omega, \tau, G)$ is compact. Then, by Proposition 3.2 and Corollary 3.3, $\{U_g \pi(A)\Omega : g \in G\}$ is totally bounded for all $A \in \mathfrak{A}$. By Proposition 4.9, with $\mathscr{S} = \{U_g : g \in G\}$, $\pi(\mathfrak{A})\Omega \subseteq H_{uds}$. Therefore, as $\overline{\pi(\mathfrak{A})\Omega} = H$ and H_{uds} is closed, it follows that $H = H_{uds}$

Conversely, if $H = H_{uds}$, then since $\pi(\mathfrak{A})\Omega \subseteq H$ and H_{uds} is the set of all elements $x \in H$ with totally bounded orbits in $\mathscr{S}$ by Proposition 4.9, it follows that

$$\{U_g \pi(A)\Omega : g \in G\}$$

is totally bounded in H for all $A \in \mathfrak{A}$. Thus $(\mathfrak{A}, \omega, \tau, G)$ is compact by Corollary 3.3. □

5. Interrelationships of ergodic properties

In this section we investigate the relationships between the ergodic properties, using the various characterizations developed in the preceding sections. In particular this will tell us which combinations of ergodic properties are impossible.

THEOREM 5.1. *If a C^*-dynamical system is weakly mixing, then it is ergodic.*

PROOF. Suppose that a C^*-dynamical system is weak mixing. Then the eigenspace of its GNS representation is one dimensional by Proposition 4.7. Thus the fixed point space of the GNS representation, which is contained in the eigenspace, is also one dimensional. Therefore the C^*-dynamical system is ergodic by Corollary 4.3. □

THEOREM 5.2. *If a C^*-dynamical system is strongly mixing, then it is weakly mixing.*

PROOF. Consider a C^*-dynamical system $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$. If $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ is strongly mixing then

$$\lim_{n\to\infty} \omega(A\tau^n(B)) = \omega(A)\omega(B)$$

for any $A, B \in \mathfrak{A}$. Thus, for any $\epsilon > 0$ there is a $K \in \mathbb{N}$ such that

$$|\omega(A\tau^n(B)) - \omega(A)\omega(B)| < \epsilon$$

for all $n > K$. For each $n \in \mathbb{N}$, let $\Lambda_n = \{1, ..., n\}$. Then (Λ_n) is a Følner sequence of $\mathbb{Z}$ relative to the counting measure μ on $\mathbb{Z}$. Suppose that

$$\sum_{j=1}^{K} \left|\omega(A\tau^j(B)) - \omega(A)\omega(B)\right| = \delta$$

and select an $M \in \mathbb{N}$ such that $\frac{\delta}{M} < \epsilon$. Let $N = \max\{K, M\}$. Then

$$\begin{aligned}
&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(A\tau^n(B)) - \omega(A)\omega(B)| \; d\mu \\
&= \frac{1}{n}\left(\delta + \sum_{j=K+1}^{n} \left|\omega(A\tau^j(B)) - \omega(A)\omega(B)\right|\right) \\
&< \epsilon + \frac{(n-K)\epsilon}{n} \\
&< 2\epsilon
\end{aligned}$$

for all $n > N$. Since $\epsilon > 0$ was arbitrary,

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(A\tau^n(B)) - \omega(A)\omega(B)| \; d\mu(n) = 0$$

follows. Since $A, B \in \mathfrak{A}$ were also arbitrary, it follows that $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ is weak mixing. $\square$

THEOREM 5.3. *A non-trivial C^*-dynamical system cannot be both weakly mixing and compact.*

PROOF. Consider a non-trivial C^*-dynamical system $(\mathfrak{A}, \omega, \tau, G)$. Thus ω is not a homomorphism so that the Hilbert space H of the GNS representation is at least two dimensional by Proposition 2.6. If $(\mathfrak{A}, \omega, \tau, G)$ is weak mixing then $\dim H_{uds} = 1$ by Proposition 4.7. Therefore $(\mathfrak{A}, \omega, \tau, G)$ cannot be compact, since otherwise $H = H_{uds}$ by Theorem 4.10, which would contradict $\dim H \geq 2$. $\square$

CHAPTER 3

Quantum Systems with Discrete Energy Spectrum

In this chapter we will study the ergodic properties of quantum systems with discrete energy spectrums. This simply means the set of possible energies of the system is countable or even finite, i.e. $E_1, E_2, \ldots$. Standard examples of such systems which are of great importance in quantum physics include the harmonic oscillator and the spin $\frac{1}{2}$ systems (qubits), as well as systems consisting of finite collections of these. We will study such systems with discrete energy spectrums in a more general context, by starting with any Hilbert space H that possesses a countable total orthonormal basis $\{h_n\}$, and making no assumption on the energy values $E_1, E_2, \ldots$ beyond it being a sequence of real numbers. Furthermore, we consider states of such systems which have physical relevance, in particular states which include the canonical ensemble in quantum statistical mechanics as a special case. This chapter therefore gives an indication of how ergodic properties fit into quantum physics, albeit in a fairly simple context.

1. Construction of the C*-dynamical system

We start by showing how, given a Hilbert space with a countable total orthonormal basis $\{h_n\}$, an abstract dynamical system can be defined on $\mathscr{L}(H)$, provided we have an appropriate state on $\mathscr{L}(H)$. Since we are interested in physical time, our group G from Chapter 2 will now be taken to be $\mathbb{R}$.

Proposition 1.1. *Let H be a Hilbert space possessing a total orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then, for each $t \in \mathbb{R}$, there is a uniquely determined operator $V_t \in \mathscr{L}(H)$ such that*

$$V_t h_n = e^{-iE_n t} h_n$$

for all $n \in \mathbb{N}$. Moreover, for each $t \in \mathbb{R}$, the operator V_t is unitary and

$$\tau_t : \mathscr{L}(H) \to \mathscr{L}(H) : A \mapsto {V_t}^* A V_t \qquad (48)$$

defines a $*$-automorphism on the C^*-algebra $\mathscr{L}(H)$. Furthermore, if $\omega : \mathscr{L}(H) \to \mathbb{C}$ is a state, such that $\omega \circ \tau_t = \omega$ for all $t \in \mathbb{R}$, then $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ is an abstract dynamical system with $\tau : t \mapsto \tau_t$.

PROOF. Let $\mathfrak{H} = \operatorname{span}\{h_n : n \in \mathbb{N}\}$, fix an arbitrary $t \in \mathbb{R}$, and consider the mapping

$$W : \mathfrak{H} \to \mathfrak{H} : \sum_{k=1}^{n} \alpha_k h_k \mapsto \sum_{k=1}^{n} \alpha_k e^{-iE_k t} h_k \tag{49}$$

on the normed subspace $\mathfrak{H}$ of H. W is clearly linear and for any $s \in \operatorname{span}\{h_n : n \in \mathbb{N}\}$, say $s = \sum_{j=1}^{m} \alpha_j h_j$, it follows that

$$\begin{aligned}
\|Ws\|^2 &= \left\| \sum_{j=1}^{m} \alpha_j e^{-iE_j t} h_j \right\|^2 \\
&= \sum_{j=1}^{m} \left|\alpha_j e^{-iE_j t}\right|^2 \\
&= \sum_{j=1}^{m} |\alpha_j|^2 \\
&= \|s\|^2.
\end{aligned}$$

Thus, $W \in \mathscr{L}(\mathfrak{H})$. Hence, as $\overline{\mathfrak{H}} = H$, there is a unique operator $\widetilde{W} \in \mathscr{L}(H)$ such that $W = \widetilde{W}$ on $\mathfrak{H}$, by [**20**, Theorem 2.7-11]. It is now clear that $V_t = \widetilde{W}$.

The mapping

$$W' : \mathfrak{H} \to \mathfrak{H} : \sum_{k=1}^{n} \alpha_k h_k \mapsto \sum_{k=1}^{n} \alpha_k e^{iE_k t} h_k$$

is in $\mathscr{L}(\mathfrak{H})$ as well, for the same reason that W is, and has the property $WW' = W'W = \mathsf{id}$. It also follows that, for any $s_1, s_2 \in \mathfrak{H}$ with, say, $s_1 = \sum_{j=1}^{n} \alpha_j h_{n_j}$ and $s_2 = \sum_{k=1}^{m} \beta_k h_{m_k}$,

$$\begin{aligned}
\langle s_1, Ws_2\rangle_{\mathfrak{H}} &= \sum_{j:n_j=m_j} \overline{\alpha_j}\beta_j e^{-iE_{m_j} t} \\
&= \sum_{j:n_j=m_j} \overline{e^{iE_{n_j} t}\alpha_j}\beta_j \\
&= \langle W's_1, s_2\rangle. \qquad (50)
\end{aligned}$$

Let $V_t' \in \mathscr{L}(H)$ be the bounded linear extension of W' to H. Since $V_t'V_t = V_tV_t' = \mathsf{id}$ on $\mathfrak{H}$, $V_t'V_t = V_tV_t' = \mathsf{id}$ on H. That is, $V_t' = V_t^{-1}$, and in particular V_t is bijective and thus possesses an inverse. By (50) and the continuity of the inner product $\langle x, V_t y\rangle_H = \langle V_t' x, y\rangle_H$ for all $x, y \in H$. That is, $V_t^{-1} = V_t' = V_t^*$ and we have that V_t is unitary.

τ_t is clearly linear, and for any $A \in \mathscr{L}(H)$, $\tau_t(V_T A V_t^*) = A$ and $\tau_t(A^*) = V_t^* A^* V_t = (V_t^* A V_t)^* = \tau_t(A)^*$. Thus τ_t defines a $*$-automorphism on $\mathscr{L}(H)$, for any $t \in \mathbb{R}$.

For any $s, t \in \mathbb{R}$ and $z \in \mathfrak{H}$, say $z = \sum_{k=1}^n \alpha_k h_{n_k}$, it follows that

$$\begin{aligned} V_s V_t z &= V_s \sum_{k=1}^n \alpha_n e^{-iE_k t} h_{n_k} \\ &= \sum_{k=1}^n \alpha_n e^{-iE_k(s+t)} h_{n_k} \\ &= V_{st}. \end{aligned}$$

Thus $V_s V_t = V_{st}$ on $\mathfrak{H}$, and hence also on H. It therefore follows, for any $A \in \mathfrak{A}$, that

$$\tau_s \circ \tau_t(A) = V_s V_t A V_t^* V_s^* = V_s V_t A (V_s V_t)^* = \tau_{st}(A).$$

Since A was arbitrary, $\tau_s \circ \tau_t = \tau_{st}$ follows for the arbitrary choice in $s, t \in \mathbb{R}$. All the conditions are now met for $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ to be an abstract dynamical system. □

For the remainder of the chapter, the notation of Proposition 1.1 will be implicitly assumed.

What remains is to construct concrete examples of the states required in Proposition 1.1. Since we require $\omega \circ \tau_t = \omega$ it makes physical sense to construct them from the energy eigenstates h_m. A key problem that needs to be addressed is to identify the GNS representation corresponding to whichever states we construct. We will then be able to determine whether the conditions of Definition 1.6 are met. Our approach will be to define a suitable state in very general terms. We will then identify two different states as special cases so that the separate C^*-dynamical systems defined by them have distinct ergodic properties. We will require a GNS representation sturdy enough for this general case, and to that end we proceed by first identifying a suitable Hilbert space.

THEOREM 1.2. *Let H be a Hilbert space and define*

$$H^{\oplus} = \left\{ (x_n)_{n\in\mathbb{N}} : x_n \in H \ \forall\ n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}$$

Then $H^{\oplus}$, with addition and scalar multiplication defined by

$$(x_n) + (y_n) = (x_n + y_n) \tag{51}$$

$$a(x_n) = (ax_n) \tag{52}$$

for all $(x_n), (y_n) \in H^{\oplus}$ *and* $a \in \mathbb{C}$, *is a vector space.* $H^{\oplus}$ *is a complex Hilbert space if equipped with the inner product defined*

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle \tag{53}$$

for all $(x_n), (y_n) \in H^{\oplus}$.

Proof. See [**8**, IV.4.19] □

We now consider a Hilbert space that is a generalization of $H^{\oplus}$, and which will be general enough for our purposes.

Proposition 1.3. *Let* H *be a Hilbert space and let* $p = (p_n)$ *be any sequence in* $\mathbb{R}$. *Then*

$$H^{p\oplus} = \left\{ (x_n) \in H^{\oplus} : x_n = 0 \; if \; p_n = 0 \right\}$$

is a Hilbert subspace of $H^{\oplus}$.

Proof. It is clear that $H^{p\oplus}$ is an inner product subspace of $H^{\oplus}$ so that we only have to show that $H^{p\oplus}$ is complete, i.e. that $H^{p\oplus}$ is closed in $H^{\oplus}$ which can be shown trivially. Consider any $z = (z_n) \in \overline{H^{p\oplus}}$ and let (x_n) be a sequence in $H^{p\oplus}$ that converges to z in $H^{\oplus}$, where $x_n = (x_{n,k})_k \in H^{p\oplus}$. Then, if we assume that $p_m = 0$, it follows that $x_{n,m} = 0$ for all $n \in \mathbb{N}$. Now, put $w_m = z_m$ and $w_n = 0$ if $n \neq m$ and consider thee sequence $(w_n) \in H^{p\oplus}$. We have

$$\begin{aligned} \|z_m\|^2 &= \langle z_m, z_m \rangle_H \\ &= \langle z, w \rangle_{H^{\oplus}} \\ &= \lim_{n\to\infty} \langle x_n, w \rangle_{H^{\oplus}} \\ &= \lim_{n\to\infty} \langle x_{n,m}, w_m \rangle_H \\ &= 0, \end{aligned}$$

since $x_{n,m} = 0$ for all n. Thus $p_m = 0$ implies $z_m = 0$ and so $z = (z_n) \in H^{p\oplus}$ as required. □

Proposition 1.4. *Let* $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ *be the abstract dynamical system defined in Proposition 1.1. For any sequence* $p = (p_n)$ *of real numbers, with* $p_n \geq 0$ *for all* $n \in \mathbb{N}$, *such that* $\sum_{n=1}^{\infty} p_n = 1$, *the mapping*

$$\omega : \mathscr{L}(H) \to \mathbb{C} : A \mapsto \sum_{n=1}^{\infty} p_n \langle h_n, A h_n \rangle$$

defines a state on $\mathscr{L}(H)$ *such that* $\omega \circ \tau_t = \omega$ *for all* $t \in \mathbb{R}$. *Furthermore, the GNS representation of* $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ *is given by* $(H^{p\oplus}, \pi, \Omega, U)$, *where* $\pi : \mathscr{L}(H) \to \mathscr{L}(H^{p\oplus})$ *with* $\pi(A) : H^{p\oplus} \to H^{p\oplus} : (x_n) \mapsto (Ax_n)$ *for all* $A \in \mathscr{L}(H)$, $\Omega = (\sqrt{p_n} h_n)$ *and*

$$U_t : H^{p\oplus} \to H^{p\oplus} : (x_n) \mapsto (e^{-iE_n t} V_t^* x_n) \tag{54}$$

for all $(x_n) \in H^{p\oplus}$ *and* $t \in \mathbb{R}$. *Finally,* $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ *is a* C^*-*dynamical system.*

PROOF. We will have to prove that:

(i) π is a $*$-homomorphism.
(ii) ω is a state, $\omega \circ \tau_t = \omega$ for all $t \in \mathbb{R}$ and $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathscr{L}(H)$.
(iii) Ω is a cyclic vector.
(iv) The GNS representation $U : t \mapsto U_t$ of τ is given by (54).
(v) $t \mapsto U_t$ is strongly continuous and $\mathbb{R}$ is a separable Abelian amenable group.

(i) Let $(x_n) \in H^{p\oplus}$ and $A \in \mathscr{L}(H)$ be arbitrary. It follows that

$$\|\pi(A)(x_n)\|^2_{H^{p\oplus}} = \sum_{n=1}^{\infty} \langle Ax_n, Ax_n \rangle \leq \|A\|^2 \sum_{n=1}^{\infty} \|x_n\|^2 = \|A\|^2 \|(x_n)\|^2_{H^{p\oplus}}.$$

As $\pi(A)$ is also clearly linear, $\pi(A)$ is therefore bounded linear and we have that $\pi : \mathscr{L}(H) \to \mathscr{L}(H^{p\oplus})$ is well defined. Let (y_n) be another arbitrary vector in $H^{p\oplus}$. It follows that

$$\begin{aligned}
\left\langle (x_n), \pi(A^*)(y_n) \right\rangle_{H^{p\oplus}} &= \sum_{n=1}^{\infty} \langle x_n, A^* y_n \rangle \\
&= \sum_{n=1}^{\infty} \langle Ax_n, y_n \rangle \\
&= \langle \pi(A)(x_n), (y_n) \rangle_{H^{p\oplus}} \\
&= \left\langle (x_n), \pi(A)^*(y_n) \right\rangle_{H^{p\oplus}}.
\end{aligned}$$

Thus $\pi(A^*) = \pi(A)^*$. Since π is also clearly linear, $\pi : \mathscr{L}(H) \to \mathscr{L}(H^{p\oplus})$ defines a $*$-homomorphism.

(ii) For any $A \in \mathscr{L}(H)$ we have that $\omega(A^*A) = \sum_{n=1}^{\infty} p_n \langle Ah_n, Ah_n \rangle \geq 0$. Since $\{h_n\}$ is an ortho*normal* set and $\sum_{n=1} p_n = 1$, it is clear that $\omega(1) = 1$ where $1 \in \mathscr{L}(H)$ denotes the identity operator. Since ω is also clearly linear, ω defines a state on $\mathscr{L}(H)$.

For any $t \in \mathbb{R}$ and $A \in \mathscr{L}(H)$ it follows that

$$\begin{aligned}\omega \circ \tau_t(A) &= \sum_{n=1}^{\infty} p_n \langle h_n, V_t^* A V_t h_n \rangle \\ &= \sum_{n=1}^{\infty} p_n \langle V_t h_n, A e^{-iE_n t} h_n \rangle \\ &= \sum_{n=1}^{\infty} p_n e^{-iE_n t} \langle e^{-E_n t} h_n, A h_n \rangle \\ &= \sum_{n=1}^{\infty} p_n \langle h_n, A h_n \rangle = \omega(A).\end{aligned}$$

Thus $\omega \circ \tau_t = \omega$.

For any $A \in \mathscr{L}(H)$ it follows that

$$\langle \Omega, \pi(A)\Omega \rangle_{H^{p\oplus}} = \sum_{n=1}^{\infty} \langle \sqrt{p_n} h_n, \sqrt{p_n} A h_n \rangle = \sum_{n=1}^{\infty} p_n \langle h_n, A h_n \rangle = \omega(A).$$

(iii) Consider any $(x_n) \in H^{p\oplus}$ and let

$$x_{n,k} = \begin{cases} x_n & \text{if } n \leq k, \\ 0 & \text{if } n > k \end{cases}$$

Thus $(x_{n,k}) = (x_1, x_2, ..., x_k, 0, ...)$ and it is easy to see that $\lim_{k \to \infty} (x_{n,k}) = (x_n)$ in $H^{p\oplus}$. For any $x, y \in H$, define an operator $x \otimes y \in \mathscr{L}(H)$ as follows:

$$x \otimes y : z \mapsto x \langle y, z \rangle$$

$A_{x,y}$ can readily be seen to be a bounded linear operator. For any $k \in \mathbb{N}$ it now follows that

$$\begin{aligned}\pi\left(\sum_{\ell=1}^{k} \frac{1}{\sqrt{p_\ell}} x_\ell \otimes h_\ell\right)\Omega &= \left(\left(\sum_{\ell=1}^{k} x_\ell \otimes h_\ell\right)\sqrt{p_n} h_n\right) \\ &= (x_{n,k}).\end{aligned}$$

Hence, since $(x_{n,k})$ converges to (x_n) in $H^{p\oplus}$, it follows that $(x_n) \in \overline{\{\pi(A)\Omega : A \in \mathscr{L}(H)\}}$. As $(x_n) \in H^{p\oplus}$ was arbitrary, it follows that Ω is a cyclic vector of the representation $(H^{p\oplus}, \pi)$.

(iv) It follows from (54) and the definition of V_t, that for any $A \in \mathscr{L}(H)$

$$\begin{aligned} U_t\pi(A)\Omega &= U_t(\sqrt{p_n}Ah_n) \\ &= (\sqrt{p_n}e^{-iE_nt}V_t^*Ah_n) \\ &= (\sqrt{p_n}V_t^*AV_th_n) \\ &= \pi(V_t^*AV_t)\Omega \\ &= \pi(\tau_t(A))\Omega. \end{aligned}$$

Thus U is the GNS representation of τ.

(v) Consider an arbitrary convergent sequence (t_n) in $\mathbb{R}$ with, say $t_n \longrightarrow t$. Let $(x_n) \in H^{p\oplus}$ also be arbitrary. We wish to show that $U_{t_j}(x_n) \longrightarrow U_t(x_n)$ in $H^{p\oplus}$ so let $\epsilon > 0$ be arbitrary. We know that $\lim_{k\to\infty}(x_{n,k}) = (x_n)$ and so there is a $K \in \mathbb{N}$ such that $\|(x_{n,K}) - (x_n)\| < \frac{\epsilon}{9\sqrt{K}}$. Since $\{h_n\}$ is total in H, for each $n \in \mathbb{N}$,

$$\|x_{n,K} - S_{n,K}\| < \frac{\epsilon}{9K}$$

for some $S_{n,K} \in \operatorname{span}\{h_n\}$. The mapping $\mathbb{R} \to \mathbb{C} : r \mapsto e^{i\beta r}$ is continuous for all $\beta \in \mathbb{R}$, and so for any $\sum_{k=1}^m \alpha_k h_k \in \operatorname{span}\{h_n\}$ and $\delta > 0$ there is an $N \in \mathbb{N}$ such that

$$\begin{aligned} &\left\| e^{iE_nt_j}V_{t_j}^*\sum_{k=1}^m \alpha_k h_k - e^{iE_nt}V_t^*\sum_{k=1}^m \alpha_k h_k \right\| \\ &= \left\| \sum_{k=1}^m \alpha_k e^{i(E_k+E_n)t_j}h_k - \sum_{k=1}^m \alpha_k e^{i(E_k+E_n)t}h_k \right\| \\ &\leq \sum_{k=1}^m |\alpha_k| \left\| e^{i(E_k+E_n)t_j}h_k - e^{i(E_k+E_n)t}h_k \right\| \\ &\leq \sum_{k=1}^m |\alpha_k| \left| e^{i(E_k+E_n)t_j} - e^{i(E_k+E_n)t} \right| \|h_k\| \\ &< \delta \end{aligned}$$

for all $n \in \{1, ..., m\}$, if $j \geq N$. In particular, there is an $N' \in \mathbb{N}$ such that

$$\left\| e^{iE_nt_j}V_{t_j}^*S_{n,K} - e^{iE_nt}V_t^*S_{n,K} \right\| < \frac{\epsilon}{9\sqrt{K}} \tag{55}$$

for all $n \in \{1, 2, ..., K\}$, if $j \geq N'$. It now follows that

$$\begin{aligned}
&\left\|U_{t_j}(x_n) - U_t(x_n)\right\|_{H^{p\oplus}} \\
\leq\ &\left\|U_{t_j}(x_n) - U_{t_j}(x_{n,K})\right\|_{H^{p\oplus}} + \left\|U_{t_j}(x_{n,K}) - U_t(x_{n,K})\right\|_{H^{p\oplus}} \\
&+ \left\|U_t(x_{n,K}) - U_t(x_n)\right\|_{H^{p\oplus}} \\
\leq\ &\frac{\epsilon}{3} + \left\|U_{t_j}(x_{n,K}) - U_t(x_{n,K})\right\|_{H^{p\oplus}} + \frac{\epsilon}{3} \\
=\ &\frac{2\epsilon}{3} + \sqrt{\sum_{n=1}^{K} \left\|e^{-iE_n t_j} V_{t_j}^* x_{n,K} - e^{-iE_n t} V_t^* x_{n,K}\right\|^2}.
\end{aligned}$$

Furthermore, by (55) and the fact that V_r^* is unitary for all $r \in \mathbb{R}$, it follows for each $n = 1, ..., K$ that

$$\begin{aligned}
&\left\|e^{-iE_n t_j} V_{t_j}^* x_{n,K} - e^{-iE_n t} V_t^* x_{n,K}\right\| \\
\leq\ &\left\|V_{t_j}^* x_{n,K} - V_{t_j}^* S_{n,K}\right\| + \left\|e^{-iE_n t_j} V_{t_j}^* S_{n,K} - e^{-iE_n t} V_t^* S_{n,K}\right\| \\
&+ \left\|V_t^* x_{n,K} - V_t^* S_{n,K}\right\| \\
\leq\ &\frac{\epsilon}{9\sqrt{K}} + \frac{\epsilon}{9\sqrt{K}} + \frac{\epsilon}{9\sqrt{K}} \\
=\ &\frac{\epsilon}{3\sqrt{K}}
\end{aligned}$$

and therefore that

$$\sum_{n=1}^{K} \left\|e^{-iE_n t_j} V_{t_j}^* x_{n,K} - e^{-iE_n t} V_t^* x_{n,K}\right\|^2 \leq \frac{\epsilon^2}{9}$$

for all $j \geq N'$. So we have that

$$\left\|U_{t_j}(x_n) - U_t(x_n)\right\|_{H^{p\oplus}} \leq \frac{2\epsilon}{3} + \sqrt{\frac{\epsilon^2}{9}} = \epsilon$$

for all $j \geq N'$. That is, $U_{t_j}(x_n) \longrightarrow U_t(x_n)$ in $H^{p\oplus}$ as required.

Finally, $\mathbb{R}$ is a locally compact group and can easily be seen to possess a Følner sequence, for example $(\{t : |t| \leq n\})_n$. Thus $\mathbb{R}$ is amenable and of course also separable and Abelian. $(\mathscr{L}(H), \omega, \tau, \mathbb{R})$ now satisfies all the criteria to be a C^*-dynamical system. □

Proposition 1.4 provides us with our first example of a C^*-dynamical system, or rather a collection of C^*-dynamical systems due to the freedom of choice in the sequence $p = (p_n)$. Note that the C^*-dynamical system is non-trivial, as H and therefore $H^{p\oplus}$ is more than one dimensional. In fact, both Hilbert spaces are infinite dimensional. It will be convenient to assign a name to these kind of systems. We adopt the habit of defining this, and all subsequent C^*-dynamical systems in complete detail.

DEFINITION 1.5. The *Discrete Energy Spectrum System* (DESS(p, E_n)) is the C^*-dynamical system $(\mathfrak{A}, \omega, \tau, \mathbb{R})$ given by

(i) $\mathfrak{A} = \mathscr{L}(H)$ where H is a complex Hilbert space with a countable total orthonormal basis $\{h_n\}_{n\in\mathbb{N}} = \mathfrak{H}$.
(ii) $p = (p_n)$ is a sequence in $\mathbb{R}$ where $p_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} p_n = 1$.
(iii) $\omega : \mathscr{L}(H) \to \mathbb{C} : T \mapsto \sum_{n=1}^{\infty} p_n \langle h_n, Th_n \rangle$.
(iv) $\tau : \mathbb{R} \to \mathrm{Aut}\,(\mathfrak{A}) : t \mapsto \tau_t$ where $\tau_t : \mathfrak{A} \to \mathfrak{A} : T \mapsto V_t^* T V_t$ and $V_t \in \mathfrak{A}$ is uniquely defined by its action on $\mathfrak{H}$:

$$V_t h_n = e^{-iE_n t} h_n$$

where $E_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. $E_1, E_2, ...$ are called the energy values of the system.

2. Ergodic properties

To investigate the ergodic properties of our first C^*-dynamical system, we will utilize the spectral characterizations, and interrelationships, of the ergodic properties derived in Chapter 2. For the examples of subsequent chapters, however, we will find cases wherein the Hilbert space characterizations are more efficient.

PROPOSITION 2.1. *DESS(p, E_n) is compact, for any p and any energy values $E_1, E_2, \ldots$.*

PROOF. Let $(H^{p\oplus}, \pi, \Omega, U)$ be the GNS representation of DESS$(p, (E_n))$ obtained from Proposition 1.4.

Let $(x_n) \in H^{p\oplus}$ and $\epsilon > 0$ be arbitrary. Then $\lim_{k\to\infty}(x_{n,k}) = (x_n)$ in $H^{p\oplus}$ where $(x_{n,k}) = (x_1, x_2, ..., x_k, 0, ...)$. Thus, there is a $K \in \mathbb{N}$ such that

$$\|(x_n) - (x_{n,k})\|_{H^{p\oplus}} \leq \frac{\epsilon}{2}\ \forall\ k \geq K.$$

Now consider the set

$$H_\Omega = \{(y_n) \in H^{p\oplus} : y_\ell = h_m \text{ for some } m, \ell \in \mathbb{N}, y_n = 0 \text{ for all } n \neq \ell\}.$$

Thus, if $z \in H_\Omega$, then z has an element of the orthonormal basis set $\{h_n\}$ in one coordinate, and zeros elsewhere. It readily follows that, for any $k \in \mathbb{N}$, there is a $q_k \in \mathrm{span}\,(H_\Omega)$ such that $\|(x_{n,k}) - q_k\|_{H^{p\oplus}} < \frac{\epsilon}{2}$. Therefore

$$\|(x_n) - q_k\|_{H^{p\oplus}} \leq \|(x_n) - (x_{n,K})\|_{H^{p\oplus}} + \|(x_{n,K}) - q_K\|_{H^{p\oplus}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that $(x_n) \in \overline{\mathrm{span}\,H_\Omega}$, and as $(x_n) \in H^{p\oplus}$ was arbitrary, we have that $\overline{\mathrm{span}\,H_\Omega} = H^{p\oplus}$. Thus, if $\mathrm{span}\,H_\Omega \subseteq H^{p\oplus}_{uds}$, then $H^{p\oplus} = H^{p\oplus}_{uds}$ and DESS(p) will be compact by Theorem 4.10.

Let $z = (z_n)$ be any element of H_Ω. Thus at some coordinate $\ell \in \mathbb{N}$, $g_\ell = h_m$ for some $m \in \mathbb{N}$, and $g_n = 0$ for all $n \neq \ell$. For any $t \in \mathbb{R}$, it follows by Proposition 1.4 that

$$\begin{aligned} U_t z &= (e^{-iE_n t} V_t^* z_n) \\ &= (0, ..., 0, e^{-iE_\ell t} V_t^* h_m, 0, ...) \\ &= e^{i(E_m - E_\ell)t}(0, ..., 0, h_m, 0, ...) \\ &= e^{i(E_m - E_\ell)t} z \end{aligned}$$

which reveals z as an eigenvector, with unimodular eigenvalues, of $\{U_t : t \in \mathbb{R}\}$. Therefore $z \in H^{p\oplus}_{uds}$ and as $z \in H_\Omega$ was arbitrary, we have that $H_\Omega \subseteq H^{p\oplus}_{uds}$. As $H^{p\oplus}_{uds}$ is a vector subspace of $H^{p\oplus}$, span $H_\Omega \subseteq H^{p\oplus}_{uds}$ as required. □

In contrast to Proposition 2.1, whether or not DESS(p, E_n) is ergodic depends on p and the energy values $E_1, E_2, ...$.

PROPOSITION 2.2. *If $p_k = 1$, for some $k \in \mathbb{N}$, then DESS(p, E_n) is ergodic if and only if $E_n \neq E_k$ for all $n \in \mathbb{N}\backslash\{k\}$.*

PROOF. Let $(H^{p\oplus}, \pi, \Omega, U)$ be the GNS representation of DESS$(p, (E_n))$ obtained from Proposition 1.4.

Since $p_k = 1$, $p_n = 0$ for all $n \neq k$, and therefore all vectors in $H^{p\oplus}$ can only have a nonzero coordinate in the k'th position. Assume first that $E_n \neq E_k$ for all $n \in \mathbb{N}$. Consider any $x \in H$ with $\langle x, h_m\rangle \neq 0$ for some $m \neq k$. That is, x "contains" an h_m term other than h_k. Let $z = (z_n) \in H^{p\oplus}$ be the vector with x as its sole coordinate. Then, for any $t \in \mathbb{R}$, it follows by Proposition 1.4 that

$$\begin{aligned} U_t z &= (e^{-iE_n t} V_t^* z_n) \\ &= (0, ..., 0, e^{-iE_k t} V_t^* x, 0, ...) \\ &= (0, ..., 0, e^{-iE_k t} V_t^* \sum_{j=1}^{\infty} \langle h_j, x\rangle h_j, 0, ...) \\ &= (0, ..., 0, \sum_{j=1}^{\infty} e^{i(E_j - E_k)t} \langle h_j, x\rangle h_j, 0, ...). \end{aligned}$$

In particular, for a $t' \in \mathbb{R}$ such that $(E_m - E_k)t \neq 2\pi n$ for all $n \in \mathbb{N}$, we would have that $e^{i(E_m - E_k)t'} \neq 1$. It is then clear that $U_{t'}(z_n) \neq (z_n)$. Hence, the only fixed point of $\{U_t : t \in \mathbb{R}\}$ in $H^{p\oplus}$ is the cyclic vector $\Omega = (0, ..., 0, h_k, 0, ...)$.

Conversely, suppose that $E_m = E_k$ for a $m \neq k$. Let $z = (z_n) \in H^{p\oplus}$ be the vector with h_m as its sole coordinate. Then, for any $t \in \mathbb{R}$, it follows by Proposition 1.4 that

$$\begin{aligned}U_t z &= (e^{-iE_n t}V_t^* z_n)\\ &= (0, ..., 0, e^{-iE_k t}V_t^* h_m, 0, ...)\\ &= (0, ..., 0, e^{i(E_m - E_k)t} h_m, 0, ...)\\ &= (0, ..., 0, h_m, 0, ...)\\ &= z.\end{aligned}$$

Thus $z = (0, ..., 0, h_m, 0, ...)$ is a fixed point of $\{U_t : t \in \mathbb{R}\}$ other than $\Omega = (0, ..., 0, h_k, 0, ...)$. Since h_m, h_k are linearly independent in H, it follows that z, Ω are linearly independent in $H^{p\oplus}$. Thus the fixed point space of U is at least 2 dimensional. Therefore, DESS(p, E_n) is not ergodic by Corollary 4.3. □

Even if the energy values $E_1, E_2, ...$ are all distinct, DESS(p, E_n) can still fail to be ergodic:

PROPOSITION 2.3. *If $p_k, p_m \neq 0$ for some $k, m \in \mathbb{N}$ with m, k distinct, then DESS(p, E_n) is not ergodic.*

PROOF. Let $(H^{p\oplus}, \pi, \Omega, U)$ be the GNS representation of DESS$(p, (E_n))$ obtained from Proposition 1.4.

Let $x = (x_n)$ be the vector in $H^{p\oplus}$ such that $x_k = h_k$, and $x_n = 0$ for all $n \neq k$. Likewise, let $y = (y_n)$ be the vector such that $y_m = h_m$ and $y_n = 0$ for all $n \neq m$. For any $t \in \mathbb{R}$ it follows that

$$\begin{aligned}U_t x &= (e^{-iE_n t}V_t^* x_n)\\ &= (0, ..., 0, e^{-iE_m t}V_t^* h_m, 0, ...)\\ &= (0, ..., 0, e^{-iE_m t}e^{iE_m t} h_m, 0, ...)\\ &= (0, ..., 0, h_m, 0, ...) = x.\end{aligned}$$

Thus x, and likewise y, are fixed points of U. Hence, as x, y are clearly linearly independent, the fixed point space of U is at least 2 dimensional. Therefore, DESS(p, E_n) is not ergodic by Corollary 4.3. □

By the two above Propositions, any discrete energy spectrum system is either ergodic and compact, or compact but not ergodic. Let us explicitly define two such systems, or at least as explicitly as is reasonable.

DEFINITION 2.4. Let (E_n) be a sequence of distinct real numbers. Let $p = (p_n)$ be a sequence in $\mathbb{R}$ such that $p_k = 1$ for some $k \in \mathbb{N}$ and $p_n = 0$ for all $n \neq k$. Let $q = (q_n)$ be a sequence of non-negative real numbers containing at least two nonzero terms and such that $\sum_{n=1}^{\infty} q_n = 1$. *Discrete Energy Spectrum System 1* (DESS1) is DESS(p, E_n) and *Discrete Energy Spectrum System 2* (DESS2) is DESS(q, E_n).

Proposition 2.5. *DESS1 is ergodic and compact.*

Proof. DESS1 is compact by Proposition 2.1 and ergodic by Proposition 2.2. □

Proposition 2.6. *DESS2 is compact and not ergodic.*

Proof. DESS2 is compact by Proposition 2.1 and not ergodic by Proposition 2.3. □

CHAPTER 4

Reduced Group C*-Algebras

It is possible to construct a C^*-algebra from any group, which we call a reduced group C^*-algebra, and given a group automorphism of the group, we can further derive a $*$-automorphism on the C^*-algebra such that a state on the C^*-algebra can readily be identified, that is invariant under the $*$-automorphism. This is all of the ingredients for a C^*-dynamical system with $\mathbb{Z}$ action, which we will call a reduced group C^*-dynamical system. We start by establishing the process by which this is done, and prove some useful results, without making any assumptions with regard to the nature of the group. By the end of the first section of this chapter we will know, upon considering any group with an automorphism, that a functioning C^*-dynamical system can be built on top of it. We will then, in the subsequent section, analyse the ergodic properties of these systems and find that exactly three distinct ergodic "profiles" can exist on them: compact but not ergodic, strongly mixing, and neither ergodic nor compact. In the final section of this chapter we construct three concrete examples of reduced group C^*-dynamical systems on which these three "profiles" are present. The group chosen for this purpose is a free group generated from a countable set.

In contrast to Chapter 3, this chapter's work does not carry any direct physical significance, however, from a C^*-dynamical perspective, it is very natural. Our interest in reduced group C^*-dynamical systems are twofold. Two of the ergodic "profiles" are new, and we will find our first concrete examples of C^*-dynamical systems with these ergodic properties among the reduced group C^*-dynamical systems. We already have an example of a system, in DESS2, that is compact but not ergodic. Secondly, the final C^*-dynamical system that we will construct, on the quantum torus, can be shown to have the same GNS representation as a reduced group C^*-dynamical system, with a particular group and group automorphism, and therefore have the same ergodic properties. We will then be spared from performing a separate ergodic analysis on this more complicated final system, and will promptly identify its ergodic properties from the results in this chapter. Specifically, this relates to the so-called cat mappings on the torus that we will use to construct a C^*-dynamical system on the quantum torus, in the next chapter.

1. Construction of the C*-dynamical System

Throughout this section Γ will represent an arbitrary group equipped with the discrete topology, and $\rho : \Gamma \to \Gamma$ an automorphism of Γ, unless stated otherwise. Our only assumption about Γ is that it is not trivial, i.e. it has more than one element. We endow Γ with a measure structure by equipping it with a counting measure μ defined on its Borel σ-algebra. Thus, all subsets of Γ are measurable and the measure μ of any set equals the number of elements in that set.

The Hilbert space $L^2(\Gamma)$ is therefore fully realised and we will define the reduced group C^*-algebra as a C^*-subalgebra of $\mathscr{L}(L^2(\Gamma))$. To isolate the operators of interest in $\mathscr{L}(L^2(\Gamma))$ we proceed by first identifying a total orthonormal basis for $L^2(\Gamma)$, so that, to characterize an operator in $\mathscr{L}(L^2(\Gamma))$ one only has to describe its action on the basis in such a manner that it is well defined when extended to $L^2(\Gamma)$.

For all $g \in \Gamma$, we define $\delta_g : \Gamma \to \mathbb{C}$ by

$$\delta_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1.1. *The collection $\{\delta_g : g \in \Gamma\}$ is a total orthonormal set in $L^2(\Gamma)$.*

PROOF. Since $\delta_g \delta_h = \overline{\delta_g}\delta_h = 0$ for all $g \neq h$ in Γ, and $|\delta_g|^2 = \delta_g$, it is clear that $\{\delta_g : g \in \Gamma\}$ is an orthonormal set in $L^2(\Gamma)$. Consider first an arbitrary integrable function $f : \Gamma \to \mathbb{R}$ with $f \geq 0$. If $s = \sum_{k=1}^{n} a_k \chi_{A_k}$ is a simple function such that $0 \leq s \leq f$ then $A = \bigcup_k A_k$ is necessarily a finite set since otherwise $\int_\Gamma f \; d\mu \geq \int_\Gamma s \; d\mu = \infty$. As each δ_g is the characteristic function of the point set $\{g\}$, we thus have that $s' = \sum_{g \in A} f(g)\delta_g$ is a simple function satisfying $s \leq s' \leq f$. Moreover, since $\sum_{g \in S} f(g)\delta_g$ is a simple function less than or equal to f for *any* finite $S \subseteq \Gamma$, it therefore follows that

$$\begin{aligned} \int_\Gamma f \; d\mu &= \sup\left\{ \int_\Gamma s \; d\mu : 0 \leq s \leq f \text{ is a simple function} \right\} \\ &= \sup\left\{ \int_\Gamma \sum_{g \in S} f(g)\delta_g \; d\mu : S \subseteq \Gamma \text{ is finite} \right\} \\ &= \sup\left\{ \int_S f \; d\mu : S \subseteq \Gamma \text{ is finite} \right\}. \end{aligned}$$

Let $\epsilon > 0$ and $f \in L^2(\Gamma)$ now be arbitrary. Hence $|f|^2 \geq 0$ is integrable so that there is a finite set $S \subseteq \Gamma$ such that

$$\int_{\Gamma \setminus S} |f|^2 \; d\mu = \int_\Gamma |f|^2 \; d\mu - \int_S |f|^2 \; d\mu < \epsilon^2$$

from which it promptly follows that

$$\begin{aligned}\|f - \sum_{g \in S} f(g)\delta_g\|^2 &= \int_\Gamma |f - \sum_{g\in S} f(g)\delta_g|^2 \, d\mu \\ &= \int_{\Gamma - S} |f|^2 \, d\mu \\ &< \epsilon^2.\end{aligned}$$

Thus $\|f - \sum_{g\in S} f(g)\delta_g\| < \epsilon$ and as $\epsilon > 0$ and $f \in L^2(\Gamma)$ were arbitrary, this show that $\{\delta_g : g \in \Gamma\}$ is a total orthonormal set in $L^2(\Gamma)$. □

Since $\{\delta_g : g \in \Gamma\}$ is a total orthonormal set in $L^2(\Gamma)$ we thus have that, for any $f \in L^2(\Gamma)$

$$f = \lim_{n\to\infty} s_n \text{ with } s_n = \sum_{g \in G_n} a_{n,g}\delta_g$$

for some scalars $a_{n,g} \in \mathbb{C}$ and finite subsets $G_n \subseteq \Gamma$. As was seen in the proof of Theorem 1.1, we may take $a_{n,g} = f(g)$ for all $n \in \mathbb{N}$ and $g \in S_n$.

The following proposition is elementary and will see multiple applications.

PROPOSITION 1.2. *For any bijective mapping $\gamma : \Gamma \to \Gamma$,*

$$\int_\Gamma f \, d\mu = \int_\Gamma f \circ \gamma \, d\mu \tag{56}$$

holds for all integrable $f : \Gamma \to \mathbb{C}$.

PROOF. Since γ is bijective, it is an invertible measure preserving transformation on Γ, so that the proposition follows promptly from elementary measure theory arguments. That is, (56) is first proven for step functions, then simple functions and then for positive measurable functions by using the Monotone Convergence Theorem. Finally, the result follows from the fact that any integrable f is the difference of two positive measurable functions. □

PROPOSITION 1.3. *For any bijective mapping $\gamma : \Gamma \to \Gamma$,*

$$\vartheta : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \gamma \tag{57}$$

determines a unitary operator in $\mathscr{L}(L^2(\Gamma))$.

PROOF. Let us first use ϑ to denote the uniquely determined linear operator on $S(\Gamma) := \text{span}\,\{\delta_h : h \in \Gamma\}$ for which (57) holds. That is,

$$\begin{aligned}\vartheta : S(\Gamma) \to S(\Gamma) : \sum_{k=1}^{n} a_k \delta_{h_k} &\mapsto \sum_{k=1}^{n} a_k (\delta_{h_k} \circ \gamma) \\ : \sum_{k=1}^{n} a_k \delta_{h_k} &\mapsto \sum_{k=1}^{n} a_k \delta_{\gamma^{-1}(h_k)}.\end{aligned}$$

For any $s \in S(\Gamma)$ it follows from Proposition 1.2 that

$$\begin{aligned}\|\vartheta s\|_{L^2}^2 &= \|(s \circ \gamma)^2\|_{L^1} \\ &= \|s^2 \circ \gamma\|_{L^1} \\ &= \|s^2\|_{L^1} \\ &= \|s\|_{L^2}^2.\end{aligned}$$

Thus $\vartheta \in \mathscr{L}(S(\Gamma))$. By the same argument, $\vartheta^{-1} \in \mathscr{L}(S(\Gamma))$, where $\vartheta^{-1} : f \mapsto f \circ \gamma^{-1}$. Hence, by [**20**, Theorem 2.7-11], ϑ and ϑ^{-1} have unique extensions in $\mathscr{L}(L^2(\Gamma))$.

For any $r, s \in S(\Gamma)$, it follows by Proposition 1.2 that

$$\begin{aligned}\langle \vartheta r, s\rangle_{L^2} &= \int_\Gamma \overline{\vartheta r}\, s \; d\mu \\ &= \int_\Gamma \overline{r \circ \gamma}\, s \; d\mu \\ &= \int_\Gamma \overline{r}\, s \circ \gamma^{-1} \; d\mu \\ &= \langle r, s \circ \gamma^{-1}\rangle_{L^2} \\ &= \langle r, \vartheta^{-1} s\rangle_{L^2}.\end{aligned}$$

By the continuity of the inner product, it now follows that $\langle \vartheta f, g\rangle = \langle f, \vartheta^{-1} g\rangle$ for all $f, g \in L^2(\Gamma)$. Thus, $\vartheta^* = \vartheta^{-1}$, i.e. ϑ is unitary. $\square$

For the bijections of our general group Γ of the form

$$\Gamma \to \Gamma : h \mapsto g^{-1} h$$

where g is some element in Γ, we denote the corresponding unitary operator given by Proposition 1.3 as $\lambda(g)$. Thus

$$\lambda(g)\delta_h = \delta_{gh}$$

for all $h \in \Gamma$. $\lambda : \Gamma \to \mathfrak{U}(L^2(\Gamma))$ is known as the left regular representation of Γ, since $\lambda(gh) = \lambda(g)\lambda(h)$ and Γ acts on itself from the left as is clear from $\lambda(g)\delta_h = \delta_{gh}$. We use this representation to generate our C^*-algebra.

DEFINITION 1.4. The *reduced C^*-algebra of* Γ, denoted $C_r^*(\Gamma)$, is the C^*-algebra generated by the collection of unitary operators $\{\lambda(g) : g \in \Gamma\}$.

A question that immediately arises is what the general form of an element of $C_r^*(\Gamma)$ is. Since $C_r^*(\Gamma)$ is the C^*-algebra generated by $\{\lambda(g) : g \in \Gamma\}$ it follows that any $T \in C_r^*(\Gamma)$ is of the form

$$T = \lim_{n\to\infty} S_n$$

where $S_n = \sum_{k=1}^{m_n} V_{n,k}$ and the $V_{n,k}$ are elements in the span of $\{\lambda(g) : g \in \Gamma\}$. An element in the span of $\{\lambda(g) : g \in \Gamma\}$ is of the form

$$\sum_{i=1}^{m}\left(a_i \prod_{j=1}^{n_i} \lambda(g_{i,j})\right)$$

for some $g_{i,j} \in \Gamma$. Thus, an operator $T \in C_r^*(\Gamma)$ is the limit of a sequence of finite sums of finite products of $\lambda(g)$'s. However, this may be simplified since $\lambda(g)\lambda(h) = \lambda(gh)$ for any $g, h \in \Gamma$. That is, an element in the span of $\{\lambda(g) : g \in \Gamma\}$ is of the form

$$\sum_{i=1}^{m} a_i\lambda(g_i) \tag{58}$$

for some $a_i \in \mathbb{C}$ and $g_i \in \Gamma$. Therefore any operator $T \in C_r^*(\Gamma)$ is the limit of a sequence of finite sums of the form (58).

To define a C^*-dynamical system with a $\mathbb{Z}$ action we still require a $*$-automorphism on $C_r^*(\Gamma)$ and a state that is invariant under the $*$-automorphism. Similar to the way in which a group Γ yields the C^*-algebra $C_r^*(\Gamma)$, a group automorphism ρ of Γ can be "expanded" to define a $*$-automorphism on $C_r^*(\Gamma)$. There are many possible states on $C_r^*(\Gamma)$, so we will identify a "canonical" one with respect to which the $*$-automorphisms created in this way are always invariant.

Proposition 1.5. *If $\rho : \Gamma \to \Gamma$ is a group automorphism, then the mapping $\tau : C_r^*(\Gamma) \to C_r^*(\Gamma) : A \mapsto UAU^*$, where $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$, is a $*$-automorphism on $C_r^*(\Gamma)$. The mapping $\omega : C_r^*(\Gamma) \mapsto \mathbb{C} : A \mapsto \langle \delta_1, A\delta_1\rangle$ is a state on $C_r^*(\Gamma)$, where 1 here denotes the identity element of Γ. Moreover, ω is invariant under τ.*

Proof. By Proposition 1.3, as ρ is bijective, U is a well defined unitary operator in $\mathscr{L}(L^2(\Gamma))$, and thus $\tau' : \mathscr{L}(L^2(\Gamma)) \to \mathscr{L}(L^2(\Gamma)) : A \mapsto UAU^*$ is a well defined mapping. τ' is clearly linear, and for any $A, B \in C_r^*(\Gamma)$,

$$\tau'(AB) = UABU^* = UAU^*UBU^* = \tau'(A)\tau'(B), \text{ and}$$
$$\tau'(A^*) = UA^*U^* = (UAU^*)^* = \tau'(A)^*.$$

Thus τ' is a $*$-homomorphism, and so to show that τ is a $*$-automorphism, we have to show that $\tau : C_r^*(\Gamma) \to C_r^*(\Gamma)$ is well defined and bijective.

To show that τ is well defined, we will show that $\tau'(C_r^*) \subseteq C_r^*$. Let $g, h \in \Gamma$ be arbitrary. It follows that

$$\begin{aligned}\tau'(\lambda(g))\delta_h &= U\lambda(g)U^*\delta_h \\ &= U\lambda(g)\delta_h \circ \rho^{-1} \\ &= U\lambda(g)\delta_{\rho(h)} \\ &= \delta_{g\rho(h)} \circ \rho \\ &= \delta_{\rho^{-1}(g)h}\end{aligned}$$

where in the final step we used the knowledge that ρ, and therefore ρ^{-1}, is a group automorphism of Γ. Thus $\tau'(\lambda(g))\delta_h = \lambda(\rho^{-1}(g))\delta_h$ for all $h \in \Gamma$, so that since span $\{\delta_h : h \in \Gamma\}$ is dense in $L^2(\Gamma)$, $\tau'(\lambda(g)) = \lambda(\rho^{-1}(g)) \in C_r^*(\Gamma)$.

Similarly for any sum of the form

$$S = \sum_{i=1}^{n} \alpha_i \lambda(g_i) \in C_r^*(\Gamma) \tag{59}$$

it follows from the linearity of τ' that $\tau'(S) = \sum_{i=1}^{n} \alpha_i \lambda(\rho^{-1}(g_i)) \in C_r^*(\Gamma)$. For a general $A \in C_r^*(\Gamma)$, $A = \lim S_n$ for a sequence of finite sums of the form (59), and as

$$\|\tau'(S_n) - \tau'(A)\| = \|US_nU^* - UAU^*\| = \|S_n - A\|$$

for all $n \in \mathbb{N}$, it follows that $\tau'(S_n) \longrightarrow \tau'(A)$. However, since $C_r^*(\Gamma)$ is closed and $\tau'(S_n) \in C_r^*(\Gamma)$ for all $n \in \mathbb{N}$, we have that $\tau'(A) \in C_r^*(\Gamma)$.

That $\tau : C_r^*(\Gamma) \to C_r^*(\Gamma)$ is bijective, follows from the preceding argument which similarly shows, by interchanging U and U^*, that $U^*AU \in C_r^*(\Gamma)$ for all $A \in C_r^*(\Gamma)$. Thus, for any $A \in C_r^*(\Gamma)$, $\tau(U^*AU) = UU^*AUU^* = A$. Thus τ is surjective, and injective since τ' is clearly injective.

The functional ω clearly defines a state on $C_r^*(\Gamma)$ as it is linear, $\omega(1) = 1$ and $\omega(R^*R) = \|R\delta_1\|^2 \geq 0$ for all $R \in C_r^*(\Gamma)$. For any $A \in C_r^*(\Gamma)$ it follows that

$$\begin{aligned}\omega \circ \tau(A) &= \langle \delta_1, UAU^*\delta_1 \rangle \\ &= \langle U^*\delta_1, AU^*\delta_1 \rangle \\ &= \langle \delta_1 \circ \rho^{-1}, A\delta_1 \circ \rho^{-1} \rangle \\ &= \langle \delta_{\rho(1)}, A\delta_{\rho(1)} \rangle \\ &= \langle \delta_1, A\delta_1 \rangle \\ &= \omega(A)\end{aligned}$$

where $\rho(1) = 1$ since ρ is a group automorphism. Thus ω is invariant under τ. □

In the proof of Proposition 1.5 we saw that

$$\tau(\lambda(g)) = \tau'(\lambda(g)) = \lambda(\rho^{-1}(g)).$$

We include this result as a separate proposition.

PROPOSITION 1.6. *For any $g \in \Gamma$,*

$$\tau(\lambda(g)) = \lambda(\rho^{-1}(g))$$

To summarize, starting with any group Γ and group automorphism we can define a C^*-algebra $C_r^*(\Gamma)$ (Definition 1.4), a $*$-automorphism τ and a state ω invariant under the τ (Proposition 1.5). Hence, we have all of the ingredients for a C^*-dynamical system with a $\mathbb{Z}$ action and since it will be convenient to assign a name to such a system, we introduce a definition, in keeping with Remarks 1.9.

DEFINITION 1.7. If Γ is a group and $\rho : \Gamma \to \Gamma$ is a group automorphism, then $\mathrm{RG}(\Gamma, \rho) = (\mathfrak{A}, \omega, \tau, \mathbb{Z})$ is a C^*-dynamical system where

(i) $\mathfrak{A} = C_r^*(\Gamma)$
(ii) $\omega : C_r^*(\Gamma) : A \mapsto \langle \delta_1, A\delta_1 \rangle$
(iii) $\tau \in \mathrm{Aut}\,(\mathfrak{A})$ is given by $\tau : C_r^*(\Gamma) \to C_r^*(\Gamma) : A \mapsto UAU^*$ with $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$

We call such a system a *reduced group C^*-algebraic dynamical system* and refer to Γ and ρ as *its group* and *its group automorphism*, respectively.

2. Ergodic properties

In this section we will set out to prove that the C^*-dynamical systems constructed in the preceding section, i.e. reduced group C^*-algebraic dynamical systems, can only have one of three ergodic "profiles":

(1) Strong mixing
(2) Compact but not ergodic
(3) Neither compact nor ergodic

Which of these a reduced group C^*-algebraic dynamical system possesses, we will find, depends on a very simple aspect of its group automorphism. This is to be expected, of course, since the group automorphism determines the $*$-automorphism group of the C^*-dynamical system. That is, the group automorphism determines the time evolution of the system.

In this section we will carry over the symbols and notations introduced in the preceding section. Specifically, Γ is an arbitrary group with accompanying group automorphism ρ, which determines a reduced group C^*-dynamical system $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$. The terms are summarized in Definition 1.7.

DEFINITION 2.1. For any $g \in \Gamma \backslash \{1\}$, the set $\{\rho^n(g) : n \in \mathbb{Z}\}$ is called the *orbit* of ρ in g. An orbit is called *finite* if it only contains a finite number of elements, or *infinite* if it contains an infinite number of elements.

Which of the three ergodic "profiles" mentioned above a reduced group C^*-dynamical system possesses, depends on whether its group automorphism has only finite orbits, only infinite orbits or both finite and infinite orbits. This is most easily seen on Hilbert space level so we proceed by first identifying a GNS representation of a reduced group C^*-dynamical system.

LEMMA 2.2. *If $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$ is a reduced group C^*-dynamical system, and ρ is its group automorphism, then $(L^2(\Gamma), \mathsf{id}, \delta_1, U)$ is the GNS representation of the system where $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$.*

PROOF. $(L^2(\Gamma), \mathsf{id})$ is a trivial representation of $C_r^*(\Gamma)$ with ω as its vector state in δ_1. Therefore, for $(L^2(\Gamma), \mathsf{id}, \delta_1)$ to be the cyclic representation of the system, we only have to show that δ_1 is a cyclic vector, i.e. that $\{\mathsf{id}(T)\delta_1 : T \in C_r^*(\Gamma)\} = \{T\delta_1 : T \in C_r^*(\Gamma)\}$ is dense in $L^2(\Gamma)$. Consider any $f \in L^2(\Gamma)$ and let $\epsilon > 0$ be arbitrary. Since $\{\delta_g : g \in \Gamma\}$ is a total orthonormal set in $L^2(\Gamma)$ by Theorem 1.1,

$$\|f - \sum_{j=1}^{m} a_j \delta_{g_j}\| < \epsilon$$

for some $a_1, ..., a_m \in \mathbb{C}$ and $g_1, ..., g_m \in \Gamma$. Since $S = \sum_{j=1}^{m} a_j \lambda(g_j)$ is an operator in $C_r^*(\Gamma)$ and $S\delta_1 = \sum_{j=1}^{m} a_j \delta_{g_j}$ we thus have that $\|S\delta_1 - f\| < \epsilon$. As $f \in L^2(\Gamma)$ and $\epsilon > 0$ was arbitrary this shows that $\{T\delta_1 : T \in C_r^*(\Gamma)\}$ is dense in $L^2(\Gamma)$.

By definition, the GNS representation of τ is given by the mapping

$$\mathsf{id}(A)\delta_1 \mapsto \mathsf{id}(\tau(A))\delta_1$$

for all $A \in C_r^*(\Gamma)$, and in particular for all $g \in G$ by,

$$\delta_g = \lambda(g)\delta_1 \mapsto U\lambda(g)U^*\delta_1 = U\lambda(g)\delta_1 = U\delta_g.$$

Thus the GNS representation of τ corresponds with U, used to define τ, on span $\{\delta_g : g \in G\}$. Hence, as $\{\delta_g : g \in G\}$ is an orthonormal basis of $L^2(\Gamma)$, the GNS representation of τ is $U \in \mathscr{L}(L^2(\Gamma))$. □

An important observation to be made from Lemma 2.2 is that, since we are excluding trivial groups consisting of single elements from consideration, no reduced group C^*-dynamical system is trivial. That is, the Hilbert space of the GNS representation, $L^2(\Gamma)$, with total orthonormal basis $\{\delta_g : g \in \Gamma\}$, is not one dimensional.

PROPOSITION 2.3. *If all of the orbits of a reduced group C^*-algebraic system's group automorphism are infinite, then the system is strongly mixing.*

PROOF. Let ρ be the group automorphism of a reduced group C^*-dynamical system $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$. By Lemma 2.2, τ has a GNS representation given by $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$.

For any $g, h \in \Gamma$ and $n \in \mathbb{N}$, we have that

$$\langle \delta_g, U^n \delta_h \rangle = \langle \delta_g, \delta_{\rho^{-n}(h)} \rangle.$$

If $h \neq 1$, then there is an $N \in \mathbb{N}$ such that $\delta_{\rho^{-n}(h)} \neq \delta_g$ for all $n \geq N$, since otherwise the orbit $\{\rho^n(h) : n \in \mathbb{N}\}$ would be a finite set. That is, if $\rho^{-m}(h) = g$ and $\rho^{-n}(h) = g$ for distinct $m, n \in \mathbb{N}$, then $\{\rho^n(h) : n \in \mathbb{N}\}$ can consist of at most $|m - n|$ elements. Since $\{\delta_g : g \in \Gamma\}$ is an *ortho*normal set by Theorem 1.1, $\langle \delta_g, U^n \delta_h \rangle = 0$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} \langle \delta_g, U^n \delta_h \rangle = 0 = \langle \delta_g, \delta_1 \rangle \langle \delta_1, \delta_h \rangle. \tag{60}$$

If $h = 1$, then $U^n \delta_h = \delta_h$ for all $n \in \mathbb{N}$ so that

$$\begin{aligned} \langle \delta_g, U^n \delta_h \rangle &= \langle \delta_g, \delta_h \rangle \\ &= \langle \delta_g, \delta_1 \rangle \langle \delta_1, \delta_h \rangle \end{aligned}$$

since $h = 1$. Thus

$$\lim_{n\to\infty} \langle x, U^n y \rangle = \langle x, \delta_1 \rangle \langle \delta_1, y \rangle \tag{61}$$

holds for $x = \delta_g, y = \delta_h$ with $g, h \in \Gamma$ arbitrary. Let $x, y \in \operatorname{span}\{\delta_g : g \in \Gamma\}$ be arbitrary with, say $x = \sum_{i=1}^{\ell} \alpha_i \delta_{g_i}$ and $y = \sum_{j=1}^{m} \beta_j \delta_{h_j}$. If $\delta_{h_j} \neq 1$ for all j, then by the preceding arguments

$$\langle x, U^n y \rangle = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \overline{\alpha_i} \beta_j \langle \delta_{g_i}, U^n \delta_{h_j} \rangle = 0$$

for n large enough. Since, $\langle \delta_1, y \rangle = 0$ in this case, (61) holds. If $h_k = 1$ for some $k \in \{1, ..., m\}$, then it again follows that for a large enough n

$$\begin{aligned} \langle x, U^n y \rangle &= \sum_{i=1}^{\ell} \overline{\alpha_i} \beta_k \langle \delta_{g_i}, U^n \delta_{h_k} \rangle \\ &= \sum_{i=1}^{\ell} \overline{\alpha_i} \beta_k \langle \delta_{g_i}, \delta_1 \rangle \\ &= \langle x, \delta_1 \rangle \beta_k \\ &= \langle x, \delta_1 \rangle \langle \delta_1, y \rangle. \end{aligned}$$

This show that (61) holds for any $x, y \in \operatorname{span}\{\delta_g : g \in \Gamma\}$. That (61) holds for all $x, y \in L^2(\Gamma)$, follows from the fact that $\{\delta_g : g \in \Gamma\}$ is dense in $L^2(\Gamma)$ and similar arguments utilizing the triangle inequality, as in the proof of Proposition 3.1. Thus $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$ is strongly mixing by Proposition 3.1 (c). $\square$

The condition in Proposition 2.3 is in fact an "if and only if" condition which we will not have to prove, as it will be an immediate corollary of a later proposition.

PROPOSITION 2.4. *A reduced group C^*-algebraic system is compact if and only if all of its group automorphism's orbits are finite.*

PROOF. Let ρ be the group automorphism of a reduced group C^*-dynamical system $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$. By Lemma 2.2, τ has a GNS representation given by $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$. We first prove a preliminary result, from which the proposition will follow readily.

Consider any $g \in \Gamma$ and suppose that $\{\rho^n(g) : n \in \mathbb{Z}\}$ is finite. Then, since $U^n \delta_g = \delta_{\rho^{-n}(g)}$, $\{U^n \delta_g : n \in \mathbb{Z}\}$ is finite and therefore trivially totally bounded in $L^2(\Gamma)$. Conversely, suppose that $\{U^n \delta_g : n \in \mathbb{Z}\} = \{\delta_{\rho^{-n}(g)} : n \in \mathbb{Z}\}$ is totally bounded in $L^2(\Gamma)$. We can then show that $\{\rho^{-n}(g) : n \in \mathbb{Z}\} = \{\rho^n(g) : n \in \mathbb{Z}\}$ must be finite. For any $g, h \in \Gamma$, $\|\delta_g - \delta_h\|_{L^2} = \sqrt{2}$ so fix an $\epsilon > 0$ such that $\epsilon < \frac{\sqrt{2}}{2}$. Since $\{\delta_{\rho^n(g)} : n \in \mathbb{Z}\}$ is totally bounded there is a finite set $M = \{z_1, ..., z_m\} \subseteq L^2(\Gamma)$ such that for any $n \in \mathbb{Z}$, $\|\delta_{\rho^n(g)} - z_k\| < \epsilon$ for some $k = 1, ..., m$ *and* $\|\delta_h - z_k\| \geq \epsilon$ for all $h \neq \rho^n(g)$, since otherwise

$$\|\delta_{\rho^n(g)} - \delta_h\| \leq \|\delta_{\rho^n(g)} - z_k\| + \|z_k - \delta_h\| < 2\epsilon < \sqrt{2}.$$

Hence, each open ϵ ball centered at z_k, for $k = 1, ..., m$, contains at most one element from $\{\delta_{\rho^n(g)} : n \in \mathbb{Z}\}$ and since every element in $\{\delta_{\rho^n(g)} : n \in \mathbb{Z}\}$ is in one of these ϵ-balls, it follows that $\{\rho^n(g) : n \in \mathbb{Z}\}$ is finite.

Now, assume that $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$ is compact. By Theorem 4.10 and Proposition 4.9, all $f \in L^2(\Gamma)$ have totally bounded orbits relative to $\{U^n : n \in \mathbb{Z}\}$ in $L^2(\Gamma)$. In particular, $\{U^n \delta_g : n \in \mathbb{Z}\}$ is totally bounded in $L^2(\Gamma)$ for all $g \in \Gamma$. Then, by the preceding argument, $\{\rho^n(g) : n \in \mathbb{Z}\}$ is finite for all $g \in G$, or in other words, the group automorphism ρ has finite orbits.

Conversely, assume that ρ has finite orbits. Then, by the preceding argument, δ_g has totally bounded orbits relative to $\{U^n : n \in \mathbb{Z}\}$ for all $g \in \Gamma$ in $L^2(\Gamma) = H$. By Lemma 4.8 and Proposition 4.9, span $\{\delta_g : g \in \Gamma\} \subset H_{uds}$. Since H_{uds} is a closed, $H_{uds} = H = L^2(\Gamma)$ by Theorem 1.1. Thus $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$ is compact by Theorem 4.10. □

PROPOSITION 2.5. *If a reduced group C^*-dynamical system's group automorphism has at least one finite orbit, then the system is not ergodic.*

PROOF. Let ρ be the group automorphism of a reduced group C^*-dynamical system $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$. By Lemma 2.2, τ has a GNS representation given by $U : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto f \circ \rho$. Suppose that ρ has a finite orbit in $h \in \Gamma \backslash \{1\}$, i.e. $\{\rho^n(h) : n \in \mathbb{Z}\}$ is finite.

By Corollary 4.3 and Proposition 1.4 we know that $(C_r^*(\Gamma), \omega, \tau_\rho, \mathbb{Z})$ is ergodic if and only if

$$Uf = f \;\forall\; n \in \mathbb{Z} \iff f \in \mathbb{C}\delta_1$$

The fact that $\{\rho^n(h) : n \in \mathbb{Z}\}$ is a finite set allows us to easily define an $f \in L^2(\Gamma)$ that is a fixed point of U, but not a constant multiple of δ_1. Suppose that $\{\rho^n(h) : n \in \mathbb{Z}\} = \{h_1, ..., h_m\}$. It is clear that $\sum_{j=1}^m \delta_{\rho^{-1}(h_j)} = \sum_{j=1}^m \delta_{h_j}$. If we set $f = \sum_{j=1}^m \delta_{h_j}$, then it follows that

$$\begin{aligned} Uf &= U\sum_{j=1}^m \delta_{h_j} \\ &= \sum_{j=1}^m \delta_{\rho^{-1}(h_j)} \\ &= \sum_{j=1}^m \delta_{(h_j)} = f. \end{aligned}$$

Therefore f is a fixed point of U and since $f \notin \mathbb{C}\delta_1$, $(C_r^*(\Gamma), \omega, \tau, \mathbb{Z})$ is not ergodic. □

Since a C^*-dynamical system that is strongly mixing is also ergodic, by Proposition 2.3, Proposition 2.5 has the following corollary.

Corollary 2.6. *If a reduced group C^*-dynamical system, with group automorphism ρ, is strongly mixing, then all of the orbits of ρ are infinite. Furthermore, a reduced group C^*-dynamical system is strongly mixing if and only if it is ergodic.*

Finally, if the group automorphism ρ has both finite and infinite orbits, then the system is not ergodic, by Proposition 2.5, and not compact by Proposition 2.4. For any reduced group C^*-dynamical system with group automorphism ρ we therefore have in summary:

ρ	has only infinite orbits	$\equiv$	strongly mixing
ρ	has only finite orbits	$\equiv$	compact but not ergodic
ρ	has both finite orbits and infinite orbits	$\equiv$	neither compact nor ergodic

3. The case of free groups

In the preceding section we learned that to completely specify a reduced group C^*-dynamical system, we only need to supply a group with a group automorphism. It was also proven that a reduced group C^*-dynamical system is either compact but not ergodic, strongly mixing or does not have any of the four ergodic properties that we are considering. We referred to these as the three different ergodic "profiles" that are possible. It is now our aim to supply examples of each of these three possibilities. With reference to the summary at the end of the previous section, it is not an altogether difficult exercise to think of three group and group automorphism pairs that will yield systems having, consecutively, each of these three ergodic "profiles". It is far more interesting, however, to consider three reduced group C^*-dynamical systems that share the same group, but with three different group automorphisms. Ergodic properties depend on the C^*-algebra, state and the time evolution. For any reduced group C^*-dynamical system, changing the group *and* the automorphism, amounts to changing the C^*-algebra, the state *and* the time evolution. It would then come as no surprise if the new system has different ergodic properties. However, if we only change the group automorphism, then all that changes, is the time evolution of the system, so we would of course then expect the ergodic properties to be at risk of changing as well. Therefore, identifying a single group with three different group automorphisms, such that the resulting three reduced group C^*-dynamical systems have these three different ergodic "profiles", would be a good example this. A group for which this is readily possible, is the free group of a countable set.

Let $S = \{..., s_{-2}, s_{-1}, s_0, s_1, s_2, ...\}$ be a countable set and let F_S be the free group generated by S. Thus, each element in S is assigned an inverse $S^{-1} = \{s^{-1} : s \in S\}$ and elements of F_S consist of finite strings of elements of $S \cup S^{-1}$ with "elimination" taken into account, i.e. any substring $s^{-1}s$ or ss^{-1} is removed. In particular, the identity 1 of F_S is the empty string. Thus

$$F_S = \{a_1...a_m \ : \ a_i \in S \cup S^{-1} \text{ and } a_i \neq a_{i+1}^{-1} \ \forall \ i = 1, ..., m-1\}.$$

For any $a, b \in F_S$ with, say, $a = a_1...a_m$ and $b = b_1...b_n$, the group operation of F_S is given by:

$$ab = a_1...a_{m-k}b_k...b_n$$

where $0 \leq k \leq \min\{m, n\}$ is the smallest number such that $a_{m-k} \neq b_k^{-1}$.

If $q : S \to S$ is a bijection then q can be used to define a group automorphism on F_S.

PROPOSITION 3.1. *If $q : S \to S$ is bijective, then*

$$Q_q : F_s \to F_S : a_1...a_m \mapsto q(a_1)...q(a_m)$$

is a group automorphism on F_S if we define

$$q(1) = 1 \textit{ and } q(s^{-1}) = q(s)^{-1} \ \forall \ s \in S.$$

PROOF. Since $q : S \to S$ is bijective, it is clear that

$$q(s_1) = q(s_2)^{-1} \ \Leftrightarrow \ s_1 = s_2^{-1} \tag{62}$$

for all $s_1, s_2 \in S \cup S^{-1}$ which shows that $Q_q : F_S \to F_S$ is well defined.

Let $a, b \in F_S$ be arbitrary with, say, $a = a_1...a_m$ and $b = b_1...b_n$. Let $0 \leq k \leq \min\{m, n\}$ be the smallest number such that $a_{m-k} \neq b_k^{-1}$. Then, by (62),

$$\begin{aligned} Q_q(a)Q_q(b) &= q(a_1)...q(a_{m-k})q(b_k)...q(b_n) \\ &= Q_q(a_1...a_{m-k}b_k...b_n) \\ &= Q_q(ab). \end{aligned}$$

Thus $Q_q : F_s \to F_S$ is a group automorphism □

Now that we know that any bijection of S defines a group automorphism of F_S it is a simple matter to identify three bijections that defines three group automorphisms of F_S that have, consecutively, infinite orbits, finite orbits and neither finite nor infinite orbits. This is simply because it is easy to identify bijections of S that have, consecutively, infinite orbits, finite orbits and neither finite nor infinite orbits, which then carry over to F_S.

PROPOSITION 3.2. *Let $m \in \mathbb{N}$. If we define*

$$q_1 : S \to S : s_i \mapsto s_{i+1} \ \forall \ i \in \mathbb{Z}$$

$$q_2 : S \to S : \begin{cases} s_i \mapsto s_{i+1} & \forall \ i \notin \{mn : n \in \mathbb{Z}\}, \\ s_{mn} \mapsto s_{m(n-1)+1} & \forall \ n \in \mathbb{Z} \end{cases}$$

$$q_3 : S \to S : \begin{cases} s_{2n} \mapsto s_{2(n+1)} & , \textit{and} \\ s_{2n+1} \mapsto s_{2n+1} & \forall \ n \in \mathbb{Z} \end{cases}$$

then $Q_{q_1} : F_S \to F_S$ has infinite orbits, $Q_{q_2} : F_S \to F_S$ has finite orbits and $Q_{q_2} : F_S \to F_S$ has both finite orbits and infinite orbits.

PROOF. The proof is elementary. First note that q_1 "shifts" each element ahead by one, q_2 shifts every element ahead by one, except if the element is in a mn position, in which case it is moved backwards by m units, and so creates an infinite series of consecutive "m-loops". Lastly, q_3 only moves every second element ahead by two.

It is clear that q_1, q_2 and q_3 have, respectively, infinite orbits, finite orbits and both finite orbits and infinite orbits in S. Therefore, we only have to understand that these orbits carry over to F_S. Consider any nonempty string in F_S, say $a = a_1a_2...a_m$. Then, for any $n \in \mathbb{Z}$, $Q_{q_i}^n(a) = q_i^n(a_1)q_i^n(a_2)...q_i^n(a_m)$.

In the case of Q_{q_1}, since for any $s \in S \cup S^{-1}$, $q_1^n(s)$ is distinct for all $n \in \mathbb{Z}$, it is clear that $Q_{q_1}^n(a) \neq a$ for any $n \in \mathbb{Z}\backslash\{0\}$. That is, Q_{q_1} has infinite orbits.

In the case of Q_{q_2}, we have that $q_2^m(s) = s$ for any $s \in S \cup S^{-1}$, so that $Q_{q_2}^{mn}(a) = a$ for all $n \in \mathbb{Z}$. That is, Q_{q_2} has finite orbits.

In the case of Q_{q_3}, if $a_1, ..., a_m \in \{s_{2n+1} : n \in \mathbb{Z}\}$, then $Q_{q_3}^n(a) = a$ for all $n \in \mathbb{Z}$. But if $a_i \in \{s_{2n} : n \in \mathbb{Z}\}$ for at least one i, then q_3 will send a_i on an infinite cycle of two "shifts" forwards or backwards so that $Q^n_{q_3}(a) \neq a$ for all $n \in \mathbb{Z}\backslash\{0\}$. That is, $\{Q^n_{q_3}(a) : n \in \mathbb{Z}\}$ is finite for some $a \in F_S$ and infinite for others. □

With the free group F_S generated from S, and the three group automorphisms q_1, q_2, q_3 of F_S we can now define the three reduced group C^*-dynamical systems using definition 1.7. For convenience, we wish to give each a name.

Definition 3.3. The reduced free group C^*-dynamical systems 1,2 and 3 (ReFG1,ReFG2 and ReFG3) are the reduced group C^*-dynamical systems with group F_S and, respectively, group automorphisms Q_{q_1}, Q_{q_2} and Q_{q_3}. That is, ReFG1 $=$ FG(F_S, Q_{q_1}), ReFG2 $=$ FG(F_S, Q_{q_2}) and ReFG3 $=$ FG(F_S, Q_{q_3}).

Proposition 3.4. *ReFG1 is strongly mixing.*

Proof. Immediate from Propositions 2.3 and 3.2. □

Proposition 3.5. *ReFG2 is compact but not ergodic.*

Proof. Immediate from Propositions 2.4,2.5 and 3.2. □

Proposition 3.6. *ReFG1 is neither ergodic nor compact.*

Proof. Immediate from propositions 2.4,2.5 and 3.2. □

CHAPTER 5

The Quantum Torus

The quantum torus is a very basic example of a noncommutative C^*-algebra that has come up again and again in the literature over the past thirty years. In particular, the quantum torus has been studied both in regard to quantum chaos and quantum ergodic theory in the physics literature. In [**23**], Heide explains that, to study and define quantum chaos, a classical chaotic system can be quantized to study what footprints of the classic chaotic behaviour remains in the quantum case, and in [**5**] it is explained that, as maps on the torus offer canonical examples of chaotic systems, it is natural to test conjectures about the quantization of chaotic systems by quantizing such maps. Our interest lies in the quantization of maps on the torus, as this yields a noncommutative dynamical system on which we can attempt ergodic analysis. In [**17**], Klimek and Leśniewski explains that quantization of dynamical systems has two components: construction of a quantized phase space of the dynamical system, given by a noncommutative algebra of observables, and defining a time evolution on the quantized phase space. In [**18**], Klimek and Leśniewski discusses how quantum ergodicity can be defined and interpreted on these quantized systems, and then proceed to study the ergodic properties of two quantized maps on the torus, the Kronecker map (translation on the torus) and Arnold's cat map (group automorphism on the torus).

In this chapter we will quantize two maps on the torus: We will first quantize the group of translations on the torus, which should not be confused with the Kronecker map wherein a dynamic is defined with a $\mathbb{Z}$ action using a single translation on the torus. Instead, we will use the group $\mathbb{R}^2$ of all translations on the torus to determine an action. For our final example we will, similar to [**18**], quantize the Arnold cat map albeit in a different manner that fits into our framework, using C^*-dynamical systems. Our approach for both examples will be to start with the Hilbert space of functions *on the torus*, $L^2(\mathbb{T}^2)$, and define two unitary operators in $\mathscr{L}(H)$, satisfying a certain commutation relation, that will act as the generators of the quantum torus $\mathcal{A}_\theta$. We will then define evolution on $\mathcal{A}_\theta$ by realizing the group of translations, and Arnold's cat map as unitary operators in $\mathscr{L}(L^2(\mathbb{T}^2))$, and then defining evolution in terms of these unitary operators. Our approach to quantize Arnold's cat map will be significantly simpler than [**18**] but our ergodic analysis will mirror the results of [**18**].

1. Construction of the quantum torus

In this section we will first define what is meant by a quantum torus, and discuss its relation to the classical torus, before we consider how a C^*-dynamical system can be constructed on top of it. Both the quantum and classical torus are constructed on top of the topological torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Its algebraic, topological and measure structure are discussed in the appendix. For the sake of brevity, and to safeguard the simplicity of the expressions to follow, we will adopt a simplified notation for functions defined on the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. For any $f : \mathbb{T}^2 \to \mathbb{C}$ and $(x, y) + 2\pi\mathbb{Z}^2 \in \mathbb{T}^2$ we will denote the value of f at $(x, y) + 2\pi\mathbb{Z}^2$ simply by $f(x, y)$. We will do the same for points on $\mathbb{T}^2$ and will write $(x, y) \in \mathbb{T}^2$ instead of $(x, y) + 2\pi\mathbb{Z}^2 \in \mathbb{T}^2$.

Furthermore, throughout this chapter, ν will denote the normalized Haar measure on $\mathbb{T}^2$, which is compact, and $L^2(\mathbb{T}^2)$ will denote $L^2(\nu)$.

In the following Theorem we define two unitary operators in $\mathscr{L}(\mathbb{T}^2)$, which will be the building blocks of the quantum torus.

THEOREM 1.1. *For any $\theta \in \mathbb{R}$, let*

$$\mathbb{U}_\theta, \mathbb{V}_\theta : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$$

be defined by

$$(\mathbb{U}_\theta f)(x, y) = e^{ix} f(x, y - \frac{\theta}{2})$$
$$(\mathbb{V}_\theta f)(x, y) = e^{iy} f(x + \frac{\theta}{2}, y)$$

for any $f \in L^2(\mathbb{T}^2)$ and $x, y \in \mathbb{R}$. Then $\mathbb{U}_\theta$ and $\mathbb{V}_\theta$ are unitary operators in $\mathscr{L}(L^2(\mathbb{T}^2))$. Furthermore

$$\mathbb{U}_\theta\mathbb{V}_\theta = e^{-i\theta}\mathbb{V}_\theta\mathbb{U}_\theta.$$

PROOF. For any $f \in L^2(\mathbb{T}^2)$ it follows by the translation invariance of ν that

$$\begin{aligned}\int_{\mathbb{T}^2} |f|^2 \, d\nu &= \int_{\mathbb{T}^2} |f(x, y - \frac{\theta}{2})|^2 \, d\nu(x, y)\\ &= \int_{\mathbb{T}^2} |e^{ix} f(x, y - \frac{\theta}{2})|^2 \, d\nu\\ &= \int_{\mathbb{T}^2} |\mathbb{U}_\theta f|^2 \, d\nu.\end{aligned}$$

Similarly, $\int_{\mathbb{T}^2} |f|^2 \, d\nu = \int_{\mathbb{T}^2} |\mathbb{V}_\theta f|^2 \, d\nu$. Thus $\mathbb{U}_\theta$ and $\mathbb{V}_\theta$, which are also clearly linear, are well defined isometries in $\mathscr{L}(L^2(\mathbb{T}^2))$. Note that by a completely similar argument, the mappings $\mathbb{U}'_\theta, \mathbb{V}'_\theta$ defined by

$$(\mathbb{U}'_\theta f)(x,y) = e^{-ix} f(x, y + \frac{\theta}{2})$$
$$(\mathbb{V}'_\theta f)(x,y) = e^{-iy} f(x - \frac{\theta}{2}, y)$$

for all $f \in L^2(\mathbb{T}^2)$, are also well-defined operators in $\mathscr{L}(L^2(\mathbb{T}^2))$. Clearly

$$\mathbb{U}_\theta \mathbb{U}'_\theta = \mathbb{U}'_\theta \mathbb{U}_\theta = 1 = \mathbb{V}_\theta \mathbb{V}'_\theta = \mathbb{V}'_\theta \mathbb{U}_\theta$$

so that, therefore, $\mathbb{U}'_\theta = \mathbb{U}_\theta^{-1}$ and $\mathbb{V}'_\theta = \mathbb{V}_\theta^{-1}$. Therefore, $\mathbb{U}_\theta$ and $\mathbb{V}_\theta$ are surjective isometries, and hence necessarily unitary by [**20**, 3.10-6 Theorem (f)].

To obtain the final assertion, let $f \in L^2(\mathbb{T}^2)$ be arbitrary. Then, for any $x, y \in \mathbb{R}$ we have that

$$\begin{aligned}
(\mathbb{U}_\theta \mathbb{V}_\theta f)(x,y) &= (\mathbb{U}_\theta(\mathbb{V}_\theta f))(x,y) \\
&= e^{ix}(\mathbb{V}_\theta f)(x, y - \frac{\theta}{2}) \\
&= e^{ix} e^{i(y-\frac{\theta}{2})} f(x + \frac{\theta}{2}, y - \frac{\theta}{2}) \\
&= e^{-i\theta} e^{i(x+\frac{\theta}{2})} e^{iy} f(x + \frac{\theta}{2}, y - \frac{\theta}{2}) \\
&= e^{-i\theta} e^{iy} (\mathbb{U}_\theta f)(x + \frac{\theta}{2}, y) \\
&= e^{-i\theta} (\mathbb{V}_\theta(\mathbb{U}_\theta f))(x,y).
\end{aligned}$$

Thus, $\mathbb{U}_\theta \mathbb{V}_\theta = e^{-i\theta} \mathbb{V}_\theta \mathbb{U}_\theta$. □

DEFINITION 1.2. For any fixed $\theta \in \mathbb{R}$, the *quantum torus* is the C^*-subalgebra of $\mathscr{L}(L^2(\mathbb{T}^2))$ generated by the unitary operators $\mathbb{U}_\theta$ and $\mathbb{V}_\theta$, and is denoted $\mathcal{A}_\theta$.

The trigonometric polynomials play a central role in the quantum torus and that they are dense in $C(\mathbb{T}^2)$ and $L^2(\mathbb{T}^2)$ is of fundamental importance. In order to make the proof of this fact more concise, we include the following lemma, which is a special case of [**6**, Proposition 7.4.2] extended to complex functions (see bottom of [**6**, p. 228]).

LEMMA 1.3. *Let μ be a regular measure on a compact Hausdorff space X. Then $C(X)$ is dense in $L^2(\mu)$.*

Since the Haar measure ν is regular, by definition [**6**, Section 9.2], and $\mathbb{T}^2$ is compact, Lemma 1.3 is applicable to $(\mathbb{T}^2, \nu)$.

THEOREM 1.4. *The trigonometric polynomials* span $\{E_{m,n} : m, n \in \mathbb{N}\}$ *where $E_{m,n}(x,y) = e^{imx}e^{iny}$, are dense both in $C(\mathbb{T}^2)$ and $L^2(\mathbb{T}^2)$ in their respective norms.*

PROOF. Let $\Gamma = \operatorname{span}\{e^{imx}e^{iny} : n, m \in \mathbb{Z}\}$. As $\Gamma \subseteq C(\mathbb{T}^2)$ and $\mathbb{T}^2$ is compact, $\Gamma \subseteq L^2(\mathbb{T}^2)$. Γ is self-adjoint in the sense that it is closed under complex conjugation. As the constant function 1 is a function in Γ, Γ vanishes at no point in $\mathbb{T}^2$. Γ also separates the points of $\mathbb{T}^2$ for if $(x_1, y_1) \neq (x_2, y_2)$ for some $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ then $f(x_1, y_1) \neq f(x_1, y_1)$ if $f = e^{ix}$ or if $f = e^{iy}$, depending on whether $x_1 - x_2 \neq 2\pi n$ or $y_1 - y_2 \neq 2\pi n$. Therefore by the Stone-Weierstrass Theorem, generalized for complex functions, [**26**, Theorem 7.33], Γ is dense in $C(\mathbb{T}^2)$.

For any $f \in C(\mathbb{T}^2)$, in particular any $f \in \Gamma$, we have that $\|f\|_2 \leq \|f\|_\infty$, so that the topology of $L^2(\mathbb{T}^2)$ restricted to Γ, is weaker than the topology of $C(\mathbb{T}^2)$ restricted to Γ. Therefore

$$\mathrm{cl}_{L^2(\mathbb{T}^2)}(\Gamma) \supseteq \mathrm{cl}_{C(\mathbb{T}^2)}(\Gamma) = C(\mathbb{T}^2) \tag{63}$$

by the density of Γ in $C(\mathbb{T}^2)$. By Lemma 1.3, $C(\mathbb{T}^2)$ is dense in $L^2(\mathbb{T}^2)$ from which it follows, by (63), that

$$\mathrm{cl}_{L^2(\mathbb{T}^2)}(\Gamma) \supseteq \mathrm{cl}_{L^2(\mathbb{T}^2)}(C(\mathbb{T}^2)) = L^2(\mathbb{T}^2)$$

which shows that $\mathrm{cl}_{L^2(\mathbb{T}^2)}(\Gamma) = L^2(\mathbb{T}^2)$. That is, Γ is dense in $L^2(\mathbb{T}^2)$ □

By [**22**, Theorem 2.1.15] we can recover $\mathbb{T}^2$ from the C^*-algebra $C(\mathbb{T}^2)$ up to a homeomorphism, and so we refer to $C(\mathbb{T}^2)$ as the classical torus, since, at least topologically, it contains all the information regarding $\mathbb{T}^2$. Hence, the classical torus $C(\mathbb{T}^2)$ is a commutative C^*-algebra as the algebraic operation is pointwise multiplication of the complex functions. The question immediately arises: In what way, if any, does the quantum torus generalize the classical torus? The answer lies in the choice of the parameter θ. As $\mathbb{U}_\theta\mathbb{V}_\theta = e^{-i\theta}\mathbb{V}_\theta\mathbb{U}_\theta$, by Theorem 1.1, it is clear that the quantum torus is commutative if and only if $\theta = 0$. Thus, the quantum torus, like the classical torus, is commutative if $\theta = 0$. More than that, not wishing to increase the suspense any further, the quantum torus and the classical torus are $*$-isomorphic if $\theta = 0$. This important result is a consequence of the fact that the trigonometric polynomials, $\sum \alpha_i e^{in_i x} e^{im_i y}$ are dense in $C(\mathbb{T}^2)$, and the observation that the norm of an operator in $\mathcal{A}_0$ is effectively the sup-norm of $C(\mathbb{T}^2)$. Before we proceed to prove that the two spaces are isomorphic, we state the latter more clearly in the following lemma.

LEMMA 1.5. *For any $g \in C(\mathbb{T}^2)$, the operator*

$$A : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto gf$$

is in $\mathscr{L}(L^2(\mathbb{T}^2))$, and

$$\|A\| = \|g\|_\infty = \sup_{(x,y)\in\mathbb{T}^2} |g(x, y)|.$$

PROOF. Let $M = \|g\|_\infty$. Thus $|g(x,y)| \leq M$ for all $(x,y) \in \mathbb{T}^2$ so that, for any $f \in L^2(\mathbb{T}^2)$

$$\|Af\|_{L^2}^2 = \int_{\mathbb{T}^2} |fg|^2 \, d\nu \leq M^2 \int_{\mathbb{T}^2} |f|^2 \, d\nu = M^2 \|f\|_{L^2}^2 \tag{64}$$

A is clearly linear, and from (64) we can see that $A : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is well defined, that $A \in \mathscr{L}(L^2(\mathbb{T}^2))$ and furthermore that $\|A\| \leq M$. If $M = 0$ we are done, so assume that $M > 0$.

As $|g|$ is continuous on $\mathbb{T}^2$, which is compact, $|g(x', y')| = M$ for some $(x', y') \in \mathbb{T}^2$. Let $0 < \epsilon < M$ be arbitrary. By the continuity of $|g|$, there is a nonempty open set $(x,y) \in A_\epsilon$ such that $|g(x,y)| > M - \epsilon$ for all $(x,y) \in A_\epsilon$. Since ν is a Haar measure, $\nu(A_\epsilon) > 0$ by [**6**, Lemma 9.2.2]. Define the function

$$f : \mathbb{T}^2 \to \mathbb{C} : (x,y) \mapsto \frac{1}{\sqrt{\nu(A_\delta)}} \chi_{A_\delta}.$$

Clearly $f \in L^2(\mathbb{T}^2)$ and $\|f\|_{L^2} = 1$. It follows that

$$\begin{aligned} \|Af\|_{L^2} = \sqrt{\int_{\mathbb{T}^2} |g \frac{1}{\sqrt{\nu(A_\epsilon)}} \chi_{A_\delta}|^2 \, d\nu} &= \sqrt{\frac{1}{\nu(A_\epsilon)} \int_{A_\epsilon} |g|^2 \, d\nu} \\ &\geq \sqrt{\frac{1}{\nu(A_\epsilon)} \int_{A_\epsilon} |M - \epsilon|^2 \, d\nu} \\ &= M - \epsilon. \end{aligned}$$

Thus $\|A\| \geq |M - \epsilon|$, and as $0 < \epsilon < M$ was arbitrary, $\|A\| \geq M$. $\square$

THEOREM 1.6. *If $\theta = 0$, the quantum torus, $\mathcal{A}_\theta$, and the classical torus, $C(\mathbb{T}^2)$, are $*$-isomorphic through the mapping*

$$\varphi : C(\mathbb{T}^2) \to \mathcal{A}_\theta$$

where $\varphi(f)h := fh$ for all $f \in C(\mathbb{T}^2)$ and $h \in L^2(\mathbb{T}^2)$.

PROOF. $\varphi : C(\mathbb{T}^2) \to \mathscr{L}(L^2(\mathbb{T}^2))$ is well defined by Lemma 1.5, and clearly determines a homomorphism of $C(\mathbb{T}^2)$ into $\mathscr{L}(L^2(\mathbb{T}^2))$. For any $f \in C(\mathbb{T}^2)$ and $g, h \in L^2(\mathbb{T}^2)$ we have that

$$\begin{aligned} \langle \varphi(\overline{f})g, h \rangle_{L^2(\mathbb{T}^2)} = \langle \overline{f}g, h \rangle &= \sqrt{\int_{\mathbb{T}^2} f\overline{g}h \, d\nu} \\ &= \langle g, fh \rangle \\ &= \langle g, \varphi(f)h \rangle \\ &= \langle \varphi(f)^* g, h \rangle. \end{aligned}$$

Hence $\varphi(\overline{f})g = \varphi(f)^* g$ for any $g \in L^2(\mathbb{T}^2)$, or in other words, $\varphi(\overline{f}) = \varphi(f)^*$. Thus $\varphi : C(\mathbb{T}^2) \to \mathscr{L}(L^2(\mathbb{T}^2))$ is a $*$-homomorphism.

Let $\Gamma = \operatorname{span}\{E_{m,n} : m, n \in \mathbb{Z}\}$. For any $m, n \in \mathbb{Z}$ and $h \in L^2(\mathbb{T}^2)$,

$$(\varphi(E_{m,n})h)(x, y) = e^{imx}e^{iny}h(x, y) = (\mathbb{U}_0^m\mathbb{V}_0^n h)(x, y).$$

Thus $\varphi(E_{m,n}) = \mathbb{U}_0^m\mathbb{V}_0^n$ so that, since φ is a homomorphism, we have that $\varphi(\Gamma) \subseteq \mathcal{A}_0$. If $f \in C(\mathbb{T}^2)$ is arbitrary then, by Theorem 1.4, $s_n \longrightarrow f$ for some sequence (s_n) in Γ. Since φ is linear and norm preserving, by Lemma 1.5, it follows that $\varphi(s_n) \longrightarrow \varphi(f)$ in $\mathscr{L}(L^2(\mathbb{T}^2))$. Therefore, since $\mathcal{A}_0$ is closed in $\mathscr{L}(L^2(\mathbb{T}^2))$, and $\varphi(\Gamma) \subseteq \mathcal{A}_0$, it follows that $\varphi(f) \in \mathcal{A}_0$. Thus $\varphi : C(\mathbb{T}^2) \to \mathcal{A}_0$ is a $*$-homomorphism. Moreover, φ is injective since for any $f, g \in C(\mathbb{T}^2)$, if $fh = gh$ for all $h \in L^2(\mathbb{T}^2)$, then $f = g$.

To show that $\varphi : C(\mathbb{T}^2) \to \mathcal{A}_0$ is surjective, let $A \in \mathcal{A}_0$ be arbitrary, and let (A_n) be a sequence in $\operatorname{span}\{\mathbb{U}_0^m\mathbb{V}_0^n : m, n \in \mathbb{Z}\}$ converging to A. Thus $(s_n) := (\varphi^{-1}(A_n))$ is a convergent sequence in $C(\mathbb{T}^2)$ converging to, say, $f \in C(\mathbb{T}^2)$. So, as before, $A_n = \varphi(s_n) \longrightarrow \varphi(f)$ so that by the uniqueness of limits in $\mathscr{L}(L^2(\mathbb{T}^2))$, we have that $A = \varphi(f)$. $\square$

It is worth noting that the $*$-isomorphism, φ, established between $C(\mathbb{T}^2)$ and $\mathcal{A}_0$ effectively does a GNS construction of $C(\mathbb{T}^2)$. That is, $(L^2(\mathbb{T}^2), \varphi, 1)$ is a cyclic representation of $C(\mathbb{T}^2)$.

2. Construction of C^*-dynamical systems

To construct a dynamical system on the quantum torus, we look to the classical torus $C(\mathbb{T}^2)$ for inspiration. One can easily imagine some sort of dynamics on the functions in $C(\mathbb{T}^2)$, and since $C(\mathbb{T}^2)$ and $\mathcal{A}_0$ are $*$-isomorphic, the dynamics' "manifestation" on $\mathcal{A}_0$ can be determined. Hopefully one could then generalize the dynamics to $\mathcal{A}_\theta$ and obtain a $*$-automorphism, or group of $*$-automorphisms. For example, if $\phi : \mathbb{T}^2 \to \mathbb{T}^2$, then it would be natural to consider the following dynamics on $C(\mathbb{T}^2)$:

$$\tau : C(\mathbb{T}^2) \to C(\mathbb{T}^2) : f \mapsto f \circ \phi. \tag{65}$$

Of course, for it to be well-defined we will have to place some conditions on ϕ, so lets assume that ϕ is a homeomorphism and measure preserving. From the former it can be shown that τ is a $*$-automorphism on $C(\mathbb{T}^2)$. If we can identify a suitable state ω, then we will have a C^*-dynamical system of the form $(C(\mathbb{T}^2), \omega, \tau, \mathbb{Z})$.

The question now is: Can we generalize (65) to $\mathcal{A}_\theta$? By Theorem 1.6, the mapping $\varphi : C(\mathbb{T}^2) \to \mathcal{A}_0$ where $\varphi(f)h = fh$, for any $f \in C(\mathbb{T}^2)$ and all $h \in L^2(\mathbb{T}^2)$, determines a $*$-isomorphism between $C(\mathbb{T}^2)$ and $\mathcal{A}_0$. So, for an arbitrary $A \in \mathcal{A}_0$, there is a unique $f = \varphi^{-1}(A) \in C(\mathbb{T}^2)$ such that $Ah = fh$ for all $h \in L^2(\mathbb{T}^2)$. Hence, we can migrate (65) to $\mathcal{A}_0$ with

$$\tau' : \mathcal{A}_0 \to \mathcal{A}_0 : A \mapsto \varphi \circ \tau \circ \varphi^{-1}(A). \tag{66}$$

Thus, for any $A \in \mathcal{A}_0$ and $h \in L^2(\mathbb{T}^2)$

$$\tau'(A)h = \varphi(\tau(\varphi^{-1}(A)))h = \tau(\varphi^{-1}(A))h = \tau(f)h.$$

That is, $Ah = fh$ and $\tau'(A)h = \tau(f)h$ which clearly shows (65)'s "manifestation" on $\mathcal{A}_0$. (66) can be written in a more useful form, if we define

$$U : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto f \circ \phi.$$

Then $U \in \mathscr{L}(L^2(\mathbb{T}^2))$ is unitary and

$$\tau' : \mathcal{A}_0 \to \mathcal{A}_0 : A \mapsto UAU^*. \tag{67}$$

The usefulness of (67) compared to (66) is that, in principle, it can be directly generalized to $\mathcal{A}_\theta$ with

$$\tau' : \mathcal{A}_\theta \to \mathcal{A}_\theta : A \mapsto UAU^*. \tag{68}$$

For (68) to be valid, we would require $U\mathcal{A}_\theta U^{-1} \subseteq \mathcal{A}_\theta$, however we would further like (68) to determine a $*$-automorphism of $\mathcal{A}_\theta$ in which case we will have succeeded in generalizing (65) to the quantum torus. So the next question is under what conditions does this occur? This question is answered, more generally, in the following proposition:

Proposition 2.1. *Let $\theta \in \mathbb{R}$. Let G be a group and U a unitary representation of G on $\mathscr{L}(L^2(\mathbb{T}^2))$ such that $U_g 1 = 1$ and $U_g \mathcal{A}_\theta U_g^* \subseteq \mathcal{A}_\theta$ for all $g \in G$. Define*

$$\omega : \mathcal{A}_\theta \to \mathbb{C} : A \mapsto \langle 1, A1 \rangle_{L^2}$$

$$\tau : G \to \mathrm{Aut}\,(\mathcal{A}_\theta) : g \mapsto \tau_g$$

$$\tau_g : \mathcal{A}_\theta \to \mathcal{A}_\theta : A \mapsto U_g A U_g^*$$

Then $(\mathcal{A}_\theta, \omega, \tau, G)$ is an abstract dynamical system. Furthermore, $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$ is its GNS representation.

Proof. $\mathcal{A}_\theta$ is a C^*-algebra by definition and it is easy to see that ω defines a positive linear functional on $\mathcal{A}_\theta$ such that $\omega(1) = 1$, i.e. ω defines a state. Since $U_g \mathcal{A}_\theta U_g^* \subseteq \mathcal{A}_\theta$ for all $g \in G$, we have that τ_g is well defined for all $g \in G$. That, $\tau_g \in \mathrm{Aut}\,(\mathcal{A}_\theta)$, $\tau_g \circ \tau_h = \tau_{gh}$ and $\omega \circ \tau_g = \omega$ for all $g, h \in G$, is established in the same way as was done for the group of automorphisms in Proposition 1.1 where the state and $*$-automorphisms are defined similarly. Thus, $(\mathcal{A}_\theta, \omega, \tau, G)$ as defined is an abstract dynamical system.

$(L^2(\mathbb{T}^2), \mathsf{id}, 1)$ is trivially a cyclic representation of $(\mathcal{A}_\theta, \omega, \tau, G)$ provided we know that $\{\mathsf{id}(T)1 : T \in \mathcal{A}_\theta\}$ is dense in $L^2(\mathbb{T}^2)$. Considering the operators in $\mathcal{A}_\theta$ that are finite sums of operators of the form $\mathbb{U}_\theta^m \mathbb{V}_\theta^n$ it is clear that, as $\mathsf{id}(\mathbb{U}_\theta^m \mathbb{V}_\theta^n)1 = \mathbb{U}_\theta^m \mathbb{V}_\theta^n 1 = e^{-\frac{m\theta}{2}} E_{m,n}$,

$$\{\mathsf{id}(A)1 : A \in \mathcal{A}_\theta\} \supseteq \mathrm{span}\{E_{m,n} : m, n \in \mathbb{N}\}.$$

Therefore, by Theorem 1.4, $\{\mathsf{id}(T)1 : T \in \mathcal{A}_\theta\}$ is dense in $L^2(\mathbb{T}^2)$. The GNS representation is now given by $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$ since $\mathsf{id}(\tau_g(A))1 = U_g A U_g^* 1 = U_g A U_{-g} 1 = U_g A 1$ as required in Definition 1.3. □

To apply Proposition 2.1 we have to supply the group of unitary operators $\{U_g : g \in G\}$. We will derive such a group from the group of translation maps on the torus, and later from a cat mapping on the torus, so that by Proposition 2.1 we will then have two abstract dynamical systems. In the case of the cat mapping on the torus, $G = \mathbb{Z}$, and so it will be immediate that the abstract dynamical system is a C^*-dynamical system. For the case of the group $\mathbb{R}^2$ of translations on the torus, however, we will have to explicitly establish that the abstract dynamical system obtained from Proposition 2.1 is a C^*-dynamical system, since then G will be equal to $\mathbb{R}^2$. We will consider the group of translations on the torus in the next section.

Proposition 2.1 reveals a remarkably convenient feature of C^*-dynamical systems on the quantum torus. If an abstract dynamical system is constructed in the way described by Proposition 2.1, and it can be shown to constitute a C^*-dynamical system for the particular choice in G and unitary operators $\{U_g : g \in G\}$, then the system has the same GNS representation for all values of θ. Recalling from section 2.3 that all the ergodic properties of a system can be completely characterized in terms of the GNS representation, on the Hilbert space of the cyclic representation, it follows that the ergodic properties of a C^*-dynamical system on the quantum torus $\mathcal{A}_\theta$, are the same for all values of θ. That is, the dynamical systems on the classical and quantum torus have the same ergodic properties and it is sufficient, therefore, to only consider the classical case, when $\theta = 0$, which is precisely what we shall do, if it simplifies the analysis.

3. Dynamics given by translation

In this section we will use the group of all translations on the torus

$$\mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (x + s, y + t)$$

to first define an abstract dynamical system on the torus using Proposition 2.1. In the classical case this dynamic reduces to the shifting of functions in $C(\mathbb{T}^2)$. We will then show that an abstract dynamical system created in this way, using translations, constitutes a C^*-dynamical system.

PROPOSITION 3.1. *For any $(s, t) \in \mathbb{R}^2$, let $\rho_{s,t} : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (x + s, y + t)$. Then*

$$U_{s,t} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto f \circ \rho_{s,t}$$

satisfies the requirements of Proposition 2.1 and the resulting abstract dynamical system $(\mathcal{A}_\theta, \omega, \tau, \mathbb{R}^2)$ is a C^*-dynamical system. Furthermore, $\tau_{s,t}(\mathbb{U}_\theta) = e^{is}\mathbb{U}_\theta$ and $\tau_{s,t}(\mathbb{V}_\theta) = e^{it}\mathbb{V}_\theta$.

PROOF. Let $\theta \in \mathbb{R}$ and $(s,t) \in \mathbb{R}^2$ be arbitrary. $U_{s,t}$ is easily seen to be linear, and by the translation invariance of ν

$$\int_{\mathbb{T}^2} |f|^2 \, d\nu = \int_{\mathbb{T}^2} |f(x+s, y+t)|^2 \, d\nu.$$

Thus $U_{s,t} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is a well defined isometry in $\mathscr{L}(L^2(\mathbb{T}^2))$. From the definition it is clear that $U_{s,t}U_{s',t'} = U_{s+s',t+t'}$ for any $(s',t') \in \mathbb{R}^2$, and in particular, $U_{s,t}U_{-s,-t} = U_{-s,-t}U_{s,t} = 1$ which shows that $U_{s,t}$ is invertible and therefore unitary, again by [**20**, Theorem 3.10-6 (f)]. This also shows that $G \to \mathscr{L}(L^2(\mathbb{T}^2)) : g \mapsto U_g$ is a unitary representation of G. $U_{s,t}1 = 1$ is immediate.

To show that $U_{s,t}\mathcal{A}_\theta U_{s,t} \subseteq \mathcal{A}_\theta$ first note that the mapping $\tau'_{s,t}$: $\mathscr{L}(L^2(\mathbb{T}^2)) : A \mapsto U_{s,t}AU^*_{s,t}$ is a $*$-automorphism of $\mathscr{L}(L^2(\mathbb{T}^2))$, as we have seen before, and is therefore isometric by [**22**, Theorem 2.1.7]. Now let $f \in L^2(\mathbb{T}^2)$ and $(x,y) \in \mathbb{T}^2$ be arbitrary. It follows that

$$\begin{aligned}
\left(U_{s,t}\mathbb{U}_\theta U^*_{s,t}f\right)(x,y) &= \left(\mathbb{U}_\theta U^*_{s,t}f\right)(x+s, y+t) \\
&= e^{i(x+s)}\left(U_{-s,-t}f\right)(x+s, y+t-\frac{\theta}{2}) \\
&= e^{is}e^{ix}f(x, y-\frac{\theta}{2}) \\
&= \left(e^{is}\mathbb{U}_\theta f\right)(x,y).
\end{aligned}$$

Thus $\tau'_{s,t}(\mathbb{U}_\theta) = e^{is}\mathbb{U}_\theta$, and similarly $\tau'_{s,t}(\mathbb{V}_\theta) = e^{it}\mathbb{V}_\theta$. Let

$$\mathcal{A}'_\theta = \operatorname{span}\{\mathbb{U}^m_\theta \mathbb{V}^n_\theta : m,n \in \mathbb{Z}\}.$$

Then it follows that

$$U_{s,t}\mathcal{A}'_\theta U^*_{s,t} \subseteq \mathcal{A}'_\theta$$

since $\tau'_{s,t}$ is a $*$-automorphism. If $A \in \mathcal{A}_\theta$, then $A_n \longrightarrow A$ for some sequence (A_n) in $\mathcal{A}'_\theta$. However, since $\tau'_{s,t}$ is isometric, it follows that $\lim_{n\to\infty} \tau'_{s,t}(A_n) = \tau'_{s,t}(A)$, so that since $\tau'_{s,t}(T_n) \in \mathcal{A}'_\theta \subseteq \mathcal{A}_\theta$ for all $n \in \mathbb{N}$, and $\mathcal{A}_\theta$ is closed, $\tau'_{s,t}(A) \in \mathcal{A}_\theta$ follows. That is, $\tau'_{s,t}(\mathcal{A}_\theta) = U_{s,t}\mathcal{A}_\theta U^*_{s,t} \subseteq \mathcal{A}_\theta$ as required.

Thus Proposition 2.1 applies, and $(\mathcal{A}_\theta, \omega, \tau, \mathbb{R}^2)$ is an abstract dynamical system with GNS representation $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$. We will now show that it is a C^*-dynamical system.

$\mathbb{R}^2$ is Abelian, separable and locally compact. An example of a Følner sequence in $\mathbb{R}^2$ is $\Lambda_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq n^2\}$. Thus $\mathbb{R}^2$ is a separable Abelian amenable group.

Consider any convergent sequence $(s_k, t_k) \subseteq \mathbb{R}^2$ with, say $(s_k, t_k) \longrightarrow (s,t)$, for some $(s,t) \in \mathbb{R}^2$. We wish to show that so-lim $U_{s_k,t_k} = U_{s,t}$ which holds if and only if $U_{s_k,t_k}f \longrightarrow U_{s,t}f$ for all $f \in L^2(\mathbb{T}^2)$. Consider first, an arbitrary trigonometric polynomial $e^{imx}e^{iny}$. It follows that

$$\begin{aligned}\|U_{s_k,t_k}E_{m,n} - U_{s,t}E_{m,n}\| &= \|e^{ims_k}e^{int_k}E_{m,n} - e^{ims}e^{int}E_{m,n}\| \\ &= |e^{ims_k}e^{int_k} - e^{ims}e^{int}|\|E_{m,n}\|\end{aligned}$$

so that, since $s_k \longrightarrow s$ and $t_k \longrightarrow t$, we have that $e^{ims_k}e^{int_k} \longrightarrow e^{ims}e^{int}$ by the continuity of $e^{imx}e^{iny}$, and therefore that

$$U_{s_k,t_k}E_{m,n} \longrightarrow U_{s,t}E_{m,n}. \tag{69}$$

For any element in span $\{E_{m,n} : m, n \in \mathbb{Z}\}$, say $\sum_{j=1}^{M} a_j E_{m,n}$, it follows that

$$\begin{aligned}&\|U_{s_k,t_k}\sum_{j=1}^{M} a_j E_{m_j,n_j} - U_{s,t}\sum_{j=1}^{M} a_j E_{m_j,n_j}\| \\ &\leq \sum_{j=1}^{M} |a_j|\|U_{s_k,t_k}E_{m_j,n_j} - U_{s,t}E_{m_j,n_j}\|\end{aligned}$$

so that, by (69) we still have that

$$U_{s_k,t_k}g \longrightarrow U_{s,t}g \tag{70}$$

for all $g \in$ span $\{E_{m,n} : m, n \in \mathbb{Z}\}$. Now, for any $f \in L^2(\mathbb{T}^2)$ and $\epsilon > 0$ there is, since the trigonometric polynomials are dense in $L^2(\mathbb{T}^2)$ by Theorem 1.4, an $f' \in$ span $\{E_{m,n} : m, n \in \mathbb{Z}\}$ such that $\|f - f'\| < \frac{\epsilon}{3}$. By (70) there is an $N \in \mathbb{N}$ such that $\|U_{s,t}f' - U_{s_k,t_k}f'\| < \frac{\epsilon}{3}$ for all $k \geq N$. It follows that

$$\begin{aligned}\|U_{s,t}f - U_{s_k,t_k}f\| &\leq \|U_{s,t}f - U_{s,t}f'\| + \|U_{s,t}f' - U_{s_k,t_k}f'\| \\ &\quad + \|U_{s_k,t_k}f' - U_{s_k,t_k}f\| \\ &= \|f - f'\| + \|U_{s,t}f' - U_{s_k,t_k}f'\| + \|f - f'\| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for all $k \geq N$. Hence, since $\epsilon > 0$ was arbitrary, $U_{s_k,t_k}f \longrightarrow U_{s,t}f$, and as $f \in L^2(\mathbb{T}^2)$ was arbitrary, so-lim $U_{s_k,t_k} = U_{s,t}$. Thus $U : (s,t) \mapsto U_{s,t}$ is so-continuous as required, and all of the requirements are met for $(\mathcal{A}_\theta, \omega, \tau, \mathbb{R}^2)$ to be a C^*-dynamical system. □

Proposition 3.1 gives us our first C^*-dynamical system on the torus. As with all other examples already considered, and yet to be considered, we define the C^*-dynamical system in full detail for easy reference.

Definition 3.2. For any fixed $\theta \in \mathbb{R}$, the Quantum Torus with Translations (QTT(θ)) is the C^*-dynamical system $(\mathfrak{A}, \omega, \tau, \mathbb{R}^2)$ where

(i) $\mathfrak{A} = \mathcal{A}_\theta$
(ii) $\omega : \mathcal{A}_\theta \to \mathbb{C} : A \mapsto \langle 1, A1\rangle_{L^2}$
(iii) $\tau : \mathbb{R}^2 \to \mathrm{Aut}\,(\mathcal{A}_\theta) : (s,t) \mapsto \tau_{s,t}$ where $\tau_{s,t} : \mathcal{A}_\theta \to \mathcal{A}_\theta : A \mapsto U_{s,t}AU^*_{s,t}$, $U_{s,t} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto f \circ \rho_{s,t}$ and $\rho_{s,t} : \mathbb{T}^2 \to \mathbb{T}^2 : (x,y) \mapsto (x+s, y+t)$

In the next section, we will investigate the ergodic properties of (QTT(θ)).

4. Ergodic properties of translation

By Proposition 2.1, the GNS representation of QTT(θ) as defined in Definition 3.2 is given by $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$.

By Corollary 4.3, to determine whether QTT(θ) is ergodic, we wish to find the dimension of the fixed point space of the GNS representation. That is, we wish to find all $f \in L^2(\mathbb{T})$ such that $U_{s,t}f = f$, or in other words, such that $f = f \circ \rho_{s,t}$, for all $(s,t) \in \mathbb{R}^2$. If all such fixed points are multiples of the known fixed point 1, the cyclic vector in $L^2(\mathbb{T}^2)$, then the fixed point space is one dimensional and we have that QTT(θ) is ergodic by Corollary 4.3. On the face of it, it certainly appears to be the case here, as how else could a function f satisfy $f = f \circ \rho_{s,t}$ for all $(s,t) \in \mathbb{R}^2$ if it is not constant. Note that, this is obviously true for a function $g : \mathbb{R}^2 \to \mathbb{C}$. However, briefly breaking an earlier rule regarding the *tacit* understanding that $L^2(X)$ is a space of equivalence classes,

$$f = f \circ \rho_{s,t} \ \forall \ (s,t) \in \mathbb{R}^2$$

strictly speaking means that

$$f_1 = f_2 \circ \rho_{s,t} \text{ a.e } \ \forall \ f_1, f_2 \in [f], (s,t) \in \mathbb{R}^2.$$

The question is thus a measure theoretic problem, and it has a non-trivial conclusion. We will instead derive a stronger condition than ergodicity for QTT(θ), and show that QTT(θ) is necessarily ergodic lest we contradict this stronger condition.

PROPOSITION 4.1. *In QTT(θ), ω is in fact the unique τ-invariant state on $\mathcal{A}_\theta$, i.e. the only state on $\mathcal{A}_\theta$ such that $\omega \circ \tau_{s,t} = \omega$ for all $(s,t) \in \mathbb{R}^2$.*

PROOF. Consider any $\theta \in \mathbb{R}$ and let ω' be any τ-invariant state on $\mathcal{A}_\theta$. Then, for any $m, n \in \mathbb{Z}$ and all $(s,t) \in \mathbb{R}^2$, it follows by Proposition 3.1 that

$$\begin{aligned} \omega'(\mathbb{U}_\theta^m \mathbb{V}_\theta^n) &= \omega'(\tau_{s,t}(\mathbb{U}_\theta^m \mathbb{V}_\theta^n)) \\ &= \omega'(e^{ims} e^{int} \mathbb{U}_\theta^m \mathbb{V}_\theta^n) \\ &= e^{ims} e^{int} \omega'(\mathbb{U}_\theta^m \mathbb{V}_\theta^n). \end{aligned}$$

Since $e^{ims}e^{int} \neq 1$ for some $(s,t) \in \mathbb{R}^2$ it follows that $\omega'(\mathbb{U}_\theta^m \mathbb{V}_\theta^n) = 0$. Since, by the same reasoning, $\omega(\mathbb{U}_\theta^m \mathbb{V}_\theta^n) = 0$ for arbitrary $m, n \in \mathbb{Z}$ it follows that $\omega = \omega'$ on span $\{\mathbb{U}_\theta^m \mathbb{V}_\theta^n : m, n \in \mathbb{Z}\}$. Hence, since span $\{\mathbb{U}_\theta^m \mathbb{V}_\theta^n : m, n \in \mathbb{Z}\}$ is dense in $\mathcal{A}_\theta$, it follows that $\omega' = \omega$. □

By Proposition 4.1, QTT(θ) has a property called *unique ergodicity*, which is yet another ergodic property with a slightly different origin than the properties that we are considering. Whereas the properties under our consideration have their origin in measure theoretic dynamics, unique ergodicity has its origin in topological dynamics.

We will now set out to show that the existence of a second fixed point of the GNS representation, that is not a multiple of 1, which is precisely what occurs in a system that is not ergodic, allows us to construct a second different invariant state, contradicting Proposition 4.1. We will require some preliminary results to construct this second state in general.

Lemma 4.2. *For any normalized $f \in L^2(\mathbb{T}^2)$ for which $|f| \neq 1$, there is an $g \in C(\mathbb{T}^2)$ such that*

$$|\langle f, gf \rangle - \langle 1, g1 \rangle| > 0.$$

Proof. Since $|f| \neq 1$, $|f|^2 \neq 1$. Therefore

$$\nu\left(\{z \in \mathbb{T}^2 : |f(z)|^2 > 1\}\right) > 0, \text{ or} \tag{71}$$

$$\nu\left(\{z \in \mathbb{T}^2 : |f(z)|^2 < 1\}\right) > 0. \tag{72}$$

First suppose that (71) holds. Since $\{z \in \mathbb{T}^2 : |f(z)|^2 > 1\} = \bigcup_{n=1}^{\infty}\{z \in \mathbb{T}^2 : |f(z)|^2 > 1 + \frac{1}{n}\}$, there is an $\epsilon > 0$ such that, if

$$S = \{z \in \mathbb{T}^2 : |f(z)|^2 > 1 + \epsilon\},$$

then $\nu(S) > 0$. Let $h = |f|^2 - 1$ so that $h > \epsilon$ on S and $-1 \leq h \leq \epsilon$ on $\mathbb{T}^2 \setminus S$. Without loss of generality we may assume that $\epsilon < 1$ so that $|h| < 1$ on $\mathbb{T}^2 \setminus S$. By the regularity of ν, for any $\delta > 0$ we can find an open set $W_\delta \supseteq S$ such that $\nu(W_\delta \backslash S) < \delta$ and since $\mathbb{T}^2$ has finite ν-measure and is Hausdorff, it follows by [**6**, Proposition 7.2.6] that there is a compact set $K \subseteq S$ such that $\nu(K) > 0$. Choose $\delta = \epsilon\nu(K)$. Utilizing [**26**, 2.12 Urysohn's Lemma] we now find a $g \in C(\mathbb{T}^2)$ such that $g : \mathbb{T}^2 \to [0, 1]$, $g = 1$ on K and $g = 0$ on $\mathbb{T}^2 \setminus W_\delta$. It follows by these properties of g that

$$\int_S hg \, d\nu \geq \int_K h \, d\nu > \epsilon\nu(K) = \delta$$

and that

$$\left|\int_{W_\delta \backslash S} hg \, d\nu\right| \leq \int_{W_\delta \backslash S} |hg| \, d\nu \leq \int_{W_\delta \backslash S} |h| \, d\nu < \epsilon\nu(K) = \delta.$$

It now follows that

$$
\begin{aligned}
\langle f, gf\rangle - \langle 1, g1\rangle &= \int_{\mathbb{T}^2} \left(|f|^2 g - g\right)\ d\nu \\
&= \int_{\mathbb{T}^2} hg\ d\nu \\
&= \int_S hg\ d\nu + \int_{\mathbb{T}^2\setminus S} hg\ d\nu \\
&= \int_S hg\ d\nu + \int_{W_\delta\setminus S} hg\ d\nu \\
&> 0.
\end{aligned}
$$

The result in the case that (72) holds follows by a similar argument, if we define $S = \{z \in \mathbb{T}^2 : |f(z)|^2 < 1-\epsilon\}$, but (71) has to be assumed not to hold so that $h = |f|^2 - 1$ is guaranteed to be bounded $\mathbb{T}^2 \setminus S$. $\square$

Any normalized fixed point f of the GNS representation can be used to define an invariant state $\langle f, (\cdot)f\rangle$ on $\mathcal{A}_\theta$, however it is not guaranteed to necessarily be different from QTT(θ)'s state $\omega = \langle 1, (\cdot)1\rangle$. If a normalized fixed point $f \in L^2(\mathbb{T}^2)$ has the property that $|f| \neq 1$, then we can use Lemma 4.2 to define a state that is different from ω, at some $h \in C(\mathbb{T}^2) \equiv \mathcal{A}_0$, which would contradict Proposition 4.1 in the case that $\theta = 0$. That such an f can be identified when QTT(0) is assumed to not be ergodic, is established in the following proposition.

Proposition 4.3. *Let $(L^2(\mathbb{T}^2), \mathit{id}, 1, U)$ be the GNS representation of QTT(θ)=$(\mathcal{A}_\theta, \omega, \tau, U)$. If f is a fixed point of U such that $f \notin \mathbb{C}1$, where $1 \in L^2(\mathbb{T}^2)$, then there is a normalized fixed point $g \in L^2(\mathbb{T}^2)$ of U such that $|g| \neq 1$.*

Proof. Without loss of generality we may assume that $\|f\| = 1$. If $|f| \neq 1$ we are done, so suppose that $|f| = 1$. That is, $\nu(f^{-1}\{1\}) = \nu(\mathbb{T}^2) = 1$. Define the following two functions:

$$
\begin{aligned}
g &= \frac{f+1}{\|f+1\|} \\
h &= \frac{f+i}{\|f+i\|}
\end{aligned}
$$

Since $f \notin \mathbb{C}1$, $f \neq -1$ and $f \neq -i$ so that $\|f+1\| \neq 0$ and $\|f+i\| \neq 0$. Thus, g and h are well defined. Furthermore, as a linear combination of the fixed points 1 and f, g and h are both fixed points of U.

Since $|f| = 1$, almost all of the function values of g lie on the circle in $\mathbb{C}$ centered at $1+0i$ and with radius $\frac{1}{\|f+1\|}$. Therefore, if $|g| = 1$ then g has at most two function values of the form $x+iy$ and $x-iy$, with $y \neq 0$, and hence f also has at most two distinct function values of the same form, say, $a+ib$ and $a-ib$ in $\mathbb{T}$. This is simply because the unit circle in $\mathbb{C}$ and a circle centered at $1+0i$ in $\mathbb{C}$ can only intersect at two points. Since $f \notin \mathbb{C}1$, $a+ib$ and $a-ib$ are the only two function values of f and they are distinct. That is, $\nu(f^{-1}(a+ib)) > 0$, $\nu(f^{-1}(a-ib)) > 0$ and $\nu(f^{-1}(a+ib)) + \nu(f^{-1}(a-ib)) = \nu(\mathbb{T}^2) = 1$.

Similarly, if $|h| = 1$, then similar reasoning shows that f has precisely two function values of the form $c + id$ and $-c + id$.

Clearly we cannot have that $|g| = 1$ and that $|h| = 1$ since f cannot have function values of the form $a + ib, a - ib, c + id, -c + id$ and simultaneously have precisely two function values. Thus $|g| \neq 1$ or $|h| \neq 1$. $\square$

Proposition 4.4. *QTT(θ) is ergodic.*

Proof. By Proposition 2.1 the GNS representation of QTT(θ) is given by $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$ which is independent of θ, so without loss of generality we may assume $\theta = 0$, since ergodicity can be characterized in terms of the GNS representation. If $f \in L^2(\mathbb{T}^2)$ is any normalized fixed point of the GNS representation, then

$$\omega_f : \mathcal{A}_0 \to \mathbb{C} : A \mapsto \langle f, Af \rangle$$

is a τ-invariant state on $\mathcal{A}_0$, as for any $(s,t) \in \mathbb{R}^2$ and $A \in \mathcal{A}_0$

$$\begin{aligned} \omega_f(\tau_{s,t}(A)) &= \langle f, U_{s,t} A U_{s,t}^* f \rangle \\ &= \langle U_{-s,-t} f, A U_{-s,-t} f \rangle \\ &= \langle f, Af \rangle \\ &= \omega_f(A). \end{aligned}$$

Suppose that, QTT(0) is not ergodic. Then, by Corollary 4.3 we can find a normalized fixed point $f \in L^2(\mathbb{T}^2)$ of the GNS representation such that $f \notin \mathbb{C}1$. By Proposition 4.3 we can find a normalized $g \in L^2(\mathbb{T}^2)$ that is a fixed point of the GNS representation such that $|g| \neq 1$. Thus ω_g is a τ-invariant state on $\mathcal{A}_0$. Hence, by Lemma 4.2 there is an $h \in C(\mathbb{T}^2)$ such that

$$\omega_g(A_h) - \omega(A_h) = \langle g, hg \rangle - \langle 1, h1 \rangle \neq 0 \tag{73}$$

where A_h is the operator in $\mathcal{A}_0$ such that $A_h f = hf$ for all $f \in L^2(\mathbb{T}^2)$. That $A_h \in \mathcal{A}_0$ follows from Theorem 1.6. What (73) shows is that ω_g is an invariant state on $\mathcal{A}_0$ which is different from ω. By Proposition 4.1 we have a contradiction. So QTT(0) is ergodic, and therefore so is QTT(θ) for all $\theta \in \mathbb{R}$. $\square$

Proposition 4.5. *QTT(θ) is compact.*

Proof. By Proposition 2.1 the GNS representation of QTT(θ) is given by $(L^2(\mathbb{T}^2), \mathsf{id}, 1, U)$. Let $\theta \in \mathbb{R}$ be arbitrary. For any $m, n \in \mathbb{Z}$

and $(s,t) \in \mathbb{R}^2$ it follows that

$$\begin{aligned} U_{s,t}E_{m,n}(x,y) &= E_{m,n}(x+s, y+t) \\ &= e^{im(x+s)}e^{in(y+t)} \\ &= e^{ims+int}E_{m,n}(x,y). \end{aligned}$$

Thus $E_{m,n}$ is in the eigenspace $L^2(\mathbb{T}^2)_{uds}$ of the GNS representation. Since $\{E_{m,n} : m,n \in \mathbb{Z}\}$ densely spans $L^2(\mathbb{T}^2)$, Theorem 1.4, it follows that $L^2(\mathbb{T}^2)_{uds} = L^2(\mathbb{T}^2)$. Hence, QTT$(\theta)$ is compact by Theorem 4.10. □

5. Dynamics given by a toral automorphism

In section 3 we used the translations $\{\rho_{s,t} : (s,t) \in \mathbb{R}^2\}$ on the torus to define the the time evolution of the C^*-dynamical system QTT(θ). In this section we will instead use a homeomorphic group automorphism φ of the torus, or *toral automorphism*, to define the time evolution. We will show that the resulting C^*-dynamical system, in the noncommutative case when $\theta = 0$, corresponds to a particular case of the the reduced group C^*-dynamical systems constructed in chapter 4, which will eliminate the need for a separate analysis of the system's ergodic properties even when considering the noncommutative case.

For the remainder of this section, $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ will be used to denote a toral automorphism. As the Haar measure ν is translation invariant on the torus, that the translations $\rho_{s,t}$ are measure preserving, is automatic. To establish the same for φ, however, we will require the uniqueness, up to normalization, of the Haar measure.

PROPOSITION 5.1. *Let ν be the normalized Haar measure on $\mathbb{T}^2$, and $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ a homeomorphic automorphism. Then*

$$\nu(\varphi^{-1}(E)) = \nu(E)$$

for all Borel sets E.

PROOF. Define a Borel measure μ on the $\mathbb{T}^2$ of $\mathbb{T}^2$ as follows:

$$\mu : \mathscr{B} \to \mathbb{R} : E \mapsto \nu(\varphi^{-1}(E))$$

Since φ is continuous and hence Borel measurable, $\varphi^{-1}(\mathscr{B}) \subseteq \mathscr{B}$, by [**26**, Theorem 1.12 (b)]. Thus μ is well defined and μ can now easily be seen to define a measure on $\mathscr{B}$. We can show that μ defines a normalized Haar measure on $\mathbb{T}^2$.

Consider any $E \in \mathscr{B}$ and let $\epsilon > 0$ be arbitrary. Then $\varphi^{-1}(E) \in \mathscr{B}$ so that by the outer regularity of ν there is an open set $V \supseteq \varphi^{-1}(E)$ such that $\nu(V) - \nu(\varphi^{-1}(E)) < \epsilon$. Since $\varphi = (\varphi^{-1})^{-1}$ and φ^{-1} is continuous we have that $\varphi(V)$ is an open set. It follows that

$$\mu(\varphi(V)) - \mu(E) = \nu(\varphi^{-1}(V)) - \nu(\varphi^{-1}(E)).$$

Thus, as $E \in \mathscr{B}$ and $\epsilon > 0$ was arbitrary this shows that μ is outer regular. To show that μ is inner regular, let V be an open set in $\mathbb{T}^2$ and let $\epsilon > 0$ be arbitrary. Then $\varphi^{-1}(V)$ is open so that by the inner regularity of ν there is a compact set $K \subseteq \varphi^{-1}(V)$ such that $\nu(\varphi^{-1}(V)) - \nu(K) < \epsilon$. Since φ is continuous, $\varphi(K)$ is compact. It follows that

$$\mu(V) - \mu(\varphi(K)) = \nu(\varphi^{-1}(V)) - \nu(K) < \epsilon.$$

Thus, as the open set V and $\epsilon > 0$ was arbitrary this shows that μ is inner regular. It is clear that μ is finite on all Borel sets, and in particular all compact sets. Hence, if μ is also invariant under the group operation of $\mathbb{T}^2$, then it will define a Haar measure, and in particular a normalized Haar measure since $\mu(\mathbb{T}^2) = \nu(\mathbb{T}^2) = 1$.

Let $z \in \mathbb{T}^2$ and $E \in \mathscr{B}$ be arbitrary. Then $z = \varphi(z')$ for some $z' \in \mathbb{T}^2$. It now follows by the invariance of ν that

$$\mu(zE) = \mu(\varphi(z')E) = \nu(z\varphi^{-1}(E)) = \nu(\varphi^{-1}(E)) = \mu(E).$$

Thus μ is invariant under the group operation of $\mathbb{T}^2$ and so $\mu = \nu$ by the uniqueness of the Haar measure up to normalization. Therefore the result follows from the last two terms in the above equality. $\square$

By [**28**, Theorem 0.15], the homeomorphic automorphism φ of $\mathbb{T}^2$ is given by a 2×2 matrix

$$A_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant ± 1, such that for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{T}^2$

$$\begin{aligned} \varphi \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \end{aligned}$$

The action of φ^{-1} is similarly given by the inverse matrix

$$A_\varphi^{-1} = \frac{1}{\det(A_\varphi)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so that for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{T}^2$

$$\begin{aligned} \varphi^{-1} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\det(A_\varphi)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{1}{\det(A_\varphi)} \begin{pmatrix} dx - by \\ -cx + ay \end{pmatrix} \end{aligned}$$

In Section 3 we used translations on $\mathbb{T}^2$ to define unitary operators satisfying the conditions in Proposition 2.1 to obtain an abstract dynamical system on $\mathcal{A}_\theta$. We will now define similar unitary operators using the toral automorphism φ with the aim of applying Proposition 2.1.

PROPOSITION 5.2. *Let $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphic automorphism with* $\det(A_\varphi) = 1$. *Then*

$$U_\varphi : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto f \circ \varphi$$

is a unitary operator in $\mathscr{L}(L^2(\mathbb{T}^2))$, $U_\varphi 1 = 1$ and, for any $\theta \in \mathbb{R}$, $U_\varphi \mathcal{A}_\theta U_\varphi^* \subseteq \mathcal{A}_\theta$. Furthermore $U_\varphi \mathbb{U}_\theta U_\varphi^* = e^{i\frac{ab\theta}{2}} \mathbb{V}_\theta^b \mathbb{U}_\theta^a$ and $U_\varphi \mathbb{U}_\theta U_\varphi^* = e^{-i\frac{cd\theta}{2}} \mathbb{U}_\theta^c \mathbb{V}_\theta^d$

PROOF. Let $\theta \in \mathbb{R}$ be arbitrary. Since φ is measurable, $\varphi^{-1}(E) \in \mathscr{B}$ for any $E \in \mathscr{B}$, by [**26**, Theorem 1.12]. Thus, for any $f \in L^2(\mathbb{T}^2)$, $f \circ \varphi$ is measurable and since φ is measure preserving, i.e. $\nu \circ \varphi^{-1} = \nu$ it follows that

$$\int_{\mathbb{T}^2} |f|^2 \, d\nu = \int_{\mathbb{T}^2} |f|^2 \circ \varphi \, d\nu = \int_{\mathbb{T}^2} |f \circ \varphi|^2 \, d\nu \tag{74}$$

Hence, if $f \in L^2(\mathbb{T}^2)$, then $f \circ \varphi \in L^2(\mathbb{T}^2)$ which shows that U_φ is well-defined. Moreover, (74) shows that U_φ is an isometry so that since $U_\varphi(f \circ \varphi^{-1}) = f$ for all $f \in L^2(\mathbb{T}^2)$, it follows that U_φ is unitary by [**20**, 3.10-6 Theorem (f)].

To show that $U_\varphi \mathcal{A}_\theta U_\varphi^* \subseteq \mathcal{A}_\theta$, first note that

$$\tau' : \mathscr{L}(L^2(\mathbb{T}^2)) \to \mathscr{L}(L^2(\mathbb{R}^2)) : A \mapsto U_\varphi A U_\varphi^*$$

is a $*$-automorphism and hence isometric. To see why $\det(A_\varphi) = 1$ is required, define $\Delta := \det(A_\varphi)$. Let $\mathcal{A}'_\theta = \operatorname{span}\{\mathbb{U}_\theta^m \mathbb{V}_\theta^n : m, n \in \mathbb{Z}\}$. For any $f \in L^2(\mathbb{T}^2)$ it follows that

$$\begin{aligned}
\tau'(\mathbb{U}_\theta) &= \left(U_\varphi \mathbb{U}_\theta U_\varphi^* f\right)(x,y) \\
&= \mathbb{U}_\theta (f \circ \varphi^{-1}) \varphi(x,y) \\
&= \mathbb{U}_\theta \left(f \circ \varphi^{-1}\right)(ax + by, cx + dy) \\
&= e^{i(ax+by)} \left(f \circ \varphi^{-1}\right)\left(ax + by, cx + dy - \frac{\theta}{2}\right) \\
&= e^{i(ax+by)} f\left(\begin{pmatrix} x \\ y \end{pmatrix} + \varphi^{-1}\begin{pmatrix} 0 \\ -\frac{\theta}{2} \end{pmatrix}\right) \\
&= e^{i(ax+by)} f\left(\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{\Delta}\begin{pmatrix} b\frac{\theta}{2} \\ -a\frac{\theta}{2} \end{pmatrix}\right) \\
&= e^{i(ax+by)} f\left(x + \frac{b}{2}\frac{\theta}{\Delta}, y - \frac{a}{2}\frac{\theta}{\Delta}\right)
\end{aligned}$$

Since

$$\begin{aligned}\left(\mathbb{U}_\theta^a \mathbb{V}_\theta^b f\right)(x,y) &= e^{iax}\left(\mathbb{V}_\theta^b f\right)\left(x, y - a\frac{\theta}{2}\right)\\ &= e^{-i\frac{ab\theta}{2}} e^{i(ax+by)} f\left(x + b\frac{\theta}{2}, y - a\frac{\theta}{2}\right)\end{aligned}$$

it is clear that $\tau(\mathbb{U}_\theta) = e^{i\frac{ab\theta}{2}}\mathbb{U}_\theta^a\mathbb{V}_\theta^b \in \mathcal{A}'_\theta$, if $\Delta = 1$. We make the observation that if $\Delta = -1$, then $\tau(\mathbb{U}_\theta) \in \mathcal{A}_{-\theta}$ which, since $\mathcal{A}_{-\theta}$ is a different algebra, gives a good indication that the result does not hold in general when $\Delta = -1$. It similarly follows that $\tau'(\mathbb{V}_\theta) = e^{i\frac{cd\theta}{2}}\mathbb{U}_\theta^c\mathbb{V}_\theta^d = \in \mathcal{A}'_\theta$ if $\Delta = 1$. Since τ' is a $*$-automorphism we therefore have that $\tau'(\mathcal{A}'_\theta) \subseteq \mathcal{A}'_\theta$. If $A \in \mathcal{A}_\theta$, then $A_n \longrightarrow A$ for some sequence (A_n) in $\mathcal{A}'_\theta$. However, since τ' is isometric, it follows that $\lim_{n\to\infty}\tau'(A_n)\tau'(A)$, so that since $\tau'(A_n) \in \mathcal{A}'_\theta \subseteq \mathcal{A}_\theta$ for all $n \in \mathbb{N}$, and $\mathcal{A}_\theta$ is closed, $\tau'(A) \in \mathcal{A}_\theta$ follows. That is, $\tau'(\mathcal{A}_\theta) = U_\varphi \mathcal{A}_\theta U_\varphi^* \subseteq \mathcal{A}_\theta$ as required.

$U_\varphi 1 = 1$ is immediate. □

By feeding the unitary operator defined in Proposition 5.2 to Proposition 2.1, we obtain an abstract dynamical system with $\mathbb{Z}$ action, which is thus a C^*-dynamical system.

As with all other examples already considered, we define our final C^*-dynamical system in complete detail, in keeping with Remarks 1.9.

Definition 5.3. For any fixed $\theta \in \mathbb{R}$, the Quantum Torus with a toral Automorphism φ, $(\mathrm{QTA}(\theta,\varphi))$ is the C^*-dynamical system $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ where

(i) $\mathfrak{A} = \mathcal{A}_\theta$
(ii) $\omega : \mathcal{A}_\theta \to \mathbb{C} : T \mapsto \langle 1, T1\rangle_{L^2}$
(iii) $\det(A_\varphi) = 1$
(iv) $\tau : \mathcal{A}_\theta \to \mathcal{A}_\theta : A \mapsto U_\varphi A U_\varphi^*$, $U_\varphi : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) : f \mapsto f \circ \varphi$ and $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is an automorphism.

6. Ergodic properties of a toral automorphism

$\mathrm{QTA}(\theta,\varphi)$ shares its ergodic properties with a type of C^*-dynamical system that we have already analysed, namely the reduced group C^*-dynamical systems. In order to notice this note that on Hilbert space level $\mathrm{QTA}(\theta,\varphi)$ is equivalent to $\mathrm{QTA}(0,\varphi)$ which in turn is precisely the system in Definition 1.7 for $\Gamma = \mathbb{Z}^2$ as we'll now explain. Namely, we will show that $C_r^*(\mathbb{Z}^2) \equiv \mathcal{A}_0$ and that ρ is obtained as the "harmonic dual" of φ, and that the respective states are naturally identified. Roughly speaking we simply show that $C_r^*(\mathbb{Z}^2)$ represents the harmonic dual of $\mathbb{Z}^2$, namely $\mathbb{T}^2$, as a C^*-algebra $C_r^*(\mathbb{Z}^2) \equiv \mathcal{A}_0$, and this identification allows us to see that $\mathrm{QTA}(0,\varphi)$ is equivalent to Definition 1.7 for $\Gamma = \mathbb{Z}^2$.

We start by identifying the linear isomorphism between $L^2(\mathbb{Z}^2)$ and $L^2(\mathbb{T}^2)$.

PROPOSITION 6.1. *The continuous extension $\tilde{\iota}$, to $L^2(\mathbb{T}^2)$, of*

$$\begin{aligned}\iota &: \text{span}\,\{E_{m,n} : m,n \in \mathbb{Z}\} \to L^2(\mathbb{Z}^2)\\ &: \sum_{j=1}^{k} a_j E_{m_j,n_j} \mapsto \sum_{j=1}^{k} a_j \delta_{(m_j,n_j)}\end{aligned}$$

is an isometric isomorphism.

PROOF. Consider any $s \in \text{span}\,\{E_{m,n} : m,n \in \mathbb{Z}\}$, say $s = \sum_{j=1}^{n} a_j E_{m_j,n_j}$. Since $\{\delta_{(m,n)} : m,n \in \mathbb{Z}\}, \{E_{m,n} : m,n \in \mathbb{Z}\}$ are orthonormal sets in $L^2(\mathbb{Z}^2), L^2(\mathbb{T}^2)$, respectively, it follows that

$$\|s\|^2 = \|\iota(s)\|^2.$$

Therefore, as ι is also linear and $\overline{\text{span}\,\{E_{m,n} : m,n \in \mathbb{Z}\}} = L^2(\mathbb{T}^2)$, it follows that the continuous extension $\tilde{\iota} : L^2(\mathbb{T}^2) \to L^2(\mathbb{Z}^2)$ is well defined by [**20**, Theorem 2.7-11] and clearly also an isometry. Wholly similar arguments reach the same conclusion for the mapping

$$\begin{aligned}\varsigma &: L^2(\mathbb{Z}^2) \to \text{span}\,\{\delta_{(m,n)} : m,n \in \mathbb{Z}\}\\ &: \sum_{j=1}^{k} a_j \delta_{(m_j,n_j)} \mapsto \sum_{j=1}^{k} a_j E_{m_j,n_j}\end{aligned}$$

which then gives the inverse of $\tilde{\iota}$. Hence $\tilde{\iota}$ is bijective. Thus $\tilde{\iota}$ is an isometric isomorphism. □

Proposition 6.1 now allows us to establish the same relationship between $\mathcal{A}_0$ and $C_r^*(\mathbb{Z}^2)$. Note that for the remainder of this section we will use $\tilde{\iota}$ to denote the isometric isomorphism defined in Proposition 6.1.

PROPOSITION 6.2. *The continuous extension $\tilde{\pi}$, to $\mathcal{A}_0$, of*

$$\begin{aligned}\pi &: \text{span}\,\{\mathbb{U}_0^m \mathbb{V}_0^n : m,n \in \mathbb{Z}\} \to C_r^*(\mathbb{Z}^2)\\ &: \sum_{j=1}^{k} a_j \mathbb{U}_0^{m_j} \mathbb{V}_0^{n_j} \mapsto \sum_{j=1}^{k} a_j \lambda(m_j,n_j)\end{aligned}$$

is an isometric $*$-isomorphism. Furthermore,

$$\tilde{\pi}(A) = \tilde{\iota} \circ A \circ \tilde{\iota}^{-1} =: \tilde{\iota} A \tilde{\iota}^{-1}$$

for all $A \in \mathcal{A}_0$.

PROOF. Consider any $A \in \text{span}\{\mathbb{U}_0^m\mathbb{V}_0^n : m,n \in \mathbb{Z}\}$, say $A = \sum_{j=1}^{N_1} a_j\mathbb{U}_0^{m_j}\mathbb{V}_0^{n_j}$, and any $s \in \text{span}\{E_{m,n} : m,n \in \mathbb{Z}\}$, say $s = \sum_{k=1}^{N_2} b_j E_{m_k,n_k}$. It follows by Proposition 6.1 that

$$\begin{aligned}
\|As\| &= \left\| \sum_{j=1}^{N_1} a_j\mathbb{U}_0^{m_j}\mathbb{V}_0^{n_j} \sum_{k=1}^{N_2} b_k E_{p_k,q_k} \right\| \\
&= \left\| \sum_{j=1}^{N_1}\sum_{k=1}^{N_2} a_j b_k e^{i(m_j+p_k)x} e^{i(n_j+q_k)y} \right\|_{L^2(\mathbb{T}^2)} \\
&= \left\| \sum_{j=1}^{N_1}\sum_{k=1}^{N_2} a_j b_k \delta_{(m_j+p_k,n_j+q_k)} \right\|_{L^2(\mathbb{Z}^2)} \\
&= \left\| \sum_{j=1}^{N_1} a_j\lambda(m_j,n_j) \sum_{k=1}^{N_2} b_k\delta_{(p_k,q_k)} \right\|
\end{aligned}$$

$$\|As\| = \|\pi(A)\iota(s)\| \tag{75}$$

It follows that this then also holds for any $s \in L^2(\mathbb{T}^2)$ and therefore $\|\pi(A)\| = \|A\|$. Therefore π extends to an isometry $\tilde{\pi} : \mathcal{A}_0 \to C_r^*(\mathbb{Z}^2)$. Wholly similar arguments reach the same conclusion for the mapping

$$\begin{aligned}
\wp &: \text{span}\{\lambda(m,n) : m,n \in \mathbb{Z}\} \to \mathcal{A}_0 \\
&: \sum_{j=1}^{k} a_j\lambda(m_j,n_j) \mapsto \sum_{j=1}^{k} a_j\mathbb{U}_0^{m_j}\mathbb{V}_0^{n_j}
\end{aligned}$$

and a moment's reflection will show that $\tilde{\pi}\tilde{\wp} = \tilde{\wp}\tilde{\pi} = \mathsf{id}$. Hence $\tilde{\pi}$ is bijective.

Recall that, for any group Γ and $\lambda(g), \lambda(h) \in C_r^*(\Gamma)$, $\lambda(g)\lambda(h) = \lambda(gh)$. Hence, for any $m,n,p,q \in \mathbb{Z}$, $\lambda(m,n)\lambda(p,q) = \lambda(m+p,n+q)$. It can therefore be easily seen that, for any $A, B \in \text{span}\{\mathbb{U}_0^m, \mathbb{V}_0^n : m,n \in \mathbb{Z}\}$, $\pi(AB) = \pi(A)\pi(B)$. Since π is also linear

$$\begin{aligned}
\tilde{\pi}(aA + bB) &= a\tilde{\pi}(A) + b\tilde{\pi}(B) \\
\tilde{\pi}(AB) &= \tilde{\pi}(A)\tilde{\pi}(B)
\end{aligned}$$

for all $A, B \in \mathcal{A}_0$ and $a, b \in \mathbb{C}$. Thus the bijective $\tilde{\pi}$ is an isomorphism. To show that it is a $*$-isomorphism, again consider any

$T \in \text{span}\{\mathbb{U}_0^m\mathbb{V}_0^n : m, n \in \mathbb{Z}\}$, say $A = \sum_{j=1}^N \mathbb{U}_0^{m_i}\mathbb{V}_0^{n_i}$. It follows that

$$\begin{aligned}\pi(A^*) &= \pi\left(\sum_{j=1}^N (\mathbb{U}_0^{m_i})^*(\mathbb{V}_0^{n_i})^*\right)\\ &= \pi\left(\sum_{j=1}^N \mathbb{U}_0^{-m_i}\mathbb{V}_0^{-n_i}\right)\\ &= \sum_{j=1}^N \lambda(-m_j, -n_j)\\ &= \left(\sum_{j=1}^N \lambda(m_j, n_j)\right)^*\\ &= \pi(B)^*\end{aligned}$$

Therefore, for any $A \in \mathcal{A}_0$, $\tilde{\pi}(A^*) = (\tilde{\pi}(A))^*$, for if (x_n) is any sequence in a C^*-algebra, then $x_n \longrightarrow x$ if and only if $x_n^* \longrightarrow x^*$, as the norm of an element and its adjoint are equal. Thus $\tilde{\pi}$ is a $*$-isomorphism.

For the final assertion first note that, since $\tilde{\imath} : L^2(\mathbb{T}^2) \to L^2(\mathbb{Z}^2)$ is an isometric isomorphism, $\tilde{\imath}A\tilde{\imath}^{-1} \in \mathscr{L}(L^2(\mathbb{Z}^2))$ for all $A \in \mathcal{A}_0 \subseteq \mathscr{L}(L^2(\mathbb{T}^2))$. For any $m, n \in \mathbb{Z}$ it follows that

$$\begin{aligned}\tilde{\pi}(\mathbb{U}_0)\delta_{(m,n)} &= \lambda(1,0)\delta_{(m,n)}\\ &= \delta_{(m+1,n)}\\ &= \tilde{\imath}E_{m+1,n}\\ &= \tilde{\imath}\mathbb{U}_0E_{m,n}\\ &= \tilde{\imath}\mathbb{U}_0\tilde{\imath}^{-1}\delta_{m,n}\end{aligned}$$

from which it follows that $\tilde{\pi}(\mathbb{U}_0) = \tilde{\imath}\mathbb{U}_0\tilde{\imath}^{-1}$ on span $\{\delta_{(m,n)} : m, n \in \mathbb{Z}\}$ and consequently on $L^2(\mathbb{Z}^2)$. Similarly, $\tilde{\pi}(\mathbb{V}_0) = \tilde{\imath}\mathbb{V}_0\tilde{\imath}^{-1}$, and so since $\tilde{\pi}$ and $\tilde{\imath}$ are isomorphisms, it follows that $\tilde{\pi}(A) = \tilde{\imath}A\tilde{\imath}^{-1}$ for all $A \in$ span $\{\mathbb{U}_0^m\mathbb{V}_0^n : m, n \in \mathbb{Z}\}$. Since $\tilde{\pi}$ and $\tilde{\imath}$ are isometric, it follows that $\tilde{\pi}(A) = \tilde{\imath}A\tilde{\imath}^{-1}$ for all $A \in \mathcal{A}_0$. □

We denote the transpose of A_φ by A_φ^T and will now show that $(A_\varphi^T)^{-1}$ is the "harmonic dual" of φ referred to earlier.

PROPOSITION 6.3. *There is a cyclic representation for $QTA(0, \varphi)$ and a cyclic representation for $RG(\mathbb{Z}^2, (A_\varphi^T)^{-1})$, with respect to which the dynamics of the two C^*-dynamical systems are represented by the same unitary operator. That is, the GNS representation of the $*$-automorphisms on $\mathcal{A}_0$ and $C_r^*(\mathbb{Z}^2)$ are given by the same unitary operator.*

PROOF. By Lemma 2.2, $(L^2(\mathbb{Z}^2), \mathsf{id}, \delta_{(0,0)})$ is a cyclic representation of $\mathrm{RG}(\mathbb{Z}^2, (A_\varphi^T)^{-1})$. Proposition 6.2 identifies a mapping $\tilde{\pi} : \mathcal{A}_0 \to C_r^*(\mathbb{Z}^2) \subseteq \mathscr{L}(L^2(\mathbb{Z}^2))$ that is a $*$-isomorphism. $(L^2(\mathbb{Z}^2), \tilde{\pi}, \delta_{(0,0)})$ is the cyclic representation of $\mathrm{QTA}(0,\varphi)$ since $\tilde{\pi}(\mathcal{A}_0)C_r^*(\mathbb{Z}^2)$ and $\delta_{(0,0)}$ is cyclic for the latter, and

$$\langle \delta_{(0,0)}, \tilde{\pi}(A)\delta_{(0,0)}\rangle = \langle \tilde{\iota}^{-1}\delta_{(0,0)}, A\tilde{\iota}^{-1}\delta_{(0,0)}\rangle = \langle 1, A1\rangle$$

for all $A \in \mathcal{A}_0$ so $\delta_{(0,0)}$ indeed gives the state of $\mathrm{QTA}(0,\varphi)$.

Next, we show that the dynamics of both systems are given by the same unitary operator U on $L^2(\mathbb{Z}^2)$. This is in fact very easy to show, provided we can navigate the sea of notation involved when simultaneously considering two C^*-dynamical systems' GNS representations. In the case of $\mathrm{RG}(\mathbb{Z}^2, (A_\varphi^T)^{-1})$, the unitary representation of the dynamics, i.e. the GNS representation of $(A_\varphi^T)^{-1}$, is given by

$$V : L^2(\mathbb{Z}^2) \to L^2(\mathbb{Z}^2) : f \mapsto f \circ \left(A_\varphi^T\right)^{-1}$$

and in the case of $\mathrm{QTA}(0,\varphi)$ the unitary representation of the dynamics is given by

$$\begin{aligned} U : L^2(\mathbb{Z}^2) \to L^2(\mathbb{Z}^2) &: \tilde{\pi}(A)\delta_{(0,0)} \mapsto \tilde{\pi}(U_\varphi A U_\varphi^*)\delta_{(0,0)} \\ &: \tilde{\iota}A\tilde{\iota}^{-1}\delta_{(0,0)} \mapsto \tilde{\iota}U_\varphi A U_\varphi^*\tilde{\iota}^{-1}\delta_{(0,0)} \\ &: \tilde{\iota}A1 \mapsto \tilde{\iota}U_\varphi A1. \end{aligned}$$

We wish to show that $V = U$. For any $m, n \in \mathbb{Z}$ it follows that

$$\begin{aligned} (U_\varphi E_{m,n})(x,y) &= E_{m,n}(\varphi(x,y)) \\ &= E_{m,n}(ax+by, cx+dy) \\ &= e^{im(ax+by)}e^{in(cx+dy)} \\ &= e^{i(am+cn)x}e^{i(bm+dn)y} \\ &= E_{A_\varphi^T(m,n)}(x,y) \end{aligned}$$

and therefore

$$\begin{aligned} U\delta_{m,n} = U\tilde{\iota}E_{m,n} &= \tilde{\iota}U_\varphi E_{m,n} \\ &= \tilde{\iota}E_{A_\varphi^T(m,n)} \\ &= \delta_{A_\varphi^T(m,n)} \\ &= \delta_{m,n} \circ \left(A_\varphi^T\right)^{-1} \\ &= V\delta_{m,n}. \end{aligned}$$

Therefore, as $U, V \in \mathscr{L}(L^2(\mathbb{Z}^2))$ and $\operatorname{span}\{\delta_{m,n} : m, n \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{Z}^2)$, it follows that $U = V$. Thus $\mathrm{QTA}(0,\varphi)$ and $\mathrm{RG}(\mathbb{Z}^2, (A_\varphi^T)^{-1})$ have the same GNS representation. $\square$

By propositions 6.3 and 2.1 we know that $\mathrm{QTA}(\theta, \varphi)$ and $\mathrm{RG}(\mathbb{Z}^2, (A_\varphi^T)^{-1})$ have the same ergodic properties for all values of θ, and by propositions 2.3, 2.4 and 2.5 we know that $\mathrm{RG}(\mathbb{Z}^2, (A_\varphi^T)^{-1})$ can have only one of three combinations of the ergodic properties:

(i) strongly mixing if A_φ^T only has only infinite orbits
(ii) compact, but not ergodic, if A_φ^T only has only finite orbits
(iii) neither compact nor ergodic, if A_φ^T has both finite and infinite orbits

It is a simple task to find three 2×2 integer matrices with determinant 1 having, respectively, only infinite, only finite and a both of finite orbits and infinite orbits. Let

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

A_1, A_2 and A_3 are group automorphisms on $\mathbb{Z}^2$. A_1 only has only infinite orbits which can be seen by inspection. A_2 only has only finite orbits as $(A_2)^6 = (A_2)^{-6} = \mathsf{id}$. A_3 has both finite orbits and infinite orbits, which can also be seen by inspection. Since A_1, A_2 and A_3 are integer matrices with determinant 1 it follows that each, together with the inverse of its transpose, determines an automorphism on $\mathbb{T}^2$. Let

$$\phi_1 : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (2x - y, -x + y)$$
$$\phi_2 : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (2x + y, -3x - y)$$
$$\phi_3 : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (x, -2x + y)$$

Then $(A_{\phi_1}^T)^{-1} = A_1, (A_{\phi_2}^T)^{-1} = A_2$ and $(A_{\phi_3}^T)^{-1} = A_3$. For ease in later reference, let us devote a final definition to the last three C^*-dynamical systems on the torus using these toral automorphisms.

Definition 6.4. For any $\theta \in \mathbb{R}$, the quantum tori with automorphism 1,2 and 3 ($\mathrm{QTA1}(\theta)$, $\mathrm{QTA2}(\theta)$ and $\mathrm{QTA3}(\theta)$) are the quantum tori with, respectively, toral automorphisms ϕ_1, ϕ_2 and ϕ_3. That is, $\mathrm{QTA1}(\theta) = \mathrm{QTA}(\theta, \phi_1)$, $\mathrm{QTA2}(\theta) = \mathrm{QTA}(\theta, \phi_2)$ and $\mathrm{QTA3}(\theta) = \mathrm{QTA}(\theta, \phi_3)$

Proposition 6.5. *QTA1(θ) is strongly mixing.*

Proof. $\mathrm{QTA1}(\theta)$ has the same ergodic properties as $\mathrm{RG}(\mathbb{Z}^2, A_1)$ which, since A_1 has only infinite orbits, is strongly mixing by Proposition 2.3. □

Proposition 6.6. *QTA2(θ) is compact but not ergodic.*

Proof. QTA2(θ) has the same ergodic properties as RG($\mathbb{Z}^2, A_2$) which, since A_2 has only finite orbits, is compact but not ergodic by propositions 2.4 and 2.5. □

Proposition 6.7. *QTA3(θ) is neither compact nor ergodic.*

Proof. QTA3(θ) has the same ergodic properties as RG($\mathbb{Z}^2, A_3$) which, since A_3 has both finite and infinite orbits, is neither compact nor ergodic by propositions 2.4 and 2.5. □

QTA(θ, φ) therefore gives the same ergodic possibilities as RG($\mathbb{Z}^2, \rho$) systems, but it is a different type of example, since on C^*-algebra level, $C_r^*(\mathbb{Z}^2)$ is "deformed" by θ, and this is of interest in C^*-algebras and even quantum physics, and therefore QTA(θ, φ) is an important example.

Further Research

The first combination that we did not find a concrete example of is a system that is weakly but not strongly mixing. There are several examples in classic ergodic theory of systems that are weakly but not strongly mixing. However, these systems do not necessarily have a readily identifiable noncommutative analog that is of interest. An example with a possible noncommutative analog is discussed in [**2**, Section 2] where a $SL(2,\mathbb{Z})$ action is studied on the 2 dimensional torus $\mathbb{T}^2$. In Chapter 5 the time evolution given by a translation T is generalized from the classical case. That is, if $\theta = 0$, then the time evolution reduces to $C(\mathbb{T}^2) \to (\mathbb{T}^2) : f \mapsto f \circ T$. The same holds for automorphisms on the torus in our QTA examples. So in principle, we could also attempt to generalize the example studied by [**2**] to $\mathcal{A}_\theta$ and find an example of a weak mixing C^*-dynamical system, that is not strongly mixing on the quantum torus. However, such a generalization will require us to modify aspects of our development of noncommutative ergodic theory. For example, $SL(2,\mathbb{Z})$ is not Abelian and therefore in particular not an Abelian locally compact amenable group. The most significant casualty in our development forgoing Abelian group actions, is the Splitting Theorem of Jacobs-Deleeuw-Glicksberg. Furthermore, as is, we would require $SL(2,\mathbb{Z})$ to be amenable, and it reasonable to expect that it is not (see [**25**, (0.7) Example, p6]). So, our formulation of the weak mixing property in terms of a Følner sequence will not be appropriate for such a generalization. Our definition of the strong mixing property, for a $\mathbb{Z}$ action, is even less appropriate. One possible way to proceed is to use the spectral formulations as the definitions themselves. For example Proposition 4.7 could be used to define a weakly mixing system. One would then have to find a spectral characterization of the strong mixing property as well.

The other combination for which we did not find a concrete example is a system that is ergodic, but does not possess the other three ergodic properties. A natural question to ask is whether, given two dynamical systems, a "combination" or "product" of the two systems can be defined to obtain a new dynamical system whose ergodic properties derives from its constituents. For example, could two systems with the same ergodic properties as DESS1 and ReFG be "combined" to form a system that is ergodic but does not possess any of the other three ergodic properties? A possible way in which this might be achieved is through the use of tensor products and joinings of W*-dynamical systems [**11**], which are subsets of C^*-dynamical systems.

Appendix : The Torus

There are different ways to define the torus $\mathcal{T}^2$. It can be defined as the quotient space $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ or as the product space $\mathcal{T} = \mathbb{S}^1 \times \mathbb{S}^1$ where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$. An incentive to use one definition over the other can be provided by the structure on the torus that is of greatest interest. For instance the measure structure on the torus has a clearer connection with the Lebesgue measure on $\mathbb{R}^2$ when the torus is viewed as a quotient space whereas the topological structure is simpler when the torus is viewed as a product space. This is in contrast to the group structure on the torus which is entirely straight forward in either case. We have a greater interest in the group and measure structure on the torus, however, we would like to know that the torus is compact and Hausdorff. To obtain the benefit of both formulations we define the torus as

$$\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$$

and show that there is a homeomorphism between $\mathbb{T}^2$ and $\mathcal{T}$. We then gather, and transfer, the compactness and Hausdorff property of $\mathcal{T}$ to $\mathbb{T}^2$ which will then allow us to equip $\mathbb{T}^2$ with its measure structure: the normalized Haar measure on its Borel σ-algebra.

Proposition 6.8. *The function $f : \mathbb{R} \to \mathbb{S}^1 : x \mapsto e^{ix}$ is an open mapping.*

Proof. Let $V \subseteq \mathbb{R}$ be an open set. Since every point of V is an interior point of V, it is possible to write $V = \bigcup_{\alpha\in\Lambda}(x_\alpha - r_\alpha, x_\alpha + r_\alpha)$ where, for all $\alpha \in \Lambda$, $x_\alpha \in V$ and $r_\alpha > 0$. Without loss of generality we may assume that $r_\alpha < \pi$ for all $\alpha \in \Lambda$. Since each of the intervals $(x_\alpha - r_\alpha, x_\alpha + r_\alpha)$ has a length less than 2π it is clear that

$$f_\alpha := f|_{(x_\alpha - r_\alpha, x_\alpha + r_\alpha)} : (x_\alpha - r_\alpha, x_\alpha + r_\alpha) \to \mathbb{S}^1$$

is bijective for all $\alpha \in \Lambda$ and that f_α^{-1} is a branch of the complex logarithm, and therefore continuous. Hence

$$\begin{aligned} f(V) &= \bigcup_{\alpha\in\Lambda} f(x_\alpha - r_\alpha, x_\alpha + r_\alpha) \\ &= \bigcup_{\alpha\in\Lambda} f_\alpha(x_\alpha - r_\alpha, x_\alpha + r_\alpha) \\ &= \bigcup_{\alpha\in\Lambda} (f_\alpha^{-1})^{-1}(x_\alpha - r_\alpha, x_\alpha + r_\alpha) \end{aligned}$$

which is a union of open sets. □

The following is an alternative proof that does not directly utilize the continuity of the logarithm. It is included purely for novelty purposes as the author found it interesting.

Proof. We start by considering the "distance" between $f(x), f(y)$ for arbitrary $x, y \in \mathbb{R}$:

$$\begin{aligned} |e^{ix} - e^{iy}|^2 &= |\cos x + i \sin x - \cos y - i \sin y|^2 \\ &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\ &= 2 - 2\cos x \cos y - 2 \sin x \sin y \\ &= 2 - 2\cos(x-y) \end{aligned}$$

We note from this result that if $|x-y| \in (\frac{\pi}{2}, \frac{3\pi}{2})$, then $|e^{ix} - e^{iy}| > \sqrt{2}$. Let $x \in \mathbb{R}$, $y \in [x-\pi, x+\pi)$ be arbitrary and consider an $0 < \epsilon \leq \frac{\pi}{2}$. Observing that $\cos x \leq 1 - \frac{8}{\pi^3}x^3$ for all $x \in [0, \frac{\pi}{2}]$ we deduce that if $\frac{\pi}{2} \geq |x-y| \geq \epsilon$ then

$$\cos(x-y) \leq 1 - \frac{8}{\pi^3}|x-y|^3 \tag{76}$$

$$2 - 2\cos(x-y) \geq 2\frac{8}{\pi^3}|x-y|^3 \geq \frac{16}{\pi^3}\epsilon^3 \tag{77}$$

$$|e^{ix} - e^{iy}| \geq \kappa \epsilon^{\frac{3}{2}} \tag{78}$$

where $\kappa = \frac{4}{\sqrt{\pi^3}}$. The converse of this result thus states that for any $x \in \mathbb{R}$, $y \in [x-\pi, x+\pi)$ and $0 < \epsilon \leq \frac{\pi}{2}$, if $|e^{ix} - e^{iy}| < \kappa\epsilon^{\frac{3}{2}}$, then $|x-y| < \epsilon$ or $|x-y| > \frac{\pi}{2}$. The latter case is impossible since, as noted earlier, if $|x-y| > \frac{\pi}{2}$ then $|e^{ix} - e^{iy}| > \sqrt{2} = \kappa(\frac{\pi}{2})^{\frac{3}{2}} \geq \kappa\epsilon^{\frac{3}{2}}$
Let $V \subset \mathbb{R}$ be open and consider any $z \in f(V)$. Thus $z = e^{ix}$ for some $x \in V$ and since V is open there is a $\delta' > 0$ such that $|x-y| < \delta' \Rightarrow y \in V$. Let us now consider an arbitrary $z' \in \mathbb{S}^1$, so $z' = e^{iy}$ where without loss of generality we may assume that $y \in [x-\pi, x+\pi)$. Suppose z' satisfies $|z - z'| < \kappa\delta^{\frac{3}{2}}$, where $\delta = \min\{\frac{\pi}{2}, \delta'\}$. By using the converse

of the earlier result it thus follows that

$$\begin{aligned}|z - z^{'}| < \kappa\delta^{\frac{3}{2}} &\Rightarrow |x - y| < \delta \\ &\Rightarrow y \in V \\ &\Rightarrow f(y) = z^{'} \in f(V)\end{aligned}$$

What this shows is that for an arbitrary $z \in f(V)$, there exist a $\delta > 0$ such that $z \in \mathbb{S}^1 \cap B(z, \kappa\delta^{\frac{3}{2}}) \subset f(V)$, where $B(w, r)$ is the *open* ball of radius r in $\mathbb{C}$ centered at w. Since $\mathbb{S}^1 \cap B(z, \kappa\delta^{\frac{3}{2}})$ is open in $\mathbb{S}^1$ we therefore have that z is an interior point of $f(V)$ in the subspace topology. Since $z \in f(V)$ was taken as arbitrary it follows that $f(V)$ is open. □

COROLLARY 6.9. *The function $\check{f} : \mathbb{R}^2 \to \mathcal{T} : (x, y) \mapsto (e^{ix}, e^{iy})$ is a continuous open mapping.*

PROOF. The continuity of $\check{f}$ follows from the continuity of

$$f : \mathbb{R} \to \mathbb{S}^1 : x \mapsto e^{ix}$$

as follows: Let U be an open set in $\mathcal{S}^1$. Thus $U = \bigcup_{\alpha\in\Lambda} U_\alpha \times V_\alpha$ for some indexed open sets U_α, V_α in $\mathbb{S}^1$. It now follows by the continuity of f that

$$\check{f}^{-1}(U) = \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha) \times f^{-1}(V_\alpha)$$

which is open in $\mathbb{R}^2$ since for each $\alpha \in \Lambda$, $f^{-1}(U_\alpha) \times f^{-1}(V_\alpha)$ is in the basis of the $\mathbb{R}^2$ topology. Thus $\check{f}$ is continuous.

To prove that $\check{f}$ is an open mapping mapping, consider any open set in $\mathbb{R}^2$ of the form $U \times V$ where U and V are open sets in $\mathbb{R}$. Since f is an open mapping it follows that $\check{f}(U \times V) = f(U) \times f(V)$ is open in $\mathcal{S}^1$. Since any open set in $\mathbb{R}^2$ can be written as the union of open sets of the form $U \times V$, with U and V open, it follows that $\check{f}(W)$ is open for any open set W in $\mathbb{R}^2$. Thus $\check{f}$ is an open mapping □

THEOREM 6.10. *The mapping*

$$\delta : \mathbb{T}^2 \to \mathcal{T} : (x, y) + 2\pi\mathbb{Z}^2 \mapsto (e^{ix}, e^{iy}) \tag{79}$$

is a group isomorphism and a homeomorphism.

PROOF. δ is clearly bijective and with the group operations

$$\mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 :$$

$$(x_1, y_1) + 2\pi\mathbb{Z}^2 \times (x_2, y_2) + 2\pi\mathbb{Z}^2 \mapsto (x_1 + x_2, y_1 + y_2) + 2\pi\mathbb{Z}^2, \text{ and}$$

$$\mathcal{T} \times \mathcal{T} \to \mathcal{T} :$$

$$(e^{ix_1}, e^{iy_1}) \times (e^{ix_1}, e^{iy_1}) \mapsto \left(e^{i(x_1+x_2)}, e^{i(y_1+y_2)}\right)$$

it is clear to see that δ defines a group isomorphism.

Consider any open set $V \subseteq \mathbb{T}^2$. By definition of the quotient topology, $p^{-1}(V)$, where $p : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ is the quotient map, is open in $\mathbb{R}^2$. Thus, by Corollary 6.9, $\check{f}(p^{-1}(V))$ is open in $\mathcal{T}$. Now we only have to note that, as for any x, y,

$$(x, y) \in p^{-1}(V) \Leftrightarrow (x, y) + 2\pi\mathbb{Z}^2 \in V, \text{ and}$$
$$\check{f}(x, y) = \delta\left((x, y) + 2\pi\mathbb{Z}^2\right) = (e^{ix}, e^{iy})$$

we have that $\delta(V) = \check{f}(p^{-1}(V))$ is open.

Conversely, consider any open set $U \subseteq \mathcal{T}$. $(\check{f})^{-1}(U)$ is open by the continuity of $\check{f}$ established in Corollary 6.9. Note that

$$p^{-1}(p((\check{f})^{-1}(U))) = (\check{f})^{-1}(U)$$

which follows simply because $(x, y) \in (\check{f})^{-1}(U)$ if and only if $(x, y) + 2\pi(m, n) \in (\check{f})^{-1}(U)$ for all $m, n \in \mathbb{Z}$. Since $p^{-1}(p((\check{f})^{-1}(U)))$ is therefore open, $p((\check{f})^{-1}(U))$ is open by definition of the quotient topology. Now we only have to note that, as for any x, y

$$(x, y) \in (\check{f})^{-1}(U) \Leftrightarrow (e^{ix}, e^{iy}) \in U, \text{ and}$$
$$p(x, y) = \delta^{-1}(e^{ix}, e^{iy}) = (x, y) + 2\pi\mathbb{Z}^2$$

we have that $\delta^{-1}(U) = p((\check{f})^{-1}(U))$ is open.

Therefore, the bijective mapping δ is an open mapping and continuous, or in other words, a homeomorphism. $\square$

In Theorem 6.10 we also showed that $\mathbb{T}^2$ and $\mathbb{T}^2$ are group isomorphisms of one another and are therefore effectively one and the same space.

Since $\mathbb{S}^1$ is compact in the plane, $\mathcal{T} = \mathbb{S}^1 \times \mathbb{S}^1$ is compact in the product topology by [**21**, Theorem 37.3 (Tychonoff's theorem)]. Hence, $\mathbb{T}^2$ is compact by Theorem 6.10. Furthermore, $\mathbb{S}^1 \times \mathbb{S}^1$ is clearly a compact group, i.e. its group operation is continuous, and so $\mathbb{T}^2$ is also a compact group by Theorem 6.10. Similarly $\mathbb{T}^2$ is Hausdorff since $\mathbb{S}^1$, and hence $\mathcal{T} = \mathbb{S}^1 \times \mathbb{S}^1$, are Hausdorff spaces. As a compact group, in particular a locally compact group, we equip $\mathbb{T}^2$ with the unique normalized Haar measure ν on its Borel σ-algebra $\mathscr{B}$.

In Chapter 5 we will encounter several instances where the uniqueness of ν and the invariance of ν under $\mathbb{T}^2$'s group operation are important.

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