Unique ergodicity in C*-dynamical systems

by

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DECLARATION

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Date: 6/09/2013

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Summary

The aim of this dissertation is to investigate ergodic properties, in particular unique ergodicity, in a noncommutative setting, that is in C*-dynamical systems. Fairly recently Abadie and Dykema introduced a broader notion of unique ergodicity, namely relative unique ergodicity. Our main focus shall be to present their result for arbitrary abelian groups containing a Følner sequence, and thus generalizing the $\mathbb{Z}$-action dealt with by Abadie and Dykema, and also to present examples of C*-dynamical systems that exhibit variations of these (uniquely) ergodic notions.

Abadie and Dykema gives some characterizations of relative unique ergodicity, and among them they state that a C*-dynamical system that is relatively uniquely ergodic has a conditional expectation onto the fixed point space under the automorphism in question, which is given by the limit of some ergodic averages. This is possible due to a result by Tomiyama which states that any norm one projection of a C*-algebra onto a C*-subalgebra is a conditional expectation. Hence the first chapter is devoted to the proof of Tomiyama's result, after which some examples of C*-dynamical systems are considered.

In the last chapter we deal with unique and relative unique ergodicity in C*-dynamical systems, and look at examples that illustrate these notions. Specifically, we present two examples of C*-dynamical systems that are uniquely ergodic, one with an $\mathbb{R}^2$-action and the other with a $\mathbb{Z}$-action, an example of a C*-dynamical system that is relatively uniquely ergodic but not uniquely ergodic, and lastly an example of a C*-dynamical system that is ergodic, but not uniquely ergodic.

Chapter 1

Introduction

In short one can think of ergodic theory as the study of the long term average behavior of some dynamical system, be it a physical system or one more abstract and mathematical in nature, [21]. The necessity for such long term averages originated with statistical mechanics, which in turn was necessitated by the study of large physical systems in which the application of classical mechanics seemed impractical; for example studying the temperature of a gas in a closed, isolated system. To take every single particle separately into account seems a tall order, but taking averages seems much more feasible.

An alternative description of ergodic theory encapsulating the essence thereof is the following:

> "In the broadest interpretation ergodic theory is the study of the qualitative properties of actions of groups on spaces." - Peter Walters([27])

The space mentioned typically refers to some set with some mathematical structure, e.g. a measure space, topological space or a differentiable manifold. These are not mutually disjoint. In fact, they overlap more often than not. Taking the structure into account, the "actions of groups" refers to some dynamics or time evolution on the space, that is, some function that transforms the space while preserving the structure of the space. For measurable systems this would be measure preserving transformations, for topological systems homeomorphisms and for differentiable systems diffeomorphisms.

One of the physicists that made some of the most notable contributions to statistical mechanics, Boltzmann, conjectured that the space mean and the time mean of a physical system should be equal everywhere, which turned out to be false in general. Nonetheless, this is exactly one of the interesting qualitative properties in ergodic theory worth looking for, and is called an *ergodic* system. Other interesting properties that have been introduced and extensively studied are unique ergodicity, weak and strong mixing, recurrence and isomorphisms between ergodic systems.[21]

The study of group actions on measure spaces started in the 1930's with some strong results by von Neumann and Birkhoff in the Mean Ergodic Theorem and the Pointwise Ergodic Theorem, respectively. Here the setting would typically be a dynamical system $(X, \mathscr{B}, \mu, T)$, consisting of a measure space X, a σ-algebra $\mathscr{B}$, a probability measure μ and a bijective transformation $T : X \to X$ with the property that $\mu(T^{-1}E) = \mu(E)$ for every $E \in \mathscr{B}$ (with μ said to be T-invariant). The next major breakthrough was due to Kolmogorov in 1958, when he introduced entropy,[27]. Since then many advances have been made in ergodic theory, and it is currently a very active research area. Many applications thereof have also been found in various areas in mathematics, for example, by Furstenberg in number theory, see for example [12], [11] and [13]. Another expansion of ergodic theory (as in many branches in mathematics) has been in the direction of a non-commutative setting. Typical spaces used in these settings are von Neumann algebras (non-commutative measurable dynamics) and C*-algebras (non-commutative topological dynamics). Again, these are not mutually exclusive. The reason for this shift to the non-commutative realm in most areas in mathematics, is (or was) mostly motivated by the advent of quantum mechanics, which is intrinsically a non-commutative theory. And, as quantum mechanics is arguably the most successful physical theory, it seems appropriate to find non-commutative versions for most physically applicable theories. See for example the Introduction of [3] for some more details and motivations on the physical relation to non-commutative mathematics.

Although there is the physical connection as mentioned above, this dissertation will deal with non-commutative topological dynamics, that is C*-dynamical systems, but from a purely mathematical point of view. In particular, we will consider unique ergodicity in C*-dynamical systems, which amounts to the existence of a unique state invariant under the automorphism in question. In 2009 Abadie an Dykema, [1], introduced a more general notion of unique ergodicity, namely relative unique ergodicity, where they relate the uniqueness to extensions of invariant states from the fixed point subalgebra under the specific automorphism. Our first goal is to present Abadie and Dykema's characterizations of relative unique ergodicity for any locally compact abelian group containing a Føner sequence, and thus generalizing their $\mathbb{Z}$-action. Secondly we aim to present examples that exhibit these ergodic notions.

We begin in Chapter 2 by developing the necessary mathematical tools required, including the Bochner integral, C*-algebras, von Neumann Algebras and an important result on the equivalence of norm one projections and conditional expectations. As a basic familiarity with these concepts are assumed many results in this chapter will be given without proofs, with the exception of Section 2.5 on norm one projections in C*-algebras, where we have tried to break the exposition, based on [25], down into more detail than appearing in the standard literature. Thus, the reader familiar with these concepts can skip Chapter 2 (with the possible exception of Section 2.5) and start with Chapter 3

In Chapter 3 we give the definition of a C*-dynamical system $(\mathcal{A}, \alpha)$, which just consists of a C*-algebra $\mathcal{A}$ and a group of automorphisms α on $\mathcal{A}$. The rest of the chapter is devoted to the construction of four examples of C*dynamical systems. These examples are chosen so as to illustrate that some C*-dynamical systems exhibit variations in the ergodic properties to be discussed later on. They include two examples of the non-commutative torus (or quantum torus), one with an action of $\mathbb{R}$ and the other of $\mathbb{R}^2$, the C*-algebra generated by the annihilation and creation operators on a deformed Fock space with an action of $\mathbb{Z}$, and lastly a shift on an infinite tensor product of C*-algebras, also with an action of $\mathbb{Z}$.

In Chapter 4 we discuss some ergodic properties. Besides a few necessary instances, we shall not give commutative (or classical) ergodic theory much attention, and will focus on the more general non-commutative setting. Classically, given a topological space X and a homeomorphism, T, the dynamical system (X, T) is said to be uniquely ergodic if there is only one Borel probability measure, say μ, invariant under the transformation. In a C*-algebraic setting the 'equivalent' of a measure is a state. Hence, a C*-dynamical system will be called uniquely ergodic if there is a unique state, invariant under the automorphism in question. Once we have introduced these ergodic properties and given some characterizations of these, we return to the examples of Chapter 3 to show the existence of C*-dynamical systems with variations in these ergodic properties, giving credibility (mathematically at least) to these definitions and results. Specifically, we show that the non-commutative torus (with an $\mathbb{R}^2$-action) and the C*-algebra generated by the annihilation and creation operators on a deformed Fock space (with the usual $\mathbb{Z}$-action) are uniquely ergodic, that the non-commutative torus (with an $\mathbb{R}$ action) is uniquely ergodic relative to the fixed point subalgebra but not uniquely ergodic, and lastly that the shift on an infinite tensor product of C*-algebras is ergodic but not uniquely ergodic.

Chapter 2

Preliminaries

In this chapter we present and develop the mathematics necessary for our exposition of non-commutative dynamical systems, in particular C*-dynamical systems. We begin with a vector valued integral, namely the Bochner integral, in the first section. The second section deals with C*-algebras and the third with locally convex topologies which are needed in the fourth section on von Neumann algebras. As mentioned earlier, the author assumes some familiarity with operator algebras (thus including functional analysis and topology) and measure theory, and will thus present only definitions and results directly relevant to our exposition, many without proofs (which can be found in detail in the given references). In the last section we present in more detail an exposition of the well known result of Tomiyama on norm one projections. This result will be vital in the main characterization of relative unique ergodicity in Chapter 4. If the reader is familiar with operator algebras, they can skip the first four sections and start with Section 2.5 (or even Chapter 3). Lastly, throughout this dissertation all inner products will be assumed to be linear in the second argument and conjugate linear in the first.

2.1 The Bochner Integral

This section is based on [4, Appendix E] where the proofs of the results stated without such, can be found in detail.

When taking any form of ergodic average over a more general group, it is necessary to replace the sum (over $\mathbb{Z}$) with an integral (over the general group being considered). But, working in C*-algebras would then require an integration theory for vector-valued functions. This section deals with such an integral, namely the Bochner integral.

Definition 2.1.1. Let (Ω, Σ) be a measurable space, X a Banach space and $\mathscr{B}(X)$ the Borel σ-algebra on X. A function $f : \Omega \to X$ is called *Borel* *measurable* if it is measurable with respect to Σ and $\mathscr{B}(X)$. It is called *strongly measurable* if it is Borel measurable and has a separable range.

Remark 2.1.2. If f is Borel measurable then the function $\omega \mapsto \|f(\omega)\|$ is Σ-measurable since the norm is continuous and hence Borel measurable, and the composition of two measurable functions is measurable.

Proposition 2.1.3. *[4, Prop E1,p.350] Let (Ω, Σ) be a measure space and X a Banach space. Then the collection of strongly measurable functions from Ω to X is closed under the formation of pointwise limits.*

Definition 2.1.4. Let (Ω, Σ) be a measure space and X a Banach space. A function $f : \Omega \to X$ is called *simple* if there exists $x_1, \cdots, x_n$ in X and mutually disjoint $S_1, \cdots, S_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{S_i}$, where $\chi_{S_i}(\omega) = 1$ if $\omega \in S_i$ and $\chi_{S_i}(\omega) = 0$ if $\omega \notin S_i$. Hence, $x_1, \cdots, x_n$ are the values attained by f on $S_1, \cdots, S_n$, respectively.

Proposition 2.1.5. *[4, Prop E2,p.351] Let (Ω, Σ) be a measure space, X a Banach space and let $f : \Omega \to X$ be strongly measurable. Then there exists a sequence (f_n) of strongly measurable simple functions such that*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega)$$

and

$$\|f_n(\omega)\| \leq \|f(\omega)\|,$$

for every $\omega \in \Omega$ and n=1,2,. . . .

Remark 2.1.6. From the Propositions above if follows that a function $f : \Omega \to X$ is strongly measurable if and only if it is the pointwise limit of a sequence of strongly measurable simple functions, and that the set of all such strongly measurable functions forms a vector space.

Definition 2.1.7. Let (Ω, Σ, μ) be a measure space and let X be a Banach space. A function $f : \Omega \to X$ is called *Bochner integrable* (or just *integrable*) if f is strongly measurable and if the function $\omega \to \|f(\omega)\|$ is integrable.

Theorem 2.1.8. *Let Ω be a compact space, X a Banach space, μ a finite measure on the Borel σ-algebra of Ω and let $f : \Omega \to X$ be a continuous function. Then f is Bochner integrable.*

Proof. Because f is continuous it is Borel measurable and $f(\Omega)$ is compact. Then, since $f(\Omega)$ is a compact metric space, it must be separable (see for example [24, Prop 2.5.8, p.54]), and thus f is strongly measurable. Also, by its continuity f is bounded on compact sets, i.e. $||f|| := \sup_{\omega\in\Omega} ||f(\omega)|| < \infty$, which implies that

$$\int_{\Omega} ||f(\omega)|| d\mu < \infty.$$

Hence f is Bochner integrable. $\square$

Below we describe how the integral of such an integrable function is defined. The procedure is fairly simple and follows a similar approach as in usual measure theory. We first state an auxiliary result.

Proposition 2.1.9. *[4, Prop 2.3.9,p.67] Let (Ω, Σ, μ) be a measure space and let $f : \Omega \to [0, +\infty]$ be Σ-measurable. If $t \in [0, +\infty]$ and $V_t = \{\omega \in \Omega : f(\omega) \geq t\}$ then*

$$\mu(V_t) \leq \frac{1}{t} \int_{V_t} f d\mu \leq \frac{1}{t} \int f d\mu.$$

Now, let (Ω, Σ, μ) be a measure space, X a Banach space and $f : \Omega \to X$ integrable and simple. Assume $x_1, x_2, \ldots, x_n$ are the non-zero values of f attained on the sets $S_1, S_2, \ldots, S_n$, respectively. As stated in Remark 2.1.2 the function $\omega \to \|f(\omega)\|$ is measurable, and using this function in Proposition 2.1.9 and by the integrability of f we see that

$$\mu(S_i) \leq \frac{1}{\|x_i\|} \int \|f(\omega)\| d\mu < +\infty, \tag{2.1}$$

showing that each S_i has finite measure. Hence we define the integral of f by

$$\int f d\mu = \sum_{i=1}^{n} x_i \mu(S_i)$$

which is meaningful by Inequality 2.1. Also, note that $f = \sum_{i=1}^{n} x_i \chi_{S_i}$ is integrable if and only if $\mu(S_i) < +\infty$ for each $i = 1, 2, \ldots, n$.

Proposition 2.1.10. *Let (Ω, Σ, μ) be a measure space, X a Banach space and $f, g : \Omega \to X$ simple integrable functions. Then the following properties hold:*

(i) $\| \int f d\mu \| \leq \int \|f\| d\mu$

(ii) $\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu,$ *with* $\alpha, \beta \in \mathbb{C}$

Proof. Let $f = \sum_{i=1}^{n} x_i \chi_{S_i}$ and $g = \sum_{j=1}^{m} y_j \chi_{O_j}$. We may assume that $\bigcup_{i=1}^{n} S_i = \bigcup_{j=1}^{m} O_j$, and that the sets $S_i, i = 1, 2, \ldots, n$ are disjoint as are the sets $O_j, j = 1, 2, \ldots, m$.

(i) Then

$$\begin{aligned} \| \int f d\mu \| &= \| \sum_{i=1}^{n} x_i \mu(S_i) \| \\ &\leq \sum_{i=1}^{n} \|x_i\| \mu(S_i) \\ &= \int \|f\| d\mu \end{aligned}$$

(ii)

$$\begin{aligned}\int(\alpha f+\beta g)d\mu &= \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha x_i+\beta y_j)\mu(S_i\bigcap O_j)\\ &= \alpha\sum_{i=1}^{n}x_i\mu(S_i)+\beta\sum_{j=1}^{m}y_j\mu(O_j)\\ &= \alpha\int fd\mu+\beta\int gd\mu\end{aligned}$$

□

Assume now that $f:\Omega\to X$ is any integrable function. Then by Proposition 2.1.5 there exists a sequence (f_n) of strongly measurable functions such that $f(\omega)=\lim\limits_{n\to\infty}f_n(\omega)$, and $\|f_n(\omega)\|\leq\|f(\omega)\|$, for every $n=1,2,\ldots$ and $\omega\in\Omega$. Moreover, since f is integrable, each f_n will be integrable as well. The dominated convergence theorem for real-valued functions applied to the sequence of functions given by
$\tilde{f}_n(\omega)=\|f(\omega)-f_n(\omega)\|$, which converges to 0, and to $g(\omega)=2||f(\omega)||$ as the dominating function, implies that

$$\lim_{n\to\infty}\int||f-f_n||d\mu=0,$$

or in other words that

$$\lim_{m,n\to\infty}\int||f_m-f_n||d\mu=0. \tag{2.2}$$

(Notation: $\lim\limits_{m,n\to\infty}a_{m,n}=0$ means that for every $\epsilon>0$ there exists an $N\in\mathbb{N}$ such that $|a_{m,n}|<\epsilon$ if $m,n>N$.)

Equation 2.2 and Proposition 2.1.10 imply that the sequence $(\int f_n d\mu)$ is Cauchy in X, since

$$||\int f_nd\mu-\int f_md\mu||\leq\int||f_n-f_m||d\mu\xrightarrow[\infty]{m,n}0.$$

The *Bochner integral* (or just the *integral*) of f is defined as the limit of the sequence $(\int f_nd\mu)$, i.e.

$$\int fd\mu=\lim_{n\to\infty}\int f_nd\mu.$$

That the value of $\int fd\mu$ is independent of the choice of the sequence of simple integrable functions (guaranteed by Proposition 2.1.5) is clear.

In the following two propositions we give some properties of the Bochner integral that will be required later on.

Theorem 2.1.11. *[4, Prop E4, E5, E11, pp.355-356] Let (Ω, Σ, μ) be a measure space, X a Banach space and $f, g : \Omega \to X$ integrable functions. Then the following properties hold:*

(i) $\| \int f d\mu \| \leq \int \|f\| d\mu$

(ii) $\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu,$ *with* $\alpha, \beta \in \mathbb{C}$

(iii) $\phi(\int f d\mu) = \int \phi \circ f d\mu,$ *for every* $\phi \in X^*$

(where X^* denotes the dual space of X, that is the set of all (norm) continuous linear functionals on X).

Remark 2.1.12. If one replaces the Banach space X with $\mathbb{C}$ in Theorem 2.1.11, then the integral in question is nothing other than the Lebesgue integral.

Let $B(X)$ denote the normed space of all bounded linear operators on X.

Theorem 2.1.13. *Let (Ω, Σ, μ) be a measure space, X a Banach space, $T \in B(X)$ and $f : \Omega \to X$ an integrable function. Then Tf is integrable and*

$$T(\int f d\mu) = \int T f d\mu.$$

Proof. Let $f : \Omega \to X$ be a simple integrable function, which takes on the values $x_1, x_2, \ldots, x_n$ on the disjoint sets $S_1, \ldots, S_n$, respectively. Then $Tf = \sum_{i=1}^{n} T x_i \chi_{S_i}$ is a simple integrable function, and

$$\begin{aligned} T(\int f d\mu) &= T(\sum_{i=1}^{n} x_i \mu(S_i)) \\ &= \sum_{i=1}^{n} T x_i \mu(S_i) \\ &= \int T f d\mu \end{aligned}$$

Assume now that $f : \Omega \to X$ is an arbitrary integrable function. Then by Proposition 2.1.5 there exists a sequence of integrable simple functions (f_n) such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$, and such that $\|f_n(\omega)\| \leq \|f(\omega)\|$, for every $n = 1, 2, \ldots$ and $\omega \in \Omega$.

By the continuity of T we have that $Tf_n, n = 1, 2 \ldots$, and Tf are measurable functions, and have separable ranges. Moreover, by the integrability of f we have that Tf is integrable, since

$$||Tf(\omega)|| \leq ||T|| ||f(\omega)||.$$

By Prop 2.1.5 we also have

$$||Tf_n(\omega)|| \leq ||T||||f_n(\omega)|| \leq ||T||||f(\omega)||.$$

Put $T\tilde{f}_n(\omega) = ||Tf_n(\omega) - Tf(\omega)||$ and $g(\omega) = 2||T||||f(\omega)||$. Then $T\tilde{f}_n(\omega) \leq g(\omega)$ for every $\omega \in \Omega$. Also note that, by the continuity of T, we have $\lim_{n\to\infty} Tf_n(\omega) = Tf(\omega)$, and hence $\lim_{n\to\infty} T\tilde{f}_n(\omega) = 0$. Now, by the dominated convergence theorem for real functions, using g as the dominating function, we have that

$$\lim_{n\to\infty} \int ||Tf_n(\omega) - Tf(\omega)||d\mu = \lim_{n\to\infty} \int T\tilde{f}_n(\omega)d\mu = 0.$$

Finally we obtain

$$\begin{aligned} || \int Tf_n(\omega)d\mu - \int Tf(\omega)d\mu|| &= || \int (Tf_n(\omega) - Tf(\omega))d\mu|| \\ &\leq \int ||Tf_n(\omega) - Tf(\omega)||d\mu \\ &\longrightarrow 0 \ \ (\text{as } n \to \infty). \end{aligned}$$

Hence

$$\int Tf d\mu = \lim_{n\to\infty} \int Tf_n d\mu = \lim_{n\to\infty} T(\int f_n d\mu) = T(\lim_{n\to\infty} \int f_n d\mu) = T(\int f d\mu),$$

by the dominated convergence theorem for X-valued functions
(see [4, Prop E6, p.353]).

□

2.2 C*-algebras

In this section we shall cover the basics of C*-algebras and also introduce some necessary results concerning representations of C*-algebras.

Given an algebra $\mathcal{A}$ over $\mathbb{C}$, we define a mapping, $a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$, called an *involution* or an *adjoint map* such that for all $a, b \in \mathcal{A}; \alpha, \beta \in \mathbb{C}$,

(i) $(a^*)^* = a$

(ii) $(ab)^* = b^*a^*$

(iii) $(\alpha a + \beta b)^* = \bar{\alpha}a^* + \bar{\beta}b^*$ (with $\bar{\alpha}$ denoting the complex conjugate.)

We call an algebra with an involution defined on it a **-algebra*. If in addition the algebra is a complete normed space such $||ab|| \leq ||a||||b||$ for every pair of elements in $\mathcal{A}$, then we call $\mathcal{A}$ a *Banach *-algebra*.

Definition 2.2.1. We call a Banach *-algebra with the property that for every $a \in \mathcal{A}$
$$||a^*a|| = ||a||^2,$$
a *C*-algebra.*

We will call a multiplicative identity element a unit, and if an algebra (or *-algebra, or C*-algebra) contains a unit, then the algebra is called unital (or said to contain an identity). All C*-algebras will be assumed unital, and the unit will be denoted by $1_{\mathcal{A}}$ for the C*-algebra $\mathcal{A}$.

Definition 2.2.2. An element in a C*-algebra is said to be *positive* if it is self-adjoint, i.e. $a^* = a$, and its spectrum lies in the positive reals. We write $a \geq 0$ to indicate that a is positive.

Note that any element a in a C*-algebra can be written as a linear combination, say $a = a_1 + ia_2$, of self-adjoint elements, where $a_1 = \frac{1}{2}(a + a^*)$ and $a_2 = \frac{1}{2i}(a - a^*)$.

Proposition 2.2.3. *[19, Thm 2.2.5, p.46] An element a of a C*-algebra $\mathcal{A}$ is positive if and only if $a = b^*b$ for some $b \in \mathcal{A}$.*

Proposition 2.2.4. *[19, Thm 2.2.5, p.46] For any two self-adjoint elements a and b in a C*-algebra $\mathcal{A}$, if $a \leq b$ then $c^*ac \leq c^*bc$ for every $c \in \mathcal{A}$.*

Proposition 2.2.5. *[15, Prop 4.2.3, p.246] Let a be a self-adjoint element of a C*-algebra $\mathcal{A}$. Then a can be expressed in the form $a = a_+ - a_-$, where a_+ and a_- are unique positive elements in $\mathcal{A}$, $a_+a_- = a_-a_+ = 0$ and $||a|| = \max\{||a_+||, ||a_-||\}$.*

Note that the proof of Proposition 2.2.5 is usually done using the Gelfand representation theorem for abelian C*-algebras, which will be stated with the other representation theorems for C*-algebras later in this section.

The Banach space of all bounded linear functionals on a normed space X will be called its *dual space* (or dual for short), and denoted by X^*. Similarly for the double dual, $X^{**} = (X^*)^*$.

Definition 2.2.6. Given a C*-algebra $\mathcal{A}$, then $\phi \in \mathcal{A}^*$ is called *positive* if for every $a \in \mathcal{A}$
$$\phi(a^*a) \geq 0.$$
If furthermore we have that $||\phi|| = 1$, then ϕ is called a *state.*

The next proposition gives the existence of a state on a C*-algebra with an added property that will prove useful later.

Proposition 2.2.7. *[15, Thm 4.3.4, p.258] Let $\mathcal{A}$ be a unital C*-algebra and let $a \in \mathcal{A}$ be normal, that is $a^*a = aa^*$. Then there exists a state, ϕ, on $\mathcal{A}$ such that $|\phi(a)| = ||a||$.*

We will let $\mathcal{A}_+^*$ denote the set of all positive bounded linear functionals on a C*-algebra $\mathcal{A}$, and $\mathcal{S}(\mathcal{A})$ the set of all states on $\mathcal{A}$. $\mathcal{S}(\mathcal{A})$ is convex.

Theorem 2.2.8. *[19, Thm 3.4.3, p.95] Let a be a self-adjoint element of a C*-algebra $\mathcal{A}$. Then a is positive if and only if $\phi(a) \geq 0$ for all positive linear functionals ϕ on $\mathcal{A}$.*

We give the following characterization of positivity that will frequently be used.

Theorem 2.2.9. *[15, Thm 4.3.2, p.256] Let $\mathcal{A}$ be a unital C*-algebra and ϕ a linear functional on $\mathcal{A}$. Then ϕ is positive if and only if ϕ is bounded and $||\phi|| = \phi(1_{\mathcal{A}})$.*

Definition 2.2.10. Let $\phi \in \mathcal{A}^*$, then we define $\phi^* \in \mathcal{A}^*$, called the *adjoint* of ϕ, by $\phi^*(a) = \overline{\phi(a^*)}$ (where $\overline{\phi(a)}$ denotes the complex conjugate of $\phi(a)$). We call $\phi \in \mathcal{A}^*$ *self-adjoint* if for $a \in \mathcal{A}$, $\phi(a^*) = \overline{\phi(a)}$. Then
$\phi^*(a) = \overline{\phi(a^*)} = \phi(a)$, i.e. $\phi^* = \phi$.

Proposition 2.2.11 (Jordan Decomposition)**.** *[25, Prop 2.1, p.120] Let $\mathcal{A}$ be a C*-algebra. Every self-adjoint $\phi \in \mathcal{A}^*$ is represented uniquely in the form*

$$\phi = \phi_+ - \phi_- \text{ and } ||\phi|| = ||\phi_+|| + ||\phi_-||$$

by some $\phi_+, \phi_- \in \mathcal{A}_+^*$.

For a proof of the uniqueness of the Jordan decomposition see [20, Thm 3.2.5] or [25, Thm 4.2, p.140].

Remark 2.2.12. Note that any $\phi \in \mathcal{A}^*$ can be written as a linear combination of two self-adjoint bounded linear functionals, say $\phi = \phi_1 + i\phi_2$, by taking $\phi_1 = \frac{1}{2}(\phi + \phi^*)$ and $\phi_2 = \frac{1}{2i}(\phi - \phi^*)$. Combining this fact with the Jordan decomposition, if follows that any $\phi \in \mathcal{A}^*$ can be written uniquely as a linear combination in the form

$$\phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4),$$

where $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{A}_+^*$.

Proposition 2.2.13. *[3, Prop 2.2.19, p.41] Let $\mathcal{I}$ be a closed two-sided ideal in a C*-algebra $\mathcal{A}$. Then $\mathcal{I}$ is self-adjoint and the quotient $\mathcal{A}/\mathcal{I}$ is a C*-algebra with the usual operations for a quotient and the norm defined by*
$||a + \mathcal{I}|| = \inf_{b \in \mathcal{I}} ||a + b||, \quad a \in \mathcal{A}$.

Proposition 2.2.14. *[3, Prop 2.3.1, p.42] If π is a *-homomorphism from a Banach *-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$, then*

$$||\pi(a)|| \leq ||a||, \text{ for every } a \in \mathcal{A}.$$

Moreover, if $\mathcal{A}$ is a C-algebra then the range, $\pi(\mathcal{A})$, of π is a C*-subalgebra of $\mathcal{B}$.*

Corollary 2.2.15. *[25, Cor 5.4, pp.22] If π is a *-isomorphism from a C*-algebra $\mathcal{A}$ onto a C*-algebra $\mathcal{B}$, then π is an isometry.*

Proposition 2.2.16. *[25, Cor 8.2, p.32]*
*If π is a *-homomorphism of a C*-algebra $\mathcal{A}$ onto another C*-algebra $\mathcal{B}$, then π induces a *-isomorphism π_0 from the quotient C*-algebra $A/ker(\pi)$ onto $\mathcal{B}$ such that $\pi = \pi_0 \circ i_0$, where $i_0 : \mathcal{A} \to A/ker(\pi)$ takes $a \in \mathcal{A}$ to the equivalence class $a + ker(\pi) \in A/ker(\pi)$.*

Definition 2.2.17. Let $\mathcal{A}$ be a C*-algebra. We will call the pair $(\mathfrak{H}, \pi)$ a *representation* of $\mathcal{A}$, if π is a *-homomorphism of $\mathcal{A}$ into the C*-algebra of all bounded linear operators $B(\mathfrak{H})$, on some Hilbert space $\mathfrak{H}$. A representation is called *faithful* if $ker(\pi) = \{0\}$.

Definition 2.2.18. Let $(\mathfrak{H}, \pi)$ be a representation of the C*-algebra $\mathcal{A}$. A vector $\xi \in \mathfrak{H}$ will be called a *cyclic vector* if the linear span of the set $\{\pi(a)\xi : a \in \mathcal{A}\}$ is dense in $\mathfrak{H}$.

We now make a quick and necessary diversion. In many instances we shall make use of infinite direct sums of Hilbert spaces. Hence, we give the basic definitions and the author assumes familiarity with the finite case. A complete and detailed exposition of the finite and infinite cases can be found in [15, pp.25-28 & pp.121-124]. Let $\{\mathfrak{H}_n\}, n \in \mathbb{I}$ for some index set $\mathbb{I}$, be a family of Hilbert spaces. The direct sum, denoted by $\bigoplus\limits_{n\in\mathbb{I}} \mathfrak{H}_n$, consists elements of the form $(\xi_n)_{n\in\mathbb{I}}$, where $\xi_n \in \mathfrak{H}_n$ and $\sum_{n\in\mathbb{I}} ||\xi_n||^2 < \infty$. Addition and scalar multiplication are given by

$$(\xi_n) + (\eta_n) = (\xi_n + \eta_n) \text{ and } c(\xi_n) = (c\xi_n),$$

respectively. The inner product and norm are given by

$$\langle(\xi_n), (\eta_n)\rangle = \sum_{n\in\mathbb{I}} \langle\xi_n, \eta_n\rangle \text{ and } ||(\xi_n)|| = (\sum_{n\in\mathbb{I}} ||\xi_n||^2)^{1/2},$$

respectively. With these operations and inner product, $\bigoplus\limits_{n\in\mathbb{I}} \mathfrak{H}_n$ is complete as an inner product space, and hence a Hilbert space. If $T_n \in B(\mathfrak{H}_n), n \in \mathbb{I}$, and $\sup\{||T_n|| : n \in \mathbb{I}\} < \infty$, then the equation $T(\xi_n) = (T_n\xi_n)$ defines an element in $B(\bigoplus\limits_{n\in\mathbb{I}} \mathfrak{H}_n)$. We call $T = \bigoplus\limits_{n\in\mathbb{I}} T_n$ the direct sum of (T_n). Addition, scalar multiplication, multiplication and an involution can be defined in a natural way for direct sums $\bigoplus\limits_{n\in\mathbb{I}} T_n$ of operators, and a norm is given by $||T|| = \sup\{||T_n|| : n \in \mathbb{I}\}$.

Returning to representations of C*-algebras, given any positive linear functional ω on a C*-algebra $\mathcal{A}$, it is possible to construct a representation $(\mathfrak{H}_\omega, \pi_\omega, \xi_\omega)$ of $\mathcal{A}$, called a *cyclic representation induced by* ω, where ξ_ω is a cyclic vector in $\mathfrak{H}_\omega$. This construction is known as the GNS-construction and gives the existence of a representation for any C*-algebra. The details of the construction can be found in [3, pp. 54-55 and Thm 2.3.16, p.56] and [25, Thm 9.14, pp. 39-40]. Moreover, the GNS construction holds for *-algebras as well, and this is evident in the construction done in [3, pp. 54-55 and Thm 2.3.16, p.56], but in this case the represented operators are not necessarily bounded. When we use this fact later on in the proof of Proposition 3.3.6, we shall do it for a finite dimensional Hilbert space, and therefore the operators will be bounded. If $\{\mathfrak{H}_\alpha, \pi_\alpha\}, \alpha \in \mathbb{I}$, is a family of representations of a C*-algebra $\mathcal{A}$, then one can form a new representation $(\mathfrak{H}, \pi)$ of $\mathcal{A}$, where $\mathfrak{H} = \bigoplus_{\alpha\in\mathbb{I}} \mathfrak{H}_\alpha$, and for every $\xi = \bigoplus_{\alpha\in\mathbb{I}} \xi_\alpha \in \mathfrak{H}$ and $a \in \mathcal{A}$

$$\pi(a)\xi = \bigoplus_{\alpha\in\mathbb{I}} \pi_\alpha(a)\xi_\alpha.$$

The interested reader can find the details in [3, p. 46] and [25, p.41]. By considering such a direct sum over all representations induced by positive linear functionals on a C*-algebra, one can prove the following result which shows that any C*-algebra admits a faithful representation, so that it can be seen as a norm closed algebraic 'copy' of a C*-subalgebra of $B(\mathfrak{H})$, without explicit reference to the Hilbert space on which it acts.

Theorem 2.2.19. *[25, Thm 9.18, p.42] A C*-algebra admits a faithful representation. Hence it is isometrically *-isomorphic to a uniformly (norm) closed *-subalgebra of operators on a Hilbert space.*

Proposition 2.2.20. *[25, Prop 2.1, p.120] Every $\omega \in \mathcal{A}^*$ is represented in the form*

$$\omega_{(\pi;\xi,\eta)}(a) = \langle \xi, \pi(a)\eta \rangle$$

by some representation $(\mathfrak{H}, \pi)$ of $\mathcal{A}$, where $\xi, \eta \in \mathfrak{H}$ and $a \in \mathcal{A}$.

Lastly we give Gelfand's representation theorem for abelian C*-algebras. Let $\Gamma(\mathcal{A})$ be the set of all non-zero homomorphism from $\mathcal{A}$ into $\mathbb{C}$. Let $\hat{a} : \Gamma(\mathcal{A}) \to \mathbb{C}$ be defined by

$$\hat{a}(\tau) = \tau(a).$$

It can be shown that if $\mathcal{A}$ is unital, then $\Gamma(\mathcal{A})$ is a compact Hausdorff space, in the weak*-topology (see for example [19, Thm 1.3.5, p.15]), and that $\hat{a} \in C(\Gamma(\mathcal{A}))$. We call the map $\mathcal{A} \to C(\Gamma(\mathcal{A})), a \mapsto \hat{a}$ the *Gelfand representation of* $\mathcal{A}$, and $\hat{a}$ the *Gelfand transform of* a.

Let $\sigma(a) = \{\lambda \in \mathbb{C} : (\lambda 1_{\mathcal{A}} - a)^{-1} \text{ does not exist}\}$ denote the spectrum of an element $a \in \mathcal{A}$.

Theorem 2.2.21. *[19, Thm 1.3.4, p.14] Let $\mathcal{A}$ be an abelian Banach-algebra. Then for every $a \in \mathcal{A}$*

$$\sigma(a) = \{\tau(a) : \tau \in \Gamma(\mathcal{A})\}.$$

Theorem 2.2.22. *[19, Thm 2.1.10, p.41] Let $\mathcal{A}$ be a non-zero abelian C*-algebra, then the Gelfand representation of $\mathcal{A}$ is an isometric *-isomorphism.*

Hence, from Theorems 2.2.21 and 2.2.22 it follows that every element in an abelian C*-algebra can be identified with a continuous function having its spectrum as its range. Specifically, there exists for every $\lambda \in \sigma(a)$ a $\tau \in \Gamma(\mathcal{A})$ such that $\tau(a) = \lambda$. This fact will be used on a few occasions.

2.3 Locally Convex Topologies

Below we give a brief overview of locally convex topologies and present some examples of such topologies that will be applicable later on. This is necessitated by the fact that we will work with von Neumann algebras, and these topologies form an integral part of the structure of (and proofs relating to) von Neumann algebras.

Definition 2.3.1. Suppose that $\mathcal{V}$ is a real or complex vector space. We call a set $Y \subset \mathcal{V}$ *convex* if $\alpha_1 y_1 + \alpha_2 y_2 \in Y$, for every $y_1, y_2 \in Y$ and α_1, α_2 positive real numbers such that $\alpha_1 + \alpha_2 = 1$.

Definition 2.3.2. Suppose that $\mathcal{V}$ is a real or complex vector space. We call $\mathcal{V}$ a *topological vector space* if $\mathcal{V}$ is also a Hausdorff topological space in which the algebraic operations, namely addition and scalar multiplication, are continuous in this topology. Furthermore, we will call $\mathcal{V}$ a *locally convex topological vector space* (or just a *convex space*) if the topology has a base consisting of convex sets, and we call such a topology a *locally convex topology.*

Definition 2.3.3. We say that a family of semi-norms $\mathcal{P}$ on some vector space $\mathcal{V}$ *separates* the points of $\mathcal{V}$ if for every $x \neq 0$ in $\mathcal{V}$, there is a semi-norm $p \in \mathcal{P}$ for which $p(x) \neq 0$.

Proposition 2.3.4. *[15, Thm 1.2.6, p.17] Suppose that $\mathcal{V}$ is a real or complex vector space, and that $\mathcal{P}$ is a family of semi-norms on $\mathcal{V}$ that separates the points of $\mathcal{V}$. Then there is a locally convex topology on $\mathcal{V}$ in which, for every $x_0 \in \mathcal{V}$ the family of all sets*

$$N(x_0 : p_1, \ldots, p_n; \epsilon) = \{x \in \mathcal{V} : p_i(x - x_0) < \epsilon, i = 1, \ldots, n\},$$

with $\epsilon > 0, p_i \in \mathcal{P}$, is a base of neighborhoods of x_0. With this topology, each of the semi-norms in $\mathcal{P}$ is continuous, and this is the coarsest such topology. Moreover, every locally convex topology on $\mathcal{V}$ arises, in this way, from a suitable family of semi-norms.

Proposition 2.3.5. *[15, Prop 1.2.8] Suppose that $\mathcal{V}$ is a locally convex space and $\mathcal{P}$ the family of semi-norms giving rise to the topology on $\mathcal{V}$.*

(i) *A semi-norm p on $\mathcal{V}$ is continuous if and only if there is a positive real number c and a finite set $p_1, \ldots, p_n \in \mathcal{P}$ such that*

$$p(x) \leq c \max_{1 \leq i \leq n} p_i(x), \qquad x \in \mathcal{V}.$$

(ii) *A linear functional f on $\mathcal{V}$ is continuous if and only if there is a positive real number c and a finite set $p_1, \ldots, p_n \in \mathcal{P}$ such that*

$$|f(x)| \leq c \max_{1 \leq i \leq n} p_i(x), \qquad x \in \mathcal{V}.$$

For completeness sake and the fact that we will work in locally convex topologies that are coarser than the norm topology, we need a concept of convergence that is independent of the norm (or metric), but still have some of the "nice" properties that we are accustomed to with convergence of sequences in metric spaces. Thus, we define for more general index sets the concept of a net and its convergence.

Definition 2.3.6. Let $\mathbb{I}$ be any set. A *preorder* on the set is any binary relation, say $\prec$, such that for any $a, b, c \in \mathbb{I}$

(i) $a \prec a$ and,

(ii) $a \prec b$ and $b \prec c$ implies $a \prec c$.

We call $(\mathbb{I}, \prec)$ (or just $\mathbb{I}$ if there is no confusion) a *preordered set*.

Definition 2.3.7. We call a preordered set $(\mathbb{I}, \prec)$ a *directed set* if for every $\alpha, \beta \in \mathbb{I}$ there exists a $\gamma \in \mathbb{I}$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. A *net* in a set S is a function from a directed set $\mathbb{I}$ into the set S.

Note 2.3.8. We will denote a net by $(x_\alpha)_{\alpha \in \mathbb{I}}$, and if no ambiguity can arise with regard to which index set is used, we shall simply write (x_α).

Definition 2.3.9. Let (X, τ) be a topological space. We say that a net (x_α) in X *converges to* a point $x \in X$, if for every neighborhood N_x of x, there exists an $\alpha' \in \mathbb{I}$ such that $x_\alpha \in N_x$ for every $\alpha \geq \alpha'$. We call x a *limit* of (x_α), and write $\lim_\alpha x_\alpha = x$ or simply $x_\alpha \to x$.

Theorem 2.3.10. *[24, Prop 3.2.14, p.76] Let (X, τ) be a topological space and $S \subseteq X$. Then the closure, $\bar{S}$, consists of those points in X that are limits of nets in S.*

Corollary 2.3.11. *[24, Cor 3.2.15, p.76] Let (X, τ) be a topological space. A set $S \subseteq X$ is closed if and only if every net in S that converges has its limit in S.*

Proposition 2.3.12. *[24, Prop 3.2.17, p.76] Let (X, τ) be a topological space. X is Hausdorff if and only if every net that converges in X has a unique limit.*

Theorem 2.3.13. *[24, Prop 3.3.17, p.82] Let (X, τ) be a topological space and (x_α) a net in X. Then $x \in X$ is an accumulation point of (x_α) if and only if there is a subnet of (x_α) converging to x.*

Theorem 2.3.14. *[24, Prop 3.3.18, p.82] Let (X, τ) be a topological space. Then X is compact if and only if every net in X has a convergent subnet.*

Proposition 2.3.15. *[24, Thm 3.2.18, p.77] Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every net (x_α) in X such that $x_\alpha \to x_0$, $f(x_\alpha) \to f(x_0)$.*

Theorem 2.3.16. *Let $\mathcal{P}$ be a family of semi-norms on a vector space $\mathcal{V}$ which generates a locally convex topology τ on $\mathcal{V}$ as described in Proposition 2.3.4. Then a net $(x_\alpha)_{\alpha \in \mathbb{I}}$ in $\mathcal{V}$ converges to a point $x \in \mathcal{V}$ if and only if $\lim_\alpha p(x_\alpha - x) = 0$ for every $p \in \mathcal{P}$.*

Proof. Firstly, assume that $x_\alpha \to x$, or equivalently, since we are working in a locally convex topology, that $(x_\alpha - x) \to 0$. Since every semi-norm $p \in \mathcal{P}$ is continuous in τ it follows directly from Proposition 2.3.15 that $p(x_\alpha - x) \to 0$ for every $p \in \mathcal{P}$.

Conversely, assume that $p(x_\alpha - x) \to 0$ for every $p \in \mathcal{P}$. Then for every $\epsilon > 0$ there exists an $\alpha_p \in \mathbb{I}$ such that $p(x_\alpha - x) < \epsilon$ if $\alpha > \alpha_p$. Consider any basic neighborhood of x, say

$$N(x : p_1, \ldots, p_n; \epsilon) = \{x \in \mathcal{V} : p_i(x - x_0) < \epsilon, i = 1, \ldots, n\},$$

with $\epsilon > 0$ and $p_i \in \mathcal{P}$. As stated above, for each $i = 1, 2, \ldots, n$, there exists an $\alpha_i \in \mathbb{I}$ such that $p_i(x_\alpha - x) < \epsilon$ if $\alpha > \alpha_i$. Being a directed set, there exists an $\alpha' \in \mathbb{I}$ such that $\alpha' > \alpha_i$ for every $i = 1, 2, \ldots, n$. Then, for every $i = 1, 2, \ldots, n$, we have $p_i(x_\alpha - x) < \epsilon$ if $\alpha > \alpha'$, and thus $x_\alpha \in N(x : p_1, \ldots, p_n; \epsilon)$ if $\alpha > \alpha'$. □

We consider a few specific locally convex topologies that will be of importance in Sections 2.4 and 2.5.

Weak topology: Given a Banach space X, we define the weak topology on X, denoted by $\sigma(X, X^*)$, as follows: We take as our set of semi-norms on X the set

$$\mathcal{P} = \{p_\phi : \phi \in X^*\}$$

where $p_\phi(x) = |\phi(x)|$ for each $x \in X$. This set is separating; for if $p_\phi(x) = 0$ for every $\phi \in X^*$, then $x = 0$ (see for example [18, Cor 4.3-4, p.223]). The locally convex topology given by Proposition 2.3.4 with the semi-norms $\mathcal{P}$

is called the *weak-topology* on X. Just to make it explicit, in the $\sigma(X, X^*)$-topology, any $x_0 \in X$ will thus have a base of neighborhoods consisting of sets of the form

$$\{x \in X : |\phi_i(x) - \phi_i(x_0)| < \epsilon, \ \ i = 1, \dots, n\}$$

where $\epsilon > 0$ and $\phi_i \in X^*, i = 1, \dots, n$.

Weak*-topology: Given a Banach space X, we define the weak*-topology on X^*, denoted by $\sigma(X^*, X)$, in a similar way as the weak topology: We take as our set of semi-norms on X^* the set

$$\mathcal{P} = \{p_x : x \in X\}$$

where $p_x(\phi) = |\phi(x)|$ for each $\phi \in X^*$. This set is separating; for if $p_x(\phi) = 0$ for every $x \in X$, then $\phi = 0$. Thus, the locally convex topology given by Proposition 2.3.4 using $\mathcal{P}$ is the *weak*-topology* on X^*. As with the weak topology, any $\phi_0 \in X^*$ in the $\sigma(X^*, X)$-topology will have a base of neighborhoods consisting of sets of the form

$$\{\phi \in X^* : |\phi(x_i) - \phi_0(x_i)| < \epsilon, \ \ i = 1, \dots, n\}$$

where $\epsilon > 0$ and $x_i \in X, i = 1, \dots, n$.

Theorem 2.3.17. *[15, Thm 1.6.5, p.45] Let X be a normed space, and $i : X \to X^{**}$ the canonical map given by $i(x)(\varphi) = \varphi(x)$ for every $\varphi \in X^*$. Then the weak* closure in X^{**} of the unit ball $\mathcal{B}_{i(X)}$ of $i(X)$ is equal to the unit ball $\mathcal{B}_{X^{**}}$ of X^{**}.*

Theorem 2.3.18 (Banach-Alaoglu)**.** *[23, Thm 3.15, p.68] If N_0 is a neighborhood of 0 in a Banach space X and if*

$$K = \{\phi \in X^* : |\phi(x)| \leq 1, x \in N_0\},$$

then K is weak-compact.*

Theorem 2.3.19. *Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{S}(\mathcal{A})$ is weak*-compact.*

Proof. By the Banach-Alaoglu theorem the unit ball $\mathcal{B}_{\mathcal{A}^*}$ of $\mathcal{A}^*$ is weak*-compact. $\mathcal{S}(\mathcal{A})$ is a convex weak*-closed subset of $\mathcal{B}_{\mathcal{A}^*}$, and hence is also weak*-compact. $\square$

Now, let $\mathfrak{H}$ be a Hilbert space and consider $B(\mathfrak{H})$, the C*-algebra of all bounded linear operators acting on this Hilbert space. The following topologies are examples of locally convex topologies on $B(\mathfrak{H})$.

Weak operator topology: Let $\xi, \eta \in \mathfrak{H}$. Then

$$p_{\xi,\eta}(A) = |\langle \xi, A\eta \rangle|, \qquad A \in B(\mathfrak{H}),$$

defines a semi-norm on $B(\mathfrak{H})$. The set $\{p_{\xi,\eta} : \xi, \eta \in \mathfrak{H}\}$ of all such semi-norms is separating; for if $p_{\xi,\eta}(A) = 0$ for every pair $\xi, \eta \in \mathfrak{H}$, then in particular choosing $\xi = A\eta$ it follows that $||A\eta|| = 0$ for every $\xi \in \mathfrak{H}$, and so $A = 0$. The locally convex topology generated by the set of semi-norms $\{p_{\xi,\eta} : \xi, \eta \in \mathfrak{H}\}$ is called the *weak operator topology* on $B(\mathfrak{H})$.

σ-weak operator topology: Let (ξ_n) and (η_n) be two sequences in $\mathfrak{H}$ such that

$$\sum_{n=1}^{\infty} ||\xi_n||^2 < \infty, \qquad \sum_{n=1}^{\infty} ||\eta_n||^2 < \infty.$$

Then

$$p_{\xi_n,\eta_n}(A) = |\sum_{n=1}^{\infty} \langle \xi_n, A\eta_n \rangle|, \qquad A \in B(\mathfrak{H}),$$

defines a semi-norm on $B(\mathfrak{H})$. An argument similar to the above shows that the set of all such semi-norms is separating, and thus generates a locally convex topology, the *σ-weak operator topology*, on $B(\mathfrak{H})$.

Strong operator topology: Let $\xi \in \mathfrak{H}$, then

$$p_{\xi}(A) = ||A\xi||, \qquad A \in B(\mathfrak{H}),$$

defines a semi-norm on $B(\mathfrak{H})$. Again, the set $\{p_\xi : \xi \in \mathfrak{H}\}$ of semi-norms is separating and generates a locally convex topology on $B(\mathfrak{H})$, called the *strong operator topology*.

σ-strong operator topology: Let (ξ_n) be a sequence in $\mathfrak{H}$ such that $\sum_{n=1}^{\infty} ||\xi_n||^2 < \infty$. Then

$$p_{\xi_n}(A) = (\sum_{n=1}^{\infty} ||A\xi_n||^2)^{1/2}, \qquad A \in B(\mathfrak{H}),$$

defines a semi-norm on $B(\mathfrak{H})$ and, the set of all such semi-norms is separating and generates a locally convex topology on $B(\mathfrak{H})$, called the *σ-strong operator topology*.

Remark 2.3.20. We have the following relations between these operator topologies:

$$\begin{array}{ccccc} norm & \supset & \sigma - strong & \supset & \sigma - weak \\ & & \cup & & \cup \\ & & strong & \supset & weak \end{array}$$

where "$\supset$" means finer than. [3, p.70] [25, p.68]

We prove two of these inclusions. The proofs of the others are similar.

Proposition 2.3.21. *The σ-strong operator topology is finer than the strong operator topology, which in turn is finer than the weak operator topology.*

Proof. Throughout this proof, keep in mind that we are working with locally convex topologies and, among other properties, we thus need only consider basic neighborhoods of the origin. This and similar properties of these topologies will not be mentioned explicitly when used.

We begin by showing that the strong operator topology is finer than the weak operator topology. Given any one of the semi-norms generating the weak operator topology, say $p_{\xi,\eta}$, where $\xi, \eta \in \mathfrak{H}$, we have

$$\begin{aligned} p_{\xi,\eta}(A) &= |\langle \xi, A\eta \rangle| \\ &\leq ||\xi||\, ||A\eta|| \\ &= ||\xi||\, p_\eta(A), \end{aligned}$$

for every $A \in B(\mathfrak{H})$, where $p_\eta(A)$ is one of the semi-norms generating the strong operator topology. Hence, by Proposition 2.3.5, each semi-norm in $\{p_{\xi,\eta} : \xi, \eta \in \mathfrak{H}\}$ is continuous in the strong operator topology. Now, consider any sub-basic neighborhood in the weak operator topology, say
$N_{0,\epsilon} = \{A \in B(\mathfrak{H}) : p_{\xi,\eta}(A) < \epsilon\}$ for some $\xi, \eta \in \mathfrak{H}$. Taking the open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$, we have by the continuity of $p_{\xi,\eta}$ that $p_{\xi,\eta}^{-1}((-\epsilon, \epsilon))$ is open in both the strong and weak operator topologies. But $p_{\xi,\eta}^{-1}((-\epsilon, \epsilon)) = N_{0,\epsilon}$, showing that every sub-basic neighborhood of the weak operator topology is open in the strong operator topology, and thus implying the required inclusion.

As a matter of interest, one could (equivalently) have argued that since: (i) each semi-norm generating the weak operator topology is continuous in the strong operator topology, and (ii) the weak operator topology is the coarsest topology in which each of these semi-norms is continuous, the required inclusion follows.

We show that the σ-strong operator topology is finer than the strong operator topology. For no reason other than some variation, we do this by showing that every net converging in the σ-strong operator topology also converges in the strong operator topology. Hence, without loss of generality, let (A_α) be a net that converges to 0 in the σ-strong operator topology. From Theorem 2.3.16,

$$\lim_\alpha p_{\xi_n}(A_\alpha) = 0,$$

for every sequence (ξ_n) in $\mathfrak{H}$ such that $\sum_{n=1}^\infty ||\xi_n||^2 < \infty$. Given any semi-norm from the generating set of the strong operator topology, say p_ξ, with

$\xi \in \mathfrak{H}$, by considering the sequence $(\xi_n) = (\xi, 0, 0, \ldots)$, we have

$$\begin{aligned} p_\xi(A_\alpha) &= ||A_\alpha \xi|| \\ &= (\sum_{n=1}^{\infty} ||A_\alpha \xi_n||^2)^{\frac{1}{2}} \\ &= p_{\xi_n}(A_\alpha) \longrightarrow 0. \end{aligned}$$

Hence $\lim_\alpha p_\xi(A_\alpha) = 0$ for every $\xi \in \mathfrak{H}$, and by Theorem 2.3.16, (A_α) converges to 0 in the strong operator topology, giving the required inclusion. $\square$

If the Hilbert space is infinite dimensional these inclusions become strict, [3, p.70]. Irrespective of whether $\mathfrak{H}$ is finite dimensional or not, it is the case that the closures of convex sets in some of these topologies coincide. Below we state, without proof, a separation form of the Hahn-Banach theorem, followed by two examples of this happening.

Proposition 2.3.22 (Hahn-Banach separation theorem)**.** *[22, Prop 5, p.29] Let $\mathcal{V}$ be a convex space, and suppose that A and B are disjoint convex sets in $\mathcal{V}$, with A open. Then there exists a continuous linear functional f on $\mathcal{V}$ such that $f(A)$ and $f(B)$ are disjoint.*

Corollary 2.3.23. *[22, Cor 1, p.30] If B is a convex subset of the convex space $\mathcal{V}$, and $a \notin \bar{B}$, then there exists a continuous linear functional f on $\mathcal{V}$ such that $f(a) \notin \overline{f(B)}$.*

As a first example we show that the norm and weak topologies on a normed space coincide on convex sets. The proof is based on the proof of [22, Prop 8, p.34], where a more general result valid for any dual pair is given. The normed space version will suffice for our needs though.

Theorem 2.3.24. *Let X be a normed space and X^* its (continuous) dual space. Then for any convex set $K \subset X$, the closure of K in the norm and $\sigma(X, X^*)$ topologies on X coincide.*

Proof. The norm topology is finer than the $\sigma(X, X^*)$-topology, so that $\bar{K} \subseteq \bar{K}^\sigma$, where $\bar{K}^\sigma$ denotes the closure in the $\sigma(X, X^*)$-topology and $\bar{K}$ the closure in the norm topology.

Now, let $x_0 \in X \backslash \bar{K}$. We show that this implies that $x_0 \notin \bar{K}^\sigma$. By Corollary 2.3.23 there exists a continuous linear functional f on X such that $f(x_0) \notin \overline{f(K)}$. Thus, there exists an $\epsilon > 0$ such that $|f(x) - f(x_0)| \geq \epsilon$ for every $x \in \overline{f(K)}$. Let $U = \{x \in X : |f(x_0) - f(x)| < \epsilon\}$, then U is a neighborhood of x_0 in the $\sigma(X, X^*)$-topology, such that $U \cap K = \emptyset$. Hence $x_0 \notin \bar{K}^\sigma$ and so $\bar{K}^\sigma \subseteq \bar{K}$. $\square$

Next we see that the closures of convex sets in some of the the locally convex topologies defined on $B(\mathfrak{H})$ coincide.

Theorem 2.3.25. *[3, Thm 2.4.7, p.71] Let $\mathcal{K} \subset B(\mathfrak{H})$ be any convex subset and let $\mathcal{B}_r$ denote the ball of radius $r > 0$ centered at 0. Then the following statements are equivalent:*

(i) $\mathcal{K}$ is closed in the σ-weak operator topology.

(ii) $\mathcal{K}$ is closed in the σ-strong operator topology.

(iii) $\mathcal{K} \cap \mathcal{B}_r$ is closed in the weak (and thus σ-weak) operator topology.

(iv) $\mathcal{K} \cap \mathcal{B}_r$ is closed in the strong (and thus σ-strong) operator topology.

Theorem 2.3.26. *[15, Thm 5.1.2, p.305] Let $\mathcal{K} \subset B(\mathfrak{H})$ be a convex set. Then the weak and strong operator closures of $\mathcal{K}$ coincide.*

2.4 von Neumann Algebras

Throughout this section $\mathfrak{H}$ will denote a Hilbert space.

Definition 2.4.1. Let $\mathcal{M}$ be a subset of $B(\mathfrak{H})$. The *commutant* of $\mathcal{M}$, denoted by $\mathcal{M}'$, is the set of all operators in $B(\mathfrak{H})$ that commute multiplicatively with every element in $\mathcal{M}$.

It is easily seen that $\mathcal{M}'$ is a subspace of $B(\mathfrak{H})$ and

$$\begin{array}{ccccccc} \mathcal{M} & \subseteq & \mathcal{M}'' & = & \mathcal{M}^{(iv)} & = & \ldots, \\ \mathcal{M}' & = & \mathcal{M}''' & = & \mathcal{M}^{(v)} & = & \ldots, \end{array}$$

where $\mathcal{M}''$ denotes $(\mathcal{M}')'$, and so on. As an example we show that $\mathcal{M}' = \mathcal{M}'''$. That $\mathcal{M}' \subseteq \mathcal{M}'''$ is clear. Let $T \in \mathcal{M}'''$, and let $S \in \mathcal{M}$. Then $S \in \mathcal{M}''$, and thus $ST = TS$. This holds true for every $S \in \mathcal{M}$, and hence $T \in \mathcal{M}'$.

Definition 2.4.2. Let $\mathcal{M}$ be a *-subalgebra of $B(\mathfrak{H})$. We call $\mathcal{M}$ a *von Neumann algebra* if $\mathcal{M} = \mathcal{M}''$.

Note that $B(\mathfrak{H})$ is an example of a von Neumann algebra.

Definition 2.4.3. Let $\mathcal{M} \subseteq B(\mathfrak{H})$ and let $\mathfrak{R} \subseteq \mathfrak{H}$. We denote by $[\mathcal{M}\mathfrak{R}]$ the closure of the linear span of the set $\{A\xi : A \in \mathcal{M}, \xi \in \mathfrak{R}\}$. A *-subalgebra $\mathcal{M} \subseteq B(\mathfrak{H})$ is called *nondegenerate* (or to *act nondegenerately on* $\mathfrak{H}$) if $[\mathcal{M}\mathfrak{H}] = \mathfrak{H}$.

Note that any unital *-subalgebra of $B(\mathfrak{H})$ acts nondegenrately on $\mathfrak{H}$.

Definition 2.4.4. Let $\mathcal{H} \subset \mathfrak{H}$ be a closed vector subspace. We call $\mathcal{H}$ *invariant* for $T \in B(\mathfrak{H})$ if $T\mathcal{H} \subseteq \mathcal{H}$. Similarly, we call $\mathcal{H}$ invariant for the set $\mathcal{T} \subset B(\mathfrak{H})$ if $\mathcal{H}$ is invariant for each $T \in \mathcal{T}$. We call $\mathcal{H}$ *reducing* for $T \in B(\mathfrak{H})$ if both $\mathcal{H}$ and $\mathcal{H}^\perp$ are invariant for T, where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$.

Remark 2.4.5. Let $P : \mathfrak{H} \to \mathcal{H}$ be the projection onto $\mathcal{H}$, let $\mathcal{H}$ be invariant for T, and let $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}^\perp$. Then $0 = \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, so that $\mathcal{H}^\perp$ is invariant for T^*. Also, if $\mathcal{H}$ is reducing for T, and we let $x = \xi + \eta$ be any element in $\mathfrak{H}$, where $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}^\perp$, then

$$PTx = P(T\xi + T\eta) = T\xi = T(P\xi + P\eta) = TPx.$$

Conversely, assume $PT = TP$ and let $\xi \in \mathcal{H}$. Then

$$PT\xi = TP\xi = T\xi,$$

which implies that $T\xi \in \mathcal{H}$. Also, for $\eta \in \mathcal{H}^\perp$ we have

$$PT\eta = TP\eta = 0,$$

so that $T\eta \in \mathcal{H}^\perp$. Hence $\mathcal{H}$ is reducing for T if and only if $PT = TP$.

Below we present the double commutant theorem of von Neumann, which gives us another characterization of von Neumann algebras in terms of closures in the locally convex topologies discussed on $B(\mathfrak{H})$. Although the theorem is stated in the more general form from [3], the proof is based on the one from [15, Thm 5.3.1, p.326].

Theorem 2.4.6. *[3, Thm 2.4.11 & Cor 2.4.15, pp.72-74] Let $\mathcal{M}$ be a unital *-subalgebra of $B(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. Then the strong, σ-strong, weak and σ-weak operator closures of $\mathcal{M}$ in $B(\mathfrak{H})$ coincide with $\mathcal{M}''$.*

We present a proof only showing that the the weak and strong operator closures (which by Theorem 2.3.26 coincide) of $\mathcal{M}$ equal $\mathcal{M}''$. Using Theorem 2.3.25 the other cases are shown similarly, and a proof can be found in [3, Thm 2.4.11, pp.72-74].

Proof. Firstly, note that $\mathcal{M}'$ is closed in the weak operator topology. Let T be in the weak operator closure of $\mathcal{M}'$, then there is a net (T_α) in $\mathcal{M}'$ such that $T_\alpha \to T$ in the weak operator topology. Given any $S \in \mathcal{M}$ and $\xi, \eta \in \mathfrak{H}$, then

$$\begin{aligned} \langle \xi, ST\eta\rangle &= \langle S^*\xi, T\eta\rangle \\ &= \lim_\alpha \langle S^*\xi, T_\alpha\eta\rangle \\ &= \lim_\alpha \langle \xi, T_\alpha(S\eta)\rangle \\ &= \langle \xi, TS\eta\rangle \end{aligned}$$

Applying this result to $(\mathcal{M}')'$, it follows that $\mathcal{M}''$ is weakly operator closed. Since $\mathcal{M} \subseteq \mathcal{M}''$, it follows that the weak (and thus the strong) operator closure of $\mathcal{M}$ is contained in $\mathcal{M}''$.

We show that $\mathcal{M}$ is strongly dense in $\mathcal{M}''$. Let $T \in \mathcal{M}''$ and $\xi_1, \ldots, \xi_n \in \mathfrak{H}$ be given. Let $\tilde{\mathfrak{H}} = \bigoplus_{i=1}^n \mathfrak{H}$, be the direct sum of $\mathfrak{H}$ with itself n times, and for $S \in B(\mathfrak{H})$, let $\tilde{S} = \bigoplus_{i=1}^n S$, i.e. given an $\tilde{\eta} = (\eta_1, \ldots, \eta_n) \in \tilde{\mathfrak{H}}$, we have $\tilde{S}\tilde{\eta} = (S\eta_1, \ldots, S\eta_n)$. Also, let $\tilde{\mathcal{M}} = \{\tilde{S} : S \in \mathcal{M}\}$. Then $\tilde{\mathcal{M}}$ is a *-subalgebra of $B(\tilde{\mathfrak{H}})$; for if $\tilde{\eta} \in \tilde{\mathfrak{H}}$ and $\tilde{R}, \tilde{S} \in \tilde{\mathcal{M}}$ are given, then

$$\begin{aligned} ||\tilde{S}\tilde{\eta}|| &= \left(||S\eta_1||^2 + \cdots + ||S\eta_n||^2\right)^{1/2} \\ &\leq ||S|| \left(||\eta_1||^2 + \cdots + ||\eta_n||^2\right)^{1/2} \\ &\leq ||S||||\tilde{\eta}|| \end{aligned}$$

$$(\lambda_1\tilde{R} + \lambda_2\tilde{S})\tilde{\eta} = \bigoplus_{i=1}^n (\lambda_1 R + \lambda_2 S)\eta_i \text{ , with } \lambda_1, \lambda_2 \in \mathbb{C}$$

$$(\tilde{R}\tilde{S})\tilde{\eta} = \bigoplus_{i=1}^n (RS)\eta_i$$

and

$$\tilde{R}^* = \bigoplus_{i=1}^n R^*$$

Hence, $\tilde{\mathcal{M}}$ is a *-subalgebra of $B(\tilde{\mathfrak{H}})$, and moreover $[\tilde{\mathcal{M}}\tilde{\xi}]$, with $\tilde{\xi} = (\xi_1, \ldots, \xi_n)$, is invariant for $\tilde{\mathcal{M}}$. If $P : \tilde{\mathfrak{H}} \to [\tilde{\mathcal{M}}\tilde{\xi}]$ is the orthogonal projection onto $[\tilde{\mathcal{M}}\tilde{\xi}]$, it follows from Remark 2.4.5 that P commutes with $\tilde{\mathcal{M}}$, that is $P \in \tilde{\mathcal{M}}'$.

We claim that $\tilde{T} \in \tilde{\mathcal{M}}''$, as we shall see this will imply the result. If $M_n(B(\mathfrak{H}))$ denotes all the $n \times n$ matrices with operators from $B(\mathfrak{H})$ as entries, then $M_n(B(\mathfrak{H}))$ is *-isomorphic to $B(\bigoplus_{i=1}^n \mathfrak{H}_i)$ (see for example [19, pp. 94-95] and [15, pp. 147-149]). Every $\tilde{S} \in \tilde{\mathcal{M}}$ can be represented as a matrix having S on the diagonal and zero everywhere else. Easy (but tedious) matrix calculations show that $\tilde{\mathcal{M}}'$ consists of those matrices that have entries in $\mathcal{M}'$, and $\tilde{\mathcal{M}}''$ consists of those matrices having one operator from $\mathcal{M}''$ on the diagonal and zeros elsewhere. Thus, $\tilde{T} \in \tilde{\mathcal{M}}''$. From the preceding paragraph it follows that $\tilde{T}$ commutes with P, and thus $[\tilde{\mathcal{M}}\tilde{\xi}]$ is invariant for $\tilde{T}$. By assumption $\mathcal{M}$ is unital, and thus $\tilde{\mathcal{M}}$ is unital which implies that $\tilde{\xi} \in [\tilde{\mathcal{M}}\tilde{\xi}]$. Hence $\tilde{T}\tilde{\xi} \in [\tilde{\mathcal{M}}\tilde{\xi}]$. Now since $\tilde{M}\tilde{\xi}$ is dense in $[\tilde{\mathcal{M}}\tilde{\xi}]$, it follows that, for every $\epsilon > 0$, there is a $\tilde{S} \in \tilde{\mathcal{M}}$ such that $||(\tilde{S} - \tilde{T})\tilde{\xi}|| < \epsilon$, that is

$$||(\tilde{S} - \tilde{T})\tilde{\xi}|| = \left(\sum_{i=1}^n ||(S - T)\xi_i||^2\right)^{1/2} < \epsilon$$

So, $||(S - T)\xi_i|| < \epsilon$ for each $i = 1, \ldots, n$, showing that T is in the strong closure of $\mathcal{M}$. $\square$

It follows that a *-subalgebra of $B(\mathfrak{H})$ is a von Neumann algebra if and only if it is weak (strong) operator closed. There are two consequences of the double commutant theorem worth highlighting. Firstly, if $\mathcal{A} \subset B(\mathfrak{H})$ is any self-adjoint set, then $\mathcal{A}''$ is the smallest von Neumann algebra containing $\mathcal{A}$, and is called the von Neumann algebra *generated* by $\mathcal{A}$. Secondly, any von Neumann algebra is also a C*-algebra. The question of when is a C*-algebra also a von Neumann algebra, gives rise to a more abstract notion of a von Neumann algebra independent of $B(\mathfrak{H})$, called a W*-algebra, but this will not be needed in this dissertation.

We now give some important results on continuous linear functionals on von Neumann algebras. The first is based on [3, Prop 2.4.6, p. 70].

Definition 2.4.7. If $\mathcal{M}$ is a von Neumann algebra, we denote by $\mathcal{M}_*$ the set of all σ-weakly continuous linear functionals on $\mathcal{M}$, and call it the *predual* of $\mathcal{M}$.

Proposition 2.4.8. *The σ-strong and σ-weak operator topologies on $B(\mathfrak{H})$ admit the same continuous linear functionals, all of the form* $\omega(T) = \sum_{n=1}^{\infty}\langle \eta_n, T\xi_n\rangle$, *where* $T \in B(\mathfrak{H})$ *and* $(\xi_n), (\eta_n) \in \mathfrak{H}$ *such that* $\sum_{n=1}^{\infty} ||\xi_n||^2 < \infty$ *and* $\sum_{n=1}^{\infty} ||\eta_n||^2 < \infty$.

Proof. Since the σ-strong operator topology is finer than the σ-weak operator topology, every σ-weakly continuous linear functional is also σ-strongly continuous. Thus, suppose that ω is a σ-strongly continuous linear functional on $B(\mathfrak{H})$. By Proposition 2.3.5 there are sequences $(\xi_{1,n}), (\xi_{2,n}), \ldots, (\xi_{k,n}), n \in \mathbb{N}$, in $\mathfrak{H}$ such that $\sum_{n=1}^{\infty} ||\xi_{m,n}||^2 < \infty$, for $m = 1, 2, \ldots, k$, and

$$|\omega(T)| \leq c \max_{1\leq m\leq k}\left(\sum_{n=1}^{\infty} ||T\xi_{m,n}||^2\right)^{1/2} \tag{2.3}$$

for every $T \in B(\mathfrak{H})$ and some constant c. Let $[k] := \{1, 2, \ldots, k\}$ and

$$\tilde{\mathfrak{H}} = \bigoplus_{(m,n)\in[k]\times\mathbb{N}} \mathfrak{H}_{m,n},$$

with $\mathfrak{H}_{m,n} = \mathfrak{H}$ for each $(m, n) \in [k] \times \mathbb{N}$, and for $T \in B(\mathfrak{H})$ let $\tilde{T} = \bigoplus_{(m,n)\in[k]\times\mathbb{N}} T$ (note that $\tilde{\xi} = \bigoplus_{(m,n)\in[k]\times\mathbb{N}} \xi_{m,n}$ is an element of $\tilde{\mathfrak{H}}$). Also, let $\mathcal{K} = \{\tilde{T}\tilde{\xi} : T \in B(\mathfrak{H})\}$. The the map $\phi : \mathcal{K} \to \mathbb{C}$, given by $\phi(\tilde{T}\tilde{\xi}) = \omega(T)$, defines a bounded linear functional on $\mathcal{K}$. To see that ϕ is well defined, let $\tilde{T}_1\tilde{\xi} = \tilde{T}_2\tilde{\xi}$, then

$$|\omega(T_1 - T_2)| \leq c \max_{1\leq m\leq k}\left(\sum_{n=1}^{\infty} ||(T_1 - T_2)\xi_{m,n}||^2\right)^{1/2} = 0,$$

implying $\omega(T_1) = \omega(T_2)$. The linearity is clear and the boundedness follows from inequality (2.3). Notice that $\mathcal{K}$ is a subspace of $\tilde{\mathfrak{H}}$, and by the Hahn-Banach theorem ϕ can be extended to the whole of $\tilde{\mathfrak{H}}$. Now, by the Riesz representation theorem there exists an $\tilde{\eta} \in \tilde{\mathfrak{H}}$ such that

$$\begin{aligned} \omega(T) &= \phi(\tilde{T}\tilde{\xi}) \\ &= \langle \tilde{\eta}, \tilde{T}\tilde{\xi} \rangle \\ &= \sum_{m=1}^{k} \sum_{n=1}^{\infty} \langle \eta_{m,n}, T\xi_{m,n} \rangle \\ &= \sum_{n=1}^{\infty} \langle \eta_n, T\xi_n \rangle, \end{aligned}$$

where $(\xi_n) = (\xi_{1,1}, \ldots, \xi_{k,1}, \xi_{1,2}, \ldots, \xi_{k,2}, \ldots)$ and similarly for η. Again, from Proposition 2.3.5 it now follows that ω is σ-weakly continuous. □

Proposition 2.4.9. *The strong and weak operator topologies on $B(\mathfrak{H})$ admit the same continuous linear functionals, all of the form* $\omega(T) = \sum_{n=1}^{k} \langle \eta_n, T\xi_n \rangle$, *where $T \in B(\mathfrak{H})$ and $\xi_n, \eta_n \in \mathfrak{H}$ for $n = 1, \ldots, k$.*

Proof. The proof is similar to that of Proposition 2.4.8, with the exception that we need only use finite direct sums.

Since the strong operator topology is finer than the weak operator topology, every weakly continuous linear functional is also strongly continuous. Thus, suppose that ω is a strongly continuous linear functional on $B(\mathfrak{H})$. By Proposition 2.3.5 there are $\xi_n \in \mathfrak{H}, n = 1, 2, \ldots, k$, such that

$$|\omega(T)| \leq c \max_{1 \leq n \leq k} ||T\xi_n|| \tag{2.4}$$

for every $T \in B(\mathfrak{H})$ and some constant c. Let $\tilde{\mathfrak{H}} = \bigoplus_{n=1}^{k} \mathfrak{H}_n$, with $\mathfrak{H}_n = \mathfrak{H}$ for each $n = 1, 2, \ldots, k$, and for $T \in B(\mathfrak{H})$ let $\tilde{T} = \bigoplus_{n=1}^{k} T$ (note that $\tilde{\xi} = \bigoplus_{n=1}^{k} \xi_n$ is an element of $\tilde{\mathfrak{H}}$). Also, let $\mathcal{K} = \{\tilde{T}\tilde{\xi} : T \in B(\mathfrak{H})\}$. We define a bounded linear functional $\phi : \mathcal{K} \to \mathbb{C}$, by $\phi(\tilde{T}\tilde{\xi}) = \omega(T)$, which is well defined; for if $\tilde{T}_1\tilde{\xi} = \tilde{T}_2\tilde{\xi}$, then

$$|\omega(T_1 - T_2)| \leq c \max_{1 \leq n \leq k} ||(T_1 - T_2)\xi_n|| = 0,$$

implying $\omega(T_1) = \omega(T_2)$. The linearity is clear and the boundedness follows from inequality (2.4). Since $\mathcal{K}$ is a subspace of $\tilde{\mathfrak{H}}$, ϕ can be extended to the whole of $\tilde{\mathfrak{H}}$ (by the Hahn-Banach theorem). By the Riesz representation theorem there exists an $\tilde{\eta} \in \tilde{\mathfrak{H}}$ such that

$$\begin{aligned} \omega(T) &= \phi(\tilde{T}\tilde{\xi}) \\ &= \langle \tilde{\eta}, \tilde{T}\tilde{\xi} \rangle \\ &= \sum_{n=1}^{k} \langle \eta_n, T\xi_n \rangle. \end{aligned}$$

Again, from Proposition 2.3.5 it now follows that ω is weakly continuous. $\square$

Remark 2.4.10. Although Proposition 2.4.8 was stated for $B(\mathfrak{H})$, it holds for any von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathfrak{H})$. This follows from the fact that any continuous (with respect to any of the topologies mentioned) linear functional on $B(\mathfrak{H})$ is also a continuous linear functional on $\mathcal{M}$ by restriction, and conversely any continuous linear functional on $\mathcal{M}$ can be extended to a continuous linear functional on $B(\mathfrak{H})$ by the Hahn-Banach theorem.

Remark 2.4.11. Given a von Neumann algebra $\mathcal{M}$, then by Proposition 2.4.8 every $\omega \in \mathcal{M}_*$ is of the form $\omega(T) = \sum_{n=1}^{\infty} \langle \eta_n, T\xi_n \rangle$, with $\xi_n, \eta_n \in \mathfrak{H}$, and $\sum_{n=1}^{\infty} ||\xi_n||^2 < \infty, \sum_{n=1}^{\infty} ||\eta_n||^2 < \infty$. But then

$$|\omega(T)| = |\sum_{n=1}^{\infty} \langle \eta_n, T\xi_n \rangle| = p_{(\eta_n),(\xi_n)}(T)$$

where $p_{(\eta_n),(\xi_n)}$ is one of the semi-norms that generates the σ-weak operator topology. Hence, by Theorem 2.3.16 a net (T_α) converges to T in $\mathcal{M}$ in the σ-weak operator topology if and only if $\omega(T_\alpha) \to \omega(T)$ for every $\omega \in \mathcal{M}_*$.

Proposition 2.4.12. *[3, Prop 2.4.18, p.75] The predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is a Banach space in the norm of $\mathcal{M}^*$, and $\mathcal{M} = (\mathcal{M}_*)^*$, via $T(\omega) = \omega(T)$ for $T \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$.*

Remark 2.4.13. In [3, Prop 2.4.3, p.68] Proposition 2.4.12 is proved for $B(\mathfrak{H})$ and it is shown that the $\sigma(B(\mathfrak{H}), B_*(\mathfrak{H}))$ and the σ-weak operator topologies are same. Proposition 2.4.12 thus justifies the terminology "predual" introduced in Definition 2.4.7.

Proposition 2.4.14. *[25, Thm 4.8, p.82] If $\mathcal{A}$ is a *-algebra of operators acting on a Hilbert space $\mathfrak{H}$, then the closed unit ball of $\mathcal{A}$ is strongly (operator) dense in the closed unit ball of the weak closure of $\mathcal{A}$.*

Remark 2.4.15. Note that by Theorem 2.3.25, the unit ball of $\mathcal{A}$, as above, is also σ-strongly, σ-weakly and operator weakly dense in the unit ball of the weak closure of $\mathcal{A}$.

We end this section with a result on the projections in a von Neumann algebra, which will prove useful later.

Theorem 2.4.16. *[19, Thm 4.1.11(1), p.119] If $\mathcal{M}$ is a von Neumann algebra, then it is the closed linear span of its projections.*

2.5 Conditional Expectations

The exposition in this section is based on [25] (although the main result is due to Tomiyama in [26]), but all the proofs are given in considerably more detail.

A key feature of this section is the relationship between the universal enveloping von Neumann algebra associated with a C*-algebra and the second dual of this C*-algebra. We will show the existence of a universal representation of a C*-algebra, and thus a universal enveloping von Neumann algebra, and also that there is an isometric *-isomorphism between the second dual of the C*-algebra and the universal enveloping von Neumann algebra, which is continuous with respect to some of the locally convex topologies on these spaces. In other words, the second dual of a C*-algebra can be identified with a von Neumann algebra through its enveloping von Neumann algebra. We end this section with the main result of this chapter, which is crucial for our goal of characterizing relative unique ergodicity, and states that norm one projections on C*-algebras are conditional expectations.

Lemma 2.5.1. *[25, Lem 2.2, p.121] Let $\mathcal{A}$ be a C*-algebra and $(\mathfrak{H}, \pi)$ a representation of $\mathcal{A}$. Let $\mathcal{M}_\pi$ denote the von Neumann algebra $\pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$. Then there is a unique linear map $\tilde{\pi}$ of the second dual space $\mathcal{A}^{**}$ of $\mathcal{A}$ onto $\mathcal{M}_\pi$ with the following properties:*

(i) The diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\pi"] \arrow[d, "i"'] & \mathcal{M}_\pi \\ \mathcal{A}^{**} \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

commutes, where i indicates the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$.

(ii) $\tilde{\pi}$ is continuous with respect to the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and the σ-weak operator topology of $\mathcal{M}_\pi$.

(iii) $\tilde{\pi}$ maps the closed unit ball $\mathcal{B}_{\mathcal{A}^{**}}$ of $\mathcal{A}^{**}$ onto the closed unit ball $\mathcal{B}_{\mathcal{M}}$ of $\mathcal{M}_\pi$.

Proof. To try and keep notation simple we shall throughout the proof just write $\mathcal{M}$ instead of $\mathcal{M}_\pi$.

Let $\mathcal{M}_*$ be the predual of $\mathcal{M}$, that is, the Banach space of all σ-weakly continuous linear functionals on $\mathcal{M}$. Let

$$\pi^t : \mathcal{M}^* \to \mathcal{A}^*$$

be the transpose of π given by $\pi^t(\omega) = \omega \circ \pi,\ \ \omega \in \mathcal{M}^*$. Let $\pi_* = \pi^t|_{\mathcal{M}_*}$ be the restriction of the transpose of π to $\mathcal{M}_*$.

Since the dual space of $\mathcal{M}_*$ is $\mathcal{M}$ (see Proposition 2.4.12) it follows that the transpose map of π_*, denoted by $\tilde{\pi}$, is a mapping

$$\tilde{\pi} : \mathcal{A}^{**} \to (\mathcal{M}_*)^* = \mathcal{M},$$

given by

$$\tilde{\pi}(\phi) = \phi \circ \pi_*,\ \ \phi \in \mathcal{A}^{**}.$$

Taking into consideration the canonical map $i : \mathcal{A} \to \mathcal{A}^{**}$ given by $i(a)(\varphi) = \varphi(a)$ for every $\varphi \in \mathcal{A}^*$, we obtain for every $a \in \mathcal{A}$ and $\omega \in \mathcal{M}_*$ that

$$\pi(a)(\omega) = \omega \circ \pi(a) = \pi_*(\omega)(a) = i(a) \circ \pi_*(\omega) = \tilde{\pi}(i(a))(\omega),$$

(where the first equality follows from Proposition 2.4.12) so that $\tilde{\pi} \circ i(a) = \pi(a)$ for every $a \in \mathcal{A}$, and hence $\tilde{\pi} \circ i = \pi$ proving (i).

We prove (ii). Suppose $(\phi_n)_{n\in\mathbb{I}}$ is a net converging to ϕ, relative to the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology, in $\mathcal{A}^{**}$. Then, given any $\epsilon > 0$ and $\omega \in \mathcal{M}_*$, there exists an $N \in \mathbb{I}$ such that for every $n > N$

$$\begin{aligned} |\omega(\tilde{\pi}(\phi_n - \phi))| &= |\tilde{\pi}(\phi_n - \phi)(\omega)| \quad \text{(by Proposition 2.4.12)} \\ &= |(\phi_n - \phi) \circ \pi_*(\omega)| \\ &< \epsilon. \end{aligned}$$

From Remark 2.4.11 it follows that $\tilde{\pi}(\phi_n) \to \tilde{\pi}(\phi)$ in the σ-weak operator topology, giving the desired result.

We prove (iii). By Proposition 2.2.14, π maps the open unit ball $\mathcal{B}_{\mathcal{A},0}$ of $\mathcal{A}$ into the open unit ball $\mathcal{B}_{\mathcal{M},0}$ of $\mathcal{M}$. By Proposition 2.2.16, $\pi = \pi_0 \circ i_0$, where π_0 is the *-isomorphism of $\mathcal{A}/ker(\pi)$ onto $\pi(\mathcal{A})$ (and thus an isometry by 2.2.15) and i_0 is the canonical mapping of $\mathcal{A}$ onto $\mathcal{A}/ker(\pi)$. This implies that π maps $\mathcal{B}_{\mathcal{A},0}$ onto $\mathcal{B}_{\mathcal{M},0} \bigcap \pi(\mathcal{A})$. Indeed, let $B \in \mathcal{B}_{\mathcal{M},0} \bigcap \pi(\mathcal{A})$, then $B = \pi(a)$ for some $a \in \mathcal{A}$ and

$$\begin{aligned} 1 &> ||B|| \\ &= ||\pi(a)|| \\ &= ||\pi_0 \circ i_0(a)|| \\ &= ||i_0(a)|| \\ &= \inf_{b \in ker(\pi)} ||a + b|| \\ &= \inf_{b \in a + ker(\pi)} ||b||. \end{aligned}$$

Hence there must be an element $b \in a + ker(\pi)$ such that $||b|| < 1$ and $\pi(b) = \pi(a) = B$, and hence $\mathcal{B}_{\mathcal{M},0} \bigcap \pi(\mathcal{A}) \subseteq \pi(\mathcal{B}_{\mathcal{A},0})$.

By the Banach-Alaoglu theorem the closed unit ball $\mathcal{B}_{\mathcal{A}^{**}}$ of $\mathcal{A}^{**}$ is compact in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology, and by the continuity of $\tilde{\pi}$ proved in (ii), it follows that $\tilde{\pi}(\mathcal{B}_{\mathcal{A}^{**}})$ is σ-weakly compact (and thus closed) in $\mathcal{M}$. Also, from

$$||i(a)|| = \sup_{\substack{\varphi \in \mathcal{A}^* \\ ||\varphi||=1}} |\varphi(a)| = ||a||,$$

i is isometric, where the second equality is a consequence of the Hahn-Banach theorem (see [18, Cor 4.3-4, p.223]), and so $i(\mathcal{B}_{\mathcal{A}}) \subseteq \mathcal{B}_{\mathcal{A}^{**}}$. Then we have that $\pi(\mathcal{B}_{\mathcal{A}}) = \tilde{\pi}(i(\mathcal{B}_{\mathcal{A}})) \subseteq \tilde{\pi}(\mathcal{B}_{\mathcal{A}^{**}})$. From Theorem 2.4.6 $\pi(\mathcal{A})$ is weakly operator dense in $\mathcal{M}$. Also, the σ-weak closure of the open unit ball is equal to the σ-weak closure of the closed unit ball, and by Proposition 2.4.14 and Remark 2.4.15 the open unit ball, $\mathcal{B}_{\mathcal{M},0} \bigcap \pi(\mathcal{A}) = \pi(\mathcal{B}_{\mathcal{A},0})$, is σ-weakly dense in $\mathcal{B}_{\mathcal{M}}$. Hence, since $\tilde{\pi}(\mathcal{B}_{\mathcal{A}^{**}})$ is σ-weakly closed, we have

$$\mathcal{B}_{\mathcal{M}} = \overline{\pi(\mathcal{B}_{\mathcal{A},0})}^{\sigma w} \subseteq \overline{\pi(\mathcal{B}_{\mathcal{A}})}^{\sigma w} \subseteq \tilde{\pi}(\mathcal{B}_{\mathcal{A}^{**}}).$$

To see the reverse inclusion, note that for $\phi \in \mathcal{A}^{**}$ and $\omega \in \mathcal{M}_*$

$$\begin{aligned} |\tilde{\pi}(\phi)(\omega)| &= |\phi(\pi_*(\omega))| \\ &= |\phi(\omega \circ \pi)| \\ &\leq ||\phi|| ||\omega|| ||\pi|| \\ &\leq ||\phi|| ||\omega|| \quad \text{(by Theorem 2.2.14).} \end{aligned}$$

Taking the supremum over all $\omega \in \mathcal{M}_*$ with $||\omega|| = 1$, we see that
$||\tilde{\pi}(\phi)|| \leq ||\phi||$, implying $||\tilde{\pi}|| \leq 1$ which give the required inclusion. Hence $\tilde{\pi}(\mathcal{B}_{\mathcal{A}^{**}}) = \mathcal{B}_{\mathcal{M}}$.

To see that $\tilde{\pi}$ is onto, we need only consider a $T \in \mathcal{M}$ with $||T|| > 1$. Then $\frac{1}{||T||}T \in \mathcal{B}_{\mathcal{M}}$. By (iii) there is a $\phi \in \mathcal{B}_{\mathcal{A}^{**}}$ such that $\tilde{\pi}(\phi) = \frac{1}{||T||}T$. Then $\tilde{\pi}(||T||\phi) = T$.

Lastly, we show the uniqueness of $\tilde{\pi}$. It follows from Theorem 2.3.17 that $\overline{i(\mathcal{A})}^{w*}$ (i.e. the weak* closure) contains the unit ball of $\mathcal{A}^{**}$ and thus the whole of $\mathcal{A}^{**}$. Hence $i(\mathcal{A})$ is weak* dense in $\mathcal{A}^{**}$. Now, let $\kappa : \mathcal{A}^{**} \to \mathcal{M}$ be a linear mapping satisfying conditions (i) and (ii). Also let $\phi \in \mathcal{A}^*$ and $(i(a_n))_{n \in \mathbb{I}}$ be a net in $i(\mathcal{A})$ converging to ϕ in the weak*-topology (see Theorem 2.3.10). Then

$$\kappa(i(a_n)) = \pi(a_n) = \tilde{\pi}(i(a_n)),$$

and

$$\kappa(\phi) = \lim_n \kappa(i(a_n) = \lim_n \tilde{\pi}(i(a_n) = \tilde{\pi}(\phi),$$

where the above two limits are taken in the σ-weak topology. $\square$

Definition 2.5.2. [25, Def 2.3, p.122] Let $\mathcal{A}$ be a C*-algebra. For every representation $(\mathfrak{H}, \pi)$ of $\mathcal{A}$, let $\mathcal{M}_\pi$ denote the von Neumann algebra $(\pi(\mathcal{A}))''$. A representation $(\mathfrak{H}, \pi)$ is called *universal* if for any other representation $(\mathfrak{K}, \rho)$ of $\mathcal{A}$, there exists a σ-weakly continuous *-homomorphism $\tilde{\rho}$ of $\mathcal{M}_\pi$ onto $\mathcal{M}_\rho$ such that $\rho(a) = \tilde{\rho} \circ \pi(a), a \in \mathcal{A}$. If $(\mathfrak{H}, \pi)$ is a universal representation, then $\mathcal{M}_\pi$ is called the *universal enveloping von Neumann algebra of* $\mathcal{A}$.

Remark 2.5.3. [25, p.122] The universal enveloping von Neumann algebra is uniquely determined up to isomorphism. For if $(\mathfrak{H}_1, \pi_1)$ and $(\mathfrak{H}_2, \pi_2)$ are two universal representations of a C*-algebra $\mathcal{A}$, then there exist two σ-weakly continuous *-homomorphisms

$$\begin{aligned} \rho_1 : \mathcal{M}_{\pi_1} \to \mathcal{M}_{\pi_2} \quad &\text{such that} \quad \pi_2 = \rho_1 \circ \pi_1, \ \ a \in \mathcal{A} \\ \rho_2 : \mathcal{M}_{\pi_2} \to \mathcal{M}_{\pi_1} \quad &\text{such that} \quad \pi_1 = \rho_2 \circ \pi_2, \ \ a \in \mathcal{A} \end{aligned}$$

Hence, $\rho_2 \circ \rho_1$ and $\rho_1 \circ \rho_2$ are the identity maps on $\mathcal{M}_{\pi_1}$ and $\mathcal{M}_{\pi_2}$, respectively, since $\pi_i(\mathcal{A})$ is σ-weakly dense in $\mathcal{M}_{\pi_i}$, $i = 1, 2$. Therefore, ρ_1 and ρ_2 are both isomorphisms and inverses of each other.

The following results are based on [25, Theorem 2.4, p.122], and firstly shows that the second dual of a C*-algebra can be identified with a von Neumann algebra, through the enveloping von Neumann algebra, and secondly the existence of a universal enveloping von Neumann algebra.

Theorem 2.5.4. *[25, Thm 2.4, p.122] Let $\mathcal{A}$ be a C*-algebra. Then there is a unique isometry $\tilde{\pi}$, from the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ onto $\mathcal{M}_\pi$, for some representation $(\mathfrak{H}, \pi)$, such that $\mathcal{A}^{**}$ can be viewed as a C*-algebra isomorphic to $\mathcal{M}_\pi$. Moreover, $\tilde{\pi}$ is a homeomorphism with respect to the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and the σ-weak operator topology on $\mathcal{M}_\pi$.*

Proof. For every $\omega \in \mathcal{A}^*_+$, let $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic representation of $\mathcal{A}$ induced by ω. Let $(\mathfrak{H}, \pi)$ be the representation of $\mathcal{A}$ as in Theorem 2.2.19, that is let $\mathfrak{H} = \bigoplus_{\omega \in \mathcal{A}^*_+} \mathfrak{H}_\omega$, and for every $\xi = \bigoplus_{\omega \in \mathcal{A}^*_+} \xi_\omega \in \mathfrak{H}$ and $a \in \mathcal{A}$ let $\pi(a)\xi = \bigoplus_{\omega \in \mathcal{A}^*_+} \pi_\omega(a)\xi_\omega$. Then by Lemma 2.5.1 there exists a unique linear map $\tilde{\pi}$ of $\mathcal{A}^{**}$ onto $\mathcal{M}_\pi$. We will again, for simplicity, just write $\mathcal{M}$ instead of $\mathcal{M}_\pi$. We show that for every $\omega \in \mathcal{A}^*$, there are $\xi, \eta \in \mathfrak{H}$ such that $\omega(a) = \langle \eta, \pi(a)\xi \rangle$. So, to this end let $\omega \in \mathcal{A}^*$, then following Remark 2.2.12 we can write $\omega = \omega_1 - \omega_2 + i(\omega_3 - \omega_4)$ where $\omega_i \in \mathcal{A}^*_+$ and ω_i induces a cyclic representation $(\mathfrak{H}_{\omega_i}, \pi_{\omega_i}, \Omega_{\omega_i})$ of $\mathcal{A}$, for $i = 1, \ldots, 4$. Then for every $a \in \mathcal{A}$

$$\begin{aligned}
&\omega(a) \\
&= \omega_1(a) - \omega_2(a) + i(\omega_3(a) - \omega_4(a)) \\
&= \langle \Omega_{\omega_1}, \pi_{\omega_1}(a)\Omega_{\omega_1}\rangle - \langle \Omega_{\omega_2}, \pi_{\omega_2}(a)\Omega_{\omega_2}\rangle \\
&\quad + i\langle \Omega_{\omega_3}, \pi_{\omega_3}(a)\Omega_{\omega_3}\rangle - i\langle \Omega_{\omega_4}, \pi_{\omega_4}(a)\Omega_{\omega_4}\rangle \\
&= \langle \Omega_{\omega_1} \oplus \Omega_{\omega_2} \oplus \Omega_{\omega_3} \oplus \Omega_{\omega_4}, \\
&\quad [\pi_{\omega_1}(a) \oplus \pi_{\omega_2}(a) \oplus \pi_{\omega_3}(a) \oplus \pi_{\omega_4}(a)][\Omega_{\omega_1} \oplus (-\Omega_{\omega_2}) \oplus (i\Omega_{\omega_3}) \oplus (-i\Omega_{\omega_4})]\rangle \\
&= \langle \eta, \pi(a)\xi\rangle,
\end{aligned}$$

where $\eta = \bigoplus_{\omega \in \mathcal{A}^*_+} \eta_\omega \in \mathfrak{H}$ if we let $\eta_{\omega_i} = \Omega_{\omega_i}, i = 1, \ldots, 4$ and otherwise $\eta_\omega = 0$ for $\omega \neq \omega_i$, $i = 1, \ldots, 4$, and similarly $\xi = \bigoplus_{\omega \in \mathcal{A}^*_+} \xi_\omega \in \mathfrak{H}$ by letting $\xi_{\omega_1} = \Omega_{\omega_1}, \xi_{\omega_2} = -\Omega_{\omega_2}, \xi_{\omega_3} = i\Omega_{\omega_3}, \xi_{\omega_4} = -i\Omega_{\omega_4}$ and $\xi_\omega = 0$ for all other $\omega \in \mathcal{A}^*$. We show that $\pi_*(\mathcal{M}_*) = \mathcal{A}^*$, where π_* is transpose map as defined in Lemma 2.5.1. Indeed, given any $\omega \in \mathcal{A}^*$, say $\omega = \omega_{(\pi;\xi,\eta)}$, and for every $a \in \mathcal{A}$

$$\omega_{(\xi,\eta)}(\pi(a)) = \omega_{(\pi;\xi,\eta)}(a)$$

defines a weakly operator continuous linear functional on $\pi(\mathcal{A})$, which can be extended by continuity to $\mathcal{M}$. By Proposition 2.4.9 $\omega_{(\xi,\eta)}$ is weakly (and thus σ-weakly) continuous, and so we have $\omega_{(\xi,\eta)} \in \mathcal{M}_*$. We show that this in turn implies that $\tilde{\pi}$ is injective. If $\phi \in \mathcal{A}^{**}$ and $\tilde{\pi}(\phi) = 0$, then for every $\omega \in \mathcal{M}_*$ we have

$$0 = \tilde{\pi}(\phi)(\omega) = \phi \circ \pi_*(\omega),$$

and because $\pi_*(\mathcal{M}_*) = \mathcal{A}^*$ we have that $\phi = 0$.

We can now show that $\tilde{\pi}$ is an isometry. By multiplying an element by the reciprocal of its norm, it will suffice to only consider elements of norm one. By considering Lemma 2.5.1(iii) and the fact that $\tilde{\pi}$ is a bijection, it follows that $\tilde{\pi}|_{\mathcal{B}_{\mathcal{A}^{**}}} : \mathcal{B}_{\mathcal{A}^{**}} \to \mathcal{B}_{\mathcal{M}}$ is a bijection. So, given any $\phi \in \mathcal{A}^{**}$ with $||\phi|| = 1$, then $||\tilde{\pi}(\phi)|| \leq 1$, so $\tilde{\pi}(\phi) \neq 0$ (because $\tilde{\pi}$ is bijective and $\phi \neq 0$) and $||\tilde{\pi}^{-1}(\frac{1}{||\tilde{\pi}(\phi)||}\tilde{\pi}(\phi))|| \leq 1$, which implies $||\phi|| \leq ||\tilde{\pi}(\phi)||$, so that

$$1 = ||\phi|| \leq ||\tilde{\pi}(\phi)|| \leq 1.$$

Hence $\tilde{\pi}$ is an isometry.

Since we want to show that $\mathcal{A}^{**}$ is *-isomorphic to a von Neumann algebra, we need to make $\mathcal{A}^{**}$ a C*-algebra. Thus, we define the multiplication operation on $\mathcal{A}^{**}$ as follows: for every $\phi, \psi \in \mathcal{A}^{**}$,

$$\phi\psi := \tilde{\pi}^{-1}(\tilde{\pi}(\phi)\tilde{\pi}(\psi)).$$

That this product is well defined is clear, and it is easily checked that the axioms of an algebra are satisfied. Furthermore, since $\tilde{\pi}$ an isometry, we have

$$||\phi\psi|| = ||\tilde{\pi}^{-1}(\tilde{\pi}(\phi)\tilde{\pi}(\psi))|| = ||\tilde{\pi}(\phi)\tilde{\pi}(\psi)|| \leq ||\tilde{\pi}(\phi)||||\tilde{\pi}(\psi)|| = ||\phi||||\psi||.$$

Hence $\mathcal{A}^{**}$ is a Banach algebra. We define an involution on $\mathcal{A}^{**}$ for every $\phi \in \mathcal{A}^{**}$ by

$$\phi^{\circledast} := \tilde{\pi}^{-1}\left(\tilde{\pi}(\phi)^*\right).$$

Indeed, for every $\phi, \psi \in \mathcal{A}^{**}$,

$$(\phi^{\circledast})^{\circledast} = \tilde{\pi}^{-1}\left(\tilde{\pi}(\phi^{\circledast})^*\right) = \tilde{\pi}^{-1}(\tilde{\pi}(\phi)^{**}) = \phi,$$

and

$$\begin{aligned}
(\phi\psi)^{\circledast} &= \tilde{\pi}^{-1}\left(\tilde{\pi}(\phi\psi)^*\right) \\
&= \tilde{\pi}^{-1}\left((\tilde{\pi}(\phi)\tilde{\pi}(\psi))^*\right) \\
&= \tilde{\pi}^{-1}\left(\tilde{\pi}(\psi)^*\tilde{\pi}(\phi)^*\right) \\
&= \tilde{\pi}^{-1}\left(\tilde{\pi}(\psi^{\circledast})\tilde{\pi}(\phi^{\circledast})\right) \\
&= \tilde{\pi}^{-1}\left(\tilde{\pi}(\psi^{\circledast}\phi^{\circledast})\right) \\
&= \psi^{\circledast}\phi^{\circledast},
\end{aligned}$$

making $\mathcal{A}^{**}$ a Banach *-algebra. Moreover, $\mathcal{A}^{**}$ is a C*-algebra by

$$||\phi^{\circledast}\phi|| = ||\tilde{\pi}(\phi^{\circledast}\phi)|| = ||\tilde{\pi}(\phi)^*\tilde{\pi}(\phi)|| = ||\tilde{\pi}(\phi)||^2 = ||\phi||^2.$$

We show that $\tilde{\pi}$ is a *-isomorphism. The linearity of $\tilde{\pi}$ follows from its definition in Lemma 2.5.1, and since $\tilde{\pi}$ is bijective, it follows from the definition of multiplication and the involution on $\mathcal{A}^{**}$ that for every $\phi, \psi \in \mathcal{A}^{**}$ we have

$$\tilde{\pi}(\phi\psi) = \tilde{\pi}(\phi)\tilde{\pi}(\psi),$$

and

$$\tilde{\pi}(\phi^{\circledast}) = \tilde{\pi}(\phi)^*.$$

Hence $\tilde{\pi}$ is an (isometric) *-isomorphism of the C*-algebra $\mathcal{A}^{**}$ onto the the von Neumann algebra $\mathcal{M}$.

To be able to identify $\mathcal{A}^{**}$ with $\mathcal{M}$, we need to show they are topologically the same, that is, $\tilde{\pi}$ is a homeomorphism with respect to the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$- and σ-weak operator topologies. We only need to show $\tilde{\pi}^{-1}$ continuous in this regard, since the continuity of $\tilde{\pi}$ was shown in Lemma 2.5.1(ii). Let $(\tilde{\pi}(\phi_n))$ be a net converging to $\tilde{\pi}(\phi)$ in the σ-weak operator topology on $\mathcal{M}$. Then, for every $\omega \in \mathcal{M}_*$ and $\epsilon > 0$ there exists a $N \in \mathbb{I}$ such that for all $n > N$ we have

$$|\omega(\tilde{\pi}(\phi_n)) - \omega(\tilde{\pi}(\phi))| < \epsilon.$$

If $\varphi \in \mathcal{A}^*$, then $\varphi = \pi_*(\omega)$ (as shown earlier) for some $\omega \in \mathcal{M}_*$ and by the definition of $\tilde{\pi}$ and Proposition 2.4.12

$$\begin{aligned} |\phi_n(\varphi) - \phi(\varphi)| &= |(\phi_n - \phi)(\pi_*(\omega))| \\ &= |\omega(\tilde{\pi}(\phi_n)) - \omega(\tilde{\pi}(\phi))| \\ &< \epsilon, \end{aligned}$$

for $n > N$. Thus, $\phi_n \to \phi$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology, that is $\tilde{\pi}^{-1}(\tilde{\pi}(\phi_n)) \to \tilde{\pi}^{-1}(\tilde{\pi}(\phi))$, so that $\tilde{\pi}^{-1}$ is continuous as required. □

Remark 2.5.5. Note that in Lemma 2.5.1 there is no algebraic structure on $\mathcal{A}^{**}$. But, in Theorem 2.5.4 we showed that $\mathcal{A}^{**}$ can be viewed as a von Neumann algebra, and hence has a *-algebraic structure. With this in mind we can now show that the $\tilde{\pi}$ (obtained from an arbitrary representation π) in Lemma 2.5.1 is a *-homomorphism with this algebraic structure. Also note that the $\tilde{\pi}$ in Theorem 2.5.4 is a special case of the $\tilde{\pi}$ in Lemma 2.5.1, and was obtained from a specific representation.

Lemma 2.5.6. *Let $\mathcal{A}$ be a C^*-algebra and $(\mathfrak{H}, \pi)$ any representation of $\mathcal{A}$. Then the unique linear map $\tilde{\pi} : \mathcal{A}^{**} \to \mathcal{M}_\pi$ given by Lemma 2.5.1 is a *-homomorphism.*

Proof. Let $\tilde{\rho} : \mathcal{A}^{**} \to \mathcal{M}_\rho$ be the * -isomorphism given by Theorem 2.5.4, obtained from the representation $(\mathfrak{H}, \rho)$. Then $\tilde{\rho} \circ i = \rho$, and thus $i = \tilde{\rho}^{-1} \circ \rho$, so that i is a *-homomorphism. By Lemma 2.5.1 $\tilde{\pi}$ is linear, and we have $\tilde{\pi} \circ i = \pi$, where $i : \mathcal{A} \to \mathcal{A}^{**}$, so that $\tilde{\pi}\big|_{i(\mathcal{A})}$ is a *-homomorphism, because both π and i are.

Since the mappings $a \mapsto ab$ and $b \mapsto ab$ are σ-weakly continuous in any von Neumann algebra (see for example [3, Prop. 2.4.2, p. 68]), and $\tilde{\pi}$ is a homeomorphism with respect to the weak* and σ-weak topologies on $\mathcal{A}^{**}$ and $\mathcal{M}_\pi$, respectively, these mappings will also be continuous in the weak*-topology on $\mathcal{A}^{**}$.

So, let $a \in \mathcal{A}$ and $\phi \in \mathcal{A}^{**}$. Since $i(\mathcal{A})$ is weak* dense in $\mathcal{A}^{**}$, there is a net $i(b_\alpha)$ in $i(\mathcal{A})$ converging to ϕ in the weak* topology. Then from the facts that $\tilde{\pi}\big|_{i(\mathcal{A})}$ is a *-homomorphism and that the mappings $b \mapsto ab$ and $\tilde{\pi}$ are weak* continuous, we have

$$\begin{aligned} \tilde{\pi}(i(a)\phi) &= \tilde{\pi}(\lim_\alpha(i(a)i(b_\alpha)) \\ &= \lim_\alpha \tilde{\pi}((i(a)i(b_\alpha)) \\ &= \lim_\alpha[\tilde{\pi}((i(a))\tilde{\pi}(i(b_\alpha))] \\ &= \tilde{\pi}((i(a))(\lim_\alpha \tilde{\pi}(i(b_\alpha))) \\ &= \tilde{\pi}((i(a))\tilde{\pi}(\phi) \end{aligned} \tag{2.5}$$

Now let $\phi, \psi \in \mathcal{A}^{**}$, then there is a net $i(a_\alpha)$ in $i(\mathcal{A})$ converging to ψ in the weak* topology, and by the weak* continuity of $a \mapsto ab$ and $\tilde{\pi}$ we have

$$\begin{aligned}
\tilde{\pi}(\psi\phi) &= \tilde{\pi}(\lim_\alpha(i(a_\alpha)\phi) \\
&= \lim_\alpha \tilde{\pi}(i(a_\alpha)\phi) \\
&= \lim_\alpha[\tilde{\pi}(i(a_\alpha))\tilde{\pi}(\phi)] \quad \text{(by Equation (2.5))} \\
&= (\lim_\alpha \tilde{\pi}(i(a)))\tilde{\pi}(\phi) \\
&= \tilde{\pi}(\psi)\tilde{\pi}(\phi).
\end{aligned}$$

Similarly, since $\tilde{\pi}\big|_{i(\mathcal{A})}$ is a *-homomorphism and the mapping $a \mapsto a^*$ is σ-weakly continuous in $\mathcal{M}_\pi$ and thus weak* continuous in $\mathcal{A}^{**}$, we have for any $\phi \in \mathcal{A}^{**}$ a net $i(a_\alpha)$ in $i(\mathcal{A})$ converging to ϕ in the weak* topology, so that

$$\begin{aligned}
\tilde{\pi}(\phi^*) &= \tilde{\pi}(\lim_\alpha(i(a_\alpha)^*) \\
&= \lim_\alpha \tilde{\pi}(i(a_\alpha)^*) \\
&= \lim_\alpha \tilde{\pi}(i(a_\alpha))^* \\
&= [\lim_\alpha \tilde{\pi}(i(a_\alpha))]^* \\
&= \tilde{\pi}(\phi)^*.
\end{aligned}$$

Hence $\tilde{\pi}$ is a *-homomorphism. □

We now show the existence of a universal enveloping von Neumann algebra as a corollary to Theorem 2.5.4.

Corollary 2.5.7. *[25, Thm 2.4, p.122] A C*-algebra $\mathcal{A}$ admits a universal representation $(\mathfrak{H}, \rho)$, hence the universal enveloping von Neumann algebra $\mathcal{M}_\rho$.*

Proof. Let $\tilde{\rho}$ be the *-isomorphism given by Theorem 2.5.4, and let $(\mathfrak{H}_\pi, \pi)$ be any other representation of $\mathcal{A}$, with $\tilde{\pi}$ the linear map of $\mathcal{A}^{**}$ onto $\mathcal{M}_\pi$ as given by Lemma 2.5.1. We define $\tilde{\pi}_0 : \mathcal{M}_\rho \to \mathcal{M}_\pi$ by $\tilde{\pi}_0 = \tilde{\pi} \circ \tilde{\rho}^{-1}$. Then $\tilde{\pi}_0$ is linear, onto and σ-weakly continuous. Also, for every $\phi \in \mathcal{A}^{**}$, we have $\tilde{\pi}(\phi) = \tilde{\pi}_0 \circ \tilde{\rho}(\phi)$, so that $\tilde{\pi}_0$ is a σ-weakly continuous homomorphism of $\mathcal{M}_\rho$ onto $\mathcal{M}_\pi$. □

We present two auxiliary results, followed by the main result of this chapter.

Lemma 2.5.8. *Let $b, p \in B(\mathfrak{H})$, with $\mathfrak{H}$ some Hilbert space, such that $0 \leq b \leq p$ and p is a projection. Then $pbp = b$.*

Proof. We begin by showing that $b(p\mathfrak{H})^{\perp} = 0$, where $(p\mathfrak{H})^{\perp}$ denotes the orthogonal compliment of the set $\{px : x \in \mathfrak{H}\} \subseteq \mathfrak{H}$. Let $x \in (p\mathfrak{H})^{\perp}$, then from the assumption that $0 \leq b \leq p$ it follows that $\langle x, (p-b)x\rangle \geq 0$, which then implies that $0 = \langle x, px\rangle \geq \langle x, bx\rangle \geq 0$. Hence $\langle x, bx\rangle = 0$, for every $x \in (p\mathfrak{H})^{\perp}$, and by the positivity of b, we have that

$$0 = \langle x, (b^{1/2})^* b^{1/2} x\rangle = \langle b^{1/2}x, b^{1/2}x\rangle = ||b^{1/2}x||.$$

Thus $bx = b^{1/2}b^{1/2}x = 0$ for every $x \in (p\mathfrak{H})^{\perp}$. Now, if $y \in p\mathfrak{H}$ and $x \in (p\mathfrak{H})^{\perp}$, then

$$\langle x, by\rangle = \langle b^*x, y\rangle = \langle bx, y\rangle = \langle 0, y\rangle = 0,$$

and so $b(p\mathfrak{H}) \subseteq p\mathfrak{H}$. For an arbitrary $x \in \mathfrak{H}$, put $y = px \in p\mathfrak{H}$ and $z = (1-p)x \in (p\mathfrak{H})^{\perp}$, so that $x = y + z$. Then $bx = by + bz = by$, and $pbpx = pby = by = bx$. Hence $pbp = b$. □

Lemma 2.5.9. *Let $p, q \in B(\mathfrak{H})$ be projections on some Hilbert space $\mathfrak{H}$ such that $p \perp q$, and let $a, b \in B(\mathfrak{H})$ be arbitrary. Then*

$$||paq + qbp|| = \max\{||paq||, ||qbp||\}.$$

Proof. Let $x \in \mathfrak{H}$ and $m = \max\{||paq||, ||qbp||\}$. Then using the inner product on $\mathfrak{H}$, and the fact $p \perp q$ we have that

$$\begin{aligned}
||(paq + qbp)x||^2 &= ||(paq)x||^2 + ||(qbp)x||^2 \\
&= ||(paq)qx||^2 + ||(qbp)px||^2 \\
&\leq ||(paq)||^2||qx||^2 + ||(qbp)||^2||px||^2 \\
&\leq m^2(||qx||^2 + ||px||^2) \\
&= m^2||(q+p)x||^2 \\
&\leq m^2||x||^2,
\end{aligned}$$

implying that $||paq + qbp|| \leq m$. On the other hand, we also have that

$$\begin{aligned}
||(paq)x||^2 &\leq ||(paq)x||^2 + ||(qbp)x||^2 \\
&= ||(paq + qbp)x||^2,
\end{aligned}$$

and taking the supremum over all $||x|| = 1$ we obtain $||paq|| \leq ||paq + qbp||$. Similarly, we get $||qbp|| \leq ||paq + qbp||$, so that $m \leq ||paq + qbp||$. Hence $||paq + qbp|| = \max\{||paq||, ||qbp||\}$. □

Definition 2.5.10. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$. A *projection* of $\mathcal{A}$ onto $\mathcal{B}$ is a linear map $E : \mathcal{A} \to \mathcal{B}$ such that $E(b) = b$ for every $b \in \mathcal{B}$. A *projection of $\mathcal{A}$ onto $\mathcal{B}$ of norm one* is projection such that $||E(a)|| \leq ||a||$ for every $a \in \mathcal{A}$.

Notice that if E is a projection of norm one, as in the above definition, then E is bounded and $||E|| \leq 1$, by taking the supremum over all $a \in \mathcal{A}$ with $||a|| = 1$. But, we also have $||E|| = ||E^2|| \leq ||E||^2$, so that $1 \leq ||E||$, and thus $||E|| = 1$ (hence the terminology).

Definition 2.5.11. A projection $E : \mathcal{A} \to \mathcal{B}$ satisfying the properties

(i) $E(x^*x) \geq 0$, for every $x \in \mathcal{A}$,

(ii) $E(axb) = aE(x)b$, for every $a, b \in \mathcal{B}$ and $x \in \mathcal{A}$, and

(iii) $E(x)^*E(x) \leq E(x^*x)$, for every $x \in \mathcal{A}$.

is called a *conditional expectation*.

We shall see shortly that condition (iii) is in fact a consequence of (ii), but we include it none the less in the definition.

Theorem 2.5.12. *[25, Thm 3.4, p.131] Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{B}$ a C^*-subalgebra of $\mathcal{A}$ such that $1_{\mathcal{A}} = 1_{\mathcal{B}}$, and let $E : \mathcal{A} \to \mathcal{B}$ be a projection of norm one. Then E is a conditional expectation.*

Proof. We first prove property (i) in Definition 2.5.11. Let $\mathcal{A}^*_+$ and $\mathcal{B}^*_+$ denote the sets of all positive linear functional on $\mathcal{A}$ and $\mathcal{B}$, respectively. Also, let $E^t : \mathcal{B}^* \to \mathcal{A}^*$ denote the transpose mapping. Then for every $\omega \in \mathcal{B}^*_+$ we have

$$\begin{aligned} ||E^t(\omega)|| &= \sup_{\substack{a\in\mathcal{A}\\ ||a||=1}} |E^t(\omega)(a)| \\ &= \sup_{\substack{a\in\mathcal{A}\\ ||a||=1}} |\omega(E(a))| \\ &\leq \sup_{\substack{a\in\mathcal{A}\\ ||a||=1}} ||\omega||||E||||a|| \\ &= ||\omega||, \end{aligned}$$

and we also have by the positivity of ω and Theorem 2.2.9 that

$$\begin{aligned} E^t(\omega)(1_{\mathcal{A}}) &= \omega(E(1_{\mathcal{A}})) \\ &= \omega(1_{\mathcal{B}}) \\ &= ||\omega||. \end{aligned}$$

Hence $||\omega|| = |E^t(\omega)(1_{\mathcal{A}})| \leq ||E^t(\omega)|| \leq ||\omega||$, which implies that $E^t(\omega)(1_{\mathcal{A}}) = ||E^t(\omega)||$, and thus, by Theorem 2.2.9, $E^t(\omega) \in \mathcal{A}^*_+$. Now, since $E^t(\omega)$ is a positive linear functional, it must be self-adjoint and then for every $\omega \in \mathcal{B}_+^*$ and every $a \in \mathcal{A}$ we have

$$\begin{aligned}\omega(E(a^*)) &= E^t(\omega)(a^*)\\ &= \overline{E^t(\omega)^*(a)}\\ &= \overline{E^t(\omega)(a)}\\ &= \overline{\omega(E(a))}\\ &= \omega(E(a)^*).\end{aligned}$$

Since, by Remark 2.2.12 every bounded linear functional can be written as a linear combination of positive linear functionals, it follows that

$$|\omega(E(a^*) - E(a)^*)| = 0,$$

for every $\omega \in \mathcal{B}^*$, and thus by the Hahn-Banach theorem $E(a^*) = E(a)^*$.

Given any $a \in \mathcal{A}_+$, from the fact that $E^t(\omega)$ is positive for every $\omega \in \mathcal{B}_+^*$, we have $\omega(E(a)) = E^t(\omega)(a) \geq 0$, so it follows from Theorem 2.2.8 that $E(a)$ is positive. Hence E maps $\mathcal{A}_+$ into $\mathcal{B}_+$. And, since $E(b) = b$ for every $b \in \mathcal{B}_+$, the map is onto.

We now prove property (ii). As was done earlier in this chapter, if we consider the double transpose mapping

$$E^{tt} : \mathcal{A}^{**} \to \mathcal{B}^{**}$$

and the universal enveloping von Neumann algebras of $\mathcal{A}$ and $\mathcal{B}$, we may as well assume that $\mathcal{A}$ is a von Neumann algebra and $\mathcal{B}$ is a weakly closed *-subalgebra (i.e. von Neumann subalgebra) of $\mathcal{A}$. Then, by Theorem 2.4.16, it will be sufficient to show for every $x \in \mathcal{A}$ and for every projection $e \in \mathcal{B}$ that

$$E(ex) = eE(x) \quad \text{and} \quad E(xe) = E(x)e.$$

By Proposition 2.2.4 if $a \in \mathcal{A}$ and $0 \leq a \leq 1_{\mathcal{A}}$, then $0 \leq eae \leq e$, and by property (i) we have $0 \leq E(eae) \leq E(e) = e$. Then by Lemma 2.5.8 we have that

$$E(eae) = eE(eae)e. \tag{2.6}$$

Moreover, Equation (2.6) holds for every $a \in \mathcal{A}$. Indeed, let $a \in \mathcal{A}$ be self-adjoint with $||a|| \leq 1$. By Proposition 2.2.5 we can write $a = a_+ - a_-$, with a_+ and a_- unique positive elements such that $a_+a_- = a_-a_+ = 0$ and $||a|| = \max\{||a_+||, ||a_-||\}$. Hence $||a_+||, ||a_-|| \leq 1$. Consider the abelian C*-algebra generated by $\{a_+, 1_{\mathcal{A}}\}$. Using the Gelfand representation (Theorem 2.2.22) we see that $0 \leq a_+ \leq 1_{\mathcal{A}}$. Similarly $0 \leq a_- \leq 1_{\mathcal{A}}$. Thus Equation (2.6) holds for both a_+ and a_-, and also for their difference, since it is linear in a. Hence Equation (2.6) holds for any self-adjoint $a \in \mathcal{A}$, since it holds for $a = 0$, and for $a \neq 0$, $\frac{1}{||a||}a$ has norm one, to which the above argument can be applied, and the required result follows by the linearity of Equation (2.6) in a. An arbitrary $a \in \mathcal{A}$ can be written as a linear combination of two self-adjoint elements, and again $\frac{1}{||a||}a$ has norm one, so that Equation (2.6) holds for every $a \in \mathcal{A}$.

Put $\tilde{x} = E(ex(1_{\mathcal{A}} - e))$ for $x \in \mathcal{A}$ with $||x|| \leq 1$. Then for every $\lambda > 0$, we have

$$\begin{aligned} ||\tilde{x} + \lambda e||^2 &= ||E(ex(1_{\mathcal{A}} - e) + \lambda e)||^2 \\ &\leq ||ex(1_{\mathcal{A}} - e) + \lambda e||^2 \\ &= ||(ex(1_{\mathcal{A}} - e) + \lambda e)(ex(1_{\mathcal{A}} - e) + \lambda e)^*|| \\ &= ||ex(1_{\mathcal{A}} - e)x^*e + \lambda^2 e|| \\ &\leq ||e||\,||x||\,||1_{\mathcal{A}} - e||\,||x^*||\,||e|| + \lambda^2||e|| \\ &\leq ||x||^2 + \lambda^2 \\ &\leq 1 + \lambda^2 \end{aligned} \tag{2.7}$$

Put $h = \frac{1}{2}(\tilde{x} + \tilde{x}^*)$ and $k = \frac{1}{2i}(\tilde{x} - \tilde{x}^*)$. Suppose that $ehe \neq 0$. Because ehe is self-adjoint, we can, by considering $-x$ instead of x if necessary, assume that the spectrum of ehe contains an $\alpha > 0$. Then

$$\begin{aligned} ||\tilde{x} + \lambda e|| &= ||e\tilde{x}e + \lambda e + (\tilde{x} - e\tilde{x}e)|| \\ &= ||e(\tilde{x} + \lambda 1_{\mathcal{B}})e + \tilde{x} - e\tilde{x}e|| \\ &\geq ||e(e(\tilde{x} + \lambda 1_{\mathcal{B}})e + \tilde{x} - e\tilde{x}e)e|| \\ &= ||e(\tilde{x} + \lambda 1_{\mathcal{B}})e|| \\ &= ||e\tilde{x}e + \lambda e|| \\ &\geq ||\frac{1}{2}((e\tilde{x}e + \lambda e) + (e\tilde{x}e + \lambda e)^*)|| \\ &= ||ehe + \lambda e|| \\ &\geq \alpha + \lambda. \end{aligned} \tag{2.8}$$

The last inequality can be seen by considering the abelian C*-algebra C generated by the self-adjoint elements ehe and e as follows: By Theorem 2.2.21 and the Gelfand representation theorem (Theorem 2.2.22) for abelian C*-algebras, this C*-algebra is isomorphic to the algebra of all continuous complex valued functions on some compact Hausdorff space, say X, where their ranges are precisely their spectra. Thus $ehe(z) = \alpha$ for some $z \in X$, and $e(z) = 1$, since e is the identity of C. Then

$$\begin{aligned} ||ehe + \lambda e|| &= \sup_{y \in X} |ehe(y) + \lambda e(y)| \\ &\geq ehe(z) + \lambda e(z) \\ &= \alpha + \lambda. \end{aligned}$$

From inequalities (2.7) and (2.8) we obtain for a λ sufficiently large

$$||\tilde{x} + \lambda e|| \geq \alpha + \lambda > (1 + \lambda^2)^{1/2} \geq ||\tilde{x} + \lambda e||,$$

which is a contradiction. Thus $ehe = 0$. Similarly $eke = 0$, and thus

$$e\tilde{x}e = 0. \tag{2.9}$$

Since this holds for every projection in $\mathcal{B}$, if we replace e with the projection $(1_{\mathcal{B}} - e)$, we obtain $(1_{\mathcal{B}} - e)E((1_{\mathcal{A}} - e)xe)(1_{\mathcal{B}} - e) = 0$. Taking adjoints, using the fact that $E(a^*) = E(a)^*$ for $a \in \mathcal{A}$ and replacing x with x^* (which is allowed, since $x \in \mathcal{A}$, with $||x|| \leq 1$ was arbitrary) we obtain

$$(1_{\mathcal{B}} - e)\tilde{x}(1_{\mathcal{B}} - e) = 0. \tag{2.10}$$

Next we suppose that $(1_{\mathcal{B}} - e)\tilde{x}e \neq 0$. Since $e\tilde{x}e = (1_{\mathcal{B}} - e)\tilde{x}(1_{\mathcal{B}} - e) = 0$, we have, by Lemma 2.5.9, for a $\lambda > 0$ large enough

$$\begin{aligned}
||\tilde{x} + \lambda(1_{\mathcal{B}} - e)\tilde{x}e|| &= ||\tilde{x} - e\tilde{x}e - (1_{\mathcal{B}} - e)\tilde{x}(1_{\mathcal{B}} - e) + \lambda(1_{\mathcal{B}} - e)\tilde{x}e|| \\
&= ||e\tilde{x}(1_{\mathcal{B}} - e) + (1 + \lambda)(1_{\mathcal{B}} - e)\tilde{x}e|| \\
&= \max\{||e\tilde{x}(1_{\mathcal{B}} - e)||, (1 + \lambda)||(1_{\mathcal{B}} - e)\tilde{x}e||\} \\
&= (1 + \lambda)||(1_{\mathcal{B}} - e)\tilde{x}e||.
\end{aligned}$$

But, since $(1_{\mathcal{B}} - e), \tilde{x}, e \in \mathcal{B}$, using Lemma 2.5.9 again, we have for a sufficiently large $\lambda > 0$,

$$\begin{aligned}
||\tilde{x} + \lambda(1_{\mathcal{B}} - e)\tilde{x}e|| &= ||E(ex(1_{\mathcal{A}} - e)) + \lambda(1_{\mathcal{B}} - e)\tilde{x}e|| \\
&= ||E(ex(1_{\mathcal{A}} - e) + \lambda(1_{\mathcal{B}} - e)\tilde{x}e)|| \\
&\leq ||ex(1_{\mathcal{A}} - e) + \lambda(1_{\mathcal{B}} - e)\tilde{x}e|| \\
&= \max\{||ex(1_{\mathcal{A}} - e)||, \lambda||(1_{\mathcal{B}} - e)\tilde{x}e||\} \\
&= \lambda||(1_{\mathcal{B}} - e)\tilde{x}e||,
\end{aligned}$$

which is a contradiction. Hence

$$(1_{\mathcal{B}} - e)\tilde{x}e = 0. \tag{2.11}$$

Then using Equations (2.9), (2.10) and (2.11) we obtain

$$\begin{aligned}
\tilde{x} &= e\tilde{x} + \tilde{x}e - e\tilde{x}e && \text{(from (2.10))} \\
&= e\tilde{x} - e\tilde{x}e + \tilde{x}e - e\tilde{x}e && \text{(from (2.9))} \\
&= e\tilde{x} - e\tilde{x}e + (1_{\mathcal{B}} - e)\tilde{x}e \\
&= e\tilde{x}(1_{\mathcal{B}} - e). && \text{(from (2.11))} \qquad (2.12)
\end{aligned}$$

Now, since we can write

$$x = exe + ex(1_{\mathcal{A}} - e) + (1_{\mathcal{A}} - e)xe + (1_{\mathcal{A}} - e)x(1_{\mathcal{A}} - e), \tag{2.13}$$

we obtain the equality

$$\begin{aligned}
eE(x)(1_{\mathcal{B}} - e) &= eE(exe)(1_{\mathcal{B}} - e) + eE(ex(1_{\mathcal{A}} - e))(1_{\mathcal{B}} - e) \\
&\quad + eE((1_{\mathcal{A}} - e)xe)(1_{\mathcal{B}} - e) + eE((1_{\mathcal{A}} - e)x(1_{\mathcal{A}} - e))(1_{\mathcal{B}} - e). \qquad (2.14)
\end{aligned}$$

Using Equation (2.6) the first term in Equation (2.14) becomes

$$eE(exe)(1_{\mathcal{B}} - e) = eeE(exe)e(1_{\mathcal{B}} - e) = 0.$$

Also, using Equation (2.11), applied to x^* instead of x, the third term in Equation (2.14) becomes

$$eE((1_{\mathcal{A}} - e)xe)(1_{\mathcal{B}} - e) = ((1_{\mathcal{B}} - e)E(ex^*(1_{\mathcal{A}} - e))e)^* = 0.$$

Lastly, again using Equation (2.6), but replacing e with $(1_{\mathcal{B}} - e)$, the fourth term in Equation (2.14) is also zero. Hence, using Equation (2.12), we have from Equation (2.14)

$$\begin{aligned} eE(x)(1_{\mathcal{B}} - e) &= eE(ex(1_{\mathcal{A}} - e))(1_{\mathcal{B}} - e) \\ &= E(ex(1_{\mathcal{A}} - e)). \end{aligned} \tag{2.15}$$

In the same way as above we can again use Equation (2.13) to obtain

$$eE(x)e = eE(exe)e + eE(ex(1_{\mathcal{A}} - e))e + eE((1_{\mathcal{A}} - e)xe)e + eE((1_{\mathcal{A}} - e)x(1_{\mathcal{A}} - e))e,$$

and $eE(exe)e = E(exe)$ by Equation (2.6), $eE(ex(1_{\mathcal{A}} - e))e = eE((1_{\mathcal{A}} - e)xe)e = 0$ by Equation (2.9) and $eE((1_{\mathcal{A}} - e)x(1_{\mathcal{A}} - e))e = 0$ from Equation (2.6) by replacing e with $(1_{\mathcal{B}} - e)$. Hence

$$eE(x)e = E(exe),$$

and finally implies (by multiplying out Equation 2.15) that

$$eE(x) = E(ex). \tag{2.16}$$

The above equation is for $||x|| \leq 1$. For an arbitrary $x \neq 0$ we apply it to $\frac{x}{||x||}$, and for $x = 0$ it is trivial. Hence $eE(x) = E(ex)$ for any $x \in \mathcal{A}$. Taking the involution on both sides of Equation (2.16) and using the fact that $E(x^*) = E(x)^*$ and replacing x by x^*, we obtain

$$E(x)e = E(xe),$$

which proves (ii).

Lastly we show property (iii). Let $x \in \mathcal{A}$, then

$$\begin{aligned} 0 &\leq E((x - E(x))^*(x - E(x))) \\ &= E(x^*x - E(x)^*x - x^*E(x) + E(x)^*E(x)) \\ &= E(x^*x) - E(x)^*E(x) - E(x^*)E(x) + E(x)^*E(x) \\ &= E(x^*x) - E(x^*)E(x), \end{aligned}$$

and hence $E(x^*)E(x) \leq E(x^*x)$ (since we already know $E(x^*) = E(x)^*$). □

It is worth mentioning that in the theorem above, the requirements that the C*-algebras be unital and that they have they same units, are in fact unnecessary, as shown in [25, Thm 3.4, p.131]. But since we will only be working with unital C*-algebras, this will suffice.

Proposition 2.5.13. *Let $E : \mathcal{A} \to \mathcal{B}$ be a conditional expectation. Then*

(i) $E(a^*) = E(a)^*$ for every $a \in \mathcal{A}$, and

(ii) $||E|| = 1$, i.e. E is a projection of norm one.

Proof. We show (i). Let $a \in \mathcal{A}$, then it can be decomposed into a linear combination of two self-adjoint elements, and by Proposition 2.2.5 further decomposed into the form $a = a_1 - a_2 + ia_3 - ia_4$, where $a_i, i = 1, \ldots, 4$ are unique positive elements in $\mathcal{A}$. Hence since E is a conditional expectation it follows that $E(a_i) \geq 0$ for each $i = 1, \ldots, 4$. Then

$$\begin{aligned}
E(a^*) &= E((a_1 - a_2 + ia_3 - ia_4)^*) \\
&= E(a_1^* - a_2^* - ia_3^* + ia_4^*) \\
&= E(a_1^*) - E(a_2^*) - iE(a_3^*) + iE(a_4^*) \\
&= E(a_1) - E(a_2) - iE(a_3) + iE(a_4) \\
&= E(a_1)^* - E(a_2)^* + (iE(a_3))^* - (iE(a_4))^* \\
&= (E(a_1) - E(a_2) + iE(a_3) - iE(a_4))^* \\
&= E(a_1 - a_2 + ia_3 - ia_4)^* \\
&= E(a)^*.
\end{aligned}$$

We show (ii). We have that $||E(a)||^2 = ||E(a)^*E(a)|| \leq ||E(a^*a)||$. But, $a^*a \leq ||a||^2 1_{\mathcal{A}}$, so $E(||a||^2 1_{\mathcal{A}} - a^*a) \geq 0$, and thus
$E(a^*a) \leq E(||a||^2 1_{\mathcal{A}}) \leq ||a||^2 1_{\mathcal{A}}$. Hence $||E(a)|| \leq ||a||$. ☐

Chapter 3

C*-Dynamical Systems

The aim of this chapter is mainly to construct examples of noncommutative dynamical systems, some of which involving more general groups actions than $\mathbb{Z}$, and in some sense generalizes classical (commutative) dynamical systems. Hence, we begin by giving some definitions and move straight on to the constructions. Classical dynamical systems will not be dealt with in any detail, but some definitions are given so as to draw on the analogy between the classical and noncommutative cases.

The examples of C*-dynamical systems are chosen to illustrate the ergodic notions discussed in Chapter 4. Our first two examples are on the noncommutative torus, one with an $\mathbb{R}^2$-action and the other with an $\mathbb{R}$-action. The third is on the C*-algebra generated by the annihilation and creation operators on a deformed Fock space, and lastly we consider a shift on an infinite tensor product of C*-algebras.

3.1 Definitions

Definition 3.1.1. Let Ω be a measure space, $\mathscr{B}$ the σ-algebra on Ω, μ a probability measure and T a group homomorphism from some group $\mathcal{G}$ into the group of all bijective mapping on Ω such that for each of these bijections, say T_g with $g \in \mathcal{G}$, we have $\mu(T_g^{-1}B) = \mu(B)$ for every $B \in \mathscr{B}$ and $T_g^{-1}(\mathscr{B}) \subseteq \mathscr{B}$. In this case we say μ is T-invariant and we call $(X, \mathscr{B}, \mu, T)$ a *measure preserving system (m.p.s.)*.

It is important to to keep in mind that, although we have given the classical definition of measure preserving systems, we will actually consider noncommutative topological dynamical systems. Classically, this means that in our definition above Ω will be a compact Hausdorff space, $\mathscr{B}$ the Borel σ-algebra and $T : \Omega \to \Omega$ a homeomorphism, and no specific measure is used. We are rather concerned with existence of a unique measure. The reason for giving the measure theoretic definition is that many of our characterizations will rely on their commuatative counterpart. We now give the noncommutative version(s).

Let $\mathcal{A}$ be a C*-algebra and let $Aut(\mathcal{A})$ denote the set of all automorphism on $\mathcal{A}$. It is easily checked that $Aut(\mathcal{A})$ is a group under composition.

Definition 3.1.2. Let $\mathcal{A}$ be a unital C*-algebra and $\alpha : \mathcal{G} \to Aut(\mathcal{A})$, a group homomorphism from some group $\mathcal{G}$ into the group of automorphisms on $\mathcal{A}$. Then we shall call α an *action of $\mathcal{G}$ on $\mathcal{A}$* and we call $(\mathcal{A}, \alpha)$ a *C*-dynamical system.*

Definition 3.1.3. Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system and let ω be a state on $\mathcal{A}$ such that $\omega \circ \alpha(g) = \omega$ for every $g \in \mathcal{G}$, then we call $(\mathcal{A}, \alpha, \omega)$ a *state preserving C*-dynamical system*, and we say that ω is α-invariant.

The existence of such an α-invariant state is shown later in Proposition 4.1.5. If no ambiguity can arise we shall refer to both $(\mathcal{A}, \alpha)$ and $(\mathcal{A}, \alpha, \omega)$ as C*-dynamical systems.

3.2 Non-commutatve torus with an $\mathbb{R}^2$-action and an $\mathbb{R}$-action

We follow the same construction as in [28, p.109] for the noncommutative torus, and we use the same dynamics on the torus as done in [6].

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \{(a,b) + \mathbb{Z}^2 : (a,b) \in \mathbb{R}^2\}$ denote the classical (or commutative) torus. We firstly construct a dynamical system on the classical torus, from which we generalize to a non-commutative version.

We shall denote the equivalence classes in $\mathbb{T}^2$ by square brackets, i.e. for $x, y \in \mathbb{R}$ we have $[x, y] \in \mathbb{T}^2$, where $[x, y] = \{(x + n, y + m) : m, n \in \mathbb{Z}\}$. With addition defined in the usual way, $\mathbb{T}^2$ is a group, and furthermore, considering the Borel σ-algebra $\mathscr{B}$ on $\mathbb{T}^2$, there exists a left translation invariant measure μ (the Haar measure) on $\mathbb{T}^2$. But, since $\mathbb{T}^2$ is a compact group, it follows that the Haar measure is both left and right translation invariant, and finite (see [4, Prop 9.3.3 and Prop 9.3.5, pp. 313-314]). Thus, normalising the Haar measure we obtain the probability space $(\mathbb{T}^2, \mathscr{B}, \mu)$. We move on to the dynamics on the torus; for each $(s,t) \in \mathbb{R}^2$ we define $T_{(s,t)} : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T_{(s,t)}[x, y] = [x + s, y + t].$$

Thus $T_{(s,t)}$ is just a translation on the torus, and is well defined for every $(s,t) \in \mathbb{R}^2$, since

$$\begin{aligned}
[x, y] = [u, v] &\iff x - u \in \mathbb{Z} \text{ and } y - v \in \mathbb{Z} \\
&\iff (x + s) - (u + s) \in \mathbb{Z} \text{ and } (y + t) - (v + t) \in \mathbb{Z} \\
&\iff [x + s, y + t] = [u + s, v + t].
\end{aligned}$$

It is clear that $T_{(s,t)}$ is bijective, with its inverse given by $T_{(s,t)}^{-1} = T_{(-s,-t)}$. Being a translation, it follows that both $T_{(s,t)}^{-1}$ and $T_{(s,t)}$ are measurable with respect to the Borel σ-algebra (since a translation on a locally compact group is a homeomorphism, see [4, Prop 9.1.1, p. 298]) and, that μ is $T_{(s,t)}$-invariant. Hence, $T = \{T_{(s,t)} : (s,t) \in \mathbb{R}^2\}$ is a group action from $\mathbb{R}^2$ on $\mathbb{T}^2$, giving the measure preserving dynamical system $(\mathbb{T}^2, \mathscr{B}, \mu, T)$.

We now move on to the non-commutative torus. We take as our Hilbert space $L^2(\mu)$ (which is short for $L^2(\mathbb{T}^2, \mathfrak{B}, \mu)$), and define for any $\theta \in \mathbb{R}$, the linear operators $V, W : L^2(\mu) \to L^2(\mu)$ by

$$(Vf)([x,y]) = e^{ix} f([x, y - \frac{\theta}{2}]) \tag{3.1}$$

$$(Wf)([x,y]) = e^{iy} f([x + \frac{\theta}{2}, y]). \tag{3.2}$$

Their inverses are given by

$$(V^{-1}f)([x,y]) = e^{-ix} f([x, y + \frac{\theta}{2}])$$

$$(W^{-1}f)([x,y]) = e^{-iy} f([x - \frac{\theta}{2}, y]).$$

That they are indeed inverses are easily seen by

$$V^{-1}(Vf)([x,y]) = e^{-ix}(Vf([x, y + \frac{\theta}{2}])) = f([x,y])$$

and

$$V(V^{-1}f)([x,y]) = e^{ix}(Vf([x, y - \frac{\theta}{2}])) = f([x,y]),$$

and similarly for W and W^{-1}. Moreover, the operators V and W are unitary, for if $f, g \in L^2(\mu)$ then by the translation invariance of μ

$$\begin{aligned}
\langle Vf, Vg \rangle_2 &= \int_{\mathbb{T}^2} (Vf)\overline{(Vg)} d\mu \\
&= \int_{\mathbb{T}^2} (e^{ix} f[x, y - \frac{\theta}{2}])\overline{(e^{ix} g[x, y - \frac{\theta}{2}])} d\mu[x,y] \\
&= \int_{\mathbb{T}^2} (f[x, y - \frac{\theta}{2}])\overline{(g[x, y - \frac{\theta}{2}])} d\mu[x,y] \\
&= \int_{\mathbb{T}^2} (f)\overline{(g)} d\mu \\
&= \langle f, g \rangle_2,
\end{aligned}$$

and similarly for W. From equations (3.1) and (3.2) we obtain

$$\begin{aligned}
(VW)f([x,y]) &= V(Wf([x,y])) \\
&= e^{ix}(Wf)([x, y-\frac{\theta}{2}]) \\
&= e^{ix+iy-i\frac{\theta}{2}}f([x+\frac{\theta}{2}, y-\frac{\theta}{2}]) \\
&= e^{-i\theta}e^{ix+iy+i\frac{\theta}{2}}f([x+\frac{\theta}{2}, y-\frac{\theta}{2}]) \\
&= e^{-i\theta}e^{iy}(Vf)([x+\frac{\theta}{2}, y]) \\
&= e^{-i\theta}W(Vf)([x,y]) \\
&= e^{-i\theta}(WV)f([x,y]).
\end{aligned}$$

Hence we have the commutation relation

$$VW = e^{-i\theta}WV. \tag{3.3}$$

Let $\mathcal{A}_\theta$ denote the C*-subalgebra of $B(L^2(\mu))$ generated by these two unitary operators, which we then call the non-commutative torus. Notice, from Equation (3.3), that for $\theta = 0$, the these unitary operators commute, and can be seen as functions in $C(\mathbb{T}^2)$ (with $V([x,y]) = e^{ix}$ and $W([x,y]) = e^{iy}$), and thus the C*-algebra generated by them is a (commutative) C*-subalgebra of $C(\mathbb{T}^2)$. In fact, one can show via the Stone-Weierstrass Theorem that this C*-subalgebra is precisely $C(\mathbb{T}^2)$. In this light we see $\mathcal{A}_\theta$ as a generalization of the continuous functions on the classical torus, and hence the non-commutative "torus".

Next we put some dynamics on $\mathcal{A}_\theta$ to obtain a C*-dynamical system that generalizes the dynamical system $(\mathbb{T}^2, \mathscr{B}, \mu, T)$. We start by defining for every $(s,t) \in \mathbb{R}^2$, a unitary operator $U_{(s,t)} : L^2(\mu) \to L^2(\mu)$ with the Koopman construction. That is, for every $f \in L^2(\mu)$, we define

$$(U_{(s,t)}f)[x,y] := f(T_{(s,t)}[x,y]).$$

That $U_{(s,t)}$ is an isometry and unitary follows from the translation invariance of μ as follows:

$$\begin{aligned}
||U_{(s,t)}f||_2^2 &= \int_{\mathbb{T}^2} |U_{(s,t)}f|^2 d\mu \\
&= \int_{\mathbb{T}^2} |f \circ T_{(s,t)}|^2 d\mu \\
&= \int_{\mathbb{T}^2} |f|^2 d\mu \\
&= ||f||^2.
\end{aligned}$$

One easily sees that $U_{(s,t)}^{-1} = U_{(-s,-t)}$, so $U_{(s,t)}$ is unitary. We use this group of unitaries to define our group action, $\alpha : \mathbb{R}^2 \to Aut(\mathcal{A}_\theta)$, as follows: for every $(s,t) \in \mathbb{R}^2$ we let

$$\alpha_{(s,t)}(a) := U_{(s,t)} a U_{(s,t)}^*.$$

It is clear that with this definition $\alpha_{(s,t)}$ is linear, multiplicative, isometric and preserves adjoints. We need to show that $\alpha_{(s,t)}$ is well defined in the sense that $\alpha_{(s,t)}(\mathcal{A}_\theta) \subseteq \mathcal{A}_\theta$, and we do this on the *-algebra generated by V and W. By all the properties of each $\alpha_{(s,t)}$ mentioned above and Equation (3.3), it suffices to only consider elements of the form $V^m W^n$, with $m, n \in \mathbb{Z}$, and show that $\alpha_{(s,t)}(V^m W^n) \in \mathcal{A}_\theta$. So, for every $(s,t) \in \mathbb{R}^2$, $[x,y] \in \mathbb{T}^2$ and $f \in L^2(\mu)$, we have

$$\begin{aligned}
\left(\alpha_{(s,t)}(V^m W^n) f\right)[x,y] &= U_{(s,t)}(V^m W^n U_{(s,t)}^* f)[x,y] \\
&= (V^m W^n U_{(s,t)}^* f)[x+s, y+t] \\
&= e^{i(s+x)m}(W^n U_{(s,t)}^* f)[x+s, y+t-\frac{\theta}{2}m] \\
&= e^{i(s+x)m} e^{i(t+y-\frac{\theta}{2}m)n}(U_{(s,t)}^* f)[x+s+\frac{\theta}{2}n, y+t-\frac{\theta}{2}m] \\
&= e^{i(s+x)m} e^{i(t+y-\frac{\theta}{2}m)n} f[x+\frac{\theta}{2}n, y-\frac{\theta}{2}m] \\
&= e^{ism} e^{itn} V^m W^n f[x,y],
\end{aligned}$$

showing that $\alpha_{(s,t)}(V^m W^n) = e^{ism} e^{itn} V^m W^n$, which is clearly in the *-algebra generated by V and W. Hence we have $\alpha_{(s,t)}(\mathcal{A}_\theta) \subseteq \mathcal{A}_\theta$ and $\alpha_{(-s,-t)}(\mathcal{A}_\theta) \subseteq \mathcal{A}_\theta$, so $\alpha_{(s,t)}(\mathcal{A}_\theta) = \mathcal{A}_\theta$. This and the fact that $\alpha_{(s,t)}$ is clearly a *-automorphism of $B(L^2(\mu))$ imply it is a *-automorphism of $\mathcal{A}_\theta$. Hence we have a C*-dynamical system on the non-commutative torus, given by $(\mathcal{A}_\theta, \alpha)$.

As a second example we consider an $\mathbb{R}$-action on the noncommutative torus. The action will be the same as in the $\mathbb{R}^2$ case, with the exception that one of the variables will be kept fixed. So, we take as our $\mathbb{R}$-action β defined by $\beta_s = \alpha_{(s,0)}$, $s \in \mathbb{R}$, giving the C*-dynamical system $(\mathcal{A}_\theta, \beta)$.

3.3 The q-commutation relations

The non-commutative torus dealt with in the previous example is an example of a deformation governed by a parameter in the specific commutation relation. The following example is also a deformation satisfying some commutation relation, namely the q-commutation relation given by

$$a(f)a^*(g) - q a^*(g) a(f) = \langle f, g \rangle I, \qquad \text{for } -1 \le q \le 1 \text{ and } f, g \in \mathfrak{H},$$

where $\mathfrak{H}$ is a Hilbert space, I is the identity operator and $a(\cdot)$ is a linear operator on $\mathfrak{H}$. We will consider the work done by Dykema and Fidaleo in [8] and look at the shift on the C*-algebra generated by the operators $a(\cdot)$ on a twisted (or deformed) Fock space. Our focus for now, will be on the twisted Fock space as a representation space, and as such our starting point will be the construction of this space as was done by Bożejko and Speicher in [2], after which we turn to the construction of a C*-dynamical system as done in [8].

A fair amount of work has been done on q-commutation relations in various areas, and thus as a matter of interest a little background is in order. Much of the study of these relations are in non-commutative probability theory and statistics, which was Bożejko and Speicher's motivation in [2], where they constructed the twisted Fock space as a way to obtain a generalized (or non-commutative) Brownian motion. They used an interpolation, via some parameter, say q, between the Bosonic and the Fermionic commutation relations corresponding to $q = 1$ and $q = -1$, respectively, which generalizes the Fock space representation of the canonical commutation relations (CCR), canonical anti-commutation relations (CAR), and Cuntz (corresponding to q=0) algebras. As the names Bosonic and Fermionic suggest, these commutation relations have their origin in physical theories, although this generalization does not necessarily have this in mind. In physics the symmetric (Bosonic) Fock space and the anti-symmetric (Fermionic) Fock space were introduced to allow as many particles as required, together with the creation and annihilation operators to accommodate the creation and annihilation of particles in quantum systems. Besides non-commutative probability theory (and statistics) and physical considerations, Dykema and Nica, and later Kennedy and Nica considered the C*-algebras generated by these relations in more detail in [9] and [17], respectively, where in [9] it was shown that for certain values of the parameter q, these C*-algebras are isomorphic up to unitary equivalence, and in [17] it was shown that this C*-algebra is exact. These examples of work done in the q-commutation relations are by no means indicative of the scope of applications thereof, but more an indication of origin and personal interest. They are also used in areas of algebra, for example combinatorics (we will shortly see the combinatorial nature of some of the proofs) and generalizations of orthogonal polynomials, to mention but two (see for example [10] and [14]).

Our aim now is to present Bożejko and Speicher's construction of the deformed Fock space as a representation space. Due to the nature of the Fock space, the author assumes familiarity with tensor products of Hilbert spaces. We will not make use of orthonormal bases of the Hilbert space in question in the construction of the twisted Fock space. Only when we specify the Hilbert space in the construction of the C*-dynamical system will this come into play. The reason for this approach is to give the explicit definitions of the creation and annihilation operators and the inner product on the twisted Fock space for arbitrary vectors. From these the definitions in terms of basis vectors should be clear. Thus, we need to fix some notation and terminology. Let $\mathfrak{H}$ be an arbitrary separable Hilbert space, $q \in [-1, 1]$, and let $f, g_j, h_j, j = 1, 2, \dots$ be elements in the Hilbert space. Now, by the full Fock space we mean the Hilbert space

$$\mathcal{F}_0 = \mathbb{C}\Omega \bigoplus \left(\bigoplus_{n=1}^{\infty} \mathfrak{H}^{\otimes n} \right)$$
$$(= \mathbb{C}\Omega \bigoplus \mathfrak{H} \bigoplus (\mathfrak{H} \otimes \mathfrak{H}) \bigoplus (\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}) \bigoplus \dots)$$

where Ω is any unit vector in $\mathfrak{H}$, denoting the *vacuum vector*. We shall denote the inner product, with respect to which $\mathcal{F}_0$ is complete, by $\langle \cdot, \cdot \rangle_0$. We will also denote the n-particle space by $\mathcal{F}^{(n)} = \mathfrak{H}^{\otimes n}$, which is spanned by all tensors of length n, with $\mathcal{F}^{(0)} = \mathbb{C}\Omega$, and all inner products on $\mathcal{F}^{(n)}, n = 0, 1, 2, \dots$, by $\langle \cdot, \cdot \rangle$. Now, let $\mathcal{F}$ denote the the linear span of

$$\{\Omega\} \cup \{f_1 \otimes \dots \otimes f_n : f_j \in \mathfrak{H},\ j = 1, ..., n,\ n = 1, 2, \dots\},$$

which is dense in the full Fock space $\mathcal{F}_0$. We will define (by linear extension) our operators and new inner product on $\mathcal{F}$, from which we can then take the completion with respect to the new inner product (to obtain the twisted Fock space $\mathcal{F}_q$) and extend the operators by continuity to the whole space (except for $q = 1$). So, for every $f, h_j \in \mathfrak{H}$ we define the creation and annihilation operators on $\mathcal{F}$, respectively, by

$$a^*(f)\Omega = f,$$
$$a^*(f)(h_1 \otimes \dots \otimes h_n) = f \otimes h_1 \otimes \dots \otimes h_n$$

and

$$a(f)\Omega = 0$$
$$a(f)(h_1 \otimes \dots \otimes h_n) = \sum_{k=1}^{n} q^{k-1} \langle f, h_k \rangle h_1 \otimes \dots \otimes \check{h}_k \otimes \dots \otimes h_n,$$

(where $\check{h}_k$ means that the term h_k must be deleted from the product). We show that the operator $a^*(f)$ is well defined. Let $\mathfrak{H}_i, i = 1, 2, \dots, n$, and $\mathfrak{K}$ be Hilbert spaces. From the universal property of tensor products we know there exists a multilinear (actually a weak Hilbert-Schmidt) mapping $p : \mathfrak{H}_1 \times \dots \times \mathfrak{H}_n \to \mathfrak{H}_1 \otimes \dots \otimes \mathfrak{H}_n$, taking $(f_1, \dots, f_n) \mapsto f_1 \otimes \dots \otimes f_n$, such that given any multilinear mapping $L : \mathfrak{H}_1 \times \dots \times \mathfrak{H}_n \to \mathfrak{K}$, then there exists a unique linear mapping $T : \mathfrak{H}_1 \otimes \dots \otimes \mathfrak{H}_n \to \mathfrak{K}$, such that $L = Tp$ (see for example [15, Thm 2.6.4., p. 132]). Now, in our context, let $L_f : \mathfrak{H}_1 \times \dots \times \mathfrak{H}_n \to \mathfrak{H}_1 \otimes \dots \otimes \mathfrak{H}_n \otimes \mathfrak{H}_{n+1}$ be defined by $L_f(h_1, \dots, h_n) =$

$f \otimes h_1 \otimes \ldots \otimes h_n$, which is clearly well defined and multilinear for every $f, h_i \in \mathfrak{H}, i = 1, \ldots, n$. Hence there exists a unique multilinear map T_f such that $L_f = T_f p$, for every $f \in \mathfrak{H}$, i.e. $T_f = a^*(f)$. Similarly, from the universal property of tensor products, we have that $a(f)$ is well defined for every $f \in \mathfrak{H}$.

Lemma 3.3.1. *[2, Lem 1] For every $f, g \in \mathfrak{H}$, the operators $a(f)$ and $a^*(g)$ satisfy the relation*

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle I.$$

Proof. Consider any $h_j \in \mathfrak{H}, j = 1, \ldots, n$. Then

$$\begin{aligned}
&[a(f)a^*(g)](h_1 \otimes \ldots \otimes h_n) \\
&= a(f)(g \otimes h_1 \otimes \ldots \otimes h_n) \\
&= \langle f, g\rangle h_1 \otimes \ldots \otimes h_n + \sum_{k=1}^{n} q^k \langle f, h_k\rangle g \otimes h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \\
&= \langle f, g\rangle h_1 \otimes \ldots \otimes h_n + q \sum_{k=1}^{n} q^{k-1} \langle f, h_k\rangle g \otimes h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \\
&= \langle f, g\rangle h_1 \otimes \ldots \otimes h_n + q \left[g \otimes \left(\sum_{k=1}^{n} q^{k-1} \langle f, h_k\rangle h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \right) \right] \\
&= \langle f, g\rangle h_1 \otimes \ldots \otimes h_n + q[g \otimes a(f)(h_1 \otimes \ldots \otimes h_n)] \\
&= \langle f, g\rangle h_1 \otimes \ldots \otimes h_n + [qa^*(g)a(f)](h_1 \otimes \ldots \otimes h_n) \\
&= [\langle f, g\rangle I + qa^*(g)a(f)](h_1 \otimes \ldots \otimes h_n)
\end{aligned}$$

□

We now move on to define our inner product on $\mathcal{F}$, and then show that with this inner product the operators $a(f)$ and $a^*(f)$ are adjoints of each other. Hence, we define the mapping $\langle \cdot, \cdot \rangle_q : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ by

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_m\rangle_q = 0 \quad \text{if } m \neq n,$$

$$\langle \Omega, h_1 \otimes \ldots \otimes h_m\rangle_q = 0 \quad \text{if } m \geq 1,$$
$$\langle \Omega, \Omega\rangle_q = \langle \Omega, \Omega\rangle = 1,$$

and otherwise recursively by

$$\begin{aligned}
&\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n\rangle_q \\
&= \langle g_2 \otimes \ldots \otimes g_n, a(g_1)h_1 \otimes \ldots \otimes h_n\rangle_q \\
&= \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k\rangle \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n\rangle_q.
\end{aligned}$$

We shall denote the corresponding norm by $||\cdot||_q$, and as we shall see because of Lemma 3.3.4 we have $||\cdot||_q = ||\cdot||$ on $\mathbb{C}\Omega \bigoplus \mathfrak{H}$. That the mapping above is well defined again follows from the universal property of tensor products, in the same way as was shown in the definition of the operators $a^*(\cdot)$ and $a(\cdot)$. To show that the mapping above is indeed an inner product, we would like to express it in terms of the usual inner product on the full Fock space, in the form $\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0$, for some linear operator $P_q : \mathcal{F} \to \mathcal{F}$. Once we establish the positive definiteness (via the strict positivity of P_q), we can deduce all the axioms of an inner product from $\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0$. To aid us in this we will use a unitary representation on the symmetric group of a set with n elements, denoted by S_n, that is the the group of all permutations on a set of n objects.

Definition 3.3.2. A *unitary representation* of a locally compact group $\mathcal{G}$ is a mapping, say ϕ, from $\mathcal{G}$ into the group of unitary operators on some Hilbert space, such that ϕ is a homomorphism and continuous with respect to the strong operator topology.

Lemma 3.3.3. *The mapping $\phi : S_n \to B(\mathcal{F}^{(n)}), \pi \mapsto U_\pi$, where*

$$U_\pi(h_1 \otimes \ldots \otimes h_n) = h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)},$$

is a unitary representation.

Proof. From the universal property of the tensor product, we have that U_π is well defined for every $\pi \in S_n$ on a dense subspace of $\mathcal{F}^n$ containing all simple tensors. We show that U_π is bounded on a dense subspace of $\mathcal{F}^n$ for every $\pi \in S_n$, so that it can be extended by continuity to $\mathcal{F}^n$. Let $\pi \in S_n$ and let $g_j, h_j \in \mathfrak{H}, j = 1, \ldots, n$. Then

$$\begin{aligned}
&\langle U_\pi(g_1 \otimes \ldots \otimes g_n), U_\pi(h_1 \otimes \ldots \otimes h_n)\rangle \\
&= \langle g_{\pi(1)} \otimes \ldots \otimes g_{\pi(n)}, h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)}\rangle \\
&= \langle g_{\pi(1)}, h_{\pi(1)}\rangle \cdots \langle g_{\pi(n)}, h_{\pi(n)}\rangle \\
&= \langle g_1, h_1\rangle \cdots \langle g_n, h_n\rangle \\
&= \langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n\rangle.
\end{aligned}$$

If we now consider any finite linear combination of simple tensors, say $\Sigma_{i=1}^m h_{1,i} \otimes \ldots \otimes h_{n,i}$, then it follows that

$$\begin{aligned}
&||U_\pi(\Sigma_{i=1}^m h_{1,i} \otimes \ldots \otimes h_{n,i})||^2 \\
&= \langle \Sigma_{i=1}^m h_{\pi(1,i)} \otimes \ldots \otimes h_{\pi(n,i)}, \Sigma_{i=1}^m h_{\pi(1,i)} \otimes \ldots \otimes h_{\pi(n,i)}\rangle \\
&= \Sigma_{i=1}^m \Sigma_{j=1}^m \langle h_{\pi(1,i)} \otimes \ldots \otimes h_{\pi(n,i)}, h_{\pi(1,j)} \otimes \ldots \otimes h_{\pi(n,j)}\rangle \\
&= \Sigma_{i=1}^m \Sigma_{j=1}^m \langle h_{1,i} \otimes \ldots \otimes h_{n,i}, h_{1,j} \otimes \ldots \otimes h_{n,j}\rangle \\
&= \langle \Sigma_{i=1}^m h_{1,i} \otimes \ldots \otimes h_{n,i}, \Sigma_{i=1}^m h_{1,i} \otimes \ldots \otimes h_{n,i}\rangle \\
&= ||\Sigma_{i=1}^m h_{1,i} \otimes \ldots \otimes h_{n,i}||^2,
\end{aligned}$$

where the third equality follows by rearranging the order of the terms $\langle h_{\pi(k,i)}, h_{\pi(k,j)}\rangle$. Hence $U_\pi \in B(\mathcal{F}^n)$.

Also, we have for every $\pi, \sigma \in S_n$ and $g_j, h_j \in \mathfrak{H}, j = 1, \dots, n$,

$$\begin{aligned} U_{\pi\circ\sigma}(h_1 \otimes \dots \otimes h_n) &= h_{\pi\circ\sigma(1)} \otimes \dots \otimes h_{\pi\circ\sigma(n)} \\ &= U_\pi(h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(n)}) \\ &= U_\pi \circ U_\sigma(h_1 \otimes \dots \otimes h_n), \end{aligned}$$

so that ϕ is a homomorphism. And

$$\begin{aligned} \langle U_\pi^*(g_1 \otimes \dots \otimes g_n), h_1 \otimes \dots \otimes h_n\rangle &= \langle g_1 \otimes \dots \otimes g_n, U_\pi(h_1 \otimes \dots \otimes h_n)\rangle \\ &= \langle g_1 \otimes \dots \otimes g_n, h_{\pi(1)} \otimes \dots \otimes h_{\pi(n)}\rangle \\ &= \langle g_1, h_{\pi(1)}\rangle \cdots \langle g_n, h_{\pi(n)}\rangle \\ &= \langle g_{\pi^{-1}(1)}, h_1\rangle \cdots \langle g_{\pi^{-1}(n)}, h_n\rangle \\ &= \langle U_{\pi^{-1}}(g_1 \otimes \dots \otimes g_n), h_1 \otimes \dots \otimes h_n\rangle, \end{aligned}$$

showing that $U_\pi^* = U_{\pi^{-1}} = U_\pi^{-1}$. The strong operator continuity is clear with the discrete topology on S_n. $\square$

We will denote the number of inversions of any $\pi \in S_n$ by $i(\pi)$, that is

$$i(\pi) = |\{(j,k) \in \{1,2,...,n\}^2 : j < k, \pi(j) > \pi(k)\}|$$

(where $|A|$ denotes the cardinality of the set A).

Now, we let $P_q = \oplus_{n=0}^\infty P_q^{(n)}$, where $P_q^{(n)} : \mathcal{F}^{(n)} \to \mathcal{F}^{(n)}$ is defined by

$$P_q^{(n)} = \sum_{\pi\in S_n} q^{i(\pi)} U_\pi.$$

Note that $P_q^{(0)} = I$ and $P_q^{(1)} = I$ (with I the identity operator on their respective spaces), so that the inner product is unchanged for $n = 0, 1$, i.e. $\langle\cdot,\cdot\rangle_q = \langle\cdot,\cdot\rangle_0$ if restricted to $\mathbb{C}\Omega \bigoplus \mathfrak{H}$. We also note that $P_q^{(n)}$ is a bounded operator on $\mathfrak{H}^{\otimes n}$ for every $n \in \mathbb{N}$. Indeed, for any $f_j \in \mathfrak{H}$, $j = 1, \dots, n$, let $m = \max_{\pi\in S_n}\{|q^{i(\pi)}|\}$. Then

$$\begin{aligned} ||P_q^{(n)}(f_1 \otimes \dots \otimes f_n)|| &= ||\sum_{\pi\in S_n} q^{i(\pi)} U_\pi(f_1 \otimes \dots \otimes f_n)|| \\ &\le \sum_{\pi\in S_n} |q^{i(\pi)}| ||U_\pi(f_1 \otimes \dots \otimes f_n)|| \\ &\le (n!)m||f_1 \otimes \dots \otimes f_n||. \end{aligned}$$

Lemma 3.3.4. *[2, Lem 3] We have for every $\xi, \eta \in \mathcal{F}$,*

$$\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0.$$

Proof. It will be sufficient to prove it for all $g_j, h_j \in \mathfrak{H}, j = 1, \cdots, n$, and every $n \in \mathbb{N}$ that

$$\begin{aligned} \langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_q &= \langle g_1 \otimes \ldots \otimes g_n, P_q^{(n)} h_1 \otimes \ldots \otimes h_n \rangle_0 \\ &= \sum_{\pi \in S_n} q^{i(\pi)} \langle g_1, h_{\pi(1)} \rangle \ldots \langle g_n, h_{\pi(n)} \rangle. \end{aligned}$$

We show this by induction. The case $n = 1$ is obvious. Now, assume the hypothesis true for $n-1$. Let $S_{n-1}^{(k)}$ denote the set of all bijections from the set $\{2, \ldots, n\}$ onto the set $\{1, \ldots, k-1, k+1, \ldots, n\}$. Let $\pi \in S_n$ and $\pi(1) = k$ for some $k \in \{1, 2, \ldots, n\}$. Then we can write π as a composition of the transposition $(1\ k)$ and the element $\pi' \in S_n$ given by

$$\pi' = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n \\ \pi(k) & \pi(2) & \cdots & \pi(k-1) & k & \pi(k+1) & \cdots & \pi(n) \end{pmatrix},$$

namely $\pi = \pi' \circ (1\ k)$. There is a one-to-one correspondence between elements $\pi' \in S_n$ with $\pi'(k) = k$ and elements $\sigma \in S_{n-1}^{(k)}$, and we also have $i(\pi') = i(\sigma)$. There are $k-1$ inversions in the transposition $(1\ k)$, hence we have that

$$i(\pi) = k - 1 + i(\pi') = k - 1 + i(\sigma).$$

Summing over all $\pi \in S_n$ therefore is the same as summing over all $\sigma \in S_{n-1}^{(k)}$, for each $k = 1, 2, \ldots, n$ and adding the n sums, i.e. symbolically

$$\sum_{\pi \in S_n} = \sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}^{(k)}}.$$

Thus, assuming the hypothesis holds for $n-1$, we have

$$\begin{aligned} &\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_q \\ &= \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k \rangle \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_q \\ &= \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k \rangle \sum_{\sigma \in S_{n-1}^{(k)}} q^{i(\sigma)} \langle g_2, h_{\sigma(2)} \rangle \cdots \langle g_n, h_{\sigma(n)} \rangle \\ &= \sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}^{(k)}} q^{k-1+i(\sigma)} \langle g_1, h_k \rangle \langle g_2, h_{\sigma(2)} \rangle \cdots \langle g_n, h_{\sigma(n)} \rangle \\ &= \sum_{\pi \in S_n} q^{i(\pi)} \langle g_1, h_{\pi(1)} \rangle \cdots \langle g_n, h_{\pi(n)} \rangle. \end{aligned}$$

□

The final task is to show the positive definiteness of the inner product, by showing that P_q is positive.

Definition 3.3.5. Let $\mathcal{G}$ be a finite group. We call a function $\phi : \mathcal{G} \to \mathbb{C}$ *positive definite* if

$$\sum_{s,t\in\mathcal{G}} \phi(s^{-1}t)\overline{\psi(s)}\psi(t) \geq 0$$

for every $\psi : \mathcal{G} \to \mathbb{C}$, and *strictly positive definite* if

$$\sum_{s,t\in\mathcal{G}} \phi(s^{-1}t)\overline{\psi(s)}\psi(t) > 0$$

for every $\psi : \mathcal{G} \to \mathbb{C}$.

The definition above can be generalized to arbitrary groups, but this is unnecessary since we will only need it in S_n.

Proposition 3.3.6. *[5] The point-wise product of two positive definite functions is again positive definite.*

Proof. Let $\mathcal{G}$ be a finite group, and $\phi : \mathcal{G} \to \mathbb{C}$ any function. Consider the set $\mathcal{A} = \{(a_g)_{g\in\mathcal{G}} : a_g \in \mathbb{C} \text{ for every } g \in \mathcal{G}\}$ of all functions from $\mathcal{G}$ to $\mathbb{C}$. Then $\mathcal{A}$ is a vector space if we define addition and scalar multiplication by

$$(a_g)_{g\in\mathcal{G}} + (b_g)_{g\in\mathcal{G}} = (a_g + b_g)_{g\in\mathcal{G}}$$
$$\lambda(a_g)_{g\in\mathcal{G}} = (\lambda a_g)_{g\in\mathcal{G}},$$

respectively. For each $a \in \mathcal{A}$, we will use the notation
$a = (a_g)_{g\in\mathcal{G}} = \sum_{g\in\mathcal{G}} a_g\delta_g$, where $\delta_g : \mathcal{G} \to \mathbb{C}$ is defined by

$$\delta_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g, \end{cases}$$

for every $g \in \mathcal{G}$. Note that $\delta_g \in \mathcal{A}$ for every $g \in \mathcal{G}$, and moreover, the set $\{\delta_g : g \in \mathcal{G}\}$ forms a basis for $\mathcal{A}$. We go further, and define a multiplication on $\mathcal{A}$ by

$$ab = (\sum_{g\in\mathcal{G}} a_g\delta_g)(\sum_{h\in\mathcal{G}} b_h\delta_h) = \sum_{g\in\mathcal{G}}\sum_{h\in\mathcal{G}} a_g b_h \delta_{gh},$$

and an involution on $\mathcal{A}$ by

$$a^* = (\sum_{g\in\mathcal{G}} a_g\delta_g)^* = \sum_{g\in\mathcal{G}} \overline{a_g}\delta_{g^{-1}}.$$

Then it is easily seen that $\mathcal{A}$ is a $*$-algebra, called the *group $*$-algebra* of $\mathcal{G}$.

Via the δ_g's we have that $\mathcal{G} \subset \mathcal{A}$, and we can thus extend ϕ linearly to $\mathcal{A}$ by defining $\phi(a) = \sum_{g \in \mathcal{G}} a_g \phi(g)$, which is well defined because $\{\delta_g : g \in \mathcal{G}\}$ is a basis for $\mathcal{A}$. Thus ϕ is a linear functional on $\mathcal{A}$, and will be positive if $\phi(a^* a) \geq 0$ for every $a \in \mathcal{A}$. But since

$$\phi(a^* a) = \sum_{g,h \in \mathcal{G}} \overline{a_g} a_h \phi(g^{-1} h),$$

it follows that $\phi : \mathcal{G} \to \mathbb{C}$ is positive definite if and only if the linear functional it induces on $\mathcal{A}$ is positive.

Now, let $\phi, \psi : \mathcal{G} \to \mathbb{C}$ be any two positive definite functions, and extend them to two positive linear functionals on $\mathcal{A}$, as above. From the GNS construction we obtain the two cyclic representations $(\mathfrak{H}_\phi, \pi_\phi, \xi_\phi)$ and $(\mathfrak{H}_\psi, \pi_\psi, \xi_\psi)$ induced by ϕ and ψ, respectively (see comment on p.13). Then $\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle$ and $\psi(a) = \langle \xi_\psi, \pi_\psi(a)\xi_\psi \rangle$ for every $a \in \mathcal{A}$. Let $\xi = \xi_\phi \otimes \xi_\psi$ and define $\pi : \mathcal{G} \to B(\mathfrak{H}_\phi \otimes \mathfrak{H}_\psi)$ by

$$\pi(g) = \pi_\phi(g) \otimes \pi_\psi(g),$$

or more precisely by

$$\pi(\delta_g) = \pi_\phi(\delta_g) \otimes \pi_\psi(\delta_g),$$

since $\mathcal{G}$ is embedded in $\mathcal{A}$ via the δ_g's and $B(\mathfrak{H}_\phi) \otimes B(\mathfrak{H}_\psi) \subseteq B(\mathfrak{H}_\phi \otimes \mathfrak{H}_\psi))$. We extend π linearly to $\mathcal{A}$ by letting

$$\pi(\sum_{g \in \mathcal{G}} a_g \delta_g) = \sum_{g \in \mathcal{G}} a_g \pi(\delta_g).$$

Then we have for every $a, b \in \mathcal{A}$ that

$$\begin{aligned}
\pi(ab) &= \sum_{g,h \in \mathcal{G}} a_g b_h \pi(\delta_{gh}) \\
&= \sum_{g,h \in \mathcal{G}} a_g b_h [\pi_\phi(\delta_{gh}) \otimes \pi_\psi(\delta_{gh})] \\
&= \sum_{g,h \in \mathcal{G}} a_g b_h [\pi_\phi(\delta_g)\pi_\phi(\delta_h)] \otimes [\pi_\psi(\delta_g)\pi_\psi(\delta_h)] \\
&= \sum_{g,h \in \mathcal{G}} a_g b_h [\pi_\phi(\delta_g) \otimes \pi_\psi(\delta_g)][\pi_\phi(\delta_h) \otimes \pi_\psi(\delta_h)] \\
&= \left(\sum_{g \in \mathcal{G}} a_g [\pi_\phi(\delta_g) \otimes \pi_\psi(\delta_g)] \right) \left(\sum_{h \in \mathcal{G}} b_h [\pi_\phi(\delta_h) \otimes \pi_\psi(\delta_h)] \right) \\
&= \pi(a)\pi(b),
\end{aligned}$$

and since $\delta_g^* = \delta_{g^{-1}}$

$$\begin{aligned}
\pi(a^*) &= \sum_{g\in\mathcal{G}} \overline{a_g}\pi(\delta_{g^{-1}}) \\
&= \sum_{g\in\mathcal{G}} \overline{a_g}[\pi_\phi(\delta_{g^{-1}}) \otimes \pi_\psi(\delta_{g^{-1}})] \\
&= \sum_{g\in\mathcal{G}} \overline{a_g}[\pi_\phi(\delta_g^*) \otimes \pi_\psi(\delta_g^*)] \\
&= \sum_{g\in\mathcal{G}} \overline{a_g}[\pi_\phi(\delta_g)^* \otimes \pi_\psi(\delta_g)^*] \\
&= \sum_{g\in\mathcal{G}} \overline{a_g}\pi(\delta_g)^* \\
&= \pi(a)^*.
\end{aligned}$$

Hence π is a representation of $\mathcal{A}$, and thus defines a positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ via $\omega(a) = \langle \xi, \pi(a)\xi \rangle$, and as shown above, its restriction to $\mathcal{G}$ is positive definite. But this restriction is given by

$$\begin{aligned}
\omega(g) \equiv \omega(\delta_g) &= \langle \xi, \pi(\delta_g)\xi \rangle \\
&= \langle \xi_\phi \otimes \xi_\psi, [\pi_\phi(\delta_g) \otimes \pi_\psi(\delta_g)](\xi_\phi \otimes \xi_\psi) \rangle \\
&= \langle \xi_\phi, \pi_\phi(\delta_g)\xi_\phi \rangle \langle \xi_\psi, \pi_\psi(\delta_g)\xi_\psi \rangle \\
&= \phi(g)\psi(g),
\end{aligned}$$

for all $g \in \mathcal{G}$. Hence the product, $\phi\psi$, of two positive definite functions is again positive definite. □

Proposition 3.3.7. *The point-wise product of two strictly positive definite functions is again strictly positive definite.*

Proof. This follows from the proof of Proposition 3.3.6 by noting that the function $\phi : \mathcal{G} \to \mathbb{C}$ is strictly positive definite if and only if the induced linear functional $\phi : \mathcal{A} \to \mathbb{C}$ satisfies $\phi(a^*a) \geq 0$ for every $a \in \mathcal{A}$. This amounts to ϕ being faithful, i.e. if for every $a \in \mathcal{A}$, $\phi(a^*a) = 0$ implies $a = 0$.

□

Let S_n still denote the symmetric group on n elements, and we let $\pi_j, 0 < j < n$, denote the transpositions in S_n such that π_j interchanges j and $j+1$, and can thus be written as the cycle $\pi_j = (j \;\; j+1)$.

Lemma 3.3.8. *[7] Let $\pi \in S_n$. If π is not the identity, there is a $j \in \{1, 2, \ldots, n\}$ such that for every $i \in \{1, 2, \ldots, j\}$ there are numbers $l(i) \in \{1, 2, \ldots, n\}$ and $r(i) \in \{0, 1, \ldots, n-2\}$ such that $l(1) < l(2) < \cdots < l(j)$ and π can be written as a product of transpositions of the form*

$$\begin{aligned}\pi &= (\pi_{l(j)}\pi_{l(j)+1}\cdots\pi_{l(j)+r(j)})(\pi_{l(j-1)}\pi_{l(j-1)+1}\cdots\pi_{l(j-1)+r(j-1)})\cdots \\ &\quad (\pi_{l(1)}\cdots\pi_{l(1)+r(1)}).\end{aligned}$$

If π is the identity, we put $j = 0$. The number of inversions of π is given by

$$i(\pi) = (r(1)+1) + (r(2)+1) + \cdots + (r(j)+1).$$

Proof. Let $l(1) \in \{1, 2, \ldots, n\}$ be the least number such that $\pi(l(1)) \neq l(1)$, and let $r(1) \geq 0$ be such that $\pi(l(1)+r(1)+1) = l(1)$. Note that if $r(1) = 0$ (or $j = 0$ in the general form stated in the hypothesis) then π is just the identity.

Let $\sigma \in S_n$ be such that

$$\pi = \sigma\pi_{l(1)}\pi_{l(1)+1}\cdots\pi_{l(1)+r(1)}. \tag{3.4}$$

If σ is the identity, then we are done. If not, let $l(2) \in \{1, 2, \cdots, n\}$ be the least number such that $\sigma(l(2)) \neq l(2)$. We claim that $l(2) > l(1)$. From (3.4) we have that

$$\begin{aligned}\sigma &= \pi(\pi_{l(1)}\pi_{l(1)+1}\cdots\pi_{l(1)+r(1)})^{-1} \\ &= \pi(\pi_{l(1)+r(1)}\pi_{l(1)+r(1)-1}\cdots\pi_{l(1)}) \\ &= \pi \circ \begin{pmatrix} l(1) & l(1)+1 & l(1)+2 & \cdots & l(1)+r(1)+1 \\ l(1)+r(1)+1 & l(1) & l(1)+1 & \cdots & l(1)+r(1) \end{pmatrix}\end{aligned} \tag{3.5}$$

(where the notation in (3.5) is just the cycle obtained from the product of the transpositions). Thus, from (3.5) we see that $\sigma(l(1)) = l(1)$ and for any $k < l(1)$, we also have that $\sigma(k) = k$. Hence $l(1) < l(2)$. Let $r(2) \geq 0$ be such that $\sigma(l(2)+r(2)+1) = l(2)$. Then for some $\varrho \in S_n$,

$$\sigma = \varrho\pi_{l(2)}\pi_{l(2)+1}\cdots\pi_{l(2)+r(2)}$$

and

$$\pi = \varrho\pi_{l(2)}\pi_{l(2)+1}\cdots\pi_{l(2)+r(2)}\pi_{l(1)}\pi_{l(1)+1}\cdots\pi_{l(1)+r(1)}.$$

If ϱ is the identity, then we are done. Otherwise we proceed inductively as we did above. The fact that $l(j) < l(j+1)$ and $\{1, 2, \ldots, n\}$ is finite, shows that the inductive process is finite, and thus so too the product of transpositions.

As for the inversions, notice that $l(j)+p < l(j)+k, 0 \leq p < k \leq r(j)+1$, and writing any product of transpositions as a cycle, i.e.

$$\pi_{l(j)}\pi_{l(j)+1}\cdots\pi_{l(j)+r(j)}$$
$$= \begin{pmatrix} l(j) & l(j)+1 & l(j)+2 & \cdots & l(j)+r(j) & l(j)+r(1)+1 \\ l(j)+1 & l(j)+2 & l(j)+3 & \cdots & l(j)+r(j)+1 & l(j) \end{pmatrix},$$

we can clearly see that the image of $l(j)$ changes order with each of the other $r(j)+1$ positions' images, and is the only one to change order. Thus, for this product of transpositions there are $r(j)+1$ inversions, and adding all such products representing an element gives the required result. $\square$

Lemma 3.3.9. *The function $f : S_n \to \{-1, 1\}$ defined by $\pi \mapsto (-1)^{i(\pi)}$ is positive definite.*

Proof. We begin by showing that f is a nonzero homomorphism. It is well known that any permutation, say $\pi \in S_n$, can be written as a product of transpositions, where the decomposition is not necessarily unique, but the number of transpositions of every such decomposition of π is either even or odd, and is related to the so called sign of the permutation. But, by Lemma 3.3.8 we can express π as product of transpositions, say $\pi = \pi_1 \cdots \pi_k$, such that the number of inversions of π is given by $i(\pi) = k$. Hence if π is decomposed into any other product of transpositions, say m transpositions, then $(-1)^{i(\pi)} = (-1)^k = (-1)^m$, since k and m are either both even or both odd. Let $\sigma \in S_n$, and let $\sigma = \sigma_1 \cdots \sigma_l$ be its decomposition into transpositions given by Lemma 3.3.8. Then the product $\pi\sigma = \pi_1 \cdots \pi_k \sigma_1 \cdots \sigma_l$ can be written as a product of $k + l$ transpositions, and thus $(-1)^{i(\pi\sigma)} = (-1)^{k+l}$. Hence

$$f(\pi\sigma) = (-1)^{i(\pi\sigma)} = (-1)^{k+l} = (-1)^k(-1)^l = f(\pi)f(\sigma).$$

We can now show that f is positive definite. Let $\psi : S_n \to \mathbb{C}$ be any function, then

$$\begin{aligned} \sum_{\pi,\sigma\in S_n} f(\pi^{-1}\sigma)\overline{\psi(\pi)}\psi(\sigma) &= \sum_{\pi,\sigma\in S_n} f(\pi^{-1})f(\sigma)\overline{\psi(\pi)}\psi(\sigma) \\ &= \sum_{\pi,\sigma\in S_n} f(\pi)f(\sigma)\overline{\psi(\pi)}\psi(\sigma) \\ &= \left[\sum_{\pi\in S_n} f(\pi)\overline{\psi(\pi)}\right]\left[\sum_{\pi\in S_n} f(\pi)\psi(\pi)\right] \\ &= \overline{\left[\sum_{\pi\in S_n} f(\pi)\psi(\pi)\right]}\left[\sum_{\pi\in S_n} f(\pi)\psi(\pi)\right] \\ &\geq 0. \end{aligned}$$

$\square$

Definition 3.3.10. Let $\mathfrak{H}$ be a Hilbert space. Then we call $P \in B(\mathfrak{H})$ *strictly positive* if $\langle x, Px \rangle > 0$ for all $x \neq 0$.

Proposition 3.3.11. *[2, Prop 1] The operator P_q is strictly positive for all $q \in (-1, 1)$.*

Proof. We first show that P_q is positive for all $q \in (-1, 1)$. It will be sufficient to consider $P_q^{(n)}$ for all $n \in \mathbb{N}$. We start by showing that $\phi_q : S_n \to \mathbb{C}; \pi \mapsto q^{i(\pi)}$ is a positive definite function on S_n, i.e.

$$\sum_{\pi,\sigma \in S_n} q^{i(\pi^{-1}\sigma)} \overline{\psi(\pi)} \psi(\sigma) \geq 0. \tag{3.6}$$

for any $\psi : S_n \to \mathbb{C}$.

We define

$$\begin{aligned} \Phi &= \{(i,j) : i \neq j, 1 \leq i, j \leq n\}, \\ \Phi^+ &= \{(i,j) \in \Phi : i < j\}, \end{aligned}$$

and for $\pi \in S_n$ and $A \subset \Phi$,

$$\pi(A) = \{(\pi(i), \pi(j)) : (i,j) \in A\} \subset \Phi.$$

Letting $|A|$ denote the cardinality of the set A, we see, since each $\pi \in S_n$ is a bijection, that $|\pi(A)| = |A|$, for $A \subset \Phi$. We claim that $i(\pi) = |\pi(\Phi^+) \backslash \Phi^+|$. To see this, note that

$$\pi(\Phi^+) = \{(\pi(i), \pi(j)) : i < j \text{ and } \pi(i) < \pi(j)\} \bigcup \{(\pi(i), \pi(j)) : i < j \text{ and } \pi(j) < \pi(i)\},$$

where the first set in the union is contained in Φ^+ and the second set (which is precisely all the inversions) is not. Moreover, we then also have that

$$i(\pi) = i(\pi^{-1}) = |\pi^{-1}(\Phi^+) \backslash \Phi^+| = |\pi(\pi^{-1}(\Phi^+) \backslash \Phi^+)| = |\Phi^+ \backslash \pi(\Phi^+)|.$$

Let $A \Delta B = (A \backslash B) \cup (B \backslash A)$ denote the symmetric difference between the sets A and B. Then

$$\begin{aligned} 2i(\pi) &= |\pi(\Phi^+) \backslash \Phi^+| + |\Phi^+ \backslash \pi(\Phi^+)| \\ &= \left| \left(\pi(\Phi^+) \backslash \Phi^+\right) \bigcup \left(\Phi^+ \backslash \pi(\Phi^+)\right) \right| \\ &= |\pi(\Phi^+) \Delta \Phi^+|, \end{aligned}$$

which in turn implies that

$$2i(\pi^{-1}\sigma) = |\sigma(\Phi^+) \Delta \pi(\Phi^+)|. \tag{3.7}$$

Let χ_A and χ_B denote the characteristic functions on the sets $A, B \subset \Phi$, respectively. Then

$$|\chi_A(x) - \chi_B(x)| = \begin{cases} 1 \text{ if } x \in A\Delta B \\ 0 \text{ if } x \in A \cap B \end{cases} ,$$

and thus

$$|A\Delta B| = \sum_{x\in\Phi} |\chi_A(x) - \chi_B(x)| = \sum_{x\in\Phi} |\chi_A(x) - \chi_B(x)|^2. \tag{3.8}$$

First consider the case $0 < q < 1$, and put $q = e^{-\lambda}$, with $\lambda > 0$. We use the notation $\exp(x) = e^x$, so as to make notation more manageable and legible. Now, from Equations (3.7) and (3.8) we have

$$\begin{aligned} q^{i(\pi^{-1}\sigma)} &= \exp(-\lambda i(\pi^{-1}\sigma)) \\ &= \exp\left(-\frac{\lambda}{2}\left|\sigma(\Phi^+)\Delta\pi(\Phi^+)\right|\right) \\ &= \exp\left(-\frac{\lambda}{2}\sum_{x\in\Phi}\left|\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)\right|^2\right) \\ &= \prod_{x\in\Phi}\exp\left(-\frac{\lambda}{2}\left|\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)\right|^2\right). \end{aligned}$$

By Proposition 3.3.6 the pointwise product of two positive definite functions is again positive definite, and thus we need only show that for any $x \in \Phi$

$$\sum_{\pi,\sigma\in S_n} \exp\left(-\frac{\lambda}{2}\left|\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)\right|^2\right)\overline{\psi(\pi)}\psi(\sigma) \geq 0,$$

for every $\psi : S_n \to \mathbb{C}$. To do this, first fix any $\pi, \sigma \in S_n$. Putting
$y_0 = 0, y_1 = 1$ we obtain the following four possible terms:

$$\begin{aligned} \overline{\psi(\pi)}\psi(\sigma) &= \exp\left(-\frac{\lambda}{2}|y_0 - y_0|^2\right)\overline{\psi(\pi)}\psi(\sigma) \text{ if } x \notin \pi(\Phi^+) \text{ and } x \notin \sigma(\Phi^+) \\ \overline{\psi(\pi)}\psi(\sigma) &= \exp\left(-\frac{\lambda}{2}|y_1 - y_1|^2\right)\overline{\psi(\pi)}\psi(\sigma) \text{ if } x \in \pi(\Phi^+) \text{ and } x \in \sigma(\Phi^+) \\ e^{-\lambda/2}\overline{\psi(\pi)}\psi(\sigma) &= \exp\left(-\frac{\lambda}{2}|y_0 - y_1|^2\right)\overline{\psi(\pi)}\psi(\sigma) \text{ if } x \in \pi(\Phi^+) \text{ and } x \notin \sigma(\Phi^+) \\ e^{-\lambda/2}\overline{\psi(\pi)}\psi(\sigma) &= \exp\left(-\frac{\lambda}{2}|y_1 - y_0|^2\right)\overline{\psi(\pi)}\psi(\sigma) \text{ if } x \notin \pi(\Phi^+) \text{ and } x \in \sigma(\Phi^+). \end{aligned}$$

For a fixed $\psi : S_n \to \mathbb{C}$, we define the function $f : \{y_0, y_1\} \to \mathbb{C}$ by

$$f(y_0) = \sum_{\substack{\pi \in S_n \\ x \notin \pi(\Phi^+)}} \psi(\pi)$$
$$f(y_1) = \sum_{\substack{\pi \in S_n \\ x \in \pi(\Phi^+)}} \psi(\pi).$$

Now, summing over all $\pi, \sigma \in S_n$ and grouping the four types of terms (as above) together, we obtain

$$\begin{aligned} &\sum_{\pi,\sigma \in S_n} \exp\left(-\frac{\lambda}{2}\left|\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)\right|^2\right)\overline{\psi(\pi)}\psi(\sigma) \\ &= \sum_{j,k=0}^{1} \exp\left(-\frac{\lambda}{2}|y_j - y_k|^2\right)\overline{f(y_k)}f(y_j). \end{aligned} \tag{3.9}$$

We show that the expression on the right-hand side of Equation (3.9) is positive. Note this is equivalent to showing that $\bar{z}^T M z \geq 0$, where

$$z = \begin{pmatrix} f(y_0) \\ f(y_1) \end{pmatrix}$$

and

$$M = \begin{pmatrix} e^{-\frac{\lambda}{2}|y_0-y_0|^2} & e^{-\frac{\lambda}{2}|y_1-y_0|^2} \\ e^{-\frac{\lambda}{2}|y_0-y_1|^2} & e^{-\frac{\lambda}{2}|y_1-y_1|^2} \end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{\lambda}{2}} \\ e^{-\frac{\lambda}{2}} & 1 \end{pmatrix}$$

That is, we need to show that M positive semi-definite. Note that the eigenvalues of M are given by $1 \pm e^{-\frac{\lambda}{2}} \geq 0$, since $0 < e^{-\frac{\lambda}{2}} \leq 1$. Hence, since M is a real symmetric matrix, and thus hermitian (or self-adjoint), and $\sigma(M) \subset \mathbb{R}^+$, it follows that M is positive semi-definite. Hence, $\phi_q : \pi \mapsto q^{i(\pi)}$ is positive definite for $q \in (0, 1)$.

We now turn to the case were $-1 < q < 0$. Then $0 < -q < 1$. For this case we again use Proposition 3.3.6, i.e. the fact that the pointwise product of positive definite functions is a positive definite function. By Lemma 3.3.9 $\pi \mapsto (-1)^{i(\pi)}$ is positive definite and so, since

$$\phi_{-q}(\pi) = (-1)^{i(\pi)}q^{i(\pi)} = (-1)^{i(\pi)}\phi_q(\pi) \implies \phi_q(\pi) = (-1)^{i(\pi)}\phi_{-q}(\pi),$$

ϕ_q is also positive definite. For $q = 0$, we have $\phi_0(1_{S_n}) = 1$ and $\phi_0(\pi) = 0, \pi \neq 1_{S_n}$, where 1_{S_n} is the identity element in S_n. It follows from this that ϕ_0 is positive definite. Hence ϕ_q is positive definite for all $q \in (-1, 1)$.

We show that $P_q^{(n)}$ is positive, that is $\langle \eta, P_q^{(n)}\eta\rangle_0 \geq 0$ for all $\eta \in \mathcal{F}^{(n)}$. First, notice that

$$\begin{aligned}\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\langle \eta, U_{\pi^{-1}\sigma}\eta\rangle_0 &= \sum_{\sigma\in S_n}\sum_{\pi\in S_n} q^{i(\rho)}\langle \eta, U_\rho \eta\rangle_0 \\ &= n!\sum_{\pi\in S_n} q^{i(\pi)}\langle \eta, U_\pi\eta\rangle_0.\end{aligned}$$

where $\rho = \pi^{-1}\sigma$ ranges over S_n as π ranges over S_n. Hence

$$\sum_{\pi\in S_n} q^{i(\pi)}\langle \eta, U_\pi\eta\rangle_0 = \frac{1}{n!}\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\langle \eta, U_{\pi^{-1}\sigma}\eta\rangle_0. \tag{3.10}$$

Now, let $\{\xi_j : j \in \mathbb{N}\}$ be a complete orthonormal system in $\mathcal{F}^{(n)}$, that is an orthonormal set with the property that, given $y \in \mathcal{F}^{(n)}$, if $\langle y, \xi_j\rangle = 0$, $j = 1, 2, \ldots$, then $y = 0$. Then

$$\begin{aligned}\langle \eta, P_q^{(n)}\eta\rangle_0 &= \sum_{\pi\in S_n} q^{i(\pi)}\langle \eta, U_\pi\eta\rangle_0 \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\langle \eta, U_{\pi^{-1}\sigma}\eta\rangle_0 \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\langle \eta, U_{\pi^{-1}}U_\sigma\eta\rangle_0 \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\langle U_\pi\eta, U_\sigma\eta\rangle_0 \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n}\sum_{j=1}^{\infty} q^{i(\pi^{-1}\sigma)}\langle U_\pi\eta, \xi_j\rangle_0\langle \xi_j, U_\sigma\eta\rangle_0 \\ &= \frac{1}{n!}\sum_{j=1}^{\infty}\left[\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)}\overline{\langle \xi_j, U_\pi\eta\rangle_0}\langle \xi_j, U_\sigma\eta\rangle_0\right] \\ &\geq 0,\end{aligned} \tag{3.11}$$

since ϕ_q is positive definite.

We show that $P_q^{(n)}$ is strictly positive for all $q \in (-1, 1)$. By considering Inequality (3.11), it will suffice to show that ϕ_q is strictly positive definite for all $q \in (-1, 1)$. Note that Inequality (3.6) can be re-written as $\bar{z}^T Az \geq 0$, with the positive semi-definite matrix $A = (q^{i(\pi^{-1}\sigma)})_{\sigma,\pi\in S_n}$ and the vector $z = (\psi(\pi))_{\pi\in S_n}$. Hence we need to show $\bar{z}^T Az > 0$, i.e that A is (strictly) positive definite. We only show this for $q \in (0, 1)$, as the case $q \in (-1, 0)$ is analogous.

Assume that ϕ_{q_0}, for some $q_0 \in (0, 1)$, is not strictly positive definite, that is, that $\bar{z}^T Az = 0$ for some $z \neq 0$ (since we know that $\bar{z}^T Az \geq 0$ by the positive definiteness of ϕ_q). By Proposition 3.3.7 it follows that $\phi_{\sqrt{q_0}}$ is also not strictly positive definite. By taking $2k$-th roots we get an infinite number of values for q for which ϕ_q is not strictly positive definite, namely $(q_0)^{1/2k}, k = 1, 2, 3, \ldots$. Because, by assumption, A is positive semi-definite but not (strictly) positive definite, it must have 0 as an eigenvalue. Thus, since $\det(A)$ is equal to the product of its eigenvalues (by diagonalizing A), we have that $\det(A) = 0$. Now since $i(\pi) > 0$ if $\pi \neq$ *identity*, it follows that $\det(A)$ is a non-constant polynomial in q, and thus has a finite number of zeros, i.e. only a finite number of values for q such that $\det(A) = 0$, a contradiction, because from the above argument there should be an infinite number of values for q, namely $q = (q_0)^{1/2k}, k = 1, 2, 3, \ldots$, such that $\det(A) = 0$. Hence ϕ_q is strictly positive definite for $q \in (0, 1)$, and similarly for $q \in (-1, 0)$.

The case where $q = 0$ is trivial, because then Inequality (3.6) becomes

$$\sum_{\pi \in S_n} q^{i(\pi^{-1}\pi)} \overline{\psi(\pi)} \psi(\pi) = \sum_{\pi \in S_n} |\psi(\pi)|^2 > 0,$$

for $\psi \neq 0$. Hence $P_q^{(n)}$ is strictly positive for all $q \in (-1, 1)$. $\square$

By Lemma 3.3.4 and Proposition 3.3.11 we see that $\langle \cdot, \cdot \rangle_q$ is an inner product on $\mathcal{F}$, since, by the strict positivity of P_q, $\langle \xi, \xi \rangle_q = 0$ implies that $\xi = 0$. We denote the completion of $\mathcal{F}$ with respect to this inner product by $\mathcal{F}_q$, and call it the *twisted (or deformed) Fock space.*

Remark 3.3.12. Note that the proof that $P_q^{(n)}$ is positive (in Proposition 3.3.11) actually holds for $q \in [-1, 1]$, but in the cases where $q = \pm 1$ the kernel of P_q is not trivial. To see this if $q = 1$, we consider
$f_1 \otimes f_2 - f_2 \otimes f_1 \in \mathcal{F}, f_j \in \mathfrak{H}$ for $j = 1, 2$, then

$$\begin{aligned} P_1(f_1 \otimes f_2 - f_2 \otimes f_1) &= \; = P_1^{(2)}(f_1 \otimes f_2 - f_2 \otimes f_1) \\ &= \sum_{\pi \in S_2} q^{i(\pi)} U_\pi (f_1 \otimes f_2 - f_2 \otimes f_1) \\ &= f_1 \otimes f_2 - f_2 \otimes f_1 + f_2 \otimes f_1 - f_1 \otimes f_2 \\ &= 0. \end{aligned}$$

Similarly considering $f_1 \otimes f_2 + f_2 \otimes f_1$ in the case where $q = -1$ we see that $\ker(P_q) \neq \{0\}$. Therefore we have to take the completion of $\mathcal{F} / \ker(P_q)$ in the case $q = \pm 1$. The cases where $q = \pm 1$ will however not be required for the dynamical system that we will consider.

It remains to show that the creation and annihilation operators are adjoints of each other, and that they are bounded on $\mathcal{F}_q$.

Proposition 3.3.13. *[2, Lem 2] For every $f \in \mathfrak{H}$ and for every $\xi, \eta \in \mathcal{F}$, we have*

$$\langle a^*(f)\xi, \eta \rangle_q = \langle \xi, a(f)\eta \rangle_q.$$

Proof. It will suffice to prove the above equation for simple tensors in $\mathcal{F}^{(n)}$, $n \in \mathbb{N}$. Given any $g_j, h_j \in \mathfrak{H}$, it then follows from the definition of $\langle \cdot, \cdot \rangle_q$ that

$$\begin{aligned}
&\langle a^*(f)(g_1 \otimes \ldots \otimes g_n), h_1 \otimes \ldots \otimes h_{n+1} \rangle_q \\
&= \langle f \otimes g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_{n+1} \rangle_q \\
&= \langle g_1 \otimes \ldots \otimes g_n, a(f) h_1 \otimes \ldots \otimes h_{n+1} \rangle_q .
\end{aligned}$$

□

Remark 3.3.14. Consider any $\pi \in S_{n+1}$ such that $\pi(k) = 1$ for some $1 < k \leq n+1$. Then from the construction in Lemma 3.3.8 we have $l(1) = 1$ and, then $\pi(l(1) + r(1) + 1) = l(1) = \pi(k)$ implies that $r(1) = k - 2$, so that we can write

$$\pi = \sigma \pi_1 \ldots \pi_{k-1},$$

where σ is uniquely determined by the choice of π and k, $\sigma(1) = 1$ and $i(\pi) = i(\sigma) + k - 1$. Thus, we can view σ as an element in S_n. Also, every $\sigma \in S_n$ defines a (necessarily unique) π in in the set $\{\pi \in S_{n+1} : \pi(k) = 1\}$, for each $k = 2, \ldots, n$, by

$$\sigma \pi_1 \ldots \pi_{k-1} = \pi.$$

The same argument applies to the case where $k = 1$, with the exception that $l(1) \neq 1$, but then π can be viewed as an element in S_n itself, i.e. $\pi = \sigma$. Now, let $F : S_{n+1} \to V$, with V any vector space, then this enables us to write

$$\sum_{\pi \in S_{n+1}} F(\pi) = \sum_{k=1}^{n+1} \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1)=1}} F(\sigma \pi_1 \cdots \pi_{k-1}),$$

where it is understood that π_0 is the identity.

Lemma 3.3.15. *Let $-1 < q < 1$, then*

$$P_q^{(n+1)} \leq \frac{1}{1 - |q|} (I \otimes P_q^{(n)}).$$

Proof. It will be sufficient to show the inequality for $\eta \in \mathcal{F}^{(n)}, n \in \mathbb{N}_0$, since such elements span $\mathcal{F}$, which in turn is dense in $\mathcal{F}_q$. Let $h_i \in \mathfrak{H}, i =$

$1, \ldots, n+1$, then by Lemma 3.3.8 and Remark 3.3.14 it follows that

$$
\begin{aligned}
&P_q^{(n+1)}(h_1 \otimes \ldots \otimes h_{n+1}) \\
&= \sum_{\pi \in S_{n+1}} q^{i(\pi)} (h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n+1)}) \\
&= \sum_{\pi \in S_{n+1}} q^{i(\pi)} (h_{\pi^{-1}(1)} \otimes \ldots \otimes h_{\pi^{-1}(n+1)}) \\
&= \sum_{k=1}^{n+1} q^{k-1} \left[\sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1)=1}} q^{i(\sigma)} (h_{(\sigma\pi_1 \cdots \pi_{k-1})^{-1}(1)} \otimes \ldots \otimes h_{(\sigma\pi_1 \cdots \pi_{k-1})^{-1}(n+1)}) \right] \\
&= \sum_{k=1}^{n+1} q^{k-1} \left[\sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1)=1}} q^{i(\sigma)} (h_{\pi_{k-1} \cdots \pi_1 \sigma^{-1}(1)} \otimes \ldots \otimes h_{\pi_{k-1} \cdots \pi_1 \sigma^{-1}(n+1)}) \right] \\
&= \sum_{k=1}^{n+1} q^{k-1} \left[\sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1)=1}} q^{i(\sigma)} (h_{\pi_{k-1} \cdots \pi_1(1)} \otimes h_{\pi_{k-1} \cdots \pi_1 \sigma^{-1}(2)} \otimes \ldots \otimes h_{\pi_{k-1} \cdots \pi_1 \sigma^{-1}(n+1)}) \right] \\
&= \sum_{k=1}^{n+1} q^{k-1} \left[\sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1)=1}} q^{i(\sigma)} U_{\pi_{k-1} \cdots \pi_1} (h_1 \otimes h_{\sigma^{-1}(2)} \otimes \ldots \otimes h_{\sigma^{-1}(n+1)}) \right] \\
&= \left[\sum_{k=1}^{n+1} q^{k-1} U_{\pi_{k-1} \cdots \pi_1} \right] \left[I \otimes \sum_{\sigma \in S_n} q^{i(\sigma)} U_\sigma \right] (h_1 \otimes h_2 \otimes \ldots \otimes h_{n+1}) \\
&= (1 + qU_{\pi_1} + q^2 U_{\pi_2\pi_1} + \cdots + q^n U_{\pi_n \cdots \pi_1})(I \otimes P_q^{(n)})(h_1 \otimes \ldots \otimes h_{n+1}),
\end{aligned}
$$

giving the equation

$$
P_q^{(n+1)} = (1 + qU_{\pi_1} + q^2 U_{\pi_2\pi_1} + \cdots + q^n U_{\pi_n \cdots \pi_1})(I \otimes P_q^{(n)}).
$$

Hence, by the positivity of $P_q^{(n+1)}$ it is self-adjoint, and we have for every $\eta \in \mathcal{F}^{(n+1)}$

$$
\begin{aligned}
0 \leq\ & \langle P_q^{(n+1)} P_q^{(n+1)} \eta, \eta \rangle \\
=\ & ||P_q^{(n+1)} \eta||^2 \\
=\ & ||(1 + qU_{\pi_1} + q^2 U_{\pi_2\pi_1} + \cdots + q^n U_{\pi_n \cdots \pi_1})(I \otimes P_q^{(n)})\eta||^2 \\
\leq\ & (||1 + qU_{\pi_1} + q^2 U_{\pi_2\pi_1} + \cdots + q^n U_{\pi_n \cdots \pi_1}||\ ||(I \otimes P_q^{(n)})\eta||)^2 \\
\leq\ & (1 + |q| + |q|^2 + \cdots + |q|^n)^2\ ||(I \otimes P_q^{(n)})\eta||^2 \\
=\ & \langle (1 + |q| + |q|^2 + \cdots + |q|^n)(I \otimes P_q^{(n)})\eta, (1 + |q| + |q|^2 + \cdots + |q|^n)(I \otimes P_q^{(n)})\eta \rangle \\
=\ & \langle (I \otimes P_q^{(n)})(1 + |q| + |q|^2 + \cdots + |q|^n)(1 + |q| + |q|^2 + \cdots + |q|^n)(I \otimes P_q^{(n)})\eta, \eta \rangle,
\end{aligned}
$$

which implies that

$$\begin{aligned}
0 &\leq P_q^{(n+1)} P_q^{(n+1)} \\
&\leq (I \otimes P_q^{(n)})(1+|q|+|q|^2+\cdots+|q|^n)(1+|q|+|q|^2+\cdots+|q|^n)(I \otimes P_q^{(n)}).
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
P_q^{(n+1)} &\leq (1+|q|+|q|^2+\cdots+|q|^n)(I \otimes P_q^{(n)}) \\
&\leq \frac{1}{1-|q|}(I \otimes P_q^{(n)})
\end{aligned}$$

(see [15, Prop 4.2.8(ii), p.250]). $\square$

Proposition 3.3.16. *[2, Lem 4] The operator $a(f), f \in \mathfrak{H}$, on $\mathcal{F}_q$ is bounded for $q \in [-1, 1)$, with the norm given by*

$$\begin{aligned}
||a(f)||_q &= \frac{1}{\sqrt{1-q}}||f|| && \textit{for} \quad 0 \leq q < 1 \\
||a(f)||_q &= ||f|| && \textit{for} \quad -1 \leq q \leq 0.
\end{aligned}$$

Proof. Let $-1 \leq q \leq 0$ and $\eta \in \mathcal{F}$. Then, from the q-commutation relation $a(f)a^*(f) - qa^*(f)a(f) = \langle f, f\rangle I$, we have

$$\begin{aligned}
\langle a^*(f)\eta, a^*(f)\eta\rangle_q - q\langle a(f)\eta, a(f)\eta\rangle_q &= \langle (a(f)a^*(f) - qa^*(f)a(f))\,\eta, \eta\rangle_q \\
&= \langle (\langle f, f\rangle)\eta, \eta\rangle_q
\end{aligned}$$

which implies that

$$\begin{aligned}
||a^*(f)\eta||_q^2 &= ||f||^2||\eta||_q^2 + q||a(f)\eta||_q^2 \\
&\leq ||f||^2||\eta||_q^2,
\end{aligned}$$

so that $||a^*(f)|| \leq ||f||$, and moreover, replacing η with Ω, we have

$$||f|| = ||a^*(f)\Omega||_q \leq ||f||||\Omega||_q \leq ||f||,$$

giving equality. Hence $||a^*(f)||_q = ||f||$ on $\mathcal{F}$, and can thus be extended to $\mathcal{F}_q$. Also, from Propositioin 3.3.13 we now know that $a^*(f)$ is the adjoint of $a(f)$, so that $a(f)$ is also bounded on $\mathcal{F}$, and can thus be extended to $\mathcal{F}_q$, with $||a^*(f)||_q = ||a(f)||_q = ||f||$.

Next consider the case where $0 \leq q < 1$. By Lemma 3.3.15 we have for every $\eta \in \mathcal{F}^{(n)}$ and $f \in \mathfrak{H}$

$$\begin{aligned}
||a^*(f)\eta||_q^2 &= ||f \otimes \eta||_q^2 \\
&= \langle f \otimes \eta, f \otimes \eta \rangle_q \\
&= \langle f \otimes \eta, P_q^{(n+1)} f \otimes \eta \rangle_0 \\
&\leq \frac{1}{1-q} \langle f \otimes \eta, (I \otimes P_q^{(n)}) f \otimes \eta \rangle_0 \\
&= \frac{1}{1-q} \langle f, f \rangle \langle \eta, P_q^{(n)} \eta \rangle_0 \\
&= \frac{1}{1-q} ||f||^2 \; ||\eta||_q^2.
\end{aligned}$$

To obtain the reverse inequality, we consider $f^{\otimes n}$,

$$\begin{aligned}
||a^*(f) f^{\otimes n}||_q^2 &= \langle f^{\otimes (n+1)}, f^{\otimes (n+1)} \rangle_q \\
&= \sum_{k=1}^{n+1} q^{k-1} \langle f, f \rangle \langle f^{\otimes n}, f^{\otimes n} \rangle_q \\
&= (1 + q + q^2 + \cdots + q^n) ||f||^2 ||f^{\otimes n}||_q^2.
\end{aligned}$$

Hence

$$\begin{aligned}
||a^*(f)||_q^2 &\geq \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} q^k \right) ||f||^2 \\
&= \frac{1}{1-q} ||f||^2.
\end{aligned}$$

As for the case $-1 \leq q \leq 0$, it follows that $a^*(f)$ is the adjoin of $a(f)$, so that both are bounded with the same norms, and can thus be extended to $\mathcal{F}_q$. □

The results from Lemma 3.3.1 and Proposition 3.3.13 can also be extended by continuity to the whole of $\mathcal{F}_q$, and in particular, we have that $a^*(f)$ and $a(f)$ are adjoints on $\mathcal{F}_q$.

We can now construct a C*-dynamical system using the C*-algebra generated by the creation and annihilation operators defined on the twisted Fock space. We shall use as our Hilbert space $\mathfrak{H} = l_2(\mathbb{Z})$, and the orthonormal basis $\{e_i : i \in \mathbb{Z}\}$, where e_i is the vector with 1 in the i-th coordinate and zero elsewhere. Thus, our deformed Fock space, $\mathcal{F}_q$, is densely spanned by the set $\{\Omega\} \cup \{e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k} : \sigma_i \in \mathbb{Z}, i = 1, 2, ..., k \text{ and } k = 1, 2, \ldots\}$, which are orthonormal vectors by the definition of the the inner product $\langle \cdot, \cdot \rangle_q$ and where $\Omega = 1 \in \mathbb{C}$. We shall only consider deformations with $-1 < q < 1$. In terms of this basis our C*-algebra is defined to be the C*-subalgebra of

$B(\mathcal{F}_q)$ generated by the set $\{a(e_i) : i \in \mathbb{Z}\}$, which we denote by $\mathcal{A}_q$. To see that $\mathcal{A}_q$ is unital, let $f = g = e_i$ in the commutation relation, then

$$a(e_i)a^*(e_i) - qa^*(e_i)a(e_i) = I.$$

The creation and annihilation operators are given, respectively, by

$$\begin{aligned} a^*(e_i)\Omega &= e_i \\ a^*(e_i)\xi &= e_i \otimes \xi, \ \ \xi \in \mathcal{F}_q \end{aligned}$$

and

$$\begin{aligned} a(e_i)\Omega &= 0 \\ a(e_i)(h_1 \otimes \ldots \otimes h_n) &= \sum_{k=1}^{n} q^{k-1}\langle e_i, h_k\rangle\langle h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n\rangle, \ h_i \in l_2(\mathbb{Z}). \end{aligned}$$

The norms are given by

$$\begin{aligned} ||a(e_i)||_q &= \frac{1}{\sqrt{1-q}} & \text{for} \quad 0 \leq q < 1 \\ ||a(e_i)||_q &= 1 & \text{for} \quad -1 < q \leq 0, \end{aligned}$$

and the q-commutation relations become

$$a(e_i)a^*(e_j) - qa^*(e_j)a(e_i) = \delta_{i,j}1_{\mathcal{A}_q},$$

where

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

We aim to define the automorphism on $\mathcal{A}_q$ as the right shift, given by

$$\alpha(a(e_i)) = a(e_{i+1}), i \in \mathbb{Z}.$$

But, to show that α is indeed an automorphism, we take another approach and first define a unitary operator on $\mathcal{F}_q$, playing the same role as the automorphism but on the Hilbert space, which we can then use to express the action of the automorphism group (similarly to what was done in the non-commutative torus). Thus, we define a (linear) operator U on $\mathcal{F}$ by:

$$U\Omega = \Omega$$

and

$$U(e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k}) = e_{\sigma_1+1} \otimes \cdots \otimes e_{\sigma_k+1}.$$

It is clear that U preserves the norm on these basis vectors of $\mathfrak{H}^{\otimes k}, k = 1, 2, \ldots$. We show that this is also the case on the linear span of these vectors, so that U is an isometry on $\mathcal{F}$ which can be extended by continuity to $\mathcal{F}_q$. Thus, we consider finite linear combinations of such orthonormal basis vectors, say $x = \sum_{i=1}^{p} \beta_i b_i$, where the β_i's are scalars and $b_i \in \{\Omega\} \cup \{e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k} : \sigma_j \in \mathbb{Z}, j = 1, 2, ..., k$ and $k = 1, 2, \ldots\}$, for each $i = 1, ..., p$. Then

$$\begin{aligned} ||Ux||^2 &= ||U(\sum_{i=1}^{p} \beta_i b_i)||^2 \\ &= ||\sum_{i=1}^{p} U(\beta_i b_i)||^2 \\ &= \sum_{i=1}^{p} ||U(\beta_i b_i)||^2 \\ &= \sum_{i=1}^{p} |\beta_i|^2 ||U(b_i)||^2 \\ &= \sum_{i=1}^{p} |\beta_i|^2 ||b_i||^2 \\ &= ||\sum_{i=1}^{p} \beta_i b_i||^2 \\ &= ||x||^2. \end{aligned}$$

Hence U is an isometry on $\mathcal{F}_q$, and thus also unitary, since we can define its inverse by

$$U^{-1}(e_{\sigma_1+1} \otimes \cdots \otimes e_{\sigma_k+1}) = e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k},$$

which will also be isometric by the same argument as above. We now define $\alpha : B(\mathcal{F}_q) \to B(\mathcal{F}_q)$ by

$$\alpha(a) = UaU^*,$$

so that α is a *-automorphism of $B(\mathcal{F}_q)$, and show that we can restrict it to $\mathcal{A}_q$, that is, to show that α maps elements of $\mathcal{A}_q$ to elements of $\mathcal{A}_q$. We begin by showing this on the generating set $\{a^*(e_i) : i \in \mathbb{Z}\}$ of basis elements of $\mathcal{F}_q$. Then

$$\begin{aligned} \alpha(a^*(e_i))(e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k}) &= Ua^*(e_i)U^*(e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k}) \\ &= Ua^*(e_i)(e_{\sigma_1-1} \otimes \cdots \otimes e_{\sigma_k-1}) \\ &= U(e_i \otimes e_{\sigma_1-1} \otimes \cdots \otimes e_{\sigma_k-1}) \\ &= e_{i+1} \otimes e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k} \\ &= a^*(e_{i+1})(e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k}), \end{aligned}$$

so that $\alpha(a^*(e_i)) = a^*(e_{i+1})$, and thus $\alpha(a(e_i)) = a(e_{i+1})$, so $\alpha(\mathcal{A}_q^o) \subseteq \mathcal{A}_q$, where $\mathcal{A}_q^o$ is the *-algebra generated by the set $\{a^*(e_i) : i \in \mathbb{Z}\}$. Aslo, since

α is bounded on $B(\mathcal{F}_q)$, we have that $\alpha(\mathcal{A}_q) \subseteq \mathcal{A}_q$. In the same way we have $\alpha^{-1}(\mathcal{A}_q) \subseteq \mathcal{A}_q$, implying that $\alpha(\mathcal{A}_q) = \mathcal{A}_q$. Hence α is a *-automorphism of $\mathcal{A}_q$ and we have the C*-dynamical system $(\mathcal{A}_q, \alpha)$.

3.4 Shift on an infinite tensor product of C*-algebras

The next example will use an infinite tensor products of C*-algebras. The aim is not to give a detailed exposition of infinite tensor products, but merely to use it as an example of a C*-dynamical system that will illustrate that there are noncommutative dynamical systems that are ergodic but not uniquely ergodic, and in a sense show that these notions are meaningful. Thus, familiarity with infinite tensor products of C*-algebras is assumed, and the interested reader is referred to [16, Chap 11] for a detailed construction thereof. We will, however, state the following result that will be directly applicable.

Proposition 3.4.1. *[16, Prop 11.4.5, p. 868] Let $\{\mathcal{B}_i : i \in \mathbb{Z}\}$ and $\{\mathcal{C}_i : i \in \mathbb{Z}\}$ be families of C*-algebras. Let $\mathcal{B}(i)$ denote the canonical image of $\mathcal{B}_i$ in $\bigotimes_{i\in\mathbb{Z}} \mathcal{B}_i$ and let $\mathcal{C}(i)$ denote the canonical image of $\mathcal{C}_i$ in $\bigotimes_{i\in\mathbb{Z}} \mathcal{C}_i$.*

*(i) If $\mathcal{B}_i$ is *-isomorphic to $\mathcal{C}_i$ for each $i \in \mathbb{Z}$, then $\bigotimes_{i\in\mathbb{Z}} \mathcal{B}_i$ is *-isomorphic to $\bigotimes_{i\in\mathbb{Z}} \mathcal{C}_i$.*

*(ii) If θ_i is a *-isomorphism from $\mathcal{B}(i)$ onto $\mathcal{C}(i)$ for each $i \in \mathbb{Z}$, then there is a *-isomorphism $\theta = \otimes_{i\in\mathbb{Z}}\theta_i$ from $\bigotimes_{i\in\mathbb{Z}} \mathcal{B}_i$ onto $\bigotimes_{i\in\mathbb{Z}} \mathcal{C}_i$ such that $\theta|_{\mathcal{B}(i)} = \theta_i$.*

Let $\mathcal{A}$ be a C*-algebra and consider the C*-algebra $\mathcal{A}^{\otimes} := \bigotimes_{i\in\mathbb{Z}} \mathcal{A}_i$, where $\mathcal{A}_i = \mathcal{A}$ for every $i \in \mathbb{Z}$.

We define $\alpha_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ by $\alpha_i(a) = a$ for every $a \in \mathcal{A}_i$ and for every $i \in \mathbb{Z}$. Since each α_i is in effect just the identity map on $\mathcal{A}$, it is clear that each α_i is an automorphism. By Proposition 3.4.1 we have the automorphism $\alpha^{\otimes} := \otimes_{i\in\mathbb{Z}}\alpha_i$ on $\mathcal{A}^{\otimes}$. So, if one considers the embeddings of the $\mathcal{A}_i$'s (and finite tensor products of the $\mathcal{A}_i$'s) into $\mathcal{A}^{\otimes}$, which form a dense *-subalgebra in $\mathcal{A}^{\otimes}$, the action of $\alpha^{\otimes}$ on any simple tensor in this dense *-subalgebra shifts the i-th component algebra to the $(i+1)$-th component algebra. That is

$$
\begin{aligned}
&\alpha^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{k-1}} \otimes a_k \otimes 1_{\mathcal{A}_{k+1}} \cdots) \\
&= \cdots \otimes 1_{\mathcal{A}_k} \otimes a_{k+1} \otimes 1_{\mathcal{A}_{k+2}} \cdots ,
\end{aligned}
$$

with $a_k = a_{k+1} \in \mathcal{A}$ (or more generally a can be replaced by an element from a finite tensor product in $\mathcal{A}$).

Hence, we have the C*-dynamical system $(\mathcal{A}^{\otimes}, \alpha^{\otimes})$.

Chapter 4

Unique and Relative Unique Ergodicity

In this chapter we consider unique ergodicity for C*-dynamical systems and follow the lead of Abadie and Dykema in [1] to the more general notion of relative unique ergodicity. Where Abadie and Dykema considered relative unique ergodicity for dynamical systems with $\mathbb{Z}$ as the acting group, we consider more general groups, specifically, any locally compact abelian group that contains a Følner sequence. In the second section we illustrate these notions using the examples from Chapter 3.

4.1 Unique and Relative Unique Ergodicity

In this section we define unique and relative unique ergodicity and give some characterizations thereof. The main result is based on [1], and deals with characterizations of the broader notion of relative unique ergodicity introduced there. Particularly interesting here is that relative unique ergodicity is equivalent to the norm convergence of ergodic averages taken over the C*-algebra, with the limit being a norm one projection onto the fixed point space under the automorphism in question, i.e. a conditional expectation by Theorem 2.5.12.

Before we begin some housekeeping is in order. Given any locally compact group, say $\mathcal{G}$, there exists a right Haar measure on $\mathcal{G}$, which we will denote by μ. Familiarity with the Haar measure is assumed, and the reader is referred to [4, Chap 9, pp.297 - 324] for a detailed exposition on the existence, uniqueness and properties of the Haar measure. Throughout this chapter let $A\Delta B$ denote the symmetric difference of two sets A and B. That is, $A\Delta B = (A\backslash B) \cup (B\backslash A)$.

Taking ergodic averages over more general groups, which in our case will be a locally compact groups, will then require a suitable substitute for the sequence of subsets $\{1, 2, \ldots, n\}$ of the integers. This will come in the form of Følner sequences.

Definition 4.1.1. Let $\mathcal{G}$ be a locally compact group. A *Følner sequence* in $\mathcal{G}$ is a sequence, (G_n), of compact subsets in $\mathcal{G}$ such that $0 < \mu(G_n)$ for every n, and for every $g \in \mathcal{G}$

$$\lim_{n\to\infty} \frac{\mu(G_n \Delta (G_n g))}{\mu(G_n)} = 0$$

It is important to mention that there is a deep underlying theory pertaining to the existence of Følner sequences in groups, and is related to the amenablility of the group. We do not need to explore this relationship, since we can show that Følner sequences do exist in the groups used in our examples.

Example 4.1.2. As our first example we show that the group of integers, $\mathbb{Z}$, with addition as the group operation, has a Følner sequence. So, consider $\mathcal{G} = \mathbb{Z}$ with the discrete topology, and let μ be the right Haar measure, which is just the counting measure. Consider the sequence (G_n), where $G_n = \{-n, \ldots, n\}$, for each $n \in \mathbb{N}$. Each G_n is compact with $\mu(G_n) = 2n + 1 > 0$. Taking an arbitrary $g \in \mathcal{G}$, we may without loss of generality assume that $|g| \leq 2n$, since we take the limit as $n \to \infty$. Then we have $\mu(G_n \Delta (G_n + g)) = 2g$, and so

$$\lim_{n\to\infty} \frac{\mu(G_n \Delta (G_n + g))}{\mu(G_n)} = \lim_{n\to\infty} \frac{2g}{2n+1} = 0.$$

Hence (G_n) is a Følner sequence in $\mathbb{Z}$.

In the same way as in the preceding example, $G_n = \{1, 2, \ldots, n\}$ and $G_n = \{0, 2, \ldots, n-1\}$ are also examples of Følner sequences in $\mathbb{Z}$ (and $\mathbb{N}$).

Example 4.1.3. For the second example let $\mathcal{G} = \mathbb{R}^2$, again with addition as the group operation, with the standard topology, and with the Lebesgue measure restricted to the Borel σ-algebra (which is, up to a constant multiple, the same as the Haar measure due to its uniqueness). As our sequence of compact subsets of $\mathcal{G}$ we let $G_n = [-n, n] \times [-n, n]$, where the compactness is clear from the Heine-Borel theorem. Then $\mu(G_n) = 4n^2$. Let $g = (g_1, g_2) \in \mathbb{R}^2$. Then, we can again assume with out loss of generality that $|g_1|, |g_2| < 2n$, and so we have that $\mu(G_n \Delta (G_n + g)) = 4n(g_1 + g_2) - 2g_1 g_2$. Thus

$$\lim_{n\to\infty} \frac{\mu(G_n \Delta (G_n) + g)}{\mu(G_n)} = \lim_{n\to\infty} (\frac{g_1 + g_2}{n} - \frac{g_1 g_2}{2n^2}) = 0$$

Hence (G_n) is a Følner sequence in $\mathbb{R}^2$.

One can show in exactly the same way as above that if $\mathcal{G} = \mathbb{R}$ and we let $G_n = [-n, n]$, then G_n is a Følner sequence in $\mathbb{R}$.

Remark 4.1.4. For the remainder, in our definition of a C*-dynamical system, let $\mathcal{G}$ be a locally compact group, where the group operation will be written multiplicatively, and such that $\mathcal{G}$ contains a Følner sequence, denoted by (G_n). Also let $\mathcal{A}$ be a C*-algebra and $\alpha : \mathcal{G} \to Aut(\mathcal{A})$, taking $g \mapsto \alpha_g$, be a group homomorphism into the group of automorphisms with $g \mapsto \alpha_g(a)$ Bochner integrable, for every $a \in \mathcal{A}$. The same assumptions are made for state preserving C*-dynamical systems. Note that with $\alpha_g(a)$ Bochner integrable, the integral $\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g)$ is well defined because the Haar measure is finite on compact sets, specifically in our case $\mu(G_n) < +\infty$. Also, since the Bochner integral will be C*-algebra-valued, we need to clarify the *-operation on such an element. An easy way to see this, without reverting back to simple functions, is to consider any $\phi \in \mathcal{A}^*$ and its adjoint $\phi^* \in \mathcal{A}^*$. Let $f : \mathcal{G} \to \mathcal{A}$ be Bochner integrable, $G \subset \mathcal{G}$ compact and μ the right Haar measure, then

$$\phi^* \left(\int_G f(g) d\mu(g) \right) = \overline{\phi \left(\left(\int_G f(g) d\mu(g) \right)^* \right)} \tag{4.1}$$

and, by Theorem 2.1.11,

$$\begin{aligned} \phi^* \left(\int_G f(g) d\mu(g) \right) &= \int_G \phi^*(f(g)) d\mu(g) \\ &= \int_G \overline{\phi(f(g)^*)} d\mu(g) \\ &= \overline{\int_G \phi(f(g)^*) d\mu(g)} \\ &= \overline{\phi \left(\int_G f(g)^* d\mu(g) \right)}. \end{aligned} \tag{4.2}$$

Now, since $\phi \in \mathcal{A}^*$ was arbitrary, it follows from Equations 4.1 and 4.2 that

$$\left(\int_G f(g) d\mu(g) \right)^* = \int_G f(g)^* d\mu(g).$$

Lastly, the following property will be used on many occasions: Let $f : \mathcal{G} \to \mathcal{A}$ be Bochner integrable, $G \subset \mathcal{G}$ compact, $h \in \mathcal{G}$ arbitrary and μ the right Haar measure, then

$$\int_G f(gh) d\mu(g) = \int_{Gh} f d\mu.$$

Much of the work that follows requires a state that is invariant under the automorphism group in question. The following result guarantees the existence of such a state.

Theorem 4.1.5. *Let $(\mathcal{A}, \alpha)$ be C*-dynamical system. Then there exists an α-invariant state on $\mathcal{A}$.*

Proof. Let ϕ be any state on $\mathcal{A}$. We define a mapping $I_{\phi,n} : \mathcal{A} \to \mathbb{C}$ by

$$I_{\phi,n}(a) = \frac{1}{\mu(G_n)} \int_{G_n} \phi \circ \alpha_g(a) d\mu(g).$$

We show that $I_{\phi,n}$ is a state for every $n \in \mathbb{N}$. Linearity follows from the linearity of ϕ, α and the Bochner (or more precisely Lebesgue) integral. Futhermore,

$$\begin{aligned}
|I_{\phi,n}(a)| &= |\frac{1}{\mu(G_n)} \int_{G_n} \phi \circ \alpha_g(a) d\mu(g)| \\
&\leq \frac{1}{\mu(G_n)} \int_{G_n} |\phi \circ \alpha_g(a)| d\mu(g) \\
&\leq ||\phi|| ||\alpha(a)|| \\
&= ||a||.
\end{aligned}$$

Hence $I_{\phi,n}$ is a bounded linear functional, i.e. $I_{\phi,n} \in \mathcal{A}^*$. Moreover, taking the supremum over all $a \in \mathcal{A}$ with norm 1 on both sides of $||I_{\phi,n}(a)|| \leq ||a||$, we see that $||I_{\phi,n}|| \leq 1$. On the other hand,

$$1 = ||I_{\phi,n}(1_{\mathcal{A}})|| \leq \sup_{||a||=1} ||I_{\phi,n}(a)|| = ||I_{\phi,n}||,$$

so that $||I_{\phi,n}|| = 1$. Also, we have that $I_{\phi,n}(1_{\mathcal{A}}) = 1 = ||I_{\phi,n}||$, so by Theorem 2.2.9, $I_{\phi,n}$ is a positive bounded linear functional, and hence a state. By Theorem 2.3.19 we have that $\mathcal{S}(\mathcal{A})$ is weak* compact. Thus, the sequence $(I_{\phi,n})_{n\in\mathbb{N}}$ must have a weak*-convergent subsequence in $\mathcal{S}(\mathcal{A})$, say $I_{\phi,n_k} \to I_\phi$, with $I_\phi \in \mathcal{S}(\mathcal{A})$ (see Theorem 2.3.14).

Let $k \in \mathcal{G}$, we show that I_ϕ is α-invariant. First, we note the following:

$$\begin{aligned}
0 &\leq |I_{\phi,n}(\alpha_k(a)) - I_{\phi,n}(a)| \\
&= \left| \frac{1}{\mu(G_n)} \int_{G_n} \phi \circ \alpha_{gk}(a) d\mu(g) - \frac{1}{\mu(G_n)} \int_{G_n} \phi \circ \alpha_g(a) d\mu(g) \right| \\
&= \left| \frac{1}{\mu(G_n)} \int_{G_n k} \phi \circ \alpha_g(a) d\mu(g) - \frac{1}{\mu(G_n)} \int_{G_n} \phi \circ \alpha_g(a) d\mu(g) \right| \\
&= \left| \frac{1}{\mu(G_n)} \int_{G_n k \setminus G_n} \phi \circ \alpha_g(a) d\mu(g) - \frac{1}{\mu(G_n)} \int_{G_n \setminus G_n k} \phi \circ \alpha_g(a) d\mu(g) \right| \\
&\leq \frac{1}{\mu(G_n)} \int_{G_n k \setminus G_n} |\phi \circ \alpha_g(a)| d\mu(g) + \frac{1}{\mu(G_n)} \int_{G_n \setminus G_n k} |\phi \circ \alpha_g(a)| d\mu(g) \\
&= \frac{1}{\mu(G_n)} \int_{G_n \Delta G_n k} |\phi \circ \alpha_g(a)| d\mu(g) \\
&\leq ||a|| \frac{\mu(G_n \Delta G_n k)}{\mu(G_n)} \\
&\longrightarrow 0 \qquad\qquad (\text{as } n \to \infty).
\end{aligned}$$

Now, again considering the sequence I_{ϕ,n_k} (of $I_{\phi,n}$) converging to I_ϕ in the weak*-topology, we have for any $\epsilon > 0$, $g \in \mathcal{G}$ and every $a \in \mathcal{A}$ that

$$\begin{aligned}
0 &\leq |I_\phi(\alpha_g(a)) - I_\phi(a)| \\
&\leq |I_\phi(\alpha_g(a)) - I_{\phi,n_k}(\alpha_g(a))| + |I_{\phi,n_k}(\alpha_g(a)) - I_{\phi,n_k}(a)| \\
&\qquad\qquad\qquad\qquad + |I_{\phi,n_k}(a) - I_\phi(a)| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon \qquad\qquad \text{for } n \text{ large enough.}
\end{aligned}$$

This holds true for every $\varepsilon > 0$, giving the desired result. □

Having established the existence of invariant states, we now give the definitions and characterizations of some ergodic notions in terms of such states.

Definition 4.1.6. A state preserving C*-dynamical system $(\mathcal{A}, \alpha, \phi)$ is called *ergodic* if

$$\lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n} \phi(a\alpha_g(b))d\mu(g) = \phi(a)\phi(b)$$

for every $a, b \in \mathcal{A}$.

Definition 4.1.7. A C*-dynamical system $(\mathcal{A}, \alpha)$ is called *uniquely ergodic* if there is only one α-invariant state on $\mathcal{A}$.

We now show that unique ergodicity is equivalent to the norm convergence of

$$\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a)d\mu(g), \qquad \text{for every } a \in \mathcal{A},$$

with the limit being a constant multiple of the unit of $\mathcal{A}$, and the constant is given by the α-invariant state evaluated at a.

Theorem 4.1.8. *$(\mathcal{A}, \alpha)$ is uniquely ergodic if and only if*

$$\lim_{n\to\infty} ||\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a)d\mu(g) - \lambda_a 1_{\mathcal{A}}|| = 0$$

for some $\lambda_a \in \mathbb{C}$ and for every $a \in \mathcal{A}$. Then, furthermore, $\lambda_a = \phi(a)$, with ϕ the unique α-invariant state of $(\mathcal{A}, \alpha)$.

Proof. First assume that

$$\lim_{n\to\infty} ||\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a)d\mu(g) - \lambda_a 1_{\mathcal{A}}|| = 0$$

for some $\lambda_a \in \mathbb{C}$ and for every $a \in \mathcal{A}$, and let ϕ be any α-invariant state on $\mathcal{A}$. Then

$$\begin{aligned}
& \left|\phi\left(\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g) - \lambda_a 1_{\mathcal{A}}\right)\right| \\
= & \left|\frac{1}{\mu(G_n)}\int_{G_n}\phi\circ\alpha_g(a)d\mu(g) - \phi(\lambda_a 1_{\mathcal{A}})\right| \\
= & \left|\frac{1}{\mu(G_n)}\int_{G_n}\phi(a)d\mu(g) - \lambda_a\right| \\
= & |\phi(a) - \lambda_a|.
\end{aligned}$$

Hence

$$|\phi(a) - \lambda_a| \leq ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g) - \lambda_a 1_{\mathcal{A}}|| \to 0, \qquad \text{as } n \to \infty$$

for every $a \in \mathcal{A}$ and ϕ an arbitrary α-invariant state. Thus proving the uniqueness of ϕ and that $\lambda_a = \phi(a)$.

Conversely, assume $(\mathcal{A}, \alpha)$ is uniquely ergodic, but that the uniform convergence fails for self-adjoint elements $a \in \mathcal{A}$. That is, there is an $\tilde{\epsilon} > 0$, such that for every $n_0 \in \mathbb{N}$ there is an $m_0 > n_0$ such that

$$||\frac{1}{\mu(G_{m_0})}\int_{G_{m_0}}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| \geq \tilde{\epsilon}.$$

Similarly, there is an $m_1 > m_0$ such that

$$||\frac{1}{\mu(G_{m_1})}\int_{G_{m_1}}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| \geq \tilde{\epsilon}.$$

Hence we can obtain a sequence (m_j) with $m_0 < m_1 < \ldots$, such that

$$||\frac{1}{\mu(G_{m_j})}\int_{G_{m_j}}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| \geq \tilde{\epsilon},$$

for $j = 1, 2, \ldots$.

But if $a \in \mathcal{A}$ is self-adjoint, it follows from Remark 4.1.4 that $\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g)$ is also self-adjoint, and hence, by the above inequality and Proposition 2.2.7, for every $m_j \in \mathbb{N}$, there exists a state ψ_j on $\mathcal{A}$ such that

$$\begin{aligned}
\tilde{\epsilon} \leq & ||\frac{1}{\mu(G_{m_j})}\int_{G_{m_j}}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| \\
= & \left|\psi_j\left(\frac{1}{\mu(G_{m_j})}\int_{G_{m_j}}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}\right)\right| \\
= & |I_{\psi_j,m_j}(a) - \phi(a)|, \qquad (4.3)
\end{aligned}$$

for $j = 1, 2, \ldots$.

As was shown in the proof of Proposition 4.1.5, $(I_{\psi_j, m_j})_{j\in\mathbb{N}}$ is a sequence of states in a weak* compact subspace, $\mathcal{S}(\mathcal{A})$, and as such must have a convergent subnet, with its limit being an α-invariant state, which must be ϕ by its uniqueness. But, by Theorem 2.3.13 ϕ is a weak* accumulation point of $(I_{\psi_j, m_j})_{j\in\mathbb{N}}$, which contradicts Inequality 4.3. Hence, the uniform convergence holds for self-adjoint elements in $\mathcal{A}$. Now, consider any $a \in \mathcal{A}$, then we can write $a = a_1 + ia_2$, where $a_1, a_2 \in \mathcal{A}$ are self-adjoint. Then for every $\epsilon > 0$

$$
\begin{aligned}
&||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| \\
&= \quad ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a_1)d\mu(g) - \phi(a_1)1_{\mathcal{A}} \\
&\qquad\qquad\qquad + i\left(\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a_2)d\mu(g) - \phi(a_2)1_{\mathcal{A}}\right)|| \\
&\leq \quad ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a_1)d\mu(g) - \phi(a_1)1_{\mathcal{A}}|| \\
&\qquad\qquad\qquad + ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a_2)d\mu(g) - \phi(a_2)1_{\mathcal{A}}|| \\
&\leq \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \quad \epsilon \qquad \text{(for } n \text{ large enough).}
\end{aligned}
$$

□

Remark 4.1.9. An interesting observation is that the proof above nowhere depended on the specific Følner sequence used. This fact will in the proof of Theorem 4.1.16, again be illuminated by the uniqueness of the conditional expectation obtained.

Theorem 4.1.10. *If the C^*-dynamical system $(\mathcal{A}, \alpha)$ is uniquely ergodic, then it is ergodic.*

Proof. Let ϕ denote the unique α-invariant state on $\mathcal{A}$. Then by Theorem 4.1.8

$$\lim_{n\to\infty} ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g) - \phi(a)1_{\mathcal{A}}|| = 0,$$

for every $a \in \mathcal{A}$. Hence given any $\epsilon > 0$,

$$\begin{aligned}
0 &\leq \Big|\frac{1}{\mu(G_n)}\int_{G_n}\phi(a\alpha_g(b))d\mu(g) - \phi(a)\phi(b)\Big| \\
&= \Big|\phi\Big(\frac{1}{\mu(G_n)}\int_{G_n}a\alpha_g(b)d\mu(g) - a\phi(b)\Big)\Big| \\
&\leq ||\phi||\ ||a\Big(\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(b)d\mu(g) - \phi(b)1_{\mathcal{A}}\Big)|| \quad \text{(by Theorem 2.1.13)} \\
&\leq ||a||\ ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(b)d\mu(g) - \phi(b)1_{\mathcal{A}}|| \\
&\leq \epsilon,
\end{aligned}$$

for n large enough. □

Definition 4.1.11. Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system. We call the set

$$\mathcal{A}^\alpha = \{a \in \mathcal{A} : \alpha_g(a) = a, g \in \mathcal{G}\}$$

the *fixed point subalgebra of* $\mathcal{A}$.

Proposition 4.1.12. *Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system. Then $\mathcal{A}^\alpha$ is a C*-subalgebra of $\mathcal{A}$ containing $\mathcal{A}$'s identity.*

Proof. That $\mathcal{A}^\alpha$ is a unital *-algebra, with the same unit as $\mathcal{A}$, follows from the fact that every $\alpha_g, g \in \mathcal{G}$, is a *-automorphism, and is clear. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}^\alpha$ that converges to $a \in \mathcal{A}$ in norm. Then, for every $g \in \mathcal{G}$, we have by the continuity of α_g that

$$\begin{aligned}
||\alpha_g(a) - a|| &\leq ||\alpha_g(a) - \alpha_g(a_n)|| + ||\alpha_g(a_n) - a|| \\
&= ||\alpha_g(a - a_n)|| + ||a_n - a|| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2},
\end{aligned}$$

so that $\alpha_g(a) = a$ and $a \in \mathcal{A}^\alpha$. Hence $\mathcal{A}^\alpha$ is a closed subspace of the C*-algebra $\mathcal{A}$, and thus a C*-subalgebra. □

Proposition 4.1.13. *Let the C*-dynamical system $(\mathcal{A}, \alpha)$ be uniquely ergodic. Then $\mathcal{A}^\alpha$ consists of constant multiples of the identity, i.e. it is one dimensional.*

Proof. Let $a \in \mathcal{A}^\alpha$ and ϕ the unique α-invariant state. Then for every $n \in \mathbb{N}$

$$a = \frac{1}{\mu(G_n)}\int_{G_n}ad\mu(g) = \frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g),$$

and taking the limit as $n \to \infty$, we obtain from Theorem 4.1.8 that $a = \phi(a)1_{\mathcal{A}}$. □

Next we define relative unique ergodicity as was done by Abadie and Dykema, and prove their result for locally compact abelian groups containing a Følner sequence. Notice that we now require the action group to be abelian, for this will allow the existence of a unique conditional expectation E onto the fixed point subalgebra such that $\alpha \circ E = E \circ \alpha = E$.

Definition 4.1.14. We call a C*-dynamical system $(\mathcal{A}, \alpha)$ *uniquely ergodic relative to its fixed point subalgebra* if every state on the fixed point subalgebra has a unique α-invariant state extension to $\mathcal{A}$.

Lemma 4.1.15. *Let $\mathcal{A}$ be a C*-algebra, α any automorphism on $\mathcal{A}$ and $\phi \in \mathcal{A}^*$ self-adjoint and α-invariant. Then ϕ_+ and ϕ_-, obtained from the Jordan decomposition of ϕ, are both α-invariant.*

Proof. From the Jordan decomposition we have that $\phi = \phi_+ - \phi_-$. But, we also have that $\phi = \phi \circ \alpha = \phi_+ \circ \alpha - \phi_- \circ \alpha$, and $||\phi|| = ||\phi \circ \alpha|| = ||\phi_+ \circ \alpha|| + ||\phi_- \circ \alpha||$. Since the decomposition is unique it follows that $\phi_+ \circ \alpha = \phi_+$ and
$\phi_- \circ \alpha = \phi_-$. □

Theorem 4.1.16. *[1, Thm 3.2] Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system, where the action group $\mathcal{G}$ is a locally compact abelian group containing a Følner sequence (G_n), and $\mathcal{A}^\alpha$ the fixed point subalgebra of $\mathcal{A}$. Then the following five statements are equivalent:*

(i) *$(\mathcal{A}, \alpha)$ is uniquely ergodic relative to its fixed point subalgebra.*

(ii) *Every bounded linear functional on $\mathcal{A}^\alpha$ has a unique bounded, α-invariant linear extension to $\mathcal{A}$.*

(iii) *The subspace $\mathcal{A}^\alpha + \{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}$ is dense in $\mathcal{A}$.*

(iv) *The sequence of ergodic averages*

$$\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g)$$

convergence uniformly as $n \to \infty$ for every $a \in \mathcal{A}$.

(v) *We have the following*

$$\mathcal{A}^\alpha + \overline{\{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}} = \mathcal{A},$$

where the closure is with respect to the norm topology.

Moreover, conditions (i) to (v) imply

(vi) There exists a unique α-invariant conditional expectation E from $\mathcal{A}$ onto $\mathcal{A}^\alpha$, and it is given by

$$E(a) = \lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g).$$

Proof. That (ii) $\Rightarrow$ (i) is clear, as is (v) $\Rightarrow$ (iii).

We show (i) $\Rightarrow$ (iii): Assume (i) to be true, but that $x \in \mathcal{A}$ and

$$x \notin \overline{\mathcal{A}^\alpha + \{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}}.$$

Then the Hahn-Banach Theorem provides us with a bounded linear functional ϕ on A such that $\phi(x) \neq 0$ and $\phi\left(\overline{\mathcal{A}^\alpha + \{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}}\right) = 0$ (see for example [15, Cor 1.6.3, p.44]). This implies that $\phi(\mathcal{A}^\alpha) = 0$, and moreover that $\phi \circ \alpha = \phi$. Indeed, given any $a \in \mathcal{A}$ and $g \in \mathcal{G}$, then

$$a - \alpha_g(a) \in \overline{\mathcal{A}^\alpha + \{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}},$$

and thus $\phi(a - \alpha_g(a)) = 0$, giving $\phi(\alpha_g(a)) = \phi(a)$. From Remark 2.2.12 we may assume that ϕ is self-adjoint, and moreover, by the Jordan decomposition and Lemma 4.1.15 we have $\phi = \phi_+ - \phi_-$ (uniquely), where ϕ_+ and ϕ_- are positive linear functionals on $\mathcal{A}$ and are α-invariant. Notice that for every $a \in \mathcal{A}^\alpha$, we have $\phi_+(a) = \phi_-(a)$, and since $1_\mathcal{A} \in \mathcal{A}^\alpha$, we have that $||\phi_+|| = \phi_+(1_\mathcal{A}) = \phi_-(1_\mathcal{A}) = ||\phi_-||$. In particular, $\phi_+ \neq 0 \Leftrightarrow \phi_- \neq 0$. So, let us consider the following two cases separately:
(1) $\phi_+ = 0$ and $\phi_- = 0$ on $\mathcal{A}^\alpha$. Then, by the positivity of ϕ_+ and ϕ_-, $0 = \phi_\pm(1_\mathcal{A}) = ||\phi_\pm||$ which implies that $\phi = 0$, a contradiction since $\phi(x) \neq 0$.
(2) $\phi_+ \neq 0$ and $\phi_- \neq 0$ on $\mathcal{A}^\alpha$. Let $\lambda = ||\phi_\pm|| > 0$, then $\frac{1}{\lambda}\phi_\pm$ are positive linear functionals with norm 1, that is they are (equal) α-invariant states on $\mathcal{A}^\alpha$. Hence by (ii) they must be equal on $\mathcal{A}$. Then $\phi = \phi_+ - \phi_- = 0$, another contradiction. Hence (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv): Let $a \in \mathcal{A}$ and $\epsilon > 0$. By (iii) there must be a $c \in \mathcal{A}^\alpha$ and a $b \in \mathcal{A}$ such that

$$||a - (c + b - \alpha_k(b)|| < \epsilon, \qquad \text{for some } k \in \mathcal{G}.$$

Then

$$
\begin{aligned}
&||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a)d\mu(g)-\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(a)d\mu(g)|| \\
&=||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a-(c+b-\alpha_k(b))+(c+b-\alpha_k(b)))d\mu(g) \\
&\quad -\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(a-(c+b-\alpha_k(b))+(c+b-\alpha_k(b)))d\mu(g)|| \\
&\leq ||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(a-(c+b-\alpha_k(b)))d\mu(g)|| \\
&\quad +||\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(a-(c+b-\alpha_k(b)))d\mu(g)|| \\
&\quad +||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(c+b-\alpha_k(b))d\mu(g) \\
&\quad -\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(c+b-\alpha_k(b))d\mu(g)|| \\
&\leq 2\epsilon+||\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(c)d\mu(g)-\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(c)d\mu(g) \\
&\quad +\left(\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_n)}\int_{G_n}\alpha_{gk}(b)d\mu(g)\right) \\
&\quad -\left(\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_m)}\int_{G_m}\alpha_{gk}(b)d\mu(g)\right)|| \\
&\leq 2\epsilon+||\left(\frac{1}{\mu(G_n)}\int_{G_n}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_n)}\int_{G_nk}\alpha_g(b)d\mu(g)\right)|| \\
&\quad +||\left(\frac{1}{\mu(G_m)}\int_{G_m}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_m)}\int_{G_mk}\alpha_g(b)d\mu(g)\right)|| \qquad (*) \\
&=2\epsilon+||\left(\frac{1}{\mu(G_n)}\int_{G_n\setminus G_nk}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_n)}\int_{G_nk\setminus G_n}\alpha_g(b)d\mu(g)\right)|| \\
&\quad +||\left(\frac{1}{\mu(G_m)}\int_{G_m\setminus G_mk}\alpha_g(b)d\mu(g)-\frac{1}{\mu(G_m)}\int_{G_mk\setminus G_m}\alpha_g(b)d\mu(g)\right)|| \\
&\leq 2\epsilon+\frac{1}{\mu(G_n)}\int_{G_n\Delta G_nk}||\alpha_g(b)||d\mu(g)+\frac{1}{\mu(G_m)}\int_{G_m\Delta G_mk}||\alpha_g(b)||d\mu(g) \\
&=2\epsilon+2||b||\left(\frac{\mu(G_n\Delta G_nk)}{\mu(G_n)}+\frac{\mu(G_m\Delta G_mk)}{\mu(G_m)}\right),
\end{aligned}
$$

which can be made arbitrarily small by taking m, n large enough, and for clarity, the inequality (*) follows from the fact that $\alpha_g(c) = c$. Hence the sequence of ergodic averages in consideration is Cauchy in $\mathcal{A}$, and must thus converge.

(iv) $\Rightarrow$ (vi): Let E be defined by

$$E(a) = \lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g).$$

We show that E is a unique α-invariant conditional expectation from $\mathcal{A}$ onto $\mathcal{A}^\alpha$. It is clear that E maps $\mathcal{A}$ into $\mathcal{A}$, and is linear. Furthermore, for $a \in \mathcal{A}^\alpha$

$$E(a) = \lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n} a d\mu(g) = a.$$

Also, for $k \in \mathcal{G}$ and every $a \in \mathcal{A}$ we have from Theorem 2.1.13 that

$$\begin{aligned}
&||E(a) - \alpha_k(E(a))|| \\
&= \lim_{n\to\infty} ||\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g) - \frac{1}{\mu(G_n)} \int_{G_n} \alpha_{kg}(a) d\mu(g)|| \\
&= \lim_{n\to\infty} ||\frac{1}{\mu(G_n)} \int_{G_n \setminus kG_n} \alpha_g(a) d\mu(g) - \frac{1}{\mu(G_n)} \int_{kG_n \setminus G_n} \alpha_g(a) d\mu(g)|| \\
&\leq \lim_{n\to\infty} \left(\frac{1}{\mu(G_n)} \int_{G_n \setminus kG_n} ||\alpha_g(a)|| d\mu(g) + \frac{1}{\mu(G_n)} \int_{kG_n \setminus G_n} ||\alpha_g(a)|| d\mu(g) \right) \\
&= \lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n \Delta kG_n} ||\alpha_g(a)|| d\mu(g) \\
&= ||a|| \lim_{n\to\infty} \frac{\mu(G_n \Delta kG_n)}{\mu(G_n)} \\
&= 0
\end{aligned}$$

Hence $\alpha_k(E(a)) = E(a)$ which shows that E is indeed a projection onto $\mathcal{A}^\alpha$. Using exactly the same technique as above, one can show that $E(\alpha_k(a)) = E(a)$, for every $k \in \mathcal{G}$ and $a \in \mathcal{A}$, giving the α-invariance of E. We show that E has norm one. Firstly we have

$$||E(a)|| \leq \lim_{n\to\infty} \frac{1}{\mu(G_n)} \int_{G_n} ||\alpha_g(a)|| d\mu(g) = ||a||,$$

and taking the supremum over all $a \in \mathcal{A}$, with $||a|| = 1$, on both sides gives the inequality $||E|| \leq 1$. Secondly, we have $1 = ||E(1_\mathcal{A})|| \leq ||E||$. Hence $||E|| = 1$. One easily shows that $\mathcal{A}^\alpha$ is a C*-subalgebra of $\mathcal{A}$, and so by Theorem 2.5.12 E is a conditional expectation onto $\mathcal{A}^\alpha$.

For the uniqueness, let E' be any other such an α-invariant conditional expectation onto $\mathcal{A}^\alpha$. Then for any $a \in \mathcal{A}$

$$\begin{aligned}
E'(a) &= \frac{1}{\mu(G_n)} \int_{G_n} E'(a) d\mu(g) \\
&= \frac{1}{\mu(G_n)} \int_{G_n} E'(\alpha_g(a)) d\mu(g) \\
&= E'(\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g)).
\end{aligned}$$

Taking the limit as $n \to \infty$ both sides yield

$$E'(a) = E'(E(a)) = E(a).$$

(iv)+(vi) $\Rightarrow$ (ii): Assume statements (iv) and (vi) and let $\omega : \mathcal{A}^\alpha \to \mathbb{C}$ be a bounded linear functional. Then $\omega \circ E$ is an α-invariant extension of ω to $\mathcal{A}$. For the uniqueness, let ϕ be any α-invariant bounded linear extension of ω. Then

$$\begin{aligned} \phi(a) = \frac{1}{\mu(G_n)} \int_{G_n} \phi(a) d\mu(g) &= \frac{1}{\mu(G_n)} \int_{G_n} \phi(\alpha_g(a)) d\mu(g) \\ &= \phi\left(\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g) \right). \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides yields

$$\phi(a) = \phi(E(a)) = \omega(E(a)),$$

so that $\phi = \omega \circ E$.

(ii)+(vi) $\Rightarrow$ (v): Given any $a \in \mathcal{A}$, we then put $a^c = a - E(a)$, and hence we can express $a = a^c + E(a)$, where $a^c \in ker(E)$. Hence we have that

$$\mathcal{A} = ker(E) + E(\mathcal{A}) = ker(E) + \mathcal{A}^\alpha.$$

Thus, it suffices to show that

$$ker(E) \subseteq \overline{\{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}},$$

with the reverse inclusion then clear. Assume, on the contrary, that $x \in ker(E)$, but $x \notin \overline{\{a - \alpha_g(a) : a \in \mathcal{A}, g \in \mathcal{G}\}}$. By the same argument as in the first implication proved, there exists by the Hahn-Banach Theorem a bounded linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(x) \neq 0$ and $\phi \circ \alpha = \phi$. From (vi) we have that $\phi \circ E$ is an α-invariant extension of $\phi|_{\mathcal{A}^\alpha}$ to $\mathcal{A}$, which by (i) should be unique. Thus, $\phi = \phi \circ E$, and so $\phi(x) = \phi(E(x)) = 0$, a contradiction. $\square$

The following corollary gives another characterization of unique ergodicity in terms of the necessarily one dimensional fixed point space.

Corollary 4.1.17. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system. The sequence of ergodic averages*

$$\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g)$$

convergence uniformly as $n \to \infty$ for every $a \in \mathcal{A}$ and $\mathcal{A}^\alpha = \mathbb{C}1_{\mathcal{A}}$ if and only if $(\mathcal{A}, \alpha)$ is uniquely ergodic.

Proof. If $(\mathcal{A}, \alpha)$ is uniquely ergodic, then the required implication follows directly from Theorem 4.1.8 and Proposition 4.1.13.

Assume the sequence of ergodic averages $\frac{1}{\mu(G_n)} \int_{G_n} \alpha_g(a) d\mu(g)$ convergence uniformly as $n \to \infty$ for every $a \in \mathcal{A}$ and $\mathcal{A}^\alpha = \mathbb{C}1_\mathcal{A}$. Then by Theorem 4.1.16 $(\mathcal{A}, \alpha)$ is uniquely ergodic relative to $\mathcal{A}^\alpha$, and so every state on $\mathcal{A}^\alpha$ has a unique α-invariant state extension to $\mathcal{A}$. But since $\mathcal{A}^\alpha = \mathbb{C}1_\mathcal{A}$, all states on $\mathcal{A}^\alpha$ are equal, and will thus have the same unique α-invariant state extension to $\mathcal{A}$, showing that there is only one α-invariant state on $\mathcal{A}$. □

4.2 Examples of unique ergodicity

We show that the noncommutative torus with an $\mathbb{R}^2$-action as well as the q-commutation relation C*-algebra with the $\mathbb{Z}$-action are both uniquely ergodic. We begin with the torus.

Non-commutatve torus with an $\mathbb{R}^2$ action: Recall that, for $\theta \in \mathbb{R}$, the non-commutative torus (denoted by $\mathcal{A}_\theta$) was defined as the C*-algebra generated by the unitary operators V and W, as defined by Equations (3.1) and (3.2), which satisfies to the commutation relation

$$VW = e^{-i\theta} WV.$$

On $\mathcal{A}_\theta$ we constructed the C*-dynamical system $(\mathcal{A}_\theta, \alpha)$, where α is the group of automorphisms defined by $\alpha_{(s,t)}(a) = U_{(s,t)} a U^*_{(s,t)}$, for $(s,t) \in \mathbb{R}^2$, and $U_{(s,t)}$ was defined via the Koopman construction from the classical torus.

We begin by showing that the automorphisms $\alpha_{(s,t)}$ are Bochner integrable. By Theorem 2.1.8 it will be sufficient to show that $(s,t) \mapsto \alpha_{(s,t)}(a)$ is continuous for every $a \in \mathcal{A}_\theta$, since we are only interested in integrating over the Følner sequence when taking any form of ergodic average.

Proposition 4.2.1. *The function $\mathbb{R}^2 \to \mathcal{A}_\theta, (s,t) \mapsto \alpha_{(s,t)}(a)$ is continuous in the in the norm of $\mathcal{A}_\theta$ for every $a \in \mathcal{A}_\theta$.*

Proof. Since $\alpha : \mathbb{R}^2 \to \mathcal{A}_\theta$ is a group homomorphism, and each $\alpha_{(s,t)}$ a *-automorphism, we have that

$$\begin{aligned} ||\alpha_{(s,t)}(a) - \alpha_{(s_0,t_0)}(a)|| &= ||(\alpha_{(s-s_0,t-t_0)} - \alpha_{(0,0)})(\alpha_{(s_0,t_0)}(a))|| \\ &\leq ||\alpha_{(s-s_0,t-t_0)} - \alpha_{(0,0)}||\ ||a||, \end{aligned}$$

and thus we only need to show continuity at the origin. On the generators, V and W we have that $\alpha_{(s,t)}(V^m W^n) = e^{i(sm+tn)} V^m W^n$ which is clearly continuous. Moreover, so will any finite linear combination of elements of the form $V^m W^n$, with $m, n \in \mathbb{Z}$, also be continuous, showing continuity on the dense *-algebra generated by V and W. Now, let $a \in \mathcal{A}$ be arbitrary.

Then given any $\epsilon > 0$ there exists a b in the dense *-algebra generated by V and W such that $||b - a|| < \epsilon$. Then we have

$$\begin{aligned} ||\alpha_{(s,t)}(a) - a|| &= ||\alpha_{(s,t)}(a) - \alpha_{(s,t)}(b) + \alpha_{(s,t)}(b) - b + b - a|| \\ &\leq ||\alpha_{(s,t)}(a) - \alpha_{(s,t)}(b)|| + ||\alpha_{(s,t)}(b) - b|| + ||b - a|| \\ &= ||\alpha_{(s,t)}(a - b)|| + ||\alpha_{(s,t)}(b) - b|| + ||b - a|| \\ &= ||a - b|| + ||\alpha_{(s,t)}(b) - b|| + ||b - a|| \\ &< \epsilon + \epsilon + \epsilon, \end{aligned}$$

for (s, t) close enough to $(0, 0)$. Hence, the function is continuous on $\mathbb{R}^2$. □

We show that this C*-dynamical system is uniquely ergodic, that is, there is only one α-invariant state on $\mathcal{A}_\theta$.

Theorem 4.2.2. *The C*-dynamical system $(\mathcal{A}_\theta, \alpha)$ is uniquely ergodic with an $\mathbb{R}^2$-action.*

Proof. Consider the mapping $\omega : \mathcal{A}_\theta \to \mathbb{C}$ defined by

$$\omega(a) = \langle \mathbf{1}, a\mathbf{1} \rangle_2,$$

where $\mathbf{1} \in L^2(\mu)$ is the constant function valued 1. Linearity follows from the inner product's linearity, and moreover, $\omega(a^*a) = ||a\mathbf{1}||_2^2 \geq 0$ so that ω is positive (and hence bounded by Theorem 2.2.9), which in turn implies that $||\omega|| = \omega(1_{\mathcal{A}_\theta}) = 1$. Hence ω is a state on $\mathcal{A}_\theta$. We show ω is α-invariant: let $(s, t) \in \mathbb{R}^2, a \in \mathcal{A}_\theta$, then

$$\begin{aligned} \omega(\alpha_{(s,t)}(a)) &= \langle U^*_{(s,t)}\mathbf{1}, aU^*_{(s,t)}\mathbf{1} \rangle_2 \\ &= \langle \mathbf{1}, a\mathbf{1} \rangle_2 \\ &= \omega(a). \end{aligned}$$

It remains to show that ω is the only α-invariant state on $\mathcal{A}_\theta$. To this end we notice that, for any non-zero $m, n \in \mathbb{Z}$ and $(s, t) \in \mathbb{R}^2$, we have that

$$\omega(V^m W^n) = \omega(\alpha_{(s,t)}(V^m W^n)) = e^{ism} e^{itn} \omega(V^m W^n),$$

implying $\omega(V^m W^n) = 0$. Now, assume ϕ is another α-invariant state on $\mathcal{A}_\theta$. Then ω and ϕ agree on the identity of $\mathcal{A}_\theta$ (and any constant multiple thereof) and the same argument as above implies that $\phi(V^m W^n) = 0$ for every non-zero $m, n \in \mathbb{Z}$. Hence $\phi = \omega$ on the *-algebra generated by V and W. But, since both ϕ and ω are continuous, we have for an arbitrary $a \in \mathcal{A}_\theta$ that $\phi(a) = \omega(a)$, showing that ω is a unique α-invariant state, and hence the dynamical system $(\mathcal{A}_\theta, \alpha, \omega)$ is uniquely ergodic. □

It is interesting to note that, since the noncommutative torus serves as a generalization of the classical torus, and our dynamics was derived using the dynamics of the classical torus via the Koopman construction, one would also expect the classical dynamical system $(\mathbb{T}^2, \mathfrak{B}, \mu, T)$ to be uniquely ergodic (in the classical sense). And this is indeed the case; intuitively, this follows from the fact that the Haar measure is a unique translation invariant Borel measure, with uniqueness up to a constant multiple, and by normalizing it, the uniqueness "up to a constant multiple" implies just uniqueness. So ω in Theorem 4.2.2 is a type of noncommutative Haar measure.

The q-commutation relations: We now turn to the q-commutation C*-algebra. This example is based on [8], and deals with the unique ergodicity of the C*-dynamical system $(\mathcal{A}_q, \alpha)$ constructed in Section 3.3. We begin by defining an α-invariant state. Let $\omega : \mathcal{A}_q \to \mathbb{C}$ be defined by

$$\omega(a) = \langle \Omega, a\Omega \rangle_q, \qquad a \in \mathcal{A}_q.$$

By definition it is well defined, linear, continuous and complex-valued. Positivity follows from $\omega(a^*a) = ||a\Omega||_q^2 \geq 0$, and by implication $||\omega|| = \omega(1_{\mathcal{A}_q}) = 1$, showing it is a state. We show ω is α-invariant.

Proposition 4.2.3. *The state defined by*

$$\omega(a) = \langle \Omega, a\Omega \rangle_q, \qquad a \in \mathcal{A}_q$$

is α-invariant.

Proof. We first consider ω on the dense *-algebra spanned by finite products in $\{a(e_i), a^*(e_j) : i, j \in \mathbb{Z}\}$. By the linearity we need only consider a single such product, and due to the q-commutation relation we can rewrite any finite product from this dense *-algebra as a linear combination of the form

$$b = \lambda_0 1_{\mathcal{A}_q} + \sum_{i=1}^{n} \lambda_i b_i,$$

where each b_i is of the form

$$a^*(e_{\sigma_1}) \cdots a^*(e_{\sigma_n}) a(e_{\rho_1}) \cdots a(e_{\rho_m}),$$

that is, a product starting with an annihilation operator. Note that ω will be zero on any such product starting with annihilation operators, since

$$\begin{aligned}
&\omega(a^*(e_{\sigma_1}) \cdots a^*(e_{\sigma_n}) a(e_{\rho_1}) \cdots a(e_{\rho_m})) \\
&= \langle a(e_{\sigma_n}) \cdots a(e_{\sigma_1})\Omega, a(e_{\rho_1}) \cdots a(e_{\rho_m})\Omega \rangle_q \\
&= 0 \qquad\qquad (\text{ if } m > 0 \text{ or } n > 0).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&\omega(\alpha(1_{\mathcal{A}_q} + a^*(e_{\sigma_1})\cdots a^*(e_{\sigma_n})a(e_{\rho_1})\cdots a(e_{\rho_m})))\\
&= \omega(1_{\mathcal{A}_q}) + \omega(a^*(e_{\sigma_1+1})\cdots a^*(e_{\sigma_n+1})a(e_{\rho_1+1})\cdots a(e_{\rho_m+1}))\\
&= 1\\
&= \omega(1_{\mathcal{A}_q} + a(e_{\rho_m})\cdots a(e_{\rho_1})a^*(e_{\sigma_1})\cdots a^*(e_{\sigma_n})).
\end{aligned}$$

Hence, by linearity ω is α-invariant on a dense *-algebra, and by its continuity the same will true on $\mathcal{A}_q$. $\square$

Let $\mathcal{A}_q^\alpha$ denote the fixed point subalgebra under α. We define $E : \mathcal{A}_q \to \mathcal{A}_q^\alpha$ by

$$E(a) = \omega(a)1_{\mathcal{A}_q}.$$

Then E is well-defined, linear, α-invariant, and, by definition maps elements of $\mathcal{A}_q$ into $\mathcal{A}_q^\alpha$, i.e. $\alpha(E(a)) = E(a), a \in \mathcal{A}_q$. Moreover, E has norm one;

$$\begin{aligned}
||E|| &= \sup_{\substack{a\in\mathcal{A}_q\\ ||a||=1}} ||E(a)||\\
&= \sup_{\substack{a\in\mathcal{A}_q\\ ||a||=1}} ||\omega(a)1_{\mathcal{A}_q}||\\
&= \sup_{\substack{a\in\mathcal{A}_q\\ ||a||=1}} |\omega(a)|\\
&= ||\omega||\\
&= 1.
\end{aligned}$$

We will now show that the ergodic averages $\frac{1}{n}\sum_{k=1}^n \alpha^k(a)$ converges uniformly to $E(a)$ for every $a \in \mathcal{A}_q$. For this we need the following two results.

Lemma 4.2.4. *[8, Lem 3.1] Let $\{\eta_j\}, j = 1,\ldots,n$, be elements in $\mathfrak{H}^{\otimes k}$, and let $\{\xi_j\}, j = 1,\ldots,n$, be an orthonormal set in $\mathfrak{H}$, then*

$$||\sum_{j=1}^n a^*(\xi_j)\eta_j||_q \le \sqrt{\frac{n}{1-|q|}} \max_{1\le j\le n} ||\eta_j||.$$

Proof. Taking Lemma 3.3.15 into consideration, we see that

$$\begin{aligned}
&\left\langle \sum_{j=1}^{n} a^*(\xi_j)\eta_j, \sum_{j=1}^{n} a^*(\xi_j)\eta_j \right\rangle_q \\
= & \left\langle \sum_{j=1}^{n} \xi_j \otimes \eta_j, P_q^{(k+1)} \sum_{j=1}^{n} \xi_j \otimes \eta_j \right\rangle_0 \\
\leq & \frac{1}{1-|q|} \left\langle \sum_{j=1}^{n} \xi_j \otimes \eta_j, (I \otimes P_q^{(k)}) \sum_{j=1}^{n} \xi_j \otimes \eta_j \right\rangle_0 \\
= & \frac{1}{1-|q|} \left\langle \sum_{j=1}^{n} \xi_j \otimes \eta_j, \sum_{j=1}^{n} \xi_j \otimes P_q^{(k)} \eta_j \right\rangle_0 \\
= & \frac{1}{1-|q|} \sum_{i,j=1}^{n} \langle \xi_i, \xi_j \rangle \langle \eta_i, P_q^{(k)} \eta_j \rangle_0 \\
= & \frac{1}{1-|q|} \sum_{i,j=1}^{n} \langle \xi_i, \xi_j \rangle \langle \eta_i, \eta_j \rangle_q \\
= & \frac{1}{1-|q|} \sum_{j=1}^{n} \langle \eta_j, \eta_j \rangle_q \\
\leq & \frac{n}{1-|q|} \max_{1 \leq j \leq n} ||\eta_j||_q^2.
\end{aligned}$$

□

Proposition 4.2.5. *[8, Prop. 3.2] Let $0 \leq k_1 \leq k_2 \leq \cdots$ be any increasing sequence of positive integers, and let $e_{\sigma_1}, \ldots, e_{\sigma_i}, e_{\rho_1}, \ldots, e_{\rho_j}$ be elements of the basis $\{e_i : i \in \mathbb{Z}\}$ of $\mathfrak{H}$. Then*

$$||\sum_{l=1}^{n} \alpha^{k_l} (a^*(e_{\sigma_1}) \cdots a^*(e_{\sigma_i}) a(e_{\rho_1}) \cdots a(e_{\rho_j}))|| \leq \sqrt{\frac{n}{(1-|q|)^{i+j}}}$$

if at least either $i \neq 0$ or $j \neq 0$.

Proof. Firstly assume that $i > 0$. We may also take a unit vector $\eta \in \mathfrak{H}^{\otimes m}$, for $m = j, j+1, \ldots$ (because for $m < j$, $a^*(e_{\sigma_1}) \cdots a^*(e_{\sigma_i}) a(e_{\rho_1}) \cdots a(e_{\rho_j})\eta = 0$). Put,

$$\eta_l := a^*(e_{\sigma_2 + k_l}) \cdots a^*(e_{\sigma_i + k_l}) a(e_{\rho_1 + k_l}) \cdots a(e_{\rho_j + k_l})\eta,$$

then, since $1 \leq \frac{1}{\sqrt{1-|q|}}$, we have from Proposition 3.3.16 that

$||\eta_l||_q \leq \frac{1}{\sqrt{(1-|q|)^{i+j-1}}}$. Using Lemma 4.2.4, we have

$$\begin{aligned}
&||\sum_{l=1}^{n} \alpha^{k_l}(a^*(e_{\sigma_1})\cdots a^*(e_{\sigma_i})a(e_{\rho_1})\cdots a(e_{\rho_j}))\eta||_q^2 \\
&= ||\sum_{l=1}^{n} a^*(e_{\sigma_1+k_l})a^*(e_{\sigma_2+k_l})\cdots a^*(e_{\sigma_i+k_l})a(e_{\rho_1+k_l})\cdots a(e_{\rho_j+k_l})\eta||_q^2 \\
&= ||\sum_{l=1}^{n} a^*(e_{\sigma_1+k_l})\eta_l||_q^2 \\
&\leq \frac{n}{1-|q|}||\eta_l||_q^2 \\
&\leq \frac{n}{(1-|q|)^{i+j}}.
\end{aligned}$$

Assuming $i = 0$ and $j > 0$, the same argument as above can be applied to

$$\begin{aligned}
\sum_{l=1}^{n} \alpha^{k_l}(a(e_{\rho_1})\cdots a(e_{\rho_j})) &= \sum_{l=1}^{n} a(e_{\rho_1+k_l})\cdots a(e_{\rho_j+k_l}) \\
&= \left(\sum_{l=1}^{n} a^*(e_{\rho_j+k_l})\cdots a^*(e_{\rho_1+k_l})\right)^*
\end{aligned}$$

□

Theorem 4.2.6. *C^*-dynamical system $(\mathcal{A}_q, \alpha)$ is uniquely ergodic.*

Proof. Let p denote a polynomial, each term (excluding the $1_{\mathcal{A}_q}$ term) consisting of a finite product in $\{1_{\mathcal{A}_q}, a(e_i), a^*(e_j) : i, j \in \mathbb{Z}\}$ starting with an annihilation operator. Then p either has a term that is a constant multiple of the identity, say $\lambda 1_{\mathcal{A}_q}$, or not. If it does not, i.e if $\lambda = 0$, then $E(p) = \omega(p)1_{\mathcal{A}_q} = 0$ (as in the proof of Proposition 4.2.3), and if it does, i.e if $\lambda \neq 0$, then $E(p) = \lambda 1_{\mathcal{A}_q}$. Furthermore, by Proposition 4.2.5, we also have

$$\begin{aligned}
||\frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(p) - E(p)|| &= ||\frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(p) - \lambda 1_{\mathcal{A}_q}|| \\
&= ||\frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(p - \lambda 1_{\mathcal{A}_q})|| \\
&\to 0 \quad (\text{ as } n \to \infty).
\end{aligned}$$

Hence the ergodic averages converge in norm on the dense *-algebra spanning $\mathcal{A}_q$. Now, let $a \in \mathcal{A}_q$, then there exists polynomials, say p_m (of the same form as the p considered above), in the dense *-algebra such that $p_m \to a$ as $m \to \infty$. Then

$$
\begin{aligned}
&||\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a) - E(a)|| \\
&= ||\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a) - \frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(p_m) + \frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(p_m) - E(p_m) + E(p_m) - E(a)|| \\
&\leq \frac{1}{n}\sum_{k=0}^{n-1}||\alpha^k(a-p_m)|| + ||\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(p_m) - E(p_m)|| + ||E(p_m) - E(a)|| \\
&\leq ||a - p_m|| + ||\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(p_m) - E(p_m)|| + ||p_m - a|| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{(for } n, m \text{ large enough).}
\end{aligned}
$$

Hence, since the ergodic averages $\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a)$ converge in norm to $E(a)$ for every $a \in \mathcal{A}_q$ and $E(a) = \omega(a)1_{\mathcal{A}_q}$, it follows from Corollary 4.1.17 that $(\mathcal{A}_q, \alpha, \omega)$ is uniquely ergodic. □

4.3 Example of relative unique ergodicity

Non-commutatve torus with an $\mathbb{R}$ action: Consider the action from $\mathbb{R}$ on $\mathcal{A}_\theta$ as defined in Section 3.2. We show that in this context the dynamical system $(\mathcal{A}_\theta, \beta)$ is uniquely ergodic relative to the fixed point subalgebra, but not uniquely ergodic. As with the action from $\mathbb{R}^2$, we have for the $\mathbb{R}$-action an β-invariant state in $\omega(a) = \langle a\mathbf{1}, \mathbf{1}\rangle$, with $\mathbf{1} \in L^2(\mathbb{T}^2)$ the constant function valued 1, but as we shall shortly see, it can not be unique.

Theorem 4.3.1. *The C*-dynamical system $(\mathcal{A}_\theta, \beta)$ is uniquely ergodic relative to its fixed point subalgebra, but not uniquely ergodic.*

Proof. Firstly, notice that in this case the fixed point sub-algebra, $\mathcal{A}_\theta^\beta$, is no longer one dimensional, because for every $s \in \mathbb{R}$, we have

$$\alpha_{(s,0)}(1_{\mathcal{A}_\theta}) = 1_{\mathcal{A}_\theta} \quad \text{and} \quad \alpha_{(s,0)}(W^m) = W^m.$$

Hence, $\mathcal{A}_\theta^\beta$ contains at least the *-algebra spanned by the set $\{1_{\mathcal{A}_\theta}, W^m : m \in \mathbb{Z}\}$, which then by Corollary 4.1.17 excludes the possibility of unique ergodicity.

Let τ be any state on $\mathcal{A}_\theta^\beta$, and assume that ϕ and ψ are two β-invariant state extensions of τ to $\mathcal{A}_\theta$. It will suffice to consider products of the form $V^m W^n, m, n \in \mathbb{Z}$. Then

$$\phi(V^m W^n) = \phi(\beta(V^m W^n)) = e^{ism}\phi(V^m W^n),$$

for every $s \in \mathbb{R}$, implying $\phi(V^m W^n) = 0$ if $m \neq 0$. And, similarly we obtain that $\psi(V^m W^n) = 0$ if $m \neq 0$. If $m = 0$, then $\beta(W^n) = W^n$, so that $W^n \in \mathcal{A}_\theta^\beta$, and thus $\phi(W^n) = \psi(W^n)$, showing that $\phi = \psi$ on the *-algebra generated by the set $\{V^m W^n : m, n \in \mathbb{Z}\}$. To see that this holds true on the whole of $\mathcal{A}_\theta$, let $a \in \mathcal{A}_\theta$, then there is a sequence (a_n) in the *-algebra generated by $\{V^m W^n : m, n \in \mathbb{Z}\}$, such that $a_n \to a$ as $n \to \infty$. Then, by the continuity of ϕ and ψ, we have

$$\begin{aligned} \phi(a) &= \lim_{n\to\infty} \phi(a_n) \\ &= \lim_{n\to\infty} \psi(a_n) \\ &= \psi(a). \end{aligned}$$

Hence, every state on the fixed point sub-algebra has a unique β-invariant state extension to the whole of $\mathcal{A}_\theta$, and so $(\mathcal{A}_\theta, \beta)$ is uniquely ergodic relative to its fixed point subalgebra. □

4.4 Ergodicity does not imply unique ergodicity

Shift on an infinite tensor product of C*-algebras: Lastly, we return to the C*-dynamical system $(\mathcal{A}^\otimes, \alpha^\otimes)$ obtained by taking an infinite tensor product of a single C*-algebra $\mathcal{A}$, and shifting the position according to a $\mathbb{Z}$-action. This example will illustrate that the concept of unique ergodicity is sensible, in as much as that it is different from ergodicity. It is an example of a C*-dynamical systems that exhibits ergodicity but not unique ergodicity.

The states that we will consider are obtained from states on $\mathcal{A}$ and made explicit in the following proposition.

Proposition 4.4.1. *[16, Prop 11.4.6, p. 869] Suppose that $\{\mathcal{A}_i : i \in \mathbb{Z}\}$ is a family of C*-algebras and that ϕ_i is a state on $\mathcal{A}_i$ for each $i \in \mathbb{Z}$. Then there is a unique state, denoted by $\otimes_{i\in\mathbb{Z}}\phi_i$, on $\bigotimes_{i\in\mathbb{Z}} \mathcal{A}_i$ and referred to as a product state, such that*

$$\otimes_{i\in\mathbb{Z}}\phi_i(a_j \otimes a_{j+1} \otimes \cdots \otimes a_{j+n}) = \phi_j(a_j)\cdots\phi_{j+n}(a_{j+n}),$$

for every $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, where $a_j \in \mathcal{A}_j$.

So, given a state, $\phi_i = \phi$, on $\mathcal{A}_i = \mathcal{A}$, we have by Proposition 4.4.1 the product state $\phi^\otimes := \otimes_{i\in\mathbb{Z}}\phi_i$ on $\mathcal{A}^\otimes = \otimes_{i\in\mathbb{Z}}\mathcal{A}_i$. We show that any such product state is $\alpha^\otimes$-invariant.

Proposition 4.4.2. *Let $\phi^\otimes$ be the product state on $\mathcal{A}^\otimes$, obtained from any state ϕ on $\mathcal{A}$, as described above. Then $\phi^\otimes$ is $\alpha^\otimes$-invariant.*

Proof. By the continuity of both $\phi^\otimes$ and $\alpha^\otimes$, it will suffice to show the invariance on a dense *-subalgebra of $\mathcal{A}^\otimes$. And, to this end, since the embeddings of finite tensor products of elements from $\{\mathcal{A}_i : i \in \mathbb{Z}\}$ span a dense subset in $\mathcal{A}^{\otimes}$, it will suffice to only consider products of the form

$$\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \cdots$$

with $i < j$. Note that we include the subscripts for $1_{\mathcal{A}}$ purely to indicate their "positions" in $\mathcal{A}^{\otimes}$.

Recall that by definition $\alpha_i(a) = a$ for every $a \in \mathcal{A}_i = \mathcal{A}_{i+i}$, and that $\phi_i = \phi_j$ for every $i, j \in \mathbb{Z}$. Then

$$\begin{aligned}
&\phi^{\otimes}(\alpha^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes 1_{\mathcal{A}_{j+2}} \otimes \cdots)) \\
&= \phi^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes 1_{\mathcal{A}_i} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+2}} \cdots) \\
&= \phi_{i+1}(a_i)\phi_{i+2}(a_{i+1}) \cdots \phi_{j+1}(a_j) \\
&= \phi_i(a_i)\phi_{i+1}(a_{i+1}) \cdots \phi_j(a_j) \\
&= \phi^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \cdots).
\end{aligned}$$

□

Corollary 4.4.3. *If* $\dim \mathcal{A} > 1$, *then the C*-dynamical system* $(\mathcal{A}^{\otimes}, \alpha^{\otimes})$ *is not uniquely ergodic.*

Proof. If $dim(\mathcal{A}) > 1$ we have as a consequence of the Hahn-Banach theorem (see for example [18, Thm 4.3-3, p. 223]) that $dim(\mathcal{A}^*) > 1$. But, from Remark 2.2.12 it follows that $\mathcal{A}^*$ is spanned by states, since every bounded linear functional in $\mathcal{A}^*$ can be written as a linear combination of positive linear functionals, each of which is just a constant multiple of a state (by dividing with its norm). Hence $\mathcal{A}^*$ has more than one state, and the product states on $\mathcal{A}^{\otimes}$ obtained from different states will differ, ensuring that there is more than one product state on $\mathcal{A}^{\otimes}$. Now, by Proposition 4.4.2 every product state on $\mathcal{A}^{\otimes}$, which was obtained from a single state on $\mathcal{A}$, is invariant under this automorphism, and thus there is no unique such state, eliminating the possibility of a uniquely ergodic C*-dynamical system. □

We show that, for each of the above mentioned product states, this C*-dynamical system is ergodic, though. We approach this from another angle, namely strong mixing. Below we give the definition of a strongly mixing C*-dynamical system and show that it implies ergodicity. Since we are working with a $\mathbb{Z}$-action, the definitions and implication will also be done for a $\mathbb{Z}$-action (also because it is much simpler!).

Definition 4.4.4. A C*-dynamical system $(\mathcal{B}, \beta, \phi)$ is called

(i) *ergodic* if for every $a, b \in \mathcal{B}$

$$\frac{1}{n}\sum_{k=1}^{n} \phi(a\beta^k(b)) \longrightarrow \phi(a)\phi(b)$$

as $n \to \infty$.

(ii) *strongly mixing* if for every $a, b \in \mathcal{B}$

$$\phi(a\beta^k(b)) \longrightarrow \phi(a)\phi(b)$$

as $k \to \infty$.

We show that strong mixing implies ergodicity.

Proposition 4.4.5. *If a C^*-dynamical system $(\mathcal{B}, \beta, \phi)$ is strongly mixing, then it is ergodic.*

Proof. Consider any $a, b \in \mathcal{B}$. Given an $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$|\phi(a\beta^k(b)) - \phi(a)\phi(b)| < \frac{\epsilon}{2}$$

for every $k > N$. Then for n large enough we have

$$\begin{aligned}
&|\frac{1}{n}\sum_{k=1}^{n}\phi(a\beta^k(b)) - \phi(a)\phi(b)| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}|\phi(a\beta^k(b)) - \phi(a)\phi(b)| \\
&= \frac{1}{n}\sum_{k=1}^{N}|\phi(a\beta^k(b)) - \phi(a)\phi(b)| + \frac{1}{n}\sum_{k=N+1}^{n}|\phi(a\beta^k(b)) - \phi(a)\phi(b)| \\
&\leq \frac{1}{n}\sum_{k=1}^{N}|\phi(a\beta^k(b)) - \phi(a)\phi(b)| + \frac{(n-N)\epsilon}{2n} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

□

Lastly we show that $(\mathcal{A}^{\otimes}, \alpha^{\otimes}, \phi^{\otimes})$ is strongly mixing and hence ergodic for every state ϕ on $\mathcal{A}$.

Theorem 4.4.6. *The C^*-dynamical system $(\mathcal{A}^{\otimes}, \alpha^{\otimes}, \phi^{\otimes})$ is strongly mixing for every state ϕ on $\mathcal{A}$.*

Proof. We show the strong mixing property for "simple" tensors of the form

$$\cdots \otimes 1_{\mathcal{A}_{i-2}} \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes 1_{\mathcal{A}_{j+2}} \otimes \cdots\ ,$$

with $i, j \in \mathbb{Z}$ and $i < j$. So, let $i, j, m, n \in \mathbb{Z}$ with $i < j$ and $m < n$. Then for a sufficiently large k (at least such that $k > j - m$)

$$
\begin{aligned}
&\phi^{\otimes}((\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes \cdots) \\
&\qquad\qquad (\alpha^{\otimes})^k(\cdots \otimes 1_{\mathcal{A}_{m-1}} \otimes b_m \otimes \cdots \otimes b_n \otimes 1_{\mathcal{A}_{n+1}} \otimes \cdots)) \\
&= \phi^{\otimes}((\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes \cdots) \\
&\qquad\qquad (\cdots \otimes 1_{\mathcal{A}_{m-1+k}} \otimes b_m \otimes \cdots \otimes b_n \otimes 1_{\mathcal{A}_{n+1+k}} \otimes \cdots)) \\
&= \phi^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes \cdots \\
&\qquad\qquad \cdots \otimes 1_{\mathcal{A}_{m-1+k}} \otimes b_m \otimes \cdots \otimes b_n \otimes 1_{\mathcal{A}_{n+1+k}} \otimes \cdots) \\
&= \phi_i(a_i) \cdots \phi_j(a_j)\phi_{m+k}(b_m) \cdots \phi_{n+k}(b_n) \\
&= \phi_i(a_i) \cdots \phi_j(a_j)\phi_m(b_m) \cdots \phi_n(b_n) \\
&= \phi^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes \cdots \otimes a_j \otimes 1_{\mathcal{A}_{j+1}} \otimes 1_{\mathcal{A}_{j+2}} \otimes \cdots) \\
&\qquad\qquad \phi^{\otimes}(\cdots \otimes 1_{\mathcal{A}_{m-1}} \otimes b_m \otimes \cdots \otimes b_n \otimes 1_{\mathcal{A}_{n+1}} \otimes \cdots).
\end{aligned}
$$

By the linearity of both the product state and the product automorphism, it is easily (but tediously) seen that this property holds for finite linear combinations of such tensors, and hence, on the dense *-algebra spanned by these tensors. We can now extend this to the whole C*-algebra $\mathcal{A}^{\otimes}$. Let $a, b \in \mathcal{A}^{\otimes}$ be arbitrary. Then there are sequences (a_n) and (b_n) in the *-algebra spanned by linear combinations of "simple" tensors (such as above), such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then

$$
\begin{aligned}
&|\phi^{\otimes}(a(\alpha^{\otimes})^k(b)) - \phi^{\otimes}(a)\phi^{\otimes}(b)| \\
&= |\phi^{\otimes}(a(\alpha^{\otimes})^k(b)) - \phi^{\otimes}(a_n(\alpha^{\otimes})^k(b_n)) + \phi^{\otimes}(a_n(\alpha^{\otimes})^k(b_n)) - \phi^{\otimes}(a_n)\phi^{\otimes}(b_n) \\
&\qquad\qquad + \phi^{\otimes}(a_n)\phi^{\otimes}(b_n) - \phi^{\otimes}(a)\phi^{\otimes}(b)| \\
&\leq ||a(\alpha^{\otimes})^k(b) - a_n(\alpha^{\otimes})^k(b_n)|| + |\phi^{\otimes}(a_n(\alpha^{\otimes})^k(b_n)) - \phi^{\otimes}(a_n)\phi^{\otimes}(b_n)| \\
&\qquad\qquad + |\phi^{\otimes}(a_n)\phi^{\otimes}(b_n) - \phi^{\otimes}(a)\phi^{\otimes}(b)| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \\
&= \epsilon
\end{aligned}
$$

for n and k large enough. Hence the C*-dynamical system $(\mathcal{A}^{\otimes}, \alpha^{\otimes}, \phi^{\otimes})$ is strongly mixing. □

Corollary 4.4.7. *The C*-dynamical system* $(\mathcal{A}^{\otimes}, \alpha^{\otimes}, \phi^{\otimes})$ *is ergodic for every state* ϕ *on* $\mathcal{A}$.

Proof. Follows directly from Proposition 4.4.5. □

Bibliography

[1] B. Abadie and K.J. Dykema. Unique erogodicity of free shifts and some other automorphisms of C*-algebras. *Journal of Operator Theory*, 61:2:279–294, 2009.

[2] M. Bożejko and R Speicher. An example of generalized Brownian motion. *Communications in Mathematical Physics*, 137:519–531, 1991.

[3] O. Bratteli and D.W. Robinson. *Operator algebras and quantum statistical mechanics 1*. Springer, second edition, 2002.

[4] D. L. Cohn. *Measure Theory*. Birkhauser, 1980.

[5] R. Duvenhage. Private communication.

[6] R. Duvenhage and A. Ströh. Disjointness and unique ergodicity of C*-dynamical systems. *arXiv:1102.4243v2 [math.OA]*, 15 August 2011.

[7] K.J. Dykema. Private communication.

[8] K.J. Dykema and F. Fidaleo. Unique mixing of the shift on the C*-algebra generated by the q-canonical commutation relations. *Houston Journal of Mathematics*, 36:275–281, 2010.

[9] K.J. Dykema and A. Nica. On the Fock representation of the q-commutation relations. *Journal Fur Reine und Angewandte Mathematik*, 440:201–212, 1993.

[10] J.F. Furlinger and J.Hofbauer. q-catalan numbers. *Journal of Combinatorial Theory, Series A*, 40, no.2:248–264, 1985.

[11] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *Journal d'analyse mathématique*, 31:204–256, 1977.

[12] H. Furstenberg. Poincaré recurrence and number theory. *Bulletin of the American Mathematical Society*, 5:211–234, 1981.

[13] Y. Katznelson H. Furstenberg and D. Ornstein. The ergodic theoretical proof of Szemerédi's theorem. *Bulletin of the American Mathematical Society*, 7:527–552, 1982.

[14] M.E.H. Ismail. Asymptotics of q-orthogonal polynomials and a q-Airy function. *International Mathematics Research Notices*, 18:1063–1088, 2005.

[15] R.V. Kadison and J.R. Ringrose. *Fundamentals of the theory of operator algebras, Vol I: Elementary Theory.* American Mathematical Society, 1997.

[16] R.V. Kadison and J.R. Ringrose. *Fundamentals of the theory of operator algebras, Vol II: Advanced Theory.* American Mathematical Society, 1997.

[17] M. Kennedy and A. Nica. Exactness of the Fock space representation of the q-commutation relations. *Communications in Mathematical Physics*, 308:115–132, 2011.

[18] E. Kreyszig. *Introductory functional analysis with applications.* John Wiley and Sons, 1989.

[19] G.J. Murphy. *C*-algebras and operator theory.* Academic Press. Inc., 1990.

[20] G.J. Pedersen. *C*-algebras and their automorphism groups.* Academic Press Inc., 1979.

[21] K. Petersen. *Ergodic Theory.* Cambridge University Press, 1989.

[22] A.P. Robertson and W. Robertson. *Topological Vector Spaces.* Cambridge University Press, 1973.

[23] W. Rudin. *Functional Analysis.* McGraw Hill Inc., 1991.

[24] V. Runde. *A taste of topology.* Springer, 2005.

[25] M. Takesaki. *Theory of Operator Algebras, Vol 1.* Springer, 1979.

[26] J. Tomiyama. On the projection of norm one in W*-algebras. *Proceedings of the Japan Academy*, 33:608–612, 1957.

[27] P. Walters. *An Introduction to Ergodic Theory.* Springer-Verlag, 1982.

[28] N. Weaver. *Mathematical Quantization.* Chapman & Hall/CRC, 2001.

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