Analysis of the Brinkman-Forchheimer equations with slip boundary conditions

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In this work, we study the Brinkman–Forchheimer equations driven under slip boundary conditions of friction type. We prove the existence and uniqueness of weak solutions by means of regularization combined with the Faedo-Galerkin approach. Next, we discuss the continuity of the solution with respect to Brinkman's and Forchheimer's coefficients. Finally, we show that the weak solution of the corresponding stationary problem is stable.

Keywords: Brinkman–Forchheimer equations; slip boundary conditions; weak solutions; continuous dependence; stability

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1. Introduction

We consider the Brinkman–Forchheimer equations for unsteady flows of incompressible fluids, i.e.

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } Q_T, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \ \text{ in } Q_T, \tag{1.2}$$

where $Q_T = \Omega \times (0, T)$, with $0 < T \leq \infty$ and Ω a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ composed of two components Γ and S satisfying $\Gamma \cap S = \emptyset$, $\partial\Omega = \overline{\Gamma \cup S}$, with $|\Gamma| \neq 0$ and $|S| \neq 0$. The motion of the incompressible fluid is described by the velocity $\boldsymbol{u}(\boldsymbol{x}, t) : Q_T \to \mathbb{R}^3$ and the pressure $p(\boldsymbol{x}, t) : Q_T \to \mathbb{R}$. In (1.1) and (1.2), $\boldsymbol{f} : Q_T \to \mathbb{R}^3$ is the external body force per unit volume, while the positive parameters ν, a, b are, respectively, the Brinkman coefficient, the Darcy coefficient and Forchheimer coefficient, and $\alpha \in [1, 2]$ is a given number. (1.2) means that the fluid is incompressible. For a complete description of the fluid partially represented by (1.1) and (1.2), proper initial and boundary conditions must be prescribed. As far as the initial condition goes, we assume that

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 \quad \text{on } \overline{\Omega}, \tag{1.3}$$

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where $\boldsymbol{u}_0 : \overline{\Omega} \to \mathbb{R}^3$ is a given function, that will be made precise later, and $\overline{\Omega}$ is the closure of Ω. Next in order to describe the motion of the fluid at the boundary having in mind the decomposition of $\partial\Omega$, we first assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma \times (0, T). \tag{1.4}$$

The more relevant physical condition $\boldsymbol{u}|_\Gamma = \widetilde{\boldsymbol{u}}$ with $\widetilde{\boldsymbol{u}} \in H^{1/2}(S)$ (at least) may also be adopted instead of (1.4). Indeed with the help of the lifting operator $\mathcal{L} : H^{1/2}(\Gamma) \to H^1(\Omega)$ such that $\boldsymbol{u} \cdot \boldsymbol{n}|_S = \mathcal{L}\widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_S$, which is continuous from $H^{s+1/2}(S)$ into $H^{s+1}(\Omega)$ for all $s \geq 0$ (the existence of such operator is established in [1], Chapter 4, Lemma 2.3) it is always possible to revert to the homogeneous boundary condition by considering the new variable $\widehat{\boldsymbol{u}} = \boldsymbol{u} - \mathcal{L}\widetilde{\boldsymbol{u}}$. One sees that $\widehat{\boldsymbol{u}}|_\Gamma = \boldsymbol{0}$.

On S, we first assume the impermeability condition

$$u_N = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } S \times (0, T), \tag{1.5}$$

where $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$. We recall that the normal component of the velocity if $(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}$, while its tangential part is $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - u_N \boldsymbol{n}$. In addition to (1.5), we also impose on S, a threshold slip condition,[2–7] which is the main ingredient of this work. The non-linear slip condition of "friction type" can be formulated with the knowledge of a positive function $g : S \longrightarrow (0, \infty)$ which is called the barrier or threshold function and the use of sub-differential to link quantities of interest. It is written as

$$-(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}} \in g\partial|\boldsymbol{u}_{\boldsymbol{\tau}}| \quad \text{on } S \times (0, T), \tag{1.6}$$

where $(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}}$ is the tangential component of the traction $\boldsymbol{\sigma}\boldsymbol{n}$ with $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{D}(\boldsymbol{u})$ being the Cauchy stress tensor and $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]$ the symmetry part of the velocity gradient. Of course $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$, with $|\boldsymbol{w}|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$. We recall that if X is a Hilbert space with $x_0 \in X$, then

$$y \in \partial\Psi(x_0) \text{ if and only if } \Psi(x) - \Psi(x_0) \geq y \cdot (x - x_0) \quad \forall x \in X. \tag{1.7}$$

Without using the sub-differential, the threshold condition (1.6) can be written as [8] (page 138–139)

$$\left.\begin{aligned} &|(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}}| \leq g, \\ &|(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ &|(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\,, \ -(\boldsymbol{\sigma}\boldsymbol{n})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S \times (0, T). \tag{1.8}$$

In [9], a generalization of the boundary condition (1.6) is formulated and analysed for the steady Stokes flow, while the case of Navier–Stokes equations has been examined in [10]. We do not propose the Brinkman–Forchheimer equations with the non-linear slip boundary conditions as a model of any real flow problem, but it should made clear that such boundary conditions are present in concrete situations such as oil flow over or beneath sand layers,[11,12] while Hervet and Leger in [13] have established the necessity to take into account frictional effects at the interfaces of a solid and a fluid. Hence, it appears that different boundary conditions describe different physical phenomena. In [14], the equations of Brinkman corresponding to (1.1) with $b = 0$ have been derived using mixtures theory, in fact a class of approximate models for flows of incompressible fluids passing porous solids have been described. *Forchheimer* [15] studied flow experiments in sandpacks and came to the conclusion that for small Reynolds numbers (Re $\approx$ 0.2), the diffusion law of Darcy is not significant. Furthermore, he found the relationship between the pressure gradient and the velocity obtained using the law of Darcy to be nonlinear. In fact for a wide range of physical experiments, he found that the nonlinear term should be quadratic. Inertial effects in the porous medium at relatively small Reynolds numbers are the cause of the nonlinear excess pressure drop observed by *Forchheimer* and others. The slip boundary conditions of friction type (1.6) can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important. In fact on the role of the boundary conditions for such problems, *Brinkman* [16] mentioned that "The flow through this porous media is described by a modification of Darcy's equation. Such modification was necessary to obtain consistent boundary conditions". While there continues to be some debate over the functionality of the Brinkman–Forchheimer model,[17] nonlinearity has been verified experimentally in [18], and some theoretical results have been obtained in [19–23]. The Brinkman–Forchheimer equation continues to be used for predicting the velocity of the flow in the porous region, while the energy equation for the porous region accounts for the effect of thermal dispersion.[24] Since, we are well aware that for such flow, there are important features at the boundary, it is therefore important to model Brinkman–Forchheimer flow by taking into account the non-trivial effect present at the boundary.

Even though flows under boundary conditions of friction type have been considered in various publications ([2–7,9,10] among others), and Brinkman–Forchheimer Equations (1.1), (1.2) with non-slip boundary conditions has been examined in [19–23], the combination of (1.1), (1.2) and (1.6) has not been presented in the literature, and it is the object of this work. The novelty of the problem, from the mathematical point of view, derives from the following features; the highly coupled and nonlinear nature of the problem, the incompressibility constraint and related pressure, and the non-linear slip boundary conditions of "friction type" (1.5) and (1.6).

Not surprisingly, flow problems involving boundary conditions of "friction type" offer significant theoretical and computational challenges. With regard to theoretical studies, the work by Hiroshi Fujita and co-authors [2–7], represent some early, contributions. These authors established existence, and uniqueness of solutions, for the equations corresponding to Stokes equations by means of semi-group approach, regularity of solutions are also examined. An interesting and related work is that by Christiaan Leroux and co-author [9,10] on Stokes and Navier Stokes equations under more general "friction type boundary conditions". As far as computational studies for flows driven by "friction type boundary conditions" are concerned, it should be mentioned that even though the literature is very rich in problems formulated in terms of variational inequalities, [25–28] not much have been done for the specific case involving mixed variational inequalities problems,[27,29–32] and we would like to explore that research direction in the future.

Problem (1.1)–(1.6) is a coupled nonlinear system of equations with a non-differentiable expression (at zero) on S due to the sub-differential term $\partial|\boldsymbol{u}_{\boldsymbol{\tau}}|$. We propose to solve the resulting system of partial differential equations using the regularization approach,[8,33] which consists of replacing the original problem by a sequence of "better behaved" approximate problems indexed by a small positive parameter ε. We then solve the regularized problems by the Faedo–Galerkin method, and finally, the solution of the original problem is obtained by passage to the limit as ε goes to zero. The difficulty in the algorithm described is to obtain the pressure. Indeed, as the problem in its weak form is formulated as a variational inequality with only one unknown, the pressure will not be obtained in the usual way (for the classical Navier–Stokes equations see e.g [34], [Theorem 2.5-1, page 54]). But, instead we first construct a regularized pressure by using the classical approach and then pass to the limit as ε goes to zero, after showing that the regularized pressures are bounded in some appropriate function space. After constructing weak solutions of the problem, we analyse some qualitative properties of the solution, namely; the continuous dependence of the solution with respect to the *Brinkman* and *Forchheimer* coefficients, and the stability of the stationary solution. The results presented, extend in some sense those obtained in [20,23] to a family of variational inequalities with non-differentiable functionals.

The remaining part of this work is organized as follows. In Section 2, we document the variational formulation associated to the problem and prove its well-posedness. Section 3 is devoted to the stability of the solutions with respect to some data of the problem. The stability of the stationary solutions is analysed in Section 4.

2. Analysis of the problem: Solvability

We introduce some preliminaries and notation for the mathematical setting of the problem. We write down a variational formulation of problem (1.1)–(1.6). Next, we derive some *a priori* estimates of its solution and obtain the existence of solutions by means of Faedo–Galerkin.

2.1. *Preliminaries/Notation*

In what follows, for $1 \le p \le \infty$, $L^p(\Omega)$, and $L^p(\partial\Omega)$ are the usual Lebesgue spaces, with norms denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^p(\partial\Omega)}$, respectively, (of course when $p = 2$, we will denoted the norm in $L^2(\Omega)$ by $\|\cdot\|$). We shall use the following notation; for the sake of simplicity, one defines them in three dimensions. Let $k = (k_1, k_2, k_3)$ denote a triple of non-negative intergers, set $|k| = k_1 + k_2 + k_3$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}}.$$

Then, for non-negative integer m, we recall the classical Sobolev space

$$H^m(\Omega) = \{v \in L^2(\Omega) \; ; \; \partial^k v \in L^2(\Omega) \quad \forall \, |k| \le m\}$$

equipped with the seminorm

$$|v|_{H^m(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^2 \mathrm{dx} \right]^{1/2}$$

and norm

$$||v||_{H^m(\Omega)} = \left[\sum_{0 \le k \le m} \int_\Omega |\partial^k v|^2 \mathrm{dx} \right]^{1/2}.$$

For $p = 1, 2, 3, \ldots$, the inner products in the spaces $L^2(\Omega)^p$, $L^2(\partial\Omega)^p$ and $H^1(\Omega)^p$ are denoted by $(\cdot,\cdot)$, $(\cdot,\cdot)_{\partial\Omega}$ and $(\cdot,\cdot)_1$, respectively. The product spaces are denoted by bold letters: $\mathbf{H}^1(\Omega) = H^1(\Omega)^3$, $\mathbf{L}^2(\Omega) = L^2(\Omega)^3$, $\mathbf{L}^{\alpha+2}(\Omega) = L^{\alpha+2}(\Omega)^3$, etc.

Here, and in what follows, the boundary values are to be understood in the sense of traces. We omit the trace operators where the meaning is direct; otherwise we denote the traces by $\boldsymbol{v}|_\Gamma$, $\boldsymbol{v}|_S$, etc. Also, all the derivatives should be understood in the sense of distribution. We also recall from [1] (Chap. I, Theorem 1.1) for instance the following Poincaré–Friedrichs inequality:

$$\text{for all} \quad \boldsymbol{v} \in \mathbf{H}^1(\Omega) \cap \{v_n|_S = 0\,,\ \boldsymbol{v}|_\Gamma = 0\}, \quad \|\boldsymbol{v}\| \leq C\|\nabla \boldsymbol{v}\|, \tag{2.1}$$

which yields the equivalence of the norms $\|\cdot\|_1$ and $|\cdot|_1$ on $\mathbf{H}^1(\Omega) \cap \{v_n|_S = 0\,,\ \boldsymbol{v}|_\Gamma = 0\}$. For any separable Banach space E equipped with the norm $\|\cdot\|_E$, we denote by $C^0(0, T; E)$ the space of continuous functions from $[0, T]$ with values in E and by $D'(0, T; E)$ the space of distributions with values in E. $L^p(0, T; E)$ is a Banach space consisting of (classes of) functions $t \longmapsto f(t)$ measurable from $[0, T] \longmapsto E$ (for the measure dt) such that

$$\|f\|_{L^p(0,T;E)} = \left[\int_0^T \|f(t)\|_E^p dt\right]^{1/p} < \infty \quad \text{for } p \neq \infty$$
$$\|f\|_{L^\infty(0,T;E)} = \text{ess}_{0<t<T} \sup \|f(t)\|_E < \infty.$$

In what follows, $\phi(t)$ stands for the function $\boldsymbol{x} \in \Omega \mapsto \phi(\boldsymbol{x}, t)$.

We assume that the data $(\boldsymbol{f}, g)$ belong to $L^2(0, T; \mathbf{L}^2(\Omega)) \times L^\infty(S)^2$, and that the datum $\boldsymbol{u}_0$ belongs to $\mathbf{H}^1(\Omega) \cap \mathbf{L}^{\alpha+2}(\Omega)$, and satisfies the incompressibility condition

$$\text{div}\, \boldsymbol{u}_0 = 0 \quad \text{in } \Omega. \tag{2.2}$$

This last condition is not necessary for all the results that follow but, since it is not restrictive, we shall assume it from now on.

2.2. *Variational formulation*

In order to write a variational form associated with (1.1)–(1.6), we retain (1.3) and we weaken the Equations (1.1), (1.2) and constraints (1.4), (1.5) using the Green's formula, while (1.6) is re-interpreted with the help of (1.7). It follows from the nonlinear term in (1.1) that $\boldsymbol{u}(t)$ and the test function $\boldsymbol{v}$ should belong to $\mathbf{L}^{\alpha+2}(\Omega)$. Then $\boldsymbol{u}'(t)$ and $|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t)$ must belong to the conjugate of $\mathbf{L}^{\alpha+2}(\Omega)$, which is $\mathbf{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)$. We then introduce the following spaces

$$\mathcal{N} = \mathbf{H}^1(\Omega) \cap \{\boldsymbol{v}|_\Gamma = 0\ v_n|_S = 0\},$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega),\ (q, 1) = 0\}.$$

One quickly observes that since $1 \leq \alpha \leq 2$, $H^1(\Omega)$ is embedded into $L^{\alpha+2}(\Omega)$. Next, we introduce the following definition of weak solutions of (1.1)–(1.6).

Definition 2.1 Given $(\boldsymbol{f}, g)$ in $L^2(0, T; \mathbf{L}^2(\Omega)) \times L^\infty(S)^2$, and $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$, satisfying (2.2). We say that $(\boldsymbol{u}, p)$ is a weak solution of (1.1)–(1.6) if and only if; $\boldsymbol{u} \in L^\infty(0, T; \mathcal{N})$, $p \in L^2(0, T; M)$, and $\boldsymbol{u}' \in L^2\left(0, T; \mathcal{N}'\right)$, and for almost all t and all $q \in L^2(\Omega)$, $\boldsymbol{v} \in \mathcal{N}$

$$\begin{aligned}&\left(\boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)\right) + \gamma\left(\nabla \boldsymbol{u}(t), \nabla(\boldsymbol{v} - \boldsymbol{u}(t))\right) + a\left(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\right)\\ &+ b\left(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\right) - \left(\text{div}(\boldsymbol{v} - \boldsymbol{u}(t)), p(t)\right)\\ &+ J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq \left(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\right),\end{aligned} \tag{2.3}$$

$$(\operatorname{div} \boldsymbol{u}(t), q) = 0, \tag{2.4}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{2.5}$$

where, $J(\boldsymbol{v}) = (g(\boldsymbol{x}), |\boldsymbol{v}_{\boldsymbol{\tau}}(\boldsymbol{x})|)_{L^2(S)}$.

Following [8], it can be shown that any solution of (1.1)–(1.6) is a solution of (2.3)–(2.5) in the sense of distributions. The converse property holds for any solution of the problem (1.1)–(1.6) that enjoys the regularity mentioned in Definition 2.1, in a sense to be made precise later on. The kernel of the bilinear and continuous form $L^2(\Omega) \times \mathcal{N} \ni (q, \boldsymbol{v}) \longmapsto (q, \operatorname{div} \boldsymbol{v}) \in \mathbb{R}$ is

$$\mathbb{V} = \{\boldsymbol{v} \in \mathcal{N}, \ \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega\}.$$

With the space $\mathbb{V}$ in mind, it is then easy to see that the function $\boldsymbol{u}(t)$ given in (2.3)–(2.5) is a solution of the simpler variational problem: Find $\boldsymbol{u} \in L^\infty(0, T; \mathbb{V})$, $\boldsymbol{u}' \in L^2(0, T; \mathbb{V}')$ satisfying (2.5) such that for almost all t and all $\boldsymbol{v} \in \mathbb{V}$

$$\begin{aligned}(\boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)) + \gamma(\nabla \boldsymbol{u}(t), \nabla(\boldsymbol{v} - \boldsymbol{u}(t))) + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\ + b(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)).\end{aligned} \tag{2.6}$$

Next, we establish the solvability of the variational problem (2.6) by means of regularization combined with Galerkin's method. We then construct a pressure p in $L^2(0, T; L^2_0(\Omega))$ such that the couple $(\boldsymbol{u}, p)$ enjoys the regularity announced in Definition 2.1, and satisfies (2.3)–(2.5).

2.3. *Existence of a solution*

In this paragraph, we discuss the solvability of (2.6) by regularization, and passage to the limit. Thus, it is obtained in several steps, that we describe below.

Step 1 *Regularized problem.*
We first recall that one of the difficulties of solving (2.6) is the fact that the functional $\boldsymbol{v} \in \mathbb{V} \longmapsto J(\boldsymbol{v}) = (g(\boldsymbol{x}), |\boldsymbol{v}_{\boldsymbol{\tau}}(\boldsymbol{x})|)_S$ is not differentiable at zero. So, to bypass that hurdle we introduce the regularized functional J_ε defined by

$$v \in \mathbb{V} \mapsto J_\varepsilon(v) = \left(g(\boldsymbol{x}), \sqrt{|v_{\boldsymbol{\tau}}(\boldsymbol{x})|^2 + \varepsilon^2}\right)_S, \quad 0 < \varepsilon \ll 1.$$

Clearly, J_ε is convex and Gateaux differentiable with Gateaux derivative K_ε defined on $\mathbb{V}$ and given by

$$\langle K_\varepsilon(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_S g \frac{\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}}{\sqrt{|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \varepsilon^2}} \, ds,$$

moreover K_ε is monotone, that is

$$\langle K_\varepsilon(\boldsymbol{u}) - K_\varepsilon(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \geq 0. \tag{2.7}$$

The regularized form of (2.6) can be written as follows: Find $\boldsymbol{u}_\varepsilon \in L^\infty(0,T;\mathbb{V})$ satisfying (2.5) with $\boldsymbol{u}'_\varepsilon \in L^2(0,T;\mathbb{V}')$ such that for almost all t and all $\boldsymbol{v} \in \mathbb{V}$

$$\begin{aligned}&(\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla(\boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)))\\ &+ a(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) + b(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t))\\ &+ J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)).\end{aligned} \tag{2.8}$$

Since J_ε is differentiable, adopting the classical arguments in [8] (see page 157–158), one can state that (2.8) is equivalent to: Find $\boldsymbol{u}_\varepsilon \in L^\infty(0,T;\mathbb{V})$ satisfying (2.5) with $\boldsymbol{u}'_\varepsilon \in L^2(0,T;\mathbb{V}')$ such that for almost all t

$$\begin{aligned}&(\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla \boldsymbol{v}) + a(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) + b(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v})\\ &+ \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v}\rangle = (\boldsymbol{f}(t), \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbb{V}.\end{aligned} \tag{2.9}$$

Before proving the existence of a solution $\boldsymbol{u}_\varepsilon(t)$ of (2.9), we first show how the pressure is constructed knowing the velocity. For that purpose, we begin by integrating (2.9) on $[0,t]$, apply (2.5), and for $\boldsymbol{v} \in \mathcal{N}$; we introduce the functional

$$\begin{aligned}\mathcal{H}(\boldsymbol{v})(t) = &\int_0^t \Big[(\boldsymbol{f}(s), \boldsymbol{v}) - \nu(\nabla \boldsymbol{u}_\varepsilon(s), \nabla \boldsymbol{v}) - a(\boldsymbol{u}_\varepsilon(s), \boldsymbol{v}) - b(|\boldsymbol{u}_\varepsilon(s)|^\alpha \boldsymbol{u}_\varepsilon(s), \boldsymbol{v})\Big]\mathrm{ds}\\ &- \int_0^t \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(s)), \boldsymbol{v}\rangle \mathrm{ds} - (\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) + (\boldsymbol{u}_0, \boldsymbol{v}), \quad \text{for all } 0 \leq t \leq T.\end{aligned}$$

One sees that $\mathcal{H}$ is linear and continuous on $\mathcal{N}$, and according to (2.9) and (2.5), it vanishes on $\mathbb{V}$. Now following [34], [Theorem 2.5-1, page 54], for each $t \in [0,T]$, there exists a unique function $\widetilde{p}_\varepsilon(t) \in L_0^2(\Omega)$ and a positive constant C such that: for all $\boldsymbol{v} \in \mathcal{N}$,

$$\mathcal{H}(\boldsymbol{v})(t) = (\operatorname{div} \boldsymbol{v}, \widetilde{p}_\varepsilon(t)), \tag{2.10}$$

$$C\|\widetilde{p}_\varepsilon(t)\| \leq \sup_{\boldsymbol{v}\in\mathcal{N}} \frac{(\operatorname{div} \boldsymbol{v}, \widetilde{p}_\varepsilon(t))}{\|\boldsymbol{v}\|_1}. \tag{2.11}$$

Finally, we take the time derivative on both sides of (2.10); and we let

$$p_\varepsilon(t) = \frac{d}{dt}\widetilde{p}_\varepsilon(t), \tag{2.12}$$

in the resulting equation. Thus, we have obtained the following variational problem: Find $\boldsymbol{u}_\varepsilon \in L^2(0,T;\mathcal{N})$, $p_\varepsilon \in L^2(0,T;L_0^2(\Omega))$ with $\boldsymbol{u}'_\varepsilon \in L^2(0,T;\mathcal{N}')$ such that for almost all t and all $q \in L^2(\Omega)$, $\boldsymbol{v} \in \mathcal{N}$

$$\begin{aligned}&(\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla \boldsymbol{v}) + a(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) + b(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v})\\ &- (\operatorname{div} \boldsymbol{v}, p_\varepsilon(t)) + \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v}\rangle = (\boldsymbol{f}(t), \boldsymbol{v}),\\ &(\operatorname{div} \boldsymbol{u}_\varepsilon(t), q) = 0,\\ &\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0.\end{aligned} \tag{2.13}$$

It is clear that the variational problems (2.9) and (2.13) are equivalent, with the regularized pressure described by (2.10), (2.11) and (2.12).

Step 2 *Faedo–Galerkin approximation.*

We let

$$\mathbb{H} = \{\boldsymbol{v} \in \mathbf{L}^2(\Omega), \quad \text{div}\, \boldsymbol{v} = 0, \quad v_n|_{\partial\Omega} = 0\} \cap \mathbf{L}^{\alpha+2}(\Omega).$$

One readily observes that $\mathbb{V}$ is compactly embedded in $\mathbb{H}$. For the slip boundary condition, we introduce the Stokes operator defined on a subspace of $\mathbb{V}$ constructed in [35] as follows; for every $\boldsymbol{f} \in \mathbb{H}$, there exists a unique $\boldsymbol{v} \in \mathbb{V}$ such that

$$(\nabla \boldsymbol{v}, \nabla \phi) = (\boldsymbol{f}, \phi), \quad \forall \phi \in \mathbb{V}. \tag{2.14}$$

Moreover, for every $\boldsymbol{v} \in \mathbb{V}$, there exists a unique $\boldsymbol{f} \in \mathbb{H}$ such that (2.14) holds. Then (2.14) defines a one-to-one mapping between $\boldsymbol{f} \in \mathbb{H}$ and $\boldsymbol{v} \in D(A)$, where $D(A)$ is a subspace of $\mathbb{V}$. Hence, $A\boldsymbol{v} = \boldsymbol{f}$ defines the Stokes operator $A : D(A) \to \mathbb{H}$. Its inverse A^{-1} is compact and self-adjoint as a mapping from $\mathbb{H}$ to $\mathbb{H}$ and possesses an orthogonal sequence of eigenfunctions $\boldsymbol{\psi}_k$ which are complete in $\mathbb{H}$ and $\mathbb{V}$;

$$A\boldsymbol{\psi}_k = \lambda_k \boldsymbol{\psi}_k. \tag{2.15}$$

Let $\mathbb{V}_m$ be the subspace of $\mathbb{V}$ spanned by $\psi_1, \ldots, \psi_m$, that is

$$\mathbb{V}_m = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_m\}.$$

We consider the following ordinary differential equation: Find $\boldsymbol{u}_{\varepsilon,m}(t) \in \mathbb{V}_m$ such that for all $\boldsymbol{v} \in \mathbb{V}_m$;

$$\begin{aligned}(\boldsymbol{u}'_{\varepsilon,m}(t), \ \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_{\varepsilon,m}(t), \ \nabla \boldsymbol{v}) + a\,(\boldsymbol{u}_{\varepsilon,m}(t), \ \boldsymbol{v})\\ + b\,(|\boldsymbol{u}_{\varepsilon,m}(t)|^\alpha \boldsymbol{u}_{\varepsilon,m}(t), \ \boldsymbol{v}) + \langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)), \ \boldsymbol{v}\rangle = (\boldsymbol{f}(t), \boldsymbol{v}),\\ \boldsymbol{u}_{\varepsilon,m}(0) \to \boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 \in \mathbb{V}_m.\end{aligned} \tag{2.16}$$

As far as the existence of $\boldsymbol{u}_{\varepsilon,m}(t)$ defined by (2.16) is concerned, we note that the mapping

$$\mathcal{K} : \boldsymbol{w} \longmapsto (\boldsymbol{f}, \boldsymbol{v}) - \nu(\nabla \boldsymbol{w}, \ \nabla \boldsymbol{v}) - a\,(\boldsymbol{w}, \ \boldsymbol{v}) - b\,(|\boldsymbol{w}|^\alpha \boldsymbol{w}, \ \boldsymbol{v}) - \langle K_\varepsilon(\boldsymbol{w}), \ \boldsymbol{v}\rangle,$$

is locally Lipschitz thanks to the nature of the operators involved. It then follows from the theory of ordinary differential equations that (2.16) has a solution $\boldsymbol{u}_{\varepsilon,m}$ defined on $[0, t_{\varepsilon,m}]$, $t_{\varepsilon,m} > 0$. Hereafter, C denotes a constant independent of m, and depending only on the data such as Ω, and whose value may be different in each inequality. Next, we derive some *a priori* estimates and deduce that $t_{\varepsilon,m}$ does not depend on ε or m. Concerning the later property, it should be mentioned as in [8,33], that it suffices to derive *a priori* estimates of the solution with the right hand side independent of m and ε.

Step 3 Some a priori estimates.
First, we let $\boldsymbol{v} = \boldsymbol{u}_{\varepsilon,m}(t)$ in (2.16). After using Young's inequality, one obtains

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}_{\varepsilon,m}(t)\|^2 + 2\nu\|\nabla \boldsymbol{u}_{\varepsilon,m}(t)\|^2 + a\|\boldsymbol{u}_{\varepsilon,m}(t)\|^2 + 2b\,\left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|^{\alpha+2}_{L^{\alpha+2}}\\ + 2\langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t))\,,\ \boldsymbol{u}_{\varepsilon,m}(t)\rangle \le \frac{||f(t)||^2}{a},\end{aligned} \tag{2.17}$$

which by integrating over $[0, T^\sharp]$ for $T^\sharp \leq t_{\varepsilon,m}$, and using (2.7), yields

$$\sup_{0\leq t\leq T^\sharp} ||\boldsymbol{u}_{\varepsilon,m}(t)||^2 + 2\nu \int_0^{T^\sharp} ||\nabla \boldsymbol{u}_{\varepsilon,m}(t)||^2 dt + a \int_0^{T^\sharp} ||\boldsymbol{u}_{\varepsilon,m}(t)||^2 dt$$
$$+ 2b \int_0^{T^\sharp} \left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|_{L^{\alpha+2}}^{\alpha+2} dt \leq \frac{1}{a} \int_0^{T^\sharp} ||\boldsymbol{f}(t)||^2 dt + ||\boldsymbol{u}_0||^2 < \infty, \tag{2.18}$$

since by assumption $\boldsymbol{f} \in L^2(Q)$. Now let $\boldsymbol{v} = \boldsymbol{u}'_{\varepsilon,m}(t)$ in (2.16). For $0 \leq t \leq T^\sharp$, Young's inequality yields

$$||\boldsymbol{u}'_{\varepsilon,m}(t)||^2 + \frac{d}{dt}\left[\nu||\nabla \boldsymbol{u}_{\varepsilon,m}(t)||^2 + a||\boldsymbol{u}_{\varepsilon,m}(t)||^2 + \frac{2b}{\alpha+2}\left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|_{L^{\alpha+2}}^{\alpha+2}\right]$$
$$+ \frac{d}{dt}\left[2K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t))\right] \leq ||\boldsymbol{f}(t)||^2,$$

which leads to

$$\int_0^{T^\sharp} ||\boldsymbol{u}'_{\varepsilon,m}(t)||^2 dt + \nu||\nabla \boldsymbol{u}_{\varepsilon,m}(t)||^2 + a||\boldsymbol{u}_{\varepsilon,m}(t)||^2 + \frac{2b}{\alpha+2}\left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|_{L^{\alpha+2}}^{\alpha+2}$$
$$+ 2K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)) \leq ||\boldsymbol{f}||^2_{L^2(0,\,T^\sharp;\,L^2)} + \Phi(0), \tag{2.19}$$

where

$$\Phi(0) = \nu||\nabla \boldsymbol{u}_0||^2 + a||\boldsymbol{u}_0||^2 + \frac{2b}{\alpha+2}\left\|\boldsymbol{u}_0\right\|_{L^{\alpha+2}}^{\alpha+2} + 2\int_S g\sqrt{|\boldsymbol{u}_0|^2+1}\mathrm{ds}.$$

It is manifest that the right hand sides of the *a priori* estimates obtained in (2.18) and (2.19) are independent of m and ε. We then conclude that $t_{\varepsilon,m}$ is independent of ε and m following the arguments discussed in length by [8,33].

Step 4 *Passage to the limit.*

We need to pass to the limit when m approaches infinity and ε approaches zero. We start by fixing ε and study the mapping $m \longmapsto \boldsymbol{u}_{\varepsilon,m}$.

Based on (2.18) and (2.19), it is clear that when $m \to \infty$,

$$\begin{aligned} &\boldsymbol{u}_{\varepsilon,m} \text{ remains bounded in } L^\infty(0, T; \mathbb{H}),\\ &|\boldsymbol{u}_{\varepsilon,m}|^\alpha \boldsymbol{u}_{\varepsilon,m} \text{ remains bounded in } L^{\frac{\alpha+2}{\alpha+1}}(0, T; \mathbf{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)),\\ &\boldsymbol{u}'_{\varepsilon,m} \text{ remains bounded in } L^2(0, T; \mathbf{L}^2(\Omega)). \end{aligned} \tag{2.20}$$

From a consequence of the result of *Dunford-Pettis* [36], it is possible to extract from $(\boldsymbol{u}_{\varepsilon,m})_m$ a subsequence, denoted again by $(\boldsymbol{u}_{\varepsilon,m})_m$ such that

$$\boldsymbol{u}_{\varepsilon,m} \longrightarrow \boldsymbol{u}_\varepsilon \text{ weak star in } L^\infty(0, T; \mathbb{H}) \tag{2.21}$$
$$\boldsymbol{u}_{\varepsilon,m} \longrightarrow \boldsymbol{u}_\varepsilon \text{ weak star in } L^\infty(0, T; \mathbb{V}_m) \tag{2.22}$$
$$|\boldsymbol{u}_{\varepsilon,m}|^\alpha \boldsymbol{u}_{\varepsilon,m} \longrightarrow \chi_\varepsilon \text{ weak star in } L^{\frac{\alpha+2}{\alpha+1}}\left(0, T; \mathbf{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)\right) \tag{2.23}$$
$$\boldsymbol{u}'_{\varepsilon,m} \longrightarrow \boldsymbol{u}'_\varepsilon \text{ weak in } L^2(0, T; \mathbb{H}). \tag{2.24}$$

Now (2.21) and (2.24) imply in particular that

$$\boldsymbol{u}_{\varepsilon,m} \text{ remains in a bounded set of } H^1(Q). \tag{2.25}$$

But from Rellich-Kondrachoff, the embedding $H^1(Q) \longmapsto L^2(Q)$ is compact. So one can extract from $(\boldsymbol{u}_{\varepsilon,m})$ a subsequence, denoted again by $(\boldsymbol{u}_{\varepsilon,m})$ such that

$$\boldsymbol{u}_{\varepsilon,m} \longrightarrow \boldsymbol{u}_\varepsilon \text{ strong in } L^2(0,T;\mathbb{H}) \text{ and a.e. in } Q. \tag{2.26}$$

Next, it follows from (2.23) and (2.26) and Lemma 1.3 in [33] (page 12) that $\chi_\varepsilon = |\boldsymbol{u}_\varepsilon|^\alpha \boldsymbol{u}_\varepsilon$. It remains to prove that

$$K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}) \longrightarrow K_\varepsilon(\boldsymbol{u}_\varepsilon) \text{ weak star in } L^\infty(0,T,\mathbb{V}_m'). \tag{2.27}$$

Firstly from (2.22)

$$K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}) \longrightarrow \beta_\varepsilon \text{ weak star in } L^\infty(0,T,\mathbb{V}_m'). \tag{2.28}$$

Passing to the limit in (2.16), one obtains

$$\begin{aligned}&(\boldsymbol{u}_\varepsilon'(t),\boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t),\ \nabla \boldsymbol{v}) + a\,(\boldsymbol{u}_\varepsilon(t),\ \boldsymbol{v}) + b\,(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t),\ \boldsymbol{v}) + \langle \beta_\varepsilon, \boldsymbol{v}\rangle \\ &\quad = (\boldsymbol{f}(t),\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbb{V}_m.\end{aligned} \tag{2.29}$$

Next, for any $\boldsymbol{w} \in L^1(0,T;\mathbb{V}_m)$, and using the fact that $K_\varepsilon(\cdot)$ is monotone (see 2.7), one gets

$$\langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)),\ \boldsymbol{u}_{\varepsilon,m}(t)\rangle \geq \langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)),\ \boldsymbol{w}\rangle + \langle K_\varepsilon(\boldsymbol{w}),\ \boldsymbol{u}_{\varepsilon,m}(t) - \boldsymbol{w}\rangle.$$

But from (2.16)

$$\begin{aligned}\langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)),\ \boldsymbol{u}_{\varepsilon,m}(t)\rangle &= (\boldsymbol{f}(t),\boldsymbol{u}_{\varepsilon,m}(t)) - (\boldsymbol{u}_{\varepsilon,m}'(t),\ \boldsymbol{u}_{\varepsilon,m}(t)) \\ &\quad - \nu(\nabla \boldsymbol{u}_{\varepsilon,m}(t),\ \nabla \boldsymbol{u}_{\varepsilon,m}(t)) - a\,(\boldsymbol{u}_{\varepsilon,m}(t),\ \boldsymbol{u}_{\varepsilon,m}(t)) \\ &\quad - b\,(|\boldsymbol{u}_{\varepsilon,m}(t)|^\alpha \boldsymbol{u}_{\varepsilon,m}(t),\ \boldsymbol{u}_{\varepsilon,m}(t)).\end{aligned}$$

Putting together the former and latter equations yields

$$\begin{aligned}(\boldsymbol{f}(t),\boldsymbol{u}_{\varepsilon,m}(t)) - \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{\varepsilon,m}(t)\|^2 - \nu\|\nabla \boldsymbol{u}_{\varepsilon,m}(t)\|^2 - a\|\boldsymbol{u}_{\varepsilon,m}(t)\|^2 - b\,\left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|_{L^{\alpha+2}}^{\alpha+2} \\ \geq \langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)),\ \boldsymbol{w}\rangle + \langle K_\varepsilon(\boldsymbol{w}),\ \boldsymbol{u}_{\varepsilon,m}(t) - \boldsymbol{w}\rangle,\end{aligned}$$

which by integration with respect to t on $[0,T]$, gives

$$\begin{aligned}&\int_0^T (\boldsymbol{f}(t),\boldsymbol{u}_{\varepsilon,m}(t))dt - \frac{1}{2}\|\boldsymbol{u}_{\varepsilon,m}(T)\|^2 + \frac{1}{2}\|\boldsymbol{u}_{\varepsilon,m}(0)\|^2 \\ &\quad - \int_0^T \left[\nu\|\nabla \boldsymbol{u}_{\varepsilon,m}(t)\|^2 + a\|\boldsymbol{u}_{\varepsilon,m}(t)\|^2 + b\,\left\|\boldsymbol{u}_{\varepsilon,m}(t)\right\|_{L^{\alpha+2}}^{\alpha+2}\right]dt \\ &\geq \int_0^T \left[\langle K_\varepsilon(\boldsymbol{u}_{\varepsilon,m}(t)),\ \boldsymbol{w}(t)\rangle + \langle K_\varepsilon(\boldsymbol{w}(t)),\ \boldsymbol{u}_{\varepsilon,m}(t) - \boldsymbol{w}(t)\rangle\right]\mathrm{dt}.\end{aligned} \tag{2.30}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u}_{\varepsilon,m}(t)$ in (2.29), and combine the resulting equation with (2.30), one obtains (after taking the limit as m approaches to infinity)

$$\int_0^T \langle \beta_\varepsilon - K_\varepsilon(\boldsymbol{w}(t)),\ \boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t)\rangle dt \geq 0\,. \tag{2.31}$$

At this juncture, we let $\boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t) = \pm \boldsymbol{q}$ with $\boldsymbol{q} \in L^2(0, T; \mathbb{V}_m)$. Thus (2.31) leads to

$$\int_0^T \langle \beta_\varepsilon - K_\varepsilon(\boldsymbol{w}(t)),\ \boldsymbol{q}\rangle dt = 0,$$

from which we deduce (2.27). We have established that as m goes to infinity, the sequence $(\boldsymbol{u}_{\varepsilon,m}(t))_m$ converges to $\boldsymbol{u}_\varepsilon(t)$ in some sense with $\boldsymbol{u}_\varepsilon(t)$, the solution of

$$\begin{aligned}(\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t),\ \nabla \boldsymbol{v}) + a\,(\boldsymbol{u}_\varepsilon(t),\ \boldsymbol{v}) + b\,(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t),\ \boldsymbol{v})&\\ + \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v}\rangle = (\boldsymbol{f}(t), \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \mathbb{V}_m.&\end{aligned} \tag{2.32}$$

Since $\cup_m \mathbb{V}_m$ is dense in $\mathbb{V}$, we can conclude that (2.32) holds true for $\boldsymbol{v}$ in $\mathbb{V}$. Therefore, we have established that there exists a function $\boldsymbol{u}_\varepsilon$ uniformly bounded with respect to ε in $L^\infty(0, T, \mathbb{H} \cap \mathbb{V} \cap \mathbf{L}^{\alpha+2}(\Omega))$ such that $\boldsymbol{u}'_\varepsilon$ is uniformly bounded with respect to ε in $L^2(0, T, \mathbb{H})$ and $\boldsymbol{u}_\varepsilon$ satisfies (2.32).

Our final task in this paragraph is to consider the limit as ε goes to zero.

First, we take the limit on both sides of (2.18) and (2.19), one has

$$\begin{aligned}\sup_{0\le t\le T} ||\boldsymbol{u}_\varepsilon(t)||^2 + 2\nu \int_0^T ||\nabla \boldsymbol{u}_\varepsilon(t)||^2 dt + a \int_0^T ||\boldsymbol{u}_\varepsilon(t)||^2 dt&\\ + 2b \int_0^T \|\boldsymbol{u}_\varepsilon(t)\|^{\alpha+2}_{L^{\alpha+2}}\, dt \le \frac{1}{a} \int_0^T ||\boldsymbol{f}(t)||^2 dt + ||\boldsymbol{u}_0||^2,&\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\int_0^T ||\boldsymbol{u}'_\varepsilon(t)||^2 dt + \nu||\nabla \boldsymbol{u}_\varepsilon(t)||^2 + a||\boldsymbol{u}_\varepsilon(t)||^2 + \frac{2b}{\alpha+2} \|\boldsymbol{u}_\varepsilon(t)\|^{\alpha+2}_{L^{\alpha+2}}&\\ \le \int_0^T ||\boldsymbol{f}(t)||^2 dt + \Phi(0).&\end{aligned} \tag{2.34}$$

Thus, we can extract from $\boldsymbol{u}_\varepsilon$ a subsequence still denoted by $\boldsymbol{u}_\varepsilon$ such that

$$\boldsymbol{u}_\varepsilon \longrightarrow \boldsymbol{u} \text{ weak star in } L^\infty(0, T, \mathbb{H}) \tag{2.35}$$

$$\boldsymbol{u}_\varepsilon \longrightarrow \boldsymbol{u} \text{ weak star in } L^\infty(0, T, \mathbb{V}) \tag{2.36}$$

$$|\boldsymbol{u}_\varepsilon|^\alpha \boldsymbol{u}_\varepsilon \longrightarrow \chi \text{ weak in } L^{\frac{\alpha+2}{\alpha+1}}\left(0, T, \mathbf{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)\right) \tag{2.37}$$

$$\boldsymbol{u}'_\varepsilon \longrightarrow \boldsymbol{u}' \text{ weak in } L^2(0, T, \mathbb{H}). \tag{2.38}$$

Arguing as before we can prove that

$$\boldsymbol{u}_\varepsilon \longrightarrow \boldsymbol{u} \text{ strong in } L(0, T; \mathbb{H}) \text{ and a.e. in } Q, \tag{2.39}$$

$$|\boldsymbol{u}_\varepsilon|^\alpha \boldsymbol{u}_\varepsilon \longrightarrow |\boldsymbol{u}|^\alpha \boldsymbol{u} \text{ weak in } L^{\frac{\alpha+2}{\alpha+1}}\left(0, T; \mathbf{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)\right). \tag{2.40}$$

Let $\boldsymbol{v} \in L^2(0, T, \mathbb{V})$, from (2.32), it follows that

$$\begin{aligned}&(\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla(\boldsymbol{v} - \boldsymbol{u}_\varepsilon(t))) + a\,(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t))\\ &+ b\,(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)) + J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_\varepsilon(t))\\ &\quad = (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)) + J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) - \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)\rangle.\end{aligned} \tag{2.41}$$

Integrating (2.41) with respect to t along $[0,T]$ and taking into account the fact that $J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) - \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)\rangle \geq 0$, one obtains

$$\begin{aligned}
&\int_0^T \big((\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla \boldsymbol{v}) + a(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) + b(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v})\big)\, dt \\
&\qquad\qquad + \int_0^T (J_\varepsilon(\boldsymbol{v}) - (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\varepsilon(t)))\, dt \\
&\geq \frac{1}{2}||\boldsymbol{u}_\varepsilon(T)||^2 - \frac{1}{2}||\boldsymbol{u}_{\varepsilon 0}||^2 + \int_0^T \left(a||\boldsymbol{u}_\varepsilon(t)||^2 + b\int_\Omega |\boldsymbol{u}_\varepsilon(t)|^{\alpha+2} dx\right) dt \\
&\qquad\qquad + \int_0^t J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) dt.
\end{aligned} \tag{2.42}$$

Since $\boldsymbol{u}_\varepsilon \longrightarrow \boldsymbol{u}$ weak star in $L^\infty(0, T, \mathbb{V})$, and J_ε is a convex and continuous functional on $\mathbb{V}$, one has

$$\liminf_{\varepsilon\to 0} \int_0^T J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) dt \geq \int_0^T J(\boldsymbol{u}(t)) dt. \tag{2.43}$$

Using (2.43), we infer from (2.42) that

$$\begin{aligned}
&\int_0^T \big((\boldsymbol{u}'(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{v}) + b(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v}) + J(\boldsymbol{v}) \\
&\qquad\qquad\qquad - (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))\big) dt \\
&\geq \frac{1}{2}||\boldsymbol{u}(T)||^2 - \frac{1}{2}||\boldsymbol{u}_0||^2 + \int_0^T \left(a||\boldsymbol{u}(t)||^2 + b\int_\Omega |\boldsymbol{u}(t)|^{\alpha+2} dx\right) dt + \int_0^T J(\boldsymbol{u}(t)) dt \\
&= \int_0^T \big[(\boldsymbol{u}'(t), \boldsymbol{u}(t)) + a\,(\boldsymbol{u}(t), \boldsymbol{u}(t)) + b\,(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{u}(t)) + J(\boldsymbol{u}(t))\big] dt
\end{aligned}$$

which by arguing as in [8] (pages 56–57), yields

$$\begin{aligned}
&(\boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)) + \nu(\nabla \boldsymbol{u}(t), \nabla(\boldsymbol{v} - \boldsymbol{u}(t))) + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\
&+ b(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \qquad \text{for all } \boldsymbol{v} \in \mathbb{V}.
\end{aligned}$$

We then conclude that

THEOREM 2.1 *The variational problem* (2.9) *admits at least one weak solution, which moreover satisfies;*

$$\sup_{0\leq t\leq T} \|\nabla \boldsymbol{u}(t)\| \leq C, \quad \int_0^T \|\boldsymbol{u}'(t)\|^2 dt \leq C, \tag{2.44}$$

where C is a positive constant depending on the data.

Having obtained the velocity, we shall indicate how the pressure is constructed. First, we recall that from $(2.13)_1$,

$$\begin{aligned}
(\operatorname{div} \boldsymbol{v}, p_\varepsilon(t)) &= (\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}_\varepsilon(t), \nabla \boldsymbol{v}) + a\,(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) \\
&\quad + b\,(|\boldsymbol{u}_\varepsilon(t)|^\alpha \boldsymbol{u}_\varepsilon(t), \boldsymbol{v}) + \langle K_\varepsilon(\boldsymbol{u}_\varepsilon(t)), \boldsymbol{v}\rangle - (\boldsymbol{f}(t), \boldsymbol{v}),
\end{aligned} \tag{2.45}$$

but since $p_\varepsilon(t) \in L^2_0(\Omega)$, following [1], one can find a positive constant C such that

$$C\|p_\varepsilon(t)\| \le \sup_{\boldsymbol{v}\in\mathcal{N}} \frac{(\operatorname{div}\boldsymbol{v},\, p_\varepsilon(t))}{\|\boldsymbol{v}\|_1}. \tag{2.46}$$

Now, putting together (2.45), (2.46) and using the continuity of operators involved, one obtains

$$\begin{aligned} C\|p_\varepsilon(t)\| &\le \|\boldsymbol{u}'_\varepsilon(t)\| + \nu\|\nabla\boldsymbol{u}_\varepsilon(t)\| + a\|\boldsymbol{u}_\varepsilon(t)\| + b\,\|\boldsymbol{u}_\varepsilon(t)\|^{\alpha+1}_{L^{2\alpha+2}} \\ &\quad + \|K_\varepsilon(\boldsymbol{u}_\varepsilon(t))\|_{\mathcal{V}'} + \|\boldsymbol{f}(t)\| \\ &\le \|\boldsymbol{u}'_\varepsilon(t)\| + \nu\|\nabla\boldsymbol{u}_\varepsilon(t)\| + a\|\boldsymbol{u}_\varepsilon(t)\| + C(b,\Omega,\alpha)\,\|\boldsymbol{u}_\varepsilon(t)\|^{\alpha+1}_{L^6} \\ &\quad + C(\Omega)\|g\|_{L^\infty(S)}\|\boldsymbol{u}_\varepsilon(t)\|_1 + \|\boldsymbol{f}(t)\| \\ &\le \|\boldsymbol{u}'_\varepsilon(t)\| + \nu\|\nabla\boldsymbol{u}_\varepsilon(t)\| + a\|\boldsymbol{u}_\varepsilon(t)\| + C(b,\Omega,\alpha)\|\nabla\boldsymbol{u}_\varepsilon(t)\|^{\alpha+1} \\ &\quad + C(\Omega)\|g\|_{L^\infty(S)}\|\boldsymbol{u}_\varepsilon(t)\|_1 + \|\boldsymbol{f}(t)\| \end{aligned}$$

which by Young's inequality and integrating the resulting inequality over $[0, T]$, yields (after utilization of (2.33) and (2.34))

$$\begin{aligned} \int_0^T \|p_\varepsilon(t)\|^2 \mathrm{dt} &\le C\int_0^T \|\boldsymbol{u}'_\varepsilon(t)\|^2 + C\int_0^T \|\nabla\boldsymbol{u}_\varepsilon(t)\|^2 dt + C\int_0^T \|\boldsymbol{u}_\varepsilon(t)\|^2 \\ &\quad + C\int_0^T \|\nabla\boldsymbol{u}_\varepsilon(t)\|^{2\alpha+2}\mathrm{dt} + C\|g\|^2_{L^\infty(S)}\int_0^T \|\boldsymbol{u}_\varepsilon(t)\|_1^2 \\ &\quad + C\int_0^T \|\boldsymbol{f}(t)\|^2 dt < \infty, \end{aligned} \tag{2.47}$$

C being a positive constant depending on the parameters and the domain of the problem. Then we can select from $p_\varepsilon(t)$ a sequence, again denoted by $p_\varepsilon(t)$, such that

$$p_\varepsilon \longrightarrow p \ \text{ weakly in } \ L^2(0, T; L^2_0(\Omega)). \tag{2.48}$$

Next, one observes that (2.13) can be re-written as

$$\begin{aligned} (\boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) + \nu(\nabla\boldsymbol{u}_\varepsilon(t), \nabla(\boldsymbol{v}-\boldsymbol{u}_\varepsilon(t))) + a\,(\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) \\ + b\,(|\boldsymbol{u}_\varepsilon(t)|^\alpha\boldsymbol{u}_\varepsilon(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) - (\operatorname{div}(\boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)), p_\varepsilon(t)) \\ + J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_\varepsilon(t)) - (\boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}_\varepsilon(t)) \ge 0, \quad \text{for all } \boldsymbol{v}\in\mathcal{N}, \\ (\operatorname{div}\boldsymbol{u}_\varepsilon(t), q) = 0 \quad \text{for all } q\in L^2(\Omega), \end{aligned}$$

which by integration over the time interval $[0, T]$ and passage to the limit (as $\varepsilon \to 0$) yields, (after utilization of the identity $(\operatorname{div}\boldsymbol{u}_\varepsilon(t), q) = 0$ for all $q \in L^2(\Omega)$)

$$\begin{aligned} &\int_0^T \big[(\boldsymbol{u}'(t), \boldsymbol{v}-\boldsymbol{u}(t)) + \nu(\nabla\boldsymbol{u}(t), \nabla(\boldsymbol{v}-\boldsymbol{u}(t))) + a\,(\boldsymbol{u}(t), \boldsymbol{v}-\boldsymbol{u}(t))\big]dt \\ &+ \int_0^T \big[-(\operatorname{div}(\boldsymbol{v}-\boldsymbol{u}(t)),\ p(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t)) - (\boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{u}(t))\big]dt \\ &+ \int_0^T b\,(|\boldsymbol{u}(t)|^\alpha\boldsymbol{u}(t), \boldsymbol{v}-\boldsymbol{u}(t))dt \ge 0, \end{aligned}$$

for all $\boldsymbol{v} \in \mathcal{N}$. Also, $(\operatorname{div} \boldsymbol{u}(t), q) = 0$ for all $q \in L^2(\Omega)$.
Finally, arguing as in [8] (see page 56–57), one obtains

$$\begin{aligned}(\boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t)) + \nu(\nabla \boldsymbol{u}(t), \nabla(\boldsymbol{v} - \boldsymbol{u}(t))) + a\,(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))\\ + b\,(|\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) - (\operatorname{div}(\boldsymbol{v} - \boldsymbol{u}(t)), p(t)) + J(\boldsymbol{v}) - J(\boldsymbol{u}(t))\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))\end{aligned} \tag{2.49}$$

for all $\boldsymbol{v} \in \mathcal{N}$. Moreover, $(\operatorname{div} \boldsymbol{u}(t), q) = 0$ for all $q \in L^2(\Omega)$.

3. Continuous dependence on the data

In this Section, our focus is to establish some qualitative properties of the weak solutions in Theorem 2.1. In particular, we show that the solutions depend continuously on initial velocity, external force as well as the Forchheimer's and Brinkman's coefficients. We recall that such results in the literature are sometimes referred to as structural stability.

We first claim that

THEOREM 3.1 *Let $\boldsymbol{u}_i$ be the solution of* (2.5) *with respect to $\boldsymbol{u}_{i0}$, $\boldsymbol{f}_i$, $i = 1, 2$. Then there exists a positive constant C, depending on a, ν and Ω such that*

$$||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)||^2 \leq e^{-Ct}||\boldsymbol{u}_1(0) - \boldsymbol{u}_2(0)||^2 + \int_0^t e^{C(-t+s)}||\boldsymbol{f}_2(s) - \boldsymbol{f}_1(s)||^2 ds. \tag{3.1}$$

This theorem implies in particular the following uniqueness result.

COROLLARY 3.1 *The problem* (2.5) *has one and only one solution.*

Proof of Theorem 3.1 The functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy respectively:

$$\begin{aligned}(\partial_t \boldsymbol{u}_1, \boldsymbol{v} - \boldsymbol{u}_1) + \nu(\nabla \boldsymbol{u}_1, \nabla(\boldsymbol{v} - \boldsymbol{u}_1)) + a\,(\boldsymbol{u}_1, \boldsymbol{v} - \boldsymbol{u}_1) + b\,(|\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1, \boldsymbol{v} - \boldsymbol{u}_1)\\ + J(\boldsymbol{v}) - J(\boldsymbol{u}_1) \geq (\boldsymbol{f}_1, \boldsymbol{v} - \boldsymbol{u}_1) \quad \text{for all } \boldsymbol{v} \in \mathbb{V}.\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(\partial_t \boldsymbol{u}_2, \boldsymbol{v} - \boldsymbol{u}_2) + \nu(\nabla \boldsymbol{u}_2, \nabla(\boldsymbol{v} - \boldsymbol{u}_2)) + a\,(\boldsymbol{u}_2, \boldsymbol{v} - \boldsymbol{u}_2) + b\,(|\boldsymbol{u}_2|^\alpha \boldsymbol{u}_2, \boldsymbol{v} - \boldsymbol{u}_2)\\ + J(\boldsymbol{v}) - J(\boldsymbol{u}_2) \geq (\boldsymbol{f}_2, \boldsymbol{v} - \boldsymbol{u}_2) \quad \text{for all } \boldsymbol{v} \in \mathbb{V}.\end{aligned} \tag{3.3}$$

Setting $\boldsymbol{v} = \boldsymbol{u}_2$ in (3.2) and $\boldsymbol{v} = \boldsymbol{u}_1$ in (3.3) and adding the resulting inequalities, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}(t)||^2 + \nu||\nabla \boldsymbol{w}||^2 + a||\boldsymbol{w}(t)||^2\\ + b(|\boldsymbol{u}_2|^\alpha \boldsymbol{u}_2 - |\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1, \boldsymbol{w}(t)) \leq (\boldsymbol{f}_2 - \boldsymbol{f}_1, \boldsymbol{w}(t)),\end{aligned}$$

where $\boldsymbol{w}(t) = \boldsymbol{u}_2(t) - \boldsymbol{u}_1(t)$ and $\boldsymbol{w}_0 = \boldsymbol{u}_{20} - \boldsymbol{u}_{10}$. Since $T(\zeta) = |\zeta|^\alpha \zeta$ is monotone then

$$(|\boldsymbol{u}_2|^\alpha \boldsymbol{u}_2 - |\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1, \boldsymbol{w}(t)) \geq 0.$$

Therefore

$$\frac{d}{dt}||\boldsymbol{w}(t)||^2 + C(\nu, a, \Omega)||\boldsymbol{w}(t)||^2 \leq C(a, \Omega)||\boldsymbol{f}_2 - \boldsymbol{f}_1||^2, \tag{3.4}$$

where Poincaré's inequality has been used. We readily deduce the desired result from (3.4) using Gronwall's lemma. □

In line of Theorem 3.1, one can state the following result.

THEOREM 3.2 *The weak solutions of problem* (2.5) *constructed in Theorem* 2.1 *depends continuously with respect to the* L^2 *norm on:*

(a) the Forchheimer coefficient b, and
(b) the Brinkman coefficient ν.
(c) the barrier function g.

The proof follows mutatis mutandis the proof of Theorem 3.1.

4. Stability of stationary solutions

Hereafter, we study the stability of stationary solutions to (2.5).

We assume that the force $\boldsymbol{f}$ is independent of time, and we consider the following stationary problem

$$\begin{cases} -\nu\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} - \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega, \\ \boldsymbol{u} = 0 \quad \text{on } \Gamma, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \text{ and } -\boldsymbol{\sigma_\tau} \in g\partial|\boldsymbol{u_\tau}| \quad \text{on } S. \end{cases} \tag{4.1}$$

Here, we always assume that $\alpha \in [1, 2]$, $\gamma, a, b > 0$. It is clear that the velocity satisfies the simpler variational problem

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V} \text{ such that for all } \boldsymbol{v} \in \mathbb{V}, \\ \nu(\nabla\boldsymbol{u}, \nabla(\boldsymbol{v}-\boldsymbol{u})) + a\,(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) \\ +b(|\boldsymbol{u}|^{\alpha}\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}). \end{cases} \tag{4.2}$$

It is clear that (4.2) is a particular case of nonlinear monotone variational problem. Hence with classical arguments (see [33], page 371, Theorem 5.2), there exists a unique $\widetilde{\boldsymbol{u}} \in \mathbb{V}$ such that (4.2) holds true, and one has the following

THEOREM 4.1 *The weak solution* $\boldsymbol{u}$ *of* (2.5) *constructed in Theorem* 2.1 *converges to the unique solution* $\widetilde{\boldsymbol{u}}$ *to* (4.2) *exponentially as t goes to infinity. More precisely, we have the following estimate*

$$||\boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}||^2 \leq ||\boldsymbol{u}_0 - \widetilde{\boldsymbol{u}}||^2 e^{-2(a+\nu)t}, \quad \textit{for all } t \geq 0. \tag{4.3}$$

Proof We let $\boldsymbol{v} = \boldsymbol{u}(t)$ in (4.2), thus

$$\nu(\nabla\widetilde{\boldsymbol{u}}, \nabla(\boldsymbol{u}(t) - \widetilde{\boldsymbol{u}})) + a\,(\widetilde{\boldsymbol{u}}, \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}) + b(|\widetilde{\boldsymbol{u}}|^{\alpha}\widetilde{\boldsymbol{u}}, \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}) \\ +J(\boldsymbol{u}(t)) - J(\widetilde{\boldsymbol{u}}) \geq (\boldsymbol{f}, \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}).$$

Next, for $\boldsymbol{v} = \widetilde{\boldsymbol{u}}$ in (2.5), one has

$$(\boldsymbol{u}'(t), \widetilde{\boldsymbol{u}} - \boldsymbol{u}(t)) + \nu(\nabla \boldsymbol{u}(t), \nabla(\widetilde{\boldsymbol{u}} - \boldsymbol{u}(t))\,) + a\,(\boldsymbol{u}(t), \widetilde{\boldsymbol{u}} - \boldsymbol{u}(t))$$
$$+\, b\,(|\boldsymbol{u}(t)|^{\alpha}\boldsymbol{u}(t), \widetilde{\boldsymbol{u}} - \boldsymbol{u}(t)) + J(\widetilde{\boldsymbol{u}}) - J(\boldsymbol{u}(t)) \geq (\boldsymbol{f}, \widetilde{\boldsymbol{u}} - \boldsymbol{u}(t)).$$

Now, putting together the two previous inequalities yields;

$$-(\boldsymbol{w}'(t), \boldsymbol{w}(t)) - \nu||\boldsymbol{w}(t)||^2 - a||\boldsymbol{w}(t)||^2 - b(|\boldsymbol{u}|^{\alpha}\boldsymbol{u} - |\widetilde{\boldsymbol{u}}|^{\alpha}\widetilde{\boldsymbol{u}}, \boldsymbol{u} - \widetilde{\boldsymbol{u}}) \geq 0, \tag{4.4}$$

where $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}$. From the monotonicity of $T(\zeta) = |\zeta|^{\alpha}\zeta$, (4.4) imply that

$$\frac{d}{dt}||\boldsymbol{w}(t)||^2 + 2(\nu + a)||\boldsymbol{w}(t)||^2 \leq 0,$$

from which the announced estimate is readily obtained via Gronwall's lemma. □

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