Design research towards improving understanding of functions: a South African case study

by

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ABSTRACT

The function concept is one of the most important concepts in the learning of mathematics (Dubinsky & Harel, 1992), yet it is considered by many researchers to be one of the least understood and most difficult concepts to master in the learning of high school mathematics (Eisenberg, 1992, Sfard, 1992). To this end, problems concerning its teaching and learning are often confronted (Mann, 2000) and few teachers know how learners come to understand functions (Yoon, 2007). As a result, most teachers teach functions using the conventional approach which starts by stating definitions followed by examples and then a few applications. The nature of this approach has not encouraged teachers to engage learners and their ways of reasoning in knowledge construction and adequately addressing their difficulties.

The purpose of this study was to use design research to improve the teaching and learning of functions at grade 11 level. This was achieved by adapting design cycles of Wademan's (2005) Generic Design Research model in which each cycle comprised different iterative APOS (Action, Process, Object, Schema) analysis, design, development and implementation of hypothetical learning trajectories (HLTs). I started by interrogating twelve grade 11 learners of a particular rural high school on the June 2011 mathematics paper 1 examination they had written to determine the APOS theory conception level each learner was operating at, and their difficulties. Learners' difficulties from initial interviews and literature were grouped under the function definition and representation. I then designed instruction based on HLTs embedded with Realistic Mathematics Education (RME) activities and two separate tasks on the definition and representation as a form of intervention to help learners move up from their initial conception levels to the next and to overcome their difficulties. After each design cycle I interviewed learners based on the task for a particular concept and learners' responses were analysed using APOS theory and used to design further instruction to help learners approximate the schema level of understanding concepts related to functions.

The major findings of this study were that the use of learners' conceptions and RME activities in designing instruction helped learners to progress smoothly through APOS theory conception levels though they did not fully reach the intended schema level. In addition, design research cycles and their HLTs implemented in a constructivist environment enabled learners to collectively derive working definitions of the function concept and to improve

their conceptual understanding of the process of switching from a graph to an equation. Another contribution of this study has been a deeper understanding of the extent to which design research can be used to improve learners' understanding of functions and an addition of some insights to the teaching and learning of functions.

Key words: Clinical Interview; Conception level; Constructivism; Design Research; Function concept; Hypothetical Learning Trajectory; Intervention; Teaching Experiment; Realistic Mathematics Education; Understanding.

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List of abbreviations

APOS- Action- Process- Object- Schema

BODMAS- Brackets- Order- Divide- Multiply- Add- Subtract

CAPS- Curriculum and Assessment Policy Statements

DoBE- Department of Basic Education

HLT- Hypothetical Learning Trajectory

HU- Hermeneutic Unit

NCS- National Curriculum Statement

NCTM- National Council of Teachers of Mathematics

PAPOS- Prefunction, Action, Process, Object, Schema

RME- Realistic Mathematics Education

CHAPTER 1
Introduction

1.1 Background of the study

The function concept "emerges from the general inclination of humans to connect two or more quantities, which is as ancient as mathematics" (Evangelidou, Spyrou, Elia & Gagatsis, 2004, p. 351) and is "all around us" (Kalchman & Koedinger, 2005, p. 351). This means that the function concept is common among phenomena in everyday life in which people connect quantities to form functional relationships where one quantity completely determines the other. For example, "a functional relationship is at play when we are paying for petrol by the litre or fruit by the gram or kilogram. We can find the amount that we need to pay for the petrol when we know the number of litres that were filled" (Pillay, 2006, p. 4). In addition, phenomena in domains from physics and economics like motion, waves, and electric current and price, demand and rate of inflation respectively, are also modelled by functional relationships (Grinstein & Lipsey, 2001). Functions are also used "extensively for modeling and interpretation of such phenomena as local and world demographics and population growth, which are critical for economic planning and development" (Kalchman & Koedinger, 2005, p. 351). Surprisingly, in most cases people deal with functional situations like these without being aware of them and use creative or informal ways which may not be well developed and not always consistent when solving functional problems.

The function concept "is central to mathematics and its applications" (Evangelidou, Spyrou, Elia & Gagatsis, 2004, p. 351). It is used in every branch of mathematics, such as arithmetic and algebraic operations on numbers, geometric transformations on points in the plane or in space, intersection and union of pairs of sets, some solution sets to equations, formulae used in mensuration (perimeter, area, and volume), and regression functions (Akkoc & Tall, 2005). In addition, displacement, velocity, and rates of change are typical mathematical topics where functions are applied and learners may learn the mathematical concepts without being aware that they are functions. Apart from their use in calculus and analysis, "functions are also widely used in the comparison of abstract mathematical structures like determining whether two sets have the same cardinality and whether two topologies are homeomorphic. Functions can also be used as elements of abstract mathematical structures such as vector spaces, rings and groups" (Carlson, 1998, p. 115).

Without doubt, functions have an important place in the secondary mathematics curriculum (Yerushlamy & Shternberg, 2001). As early as 1921, "the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that the study of functions be given central focus in secondary school mathematics" (Cooney & Wilson, 1993, p. 17). With specific reference to the South African mathematics curriculum, function-related activities start as early as the fourth grade and continue through to the high school mathematics curriculum. In addition, Froelich, Bartkovich, and Foerester (1991) said "the idea of function is inherent in many parts of today's algebra and geometry programs" (p. 1), making the concept of function an important part of the school mathematics curriculum. The National Council of Teachers of Mathematics (NCTM) proposed in the Curriculum and Evaluation Standards for School Mathematics that "one of the central themes of school mathematics is the study of patterns and functions" (NCTM, 1989, p. 98).

1.2 Statement of the problem

Within the function concept in modern mathematics and related fields, problems concerning its teaching and learning are often confronted (Mann, 2000) and internationally its difficulty is acknowledged (Tall & Bakar, 1992). Eisenberg (1992) argues that the function concept is "... one of the most difficult concepts to master in the learning of school mathematics" (p. 140) and has proved to be "subtle and elusive whenever we try to teach it in school" (Tall & Bakar, 1992, p. 1). Despite its importance few teachers know how learners come to understand functions (Yoon, 2007) and how activities can be designed to improve learners' understanding of the function concept.

The understanding of the function concept does not appear to be easy for two main reasons. Firstly, many learners do not sufficiently understand the abstract but comprehensive meaning of the function concept (Mann, 2000). This may be caused by many different definitions of the function concept that appear in our high school textbooks. As a result teachers are compelled to give simple and understandable definitions of the function concept which do not integrate the core characteristic of the function concept, which sometimes creates misconceptions for learners (Mowahed, 2009). Secondly, because of the diversity of representations associated with the function concept, many learners do not understand the connections between different representations of the same function (Yamada, 2000). As a result, a substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (for example, Akkoc &

Tall, 2005; Thomas, 2003; Zazkis, Liljedahl, & Gadowsky, 2003). However, these studies have done little to demonstrate the connections between the different definitions of the function concept and the different representations.

Moreover, the literature (Reed, 2007; Jones, 2006; Abdullah & Saleh, 2005 Akkoc & Tall, 2005; Cunningham, 2005; Bayazit & Gray, 2003; Knuth, 2000; Hitt, 1998; Sfard, 1992; Eisenberg& Dreyfus, 1991; Vinner & Dreyfus, 1989; Markovits, Eylon & Bruckheimer, 1986) on the function concept focuses mainly on the 'bigger picture' issues like finding, enumerating and explaining learners' difficulties and misconceptions about the function concept. Insufficient attention has been given to the dynamics of how learners construct knowledge about the function concept which might have an impact on their understanding it. The literature cited above is also silent on how instruction can be designed to overcome learners' difficulties and misconceptions because there was no attempt to analyse learners' interpretations in order to design instruction to improve learners' understandings of functions. Alluding to this silence, Markovits, Eylon and Bruckheimer (1986) said that many of the difficulties noted in their study about the function concept can be overcome if "we are willing to study them further, analyse carefully what we expect of a learner who understands, and experiment with various treatments" (p. 191). This apparent gap in the literature also has been echoed by Kalchman and Koedinger (2005) when they indicated "we need an instructional plan that deliberately builds and secures conceptual understanding and facility with representing functions in a variety of ways" (p. 364). The implication is that unless teachers use learners' difficulties and misconceptions to design further instruction, they are going to be less likely to forge an understanding of the function concept. This prompted me to revisit the teaching and learning of functions with the aim of designing instruction by considering further research regarding eliciting, analysing and using the weaknesses in understanding of the function concept by grade 11 learners. A study of this nature would add to the literature on both learning and teaching of the concepts related to functions because it will inform the mathematics education community of learners' reasoning processes as they engage in activities involving the function concept. This could include the planning and application of appropriate and efficient teaching and learning activities at high school level, for improving learners' comprehension of functions.

Ensor and Galant (2005) point out that a "considerable amount of educational research in South Africa points to a crisis in mathematics teaching and learning - many teachers are deemed to be failing to teach adequately, and learners are failing to perform" (p. 301). Similarly, the literature (Kazima, Pillay & Adler, 2008; Maharaj, 2008) on the teaching and learning of functions in South Africa reveals that our classrooms are still characterised by traditional pedagogic practice in which learners are presented with concepts related to functions, provided examples of how to carry out procedures to solve related tasks, thereafter learners practise these through textbook set exercises. Such teaching approaches assume that learners will understand the presented concepts as intended. These approaches can create conceptual gaps in learners' understanding of concepts resulting in their failure to perform.

In summary, internationally there is a gap in research on the learners' understanding of functions and in South Africa there is a need to focus on how teachers can teach the concepts related to functions competently. The intention of this study is to support both the teacher who is failing to teach adequately and the learner who is failing to understand and perform by designing and developing empirically grounded hypothetical learning trajectories and instructional activities. It is hoped that teachers will be able to use this in their classrooms to improve learners' understandings of functions.

1.3 Purpose of the study

The purpose of this study was to interrogate grade 11 learners' conceptual understanding of functions and to use design research to improve the teaching and learning of functions. In line with this purpose, Depaolo (2009) asserts that "understanding how learners learn complex mathematical concepts is essential for teaching and defining curricula and this is especially true on the elementary level" (p. 7). Moreover, I sought to understand and develop the connections learners make between the concepts related to functions. Due to the difficulties and misconceptions associated with functions this study focused on 'seeing' functions through the cognitive lenses of grade 11 learners of a particular rural high school. This was achieved by eliciting and analyzing the development of concepts related to functions by these high school learners as they engaged in activities involving the function concept. This helped me to understand learners' cognitive processes and their weaknesses when learning the concepts related to functions. Such understanding enabled me to design and develop empirically grounded instructional activities to overcome these weaknesses in learners' understanding of the function concept. These instructional activities specify patterns in learners' learning of the concepts related to functions as well as the approach supporting that learning. To do the study I needed a conceptual understanding of the function concept, an understanding of how learners learn the function concept and the learning difficulties which culminated in the research questions that follow.

1.4 Research questions

This study was guided by the following research questions:

1 How do learners understand the function concept?

Sub-questions:

i) What are grade 11 learners' current understanding of functions?

ii) What are the weaknesses in the learners' understanding?

2 How can instruction be designed to improve learners' understanding of the function concept?

1.5 Rationale for the study

The rationale for this study hinges on my personal interest to improve the grade 12 pass rate in mathematics by developing a firm foundation in grade 11 and to reduce the difficulties in the teaching and learning of concepts related to functions. This brings the idea of teaching functions for understanding which Kalchman and Koedinger (2005) say "requires a set of instructional strategies for moving students along a developmental pathway and for addressing the obstacles and opportunities that appear most frequently along the way" (p. 373). Kalchman and Koedinger believe that, these instructional strategies could assist learners to develop an understanding of the concepts related to functions, the ability to represent the function concept in a variety of ways, and flexibility in moving among multiple representations of the function concept.

The importance of this study is the need to design instructional activities based on learners' current understanding of the concepts related to functions, since "it is important to understand the world of children (learners) through their own eyes rather than those of the adult (teacher)" (Arksey & Knight, as cited in Cohen, Manion & Morrison, 2007, p. 48). In support of the above proposition, Kalchman and Koedinger (2005) suggest "the importance of building new knowledge on the foundation of learners' existing knowledge and understanding" (p. 352). In this study I explored and reveal why teachers need to know and understand how learners understand the concepts related to functions and the obstacles and misunderstandings of learners.

A further motivation emerged from the importance of the function concept in mathematics, the mathematics curriculum, and its diverse applications in the sciences and related areas. In line with my motivation Lloyd, Beckmann and Zbiek (2010) declare that functions are one of the most important topics in secondary school mathematics because of their connection to other mathematical topics. Since my long term goal is to improve the grade 12 pass rate initially at my school and then, hopefully, both provincially and nationally, it is important at this point to show the links that the function concept has with other mathematics topics. On the one hand, areas in algebra and equations are closely linked to functions as algebra allows us to represent general relationships with algebraic formulas while equations define the relationship between one or more variables (Grinstein & Lipsey, 2001). The authors added that patterns and sequences are connected to functions as they are generally formed according to a rule that can be represented by an algebraic formula or equation. Grinstein and Lipsey (2001) also noted that, topics on finance, growth and decay and differential calculus mainly deal with the concept of rate of change of one quantity with respect to the other in a relationship which is also a function. Thus, for learners to be able to understand these related topics they have to have a good understanding of the function concept since it serves “a coalescing role” in learners’ understanding of these topics (Selden & Selden, 1992, p. 87). The views above indicate that the function concept ties algebra, geometry and calculus together. On closer interrogation of the new South African National Curriculum Statement (CAPS) and specifically the topics for the grade 12 mathematics paper 1 examination, one would observe that the topics discussed above are grouped together and contribute 135 marks of the total of 150 marks. The mark distribution is shown in Table 1.

Table 1: CAPS Mark Distribution for Mathematics Paper 1

Description	Grade 10 Recommended marks	Grade 11 Recommended marks	Grade 12 Recommended marks	
Algebra & Equations (and inequalities)	30 ± 3	45 ± 3	25 ± 3	**135 marks** (Algebra & Equations to Differential Calculus)
Patterns & Sequences	15 ± 3	25 ± 3	25 ± 3	
Finance and Growth	10 ± 3			
Finance, growth & decay		15 ± 3	15 ± 3	
Functions & Graphs	30 ± 3	45 ± 3	35 ± 3	
Differential Calculus			35 ± 3	
Probability	15 ± 3	20 ± 3	15 ± 3	
TOTAL	**100**	**150**	**150**	

Source: CAPS Grades 10-12 (DoBE, 2011, p. 8)

Since the function concept unifies and forms the foundation of the related topics, marked by the right brace in Table 1 above, learners' difficulties with the concepts related to functions should be identified and analyzed in order to design instruction that helps learners to improve their understanding of functions. Such an approach could improve their grade 12 mathematics results in the long term.

Furthermore, identification and analysis of difficulties learners experience with understanding concepts in the context of the function concept is necessary for guiding the design and development of intervention programmes to ease the difficulties in the teaching and learning of functions (Kalchman & Koedinger, 2005). Without such analysis, intervention programmes stand little chance of being guided by informed judgments of how learners acquire understanding of essential function components. The National Research Council (2005) alluded to this view by asserting that "new learning is built on the foundation of existing knowledge and preconceived understandings regarding the subject-matter and that such learning is enhanced when these understandings are drawn out" (p. 4). With such a cycle of learning in place, new knowledge can be directly tied to what is already known. When they are not better understood, learners can be made aware of how their existing conceptions fall short and be provided with more robust alternatives (National Research Council, 2005). To provide these alternatives, teachers need to be aware of and be guided by the weaknesses in learners' understanding of mathematical concepts when planning their lessons. This study's main thrust was to help teachers to develop an understanding of how their learners' thinking about the function concept could form the basis for the development of informed instructional strategies. This was achieved by eliciting, analyzing, refocusing, reinventing (re-creating) and reinforcing learners' conceptions. From the misconceptions and difficulties I designed and developed learning activities that challenged their understanding, and that worked with unpredictable preconceptions. These learning activities will be shared with teachers through the provision of easily modifiable activities that teachers can adapt for use in their own situations.

1.6 Research design and methodology

In order to answer the research questions for this study I needed access to situations in which learners would get the opportunity to explain their thinking and share ideas. Teaching experiments provided these situations and helped me to generate an in-depth case study of learners in a rural high school. The word experiment in teaching experiment refers to "an experimental classroom setting that is created as a result of the designed and developed HLTs and instructional activities" (Gravemeijer & Cobb, 2001, p. 14). Its primary purpose was to experience learners' learning and reasoning first-hand, and it thus served the purpose of eliminating the separation between the practice of research and the practice of teaching (Steffe & Thompson, 2000). In the teaching experiment the instructional sequences were carried out by teaching and engaging learners in RME activities with the intention of understanding and improving the initial design on the basis of learners' reasoning with respect to the created educational setting. During a teaching experiment, researchers and teachers take their 'best bets', as Lehrer and Schauble (2001) call it. That is, they use activities and types of instruction that seem most appropriate at that moment according to the planned goal.

A sample of 12 grade 11 learners of mixed ability was purposively selected using their teacher's record of marks and from those who were willing to participate in all the phases of design research for this study. Intervention was conducted after confirming learners' difficulties and their APOS (Action, Process, Object, Schema) conception levels from the initial task-based interviews. I conducted the conjecture-driven teaching experiments with grade 11 learners and I taught all the sessions. Since I did not normally teach these learners I met them daily after school hours in a program I termed ‘Extra Mile’. This program began on 15 August 2011 and ended on 17 September 2011. During the course of the experiments, the learners passed through hierarchical levels of conceptual development of the function concept that were informed by both my personal experience with grade 11 learners and by the literature. During level one of the teaching experiment, learners were engaged with the development of the key idea of the function concept which led to a definition. In level two, they dealt with representations of the function concept and in the third level they extended their knowledge by learning about the inverse of the function. Within each hierarchical level of conceptual development learners were assisted to move from their initial level towards the desired schema level.

In the first hierarchical level, learners were introduced to the function concept through activities that compelled them to develop an understanding of the key idea behind the function concept which is the dependence relationship between variable quantities. Kalchman and Koedinger (2005) concur with this approach by suggesting that learners should understand the core concept of a functional relationship. In addition, Tall, McGowen and DeMarois (2000) warn that if an understanding of the function concept does not depend on its key idea then it challenges further understanding of the concept. In this study, learners’ understanding of the key idea of the function concept helped them to evaluate different definitions of the function concept and also to derive their own working definitions. This enabled learners to move from lower conception levels to approximate the schema level.

The second hierarchical level focussed on representations of the function concept as prescribed by the South African National Curriculum Statement (CAPS) namely tables, graphs, words and formulae. Learners are expected to move flexibly between these representations. As a result I engaged learners in activities that helped them to understand the links between these representations and the translation process from one representation to the other. This improved and strengthened learners’ construct of the meaning of the function concept while at the same time developing a rich network of associations among different representations. Learners were introduced to the inverse of the function concept in the third hierarchical level to extend their understanding of the representations and the translation process and to move them towards the desired schema level.

Each level described above had its own evolving hypothetical learning trajectory (HLT) and spiralling design research cycles which I adapted from Wademan's (2005) Generic Design Research model. Each cycle comprised different iterative stages namely design and development of HLTs, implementation of the HLTs and APOS analysis. It is important at this point to describe an HLT, the stages of each design cycle. The APOS analysis of the data is explained in Chapter 3.

Based on the ideas of Bakker (2004) I have defined a hypothetical learning trajectory (HLT) as a learning path or an instructional sequence I imagined. This was after considering the literature review and learners' current understanding of concepts related to functions to help them overcome their difficulties and improve their understanding of functions. The HLT clarifies the learning goal, defines the path and direction to be followed, includes the learning activities that guide learners towards the learning goal and specifies the criteria for evaluation of the teaching experiment. It also guides the design and development of learning activities, the teaching experiment and the retrospective analysis. The HLT underwent refinement as it evolved during each of the design cycles that I carried out in this study.

Each design cycle for problem situations identified in this study comprised the following six stages listed below.

1) A literature study about teaching and learning of functions was carried out to assist me to understand the development of the function concept, difficulties likely to be met along the way and the teaching approaches to be used to forge an understanding of the function concept.
2) An analysis of the existing HLTs I identified from the literature study and my teaching experience, for example, the conventional HLT of starting by stating definitions, followed by examples and then providing a few applications when teaching the function concept. This has become ritualistic in our classrooms and is challenged in this study.
3) I elicited and analyzed grade 11 learners' understanding of concepts related to functions. This was achieved by conducting task-based clinical interviews with learners in the sample. I coded learners' transcribed interview responses and then used indicators of APOS theory conception levels to determine the level of conception at which each learner was operating. This enabled me to describe learners' concept images and reasoning about concepts related to functions which revealed their level of understanding and accompanying weaknesses.

4) I developed HLTs based on the three levels described above together with the instructional activities to help learners to overcome their difficulties and then to progress from their current APOS theory conception level to the next, thereby improving their understanding of the function concept.
5) I implemented the developed HLTs in a teaching experiment which engaged learners in Realistic Mathematics Education activities and compelled them to construct the required knowledge. My main role in the teaching experiment was to guide learners along the designed learning path towards the learning goal.
6) At the end of each design cycle I conducted a retrospective analysis of the teaching experiment which enabled me to compare the HLT with learners' actual learning. This stage marked the end of the one cycle and its outcomes then fed into the first stage of the next cycle.

My research questions on teaching and understanding of the concepts related to functions were answered by these teaching experiments. I tested my ideas and conjectures about the teaching and learning processes of the function concept through the instructional sequences in classroom situations because I was interested in the development of learners' conceptions in relation to the teaching processes.

1.7 Ethical considerations

I obtained ethical clearance from the University of Pretoria's Faculty of Education Ethics Committee prior to data collection (See Appendix 6 for the Ethics Clearance Certificate). I obtained informed consent from participants by providing to them the purpose of the study, explaining that their participation is voluntary and assuring them that they could withdraw at any time if they chose to do so. I requested participants and their parents or guardians to sign letters of consent and assent respectively prior to commencing the data collection. I avoided potential risks to participants by ensuring that my "methods were free of any form of deceit, duress, unfair inducement or manipulation" (Berg, 2001, p. 56). I used learners' preferred pseudonyms when reporting data and the real names of my participants are never mentioned in order to protect their privacy and confidentiality (Denzin & Lincoln, 2000, p. 139). The name of the school has also not been mentioned to ensure privacy and confidentiality.

1.8 My roles in this research

In this research study, I designed function tasks and the interview schedules which I used to conduct clinical interviews. I then analysed learners' interview responses and used them to design and implement interventions. I also taught during the teaching experiments and evaluated my teaching by comparing the envisioned HLT and the learners' actual learning which assisted me to adopt or modify the HLTs of each design cycle.

1.9 Definition of terms

This section explains the terms and phrases used in the previous and subsequent sections to clarify the nature of the research problem.

Activity in this study refers to an educational procedure intended to stimulate learning.

Function related concepts in this study refer to the definition, representations, inverse, intercepts, asymptotes, turning points, domain, range, variable, dependent and independent variable, relation, one-to-one, many-to-one, transformation of functions and vertical line test.

The **hypothetical learning trajectory** (HLT) is "a path imagined by the teacher about how the thinking and learning, in which the learners might engage as they participate in certain instructional activities, relate to the chosen learning goal" (Bakker, 2004, p. 9). It is based on the actual situation in the classroom (where learners have difficulty in understanding the concepts related to functions) and assists learners to move from their difficulties towards the ideal situation (where they meet the requirements prescribed by CAPS).

Instructional activities are topic specific, well-sequenced mathematical activities together with the ways of using them in teaching to help learners understand concepts which otherwise would be difficult for them.

An **inverse** of a function is a function which "undoes what a function does" (Bayazit & Gray, 2003, p. 2).

A **representation** may be regarded as "something that stands for something else" (Janvier, 1987, p. 10).

A mathematical **task** is a set of problems that focuses learners' attention on a particular mathematical idea and/or provides an opportunity to develop or use a particular mathematical habit of mind.

A **teaching experiment** is "a planned intervention that takes place over a significant period of time in a classroom where a continuing course of instruction is taught" (Gravemeijer & Cobb, 2001, p. 69).

Translation refers to "the process involved in changing from one form of representation to another" (Janvier, 1987, p. 11).

Working definition is used in this study to mean a definition of the function concept which learners could easily use to formulate examples or non-examples, identify independent and dependent variables and to determine if a given relationship is a function or not.

1.10 Layout of the study

Chapter 1: Introduction

This chapter introduces the present study by describing its background, statement of the problem, purpose, research questions, rationale, context and ethical considerations. Also included in this chapter are the definitions of terms and the layout of the entire study.

Chapter 2: Literature review

This chapter examines the general views on learning and understanding of mathematics, but focuses mainly on the definitions, representations and inverse of the function concept. It includes an analysis of definitions, representations and students' difficulties in learning and understanding the function concept and the approaches that are commonly used to teach it.

Chapter 3: Theoretical framework

This chapter explains the constructivist paradigm in which the present study is immersed and examines Piaget's theory of cognitive development and APOS theory to provide a theoretical framework for analyzing learners' understanding of the concepts related to functions. The Realistic Mathematics Education (RME) theory which informs instruction is explained in detail and lead to the merging of the three theories.

Chapter 4: Research design and methodology

This chapter describes the underlying ontological and epistemological assumptions for this study, study site and setting, the sample and sampling techniques, ethical issues, research design, data collection and analysis methods. Measures to ensure rigor and trustworthiness of the entire research process are explained in the last section of this chapter.

Chapter 5: Presentation and analysis of data

Chapter 5 presents and discusses the findings of this study within the context of the literature review and theoretical framework. Patterns or themes emerging from the results are discussed for their relevance to the research questions.

Chapter 6: Summary, conclusions and recommendations

In this chapter, I present a synthesis of the foregoing chapters, reflect on my research design and methodology, revisit my data, draw conclusions based on my findings and discuss the implications of the developed HLTs and instructional activities for classroom practice. Based on my study, I also make recommendations for classroom practice and future research.

1.11 Concluding remarks

While this chapter has provided the reader with an overview of the study, what follows seeks to provide an in-depth discussion of each stage of my journey. The next chapter examines the literature related to the teaching of functions.

CHAPTER 2

Literature Review

2.1 Introduction

This chapter presents a review of the literature on studies related to the teaching of functions. I discuss the general views on learning and understanding mathematics, but focus particularly on the function concept with an analysis of definitions and representations. Learners' difficulties in learning and understanding concepts related to the function concept and the approaches that are commonly used to teach these are included. This chapter is followed by the literature study that has informed my theoretical framework.

2.2 Learners' understanding of mathematics

The main purpose of this study was to improve learners' understanding of concepts related to functions. As such, it is necessary to begin this section by clarifying the phrase "learners' understanding of mathematics" in general and the role of the teacher in that process. The clarification is meant to serve as a guiding framework in describing the learning and understanding of the function concept in Section 2.4 and for designing instruction to improve the understanding of functions in Chapters 4 and 5. My explanation of learners' understanding of mathematics was guided by the question: What knowledge and skills or competencies must a learner demonstrate or display about a mathematical concept so that the teacher may say that he/she understands the mathematical concept? To answer this question I refer to Usiskin's (2012) assertion that a learner has a:

> full understanding of a mathematical concept if he/she can deal effectively with the skills and algorithms associated with the concept, with properties and mathematical justifications (proofs) involving the concept, with uses and applications of the concept, and with representations for the concept (p. 19).

The aspects of a mathematical concept summarised in the above quotation are referred to as dimensions of understanding. "Each aspect can be mastered relatively independently of the others, but these aspects are obviously connected when attached to a particular concept" (Usiskin, 2012, p. 19). As such, understanding is about being able to connect these aspects or ideas together, rather than simply knowing them as isolated facts. It is therefore important for learners to be able to connect together the different dimensions of understanding a mathematical concept because, according to Usiskin (2012), understanding can be measured by the quality and quantity of connections that a learner makes among these dimensions.

Similarly, Barmby, Harries, Higgins and Suggate (2007) regard learners' understanding of a mathematical concept as the resulting network of connections that learners make between their mental representations associated with that mathematical concept. This implies that the greater the number of appropriate connections to a network of ideas, the better the resultant understanding (Haylock, 2008). Learners' understanding can be measured in situations where they can demonstrate both verbal and written proficiency as well as competency in the dimensions of understanding cited above, together with the ability to identify and utilise the connections among these dimensions in problem solving.

In the teaching process, teachers need to determine how learners understand the mathematical concepts that they teach and use learners' understanding to plan further instruction which is the main focus of this study. According to Hiebert and Carpenter (1992), the main concern for mathematics education is to help learners understand mathematics. This implies that mathematics teachers should also be concerned "with the learners' construction of schemas or networks for understanding mathematical concepts" (Dubinsky, 1991, p. 119). Bringing together Usiskin and Dubinsky's ideas leads to the idea that when Usiskin's dimensions of understanding are well connected they construct Dubinsky's networks or schemas for understanding. As such, the teacher's main focus should be on guiding learners to make these constructions. This indicates that teachers need to be clear about what learners' understanding of mathematics entails to enable them to design instructional activities that promote an understanding of mathematical concepts. Therefore, it is necessary to examine the role of the teacher in learners' understanding of mathematical concepts.

2.3 The teacher's role in increasing learner understanding of mathematics

Sowder (2007) regards effective teachers as teachers who can predict what mathematics learners will understand, how they will understand it, and the potential for misunderstandings. They consider the following questions in planning instruction for their learners: "What is the structure of the mathematical concept to be understood? What forms or ways of understanding exist for each concept? What are the possible and desirable aspects or components of mathematical concepts for learners to learn at a given time and under certain circumstances? How are these components developed?" (Godino, 1997, p. 2). Sierpinska (1994) concurs when in her book on understanding in mathematics uses similar words: "…how to teach so that learners understand? What exactly don't they understand? What do they understand and how?" (p. xi). She claims that effective teachers are guided by these questions in their teaching. In addition, Sowder (2007) states that teachers can plan more effectively by anticipating learners' difficulties, then by knowing what prior knowledge must be present to understand something new and finally by knowing how to scaffold knowledge to assist learners in developing understanding. This implies that teachers may need to continuously reshape their plans to take account of what their learners are thinking and understanding.

Sowder's observations above put into perspective the belief that teachers can effectively influence their learners' understanding of mathematical concepts in a more indirect manner. To do so, teachers may need "to put themselves in the position of their learners by shifting from a teacher's point of view to that of a learner" (Cobb, Yackel & Wood, 1992, p. 47). However, the challenge for the teacher is to try to see mathematical concepts through the cognitive 'eyes' of the learner in order to plan instructional activities that may foster learner understanding of mathematical concepts. This indicates that teachers need to shift their emphasis away from what they can do towards what learners know and do.

Gravemeijer (2004) asserts that reform in mathematics education should aim at "shifting away from teaching by telling and replacing it by guiding learners to construct or reinvent knowledge" (p. 67). However, the challenge that arises is how to guide learners to construct or reinvent that knowledge. This indicates the need for instructional design strategies that use learners' current understanding of concepts to create new knowledge through thorough planning. This planning would be much easier if teachers had some exemplary instructional activities at their disposal. However, Gravemeijer (2004) warns that these instructional activities should be open-ended to elicit a variety of learners' interpretations and solutions, which can be used to plan further instruction. I agree with Gravemeijer's use of instructional activities in knowledge construction provided there is a frame of reference, or a theory to guide the teacher's planning and implementation of instruction. Providing both is exactly the objective of this study in which I sought to develop instructional sequences consisting of a series of instructional activities or tasks that are supported by specific learning theories.

In relation to instructional sequences mentioned above, Bakker (2004) speaks of a hypothetical learning trajectory (HLT) which he define as an imagined learning path that learners can follow in order to understand a particular concept. It also clarifies "the thinking and learning, in which the learners might engage as they participate in certain instructional activities related to the chosen learning goal. Apart from the aspect of anticipating the mental activities of the learners, a key element of the notion of an HLT is that the teacher needs to investigate whether the thinking of the learners actually evolves as imagined" (Bakker, 2004, p. 42). The learning trajectory may need to be revised or adjusted depending on the teacher's findings that could be the result of cyclic conditions. This would be similar to what Simon (1995) refers to as a mathematical teaching cycle or to what Freudenthal (1973) regarded as "thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment" (p. 374).

The above approach can lead to the role of the teacher being that of helping learners to understand mathematics rather than merely instructing them. In order to perform this role effectively teachers need support in the form of modifiable instructional activities that can be used in constructing and revising hypothetical learning trajectories of mathematical concepts. This could enable teachers to consider a wider range of activities to help learners to learn and understand mathematics.

2.4 Learning and understanding of the function concept

There are two different views regarding understanding of the function concept. On one hand, Sajka (2005) views understanding of the function concept as *having knowledge* of: the definition, its origin, key idea, its basic properties; representations and different languages related to the concept. The knowledge aspects identified by Sajka (2005) are similar to the dimensions of understanding mentioned by Usiskin (2012) in Section 2.2 of this study. Sajka's (2005) view equates understanding of functions to having knowledge of functions. In addition to having knowledge, I believe that understanding also involves the appropriate use of knowledge an individual has about functions to solve problems. While Sajka (2005) views understanding the function concept as '*having knowledge*' of its aspects, Markovits, Eylon and Bruckheimer (1986) view understanding the function concept as '*being able*' to: interpret, manipulate and use the concept in fields other than mathematics, together with the use of the concept in different contexts within mathematics itself. Markovits et al. (1986) regard these abilities as components of understanding. In this study I subscribe to the view of '*being able to*' provided all the aspects mentioned by Sajka (2005) are included because of my intention to determine and improve learners' understanding of these aspects. In understanding, learners should also be able to interpret and manipulate the related concepts of domain and range because it is from this knowledge that the definition of the function concept is built (National Council of Teachers of Mathematics, 2000). Furthermore, understanding the function concept is not complete without the ability to interpret the inverse function (NCTM, 1989). However, I am aware that a learner might be able to demonstrate a mathematical skill without a deep understanding of that concept, namely, having procedural understanding. For example, a learner can easily determine the inverse of a function by switching variables but cannot explain the switching of variables. To minimise such instances in this study, I compelled learners to demonstrate and explain their skills or abilities or competencies, namely, their conceptual understanding. From the above discussion and Usiskin's (2012) general definition of understanding a mathematical concept discussed in Section 2.1, I can describe the understanding of functions as an ability to deal effectively with the skills and algorithms associated with the function concept, including its properties, classification of relations as functions and non-functions, uses and applications of functions, and representations of the function concept.

Based on Sajka's (2005) view on understanding the function concept, the requirements from the CAPS mathematics document and from my teaching experience I assume that learners understand the function concept if they are able to:

- explain the key idea of the function concept which is a dependent relationship in which the value of one variable is dependent on the value of another;
- derive a working definition of the function concept based on the key idea and other definitions from different textbooks;
- explain and use the basic properties of the function concept which are univalence and arbitrariness;
- classify relations into functions and non-functions;
- give examples of relations which are functions, and relations which are not;
- transfer from one representation to another;
- identify key features in graphs and tables including intercepts, asymptotes, symmetries, maximum and minimum, intervals where the function is increasing, decreasing, positive or negative;
- Calculate or determine key features from given equations; and
- Sketch the graph using the given key features.

These abilities coincide with indicators of APOS theory (see Sections 3.5.1 to 3.5.5). Thus, each of these abilities is an indicator of the APOS level in which the learner is operating. In this study I grouped these abilities under the function definition and representation. Then, I developed instructional activities for the function definition and representation to compel learners to move from their initial APOS level towards the desired schema level. The ability to determine aspects of the function concept can be taught and understood in levels. These levels developmentally rely on each other in such a way that what learners learn in one level should propel them to the next. This is because these related aspects of the function concept connect with each other as they build to its conception. Kalchman (2001) concurs by remarking that "the levels of development of the function concept should accommodate the development of sub-concepts that also progressively build a concept" (p. 11).

Markovits, Eylon, and Bruckheimer (1986) observe that in learning and understanding the function concept, learners are expected to pass through three stages. First, they learn that the function concept is composed of three sub-concepts: domain, range and rule of correspondence. These sub-concepts of the function concept are related to its formal definition. As a result, this stage is in line with the CAPS curriculum which requires that the learner should first demonstrate knowledge of a formal definition of the function concept. In Markovits et al's (1986) second stage, learners learn that functions can be represented in several forms, such as arrow diagrams, verbal, graphical and algebraic representations. Thirdly, they learn that the same function can be represented by each of the above representations, so they have to learn to translate a given function from one representation to another (Markovits et al., 1986) including the inverse. This implies that learners should be able to make connections between the function definition and the different representations and between the representations themselves. At the end of these three stages learners will be expected to give examples of a function, convert flexibly between the different representations and find the inverse and relate it to the original function. This is also in line with the grades 10-12 South African Curriculum and Assessment Policy Statements (CAPS) document (Department of Basic Education, 2011) which requires that the learner demonstrates the ability to work with various types of functions and relations by converting flexibly between numerical, graphical, verbal and symbolic representations. I agree with the stages suggested by Markovits, Eylon and Bruckheimer (1986) provided the first stage starts by helping learners to understand the key idea forming the concept, which is the dependence relationship between variable quantities, before developing an understanding of the definition because the definition of the function concept is built upon this key idea.

It is important at this point to explain the key idea behind the function concept because it helps learners to develop an understanding of the definition of the function concept in the next section. Hence, the study of the function concept can be regarded as the study of relationships and their properties (Ronda, 2009). Thus, it is important for learners to identify the changing and unchanging quantities in a relationship and to determine the effect of the change of one quantity over others. A quantity is called a function "only if it depends on another quantity in such a way that if the latter is changed then the former undergoes change itself" (Sfard, 1991, p. 3). This implies that learners experience the function concept whenever they consider how change in one variable can cause or have a corresponding effect on another. As a result, the initial learning and understanding of the function concept can focus on identifying change and what changes (Sierpinska, 1992). This idea of change can be introduced at an early stage through the study of many examples of variable quantities such as area, volume, motion, and growth. Learners first recognize quantitative change within a variable. Later they observe change between variables, connect these changes and then look for relationships (Sfard, 1991).

According to Maharaj (2008), the function concept always involves a relationship between two or more variables. The variable concept itself often requires a complex process to learn and to understand and should be well developed before functions are introduced. The most important interpretation of a variable is 'a varying quantity'. For example, related quantities that change together, like x and y in the equation, $y = 2x + 3$, are called variables. Thus, when one variable depends on another for its value, we say that it is a function of the other. As such, learners should be able to determine which variable uniquely describes the other, thus differentiating the independent and dependent variables. Ideally, this implies that, as the learners observe the change in the independent variable they should also be able to observe how that change affects the dependent variable in order to establish a functional relationship. Understanding the key idea behind the function concept might help learners to evaluate different definitions of the function concept that appear in various textbooks. However, for teachers to enable learners to evaluate these different definitions of the function concept they need to understand how these definitions of the function have developed. In the following section the development of definitions of the function concept is discussed followed by some definitions of the function concept that have been provided in some high school mathematics textbooks to determine their connection with the key idea and related concepts described above. It should be mentioned, however, that this list is not exhaustive but sufficient to provide insights into the definitions of the function concept.

2.5 Development of the function concept using its definitions

According to Markovits, Eylon and Bruckheimer (1986) learners move through three stages of conceptual development of the function concept. Sfard (1992) asserts that there are different definitions that should be used at different stages of conceptual development of the function concept. The first stage is where the function concept is encountered, usually as a result of mathematical processes that generate it. Then learners begin to think of the concept as an entity separate to the processes that generated it and finally they are able to use it to solve problems (Sfard, 1992). In the first stage, genetical definitions are sufficient in developing the key idea underlying the function concept since they are related to the origin of the function concept. In this study learners first learnt about the key idea before being introduced to these genetical definitions. Insook (1999) gives the following examples of genetical definitions:

- "a function is a relationship between two variables such that changes in one variable result in changes in the other; and
- a function is a relationship between two variables in which the value of the independent variable uniquely determines the value of dependent variable" (p. 50).

Similarly, Laridon, et al (2007) stated the following:

- "a function *f* is a relationship between two sets A and B where every element of A (the input set-domain) is mapped to only one element of B (output set-range)" (p. 42).

These definitions allow learners to see the connection between the definition and key idea underlying the function concept, namely, that of dependent relationships. However, definitions cannot help separate functions and non-functions and can cause confusion, for instance, with constant functions where there is only one variable. Thus, separately, these definitions do not give a holistic picture of the function concept and they need to be complemented by other definitions that characterise the second stage.

At the second stage learners think of the function concept as a separate entity that performs a specific operation. As a result teachers can use analogies to explain the function concept. According to Insook (1999) there are two types of analogies, namely, expression and process which characterise the function concept as formulas or equations and as operations or machines. Insook (1999) gives the following examples:

- "a function is a machine with a little elf inside of it that changes what you input into the machine before he throws it back out of the machine; and
- a function is an equation that assigns a value to a variable by using several mathematical properties" (p. 50).

Laridon et al (1987) provides the following definition:

- "a function is a rule that assigns to each member of which each element of a first set (called the domain) is associated with only one element of a second set" (p. 96).

These definitions help learners to liken the complex function concept to a simple phenomenon. However, they are only helpful in explaining and illustrating what the function concept does but do little to explain the real nature of the concept. They only reveal one aspect of the function concept, therefore, these definitions need to be augmented using other definitions that portray all aspects of the function concept.

During the final stage learners are able to use the definition of the function concept to solve problems and teachers may put more emphasis on formal or logical definitions which enable recognition and identification of functions and non-functions. However, such recognition and identification does not necessarily explain the true nature and the real character of the function concept. The following are two examples of logical definitions.

- "a function is a correspondence between two sets of elements such that to each element in the first set, there corresponds one and only one element in the second set" (Yang, 2011, p. 1).
- "a function is a set of an ordered pair (x, y) for which there is never more than one value of y for any given value of x" (Foerster, 1984, p. 568).

Logical definitions do not allow learners to connect the function concept to its nature and the practical contexts in which learners can identify relationships and recognize changes within a relationship. Many teachers and textbooks appear to emphasise the use of logically derived definitions which are used to determine whether a given relation is a function or not.

The definitions described above concentrate on different aspects of the function concept and have unique advantages and disadvantages. It should be emphasized that all the above definitions are important in developing a deep approach to understanding of the function concept. Teachers need to be familiar with these definitions of the function concept and show how they are linked in order to help learners develop a working definition. The working definition would combine essential components of these types of definitions for the sake of helping learners to understand the true nature of the function concept. They also bring out essential characteristics of the function concept, namely, dependence, arbitrariness and univalence (Freudenthal, 1983). Since the dependence characteristic of the function concept has been discussed as the key idea underlying the function concept, the following section will discuss arbitrariness and univalence as the main characteristics of the function concept.

2.6 Characteristics of the function concept

According to Yoon (2007), the arbitrariness of a function refers to:

> both the character of the relationships between the two sets on which the function is defined and the sets themselves. In terms of the relationship between the two sets, this means that the function does not have to exhibit some regularity, be described by any specific expression or particular shaped graph. In this case two sets means that functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers (p. 578).

The arbitrariness property of functions “expanded the definition to include many relationships that were not previously considered functions” (Leinhardt, Zaslavsky & Stein, 1990, p. 7), for example, “functions defined on split domains, discontinuous functions, functions defined by a graph, functions composed of arbitrary correspondences, and functions defined by more than one rule” (Malik, 1980, p. 176).

The univalence characteristic of the function concept refers to the part of the definition that states that for each element in the domain there is only one element (image) in the range (Even, 1992). Thus, in terms of a relation between two sets, a relation in which each single element in the domain is mapped to its own single element in the range, as in a one-to-one relation, or a relation in which more than one element in the domain is mapped to the same single element in the range as in a many-to-one relation, could be a function. In contrast, a relation in which each single element in the domain is mapped to more than one element in the range, as in a one-to many relation, or a relation in which more than one element in the domain is mapped to more than one element in the range, as in a many-to-many relation could not be considered a function. The univalent nature of the function concept is used in this study to allow learners to determine whether relations are functions or not. Unfortunately, many teachers and learners prefer to use the vertical line test to determine whether a drawn graph is a function or not.

Many learners use the vertical line test without any understanding of why it works (Clement, 2001). The vertical line test uses the logical definition of the function concept which "is basically a set of points in which each element from some set, called the independent variables, is uniquely mapped to elements of another set, called the dependent variables" (Akkoc, 2006, p. 16). Therefore one cannot have two values for one independent variable. When a vertical line is drawn and it touches the curve twice, this indicates that one of the independent variables is indeed being mapped to two values. That is why the vertical line test works in determining what is not a function. Any vertical line will intersect with only one point of the graph of a function.

2.7 The role of the definition in teaching the function concept

Teachers often introduce the function concept by first giving the definition of the function concept and then teaching how to use the definition to identify examples and non-examples of functions (Kwari, 2007). As a result, learners cannot see the connection between functions and the real world which makes the concept too abstract and difficult to understand. This method of teaching functions starting with the definition has also been criticized by Vinner (1992) when he said "Before suggesting definitions to the learners, suggest examples, manipulating and other experiential opportunities as a concept definition does not guarantee understanding of the concept" (p. 196). In line with this thinking, Kwari (2007) proposed that learners need to be given an opportunity to develop early experiences in dealing mathematically, with the many situations in which functions occur before being given the formal definition of the concept. Kwari believes that, these early experiences of functions from many sources contribute towards the development and understanding of the function concept that will help learners to formulate a definition later. Without a deep approach to understanding the definition of the function concept a learner may not know what conditions are necessary for a relation to be a function and how to use it when solving problems. It is, therefore, pertinent to determine how one defines the function concept in order to begin to describe one's understanding of a function which is based not only on the learners' definition, but also on their ability to identify it in its different representations. The following section examines the different representations of the function concept.

2.8 Representation of the function concept

A representation can be regarded as "something that stands for something else" (Janvier, 1987, p. 10) and it allows for an illustration of the function concept in coherent and consistent ways that facilitate acquisition and retention of the concept (Bayazit, 2011). Representations can be external or internal (Goldin, 2001). External representations refer to visible symbolic entities that can be written or spoken, for example, algebraic expressions, Cartesian graphs, arrow diagrams, tables, sets of ordered pairs, and situations from everyday life (Yerushalmy & Shternberg, 2001). Internal representations, on the other hand, refer to mental images of the function concept developed by the learners through their interactions with and reflections upon the external representations (Goldin, 2001). Since learners construct knowledge differently their internal representations of the function concept are likely to be different. This implies that their understanding of these representations varies from one learner to the other and from one representation to the other.

In the classroom situation, many teachers use external representations because they can be written, spoken, observed and easy to assess (Yerushalmy & Shternberg, 2001). However, Moschkovich, Schoenfeld, and Arcavi (1993) found that learners can express a certain understanding of the function concept in one representation, yet express a different understanding of the function concept in another representation. This implies that it is important to examine the impact of each of the representations of functions on understanding the function concept. Eisenberg (1991) discovered that some success can be achieved by introducing the function concept in a variety of external representational contexts to aid the achievement of internalisation and so promote understanding. Eisenberg (1991) also argues that "... the unwillingness to stress the visual aspects of mathematics in general and of functions in particular, is a serious impediment to learners' learning" (p. 152). The teaching implications are that instruction should focus on using different representations to help learners understand concepts. Furthermore, learners should be exposed to activities that require them to switch from one representation to the other and to focus on different formulations of mathematical statements.

Representation of relationships between variables allows the translation of the real world situation into the mathematical world (Yerushalmy & Shternberg, 2001). These representations also help learners to organise, create, record, understand and communicate mathematical ideas. In practice, "learners are often introduced to examples in various forms: a verbal representation of a function in formal or everyday language, a set diagram (representing a function by two sets and arrows between them), a function box (representation of an input-output relationship), a set of ordered pairs, a table of values, a graph and a formula" (Akkoc & Tall, 2005, p. 9). Goldenberg (1988) proposes that flexibility with these various representations will aid learners in achieving a deeper approach to understanding the function concept. However, each of these different representations also has its own peculiarities and is interpreted by learners in subtly different ways that add to the complication of the learner's concept image (Akkoc & Tall, 2005). This has led to a growing criticism of these representations with respect to what they actually represent and how they are linked cognitively (DeMarois & Tall, 1996). For example, Thompson (1994) criticised the generally accepted meanings of representations:

> ...the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation. ...the core concept of 'function' is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produce a subjective sense of invariance (p. 39).

Thompson sees invariance in the function concept as it moves across the representations simply because one representation can only convey part of the meaning of the concept. As a result "learners will see each representation as a 'topic' to be learned in isolation of the others" (Thompson, 1994, p. 23). Eisenberg (1992) asserts that "an ability to make connections among representations of the function concept is the main component of a robust understanding of the function concept" (p. 76). This ability to make connections among different representations of the function concept is referred to as a mathematical translation. Thus, understanding of the function concept may be enhanced by "designing instructional activities that are not restricted in certain types of representation, but involve recognition and transformation activities of the notion in various representations" (Duval, 2002, p. 4). This idea was used when designing instructional activities in my project. Since the various representations represent the same function it is important to identify and analyse the connections among them. These connections are necessary in helping learners to flexibly move from one representation to the other.

2.9 Connections between different representations of the same function

Translation is "the ability to move from one representation of a function to another or to recognise the same function in different representational forms" (Demana, Schoen & Waits, 1993, p. 13), for example, from the graphic form to an equation, or from table form to the graphic form. The ability to flexibly move from one form of representation to another, "allow learners to see rich relationships, develop a better conceptual understanding and enhance their ability to solve functional problems" (Gagatsis & Shiakalli, 2004, p. 654). As such, learners need to understand the different types of representations in order for them to successfully answer questions involving these representations. Knuth (2000) added that since different representations emphasize different features of the function concept, the ability to move flexibly among representations is critical for learners to be able to choose the representation that will facilitate their ability to most efficiently solve a functional problem. In many instances learners' limited concept formation and accompanying difficulties may be traced to the use of one or another representational setting of the function concept in isolation. In most cases learners experience difficulties translating among these parallel representations because the translation process is overlooked and, as a consequence, learners exhibit almost no flexibility whatsoever (Goldenberg, 1988). Thus their ability to translate between representations of the function concept in a given problem situation is important for every learner for the problem solving process to be successful. A learner needs to understand the equivalence between different representations of the same function. Van Dyke and Craine (1997) show diagrammatically the twelve directions for translating the four main representations in Figure 1 below. These are in line with those prescribed in the South African National Curriculum Statement and is important for learners to understand them.

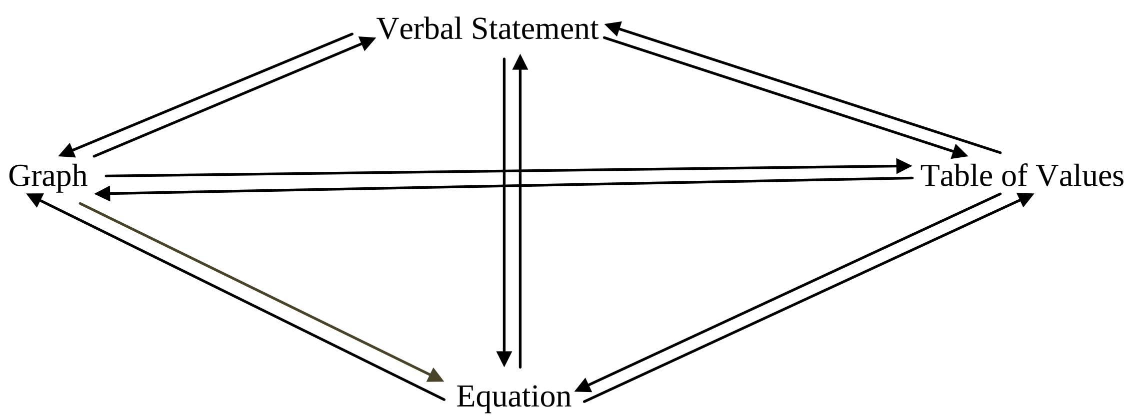


Figure 1: Van Dyke and Craines' 12 directions of equivalence (1997, p. 616)

Flexibility to move between these representations is a mathematical ability that helps learners to realize their equivalence and connection which they might utilize during problem solving. However, in most of our classrooms the connectivity between these representations is often absent (Knuth, 2000). It is important for learners to see these representations as 'informationally equivalent' and when this equivalence is evident it demonstrates a deeper approach to understanding the relationship between these different representations which will enable them to use these representations interchangeably.

The above discussion implies that meaningful learning of the function concept could be attained when a variety of representations have been developed and connections are established amongst them. "Establishing connections among the representations could produce a more coherent and unified understanding" (Knuth, 2000, p. 53). When planning instructional activities I will provide learners with an opportunity to experience the function concept across the representations since one representation can only convey part of the meaning of the concept. This flexibility with different representations of the function concept is aimed at helping learners to grasp the idea of the inverse of a function which can also be a function itself though not always so.

2.10 Inverse function

The Mathematics Modlin Dictionary (2006) defines an inverse function "as a function which 'does the reverse' of a given function" (p. 106). For example, if we have the function $f\colon x \to 3x + 2$, its inverse is found by reversing the operations that constructed it (reverse of multiplication is division and reverse of addition is subtraction) and the rules of precedence (BODMAS) are also reversed . The reversal of operations and rules of precedence gives the inverse denoted by $f^{-1}\colon x \to \frac{(x-2)}{3}$. On the other hand if we consider again the function $f(x) = 3x + 2$, we can evaluate f at 2 by substituting x by 2 in the function: $f(2) = 3(2) + 2 = 8$ and where it would help to think of f as transforming a 2 into 8 (Laridon et al., 2007). So if we think of f as 'acting on' numbers and transforming them, we define the inverse of f as the function that 'undoes' what f did. In other words, the inverse of f needs to take 8 back to 2. This can also be demonstrated using the function machine. Thus, if f is a one-to-one function with domain A and range B, then for each y in B there exists a unique x in A such that $y = f(x)$. This x is specified uniquely by the corresponding y, so there is a new function with domain B and range A (Laridon et al., 2007). Thus, it is important to stress the fact that the domain and the range of the function correspond, in this order, to the range and the domain of the inverse function. In essence, the existence of the inverse function is related to the definition of a function, and demands an analysis regarding the role of the domain and the range in finding the inverse function. In addition, Cetiner, Kavcar and Yildiz (2000) present the concept of an inverse function through a symbolic definition: $f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$. This means that, if the inverse of $f(x)$ is $g(x)$ it implies that the inverse of $g(x)$ must be $f(x)$. Again this definition involves the idea that "an inverse function undoes what a function does" (Bayazit & Gray, 2004, p. 2). Thus, the property of 'one-to-one and onto' is the basic criterion that a function must meet in order to be reversed. According to Benson and Buerman (2007) the inverse function "should be understood as a way of breaking something down by reversing the operations performed in the relationship" (p. 5), for example, that learners will have a greater understanding of inverse if they discuss reciprocals as multiplicative inverses and opposites as additive inverses.

There is a connection between the Cartesian graphs of functions and their inverses which brings up the "horizontal line test". The horizontal test verifies if a function is one-to-one and if it has an inverse function. At the same time, the graphs of the two functions are symmetrical with regard to the line $y = x$. In the case of the inverse function, a display of conceptual knowledge would be the ability to explain why the procedure of switching the x and y works in the process of calculating the inverse and by the questioning or testing the existence of the inverse function. Procedural knowledge implies the mechanical computation of the inverse, following the algorithmic steps of switching x and y, and then solving for y. It is this switching of the variables on which I will focus by mentioning that for the inverse function, the range of the original function becomes its domain while its range is the domain of the original function.

However, the learning and understanding of the function concept in terms of its definition, representation and inverse is beset with both conceptual and pedagogical difficulties. These difficulties may hinder understanding of the function concept if they are not dealt with from the learner's point of view. The next section discusses research on some of the difficulties that learners face when learning and understanding the function concept.

2.11 Learners' difficulties with the function concept

The nature of the function concept with its different definitions and representations as discussed in the previous sections presents some challenges for learners when they attempt to learn and understand it. Research on learners' understanding of functions (for example Tall, 1996; Markovits, Eylon & Bruckheimer, 1988) has shown that it is one of the least understood topics. Many researchers who have conducted studies on learners’ understanding of the function concept concur that learners have misconceptions and difficulties in learning it (for example Reed, 2007; Jones, 2006; Abdullah & Saleh, 2005; Akkoc & Tall, 2005; Cunningham, 2005; Bayazit & Gray, 2003; Knuth, 2000; Hitt, 1998; Sfard, 1992; Eisenberg & Dreyfus, 1991; Vinner & Dreyfus, 1989). These studies have documented learners’ difficulties in learning and understanding the function concept in terms of its definition, different representations and inverse of a function separately. However, this study examines the difficulties and misconceptions associated with the function concept in order to determine their sources and effects in the design of instructional activities to address them.

Research on the relationship between learners’ concept image and the formal definitions of the function concept has revealed some serious difficulties that emanate from the definition itself (Polaki, 2005). The formal function definitions may seem straight forward, but learners often exhibit strong tendencies to recall from their experiences in the classroom, rather than focusing on the definition of the function concept itself. Another reason for this difficulty might be that teachers only use the formal or logical definition instead of augmenting it with the genetical and analogical definitions as discussed in Section 2.4. Vinner (1992) also asserts that a concept definition does not guarantee understanding of the concept because the first examples of functions that learners encounter constitute, in the learner’s mind, the prototype of a function. As a result “many learners base their understanding of a function on their reservoir of examples rather than on definitions they have been taught. Furthermore, when learners are faced with a function different from the examples that they have been taught, they hesitate before accepting it as a function (Vinner, 1992, p. 257). This implies that learners need to understand the key idea underlying the function concept first and then use that understanding to generate a working definition. Using Vinner’s assertion above it means an understanding of the concept is what guarantees understanding of a definition. In this study learners were engaged in activities that helped them first to understand the key idea of the function concept and then to derive a working definition that they could use to determine when a given relation is a function or not and to provide examples of functions and non-functions.

Dubinsky and Harel (1992) note that learners have difficulty with the unique nature of the function definition and confuse it with the notion of one-to-one correspondence of elements in the domain set to those in the range set. This leads to their general neglect of domain and range thereby indicating learners' lack of ability to visualize functions. While learners may remember and understand the vertical line test which instructs them to slide a vertical line across the graph they are testing, Dubinsky and Harel observed that, it does not prevent learners, even when they are older, from confusing domain and range values. If the line ever crosses two points of the graph at once, the graph is not a function. Though this technique is a helpful aid for learners to comprehend what it means for the function to have uniqueness, data from Hitt's (1998) study indicates that learners were still not able to anticipate sub-concepts of the function definition, for example, domain and range. Breidenbach, Dubinsky, Hawks, and Nichols (1992) also note confusion between the requirement for being a function and the definition of a one-to-one function. Learners thought that all functions are one-to-one and onto, graphs of functions are 'nice' namely, smooth and with no sharp corners, and tended to use the vertical line test as a rule and not a principle to follow. They did not consider constant functions, those with split domains, or a function obtained by composition to be a function.

In his research, Hitt (1998) discovered that "most learners' knowledge of functions was limited to functions that are continuous and defined by a single symbolic expression" (p. 42). Similarly, Clement (2001) reports the tendency by learners to regard a function "as something that can be defined in terms of a simple rule, a relation whose graph is continuous, and a relation that is one-to one" (p. 9). This indicates a very narrow understanding of function, because some functions can neither be represented in the form of a symbolic rule nor in the form of a graph. Moreover, some functions are not continuous, and others are onto.

Sfard (1992) observes that learners are unable to connect different representations of functions (graphical, symbolic, and tabular). Cunningham (2005) alludes to this observation by stating that many learners have difficulties in comprehending concepts associated with the function concept in its different representations and find it difficult to link these representations in given problem situations. Learners' difficulties in linking these different representations of the function concept have also been documented in several studies (for example, Abdullah & Saleh, 2005; Akkoc & Tall, 2005; Gagatsis & Shiakalli, 2004; Knuth, 2000). Making these links, however, "have become associated with a foundational understanding of the function concept" (Eisenberg & Dreyfus, 1991, p. 87). This implies that for learners to be able to successfully solve functional problems they need to have internalised rich connections between the different representations of the function concept.

Knuth (2000) has also discovered that learners experience difficulties in translating from the graph to symbolic form and as a result fail to identify the link between these representations in a given problem situation. In this regard "learners are faced with the difficult task of not only understanding how each representation encodes and presents information but also of understanding how these representations relate to the concept they represent and to each other" (Ainsworth, 1999, p. 12). Thus, the inability to move flexibly among representations of the function concept can cause a conceptual gap that would keep a learner from progressing further until the gap is identified and bridged (Knuth, 2000).

Sierpinska (1992) has found that learners have difficulties in interpreting graphs. This was indicated by learners' failure to recognize the underlying equivalence between the graph and the equation of the graph, the verbal context or application and the table of values that the equation and graph represent. Learners' difficulties with graphical contexts were also noted in Sierpinska's (1992) study. Learners had problems reading the Cartesian coordinates of the graph and relating the equations to Cartesian coordinates, while other learners read coordinates (x, y) by looking at the furthest corner of the graph. Learners' limited understanding of the connection between equations and their graphs was also observed by Knuth (2000) who found that learners do not understand that the 'function value' refers to the y-value, assuming conventional labelling of axes. For example, some learners struggled when relating equation $y = f(x)$ with coordinates $(x, f(x))$ and what it means for one quantity to be a function of another (Carlson, 1998). A similar weakness emanated from a qualitative analysis of learners' responses in a study by Abdullah and Saleh (2005) which indicated learners' difficulties with symbols of $f(x)$. "A few learners were unaware that x is a variable of $f(x)$. They retain $f(x)$ in the formula even though x has already been given a value. The difficulty in understanding that the symbol x is a variable in $f(x)$ caused some learners to equate $p(x\text{-}4) = p(x)\text{-}4$" (Abdullah & Saleh, 2005, p. 3). In essence, learners have difficulty with function notation, that is, the role of the parentheses in the function representation.

Similarly, Vinner and Dreyfus (1989) observe that learners lack the ability to explicitly relate a function and its graph. Learners are not able to move from the one to the other identifying domain and image of function. Rigid and stereotyped ideas are often related to functions and their graphs (Markovits, Eylon & Bruckheimer, 1986). For example, learners have problems to grasp the idea of function as a relationship between variables, namely where one depends on the other. This indicates that learners have a discrete view of a function relating separate pairs of numbers, where each number may be considered as an input giving another number as an output. Although learners believe that there is a relationship between numbers, they conceive a relationship separately for each pair. Moreover, the relationship of dependency between the two variables is not visible in the graph and remains a static representation of the pair (x, y) and does not afford the meaning of dependency between the two variables (Carlson, 1998).

Eisenberg (1992) asserts that learners have a strong tendency to think of functions in terms of formulas rather than as visual representations which can be helpful. The reason for learners' reluctance to use visual representations is that:

> visual processing requires a higher level of skills than analytical processing. While analytical processing often involves only one degree of abstraction from an expression to concrete numbers, visualization requires the ability to evaluate an expression, develop trends, and transfer all the knowledge into a visual format (Eisenberg, 1992, p. 34).

If learners are not exposed to graphs and other visual representations they will be reluctant to attempt questions with graphs which will disadvantage them in an examination.

Sfard (1992) also found that learners thought that the 'function is its representation' as indicated by their perception that a graph or an equation is a function. This, clearly, is a misconception that has led many learners to think that a computational formula is a necessary condition for a function or that variables must be present to indicate input and output (Briedenbach, Dubinsky, Hawks, & Nichols (1992). Sfard (1992) concurs that "some learners believe that functions must have a rule or algorithm behind them. Learners tend to see formulas as things in themselves rather than as representations of other entities and believe that there must be an algorithm corresponding to the function in order for it to be valid" (p. 378). Thus, learners who are used to working with functions in algebraic form may find it difficult to understand that functions may be constructed arbitrarily.

Sfard (1992) has also noted that learners find it difficult to accept that algorithms that look different yet produce the same values, are actually the same function. This has resulted in learners giving independent solutions to basically equivalent problems when only slight changes in notation were introduced. For example, "when presented with the following algorithmically different functions: IN *to* IN: $g(x) = x^2$ and the recursively defined $g(0) = 0$, $g(x + 1) = g(x) + 2x + 1$, learners could not understand the equivalence between these functions though they produced the same values" (Sfard, 1992, p. 387). This misconception is a result of learners' unfamiliarity with the ordered pair definition of function. Such learners cannot comprehend that two different algorithms that produce the same set of ordered pairs are in fact the same function because they are unable to modify their understanding of functions as rules.

According to Vinner (1992) learners also show some degree of compartmentalization concerning the concept of function. He defined compartmentalization as the existence of incompatible pieces of knowledge in a learner's mind without the learner being aware of it. For example, some learners defined a function as a correspondence between two sets, but they claimed that a graph does not represent a function because there was no rule to describe it. In line with the learner handicap described above, Monk (1992) noted learners' tendency to trace back to each axis of a graph rather than look at its global characteristics. He differentiated between a point-wise analysis of a graph and an across-time analysis. A point-wise question asks for values of a function for a specific input value. An across-time analysis involves asking learners to describe a pattern of change in the value of a function that results from a pattern of change in the input values. Monk (1992) maintains that one source of difficulty that learners have with across-time analysis is their incomplete understanding of relevant concepts. For example, learners may have difficulty with the related concepts of speed, distance, and time. Given a graph of speed versus time of three runners in a race, learners may be unable to completely integrate the three. They are able to use the concepts for some purposes (for example, identifying the winner by point reading) but unable to correctly differentiate change in position with change in speed. Thus, learners have difficulty in interpreting rate of change information within a situation and demonstrating an awareness of the impact of change that one variable has on another.

An error mentioned frequently in research is the tendency of learners to treat a graph as a literal picture of the problem situation (Sierpinska, 1992). For example, in Clement's (2001) study, learners applied the vertical line test directly to the path of the caterpillar as it crawled on a piece of graph paper to determine whether the caterpillar's location was a function of time. Goldenberg, Lewis, and O'Keefe (1992) added that learners do not have a sound understanding of graphs of functions because they do not really understand or even see the varying nature of the variables. For this reason, learners had difficulties in interpreting and graphically representing covariant aspects of a real world situation.

Dubinsky and Harel (1992) discovered that some learners have difficulty in understanding the inverse of a function. They memorised the algorithm of switching x and y in an algebraic formula which made it difficult for them to find the inverse when the function was given in other different representations or without an algebraic formula. This is because "the algebraic expressions tend to shift the focus of attention from the notion of 'undoing' to the idea of an 'inverse operation' entailing the inversion of a sequence of algorithms in the process of a function by going from the end to the beginning" (Bayazit & Gray, 2003, p. 104). The implication is that understanding the inverse function cannot be limited to 'undoing' and can only be used as an informal conception leading to more formal knowledge.

In conclusion, as learners attempt to categorize and synthesize new knowledge about the function concept with their existing knowledge base, they have the tendency to develop incorrect assumptions and conceptions (Jones, 2006). The difficulties and misconceptions discussed above are the result of learners' incomplete understanding of the function concept as they obscure learners' perceptions of functions. Research in mathematics education has indicated the need to focus on the anticipation of learning problems and needed knowledge issues before they become impediments to learners' progress (English, 2002). Thus teachers need to be aware of past obstacles in the teaching and learning of the function concept to be able to eliminate the kinds of problems that learners are currently experiencing. In this study I used the difficulties identified in previous research to plan my hypothetical learning trajectories and the instructional activities because my aim is to improve learners' understanding of functions by overcoming these difficulties. I also referred to these difficulties when analyzing learners' responses in Chapter 4 by checking whether they still existed after each teaching experiment. Figure 2 summarizes learners' difficulties identified from different research discussed in this section.

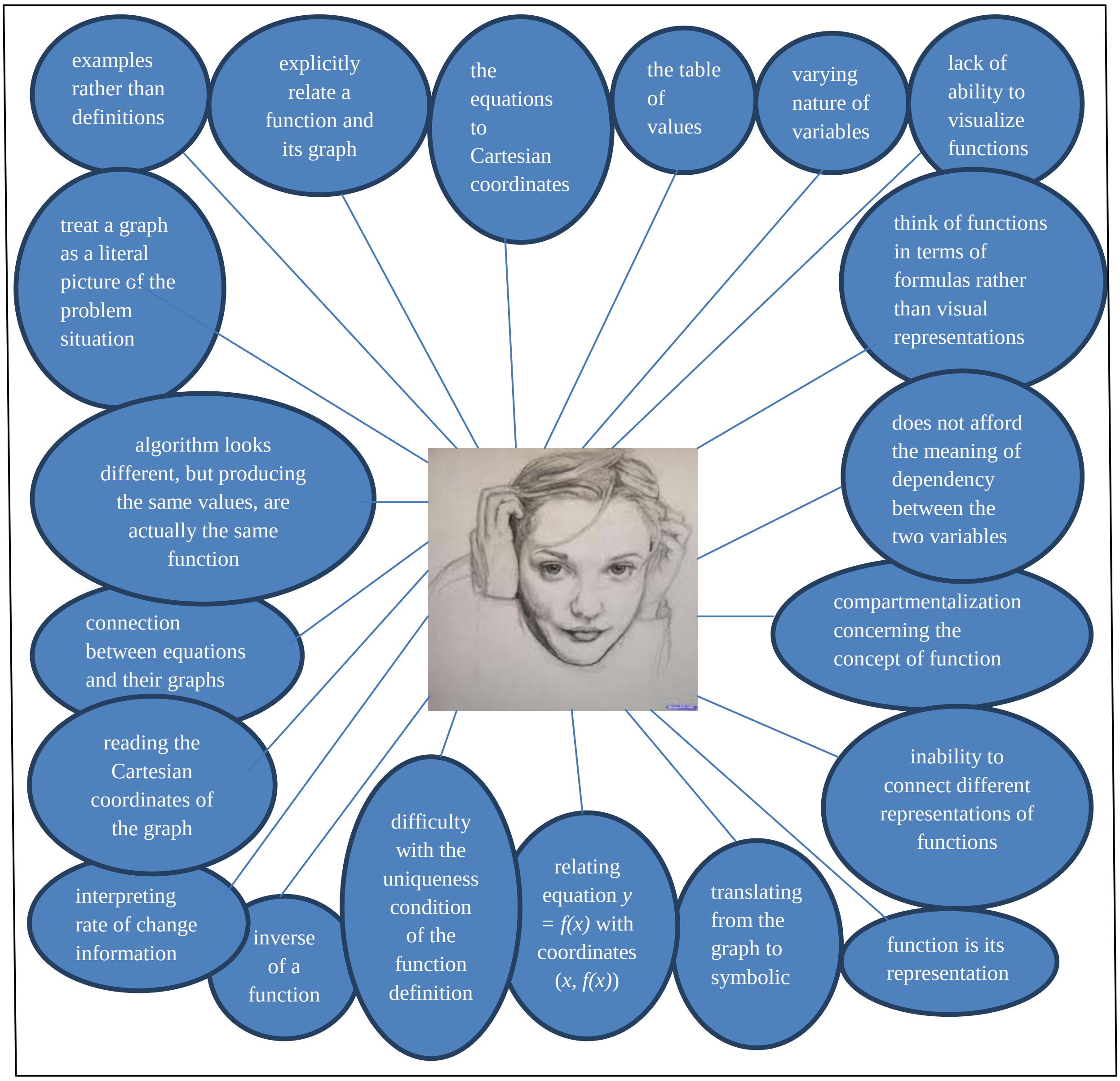


Figure 2: Balloons summarizing learners' difficulties with the function concept

The difficulties in Figure 2 are indicators of the APOS (Action, Process, Object, Schema) theory conception levels. That means learners facing these difficulties can be classified into relevant APOS conception levels. While misunderstanding is to be expected as learners learn function concepts, the goal of teachers is to choose teaching approaches that help learners to overcome their difficulties in order for them to attain the highest possible level of understanding or schema level, in the shortest amount of time.

2.12 Teaching approaches to the function concept

There are many approaches that can be used to teach functions and help learners develop a deeper approach to understanding the function concept and to overcome known learner difficulties when learning this. What follows is a description of commonly used approaches. These approaches were chosen because they are widely used by teachers and they are the least understood by learners. As a result of their use there are often misconceptions regarding the function concept.

2.12.1 The situation approach

Most textbooks and teachers introduce the function concept using a situation with related quantities already identified (Ronda, 2009). In this approach teachers mainly require learners to set up and represent relationships in tables, graphs, and equations. Ronda emphasises that teachers should let learners identify and determine which of the quantities in the situation are related. This helps learners to get a sense of what a function really is and what it is for. In this study I used realistic situations to assist learners to realize that the function is not the graph, not the table of values, and not the equation but the relationship represented by these. This implies that understanding of the function concept is the understanding of these relationships and their properties. Ronda (2009) proposes that "it is useful to use a situation where learners themselves will identify the changing and unchanging quantities, determine the effect of the change of one quantity over the others, describe the properties of the relationship and think of ways of describing and representing these relationships" (p. 4). This approach would be helpful in introducing the key idea behind the function concept in the first stage of conceptual development.

2.12.2 The example and non-example approach

Bakar and Tall (1992) noted that, in many instances the function concept is taught through examples and non-examples which "lead to mental prototypes which sometimes give erroneous impressions of the general idea of a function by conflicting with the formal definition" (p. 13). On the other hand:

> the learner cannot construct the abstract concept of function in the absence examples of the function concept in action. Accordingly, they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept" (Bakar & Tall, 1992, p. 13).

Van de Walle (2004) proposes that the teaching of functions should focus on studying change, relationships, rules, patterns and laws in contexts that are meaningful and interesting to learners. The teacher should help learners to construct the notion of function and provide learners with experiences of a function, as well as criteria by which to recognize a function and be able to respond appropriately to tasks involving functions (Pillay, 2006).

2.12.3 The pattern approach

Kwari (2007) describes the pattern approach as an approach to the development of the function concept that requires learners first understand the idea of a pattern. "Patterning activities have long been recommended as a means of supporting learners in developing an understanding of the relations among quantities that underlie mathematical functions" (Beatty & Bruce, 2004, p. 1). Patterns can "help learners to observe changes, especially what changes and how; that is: identify functional relationships between variables, obtain a rule or formula, algebraic expression or equation to describe the relationship and make predictions using a rule or formula" (Van de Walle, 2004, p. 441). The advantage of this approach is that it offers a visual representation of the function concept (Kwari, 2007). However, a problem that could arise from the pattern approach is that "most mathematical patterns generate numbers and learners might think that functions are sequences yet sequences are only a special type of function" (Sierpinska, 1992, p. 218). As a result:

> learners do not perceive the need to understand the mathematical structures and relationships underpinning pattern rules. This numeric approach to pattern learning diminishes the potential for learners to recognize commonalities in mathematical relationships across multiple representations, and obscures the underlying functional relationship of the pattern because the pattern rule becomes a sequence of arithmetic operations derived numerically in isolation from the context of the problem (Beatty & Bruce, 2004, p. 2).

2.12.4 The function machine approach

Selden and Selden (1992) suggest that using the idea of a function machine helps learners view a function as a process. The function machine accepts an input and produces an output. There is no need to know the contents of the machine. They note that it is useful in assisting learners to understand the function as a process, but does not provide a complete notion of function. Tall, McGowan, and DeMarois (2000, p. 255) recommend the "use of function machine as a cognitive root to the development of the function concept". The approach emphasises both the co-variation and correspondence relationships and develops the aspect rule (Kwari, 2007). In addition, the various representations, verbal, tabular, arrow diagram can be easily connected through this approach. The shortcoming of this approach is that it gives rise to an erroneous belief that all functions are given by a formula (Sierpinska, 1992) and this calls for the need to use a variety of approaches when introducing functions. Despite this shortcoming, Tall et al. (2000) report that the use of a function machine representations can help learners form a rich, foundational concept of functions. The function machine can also be used to teach the inverse of a function by reversing the process that transforms the inputs to yield the outputs, for example multiplication would become division and addition would become subtraction.

2.12.5 Covariational approach

Covariation was introduced by Carlson, Jacobs, Coe, Larsen and Hsu (2002) as a more natural notion of functions. They thought of a function as defining how two variables vary with each other rather than a set theoretic construct. They proposed five levels of covariation reasoning, and five mental actions that characterize these levels. The five mental actions that characterized these levels are:

> Coordinating the value of one variable with changes in the other, coordinating the direction of change of one variable with changes in the other variable, coordinating the amount of change of one variable with changes in the other variable, coordinating the average rate-of-change of the function with uniform increments of change in the input variable, and coordinating instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function (Carlson et al., 2002, p. 357).

Their ideas were propagated by Confrey and Smith (1994) who used a covariational approach to teach functions in which they built on learners' strong intuitive understandings of change to generate functional relationships. The focus was on rates of change in the variables rather than the more common approach that builds a correspondence between the variables. Covariation entails being able to move from y_m to y_{m+1} and coordinate that move with movement from x_m to x_{m+1}. Confrey and Smith (1994) used the tabular and graphic modes of functions in a language rich environment to have even young children exploring functional thinking using rates of change.

2.12.6 The word problem approach

The use of word problems provides activities that relate to what learners experience in their daily lives (Kwari, 2007). Kwari (2007) believes that, word problems provide the opportunity to use functions as models of real world situations and these models can be represented in many forms and have the potential to allow learners to move through different representations, from words to diagrams or tables, graphs to symbolical representations. The major obstacle in this approach is the problem of language. In South Africa mathematics is taught in English which is a second language to most learners. Everyday experiences for most learners are usually expressed in their vernacular languages. Kwari (2007) also observes that, the problem of language in the learning of mathematics is likely to negatively affect the development of the function concept particularly in the interpretation of the word problem. So, there is need to ensure that learners understand the word problem before they answer it. There are different language issues. First, especially for learners who receive instruction in a language other than their mother tongue, it is the language of instruction. Secondly it is also the mathematical jargon. For example, the word volume has the meaning of length times width times height for a box in mathematics, but could also mean softer and louder on a phone or radio. The teacher may even combine the learners' mother tongue and the language of instruction to help learners understand word problems. This move might cause learners to first think in their mother tongue and then translate to the language of instruction, an approach which may backfire later if they fail to translate some technical mathematical words to their home language.

2.12.7 Property-oriented approach

According to Nemirowsky and Rubin (1992) a function can be introduced and described with reference to its properties. Since this study is dealing with grade 11 learners I refer to properties of some functions they learn at that grade level namely linear and quadratic functions. A linear function whose general form is $f(x) = mx + b$ can be taught by referring to the properties of its graph namely its slope (m) and *y*-intercept (b). The property of slope can be used to determine if the function is increasing, decreasing or constant. On the other hand, the quadratic function with equation $f(x) = ax^2 + bx + c$, where $a \neq 0$, has the following properties: vertex = $(\frac{-b}{2a}, f(\frac{-b}{2a}))$; axis of symmetry: the line $x = \frac{-b}{2a}$; parabola opens up if $a > 0$ (the vertex is a minimum point); parabola opens down if $a < 0$ (the vertex is a maximum point). An understanding of these properties can assist learners in classifying different functions and in establishing connections between different representations (Monk & Nemirowsky, 1994). In this study these properties are important when translating from graphical to symbolic representations of functions and *vice versa*.

I have established links between each of the three stages of conceptual development of the function concept to particular definitions, representations and teaching approaches from the preceding sections. My intention for using definitions, representations and approaches appropriate for each stage is to help learners to develop a deeper conceptual understanding of the function concept and to encourage teachers to move away from the conventional approach of stating definitions followed by examples and then a few applications. Table 2 shows each particular stage of conceptual development of the function concept with specific definitions, representations and teaching approaches that I used in my teaching experiments.

Table 2: Stages of conceptual development of the function concept and their links with definitions, representations and teaching approaches

STAGE	TYPE OF DEFINITION	REPRESENTATION	TEACHING APPROACH
1 Process (Sfard, 1992)	Genetical definitions (Insook, 1999, p. 50) e.g. a function is a relationship between two variables such that changes in one variable result in changes in the other.	Verbal statement Table of values (Van Dyke & Craine, 1997)	Situation approach (Ronda, 2009) Pattern approach (Kwari, 2007)
2 Entity (Insook, 1999)	Analogical definitions (Insook, 1999, p. 50) e.g. a function is a machine with a little elf inside of it that changes what you input into the machine before he throws it back out of the machine.	Equations Table of values (Van Dyke & Craine, 1997)	Function machine approach (Selden & Selden, 1992) Example and non-example approach (Bakar & Tall, 1992)
3 Structural (Sfard, 1992)	Logical definitions (Yang, 2011, p. 1) e.g. a function is a correspondence between two sets of elements such that to each element in the first set, there corresponds one and only one element in the second set.	Table of values Graphs (Van Dyke & Craine, 1997)	Covariational approach (Carlson *et al.*, 2002) Word problem approach (Kwari, 2007) Property-oriented approach (Nemirowsky & Rubin, 1992)

My choice of definitions, representations and teaching approaches for each stage was influenced by the need to develop a conceptual understanding of the function concept that will enable learners to generate a working definition of it in order to switch flexibly from one representation to the other. From my review of the literature on learning and teaching of functions I want to propose that, learners' understanding of functions can be improved when learners are helped to connect all the dimensions of understanding functions following similar stages as described by Markovits, Eylon and Bruckheimer (1986) and Sfard (1992) discussed in this section. These dimensions mainly include the key idea underlying the function concept, its definitions and representations.

Chaiklin (2002) states that if you want to understand something you have to change it and if you want to change something you have to understand it. This approach has led to a realisation that if I am to understand how learners grasp concepts related to functions then I have to change my own understanding in the context of the literature review and learners' situations. For me to change their understanding I also had to grasp their understanding of concepts related to functions by 'seeing' these concepts through their cognitive lenses. The choice of the teacher's approach at each stage described above may also depend on the teacher's framework. Similarly, the design of instruction in this study has been theoretically and practically guided by specific theories that formed my theoretical framework as extrapolated in the next chapter.

CHAPTER 3

Theoretical Framework

3.1 Introduction

This chapter describes the literature that informed my theoretical framework. It discusses specific theories that helped me to establish a perspective and a set of lenses through which I viewed this study. Theories influence what one sees and what one does not see (Nkambule, 2009) and "making progress in any scientific field is difficult without explicit theories" (National Research Council, 2002, p. 468). Theories are also useful because "they direct the researcher's attention to particular relationships, provide meaning for the phenomena being studied, rate the relative importance of the research questions being asked, and place findings from individual studies within a larger context" (Hiebert & Grouws, 2007, p. 63). This study is immersed in the constructivist paradigm and guided by APOS theory which "can be thought of as an extension of the last stage of Piaget's theory, the formal operation stage, which takes place at about age sixteen" (Weyer, 2010, p. 9). In addition, Realistic Mathematics Education (RME) theory (Gravemeijer, 1994) was used to guide the intervention stage because of its close links to constructivism. These theories are described separately and then brought together to justify their position in this study. Firstly, the constructivist paradigm in which this study is immersed is discussed together with APOS theory to provide a theoretical framework for investigating learners' interpretations of the function concept. The Realistic Mathematics Education theory is also explored to provide guidance for the design of instructional sequences and activities to teach the function concept.

3.2 Constructivist paradigm

A paradigm may be viewed as a set of basic beliefs that represent a worldview that defines the nature of the 'world' (Guba & Lincoln, 1994). It is a framework within which theories are built, that fundamentally influence how one sees the world, determines one's perspective, and shapes one's understanding of how things are connected. This study is located within a constructivist paradigm (Mertens, 2005) which makes provision for the fact that each individual person interprets and makes sense of the world in his or her own way hence the notion that "no one true reality exists, only individual interpretations of the world" (Clements & Battista, 1990, p. 34). Thus, it should be understood that there are multiple realities through which one can make sense of the world, and construct reality from one's own experiences. This worldview is embedded in the qualitative research approach that I chose for this study. In this paradigm we are able to explain how we know what we know and make sense of what learners see, think, and do. Within a constructivist teaching and learning environment, the learners should be able to make sense of a real-world problem in their own way and their interpretations will depend on their experience and upon social interaction with other people. Learners acquire knowledge when they incorporate new experiences into existing mental structures and reorganize those structures to handle more problematic experiences (Kilpatrick, 1998). Constructivist views of learning posit that learners construct their own learning which may differ from formal mathematics taught by teachers in the classrooms. It is therefore crucial that constructivist aligned teachers assess learners' conceptions formatively, and identify errors or difficulties associated with the process of learning.

Gray (2007) regards constructivism as "a view of learning based on the belief that knowledge is not a thing that can be simply passed on" (p. 8) by the teacher in the front of the room to learners at their desks. Gray believes that, knowledge is constructed by learners through an active, mental process of development allowing learners to be the builders and creators of meaning and knowledge. Constructivism draws on the developmental work of Piaget and is guided by four principles:

> learning, in an important way, depends on what we already know; new ideas occur as we adapt and change our old ideas; learning involves inventing ideas rather than mechanically accumulating facts; meaningful learning occurs through rethinking old ideas and coming to new conclusions about new ideas which may conflict with our old ideas (Gray, 2007, p. 13).

The implication according to Cobb, Yackel and Wood (1992) is that, for a constructivist classroom to be productive, "instruction should be learner-centred in such a way that the teacher provides learners with experiences that allow them to hypothesize, predict, manipulate objects, pose questions, research, investigate, imagine, and invent" (p. 35). The teacher's role is to facilitate this process.

Constructivism embraces the idea that learners come into the classroom, not as empty vessels, but at various stages of conceptual understanding (Von Glaserfeld, 1987). Individual learners will continue to build their mathematical frameworks from the point where they started; therefore, at the end of a particular lesson or topic, they will still be at different stages of understanding or knowing. From the constructivist perspective, Van Glaserfeld sees mathematical learning as a reorganization of ideas already held to incorporate new information, thereby adding to the framework and building conceptual knowledge.

The basic tenets of constructivism, according to Gray (2007) include the following: knowledge is actively constructed by individuals as they make sense of the world based on their experiences; knowledge is not passively received by the individual from others, or from authoritative sources. The function of cognition becomes that of adapting and serving the individual to organize the experiential world (Von Glaserfeld, 1987). Developing the learner's personal mathematical ideas is very important to the constructivist teacher who encourages learners to use various methods for solving problems. One of the most essential skills for a constructivist educator to embrace is that of approaching "an unexpected response with a genuine interest in learning its character, its origins, its story and its implications" (Confrey, 1990, p. 108). Furthermore, attempting to see a situation as perceived by another human being should be imbued "with the assumption that the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990, p. 108). According to Van de Walle (2007) a commonly accepted goal among mathematics educators is that learners should understand mathematics, and constructivism suggests that learners must be active participants in the development of their own understanding. He added that constructivism provides us with insights concerning how learners learn mathematics and guides us to use instructional strategies that begin with learners rather than with ourselves. He further points out that constructivism rejects the notion that learners are blank slates and that they do not absorb ideas as teachers present them, rather, he regarded learners as creators of their own knowledge.

Within the constructivist paradigm, I will provide opportunities for learners to create new mathematical knowledge by reflecting on the things that they do, that is, their physical actions and the ways that they think, that is, their mental actions. Learners need to do more evaluating of their own ideas and teachers need to create opportunities where this evaluation can productively occur. This indicates that reflective ability is a major source of knowledge at all levels of mathematics as von Glasersfeld (1991) puts it: "To verbalize what one is doing ensures that one is examining it. And it is precisely during such examination of mental operations that insufficiencies, contradictions, or irrelevancies are likely to be spotted" (p. xviii).

Constructivism mainly focuses on the mental processes that construct meaning. According to Van de Walle (2007) the general principles of constructivism are based largely on Piaget's processes of assimilation and accommodation, where assimilation refers to the use of existing schemas to give meaning to experiences while accommodation is the process of altering existing ways of viewing things or ideas that contradict or do not fit into existing schemas. Constructivism stresses the idea that "learning is an active process in which learners learn from previous knowledge as well as from information provided by teachers" (Weyer, 2010, p. 9). Weyer (2010) considers knowledge as "a gradually built individual construction and described understanding in terms of building mental structures on previously built structures which also affected subsequent constructions. Mental activities of the learner such as constructing relationships, articulating what one knows, extending and applying mathematical knowledge and reflecting about experiences promote understanding (Hansson, 2006).

Dewey and Rousseau (as cited in Van de Walle, 2007) criticize the view of learning as stocking up of knowledge and of teaching as transferring such knowledge to learners who are empty vessels. The implication of their criticism in this study is that I needed to regard learning as an active process of constructing rather than receiving knowledge, and instruction as a process of supporting meaningful construction of knowledge rather than transmitting it. As such, I designed instructional sequences and activities to guide learners to construct knowledge about the definition of the function concept and to develop connections between different representations of the same function. Thus, within constructivism, I acted as a facilitator of knowledge and ensured that the learners construct knowledge through the process of discovery and problem-solving (Murphy, 1997). To this end there are certain preferred characteristics.

3.3 Characteristics of teaching approaches that encourage a constructivist way of learning

In order to be considered a teacher who draws from constructivism Brooks and Brooks (1993) said that the teacher should "design instructional activities that will compel learners to construct the requisite knowledge and also to challenge previous conceptions of their existing knowledge" (p. 43). They added that the teacher should allow learner responses to drive lessons and seek elaboration of learners' initial responses and also to allow learners some thinking time after posing questions. Brooks and Brooks (1993) believe that, "when learners can communicate their understanding, then they have truly learned" (p. 47). In this study I used learners' responses in the June examination and difficulties noted from my literature study to design instructional sequences and activities to help learners overcome the identified difficulties.

Jonassen (1991) suggests that teachers can create constructivist learning environments for their learners "by creating real-world environments that employ the context in which learning is relevant and focuses on realist approaches in solving real-world problems" (p. 58). The teacher should act as a coach and analyzer of the strategies used to solve these problems. In line with his suggestions I drew attention to conceptual interrelatedness by providing multiple representations or perspectives on the content. Although Jonassen (1991) advocates that instructional goals and objectives should be determined by learners' needs and not imposed by the teacher, and that evaluation should serve as a self-analysis tool, the reality in the last two years of schooling in South Africa is that learners need to be prepared to write an external examination. Bringing constructivism into the classroom means that "the teacher provides tools and environments that help learners interpret the multiple perspectives of the concepts and learning should be internally controlled and mediated by the learner" (Jonassen, 1991, p. 60).

The above principles and characteristics of constructivism were used in the teaching experiments for this study as they provide a framework for a conducive learning environment, productive learning process and its meaningful assessment. In such an environment learners can maximize their potential of learning any mathematical concept since the constructivist environment allows them to be involved in the production of knowledge and their understanding drives the constructivist teaching path followed by the teacher.

3.4 Constructivist mathematics teaching

White (1988) asserts that learners do not receive information from a teacher or textbook intact the way a radio receives transmitted radio signals, but, instead, all information received by learners must pass through the filter of their prior knowledge and experience. This implies that knowledge is not passively received from an external source but is actively constructed by the learner. In support of this belief, Muijs and Reynolds (2005) said that the truth is not out there, but is within each learner. The implication of this belief is that it is not possible to say something to someone and expect them to understand exactly what was intended to be conveyed. In classroom terms, just because a teacher has said something to a learner clearly and precisely, the teacher cannot infer that the learner has received the message with the intended meaning. The learner's focus might be quite different from the teacher's. Learners might fit the teacher's words into their own resulting in a meaning different from what the teacher tried to convey. This suggests that a teacher needs to find ways of knowing what sense learners make of mathematical concepts they teach, in order to plan further instruction. Task-based interviews conducted before and after the teaching experiments provided me opportunities to solicit learners' reasoning which I used to revise my HLTs and to design further instruction.

Constructivism describes how learners learn but cannot prescribe a method of teaching (Simon, 1995). Simon utilized the characteristics of effective teaching to formulate the Mathematical Teaching Cycle. This cycle consisted of four broad parts, namely, (1) *assessing* what learners know; (2) identifying the *learning* goal (mathematical concept); (3) *hypothesizing* a path by which learners will come to understand that goal (Hypothetical Learning Trajectory) and *planning activities* that are likely to bring concept formation; and (4) *implementing* them (planning and teaching). At the end of this step, the cycle reverts to step one.

The Mathematical Teaching Cycle in Figure 3 compels teachers to be aware of the dynamics of knowledge construction by their learners before they change their current teaching practices which are teacher-centred. It provides an alternative practical framework for the teacher to understand and evaluate learner learning in the classroom in order to make learner-based instructional decisions. Simon's (1995) framework is in line with Wademan's (2005) design research phases on which I modelled my teaching experiments for this study which are also cyclical in nature.

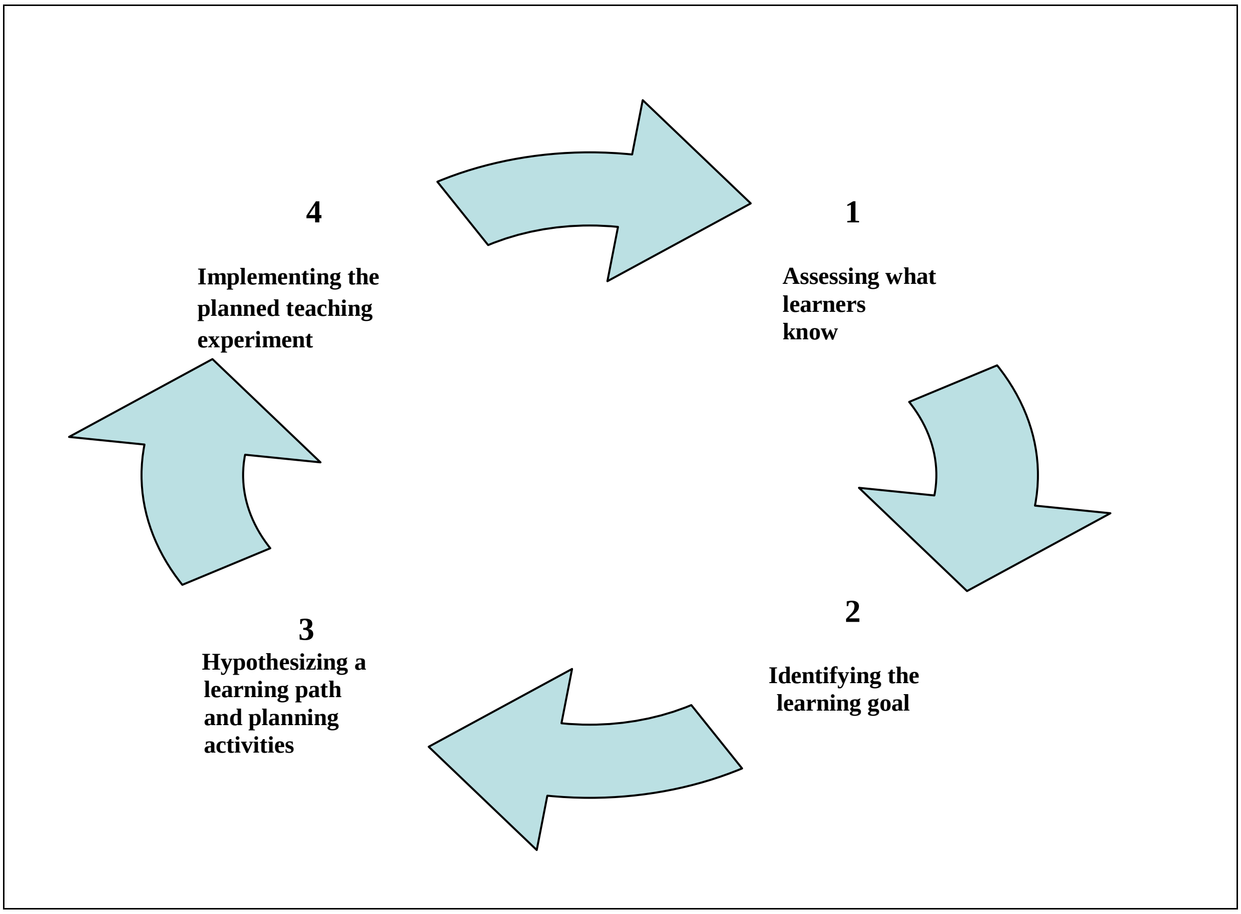


Figure 3: Mathematical Teaching Cycle (Simon, 1995, p. 123)

According to Orton (2004), constructivism appears to suggest that the teacher needs to provide the 'scaffolding' which allows the learner to progress, and it requires great skill to provide the best scaffolding for each learner. Echoing similar sentiments, Murphy (1997) observes that an important concept of constructivism is that of "scaffolding which is a process of guiding the learner from what is presently known to what is to be known" (p. 17). She points out that scaffolding allows learners to perform tasks that would ordinarily be slightly beyond their ability without the assistance and guidance from the teacher. I subscribe to the notions held by Doerr (2007) and Hansson (2006) that learners should be provided with mental activities to enable them to construct the required knowledge.

Piaget (1977) notes that differences between the teacher's intended meaning and the learner's constructed meaning are:

> … not simply reducible to missing pieces or absent techniques or methods. He argues that learners' ideas possess a different form of argument, are built from different materials, and are

> based on different experiences. Their ideas can be qualitatively different, which can sometimes mean that they make sense only within the limited framework of the learner and can sometimes mean they are genuinely alternative, wonderfully viable and pleasing to the learner (p. 20).

As such, they will not be displaced by any simple provision of the 'correct method', for, by their existence for the learner, because they must have served some purpose. However, learners sometimes hold conceptions that are not mathematically sound which in turn affect their level of achievement in mathematics. Such unmathematical ideas must be identified and addressed. Before learners can change such beliefs, they must be persuaded that their ideas are no longer effective or that another alternative is preferable.

Boaler (2009) asserts that our brains grow the most when we make mistakes. Scientists have found that when learners make a mistake in mathematics synapses spark, and there is activity in the brain that is absent when learners get work correct (Boaler, 2009). This means that when a learner gets a question right, nothing happens in his/her brain, but when he/she gets that question wrong, his/her brain grows. It is really important for learners to take risks, engage in 'productive struggle,' and make mistakes. Struggle is really important because it will make their brains grow. Teachers have long known that learners who experience 'cognitive conflict' learn deeply and that struggling with a new idea or concept is very productive for learning (Piaget, 1977). Boaler (2009) argued that open mathematics tasks encourage the opportunity for important learning and for viewing mathematics as a *learning subject*. Tasks that are narrow and closed encourage learners to believe that mathematics is a *performance subject* that is, they are in mathematics class to show what they know. Many learners think that they come to mathematics classes to answer questions correctly, not to learn. If we are serious about encouraging learners to develop growth mindsets we need to provide open tasks that have the space within them for learning, not short tasks that students are meant to get right or wrong. Tasks are made more open when they have or encourage:

- Multiple entry points
- Multiple ways of seeing
- Multiple pathways and strategies for solutions

The importance of mistakes suggest that we need mathematics environments in which learners are given open tasks and challenging work that causes them to struggle, experience cognitive conflict and make mistakes. Teachers should support or even reward students for making mistakes so that they feel comfortable doing so. Mistakes are often a result of misconceptions or carelessness.

In addition, Olivier (1989) in his study on handling learners' misconceptions, states that in the constructivist perspective 'errors and misconceptions' are seen as natural results of learners' efforts to construct their own knowledge and those misconceptions are intelligent constructions based on incorrect or incomplete previous knowledge. As such, making errors or having misconceptions cannot be avoided since in itself it forms part of the learning process. Olivier (1989) suggests that learning is a process of replacing misconceptions with appropriate expert knowledge. In this study I viewed learning as a careful modification or restructuring of these misconceptions together with the learners, to become acceptable and meaningful knowledge. I firmly believe that learners easily embrace or accept knowledge they participate in creating or re-creating with their teacher and peers using their prior knowledge and interpretations. This is because their knowledge is what they construct not what they directly receive from the teacher. A study by Clement (1982) revealed that misconceptions are resistant to traditional forms of instruction which are currently in use in many of our classrooms. This points to the need to design and develop an alternative form of instruction that uses these misconceptions as a starting point.

Dubinsky and McDonald (2001) built on the ideas of Piaget to develop a theory of learning mathematical concepts which claims that learners construct concepts through a standard set of steps namely action, process, object and schema hence the acronym APOS. Weyer (2010) regards this theory as an extension of the last stage of Piaget's theory, the formal operation stage, which takes place when learners are about age sixteen. I chose APOS theory as being the most suitable theory on which to base my teaching experiments.

3.5 APOS theory

APOS theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept (Maharaj, 2010). As such, it describes a possible process by which a mathematical concept can be learned. APOS theory attempts to analyze the internal mental structures and mechanisms constructed and used by individuals as they are thinking about a mathematical concept (Dubinsky & Wilson, 2013). APOS theory has been used effectively to develop learners' process conception of the function concept (Breidenbach, Dubinsky, Hawks & Nichols, 1992) and to evaluate learners' understanding of the function concept (Nyikahadzoyi, 2006; Dubinsky & Wilson, 2013). APOS theory can also be used both to evaluate and develop learners' understanding of the function concept

(Weller, Arnon & Dubinsky, 2011) an approach which I adopted for this study. In the present study APOS theory was used initially to predict the likely mental structures that are required to learn the function concept and to detect learners' current understanding of the function concept or evaluation of learners' understanding, which informed the design and implementation of suitable learning activities to support the construction of these mental structures or development of learners' understanding. Though Dubinsky and Wilson (2013) worked with high school learners similar to mine in terms of age, their study paid very little attention to the object and schema conceptions of the function concept. They claimed that the action and process levels are the most important for studying functions at this school level.

Similarly, Brijlall and Ndlovu (2013) used APOS theory conception levels up to the object level to evaluate grade 12 learners' understanding of linear programming a topic closely related to the function concept. In another instance, Weyer (2010) applied APOS theory to the function concept and characterised learners' ways of understanding the function concept from the action level up to the schema level. This means that learners' understanding of functions need not be limited to the action and process levels. Breiteig and Grevholm (2006) concurred when they asserted that it is possible to use APOS terminology in a slightly different way than that of Dubinsky and Wilson (2013) by characterising learners' ways of explaining to span across all the conception levels but what matters is to improve learners' understanding of the mathematical concept from their initial conception. The present study used all the four levels of APOS theory and an additional pre-function level to describe learners' understandings of the function concept in terms of its definition, representations and inverse. The addition of the pre-function level to the four levels of APOS theory could result in a new (P)APOS theory used in this study. APOS theory and the pre-function level are discussed separately but used together.

APOS theory and its application to the teaching of mathematics are based on two psychological assumptions listed below.

- Learners' mathematical knowledge is their "tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context, and constructing or reconstructing mathematical actions, processes and objects and organising these in schemas to use in dealing with situations" (Dubinsky, 2001, p. 11).

- Learners do not learn mathematical concepts directly. "They apply mental structures to make sense of a concept" (Piaget, 1977, p. 178).

These assumptions imply that "the goal for teaching should consist of strategies to help learners build appropriate mental structures, and to guide them to apply these structures to construct their understanding of mathematical concepts" (Maharaj, 2010, p. 42).

Learning is facilitated if learners possess mental structures appropriate for a given mathematical concept. If appropriate mental structures are not present, then learning the concept is almost impossible. In APOS theory, the mental structures are actions, processes, objects, and schemas. In this study I used the four levels of APOS theory together with an additional mental structure that precedes the action level called the pre-function level, hence (P)APOS theory. The following classification scheme details the method I used to determine if a learner was experiencing the conceptions according to APOS theory chosen for this study. For each conception, I briefly describe the conception and then list the item responses that indicate the existence of the conception. In general, for learners to be considered as displaying a given concept, they must have more of these indicators for that concept. If a learner is a borderline case according to this coding procedure, then other responses may be taken into account as weak indicators and these are also specified below. I also examined carefully the learners' responses for other written indicators showing that they may or may not have had the concept. Often this was the deciding factor in coding the conception of the learner.

According to Cotrill, Nichols, Schwingendorf, Thomas and Vidakovic (1996) and Dubinsky and Wilson, (2013) learners have a pre-function conception of a function if they indicate little or no conception about the function concept. Moreover, whatever little understanding learners have, it is not very useful in performing the tasks that are called for in mathematical activities related to functions (Breidenbach, Dubinsky, Hawks & Nichols, 1992). Learners at this level display the following indicators:

- give responses like " I don't know" (Breidenbach et al., 1992);
- regard a function as a social gathering (Breidenbach et al., 1992);
- give a mathematical statement that describes something or a mathematical equation with variables(Breidenbach et al., 1992); and
- regard a function as an equation (in x) with no y values (Nyikahadzoyi, 2006).

As the learners had already encountered functions in their normal classroom I have chosen not to use the pre-function level. Where the level of pre-function occurs I will indicate this as (P)APOS. In other instances I use the APOS theory. I now explain these four conception levels and their indicators with particular reference to the function concept.

3.5.1 Action level of a function

An action conception is "a form of understanding of a concept that involves a mental or physical transformation of mental or physical objects in reaction to stimuli that the subject perceives as relatively external" (Dubinsky & Harel, 1992, p. 17). At the action level, the transformation of objects is thought of as external, and the learner only knows how to perform an operation from memory or from clearly given instructions (Dubinsky & McDonald, 2001). It requires specific teaching, and the need to perform each step of the transformation explicitly (Maharaj, 2010). The following are indicators of learners who are operating at the action level.

- They typically understand the most basic ideas behind the function concept, for example, see a function as a relationship between two sets; domain; range (Dubinsky & Harel, 1992).
- They are capable of substituting numbers into a function expressed algebraically, and then doing a calculation to obtain an answer (Dubinsky & Harel, 1992).
- They think about the problem in a step by step manner and look at one step at a time (Dubinsky & McDonald, 2001), for example, when given the function $f(x) = (x + 1)^2$ and asked to solve the function for when $x = 1$, learners operating at this level would go through the following steps to get an answer: $f(x) = (x + 1)^2$ then $f(1) = (1+ 1)^2 = 2^2 = 4$. The learner would be able to understand that 1 is the input, the expression $(x + 1)^2$ represents the procedure and 4 is the output.
- They exhibit strong tendencies to recall verbatim the definitions (Breidenbach, Dubinsky, Hawks & Nichols, 1992).
- They regard a function as "some rule that relates the first number to the second" (Weyer, 2010:93) or an equation or expression that will evaluate something when either variables or numbers are plugged into the function (Breidenbach et al., 1992).
- They can substitute numbers into a function expressed algebraically, and then calculate it to obtain an answer (Weyer, 2010).

- They can think about the problem in a step-by-step manner and look at one step at a time (Weyer, 2010). For example, when given the function $f(x) = (x + 2)^2 - 4$ and asked to find the *x* and *y* intercepts learners in this level would go through the following work to get an answer:

x-intercept $y=0$	*y*-intercept $x = 0$
$0 =(x+2)^2 – 4$	$y = (0+2)^2 - 4$
$4 = (x + 2)^2$	$y = 4 - 4$
$2 = x + 2$ or $-2 = x + 2$	$y = 0$
$x = 0$ or $x = - 4$	

  The learners are able to understand that 0 is the input, the expression $(x+2)^2 - 4$ represents the procedure and $x = 0$, $x = - 4$ and $y = 0$ are outputs.
- They respond to the table of values by trying to look for "some rule that relates the first number to the second" (Dubinsky & Harel, 1992, p. 93).

3.5.2 Process level of a function

A process conception is defined as "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992, p. 19). At the process level, learners can perform the same action or transformation without external stimuli. Basically, they have internalized the procedure. Learners at this level can also "think of performing a process without actually doing it" and think about reversing the process as well as using it with other processes (Dubinsky & McDonald, 2001, p. 3). Breidenbach, Dubinsky, Hawks and Nichols *(*1992) added that a process conception is required for an individual to understand inverses of functions and other function properties. Learners operating at the process level can do the following listed below.

- Think of the transformation as an entire activity and internalize the procedure that is going on (Dubinsky & Harel, 2001).
- Look at the word 'function' as a verb and sees a function as doing something. For example, when considering equations, learners at this level can look at an equation and see the procedure as a whole without having to plug in all the specific values (Dubinsky & Harel, 2001).
- Draw the graph the function $f(x) = x^2$ by plotting only a few of the points and can also make general arguments about the function (Weyer, 2010).

- Able to describe and relate functions' properties or behaviour in terms of comparing shapes or contours and looking at a number of graphs several times to compare coefficients or algebraic terms. This will enable the learners to investigate graphs and make predictions from previous graphs (Reed, 2007).
- Begin to see how one variable changes while predicting changes in the other. An example of this concept is the relationship between the height of water in a bottle and its volume (Depaolo, 2009).
- See a function as an operation that accepts a given value and returns a corresponding value (Breidenbach, Dubinsky, Hawks & Nichols, 1992).
- Understand the difference between being a function, being one-to-one, being onto, and having a one-to-one correspondence (Reed, 2007).
- Look at the ordered pairs and see the relationships without drawing arrow diagrams.
- See a relationship when the relationship is not stated (Weyer, 2010).
- See a function as some sort of input being processed, a way to give some sort of output (Breidenbach et al., 1992).
- Talk about a general set of numbers going in resulting in numbers coming out (Weyer, 2010).
- Give a definition of a function that looks at the procedure as a whole with inputs, a process, and outputs (Weyer, 2010).
- Look at an equation and sees the procedure as a whole without having to plug in all the specific values (Dubinsky & Harel, 1992).

3.5.3 Object level of a function

An object conception is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992, p. 19). The learner at this level sees the procedure as a whole and understands that transformations can be performed on it (Dubinsky & McDonald, 2001). Encapsulation is the term used to describe "the mental construction of a process (transformed by some action) into a cognitive object that can be seen as a total entity (or coherent totality) and which can be acted upon (mentally) by actions or processes. The only way to mentally construct a mathematical object" (Dubinsky & Harel, 1992, p. 18). Learners operating at the object level can do the following:

- regard the word "function" as a noun (Dubinsky & Harel, 1992) and see a function as something that is being acted on (Dubinsky & McDonald, 2001);

- when considering the graph of the function $f(x) = x^2 + 1$, learners at the object level would see this representation of a function as taking the graph of the function $f(x) = x^2$, as if it were an object, and shifting the whole graph up one unit to obtain $f(x) = x^2 + 1$ (Weyer, 2010); and
- carry out actions, resulting in some kind of transformation on a function (Dubinsky & Harel, 1992).

3.5.4 Schema level of a function

At the schema level a learner has "a collection of actions, processes, objects and other schemas, together with their relationships, that the individual understands" in connection with functions (Dubinsky & Harel, 1992, p. 20). The learner at the schema level "is able to jump back and forth between the levels of action, process, object and schema in relation to the function concept" (Weyer, 2010, p. 10). The learner is able to link graphic and symbolic forms to construct a precise symbolisation for the information available in the given graph and to have the whole understanding of the concept of how all multiple representations of functions link together (Bennet, 2009). I also developed and compiled a list of other indicators for the action, process, object and schema of APOS theory that I used when analysing learners' responses in this study (see Section 6.6).

The four conception levels of APOS theory were used to investigate what learners understand of concepts related to functions and helped to determine the level at which a particular learner was operating. In this study, I used APOS theory to describe and analyze learners' mental constructions that characterize their understanding of the concepts related to functions at different levels as observed in clinical interviews. These mental constructions guided the coding and analysis of the interview data. Learners' responses were categorised into APOS levels. There are several reasons why, for most conceptions, not all of the indicators are required in order to assign attributes. For instance, learners will often misread an item or two on an assessment task, and even the best item still leaves some ambiguity in learners' thinking. Relying on multiple items to describe a concept diminishes the effects of assessment error.

Since the initial part of my study was to investigate what learners understand of concepts related to functions, APOS theory as a theory of how learning a mathematical concept might take place (Dubinsky & McDonald, 2001) together with the pre-function level indicators, provided a framework for understanding how learners understand the concepts related to functions. By subdividing the understanding of concepts related to functions into conception levels corresponding to specific mental constructions that a learner might make in order to develop their understanding of that concept, indicators of APOS theory conception levels helped to determine the level at which a particular learner was operating, depending on available evidence. The available evidence in the form of the learner's responses to task questions and clinical interview questions also helped planning instructional strategies to move the learner from one conception level to the other. These conception levels were also used to design task questions and to analyse learners' responses to task and clinical interview questions.

While it is useful to think of the levels of APOS theory in this order, in reality these constructions are not made in a linear sequence. Instead, they are made in a partially ordered sequence (Dubinsky & McDonald, 2001). As such, these conception stages only served as the guiding framework. Dubinsky & McDonald state that, "APOS theory makes testable predictions that if a particular collection of actions, processes, objects and schemas are constructed in a certain manner by a learner, then this individual is likely to be successful in certain problem situations" (p. 2). However,

> Explanations offered by an APOS analysis are limited to descriptions of the thinking which an individual might be capable. It is not asserted that such analyses describe what 'really' happens in an individual's mind, since this is probably unknowable. The main use of an APOS analysis is to point to possible pedagogical strategies. Data is collected to validate the analysis or to indicate that it must be reconsidered (Maharaj, 2010, p. 43).

In the schema level of APOS theory, a learner has "a collection of actions, processes, objects, and other schemas, together with their relationships, that the individual understands" in connection with functions (Dubinsky & Harel, 1992, p. 20). Since the learner is able to jump back and forth between the levels of action, process, object, and schema in relation to the concept of function, if a learner is operating at the object level s/he can jump back and operate at both the process and action levels when the need arises. But a learner operating at the action level will not be able to jump forth and operate at a higher level without any form of intervention. Therefore, as a learner approaches the schema level the number of connections among his/her conception levels also increases as shown in Figure 4. The more the connections a learner has, the deeper is his/her understanding.

Number of connections between APOS conception levels as a learner approaches the schema level

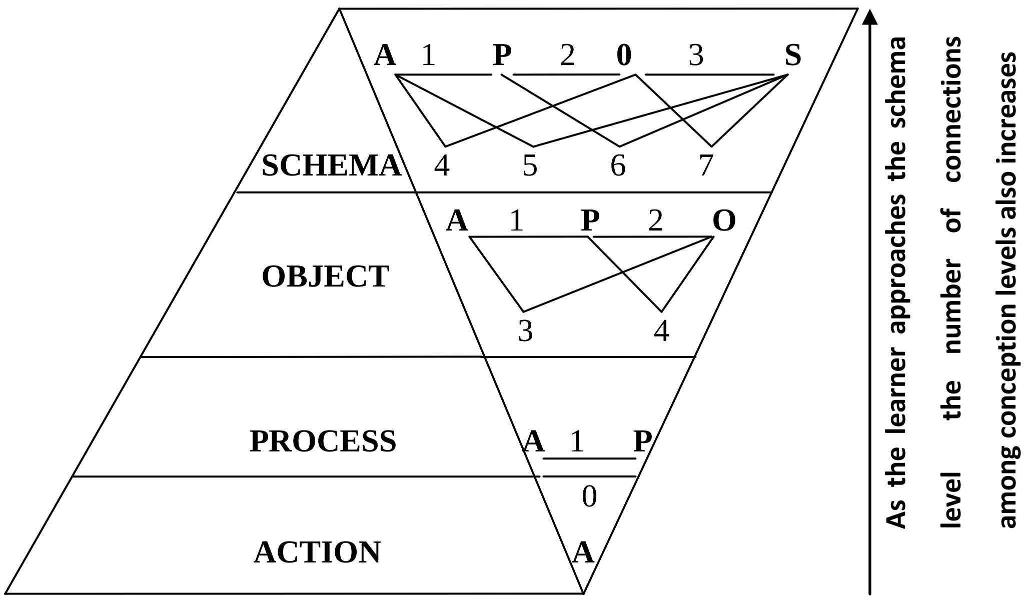


Figure 4: Relationship between the number of APOS connections and understanding

The implication of Figure 4 above is that learners should be helped to reach the schema level so that they are able to operate at any level of conception demanded by different questions on the function concept.

3.6 Realistic Mathematics Education (RME)

The RME theory was originally developed in the Netherlands and "it emphasises the idea that mathematics is a human activity which must be connected to the reality of the learner using real-world context as a source of concept development and as an area application, through the process of mathematisation both horizontal and vertical" (Gravemeijer, 1994, p. 17). In a constructivist approach learners direct the course of lessons and are no longer seen as receivers of knowledge but the makers of it, and the role of the teacher has shifted to that of a facilitator.

3.6.1 Characteristics of Realistic Mathematics Education

I discuss separately characteristics of RME that involve real life situations and those that are real or imaginable to the learners. According to Meyer (2001), RME involves the use of real life contexts as a starting point for learning that should appeal to learners' real life, allowing them to immediately engage in the situation. RME also uses models as a bridge between abstract and real life to help learners learn mathematics at different levels of abstraction. Initially the model emulates a real life situation that is familiar to learners. This is in line with the theory of constructivism. "By a process of generalising and formalising, the model eventually becomes an entity on its own and is possible to be used as a model for mathematical reasoning" (Meyer, 2001, p. 19).

Meyer (2001) also draws attention to the use of the learner's own strategy as a result of doing a real life mathematical problem. By answering questions based on real life situations in their own way, "learners are guided to reflect on the path they themselves have taken in their learning process and, at the same time, to anticipate its continuation" (Meyer, 2001, p. 20). Moreover, RME provides interaction between the teacher and learners, and between learners themselves which is essential in learning mathematics. Meyer (2001) regards "explicit negotiation, intervention, discussion, cooperation, and evaluation as essential elements in a constructivist learning process where the learner's informal methods are used to access the formal ones" (p. 20).

According to Gravemeijer (2004), RME initially presents knowledge within a real life or concrete context that allows learners to first develop informal strategies and then gradually to progress to more formal, abstract and standard strategies through the process of guided mathematisation. Gravemeijer identifies two kinds of mathematisation, namely, horizontal and vertical which he defines as follows. "Horizontal mathematisation is when the learners discover mathematical tools which can help them to organise and solve a problem located in a real-life situation while vertical mathematisation refers to the process of reorganisation within the mathematical system itself" (Gravemeijer, 2004, p. 19). An example would be refining and adjusting models or generalising to create more challenging mathematics and hence a greater use of abstract strategies.

Some of the views on the nature of mathematics as outlined in the South African National Curriculum Statement (DoE, 2003) are closely linked to the Realistic Mathematics Education

Theory. These views include the notion that mathematics is seen as a human activity and that mathematical problems should include real-life situations. According to Vaid (2004), "mathematics must be connected to reality, stay close to children's experience and be relevant to society, in order to be of human value" (p. 34). Benson (2004) concurs by adding that RME involves putting mathematics into recognisable, real life contexts to allow the learners to engage with the mathematics and generate solutions in a variety of forms. This encourages discussion in a more informal atmosphere whilst moving towards a more formal solution. This model is based on the principle that learners see meaning in their schoolwork when they connect information with their own experience. It was appropriate to embody RME in my planning of the teaching experiments because learners were preparing to write the school-leaving examination based on the South African National Curriculum Statement (2003).

Further justification for using RME comes from Treffers and Beishuizen (2000) who illustrate the applicability and relevance of mathematics in real-world situations. For them realistic mathematics involves taking realistic context situations as the starting point or as the source for learning mathematics. They point out that this RME viewpoint does not mean adding a few application problems to mathematics lessons, but rather to have a complete reversal of the teaching and learning process. Then, the emphasis is no longer on the teacher transmitting knowledge and concepts, but on learners finding mathematical patterns and structures in realistic situations, and becoming active participants in that teaching and learning process. At this point one can draw parallels with constructivism where 'realistic' also relates to mathematical activities which are experientially real to a child. Similarly Zulkardi (2003) notes that mathematics must be close to children and relevant to everyday life situations.

Given the misconceptions and difficulties that learners face in learning the function concept, it is evident that teaching this function concept to learners is challenging. Teachers need to use strategies that build learners' understandings of the concept of a function through everyday experiences. They should start with concrete examples and gradually move to abstract ideas. Additionally, teachers need to continually challenge learners' initial understandings of the function concept to help them progress from one lower cognitive level to the next. "Learners should be compelled to work with various functions in meaningful contexts and interpret important features of functions and their graphs for example, increasing, decreasing, maximum values, increasing and decreasing rates and working flexibly between various representations of functions" (Ojose, 2008, p. 7).

On the other hand, Mudaly (2004) argues that realistic mathematics teachers place much emphasis on making a mathematical idea real in the mind of the learner. According to Zulkardi (2003) 'realistic' refers not just to the connection with the real world, but also refers to problem situations which are real in learners' minds. In addition, Bottle (2005) points out that '*realistic'* does not just mean real-life situations. She explains that the term 'realistic' is taken from the Dutch word '*zich REALISEren'* which can mean 'to realise' or 'to imagine'. Therefore it means including contexts that may be imaginary but that are realistic to children. RME as a teaching and learning theory represents a significant departure from traditional ideas about teaching and learning mathematics. According to Meyer (2001) "perhaps the most obvious difference between an instructional sequence based on RME and a more traditional sequence is their starting point. Instead of starting in an abstract realm and moving toward a concrete application, the mathematics starts in contexts and gradually progresses to formal symbolism" (p. 67). This shift allows learners to engage in meaningful pre-formal activities in lessons at an earlier stage than they traditionally have been doing. Through a structured instructional sequence, learners can explore and rediscover significant mathematics that anticipates the more formal representations.

In addition to the above characteristics, Vaid (2004) asserts that the term 'realistic' refers to situations which can be imagined by the learners and draws attention to learners' understanding of processes rather than learning and memorising algorithms. As a result, learners should be helped to 'discover' the mathematics for themselves by encouraging and valuing different solutions. Hence, the teacher should begin with a range of informal strategies provided by learners, and build on these to promote the conceptualisation of more sophisticated ways of symbolising and understanding. Vaid (2004) suggests "allowing the learners to begin at the basics, using informal strategies and constructing the mathematics for themselves, simulates the discovery of the mathematics and allows them to appreciate its complexity" (p. 37). RME incorporates the use of effective models like the function machine and graphs to provide a more visual process of learning functions.

The present study was interventional in nature, its emphasis was on eliciting learners' different ways of understanding concepts related to functions so that those understandings, if not accurate, could be used as starting points in the design of interventions to identify and rectify the inaccurate understandings. To better guide the design of these interventions the instructional principles and associated design methods which are tenets of the theory of RME, were used. The theory of RME was selected as the vehicle to drive the design and implementation of the intervention for the following three reasons. Firstly, it provided a basis from which to work with the misconceptions (Gravemeijer, 1994) and difficulties elicited from the previous chapter in this research. Secondly, it placed learners' mathematical reasoning at the centre of the design process while simultaneously proposing the specific means by which the development of their reasoning could be systematically supported (Cobb, Zhao & Visnovska, 2008). In this study learners created meaningful mathematical ideas as they engaged in challenging tasks on concepts related to functions. In this process, "mathematisation, symbols, algorithms, and definitions were built from the bottom up through a process of suitably guided reinventions" (Rasmussen, Zandieh, King & Teppo, 2005, p. 26). Guided reinventions speak to the need to locate instructional starting points that are experientially real to learners and that take into account learners' current mathematical ways of knowing. This calls for an examination of learners' informal solution strategies and interpretations that might suggest pathways by which more formal mathematical practices might be developed. Thirdly, RME places emphasis on understanding processes, rather than merely learning algorithms so that the focus is on the growth of knowledge and understanding of mathematical concepts. Gravemeijer (1994) adds that "for mathematical concepts to be understood, they must be connected to reality, stay close to learners and be relevant to society" (p. 358). As such, teachers need to establish a link between learners' own understandings and the correct mathematical ideas. The RME theory was applied in the design of the intervention, where the aim was to assist low attaining learners in a remedial programme conducted during school hours (Cobb, Zhao & Visnovska, 2008).

3.6.2 RME's learning and teaching principles

- *Learning mathematics is a constructive activity:* this principle contradicts the idea that learning involves absorbing knowledge which is presented or transmitted (Treffers, 1991). Regarding teaching, "the instruction should start with a concrete orientation. In RME, the starting point of instructional experiences should be `real' to the learners; allowing them to immediately become engaged in the situation" (Treffers, 1991, p. 79). This means that instruction should not start with the formal system. The phenomena by which the concepts appear in reality should be the source of concept formation. The process of extracting the appropriate concept from a concrete situation is stated by De Lange (1996) as 'conceptual mathematisation'. This process will compel learners to explore the situation, find and identify the relevant mathematics and develop a 'model' resulting in a mathematical concept.

- *Models develop learning through levels of abstraction:* according to Treffers (1991), the term model "refers to situation models and mathematical models that are developed by the learners themselves" (p. 78). This means that learners develop models in solving problems. At first, the model is a model of a situation that is familiar to the learners. By a process of generalising and formalising, the model eventually becomes an entity on its own. It becomes possible that it is used as a model for mathematical reasoning. In this principle, the learning of a mathematical concept or skill is viewed as "a process often stretched out over the long term and which embodies various levels of abstraction, namely, from informal to formal and from the intuitive level to the level of subject-matter systematic" (Treffers, 1991, p. 80). Gravemeijer (1994) advocates paying attention to visual models, model situations, and schemata that arise from problem solving activities as these will help learners to move through these various levels.
- *The use of learners' own productions and constructions:* learners should be asked to 'produce' more concrete things. De Lange(1996) stresses the fact that, "by making 'free production', learners are compelled to reflect on the path they themselves have taken in their learning process" (p. 25).
- *Social context and interaction:* learning is not a solo activity but it occurs in a society and is directed and stimulated by the socio-cultural context (Treffers, 1991). By working in groups, for example, learners have the opportunity to exchange ideas and arguments so that they can learn from each other.
- *Structuring and intertwining of learning strands:* learning mathematics does not consist of absorbing a collection of unrelated knowledge and skill elements, but is the construction of knowledge and skills within a structured entity (Treffers, 1991). This brings in the idea of a holistic approach advocated for by Gravemeijer (1994) which places emphasis on applications and *intertwining* of learning strands.

3.6.3 Using RME principles in designing lessons about functions

Streefland (1991) developed mathematics lessons based on the characteristics and principles of RME and focused on constructivism through horizontal mathematisation. First, he started by introducing open material into learning situations and provided opportunities for carrying out free productions, using own solution strategies. In applying the characteristics and principles of RME in his lesson, Streefland (1991) started from meaningful contexts which guided learners to construct the intended concepts. To help learners learn through constructivism "the teacher needs to arrange activities for learners, so they can interact with each other, discuss, negotiate, and collaborate. By this way, the learners' contribution to their own learning path can be guaranteed" (Streefland, 1991, p. 32). The learners can be encouraged to follow this kind of constructivist activity by giving them an assignment which leads to free productions (the use of own creative strategies).

In addition to the above, in order to design RME lessons, the components of a lesson plan are identified and connected to RME. These components are goals, content (materials), methodology (activities), and assessment. In RME intended goals are not always immediately clear for both the teacher and learners but emphasis is placed on the reasoning skills, communication and the development of a critical attitude. These are popularly called 'higher order' thinking skills (De Lange, 1996). Focus is on making connections between the different concepts and solving simple problems without unique strategies. To achieve the intended goals the teacher needs materials to manipulate and construct knowledge.

The roles of the RME teacher in the classroom include being a facilitator, an organizer, a guide, and an evaluator (De Lange, 1996). As the starting point the teacher gives the learners a contextual problem that relates to the topic. During the interaction activity, the teacher gives learners a clue, for instance, by drawing a table on the board when teaching the function concept, guiding the learners individually or in small groups in case they need help and encouraging learners to compare their solutions in a class discussion. The discussion is based the contextual problem and the focus is on problem interpretation and production of individual solutions. As learners are allowed to find their own solutions, they are free to make discoveries at their own level, to build on their own experiential knowledge, and perform shortcuts at their own pace. They are given another problem in the same context.

According to De Lange (1996), the primary purpose of assessment is to improve learning and teaching by measuring learners during the teaching and learning process and at the end. Methods of assessment should enable the learners to demonstrate what they know rather than what they do not know. It can be conducted by having problems that have multiple solutions with multiple strategies. Learners should be assessed in order to see whether they really understand the problems.

3.7 Merging the theories

Pegg and Tall (2005) discuss the importance of theoretical triangulation, "the process of compiling relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses, and appropriateness, and using some subset of these as the focus of empirical investigation" (p. 43). This implies that similar characteristics of different psychological theories of learning mathematics can be merged to complement each other and still maintain their own identity. As such, the theories of Piaget, APOS and RME were brought together within the constructivist paradigm in this study to take advantage of their complementary themes and characteristics as illustrated by an integrated theoretical framework in Figure 5.

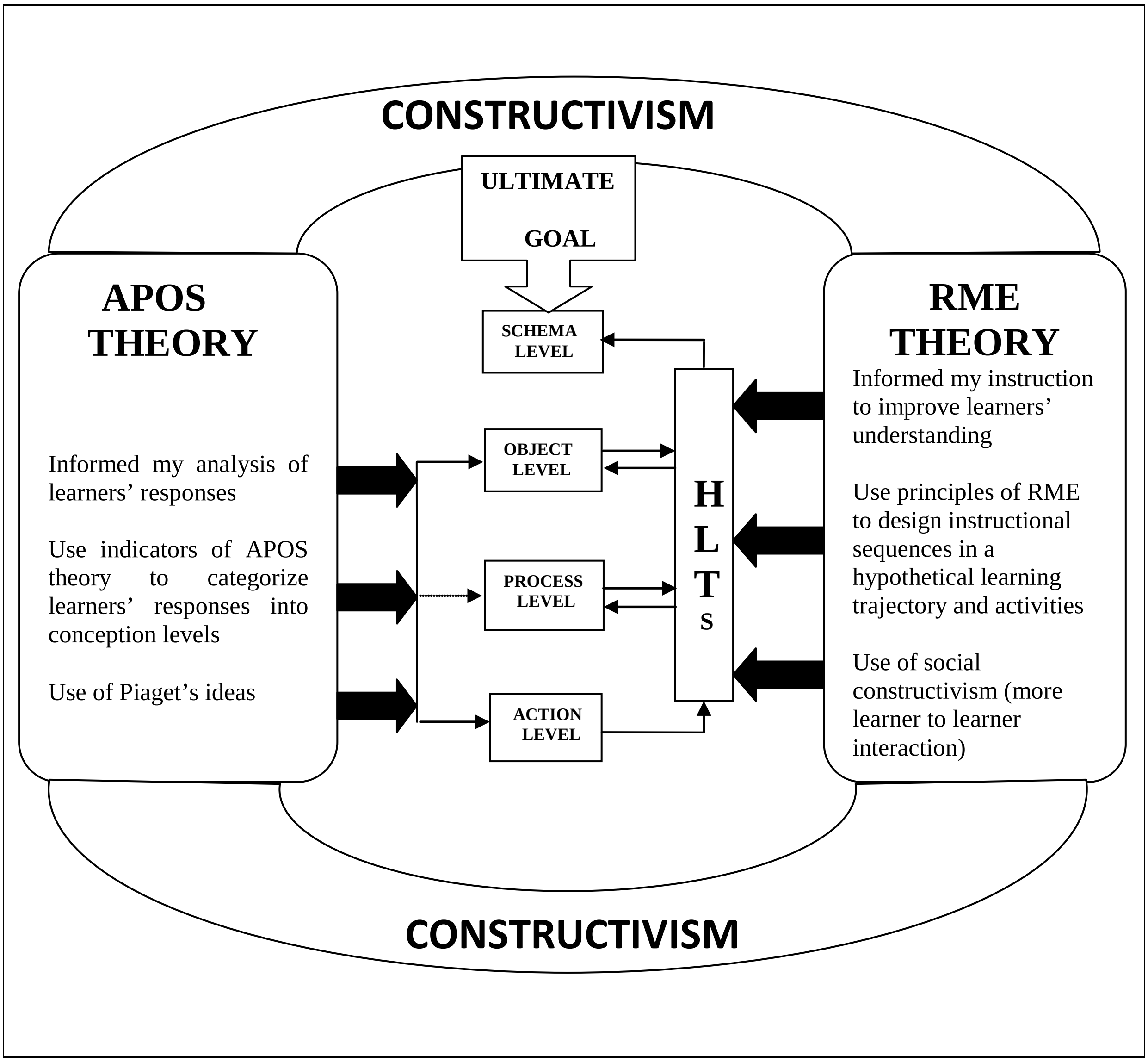


Figure 5: An integrated theoretical framework

I focused on what I saw as their main agenda, namely their focus on building mathematical concepts over time and a fundamental cycle underlying the development of concepts that characterise different ways of thinking during learning mathematics.

Regarding the theories of Piaget, APOS, and RME the overall processes that they describe are broadly similar, namely, to begin with actions on known or real physical or mental objects. According to Piaget (1977), "these actions are practised until they become routine step-by-step procedures that are seen as a whole process" (p. 257). Then they become embedded as independent entities on which the learner can operate at a higher level to generate a further cycle of construction. What these theories have in common is that they involve "a shift in focus from *actions* on already known objects to thinking of those actions as manipulatable mental *objects*" (Pegg & Tall, 2005, p. 47). For this study the initial action is at the concrete operational stage of Piaget where a learner needs concrete materials to manipulate in order to understand and use them to solve a particular functional problem. The concrete operational level uses alternative procedures that are not seen as interconnected and hence remain at the action level of APOS.

RME complements my study by initially presenting knowledge within a concrete context, allowing learners to develop informal strategies, but through the process of guided 'mathematisation', gradually allows learners to progress to more formal, abstract, standard strategies. It involves taking realistic contexts as the starting point or source for learning the function concept. It also speaks to the need to locate instructional starting points that are experientially real to learners and that take into account the learners' current mathematical ways of knowing. This calls for an examination of learners' informal solution strategies and interpretations that might suggest pathways by which more formal mathematical practices might be developed. RME also stresses understanding processes, rather than learning Piaget's routine step-by-step procedures and the focus is on the growth of the learners' knowledge and understanding of mathematical concepts. Gravemeijer (1994) adds that if mathematical concepts are to be understood, they must be connected to reality, stay close to the learners and be relevant to society. As such, there is need to establish a link between the learners' own understandings and mathematical ideas.

The conception levels of APOS theory helped in determining at which stage a particular learner was operating, depending on the available evidence in the form of the learner's responses to task questions and clinical interview questions. After classifying learners in these conception levels I used the principles of RME, in designing level appropriate and realistic activities that moved the learner from one conception level to another. As learners were working on these activities they interacted with each other, sharing ideas and comparing solutions. To move a learner from APOS theory's action level to the process level I used the principles in Piaget's concrete operational stage because I regarded the development as an effect of operations on concrete materials known to the learner. The theory of RME was selected as the vehicle to drive the design and implementation of the instructional activities.

Moreover, RME provided a firm basis from which to work with the misconceptions (Gravemeijer, 1994) and difficulties that were elicited in this research. It placed learners' mathematical reasoning at the centre of the design process while simultaneously proposing specific means by which the development of their reasoning could be systematically supported (Cobb, Zhao & Visnovska, 2008).

A combination of these theories was used as a sorting-tool to analyse a large amount of interview and observation data, and to design and implement teaching experiments that helped learners overcome their difficulties in understanding concepts related to functions. Learners in this study tended to refer to elements that were associated with these theories. This suggests the importance of determining where a learner is before presenting new information to fully address whether a learner is capable of understanding the new material based on their current developmental stage. So, understanding how a learner moves through this developmental process can enhance our understanding of how learners learn and therefore increase the chances of helping them to understand new complex ideas.

It can be seen from the discussions in this chapter that APOS theory provided a broad description of concept images of the function concept possessed by learners and their functional reasoning. These concept images informed my analysis of learners' interview responses. I used indicators of APOS conception levels to categorize learners' responses. Learners' APOS conception levels were used to design instructional strategies to help learners move up the APOS theory conception hierarchy. Complementing APOS theory were Piaget's theory and Realistic Mathematics Education theory which guided the design and development of stage-appropriate and realistic instructional materials. The theoretical framework influenced my methodology and research design for this study. The next chapter describes in detail this methodology and research design.

Table 3 summarizes the links between the three theories examined in this chapter and used in this study and how they informed my analysis of learners' responses and the development of instructional activities.

Table 3: Summary of three theories and their links to functions

APOS THEORY LEVELS	PIAGET'S THEORY STAGES	RME THEORY
⟵ Informed analysis ⟶		⟵ Informed instruction ⟶
ACTION ↓ student has internalized an action to form a process. Use a given equation to substitute values e.g. when finding intercepts ↓ PROCESS	Piaget spoke of three modes of abstraction: *Empirical abstraction* from objects of the environment (concrete operational)	emphasize the idea of making a mathematical concept real in the mind of the student. Help learners to understand why at the x-intercept y=0 and at the y-intercept x=0
↓ PROCESS student sees a function as an object that is being acted on. ↓ OBJECT	*Pseudo-empirical abstraction* from actions on objects in the environment for example transforming a graph of function.	emphasizes the understanding of processes (transformations), rather than learning algorithms.
↓ OBJECT student has a collection of actions and processes on functions (objects) ↓ SCHEMA	*Reflective abstraction* from mental objects (formal operational) processes are performed on already known objects moving to the formation of schemas.	emphasizes understanding of the effects of actions and processes on known and unknown objects

CHAPTER 4
Research Design

4.1 Introduction

This chapter describes the research design and methodology for answering the research questions for this study. Table 4 below summarizes the relationship between the research questions, design research phase, data collection instruments and techniques as well as the method of analysis for all the phases of design research in this study.

Table 4: Relationship between research questions, design research phase, data collection instruments/ techniques and method of analysis

Research Question	Phase	Data collection instruments/ techniques	Purpose of phase before and after intervention	Method of analysis
1. How do learners understand the function concept? Sub-questions: i) What are grade 11 learners' current understanding of functions? ii) What are the weaknesses in the learners' understanding?	Phase1: Problem identification (for the definition, representations and inverse of the function concept separately)	1. June 2011 mathematics examination: Question 7 & 8 2. Learners' answer scripts 3.Clinical task-based interview 4. Learners' transcribed interview responses 5. Group Interviews 6. Observation	1. To determine learners' understanding of the definition, representation and related difficulties 2. Learners' responses address both sub-question i and ii of the first research question	1. APOS and thematic analysis of learners' written responses, explanations, justifications of solutions and their strategies was used to determine the APOS level of each learner before and after intervention and also to allow patterns, themes and categories to emerge from the data. 2. Constant comparative method of data analysis was used to compare what students do on paper and what they say they do in oral interviews to identify similarities, differences and general patterns.

2. How can instruction be designed to improve learners' understanding of the function concept?	Phase 2: Development of interventions (for the definition, representation and inverse of the function concept separately) Phase 3: Tentative products and theories (development of teaching experiments for the definition, representations and inverse of the function concept separately) Phase 4: Teaching experiment and theory refinement	1. Hypothetical Learning Trajectories (HLTs) 2. Assessment activities on the definition, representation & inverse of functions 3. Task-based clinical interviews 4. Researcher's own journal	1. To design and develop interventions informed by the theoretical framework 2. To identify tentative products and design principles 3. To apply the tentative products and theories (HLTs, RME activities and APOS conception levels) 4. To refine the tentative products (HLTs & RME activities)	1. Retrospective analysis in which the researcher compares the HLT with students' actual learning 2. Constant comparative analysis and retrospective analysis. The researcher will look back and reflect upon the whole research process and check whether what was intended by the study was achieved and to what extent

The underlying philosophical assumptions (ontological and epistemological) that I brought into this study are discussed first. A description of the study site and setting, the sample and sampling techniques follows. The research questions for the study favour design research because of the intervention strategies that had to be designed and refined to answer them. I give a brief summary of the phases of Wademan's (2005) generic design research model and then explain in detail how I adapted Wademan's (2005) model to suit the context of my research. The data collection procedures and methods used for the study are explained together with how the collected data was analyzed. Coding of learners' responses and strategies as well as indicators of APOS conceptions are also described. Ethical issues, measures to ensure rigor, trustworthiness and limitations of the entire research process are explained in the last section of this chapter.

4.2 Ontological and Epistemological assumptions

"The deeper meaning behind research questions lies in the ontological and epistemological perspectives" (Trede & Higgs, 2009, p. 18). Trede and Higgs state that, ontology is concerned with the nature of knowledge and epistemology is concerned with the theory of knowledge. I believe that learners' construction of reality lies in their sense-making of what they are taught and what they already know. This belief prompted me to select the qualitative research paradigm which "assumes and recognizes multiple constructed realities that are grounded in learners' different attributions of meanings to taught concepts" (Trede & Higgs, 2009, p. 21). In this study, I recognized and valued multiple interpretations by using rich descriptions (Bogdan & Biklen, 2003) of students' interpretations of the concepts related to functions in the form of quotations from the transcribed interviews and field notes of observation data.

I believe that knowledge is constructed by individuals in their mind when trying to make sense of information from teachers and textbooks in the light of their previous knowledge. This implies that learners do not passively receive knowledge from an outside source but they actively build it up (Trede & Higgs, 2008). Similarly, Hansson (2006) considered knowledge "as a gradually built individual construction and understanding as building of mental structures where previously built structures affect subsequent constructions" (p. 14). In this study the construction of knowledge was promoted by engaging learners in function-related tasks and activities and asking them to explain their solutions. This ensured that learning was "an active process involving learners constructing rather than acquiring knowledge, and instruction or teaching was a process of supporting that construction rather than communicating knowledge" (Clements & Ellerton, 1995, p. 4). The data in this study, which comprised mainly written solutions and transcribed explanations of learners, might not completely reflect how learners solved the problem. For example, some learners might fail to show or explain their solutions. To minimize this concern, before the commencement of tasks and interviews I explained to learners that answers without explanations would not help me understand how their minds worked in solving problems. Because of convenience I worked with only 12 learners whereas a teacher generally has about 50 learners in class. The reason for my smaller sample was that I wanted to do an in-depth study of learners' understanding of functions and to see if the use of design research made any difference in learners' understanding. I was also able to 'experiment' with my teaching intervention as supplemental to the work being done by the teacher without disadvantaging the learners in any way.

4.3 Context

The research was conducted at a rural, day high school in the Ehlanzeni district located in Mpumalanga Province of South Africa. The school was chosen for convenience as I was a fulltime mathematics teacher at this school teaching grades 9 and 10 at the time of the data collection.

4.4 Sample and sampling techniques

A sample of twelve grade 11 learners of mixed ability was purposively selected using their teacher's record of marks and from those who were willing to participate in all the phases of design research for this study. Grade 11 learners were chosen for the study typically because they would have gone through many aspects of this topic from primary school up to grade 11 and are expected to have developed more mental images about the selected aspects of the function concept. The learners' average age was fifteen years. Gender balance was also purposively sought from the group of volunteers. Purposive sampling offered me "a degree of control rather than being at the mercy of any selection bias inherent in pre-existing groups" (Mays & Pope, 2000, p. 17). The sample may not be representative and their interpretations may not be generalisable, because this is not the primary concern of such sampling, rather the concern is to acquire in-depth information. Thus, with purposive sampling, I deliberately sought to include extreme or deviant cases conventionally discounted in quantitative approaches which focus on average or normal cases (Barbour, 2001).

4.5 Research questions

1 How do learners understand the function concept?

Sub-questions: i) What are grade 11 learners' current understanding of functions?

ii) What are the weaknesses in the learners' understanding?

2 How can instruction be designed to improve learners' understanding of the function concept?

To answer these questions I created situations in which learners would get the opportunity to explain their understanding and share their ideas.

4.6 The research design

Since learners' understanding varies, it means that, to explore such understanding requires an in-depth study of a few learners. As such, the case study research method was deemed appropriate for this study. According to Harling (2002) a case study:

> is a holistic inquiry that investigates a contemporary phenomenon within its natural setting. It involves a collection of in-depth and detailed data that was rich in content from multiple sources of information including direct observation, interviews and documents. The multiple sources of information provide the wide array of information needed to provide an in-depth picture (p. 29).

I was able to capture evidence of and synthesize dimensions of learners' interpretations through inquiry into the origins of their interpretations and the meanings they seem to hold. In this study I used a case study of a particular high school located in rural Mpumalanga Province of South Africa. An important advantage of using a case study was that, it offered me a multi-perspective analysis in which I used the voices and perspectives of all the participants in the study (Nieuwenhuis, 2010).

The purpose of this research was to understand grade 11 learners' conceptual understanding of functions and to use design research to improve the teaching and learning of functions. To this end, I developed and administered instructional activities and tasks that were suitable for the current CAPS grade 11 curriculum and located in the South African context. This means that designing of instructional sequences and activities was crucial in the research and as a result my methodology falls under design research which is described in detail in the next section.

4.7 Design research

Plomp (2006) defines design research as:

> a systematic study of designing, developing and evaluating educational interventions (such as programs, teaching-learning strategies and materials, products and systems) as solutions for complex problems in educational practice, which also aims at advancing our knowledge about the characteristics of these interventions and the processes of designing and developing them. (p. 9)

Reeves (2006) holds forth that design research "investigates the development of solutions to practical problems in learning environments with the identification of reusable design principles. He argues that design research aims at developing optimal solutions for problems in context" (p. 52). Similarly, Wang and Hannafin (2005) view design research as a:

> systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories (p. 6).

The definitions above indicate that design research is applicable to complex educational problems which require intervention at the classroom level.

In line with this study and from the definitions, design research begins with "*learners' difficulties in understanding functions*" namely, complex problems, which require a process of practical intervention or solutions which, according to Plomp (2006), is informed by review of relevant literature to provide design principles. I subscribe to Wang and Hannafin's (2005) definition because it speaks of an iterative methodology that suits the nature of my study. My aim was to use design research to improve learners' understanding of functions through iterative analysis, design, development, and implementation of interventions which I achieved by adapting design cycles of Wademan's (2005) Generic Design Research model shown in Figure 6 that follows.

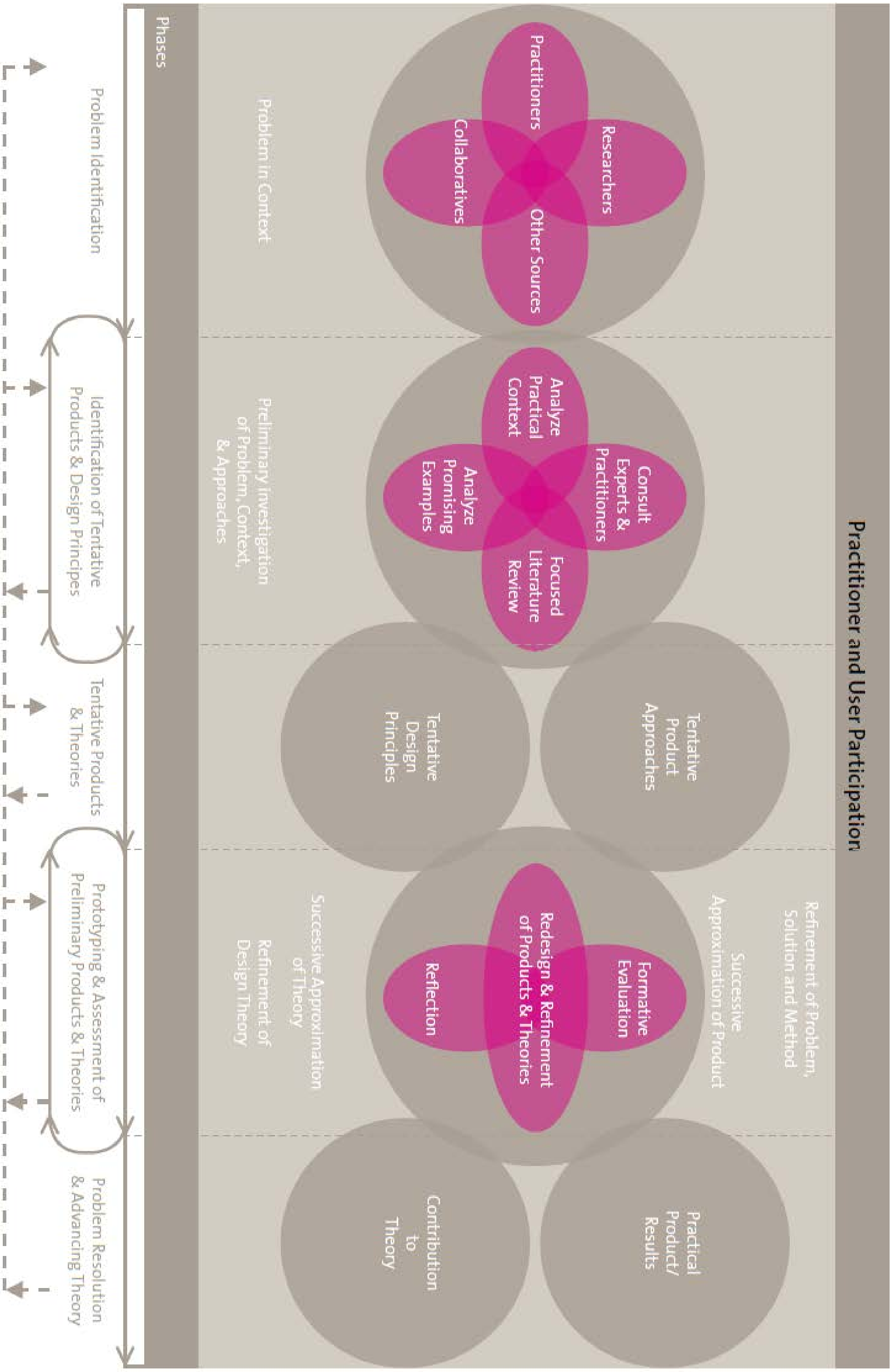


Figure 6: Generic Design Model (Wademan, 2005)

Wademan's (2005) model in Figure 6 captures the main features and characteristics of design research which I adapted in this study. I briefly explain the process and the phases of Wademan's model first, to help the reader understand my adapted model for this study.

Phase 1: Problem identification

- *Stakeholders*: researchers, practitioners and other sources.
- *Process*: in this phase the context is studied to identify the problem. The stakeholders interact with each other and arrive at a common conclusion.

Phase 2: Identification of tentative products and design principles

- *Stakeholders*: experts and practitioners in the specified domain.
- *Process*: after the problem is identified, a preliminary investigation of the problem, context, and approaches is done in consultation with the experts and the practitioners. This is supported by conducting a focussed literature review, and analyzing the practical context. Promising examples, which represent the problem, are analyzed. This process helps to identify a tentative list of products and design principles, emanated from specific theories to be applied in the study.

Phase 3: Applying tentative products and theories

- *Stakeholders*: researchers, practitioners and end-users.
- *Process*: tentative products are created using the tentative list of design principles or specific theories. These are introduced to the users and the results are captured.

Phase 4: Prototyping and assessment of preliminary products and theories

- *Stakeholders*: researchers, practitioners and end-users.
- *Process*: Redesign and refinement of the problem, solutions (created in phase 3) and method are done based on the feedback received in phase 3, namely, the interplay between theory and practice. Formative evaluation is used along with the reflection of the feedback, to do the redesign and refinement. This is an iterative stage where, the refinement is done for achieving successive approximation of theory, and refinement of design theory.

Phase 5: Problem resolution and advancing theory

- *Stakeholders*: researchers and end-users
- *Process*: practical products are created, after the iterations conducted in phase 4.

These products have certain aspects which contribute to the existing theory.

4.8 Adaptation of my research to the generic design research model

Phases and the stakeholders mentioned in Wademan's (2005) model coincide with the phases and stakeholders of my research. As the model cannot be adopted as it is, I adapted it to suit my context. Figure 7 below shows the adaptation of Wademan's (2005) model to my research.

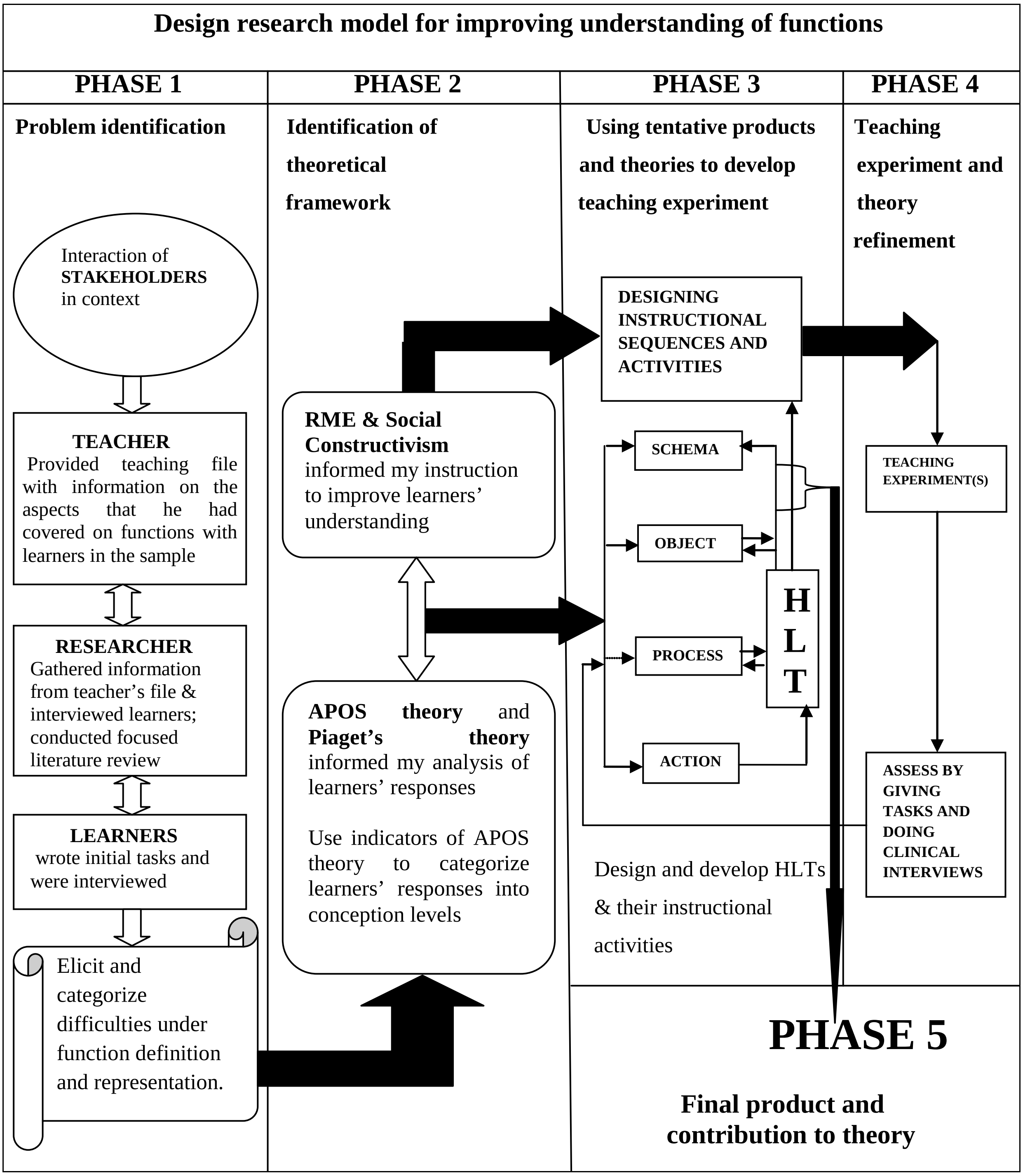


Figure 7: Adaptation of Wademan's (2005) model for my research

The phases are the same but the details have been changed according to my context. A summary of phase-wise modification is presented here. The details of the actual implementation are given in the next chapter.

4.8.1 Phase 1: Problem identification

The role of different stakeholders

Researcher: conducted focused literature review; interviewed learners and their teacher; analysed learners' responses and used them to design HLTs; taught in all teaching experiments.

Teacher: provided his teaching file with lesson plans on functions which helped me to identify the aspects that he taught, the objectives and teaching methods that he used. This gave me a clear background on how the function concept was developed.

Grade 11 learners: wrote initial tasks and formative assessment tasks and were interviewed.

Process

Prior to this study learners in the sample had learned about functions. To elicit learners' conceptual understanding of functions I used two instruments listed below.

1. June 2011 mathematics examination: question 7 and 8.
2. Formative assessment tasks on the definition and representation of functions.

Table 5: Relationship between instruments, data collection procedures and method of analysis

Instrument	Data collection procedure	Analysis
June 2011 mathematics examination	Learners do questions 7 & 8 and are clinically interviewed	Constant comparative and APOS analysis of transcribed interview responses
Formative assessment tasks	Learners write these tasks and their written responses give feedback to the researcher	APOS analysis of learners' written responses

Learners had also written the June 2011 mathematics examination which had two questions that were testing concepts related to functions. Thus, learners were solving the same problems for the second time. Question 7 in Figure 8 was testing learners' understanding of the calculation of intercepts (action and process conceptions), determination of asymptotes from a given equation (process conception) and switching from the symbolic representation (equation) to the graphical representation (action and process conceptions).

QUESTION 7

Given $f(x) = \frac{1}{x-4} + 2$

7.1	Calculate the co-ordinates of the *x* and *y* intercepts of *f*.	(4)
7.2	Determine the equations of the asymptotes of *f(x)*.	(2)
7.3	Sketch the graph of *f(x)* showing all the critical points.	(4)
		[10]

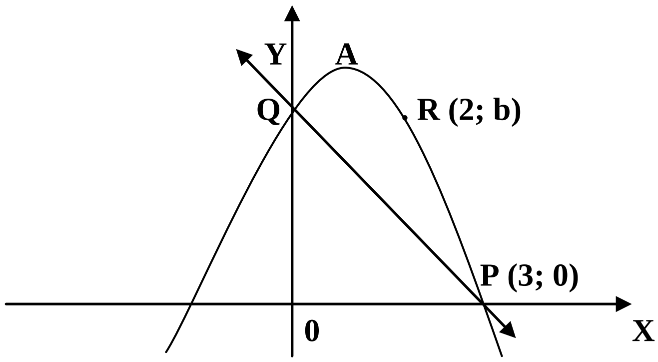
Figure 8: Taken from the Mathematics Paper 1 NSC June 2011

On the other hand, the purpose of question 8 shown in Figure 9 was to elicit learners' understanding of the coordinates of the intercept at Q, gradient of a straight line given two points on the line, the coordinates of a turning point from a given equation, in this case the action conception and being able to switch from the graphical representation of a function to the symbolic representation, namely, process, object and schema conceptions.

QUESTION 8

The sketch below, not drawn to scale, shows the graphs of the functions defined by: $h(x) = -2(x-3)(x+1)$ and $g(x) = mx+c$.

where A is the turning point of $h(x)$ and R(2;b) is a point on $h(x)$



8.1 Calculate the coordinates of the turning point A. (3)

8.2 Calculate the coordinates of Q. (1)

8.3 Determine the numerical values of m and b. (4)

8.4 Write down the equation of $g(x)$. (1)

[9]

Figure 9: Taken from the Mathematics Paper 1 NSC June 2011

Table 6: Content analysis of instrument 1 (June examination)

Question	What concepts are covered?	APOS level
7.1	Calculation of the coordinates of the *x* and *y* intercepts	A & P
7.2	Determination of equations of asymptotes	A & P
7.3	Sketching of graph	A, P & O
8.1	Calculation of coordinates of turning point	A & P
8.2	Determination of the coordinates of the y-intercept	A
8.3	Calculation of gradient of straight line	A
8.4	Determination of equation of line from drawn graph	P

Key: A- Action, **P-** Process, **O-**Object, **S-**Schema

Table 7: Content analysis of instrument 2 (Task 1 and Task 2)

Task	What concepts are covered	APOS level
1	Determination of whether a given relation is a function or not Finding the domain and range	P & O
2	Use of vertical line test Explaining why a graph which passes the vertical line test represents a function	A & P O
3	Sketching graphs of given equations Using given graphs to determine their equations	A & P O & S
4	Inverse of a function Relationship between graphs of functions and their inverses	A & P O & S

The learners were under no time constraints to complete the tasks. I gave each learner a copy of the first question, namely, task and instructed them to answer the questions in their own way. I provided them with only pencil and paper. Once I had estimated that learners had finished their solutions, I interviewed them individually regarding the manner in which they approached the questions, namely, tasks. I also asked them to think about other ways to answer the same questions. The procedure for the first question was repeated for the second question in the June examination.

These two questions, as initial tasks, provided opportunities for me to:

(i) analyse learners' written work which gave me initial clues about their understanding of intercepts, asymptotes, turning points and transformations of graphs of functions, how they reason and which counted as evidence of what they understood;

(ii) listen to learners explain and justify their answers and their methods on the tasks which revealed, how they reason as they solve a problem, which may not be apparent in their written work; and

(iii) decide on the instructional options based on the analysis of learners' written work and transcribed recordings of their interviews.

Data collection strategies and instruments

The individual in-depth, open-ended, task-based qualitative ***clinical interviews*** with learners in the sample were my primary data collection method. My choice of clinical interviews was influenced by the nature of my research questions and the recent calls by various mathematics educators, professional organizations and curriculum reformers both local and international (Bansilal, 2009; NCTM, 2000; Ginsburg et al., 1998) for the use of clinical interviews by teachers in their classroom instruction. Ginsburg in Bishop (2010) defines a clinical interview as "a flexible method of questioning intended to explore the richness of learners' thought, to capture its fundamental activities, and to establish the learner's cognitive competence" (p. 478). I believe that clinical interviewing would result in more thinking and reflection in the classroom. As a result, I conducted initial clinical interviews based on the two questions in the June 2011 examination paper using a prepared interview schedule (Appendix 1) to elicit learners' understanding of concepts related to functions and encouraged their reflection.

These clinical interviews were used in each cycle of the design research phases and began with a task from which I then asked further questions, depending on the learner response, to elicit learner reflection. This enabled me to discover the cognitive activities, specifically the structures, processes, and thought patterns, and to evaluate the levels of competence of the learner. Thus, I aimed to discover and evaluate learners' interpretations which were a result of their cognitive activities. Asking learners to interpret the questions, explain and justify their methods of answering the questions was a rich source of information regarding their reasoning and development of rational ideas. By listening to their interpretations, explanations, justifications as well as observing them answering the questions in this study, I was able to learn about learners' mathematical understanding of functions, their concept images, difficulties and misconceptions.

Clement (2000) noted the strengths of the clinical interview over other data collection techniques which include:

> The ability to collect and analyze data on mental processes at the level of the subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques (p. 547).

As such, clinical interviews enabled me to see the concepts related to functions through the 'eyes' of understanding of the participants and to obtain rich descriptive data that also helped me understand the participant's construction of knowledge and social reality (Nieuwenhuis, 2010). Hunting and Doig (1997) state that centering the dialogue on a task or problem gives the subject every opportunity to display behavior from which to infer which mental processes are being used when thinking about the task or solving that problem. I used a prepared interview guide to give structure to the interviews which made it easy for me to organize and analyze interview data. It will also help the readers of this research report to judge the quality of the interviewing methods and instruments that were used. Interview sessions were audio-taped, transcribed and coded in Atlas.ti for use in the analysis.

To augment and triangulate data from clinical interviews I conducted a ***follow-up group interview*** of all the six learners in the sample. Since learners in the sample were of mixed ability I ensured that the groups are balanced by having two above average, two average and two below average learners each. This was because group interviews can generate a wider range of responses than in individual interviews (Cohen, Manion & Morrison, 2007). Group interviewing enabled learners to challenge each other and participate in a way that may not happen in a one-to-one, teacher-learner interview and using language that the learners themselves use (Cohen et al., 2007). In such a situation participants interact with each other rather than with the interviewer, so that views of participants can emerge and participants are also empowered to speak out, and in their own words (Cohen et al., 2007). Group interviews were audio-taped and transcribed and were useful in triangulating data from my observations.

Observation allowed me to hear, see and begin to experience reality from the view of participants in the research group (Nieuwenhuis, 2010). Using observation can also lead to deeper understandings compared to the sole use of interviews, because it provides knowledge of the context in which events occur. In this study it enabled me to see things that participants themselves were not aware of, or that they were unwilling to discuss (Patton, 2002). The other advantage of using observation in this research is, as Robson (2002, p. 27) puts it, "what learners do may differ from what they say they do, and observation provides a reality check". This allowed me to triangulate the learners' responses from the clinical interviews and what they discussed with their group members. I first observed individual learners as they worked on the given tasks, asking questions where necessary, a process referred to as participant observation. These observations were carried out during the period individual learners were working on tasks, learners were discussing issues in groups and also when they were being individually interviewed. I observed the amount of time each learner took to complete a task, the errors they made, how they interacted in their groups, how they asked and answered each other's questions and what they did when facing a difficulty. I noted these observations in my journal immediately after the sessions with the learners. My observations were supported by learners' written work on the administered tasks which after marking revealed valuable information about their understanding of concepts related to functions. This is in line with Swanson et al's (1981) view that learners' reasoning and understanding of mathematical concepts can be effectively assessed by giving learners tasks to complete.

Categorising learners' understanding from initial interviews and literature

As learners took me through their solutions to the two questions in the June 2011 examination paper I asked them to explain to me the meanings of intercepts, asymptotes and turning points, and how they calculated them. I also asked them to explain how they switched from a sketched graph to an equation and *vice versa*. New or similar difficulties may emerge from learners' explanations on these function-related concepts during individual and focus group interviews. These difficulties corresponded to those I found in the literature review. I grouped these difficulties into two categories namely function definition and representations of the function concept. It was my intention to include the inverse at this point but because I found that it had not been taught I left this to a later stage.

Eliciting learners' specific difficulties with concepts related to the definition and representations

I then developed two tasks based on the definition and representations of the function concept to confirm difficulties and misconceptions from initial tasks and the literature review which provided further evidence for judging the strength of learners' understanding of concepts related to functions that they would have already covered in class. Of the two tasks, one was on the definition of the function concept and the other on representations (Appendix 2). Questions on these two tasks included among others the recognition of functions, given in various types of representation, namely, verbal expressions, graphs and mapping diagrams or algebraic expressions. This was followed by clinical interviews. In order to maintain consistency among interviewees and to elicit rich information as much as possible about learners' understanding of the concepts related to functions I used prepared task-based interview schedules (Appendices 3 & 4). Learners had learned about the definition and representations of the function concept with their teacher and in earlier grades before this study. I had to establish how they understood these definitions and the different representations of the function concept as well as difficulties and misconceptions they might be having. I analyzed learners' solution strategies (both oral and written) using APOS theory to elicit the difficulties they faced when answering questions on concepts related to functions. In order to overcome these difficulties, I considered these when designing the instructional sequences and activities.

4.8.2 Phase 2: Development of interventions informed by theoretical framework

After eliciting, re-confirming and categorizing learners' difficulties summarized in the "Balloon diagram" in Figure 2 (Chapter 2) under function definition and representation, I designed and developed interventions in the form of HLTs and their instructional activities which were informed by the principles of RME and constructivism. I began by transcribing all interviews and coding them using Atlas.ti in the context of the research questions and theoretical framework. A code in qualitative research is most often a word or a short phrase that symbolically assigns a summative, salient and evocative attribute for a portion of visual data (Saldana, 2008).

Application of Atlas.ti

Atlas.ti is powerful software program that allowed me to handle large amounts of data (28 interview transcripts) in order to group learners' interview responses into code families. This program made it easy for me then to use APOS theory indicators to classify learners' responses under action level, process level, object level and schema level.

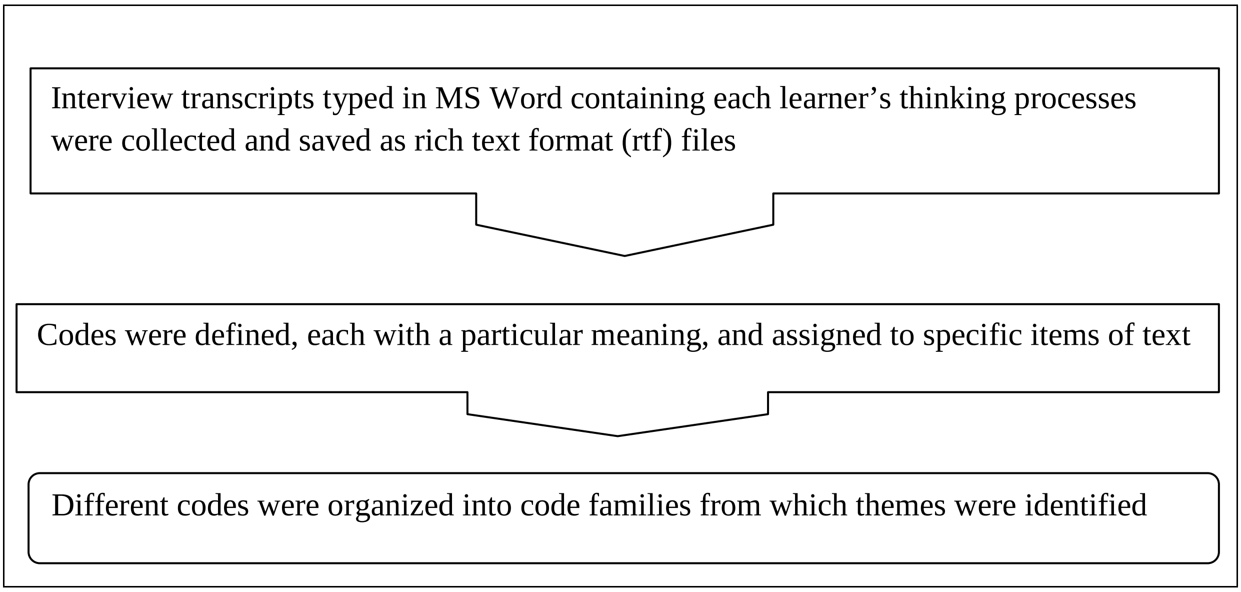


Figure 10: Steps in the analysis of learners' understanding of functions with Atlas.ti

In this thesis I worked with 28 interview transcripts, which are called *primary documents* (PD) in *Atlas.ti*. I highlighted some 308 text segments, referred to as *quotations*, which yielded 309 labels, which are called *codes* in the software. I then grouped the codes into 6 code clusters, that is, *code families*.

```
HU:  PhD MATHEMATICS EDUCATION
File:  [C:\Users\user\Desktop\TINODA\CODING DATA FOR ANALYSIS.hpr6]
Edited by: Super
Date/Time: 12/04/20 09:20:57 AM
--------------------
Codes-quotations list
Code-Filter: All [29]
--------------------
Code: Definition: as a correspondence relation {1-0}

P 3: INTERVIEW 3 DEFINITION INTERVIEW TRANSCRIPTS c.doc - 3:10 [Like
here, like, when you are  ...] (81:81)   (Super)
Codes:  [Definition: as a correspondence relation]
```

Figure 11: Section of an Atlas.ti data analysis

Figure 11 illustrates a small section of the Atlas.ti data analysis from which I explain the following concepts. According to Smit (2002):

> The *Hermeneutic Unit*, (HU) in *Atlas.ti* refers to the complete project or research for example a thesis or a dissertation. The file reference indicates the location where the project is saved. The word *Super* refers to the researcher who actually does the analysis, and time and date are given for further reference. *Codes-quotations list* means that this particular information shows a particular *code*, with the relevant *quotation*, that is the verbatim evidence given by the respondent. The *code-filter*: shows that this particular list was filtered by using all the *primary text*, also referred to *primary documents*, which simply means all the interviews. P3 would then represent the third interview. 3:10 stands for the third interview, 10th code. The {1-0} refers the number of codes, and how often this code has been linked to another (p. 71).

```
Code: Definition: as a correspondence relation {1-0}
Code: Definition: as a rule {0-0}
Code: Definition: as a symbolic expression or equation {1-0}
Code: Definition: as relationship between variables {5-0}
Code: Definition: as set-theoretical {0-0}
Code: Definition: other {2-0}
Code: Difference: between a function and non-function {6-0}
Code: Difficulties: critical points and sketching {19-0}
Code: Difficulties: inverse {7-0}
Code: Example: ambiguous relation {3-0}
Code: Example: equation in verbal or symbolic form {1-0}
Code: Example: one-to-one function {3-0}
Code: Example: outside mathematics {5-0}
Code: Example: use discrete elements of sets {1-0}
Code: Explanation: asymptotes {14-0}
Code: Explanation: calculating asymptotes {17-0}
Code: Explanation: calculating intercepts {16-0}
Code: Explanation: calculating inverse {16-0}
Code: Explanation: calculating turning points {49-0}
Code: Explanation: domain and range {11-0}
Code: Explanation: intercepts {14-0}
Code: Explanation: inverse {15-0}
Code: Explanation: non-example {6-0}
Code: Explanation: relationship between function and its inverse {24-0}
Code: Explanation: sketching graph {16-0}
Code: Explanation: switching from graph to equation {26-0}
Code: Explanation: turning point {10-0}
Code: Representations: aware of {12-0}
```

Figure 12: Codes used and their frequencies

Codes in the preceding Figure 12 were categorised in the following 6 code families that subsequently became themes: definition (6 codes), differences (1 code), difficulties (2 codes), examples (5 codes), explanation (14 codes) and representations (1 code). After learners' responses were grouped into these code families I used the indicators of APOS theory conception levels described in section 3.5 to classify learners' responses into these levels instead of using a focused code network.

4.8.3 Phase 3: Using tentative products and theories

I analysed learners' difficulties and their APOS theory conception levels together with some promising examples I found in textbooks to identify possible starting points of hypothetical learning trajectories (HLTs). HLTs and instructional activities make up my tentative products which I used in the intervention(s). These tentative products were shaped by my theoretical framework and refined by the feedback from teaching experiments until they yielded an improved understanding of function related concepts. Bakker (2004) describes a hypothetical learning trajectory as a learning path imagined by the teacher and based on the actual situation in the classroom. For example, in this study where learners are having difficulty in understanding concepts related to functions, moving learners from a situation of difficulty to an ideal situation where they attain the desired goal. In this study it would be represented by where they meet the requirements prescribed by the work schedule. Figure 13 illustrates the HLT.

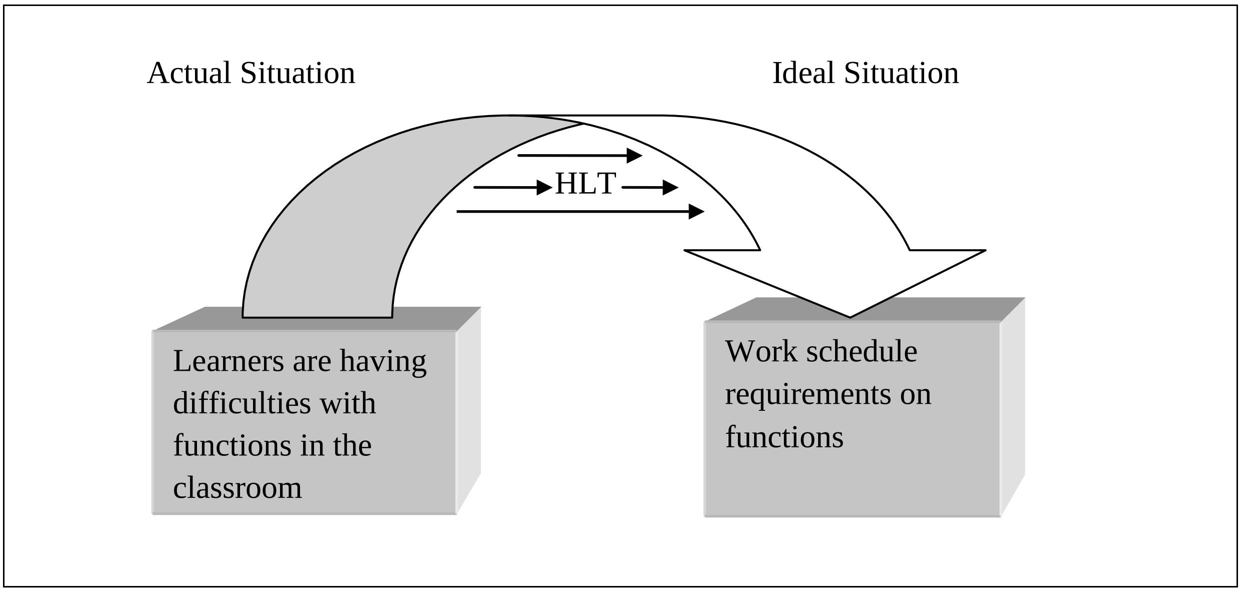


Figure 13: Hypothetical learning trajectory

The HLT, according to Simon (1995) is "made up of three components: the learning goal that defines the direction; the learning activities; and the hypothetical learning process which is an evaluation of how the learners' thinking and understanding evolved in the context of the learning activities" (p. 115). The HLT is the link between an instruction theory and a concrete teaching experiment. It is informed by general domain-specific and conjectured instruction theories (Gravemeijer, 1994), and it informs researchers and teachers on how to carry out a particular teaching experiment. After the teaching experiment, it would guide retrospective analysis. The interplay between the HLT and empirical results forms the basis for further planning and adjusting the HLT. This means that an HLT, after it has been mapped out, has different purposes depending on the phase of the design research. Also, it continually develops through the different phases and can even change during a teaching experiment.

In the teaching experiment, the HLT functioned as a guideline for what to focus on in teaching, interviewing, and observing. During the retrospective analysis, the HLT functioned as a guideline determining my focus in the analysis. As I made assumptions about learners' learning, I also needed to compare those with the observations I had made during the teaching experiments. Such an analysis of the interplay between the evolving HLT and empirical observations formed the basis for developing instructional activities. After retrospective analysis, an HLT can be reformulated, in an often more drastic way than during the teaching experiment, and the new HLT can guide a next design phase. An HLT in this case can be seen as a concretization of an evolving instruction theory. Conversely, the instruction theory is informed by evolving HLTs. For example, if patterns of an HLT stabilize after a few macro-cycles, these generalized patterns in learning or instruction and the insights of how these patterns are supported by instructional means can represent new knowledge.

The HLT is conjecture-driven and entails attempts to ensure equal opportunities for all learners to participate in and succeed at learning the concepts related to functions. A conjecture is an inference based on inconclusive or incomplete evidence (Confrey, 1995). It was a means to reconceptualise ways in which to approach concepts related to functions in terms of their content and the pedagogy. Thus, a conjecture has both a mathematical content dimension and a pedagogical dimension. The mathematical content of the conjecture answers the question: what should be taught? While the pedagogical dimension is linked to the content dimension and answers the question: how should this be taught? These two dimensions guided me in organising my classroom for instruction and the kinds of tasks, activities, and resources that I needed to provide for the content.

A conjecture is necessarily situated in a theory which serves to structure the activities and methodologies in the teaching experiment. The theory helps to weave together the content and pedagogical dimensions (Confrey, 1995). In this study, the pedagogy of the conjecture was embedded in a constructivist paradigm increasingly informed by a socio-cultural perspective and sensitivity to learners' voices. The purpose of this study was to use design research to improve learners' understanding of functions. I developed instructional activities for learning and understanding concepts related to functions. Such instructional activities specify patterns in learners' learning of the concepts related to functions under the broad categories of the function concept, namely, its definition and representation as well as the means supporting that learning. This implies that the development of such instructional activities had to include both the design of instructional means and research of how these means support successive patterns in learners' reasoning when engaged in tasks related to functions. In general, I had to create the conditions in which I could develop and test the instructional activities, but to create those conditions I also needed to do research. Design and research were deeply intertwined in developing the instructional activities. In this research I was especially interested in how learners can learn to reason about concepts related to functions in constructivist and RME-oriented learning. This implied the need to design an instructional environment that supports such learning and anticipating successive patterns in learners' reasoning that could lead to the achievement of specified end goals thereby improving learners' understanding of functions.

I designed HLTs that had a dual purpose of helping learners move from their current conceptual level to the next in a constructivist environment while at the same time assisting them to overcome their difficulties. The HLTs I formulated guided the design of instructional activities that had to be developed (Drijvers, 2002). I used the principles and characteristics of constructivism in the design and teaching experiments for this study as they provide a framework for a conducive learning environment, productive learning process and meaningful assessment of the learning process. This environment enabled learners to maximize their potential of learning the concepts related to functions since the constructivist environment allowed them to be involved in the production of knowledge and their understanding was driving the constructivist teaching path I followed in the teaching experiment. I also applied characteristics of RME to the lessons by situating the intended instructional activities in reality, which served as source and as an area of application, starting from meaningful contexts having the potential to produce a better understanding of functions.

I discovered that learners in the sample were not taught the inverse of the function concept. However, I had planned to use the inverse function to extend the learners' understanding of representations because an inverse of a function can also be a function itself. Moreover, the process of reversing operations when finding the inverse can push learners from the action conception to a process conception of the function. Therefore, I had to use learners' difficulties in understanding the inverse function found in the literature and my experience as a mathematics teacher in order to produce a hypothetical learning trajectory for learning the inverse of a function. From experience I discovered that learners are taught the inverse of a function using the rule, 'interchange the positions of x and y and make y the subject of the formula'. The implication of this approach to learners is that the inverse does not exist if there is no formula connecting the variables. In most cases learners can easily follow this procedure correctly but without any understanding of why they are interchanging the positions of x and y. In addition, while many teachers use this approach because it is easy to use but in most cases they do not explain to learners why the positions of x and y were interchanged. This approach is also common in many textbooks without their providing a complete explanation (Buerman, 2007). Buerman also points out that, the property of 'one-to-one and onto' is the basic criterion that a function must meet so that it may be reversed and is also often misunderstood by learners.

Dubinsky and Harel (1992) discovered that some learners have difficulty in understanding the inverse of a function. They believed that this cognitively simple mathematical idea was made difficult for many learners by the peculiarity of the representations and the absence of an algebraic formula. In essence, "the algebraic expressions tend to shift the focus of attention from the notion of 'undoing' to the idea of an 'inverse operation' entailing the inversion of a sequence of algorithms in the process of a function by going from the end to the beginning" (Bayazit & Gray, 2003, p. 2). However, understanding the inverse function as "undoing" has insufficient information. So, "a solid understanding of the concept of inverse function cannot be limited to 'undoing'. Teachers need to have an informal conception as well as more formal knowledge" (Buerman, 2007, p. 32). I used a prepared interview guide (Appendix 5) to determine learners' understanding of the inverse function after the teaching experiment.

I then designed instruction using the HLTs I had developed, RME activities and three separate tasks on the definition, representation and inverse as a form of intervention to help learners move up from their initial conception levels to the next and to overcome their difficulties. After each design cycle I interviewed learners based on the task for a particular concept. Learners' responses were analysed using APOS theory and then used to design further instruction to help learners approximate the schema level of understanding concepts related to functions.

Analysis determined both the starting points and learning goals of the teaching experiments in which I taught the learners and then used the designed tasks to test the effectiveness of the intervention. The last stage involved designing and using the instructional activities to help learners improve their understanding of concepts related to functions and to progress to the next level of APOS theory.

In this study I conducted conjecture-driven teaching experiments with grade 11 learners and I taught all the sessions. During the course of the experiments, the learners passed through stages of learning and understanding concepts related to functions, that is, definition stage, representation stage and the inverse stage. These stages are a result of my analysis of the conceptual development of the function concept and the identified difficulties in Chapter 2 as well as from clinical interviews. The reason for these stages is that learners should understand the basic idea of the function concept first before representing it in its various forms. That ability to manipulate representations back and forth is a sign of a robust understanding of the function concept. In the first stage, the learners were introduced to the function concept through a variety of activities linked to RME (see next Chapter) that were aimed at helping learners to grasp the key idea, as explained in Chapter 2, behind this concept. The activity helped them to understand and to evaluate different definitions of the function concept.

In the second stage, instruction focused on the representation of the function concept in order to strengthen learners' construct of its meaning. While studying representations, the learners worked in such contexts as translating real world problems to mathematical problems and also translating from one representation another. After having developed a rich network of connections among different representations, learners were introduced to the inverse of the function concept during the third stage.

Observations in one lesson and theoretical arguments from multiple sources can influence what is done in the next lesson. In this study the focus was on designing and developing instructional activities that could be tested and revised by teachers and researchers. For a careful retrospective analysis, it was necessary to keep track of changes in the HLT and of learners' learning in my journal. After teaching the learners I had to assess them to check whether they had made the requisite constructions to move up the conceptual ladders and whether instructional activities had helped learners overcome their difficulties. Figure 14 illustrates how learners' understanding was scaffolded through APOS theory conception levels. Scaffolding of learners' understanding consist of three steps labelled A, B and C in Figure 14. In step A I carried out an APOS analysis of learners' understanding of functions based on given tasks and the literature review to elicit difficulties and APOS conception level. In step B I designed HLTs and used RME instructional activities to assist learners to overcome their difficulties and move to the next conception level. Then testing, reflecting and refining the instructional activities to check whether the learner had moved to the next conception level in step C. The cycle was repeated until the desired goals were approximated and learners had made progress.

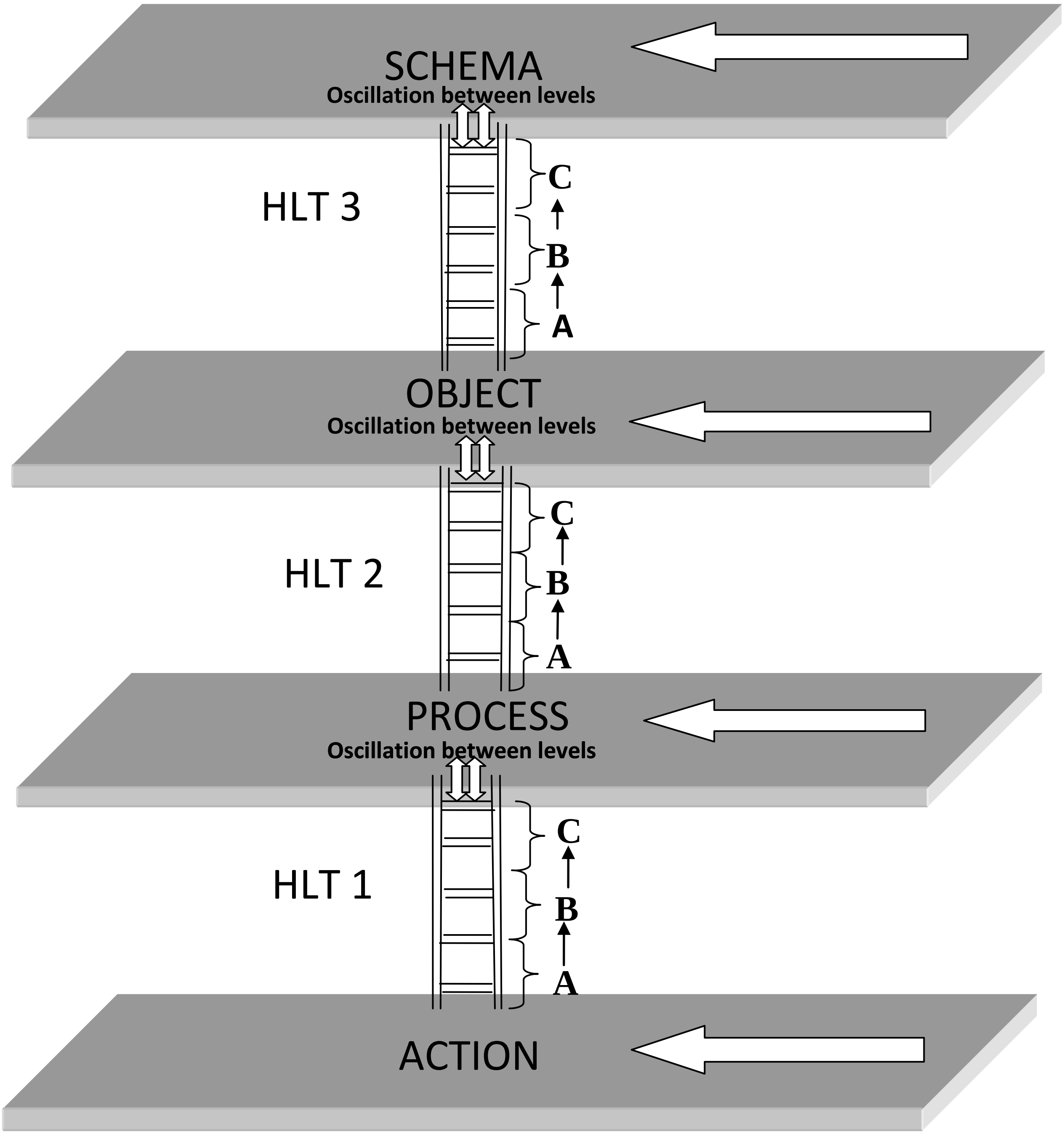


Figure 14: Scaffolding learners' understanding through APOS theory levels of conception

4.8.4 Phase 4: Product and theory refinement

According to Wademan (2005) reflection after the first teaching experiment leads to adaptation of HLTs and the teaching sequences, which becomes the starting point for a second teaching experiment. This iterative process follows specified HLTs and uses the designed instructional activities to provide empirically grounded answers to the research questions. The sequence, therefore, should be tried out and analysed in various situations, as well as be discussed with other parties who play a role in educational innovation, such as teacher training institutes and educational publishers. In this phase it was important to specify my role, the nature of classroom interactions, assessment, and method of data collection which are explained below.

Role of the teacher: major responsibility is listening and creating an environment where learners can construct knowledge. The teacher also acts as an expert resource, a facilitator, a discussion leader, coach and an evaluator.

Classroom interactions: instruction was structured in a way that allowed learners to construct knowledge from real contexts sequenced from simple to complex. Learners first worked individually on worksheets, then in two small groups of three. First I explained important aspects of the function concept then I gave the learners some assessment activities on which I based my clinical interviews.

Assessment: the assessment activities used to evaluate learners' learning and progress were informed by and consistent with the content, namely, the definition, representation and inverse of the function concept, pedagogy, and the theoretical framework that was a merger of constructivism, Piaget's theory, APOS theory and RME of the conjectures along with the other components of the intervention. The results of these assessments provided outcome data related to the impact of the intervention and allowed an ongoing formative evaluation of the intervention process. The formative assessment continuously guided the evolution of several conjectures (HLTs) for this study and the related instructional activities. These assessment tasks (see Appendix 6) were varied both in terms of their form, that ranged from fill-in, short questions, open-ended questions, homework, in-class, non-routine, and their skills development. From these assessment tasks learners were allowed to present and defend their solutions to problems.

Data collection: I collected data about my thinking and actions during the intervention because much of the teaching experiment depends on what emerges from the classroom interactions. As a result I made careful notes of classroom observations in my journal which I referred to at the analysis stage. The assessment activities provided data on what learners had learned but an in-depth understanding of learners' learning and development was obtained through the task-based clinical interviews. The number and form of these interviews was determined by the content and theoretical framework of conjectures. My following the groups was also a valuable source of information because it allowed for articulation of learners' voices which was rich and revealed their conceptions. I kept field notes of my thoughts and reflections of the interventions as they progressed. On some occasions, learners in the sample shared their concerns about the difficulties they faced in accomplishing a task. This was a source of significant data. Other sources of information included the learners' written assessment activities and the verbal responses gathered during intermittent interviews. Written work was given, collected, and corrected each week for three weeks in order to maintain a continuous record of learners' conceptual development. Written work was used to gauge the depth and breadth of learners' understanding.

Gravemeijer and Cobb (2001) define the word 'experiment' in 'teaching experiment' as "an experimental classroom setting created as a result of the innovative teaching materials provided. The main goal for the experiment is to understand and improve the initial design on the basis of learners' reasoning with respect to the newly created classroom setting" (p. 4). My HLTs and instructional activities for the learning of concepts related to functions were implemented in classroom situations. I was interested in the development of learners' conceptions in relation to the teaching processes. Did learners notice the relation between the function concept and the rule representing it? Did they see the relation between these representations and the need for translating among them? Did this evoke the need for two-dimensional graphs? Was I, as teacher capable of guiding the discussion without suggesting intended directions? These questions were answered by the teaching experiments. To gain insight into major shifts in learners' reasoning, I collected data that reflected their thinking and the role of the teacher. I audio recorded all individual task-based interviews and also copied all written materials during activities and the learners' final test (Appendix 7).

4.8.5 Phase 5: Final product and contribution to theory

After multiple iterations of the intervention, the refined HLTs and their instructional activities become the product of the intervention, and therefore could be used by teachers to help their learners understand the function related concepts. The theories which have been operationalized in the teaching experiments are redefined and guidelines prepared. In Chapter 5 on actual implementation of the interventions, details of the model presented in this chapter are discussed in detail.

It is apparent that different learners might have quite different schemas for the function concept. This study attempts to describe learners' different ways of understanding which include individual cognitive constructs, their origins, and their connections to one another. According to APOS theory, a learner's cognitive development of the function concept follows the order in the acronym APOS (Weyer, 2010). "Initially, learners have little or no knowledge about functions, and then first, they must perform actions implementing a concept. Next, these actions are internalised into processes" (Dubinsky & Wilson, 2013, p. 96). The resulting processes are, in turn, encapsulated into objects (Dubinsky, 1991). Finally, the individuals coordinate these mental constructs into schema for a concept. In practice, mental constructions rarely-if ever-occur in a simple logical sequence. The individuals' early schema may be disorganized, incomplete, and contain inconsistencies. As they experience disequillibration resulting from conflict between expectations and results and they then engage in serious reflection, maturation as described by the APOS framework may take place (Weyer, 2010).

It follows from the constructivist point of view in general and the APOS theoretical view in particular, that the role of teachers is not to transfer their understanding of the concepts related to functions to their learners. Instead, "the role of teachers is to create situations in which learners are likely to construct these actions, processes, objects and schemas for themselves" (Weyer, 2010, p. 10). This is not to say that the students are expected to discover all, or even most, of the mathematics for themselves. Rather, a teacher implementing a pedagogical approach based on APOS theory would structure activities intended to provide learners with a base of experience working at the action, process, and object levels of the concepts related to functions. This would be an attempt to help learners build elementary mental constructs and organize these into a coherent schema. Thus from the point of view of APOS theory, my role in this study begins with an attempt to identify the relevant cognitive structures which must be constructed in order to learn the chosen aspects of the function concept.

4.9 Limitations of the design research model

Collins, Joseph and Bielaczyc (2004) note the following fundamental limitations of design experiments. Because they are carried out in the fuzziness of actual learning environments, such as classrooms or afterschool settings, there are many variables that affect the success of the design, many of which cannot be controlled. "Design researchers usually collect large amounts of data, such as video and audio records of the intervention and outputs of the learners' work, in order to understand events in detail. Hence, they are swamped with data, and given the data reduction problems; there is usually not enough time or resources to analyze all data collected" (Collins et al., 2004, p. 67). It also requires adequate resources to collect so much data, so design experiments tend to be large endeavours with many different participants, all of whose work needs to be coordinated. All these factors make design experiments difficult to conduct and conclusions sometimes uncertain (Collins et al., 2004).

However, the main result:

> is not merely a design that works, but the reasons how, why and to what extent it works. Firstly, an initial instructional design is developed, and educational settings are created for investigating and generating theoretical conjectures. Secondly, depending on questions to be considered, the analysis of the teaching experiments focused on various elements of the design, such as learners reasoning with the tools provided, classroom discussions, collaborative work, or the development of specific classroom norms (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003, p. 8).

The initial instructional design for the teaching experiment aims at a conjectured learning process based on prior research and theory. However, during the design research, initial conjectures may be refuted or adapted, and new conjectures can be generated and tested. Design research in this sense "has both a hypothetical and a reflective side, which leads to a delicate and iterative process of testing, reflection, and redesign. The testing takes place in teaching experiments, and the reflection in most cases is based upon qualitative data analyses" (Cobb et al., 2003, p. 10).

4.10 Rigor and trustworthiness

In the research process it is important to "provide checks and balances to maintain acceptable standards of scientific inquiry by addressing the need for rigorous data collection and methods of analysis" (Bowen, 2005, p. 214). In this study, I enhanced rigor and trustworthiness by using triangulation in which I employed three methods in collecting data: task-based clinical interviews, participant observation and document reviews in the form of learners' written work.

> Producing similar findings from different methods provides corroboration or reassurance. The absence of similar findings will not, however, provide grounds for refutation. This is because different methods used in qualitative research furnish parallel datasets, each affording only a partial view of the whole picture. Triangulation relies on the notion of superior explanation,

against which other interpretations can be measured (Barbour, 2001, p. 1117).

Qualitative research, however, is usually carried out from a relativist perspective, which acknowledges the existence of multiple views of equal validity (Mays & Pope, 2000). Although the interviews were time-consuming, they were my main data-gathering method. Observations were done while learners were working on tasks and during group discussions. Document reviews were done at the end when comparing learners' interview responses, what I had observed and what I had seen in learners' written work.

The validity (both internal and external) of data collection methods and procedures, methods of analysis and conclusions are important for this study. Internal validity refers to the quality of data collections and soundness of reasoning leading to the conclusions (Gravemeijer & Cobb, 2001). I used the following methods to improve the internal validity of this study. During the retrospective analysis, I tested conjectures that I generated at specific episodes using my field notes, tasks, and other learner work. During this testing stage I searched for counterexamples of my conjectures. The succession of different teaching experiments made it possible to test these conjectures developed in earlier experiments, in later experiments. I analyzed important episodes with multiple theoretical instruments of analysis in other words, theoretical triangulation. Theoretical claims are substantiated where possible with transcripts to provide rich and meaningful context.

External validity is mostly interpreted as the transferability of results (Confrey, 2003). The question is how we can transfer the results from specific contexts so as to be useful for other contexts. The challenge is to present the results, particularly the HLT and instructional activities, in such a way that others can adjust them to their local contexts (Barab & Kirshner, 2002). If lessons learned in one experiment are successfully applied in other experiments, this is a sign of successful transferability. This implies that the transferability and viability of the results of this study can better be judged in the future if applied in other situations.

4.11 Ethical issues

I first obtained permission from the District office of the Department of Education and the principal of the school. I then arranged with the grade 11 teacher involved to meet with the grade 11 learners from which the participants were selected. Before learners volunteered to participate in the study, I briefed them on the purpose of the study, benefits of the study to them and education as well as the fact that this was part of my PhD study, the duration of their involvement in the study, general procedures of the study, possible risks such as loss of study time and discomfort of answering task and interview questions, the use of pseudonyms to guarantee their anonymity and their rights to volunteer to participate or to withdraw their participation from the study at any given time without being penalized. I also informed learners that they would first work on the tasks as individuals and then as groups. Thus, learners volunteered to participate from an informed position. The written parental permission and informed assent letters were signed by parents/ guardians and learners respectively (Appendix 7).

4.12 Summary of chapter

In this chapter the research design and methodology for answering the research questions for this study were described together with the underlying philosophical assumptions, both ontological and epistemological that I brought into this study as well as its limitations. The study site and setting, the sample and sampling techniques and ethical issues were explained and justified. The research questions for the present study were stated and I used the design research method in which I adapted Wademan's (2005) generic design research model which was described together with its five phases. The hypothetical learning trajectory was also examined because of its important role in all the five phases. The data collection and analysis methods used for the present study were also described and justified. Measures to ensure rigor and trustworthiness of the entire research process were explained in the last section of this chapter.

The next chapter summarizes the findings that came from analyzing learners' responses to questions asked in the interviews with respect to the learner's response rather than the intended effect. I needed to determine that learners could understand these concepts related functions at many levels and I expected them to respond at various levels and at various points throughout the interviews. I began with a theoretical analysis of the elements perceived as necessary for development of conceptual understanding. I then designed instructional activities to compel learners to make the requisite constructions or that would help learners to progress from one conception level of APOS theory to the next.

CHAPTER 5

Presentation and Analysis of Data

5.1 Introduction

This chapter presents and analyzes data from the five design research phases I adapted for this study. The first phase (*Problem Identification)* addresses my first research question:

- How do learners understand the function concept?

and its two sub-questions:

1) What are grade 11 learners' current understandings of functions?

2) What are the weaknesses in learners' understanding?

Phases 2, 3 and 4 address my second question:

- How can instruction be designed to improve learners' understanding of functions?

Each phase presents a teaching experiment based on the design principles of RME. Phase 1 focused on learners' understanding of functions and the literature reviewed in chapter 2. In this phase I transcribed all interviews and coded them using ATLAS.ti in the light of my research questions and theoretical framework as described in Section 4.8.2. Phase 2 (*Development of interventions)* provided design guidelines for the development of specific interventions. Phase 3 (*Using tentative products and theories)* is a progression from phase 2 and examines how I used the feedback from phases 1 and 2 to design and develop specific HLTs and instructional activities which I used in each intervention in the form of teaching experiments. This leads to phase 4 (*Product and theory refinement)* where the HLTs and instructional activities are refined based on the results of the feedback from formative retrospective analyses of their use and adjustment in the teaching experiments which leads to the final products in phase 5 (*Final product and contribution to theory)*.

The purpose of this study was to use design research to improve the teaching and learning of functions which I purposefully split into the definition, representations and inverse of the function concept to ensure a thorough coverage of these aspects without separating them conceptually. I designed and developed empirically grounded instructional sequences and activities to optimise learning of these selected aspects of the function concept. The focal point of this chapter is where the collected data is presented and analyzed in the light of the research questions for this study in order to forge ways of improving learners' understanding of functions.

5.2 Data analysis for all phases

The basic techniques I used in analyzing the qualitative data in this study are thematic analysis combined with constant comparative narratives and dialogic reporting. Thematic analysis is “a process for encoding qualitative information” (Boyatzis, 1998, p. vi), which was the mode of data treatment I employed in identifying the emerging themes related to the primary code of interest. In the context of thematic analysis, the advantage of the present research is that the analysis was centered on the APOS theory which guided the analysis process. This centrality of examination provided a more meaningful assessment of the interacting and related codes, in identifying the themes that constituted the learners’ understanding of the function concept.

Guided by themes, I present in this chapter more detailed, fluid descriptions to help the reader see and possibly, feel, how learners understand the function concept in terms of its definition, representation and inverse. An important qualitative analytical technique utilized in this research was the dialogic style of data reporting. Key to this technique was the use of research dialogue between the researcher and the learners in the sample in data presentation and analysis. This technique was extensively used by Bourdieu, Chamboredon, & Passeron (2000) in their qualitative work. In this work, learners’ written work and their interview transcripts were reported illustrating how research information was shaped and influenced by both the researcher and respondents. In the same manner, conscious of my power as a researcher, particularly in interviewing learners, I chose to show selected portions of actual transcripts to illustrate the research dialogue between myself and the learners. In a self-reflexive manner, this allowed the presentation of multiple voices from learners. In so doing, data interpretation was not centered on my relatively powerful views as a researcher but rather, through research dialogue, I was able to diffuse the power of interpretation by allowing the data to speak as they were gathered, thereby preserving the naturalness of conversation. I considered this type of research reporting as enabling better discourse of research insights reflecting my intention to let the voices of learners be heard and felt in the text of this thesis.

5.3 Phase 1: Problem identification

5.3.1 The analyses of learners’ initial individual tasks and task-based interviews

These interviews were done in two sessions to allow learners to prepare themselves for the next session. Session 1 was based on question 7 and session 2 on question 8 of the grade 11 mathematics examination paper 1 of 2011 (see Figures 8 and 9). These two questions focused on understanding the transformation of graphs by translating from the equation to the graph and from the graph to the equation respectively. As learners took me through their solutions to these two questions I asked them to explain to me the meanings of function-related concepts involved in the translation process namely the intercepts, asymptotes and turning points, and how they calculated them. The following section presents a *case analysis* of each student supported by *actual examination item answers* and *interview excerpts* data to provide evidence of the APOS level at which the learner is operating in terms of understanding the *function-related concepts* (intercepts asymptotes, turning points and sketching of graphs). This analysis is based on my theoretical framework and reviewed literature to enable me to classify each learner according to APOS theory conception levels.

Case 1: Diva's understanding of the function-related concepts

Question 7 asked learners to calculate the intercepts and asymptotes from a given equation and to sketch the graph of the given function. To answer this question correctly learners should be operating at the action, process and object levels of APOS theory. This question allowed me to analyse learners' understanding of representations of functions in terms of the APOS levels of conception.

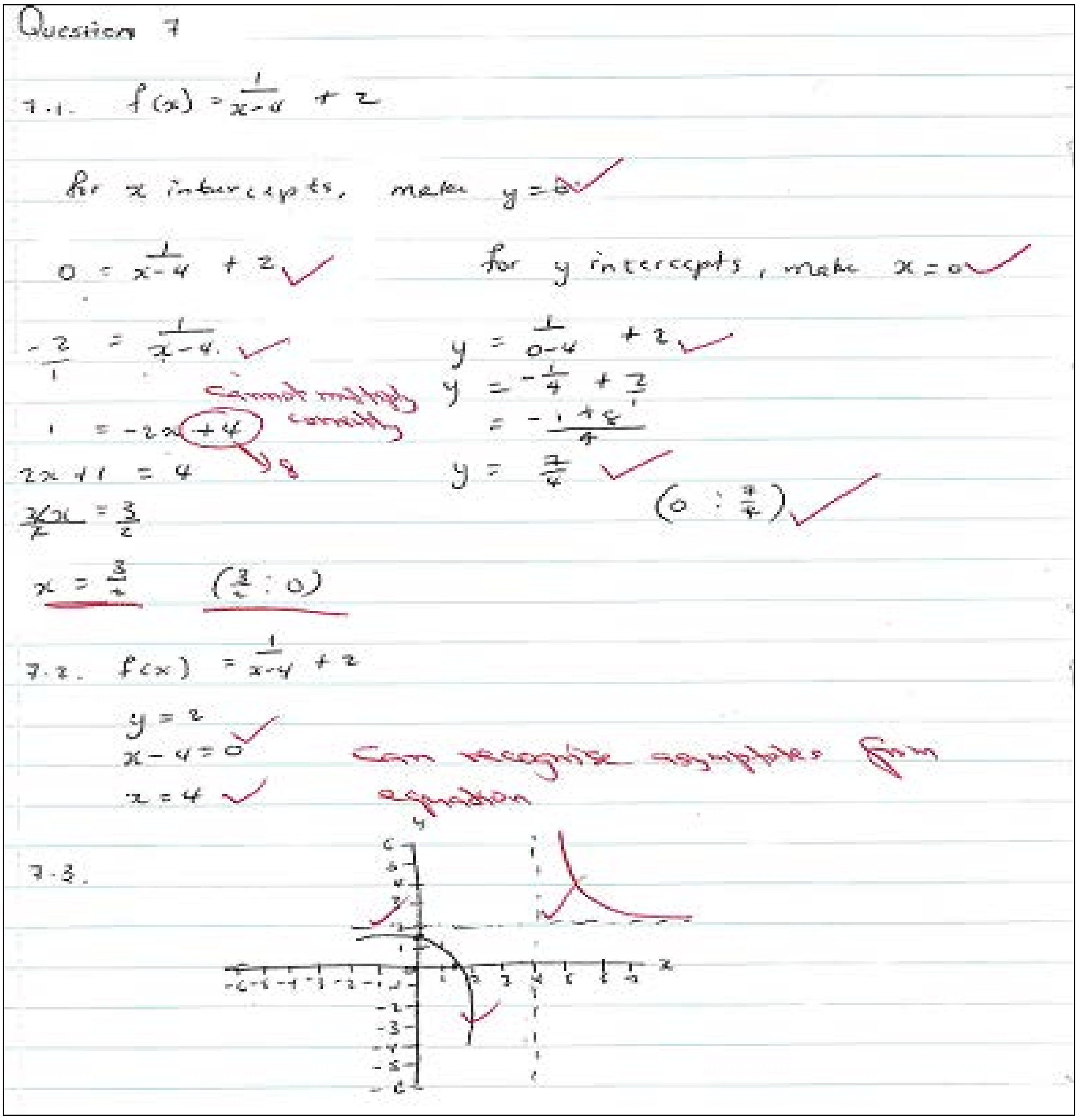


Figure 15: Diva's written examination answers for question 7

In my analysis of Diva's written responses in Figure 15 above I refer to APOS theory indicators described in Section 3.5 and 6.6. Calculation of critical points is an important skill that learners should acquire to be able to translate from one representation to the other. Diva was able to substitute $x = 0$ and $y = 0$ into the given equation to calculate the y and x-intercepts respectively (action level). She could also determine the asymptotes. Though Diva made a mistake in cross-multiplying when finding the x-intercept she could follow procedures of finding the asymptotes and intercepts step-by-step but could not explain her procedures which are indicators of the action level described in Chapter 3. In addition the learner was able to sketch the graph correctly using the critical points which is an indicator of APOS theory's action level. My analysis of Diva's written responses to question 7 was also supported by her responses to interview questions where she explained her solution strategies as shown in the interview excerpts that follow:

Interviewer: How do you explain an intercept?

Diva: Intercepts are the points you are supposed to plot so you can get your graph.

Interviewer: How do you calculate these intercepts?

Diva: I want to find the *y*-intercept first by putting $x = 0$ and then I'm going to find the *x*-intercept by putting $y = 0$ and calculate.

Interviewer: Why put $x = 0$ on *y*-intercept and $y = 0$ on *x*-intercept?

Diva : Sir because our teacher said when you calculate intercepts you must always let *y* or *x* be equal to zero.

Interviewer: You managed to find the asymptotes; can you explain to me what these are?

Diva: An asymptote, I think is the line where …, which shows us that the graph can only approach, not mean to touch or cross.

Interviewer: Can you explain how you obtained these asymptotes?

Diva: I don't know how to explain to someone how to find it. I say zero is equated to the denominator umm..., I forgot how I calculate like that....$f(x) = \frac{1}{x-4} + 2$, ok, for *x*, I will take this one, and I say $x - 4 = 0$.

Interviewer: Why do you equate $x - 4$ to 0?

Diva: This is what we were told!

Interviewer: Tell me, how did you sketch this graph?

Diva: I want to show you my axis before writing a …, is like this as I have plotted, and then it will be *q*, this is *y*-asymptote. Here I put *x*, *x* - axis and the *y* - axis, then I look for $y = 0$ is here, and $x = 1.75$, I'm going to put it here and for $x = 0$, $y = 3.5$, I think is here, so I check my asymptote, for *y* is 2, I write a dotted line, for $x = 4$, and for *y*, so then I join my points.

Analysis of the above dialogue reveals that Diva cannot explain what an intercept is, but she could explain the procedure of calculating the intercepts indicating procedural understanding. However, she failed to explain why $x = 0$ on the *y*-axis and $y = 0$ on the *x*-axis indicating that she had an incomplete understanding of the procedure of calculating the intercepts which might cause problems later when obtaining these intercepts from a drawn graph. Similarly, Diva determined the asymptotes correctly but could not explain the procedure indicating that she has insufficient knowledge related to the concept of asymptote. Diva could explain how to sketch the graph using the critical points she had calculated. Diva's inability to explain concepts and why particular procedures work indicates that she is operating at the action level where she can just follow a procedure without understanding it as also documented in literature (Polaki, 2005).

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.

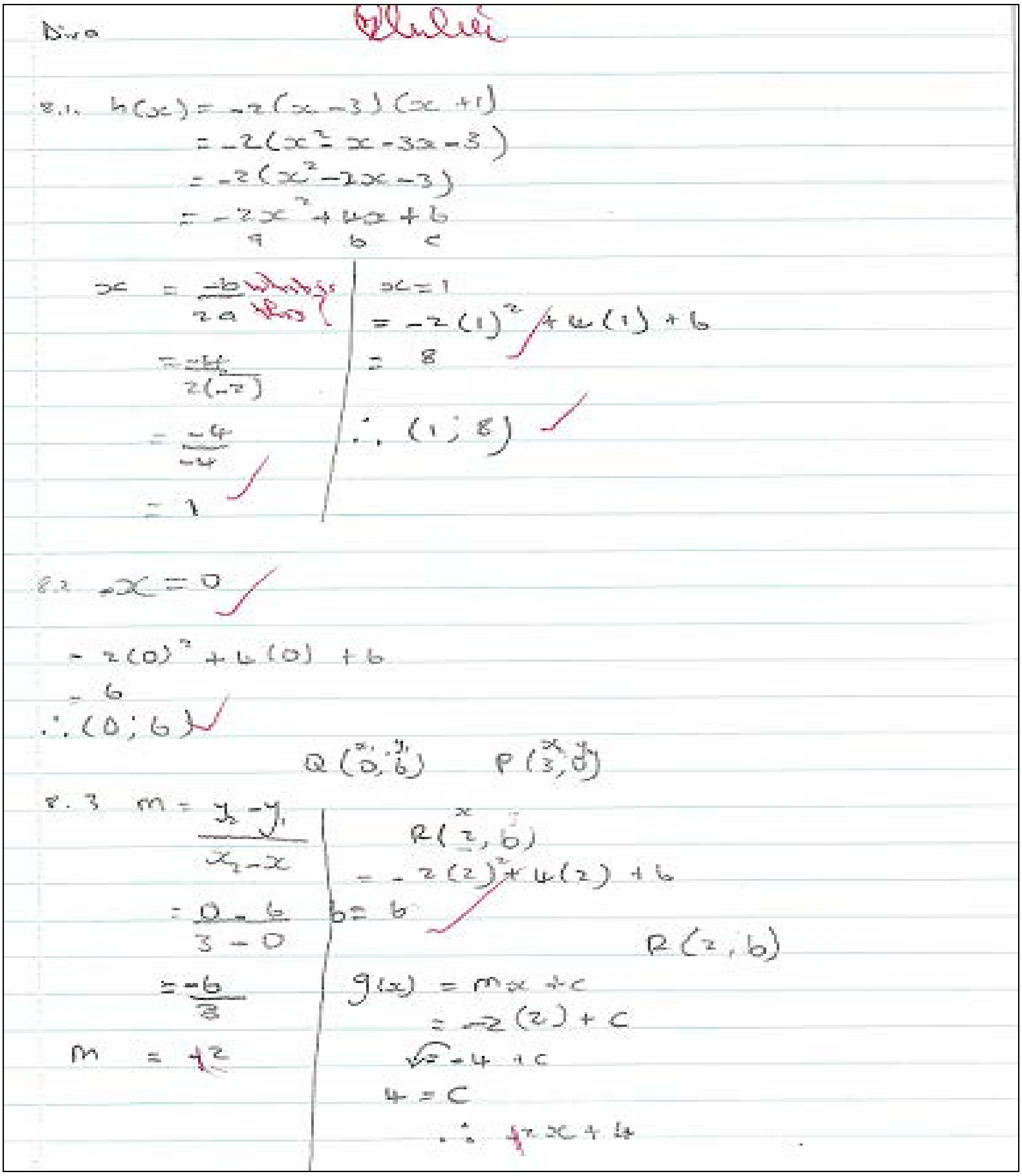


Figure 16: Diva's written examination answers for question 8

Diva correctly calculated the coordinates of the turning point A, the coordinates of Q, the numerical value of m (gradient) and the value of b. However, she did not realise that 6, the *y*-coordinate of Q (*y*-intercept) was the value of c which resulted in her giving an incorrect equation of *g*(*x*). Diva's written solutions show her ability to use procedural knowledge as shown by her use of $x = \frac{-b}{2a}$ and $m = \frac{y_2-y_1}{x_2-x_1}$. This substitution of numerical values in known formulas according to the indicators for function representations indicates that the learner is operating at the action level of APOS theory (Dubinsky & Harel, 1992). To further authenticate my claim that Diva was operating at the action level, I interviewed her on the solution strategies that she gave above and her responses are shown in the following interview excerpts:

Interviewer: Looking at your solutions to question 8, I can see that you calculated the coordinates of the turning point correctly. What is your meaning of a turning point of a graph?

Diva: Turning point is where my graph turns, goes back where it comes from or same direction where it originates.

Interviewer: Explain to me how you calculated the coordinates of a turning point.

Diva: To calculate the turning point, I used the formula $x = \frac{-b}{2a}$.

Interviewer: Where does this formula come from and what does it say?

Diva: Yes, from the quadratic formula but I'm not sure but this is the *x*-coordinate.

Interviewer: What is Q on the diagram and how did you calculate its coordinates?

Diva: Q is a point on the *x*-axis, we know that on the *y*-axis, the value of *x* is 0 'umm' so we substitute this value into one of these equations, on this, on coordinates of *x* to find the value of *y*, so the coordinate of Q is (0; 6).

Interviewer: Correct. In 8.3 what is this m and b? How did you find them?

Diva: 8.3 numerical value of m and b....m is the gradient of this line b is on this point and then … the formula of the gradient $m = \frac{y_2-y_1}{x_2-x_1}$, the value of b which is 6, the value of m is -2.

Interviewer: Now how do you find the equation of *g(x)*?

Diva: I already calculated the gradient which is -2, so, usually this point b for *y*, it will be 2 for *x*, to find the value of c is 4, which means g (*x*) = -2*x* + 4.

Based on the above responses I can conclude that Diva has an idea of what a turning point is, she just memorized that the *x*-coordinate of the turning point is $\frac{-b}{2a}$. However, Diva's explanation of Q as a y-intercept shows the positive effects of the first interview session on question 7. Diva could calculate the turning point by substituting numbers into a formula but could not explain why $x = \frac{-b}{2a}$ works and how it came about which are indicators of the action level of APOS theory (Dubinsky & Harel, 1992). The discussions on Diva's written and oral responses to questions 7 and 8 and the indicators of APOS theory conception levels for function representations, lead to the conclusion that she is operating at the action level. Her difficulties are not unique as they were also documented in the literature (Gagatsis & Shiakalli, 2004; Cunningham, 2005).

Case 2: Coco's understanding of the function-related concepts

Question 7 was testing learners' ability to calculate intercepts and asymptotes from a given equation and to use them to sketch the graph of a given function.

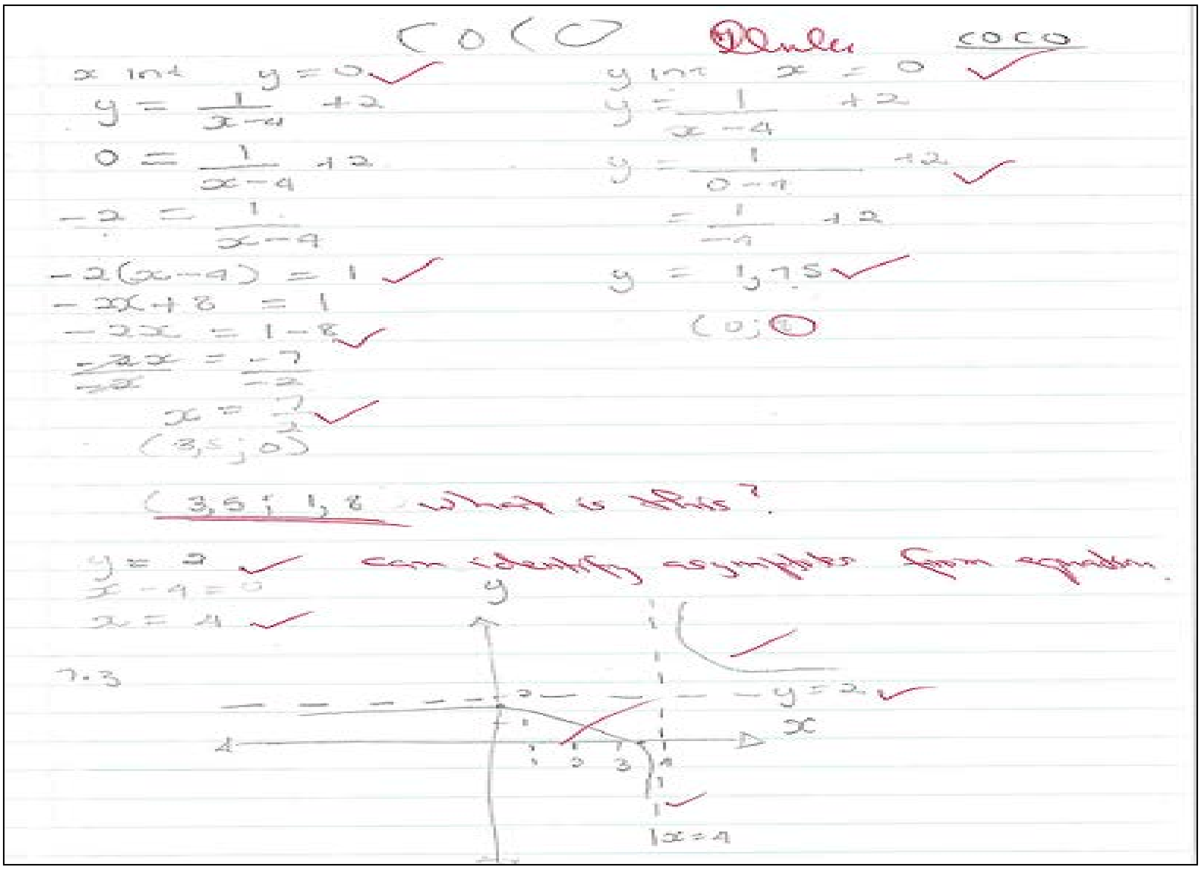


Figure 17: Coco's written examination answers for question 7

Coco correctly calculated the intercepts and asymptotes but wrongly took the *x* and *y*-intercepts as coordinates of an unspecified point. The learner also correctly sketched the graph showing all the critical points. This ability to carry out procedures has been placed at the action level of APOS theory (Dubinsky & McDonald, 2001). To verify my claim I interviewed the learner to triangulate her written and oral responses:

Interviewer: Question 7.1 asks you to calculate the intercepts, what is your understanding of an intercept?

Coco: The point at which the graph will cut the axes.

Interviewer: How do you calculate the coordinates of the intercepts?

Coco: I'll change the *x* into zero for the *y*-intercept and *y* into zero for the *x*-intercept. That is what I will do.

Interviewer: Why change *x* and *y* into zero at the intercepts?

Coco: This is what we were told. I don't know why.

Interviewer: You also calculated the asymptotes correctly. Can you explain to me what an asymptote is?

Coco: Is the point in the graph that does not have to touch the line of the asymptote.

Interviewer: How do you calculate the asymptotes?

Coco: The constant 2 is my *y*-asymptote and I equate the denominator *x*-4 to zero to get the *x*-asymptote

Interviewer: Can you explain to me why your procedure works?

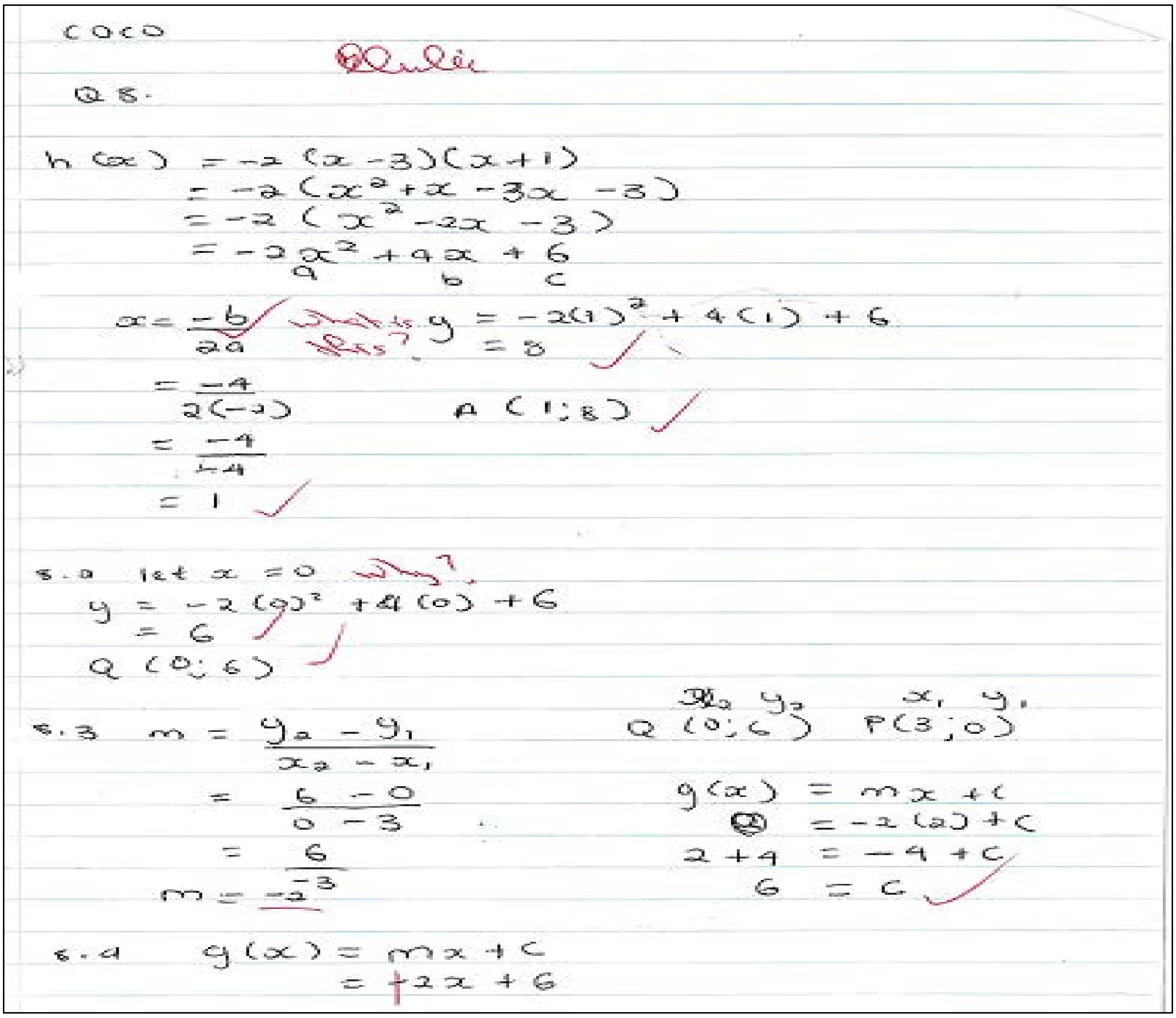
Coco: I am not sure but this is what we were told by our teacher.

Interviewer: How did you sketch the graph?

Coco: Okay for *y*, okay I am going to use 4 for *y* because 3.5 is where, I am looking for the asymptote, it is the point where it can't touch or pass or go beyond. So 3.5 cannot go beyond 4. So I don't know what happened here. This *y* is supposed to be 4 and then *x* is supposed to be 3 on this one. Then I am going to plot 3.5, then the 2.25 and then I was told if you have this graph on this third quadrant you have to have the same graph on the first quadrant and when you have it on the second you have to have the same graph on the fourth quadrant. So this means I will have another one here. 'Ah' I'm not sure whether I use the asymptote where the graph that is to cut and then the intercept.

Just like Diva, Coco could carry out all the procedures correctly but failed to explain why those procedures work which confirmed my claim from her written responses that she was operating at the action level. Her ability to use rules without reasons is an indicator of operating at the action level and limits the learner's ability to manipulate concepts and justify their answers.

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.



COCO

Q8.

$h(x) = -2(x-3)(x+1)$
$= -2(x^2+x-3x-3)$
$= -2(x^2-2x-3)$
$= -2x^2+4x+6$

$x = \frac{-b}{2a}$ $\quad y = -2(1)^2+4(1)+6 = 8$
$= \frac{-4}{2(-2)}$ $\quad A(1;8)$
$= \frac{-4}{-4}$
$= 1$

8.2 let $x = 0$ why?
$y = -2(0)^2 + 4(0) + 6$
$= 6$
$Q(0;6)$

8.3 $m = \frac{y_2-y_1}{x_2-x_1}$ $\quad Q(0;6)$ $P(3;0)$
$= \frac{6-0}{0-3}$ $\quad g(x) = mx+c$
$= \frac{6}{-3}$
$m = -2$

8.4 $g(x) = mx + c$
$= -2x + 6$

Figure 18: Coco's written examination answers for question 8

Coco managed to determine all the critical points through the use of procedures which she successfully carried out. To verify Coco's understanding of these procedures and confirm whether she was still operating at the action level I interviewed her and the following interview excerpts reveal her understanding of function-related concepts embedded in this question:

Interviewer: In your own words how do you explain a turning point?

Coco: A turning point is where the graph is cutting, like in a parabola graph.

Interviewer: How do you calculate its coordinates?

Coco: So, before I get the value of b and the value a, I have to simplify this equation. Let's say I should calculate the turning point of A, I'm going to use this equation, h(*x*) = - $2x^2$ + 4*x* + 6, because, minus and minus is positive, this is the new equation, for the turning point, I have to use this formula $x = \frac{-b}{2a}$, *y*, then -*2*, this one is our a and this 4 is our b, it will be, 'ja' it will be 4, is $\frac{-b}{2a}$, all this, is equal to $\frac{-4}{-4}$, which gives me 1, *x* = 1, then from there, I have to get the *y* equation, substitute all the *x* values to get *y*. Because I want the *y* values. It will be $y = -2x^2 + 4x + 6$, this is equal to $-2(1)^2 + 4(1) + 6$, and this will give me -2 + 4 + 6 = 8 and then this gives me +8, then my turning point is (1; 8)

Interviewer: Correct. But tell me, how does this formula $x = \frac{-b}{2a}$ work?

Coco: We were just told that this is the *x*-coordinate of the turning point.

Interviewer: How did you calculate the coordinates of Q?

Coco: I will say at Q, let *x* = 0 ...I'm not quite sure why *x* = 0. I want the value of *y*, and if you check the point at Q, *x* is already there, which is 0, so you need to calculate the *y*-value, so it will be like, $y = -2(0)^2 + 4(0) + 6$ then -2×0 it will give me 0 because any number ×0 is 0, then q = (0;6) then, the other point you determine the numerical value of m and b, which is, If I want to determine the numerical value, we use the point Q, because they are given and they are on the straight line, which means m is the gradient, so I need to calculate *x*, so m = $\frac{y_2-y_1}{x_2-x_1}$; and then, Q (0; 6), then P(3; 0), it will be …, I substituted *x* from the given coordinates will be y_2 is 0, y_1 is 6, x_2 is 3, then x_1 which is equal to 0, then $\frac{-6}{3}$, because we cannot say this minus the total of this, so m will be 2, will be -2, then the value of b, 'eish' this one is tough, I'm not quite sure how to calculate this, because there is this thing that r is the coordinate of (2; b), so b is the y -axis?

The learner failed to explain a turning point and why the procedures that she used work. However, Coco could describe the steps followed in each procedure precisely. Her understanding is limited to blindly following the procedures without any slightest understanding of why they work. These are indicators that Coco is operating at the action level of APOS theory.

Case 3: Monga's understanding of the function-related concepts

Question 7 was testing learners' ability to calculate intercepts and asymptotes from a given equation and used them to sketch the graph of a given function.

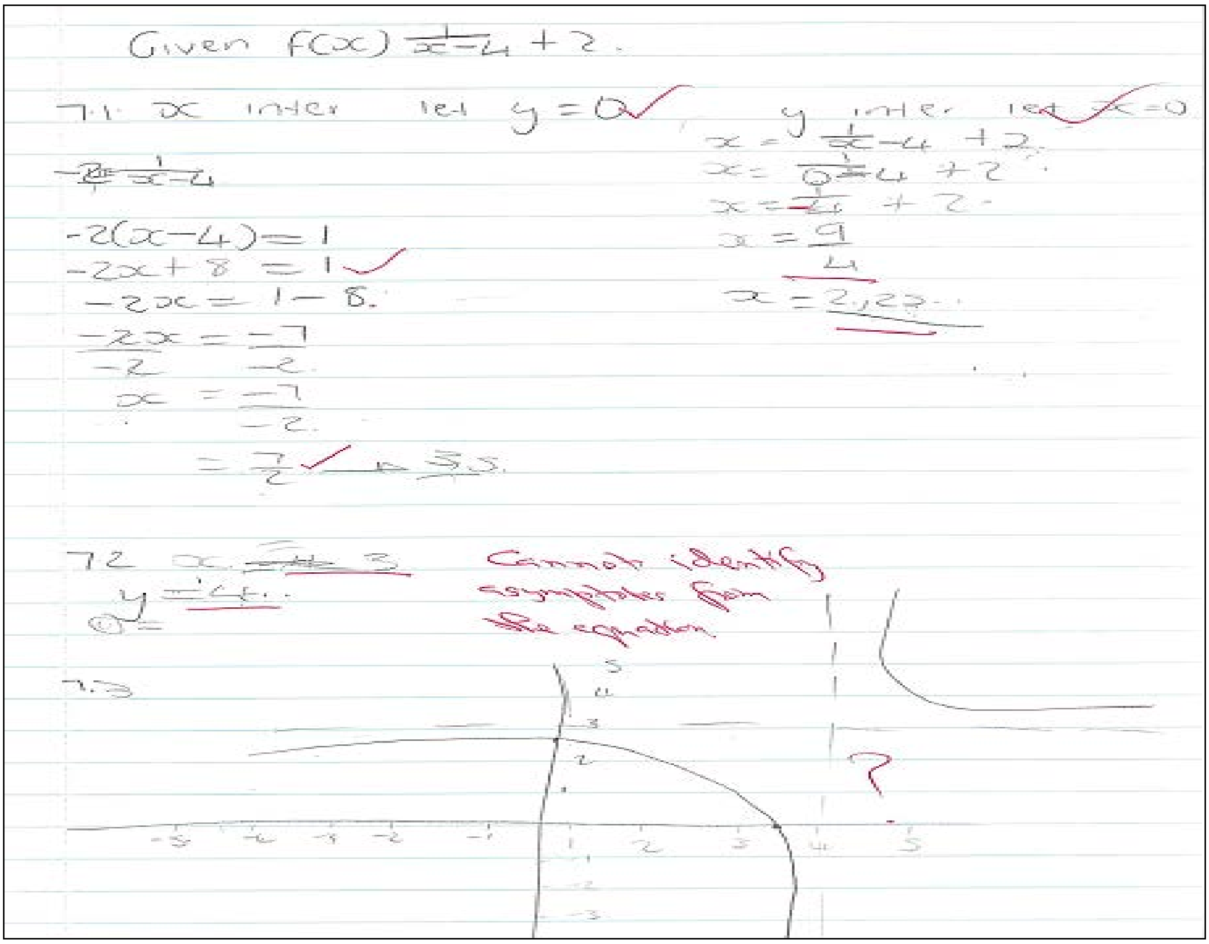


Figure 19: Monga's written examination answers for question 7

Monga managed to calculate the *x*-intercept but failed to calculate the *y*-intercept because he made an arithmetic mistake and also substituted *y* with *x*. The learner could not determine both asymptotes and as a result he drew a wrong sketch. Monga used the procedures unsuccessfully maybe because he did not understand them. When referring to the indicators of (P)APOS theory I place him at the pre-function level. To verify my claim I then interviewed him to explain his solutions and strategies. The interview excerpts that follow capture our dialogue:

Interviewer: What does an intercept mean to you?

Monga: The *y* and the *x*-axis.

Interviewer: Tell me about your calculations here (pointing at his written solutions).

Monga: I just remember that at first I substitute *x* with 0 and then *y* with 0 in the equation but I don't know exactly what will be happening here I just substituted.

Interviewer: Ok. How do you explain an asymptote?

Monga: I don't know!

Interviewer: How did you find these values you wrote?

Monga: Ok! The y-asymptote is this 2 standing on its own. I made a mistake to write 4 here, now I remember. This is what we were taught. For an *x*-asymptote I'm not sure but we were taught in class. For the *x*-asymptote you have to look for the number below, Ok! Like here (pointing at $\frac{1}{x-4}$), the first fraction, we have the number below, for a denominator , we look at the denominator... make it become 0, that is 4, this is my *x* asymptote. I confused my *x* and *y* here but I was told like that. Even my graph is wrong I was using wrong things, eish I made a mistake!

Monga failed to explain intercepts and asymptotes. He could not explain clearly how to calculate intercepts indicating a lack of the basic knowledge about the function-related concepts. He admits making mistakes and also confesses that he does not know these concepts which indicate that Monga is operating at the pre-function level (Breidenbach et al., 1992). At last Monga remembers the procedure of calculating the asymptotes which he explained well but without understanding why it works. This is enough evidence to place Monga somewhere between the pre-function and the action level of (P)APOS theory (Dubinsky & Harel, 1992).

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.

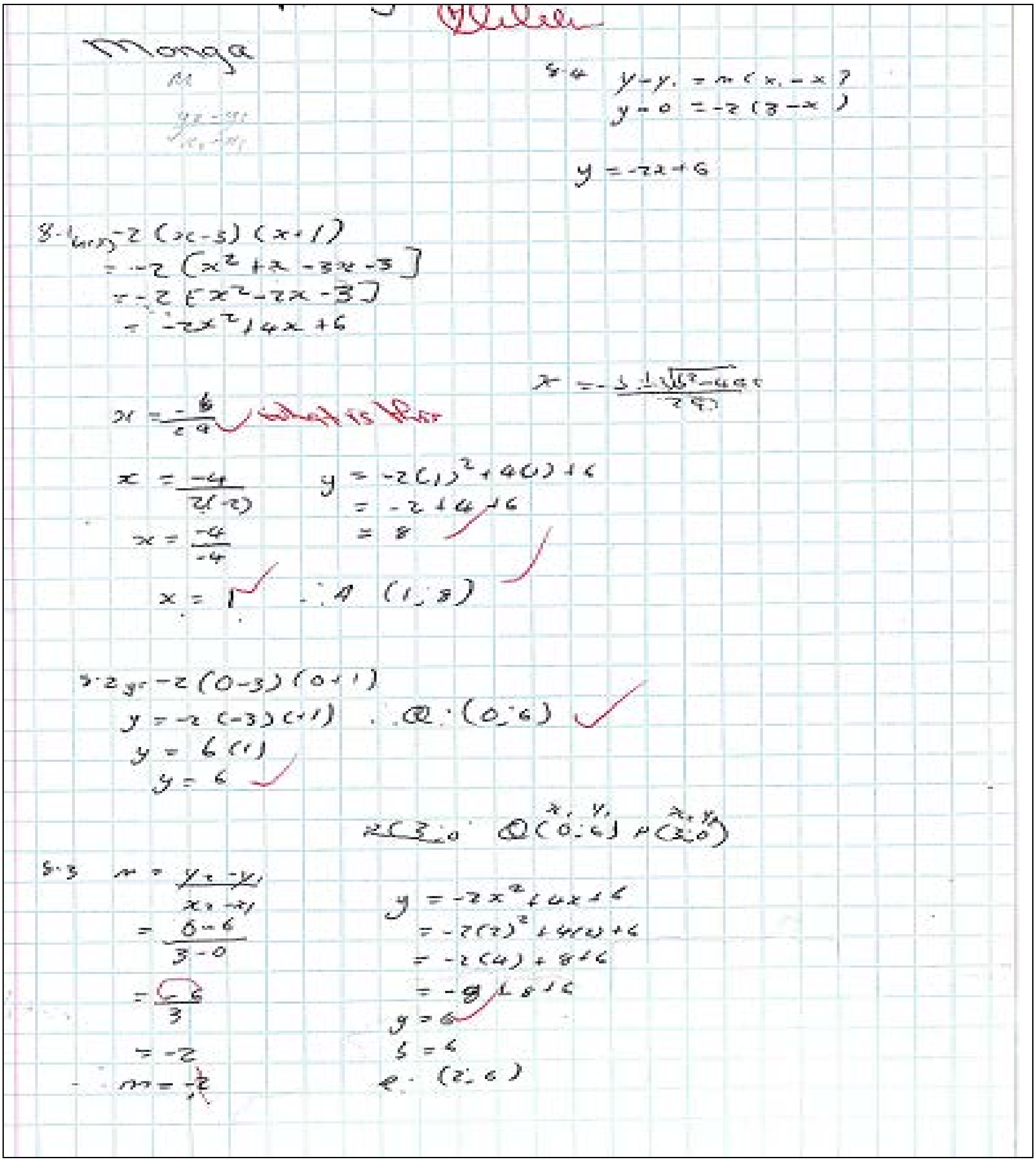


Figure 20: Monga's written examination answers for question 8

Monga's written work was correct and well done. He followed the procedures correctly obtaining correct answers showing that he had mastered these calculations. There is evidence that the learner can refer to a graph to answer questions, though the learner could have memorized these procedures. What the learner wrote on paper and according to the indicators of APOS theory it is understandable to place him at the action level (Dubinsky & McDonald, 2001). However, to confirm the learner's understanding of these concepts and my claim that he is at the action level I interviewed him as shown in the interview excerpts that follow:

Interviewer: How do you explain a turning point?

Monga: A turning point of a parabola is a point whereby your graph turns, whether negative or positive.

Interviewer: Tell me how you calculated the coordinates of the turning point.

Monga: I will be using the turning point formula which is: $x = \frac{-b}{2a}$, first of all …, It originates from the quadratic formula. First of all before I go any further, I collect my data, where I have my a; b; and my c, but here I will be using my b and my a. My b is 4,then I will write the 4,all over the 2 from an original formula multiply by a, my a is -2, then -4 all over -4, then, this will cancel out, I will remain with 1. Using my *x*, where $x = 1$, I will use my original formula to find my *y*, $y = -2x^2 + 4x + 6$ where there is *x*, I will replace *x* by 1, $-2(1)^2 + 4(2) + 6$, then I will multiply out, I mean I add … because here, they stated they will allocate me 3 marks, no need to go step by step when calculating this, because my marks will be 3marks, unless if it was 5 marks, I will go step by step, therefore the turning point is (1;8), this 1 is for *x*, this 8 for *y*-axis.

Interviewer: How did you calculate the coordinates of Q?

Monga: These last questions I was just writing what I had memorized I cannot actually explain how I got them.

Putting together Monga's written and verbal responses brings a similar pattern among learners in my sample that they can carry out a procedure without understanding it as evidenced by his failure to explain how and why the procedures work. The learner failed to explain basic concepts of intercept, asymptote and turning point. My claim was right that the learner could have memorized the solutions to these questions which indicate that he is still operating at the action level of APOS theory. Learners' ability to carry out procedures without understanding them was also reported in a study by Polaki (2005).

Case 4: Teko's understanding of the function-related concepts

Question 7 was testing learners' ability to calculate intercepts and asymptotes from a given equation and used them to sketch the graph of a given function.

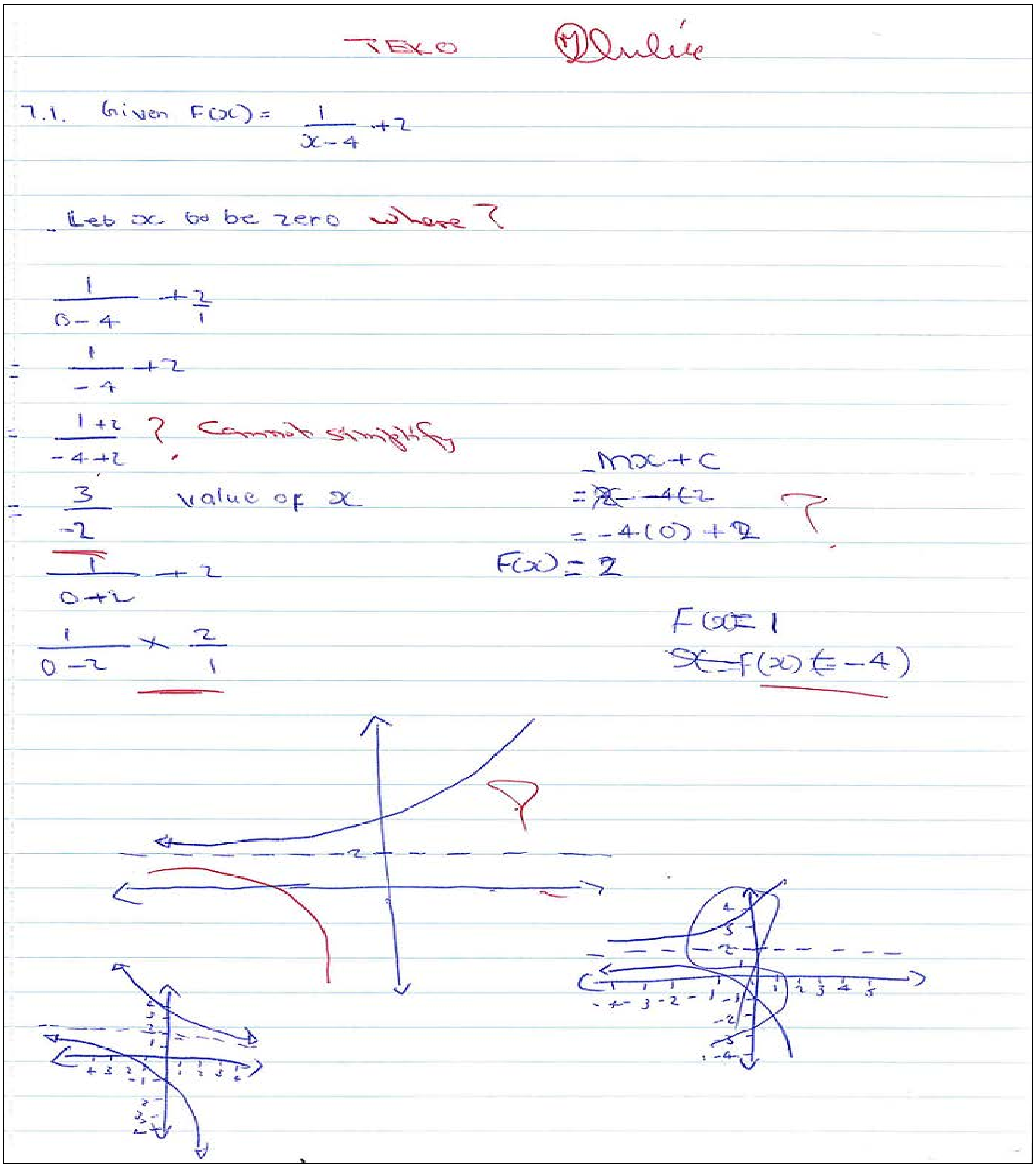


Figure 21: Teko's written examination answers for question 7

Teko could not make a single correct calculation. All his calculations were incorrect, he even failed to use a calculator to add or subtract fractions. The learner did not understand these concepts at all. He has incomplete ideas about intercepts and asymptotes. These are indicators of the pre-function level of (P)APOS theory (Breidenbach et al., 1992). Teko really needs assistance with these concepts for it seems as if he did not understand them when they were taught. I interviewed Teko to further check how he understood these concepts and the difficulties that he was facing. The interview excerpts below were a dialogue I had with him:

Interviewer: Can you explain to me the meaning of an intercept?

Teko: I know it as I calculate it here, but I don't know the meaning of it? I think the intercepts are the points of *y*.

Interviewer: What is an asymptote?

Teko: The asymptote I should find a point where the intercept cannot touch or go beyond.

Interviewer: How do you calculate an asymptote?

Teko: It's going to be 4 ..... (pause) because..... (pause) I don't know.

Interviewer: I can see three graphs here, how did you draw these graphs?

Teko: I will start by drawing the asymptote of *y*, the dotted line $y = 4$. I was told to use a dotted line, so I can easily plot the graph.

Teko's responses above confirm that he did not understand the critical points as they were taught. He has no idea of the meanings and procedures of calculating these critical points. Teko confesses that he does not know them. His explanation of drawing the graph shows that he remembers only a little of what was taught in class. All these are indicators that Teko is operating at the pre-function level of (P)APOS theory (Breidenbach et al., 1992).

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.

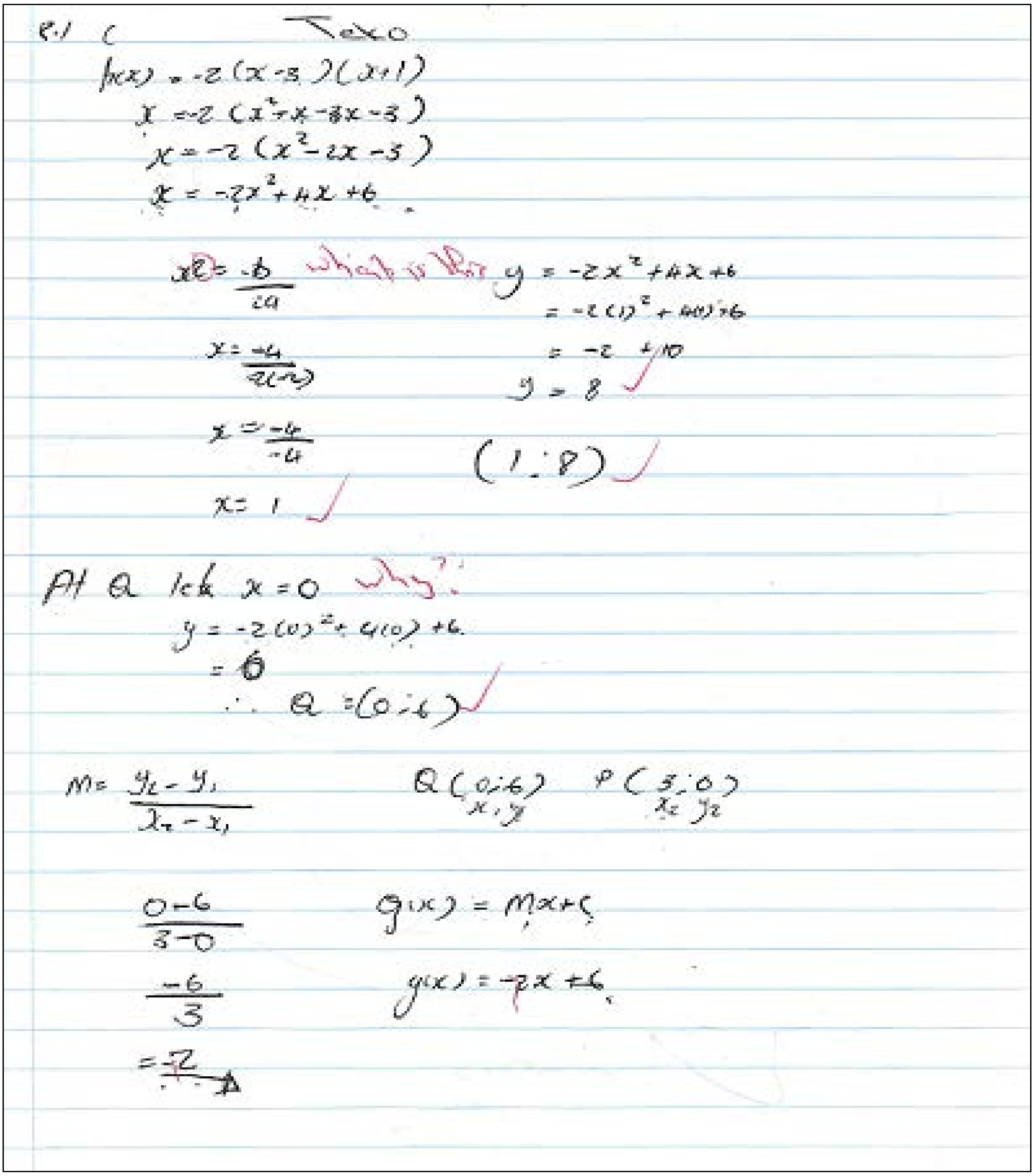


Figure 22: Teko's written examination answers for question 8

Teko managed to answer all the questions successfully except for the value of b which he did not even attempt. He miscopied the formula for the *x*-coordinate of the turning point, he wrote x^2 instead of *x* but went on to use *x* correctly and his work looks a bit organized. The first interview could have impacted on him positively. Teko's ability to follow known procedures is an indication that the learner is operating at the action level of APOS theory (Dubinsky & McDonald, 2001). I interviewed him to check his understanding of the procedures that he followed correctly in his written solutions. My dialogue with him is shown in the interview excerpts that follow:

Interviewer: Well done for getting most of the solutions correct this time. Ok, how do you explain a turning point?

Teko: The turning point is the maximum point that the graph can reach. It is the maximum and the minimum points that the graph can reach.

Interviewer: How did you calculate the coordinates of this turning point?

Teko: Ok! Then I say this -2 is a, from the original formula. This -2 will be my a and the 4 my b and the 6 will be my c. Ok! $\frac{-b}{2a}$ is $\frac{-4}{2}$ = -2, this is equal to $\frac{-4}{-4}$ then *x* = 1, ok! Now I'm calculating the y. Ok! I must calculate here, $-2(1)^2 + 4(1) + 6 = 8$, therefore, ok! the turning point of A (1;8).

Interviewer: Good! Take me through your solution of 8.2.

Teko: Ok! 8.2 is saying calculate the coordinate of Q. Before I calculate, I'm given *x* = 0, so I used the original formula again, then say $-2x^2 + 4x + 6$, ok! I will say where there is *x* I will put 0, ok!
$-2(0)^2 + 4(0) + 6$, then I punch this on the calculator, ok! I don't have to use the calculator because anything times by 0 is 0, so, I will say 'eh' Q(0;6). 'Q' is on the *y* -axis.

Interviewer: Why is *x* = 0 at Q?

Teko: We were told it's always zero there!

Interviewer: Ok. How did you determine the values of m and b?

Teko: This was simple! I used the coordinates of Q I got in 8.2 and coordinates P given and I used the formula for the gradient. For b, eish this one, I don't know. But for the equation of g(*x*) I used m above and for c I used the *y*-value for Q since it is the *y*-intercept.

Teko was beginning to show some indications of understanding the function-related concepts as he showed conceptual understanding of these concepts as evidenced by his explanations of procedures involved in calculating the coordinates of the turning point, gradient and extracting information from the graph and using it in his calculations. He admits that he cannot find b and could not explain why, on the *y*-axis *x* = 0. These are indicators that he is now operating at the pre-function level and the action level.

Case 5: Mat's understanding of the function-related concepts

Question 7 was testing learners' ability to calculate intercepts and asymptotes from a given equation and to use them to sketch the graph of a given function.

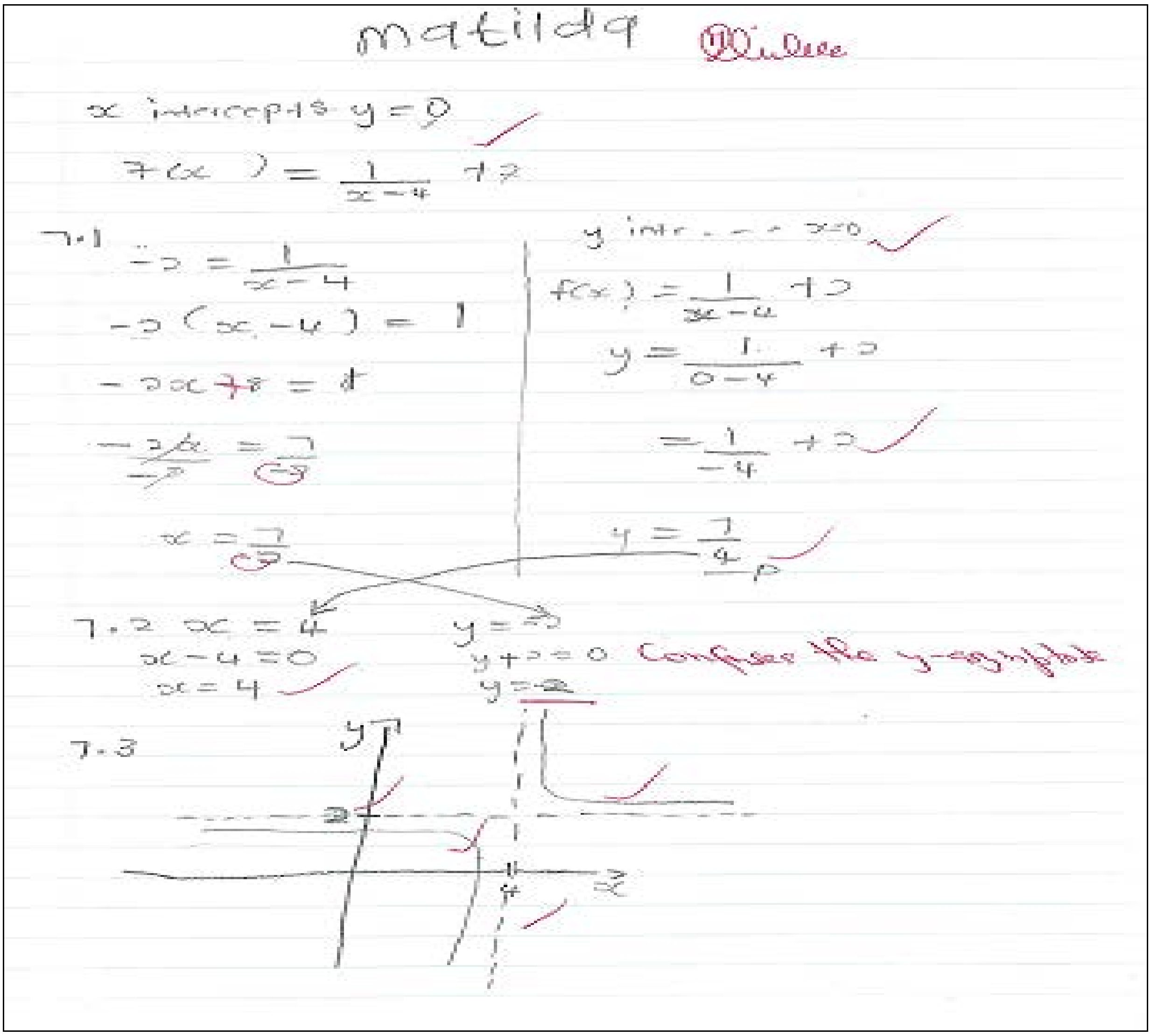


Figure 23: Mat's written examination answers for question 7

Mat wrongly calculated the *x*-intercept using the correct procedure. He also got confused when calculating the *y*-asymptote but on the graph the learner used the correct value of the *y* - asymptote. It seems Mat was remembering his previous solutions from the examination but had forgotten the precise details maybe that is why his graph is correct but some critical points were wrongly calculated. This is an indicator of the person operating at the pre-function level (Breidenbach, Dubinsky, Hawks & Nichols, 1992). I interviewed Mat to confirm my claims and the excerpts below captured our dialogue:

Interviewer: How do you explain an intercept?

Mat: Is a line in a Cartesian plane where there are *x*-axis and *y*-axis, as we are told.

Interviewer: How do you calculate the coordinates of these intercepts?

Mat: Where there is *x*, you have to put 0 to calculate for a *y*-intercept, *x* has to be equal to 0.

Interviewer: Why put $x = 0$ at *y*-intercept?

Mat: I think it is because I didn't try to understand, but then I cram.

Interviewer: What do you understand by an asymptote?

Mat: That is when there is *x*, when I meet the intercept, asymptote of *y*, I can close this one $\rightarrow \boxed{\frac{1}{x-4}} + 2$

$\uparrow$

and that one is the asymptote of *y* (pointing at +2), when I need the asymptote for *x*, I close this one in box $\frac{1}{x-4}\boxed{+2}$ and take the denominator of $\frac{1}{x-4}$ which *x*-4 and equate it to zero then I solve for *x*.

Interviewer: Can you explain to me why you equated *x* - 4 to zero?

Mat: I just know how to write them down from the equation.

Interviewer: Show me how did you draw your graph?

Mat: I make sure that I always put the asymptotes first and make sure the asymptotes and the intercepts don't touch each other....Cartesian plane, they only need *x* and *y*, then when *x* is 4, *y* is here, 'oh', I first have to show the asymptote, *y* is 2, $y = 0$, then $x = 4$.

Mat also failed to explain and justify the procedures that he used claiming that it was how they were taught and that he just crammed. His explanations were quite limited and seemed to have been rehearsed with little understanding. According to the indicators of APOS theory Mat is operating at the action level (Dubinsky & McDonald, 2001).

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.

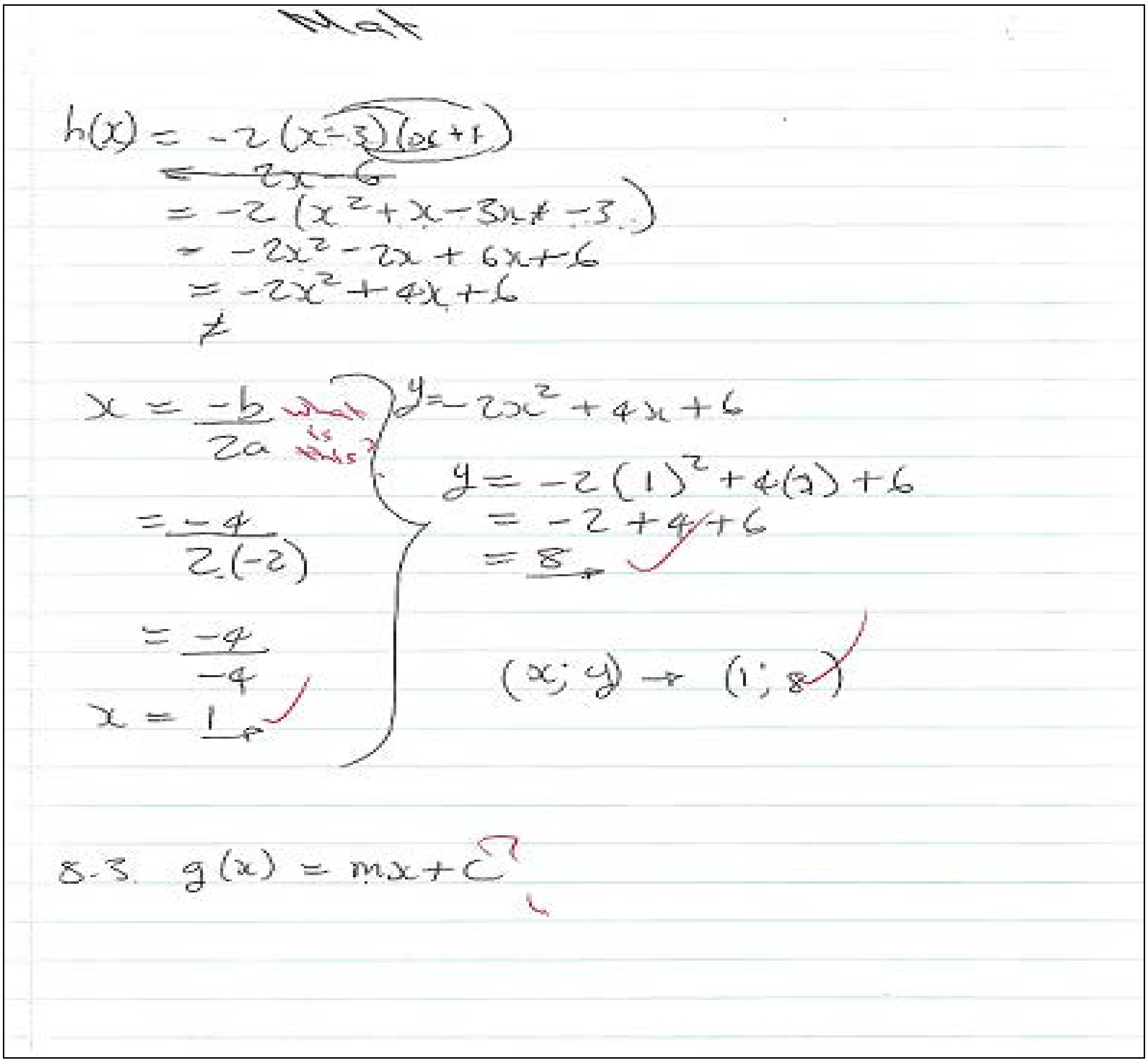


Figure 24: Mat's written examination answers for question 8

Mat only did 8.1 on finding the coordinates of the turning point and just could not continue. He could not read and use the graph to extract information to use in answering 8.2, 8.3 and 8.4. I interviewed him to find out how he reasoned as he solved 8.1 and why he did not attempt the other parts of the question. This is an indication that the learner lacks the basic concepts of gradient and intercept which are used when determining the equation of a straight line which is an indicator that the learner is operating at the pre-function level (Breidenbach, Dubinsky, Hawks & Nichols, 1992). I interviewed the learner to confirm my claim that he is operating at the pre-function level of (P)APOS theory as shown in the interview excerpts below:

Interviewer: What is your meaning of a turning point?

Mat: The turning point is where the graph curves, that means where it changes directions.

Interviewer: How do you calculate the coordinates of the turning point?

Mat: What are we going to do here is that we 'gonna' multiply the things in brackets and then, 'ja', let's do it! We 'gonna' take the -2 as it is and put it down here, then open brackets, then I will say x times x is x^2, then I will say, ok! This is positive, then I will say x times 1 is x, then -3 times x is-3x, then -3 times 1 is -3 close [brackets], then next step, the -2 will remain as it is, I will put it down here, oh! Here there are like terms, the x^2 then, x - 3x is -2x, then the -3 close [brackets]. Now we 'gonna' multiply, ok! The -2, (-2x times x is -2x^2), (-2 times -2x is +4x), (-2 times -3 is +6), so, this will be our original formula, ok! We 'gonna' calculate the turning point, we 'gonna'say $x = \frac{-b}{2a}$

Interviewer: Where is this formula coming from?

Mat: I just memorized it!

Interviewer: You did not attempt the other questions, why?

Mat: …eeh I can't proceed.

Mat had a vague idea of what a turning point is. His explanation of calculating its coordinates was pleasing though at the end he talked of memorizing this procedure and the use of a formula $x = \frac{-b}{2a}$. The learner failed to do the other questions saying that he cannot proceed. The implication of this is that maybe he forgot what he had memorized because at least he was supposed to show some reasoning and attempt something even though it would be wrong. This memorization of procedures by Mat places him at the pre-function level of (P)APOS theory (Breidenbach et al., 1992).

Case 6: Edy's understanding of the function-related concepts

Question 7 was testing learners' ability to calculate intercepts and asymptotes from a given equation and used them to sketch the graph of a given function.

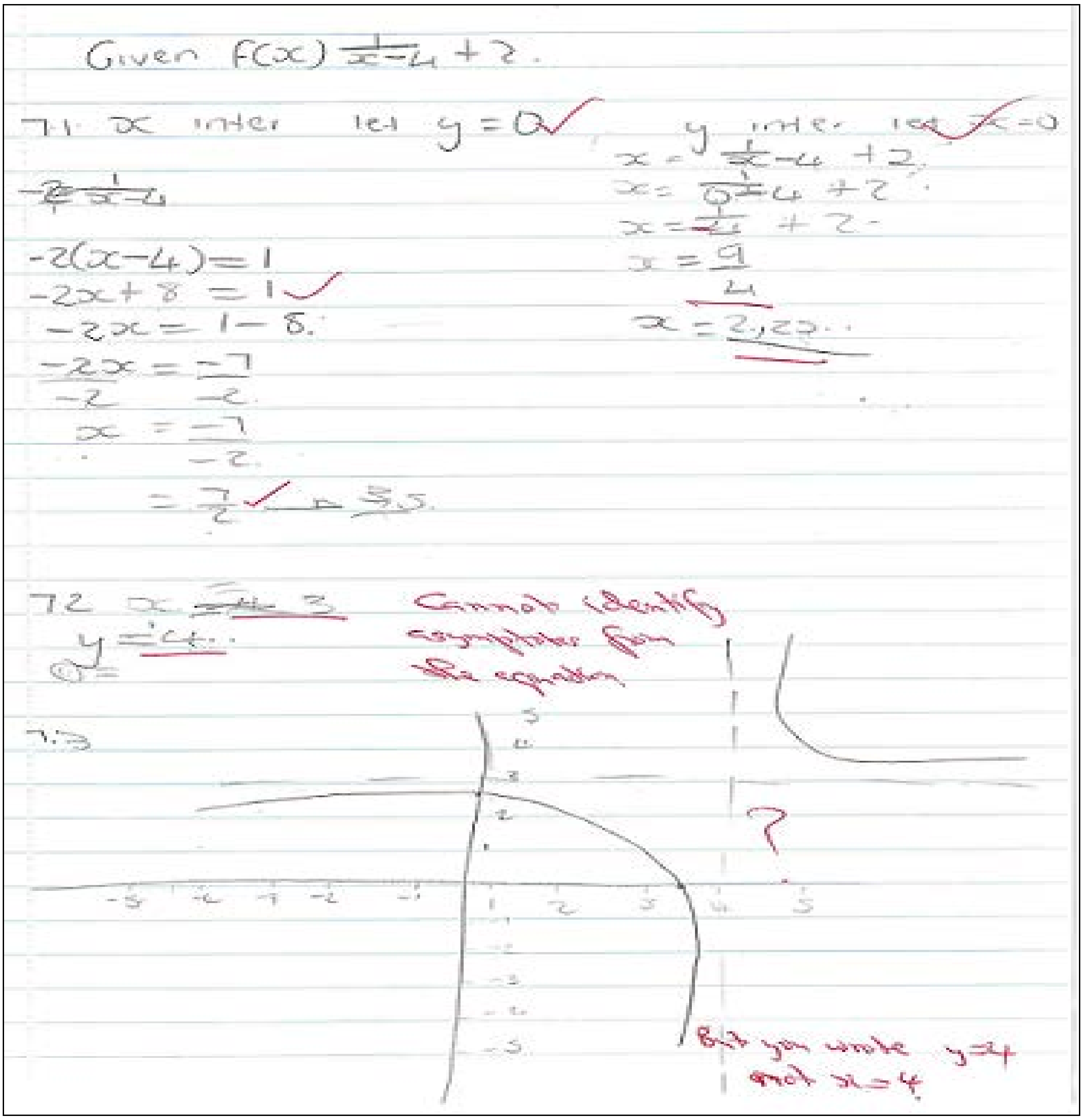


Figure 25: Edy's written examination answers for question 7

Edy failed to calculate the *y*-intercept and both asymptotes showing little understanding of these concepts. The asymptotes that he drew were different to what he had calculated showing that the learner had memorized these solutions preparing for this interview session. This memorization of procedures without understanding them by Edy makes him to be placed at the pre-function level of (P)APOS theory (Breidenbach et al., 1992). To elicit the learner's understanding of procedures and concepts involved in this question I interviewed him and our dialogue was captured in the following interview excerpts:

Interviewer: In your own words how do you describe an intercept?

Edy: The point where the *y* or the *x* is, that is the intercept. In the *y*-intercept, I think is the point y and in the *x*-intercept, I think is the point *x*.

Interviewer: How do you calculate the coordinates of the intercepts in 7.1?

Edy: I will put $y = \frac{1}{x-4} + 2$, so *y*-intercept, $x = 0$, then $y = \frac{1}{0-4} + 2$

Interviewer: Why is $x = 0$ at the *y*-intercept?

Edy: 'Oh' I don't know, the only thing I know is that, when you are calculating the *y* -intercept, $x = 0$, and for the *x*-intercept, $y = 0$.

Interviewer: Ok. Explain to me what an asymptote is, as you understand it?

Edy: The asymptote is the marginal line where the graph will be, as are those ones we have drawn, usually plotted with dotted lines.

Interviewer: How do you calculate the asymptotes?

Edy: I don't know how to find the asymptote but I just think of cramming.

Interviewer: Tell me what did you cram?

Edy: I forgot sir!

Interviewer: Ok. How did you draw the sketch then?

Edy: When I plot my graph, it must not touch the asymptote points. 'Oh' the critical points, ok! we 'gonna' label the graph, here is the *x* -axis and here is the *y* -axis, ok! first I 'gonna' put my asymptote before plotting the other points on the *y* -axis, my asymptote is +2 and then, plot a broken line and then on the *x* -axis, my asymptote is 4, which I plot a broken line and then, already this one say: $y = 3$ and here $x = 4$.

Interviewer: But you wrote $x = 3$ and $y = 4$. Are you changing it now?

Edy: I made a mistake when I was writing.

From Edy's responses above it shows that the learner does not know the meaning of an intercept and asymptote. He only remembers that at the *y*-intercept $x = 0$ and at the *x*-intercept $y = 0$ without understanding why it is so. The asymptotes that he wrote are now different to the ones he put on his sketch graph and admitted to having made a mistake. Edy's description of sketching the graph is not based on understanding where to put the critical points first and then drawing. His ideas are still vague indicating that he is operating at the pre-function level of (P)APOS theory (Breidenbach et al., 1992).

Question 8 was testing learners' ability to use a graphical representation and to translate from the graphical to the symbolic representation.

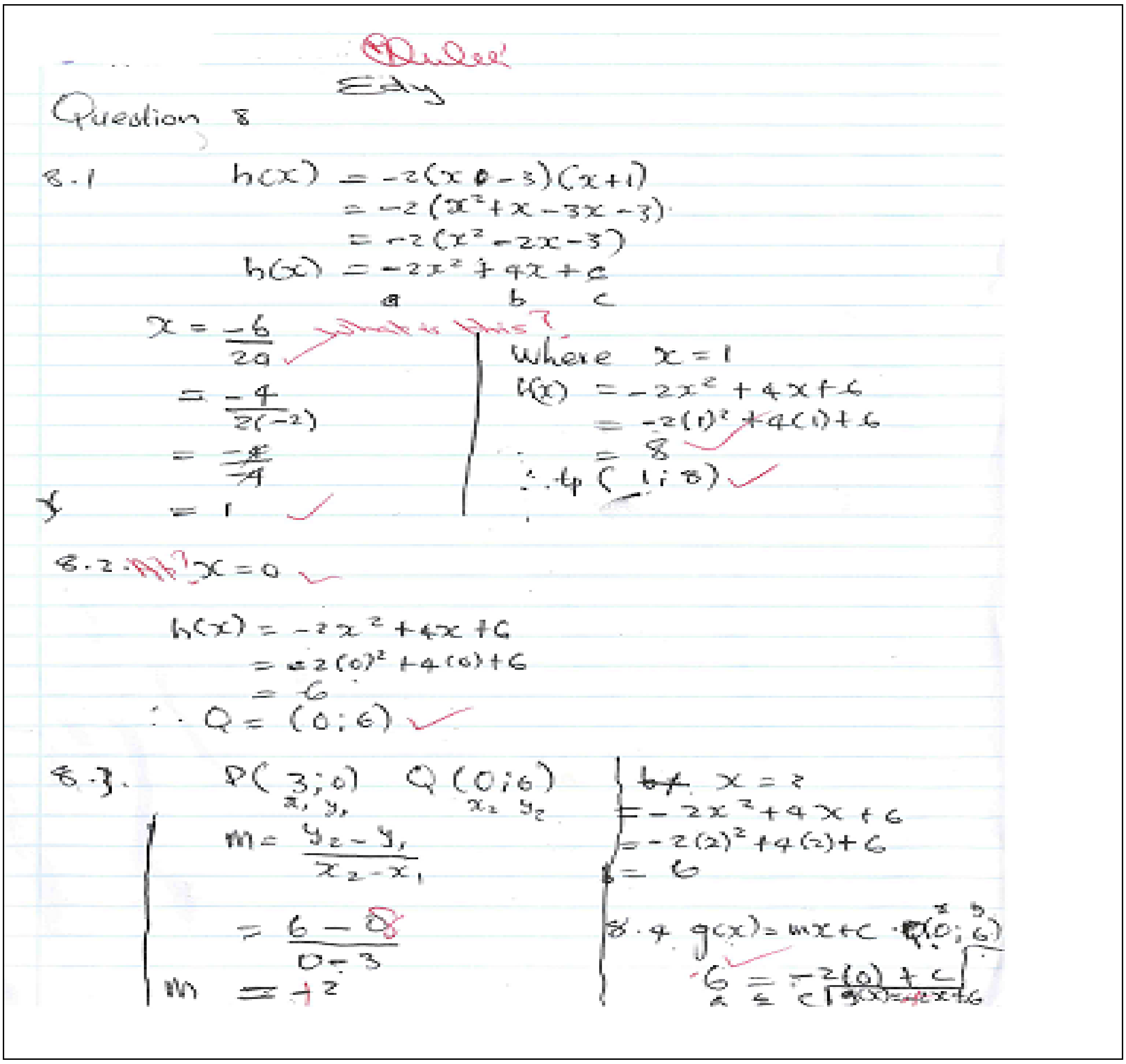


Figure 26: Edy's written examination answers for question 8

Edy's written answers for question 8 were correct though he did not find the value of b in 8.3. The presentation of the work shows that he can follow accurately the procedures involved in solving these questions. The ability to follow procedures is an indicator that a learner is operating at the action level of APOS theory (Breidenbach et al., 1992). To confirm this claim and whether the learner understood these procedures I interviewed him as shown the interview excerpts that follow:

Interviewer: Please go through your solution to 8.1 with me I want to understand how you calculated the coordinates of the turning point.

Edy: I will use the formula $x = \frac{-b}{2a}$ where b = 4 and a = -2 from h(x) = $-2x^2 + 4x + 6$, we were told that *x* is the *x*-coordinate of the turning point then we substitute this value in the original equation to find the corresponding value of *y*.

Interviewer: How did you calculate the coordinates of Q?

Edy: I will replace the *x* by 0. This is obvious that this will be equal to 0 because we are on the y-axis and then my answer is 6, therefore I will answer the question, therefore Q is equal to (0;6). So my c the *y*-intercept is 6 already I will write the original formula g(*x*) = m*x* + c, then I will choose one point , is either P or Q, then, I have decided to take Q, whereby (0;6) and I will write *x* and write *y* there, and where there is *y*, 'oh' where there is g(*x*) I will put 6 =, my m, is -2, my *x* is 0, now I want the value of c, 6 = c, which means c = 6, then write g(*x*) = -2*x* + 6, then, this has been proven.

Interviewer: But you did not find the value of b. Why?

Edy: That one I don't even know where to start sir. I thought b was c at first but eh its not.

Edy's responses in this interview showed an understanding of the procedures as he could explain clearly what he had done and why he was doing it in that way. Though he did not answer all the questions, Edy's ability to use and explain procedures that he used places him at the action level (Breidenbach et al., 1992).

5.3.2 Cross case analyses of learners' calculations and explanations of the critical points

The examination question 7 had asked learners to calculate the *x* and *y* intercepts. My interview questions were meant to compel learners to reveal their conceptual understanding of the *x*-intercepts and *y*-intercepts and the procedure of calculating them. The intention was to determine the conception levels learners were operating at and then help them to move up towards the schema level. Discussions in Section 5.3.1 indicate that on question 7 two learners were operating at the action level, two learners were operating between the pre-function and the action level and the other two learners were entirely at the pre-function level. Two out of the six learners did not calculate the coordinates of the intercepts correctly. One out of the six learners gave the correct explanation of an intercept while the other five learners could not give a precise and complete explanation of intercepts. Based on both learners' written and verbal responses from cases 1 to case 6, I can conclude that learners can calculate the intercepts but do not understand the procedure. They cannot explain why $x = 0$ on y-intercept and $y = 0$ on the x-intercept. They can carry out the procedure without an idea of why and how the procedure works. This is also documented in literature (Polaki, 2005).

After finding the critical points in question 7, learners were required to sketch the graph. To sketch the graph they needed to know where to plot each critical point and how to join the plotted critical points. As a result I asked them to explain this procedure and their responses revealed that 4 out of the 6 learners could explain how they sketched the graph using the critical points they had calculated. Coco and Monga were not sure as to how to start to sketch the graph. I discovered that the other four learners can correctly sketch the graph but could not explain precisely what they did. This indicates that they were operating at the action level while Coco and Monga were operating between Dubinsky and Wilson's (2013) pre-function level and the action level.

In question 8 learners were asked to determine the equations of the asymptotes of the given equation of a function. During the interviews I asked the learners to explain the meaning of an asymptote before determining the equations. From their responses, discussed in cases 1 to 6, no learner managed to correctly explain an asymptote leading to the conclusion that learners do not have a clear understanding of an asymptote. In addition, responses of all the six learners show that learners are not aware of the conditions of a line being an asymptote. They just memorized the procedure of finding the asymptote. This indicates that learners have insufficient knowledge related to the concept of asymptote and are still operating at the action level.

Question 8 had also asked learners to sketch the graph showing all the critical points. The other critical points had been found in the earlier questions and only the turning point was remaining. I asked the learners to reveal their conceptual understanding of the turning point and the calculation of its coordinates which helped me to determine their conception levels. Based on their responses I can conclude that all six learners had a vague idea of what a turning point is, and their ideas were not complete, and needed to be refined. Their responses also indicated that they just memorized that the *x*-coordinate of the turning point is $\frac{-b}{2a}$ without understanding where this formula comes from and how it works. Although all six learners could calculate to some extent the coordinates of the turning point they failed to explain clearly how they were calculating them and why they were calculating them in the way that they were doing indicating that they were operating at the action level.

The two questions on functions in the June 2011 mathematics examination tested learners' understanding of the process of translating from equation to graph and from graph to equation respectively. Thus, for learners to succeed in this two-way translation they needed to understand the critical points because they connect these two representations of the function concept. Analyses of learners' individual written and verbal responses have revealed that all six learners had difficulties in answering questions that refer to the drawn graph as they could not deduce the critical points from the sketched graph and use them to determine the required equations. This indicates that learners have insufficient knowledge on the connection between the graph and the equation as different representations of the same function. This is also documented in literature (Sfard, 1992; Sierpinska, 1992; Knuth, 2000; Gagatsis & Shiakalli, 2004; Abdullah & Saleh, 2005; Akkoc & Tall, 2005; Cunningham, 2005).

5.3.3 Focus group interviews

I had planned to group learners in the sample into two groups of six each to take advantage of learner to learner interaction, but since some learners had dropped out and others were not consistent in their attendance I was left with one group of six learners. However, on the day of focus group interviews again one of the six learners in the sample was absent. I allowed the five learners who were present to share their difficulties, to triangulate data from the initial individual interviews. I initiated group discussions as follows:

Interviewer: Discuss as a group the difficulties that you faced when you were answering the two questions in the June examination. Tell others about the difficulties that you faced and the kind of assistance that you need.

Edy: Guys I get confused when asked to sketch the graph, I don't know whether to use the asymptote or intercept for this one (question 7). When exchanging this, when you are finding the intercept, usually is difficult for me because in 7.3, it is easy for you is to just take the points and allocate them.

Teko: Ya I have the same problem on finding the intercepts. Because, sometimes I think I know what I'm doing, but I got lost, like here, I got lost from the beginning, so which means the whole answers there went very wrong.

Mat: 'Ja' like this point here, when they say: calculate the value of b, already they have given you the other points, I don't know where should I substitute from? Or should I use g(x) = mx + c, I think …which is x, and b is suppose to be the y-value, what gives me the problem is that, why did they give me the 2, maybe if they gave me , there may be the m, the m must be maybe here.

Coco: Maybe 8.3, 8.3 is a bit tricky because, here, you cannot find m on your sketch, because, here it says that the sketch below, not drawn to scale, then you will now understand that you will be using your sketch to answer the questions that follow. 'Oh' here is because, can't write … I'm not sure if I can write the graph for the equation.

Monga: I don't know whether to use the asymptote for the dotted lines or for the graph. It is difficult to find the coordinates of Q and to find the values of m, because we don't have the coordinates of Q …I have a problem with mixing graphs. I have problems with these drawn graphs I don't even know where to start.

Learners' responses presented in the interview excerpts above confirm that the five learners had difficulties in extracting information from a drawn graph; sketching the graph showing all intercepts, asymptotes, turning points, and to translate from graph to equation. These difficulties support findings in the studies carried out by Sierpinka (1992) and Knuth (2000). The following major theme emerged when I matched the difficulties from these initial interviews and group interviews with the ones from my literature review: *Learners cannot flexibly translate from the graphical to the symbolic (equation) representation of the function concept*. Moreover, the interviews I carried out with grade 11 learners in my sample also revealed that the learners had a limited understanding of the critical points which made it difficult for them to translate from graphical to symbolic representation and vice versa. Making these links have however become associated with a foundational understanding of the function concept (Eisenberg & Dreyfus, 1991) which includes its key idea and characteristics. This means that learners cannot manipulate representations of the function concept without first understanding its key idea and characteristics which are closely linked to the definition of the function concept (Polaki, 2005). As a result I divided the major theme into two problem areas: *Problem Area 1*: Learners did not understand the key idea (that of a dependence relationship) of the function concept. *Problem Area 2*: Learners cannot flexibly translate from the graphical to the symbolic (equation) representation of the function concept. According to Even (1988) a situation which enables the transition between multiple representations of functions geometric, numeric, and symbolic may lead to conceptual understanding. The nature of *problem area 1* and *problem area 2* compelled me to classify learners' difficulties under the definition of the function concept and representations. The classification of learners' difficulties was not meant to separate them conceptually but to allow an in-depth study of these difficulties with the intention of forging strategies of overcoming them and improving learners' understanding of the function-related concepts in the classroom. As a result I dealt with, and presented these two problem areas separately starting with the key idea of the function concept and then its representation.

5.4 Problem Area 1: Understanding of the function concept

5.4.1 Phase 1: Problem identification

The following section presents *problem area 1* which is closely related to the key idea of the function concept. Learners in the sample had been taught about the function concept by their teacher prior to the initial interviews which indicated that learners had difficulties with the aspects related to the function concept. This prompted me to conduct individual interview session 3 to determine: what learners understand by the function concept, reasoning, and weaknesses in learners' understanding. Based on this I looked at how instruction can be designed to improve learners' understanding of the function concept.

Individual interviews (session 3) on the definition of the function concept

Core interview questions related to the definition of the function concept were as follows:

- What do you think of when you hear the word function in mathematics?
- Using your own words and any diagrams you need to express your ideas, explain the meaning of the word function.
- List and explain any special properties of a function that you can recall and explain how you would illustrate them.
- Give me two examples of a function and two examples of non-functions.
- How do you distinguish a function from a non-function?

- Explain in your own words what you understand by an independent variable and a dependent variable?
- How do you identify the independent and dependent variables in a given functional relationship?
- A function has a domain and a range (co-domain). Explain in your own words the meaning of a domain and range.
- Where do you use functions in real life? You can use an example to explain the application of functions in real life.

Note: These questions were guided by Markovits, Eylon and Bruckheimer (1986) and Sajka's (2005) knowledge and skills that a learner ought to have to be considered as understanding the definition of function. I had also confirmed with the mathematics teacher that learners were taught these function definition related concepts in class.

Research question 1: How do learners understand the function concept?

This research question was answered by analyzing learners' responses with respect to what they said a function was, the examples and non-examples that they formulated, the ways they used to identify the dependent and independent variables of a function. Examples of functions and non-functions that learners gave could indicate the extent to which they understood the function concept. If they are able to formulate examples of functions and non-functions it implies that they can recognize a function using its properties. This ability can in turn help identify the dependent and independent variables in a given function and then distinguish a function from a non-function. The following excerpts provide the evidence of learners' level of understanding of the definition of the function concept. When I asked learners to tell me what comes to their mind when they hear the word function or to explain the meaning of the word function in mathematics, they gave me the following responses:

Coco: ...something having an input, the output must have a relationship, maybe - when you, like, when you are working somewhere, when you get paid, it must have a corresponding, this two must correspond the hours, the hours and the money that you get paid.

Diva: Function is a relationship between two things or variables, for an example a stove and a pot there is a relationship between them.

Edy: If I hear the word function it is a relationship between two points.

Mat: Oh as I have said earlier it actually means the relationship between two variables, ja relationship.

Monga: I think it is a relationship between two friends or intercepts.

Teko: I think function is a number which represents the range and domain. Function is something that tells us about range and domain.

Learners gave the following examples of functions and non-functions:

Coco: when you are working somewhere, when you get paid, it must have a corresponding, this two must correspond the hours, the hours and the money that you get paid.

Diva: Okay a cell phone has one number. One cell phone has one number. You can't share one number with two phones... you can't use one number for the two cell phones.

Edy: I remember we were told that area of a rectangle is a function of its length and breadth.

Mat: Oh a number that is depending on two numbers like $6 = 2 \times 3$

Monga: The birth of a child is a function.

Teko: Two examples of a function. A shop owner depends on the customers for buying.

How learners distinguished a function from a non-function:

Coco: You can distinguish a non function from a function; a function like a domain can only sometimes correspond to the other element or two, both domain can correspond to one, they can share the element but two of the …, ok! 1-1, and then …, but …, one domain cannot have other two elements.

Diva: I will say here is a graph, then if it is a true function it has to cut in one, but if it is a non-function this would be like maybe into the circle, then it will cut in two points like this.

Edy: I am not sure but we were taught in class.

Mat: A function takes one, and then non function takes many.

Monga: For a non-function one component open an event and appear maybe like more than once, then for the other one appear only once for a function, the first component to appear once.

Teko: A function has one domain but can share ranges. A non-function it can have a domain and 2 ranges.

Learners' explanations of dependent and independent variables

Coco: An independent variable, I think is the variable that, even if you can change something, it does not affect it, but the dependant variable, if you affect the independent, also get affected because the dependent depend on the independent variables.

Diva: I think the …, dependent variable is the one that changes and the independent variable do not change. An independent I think it is something that stand alone, it doesn't depend on anything.

Edy: Independent variable, okay ja let me talk of a tree, the independents in a tree are the water and the soil. The tree depends on the soil. That is why it is independent. Okay that is the independent. The tree cannot live without the water.

Mat: A dependent is something that cannot really work without maybe the help of something. It depends on something else in order for it to work. Okay on the independent I think it is - okay maybe this thing doesn't need anything else; it is just okay, when it doesn't want any help, and a dependent …

Monga: I think an, okay an example will be like plants needed sunlight for growing so I will take plants as the independent value, it rely on the sun and the sun is the dependent because it rely on itself.

Teko: The first one that the student depends on his parents for money. The other one is independent or a wife does not depend on her husband for money.

Learners' identification of dependent and independent variables in a function

Coco: Because both of us we are dependent on the name, maybe someone want to say… maybe that is property, like you can refer, someone is looking for me -by my name. Maybe we will have different surnames but we can share my name … Ok! I will be like a function, someone like me, I only have the Identity number, I am dependant on the Identity number, which is how you can find me, with the identity number or a car and the engine, the car is dependent on the engine.

Edy: I think it is based on the independent and dependent variables like one for example, we as people we depend on water because you can't live without it and water can't depend on us as people.

Mat: Another example can be a tree. It can also depend on water or soil.

Monga: The apples are dependent on the tree because they can't live on their own but they need a tree.

Teko: A shop owner depends on the customers for buying.

The learners' responses address both sub-questions i and ii of the first research question:
i) *What are grade 11 learners' current understanding (concept images and reasoning) of the function concept?*
Learners' concept images of a function that emanated from the above excerpts include:

- A relationship between things, points, friends or variables,
- A correspondence between input and output,
- A connection between input and output,
- Something that tells us about domain and range and
- A graph cut in one place by a vertical line.

Learners' reasoning about the definition was depicted in the examples and non-examples that they gave and in their inability to identify dependent and independent variables in a function. The above interview excerpts indicate that:

- Learners could only give examples of functions that are narrow, vague or inaccurate,
- Learners could not give examples of non-functions and
- Learners had a narrow view of dependent and independent variables and could not differentiate the two in their examples.

These concept images and reasoning of learners are also reported in literature (Dubinsky & Harel, 1992; Breidenbach et al., 1992; Hitt, 1998; Polaki, 2005). They indicate that learners do not understand the key idea (that of dependence relationship) of the function concept.

Learners' responses were categorised into (P)APOS levels depending on the indicators they displayed in their oral and written responses and in terms of what they could do and what they could not do. These indicators were explained in detail in Chapter 3 Section 3.5. The table below shows that, for questions 7 and 8, 1 out of 6 of the learners in the sample was operating at the pre-function level, three learners were between the pre-function and the action level while two learners were at the action level because of the indicators they exhibited in their responses.

Table 8: Learners' initial (P)APOS theory conception levels on the function concept

<table>
<tr><th>LEARNER</th><th>(P)</th><th>A</th><th>P</th><th>O</th><th>S</th><th>INDICATORS OF PAPOS THEORY CONCEPTION LEVELS</th></tr>
<tr><td>Monga</td><td colspan="2">X</td><td></td><td></td><td></td><td>Has the basic idea of a relationship
Cannot formulate example and a non-example
Cannot explain and identify dependent and independent variables in his own example</td></tr>
<tr><td>Coco</td><td></td><td>X</td><td></td><td></td><td></td><td>Can use the one-to-one correspondence property
Formulated reasonable examples of functions and non-functions
Can explain and identify dependent and independent variables</td></tr>
<tr><td>Mat</td><td>X</td><td></td><td></td><td></td><td></td><td>Basic idea of a relationship
Cannot formulate examples of functions and non-functions
Cannot explain and identify dependent and independent variables</td></tr>
<tr><td>Diva</td><td></td><td>X</td><td></td><td></td><td></td><td>Basic idea of a relationship
Gave vague example of a function but used vertical line test to give an example on a non-function
Shallow explanations of dependent and independent variables</td></tr>
<tr><td>Edy</td><td colspan="2">X</td><td></td><td></td><td></td><td>Basic idea of a relationship
Recall verbatim examples of functions and non-functions
Cannot clearly explain and identify dependent and independent variables</td></tr>
<tr><td>Teko</td><td colspan="2">X</td><td></td><td></td><td></td><td>Recall some basic definition aspects
Gave vague examples and non-examples of functions
Vague explanations of dependent and independent variables</td></tr>
</table>

What are the weaknesses in learners' understanding of the function concept?

I was able to detect that there are weaknesses in learners' understanding because their answers showed that they do not have indicators to show the schema level. Analysis of learners' interview responses above revealed the following weaknesses, difficulties and misconceptions:

- 4 out 6 learners can only mention that a function is a relationship without explaining the nature of the relationship indicating that they were operating at the action level for that particular question;
- Their explanations are based on examples of which some of the examples are vague like the one of a stove and pot indicating conceptions at the action level;
- 5 out of the 6 learners could not identify the variables (dependent and independent) in their examples of a function;
- All the six learners could not use their definition of the function concept to formulate examples and non-examples of functions;
- They cannot use their function definition to determine whether given relationships are functions or non-functions; and
- They confuse the uniqueness condition of the function definition with the notion of one-to-one correspondence.

The literature also revealed that it is common for learners to have the weaknesses, difficulties and misconceptions listed above (Sfard, 1991; Dubinsky & Harel, 1992; Breidenbach et al., 1992*;* Hitt, 1998) which indicate that there is need for intervention to help learners overcome these obstacles to their understanding. Intervention was done in phase 3 of my adapted model (Figure 7) through the use of tentative products and theories.

5.4.2 Phase 2: Development of interventions

Phase 2 addresses research question 2:

How can instruction be designed to improve learners' understanding of the function concept?

According to the theoretical framework for this study I brought in and used the following RME's learning and teaching principles as propounded by Treffers (1991):

- learning mathematics is a constructive activity and instruction should start with *concrete or everyday activities that are experientially real to learners;*
- learning of a mathematical concept or skill is viewed as a process which is often stretched out over the long term and which moves at various levels of abstraction and problem solving activities help learners to move through these various levels;

- serious attention has to be paid to a *learner's own constructions* and *productions* and the learners must constantly have the opportunity to reflect on learning strands that have already been encountered;
- working in groups is important as learners have the opportunity for the exchange of ideas and arguments so that they can learn from others;
- learning strands should be intertwined and cannot be dealt with as separate entities and *intertwining* of learning strands is exploited in problem solving; and
- learning should focus on understanding of processes (procedures) rather than memorizing them.

A central construct in RME is progressive mathematisation. Mathematicians take subject matter from reality and organise it according to mathematical patterns in order to solve problems from reality (Gravemeijer, 1994). “There is no mathematics without mathematising” (Freudenthal, 1973, p. 134). There are two types of mathematisation: “horizontal mathematisation, which refers to modelling a problem situation into mathematics and *vice versa*, and vertical mathematisation, which refers to the process of reaching a higher level of mathematical abstraction” (Drijvers, 2002, p. 192). The idea that moving from the world of life to the world of symbols was horizontal mathematisation while operating within the world of symbols is vertical mathematisation was emphasised by Freudenthal (1991). I then related actual examples in my lessons to the learners’ needs which I identified above in the form of weaknesses, difficulties and misconceptions and their various conception levels. These weaknesses, difficulties, misconceptions and their conception level indicated that there was need for intervention in order to help learners reduce their identified difficulties and misconceptions and help them to progress up the conceptual ladders of (P)APOS theory from the current pre-function and action levels to the schema level. To improve learners’ understanding of these concepts I developed my teaching experiments following the stages of conceptual development suggested by Markovits et al (1988). Learners’ concept images and reasoning indicate that they do not understand the basic idea (that of dependence relationship) of the function concept, they could not identify dependent and independent variables in a given functional relationship and they could not formulate their own examples of functions and non-functions. I formulated the following HLT for the first teaching experiment of a series of teaching experiments to help learners understand the function concept:

5.4.2.1 Teaching experiment 1

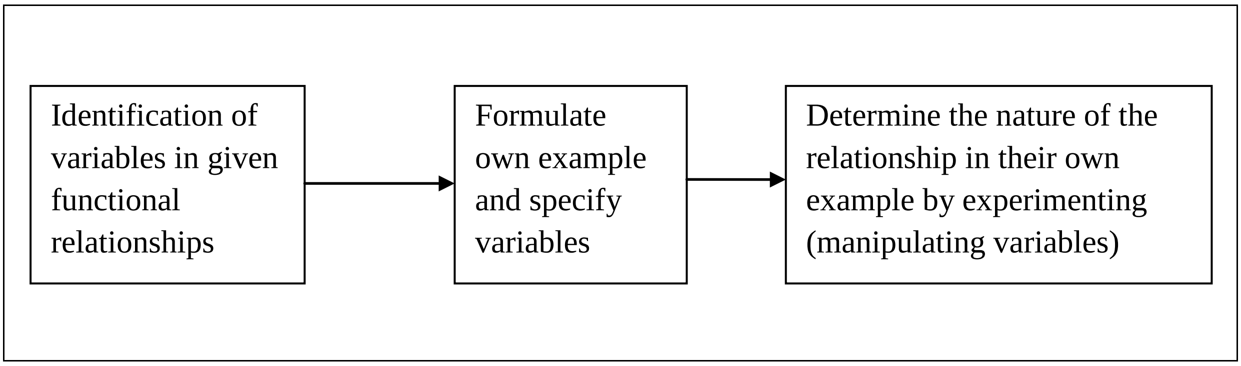


Figure 27: HLT for developing an understanding of the function concept

Evidence from excerpts above suggests that five of the six learners in the sample were operating at the action level of APOS theory. I designed activities (instructional material) based on the principles of RME to help shift learners from the action level to the schema level which is the main aim of this study.

At the end of the intervention learners should be able to:

i) explain the basic idea behind the function concept;
ii) state and explain examples of variable quantities;
iii) identify the dependent and independent variables from given functions; and
iv) explain in their own words, what a function is.

My aim was to assist learners to understand how the concept of a function exists in real life outside of the mathematics classroom. I used some real world examples of functions and some examples with numbers using those functions. It was important to make the examples as applicable and understandable as possible as this was the introduction. Since this was a form of intervention I was aware of learners' current understanding of the definition of the function concept and I was aware of their weaknesses (difficulties) and misconceptions. This gave me the opportunity to formulate and choose activities that could help learners to overcome their weaknesses and misconceptions referred by Sowder (2007) as scaffolding of knowledge. To develop learners' understanding of the basic idea of the function concept I designed the activities 1 and 2 which took us into Phase 3.

5.4.3 Phase 3: Prototype 1

In this activity I gave learners functional relationships and asked them to work in pairs and identify and explain the dependent and independent variables, and to formulate an expression connecting these variables. RME is embedded in these examples to help learners connect functions to real life. This means that learners need real life examples to enable them to mathematise and understand the concepts embedded in real life situations. This activity assists learners to construct a process conception of the function concept. As such I used the example approach advocated for by Van de Walle (2004):

Activity 1: Identifying and explaining variables

Work in pairs and identify the variables

i. The amount of hours someone works to the amount they get paid.
ii. The amount of petrol put in the tank and the amount of money paid by the motorist.
iii. The number of loaves of bread and the amount paid by the customer.
iv. The number of nights spent at a hotel and the amount of money the guest has to pay..
v. The number of patients admitted in a hospital and the number of beds in the hospital.

At the end of this activity I interviewed learners in pairs and asked them to explain to me how they identified the variables in the given relationships. The following are excerpts from these interviews:

Monga & Edy: We have seen that there is a fixed variable and a variable that changes because of a change in the fixed variable (Monga reporting).

Interviewer: Of these two, which one is the dependent and which one is the independent variable?

Monga & Edy: The fixed variable is the independent and the one changed by the fixed variable is the dependent variable (Edy reporting).

I asked the same questions to (Teko & Diva) and (Mat & Coco) and they gave the following responses:

Teko & Diva: The variable which can affect the other is the independent variable and the affected variable is the dependent variable (Diva reporting).

Mat & Coco: A variable which depends on another variable for its value is the dependent variable and the one which is depended on is the independent variable (Coco reporting).

The use of real life relationships helped the learners to correctly identify the variables in the given relationships. They could imagine how the independent variable affects the dependent variable. This indicates that learners were now operating between the process level and the object level which was an improvement from the initial interviews. To consolidate learners' knowledge on the identification of variables in a given relation I designed another open-ended activity (Gravemeijer, 2004) in which learners could formulate their own relationship which they could manipulate to see how a change in one variable would affect the other. This is similar to Confrey and Smith's (1994) covariational approach. It helped learners to understand the dependence relationship between the dependent and independent variables.

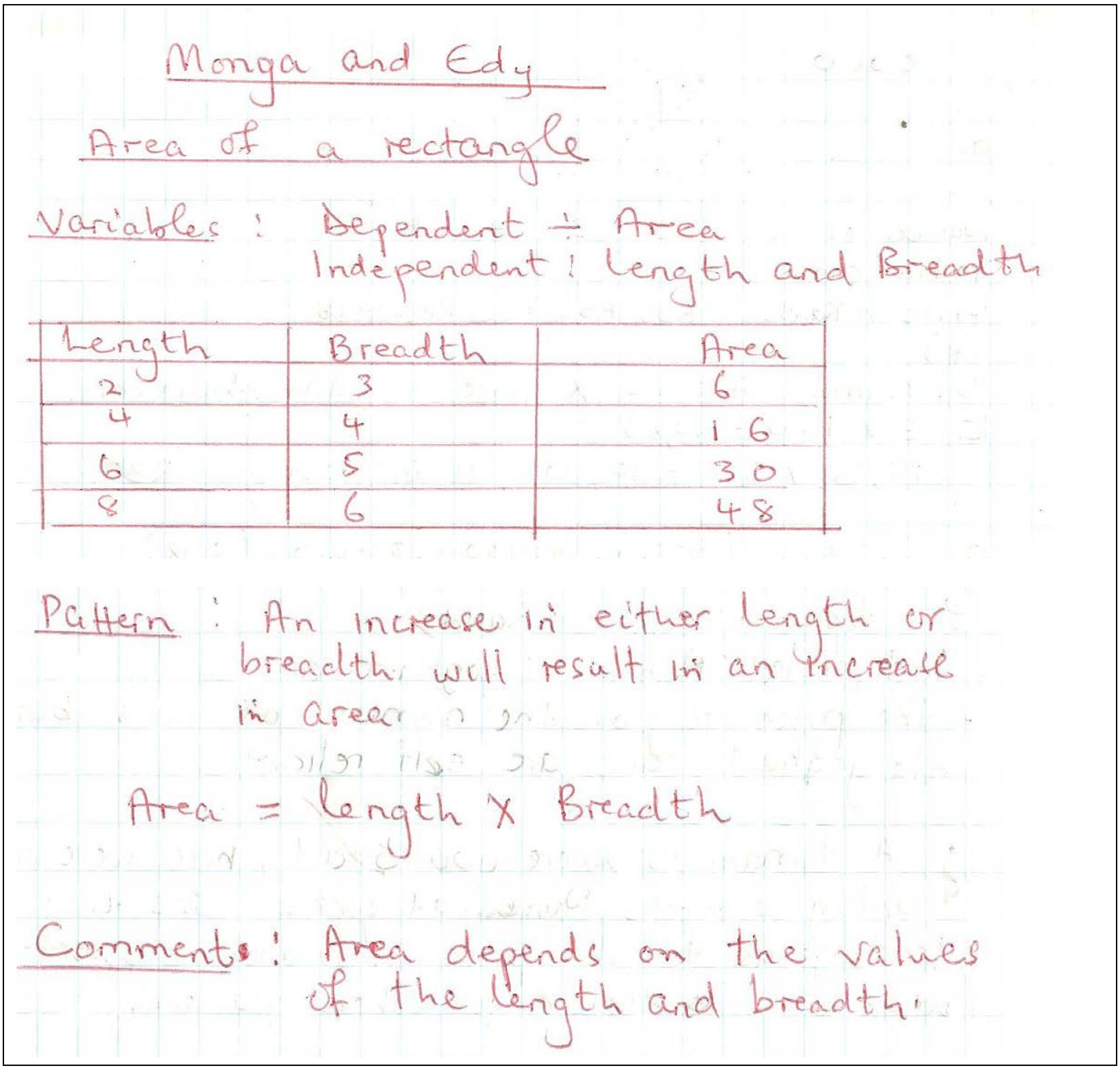
Activity 2: Applying the knowledge

This activity was meant to allow learners to work in pairs to formulate their own relationships and manipulate the variables in those relationships for them to see how one variable changes while imagining changes in the other. I gave them an example of the relationship between the height of water in a bottle and the volume of the water. Charles (1990) contends that learners must be able to recognize change in order to understand functions. This activity was meant to help learners to construct the object and schema conceptions of the function concept.

Procedure for activity 2:

1. Learners formulate their own relationship which they will be able to manipulate in an activity by varying the quantities.
2. Identify quantities that vary in the course of the activity and focus on the relationship between those variables.
3. Create a record of the corresponding values of the varying quantities by using a table or graph.
4. Identify patterns in the records created.
5. Create a representation of the identified pattern in the relationship.
6. What can you say about the representation you created in 5?

5.4.3.1 Discussion of learners' responses on activity 2



Monga and Edy

Area of a rectangle

Variables : Dependent ÷ Area
Independent : Length and Breadth

Length	Breadth	Area
2	3	6
4	4	16
6	5	30
8	6	48

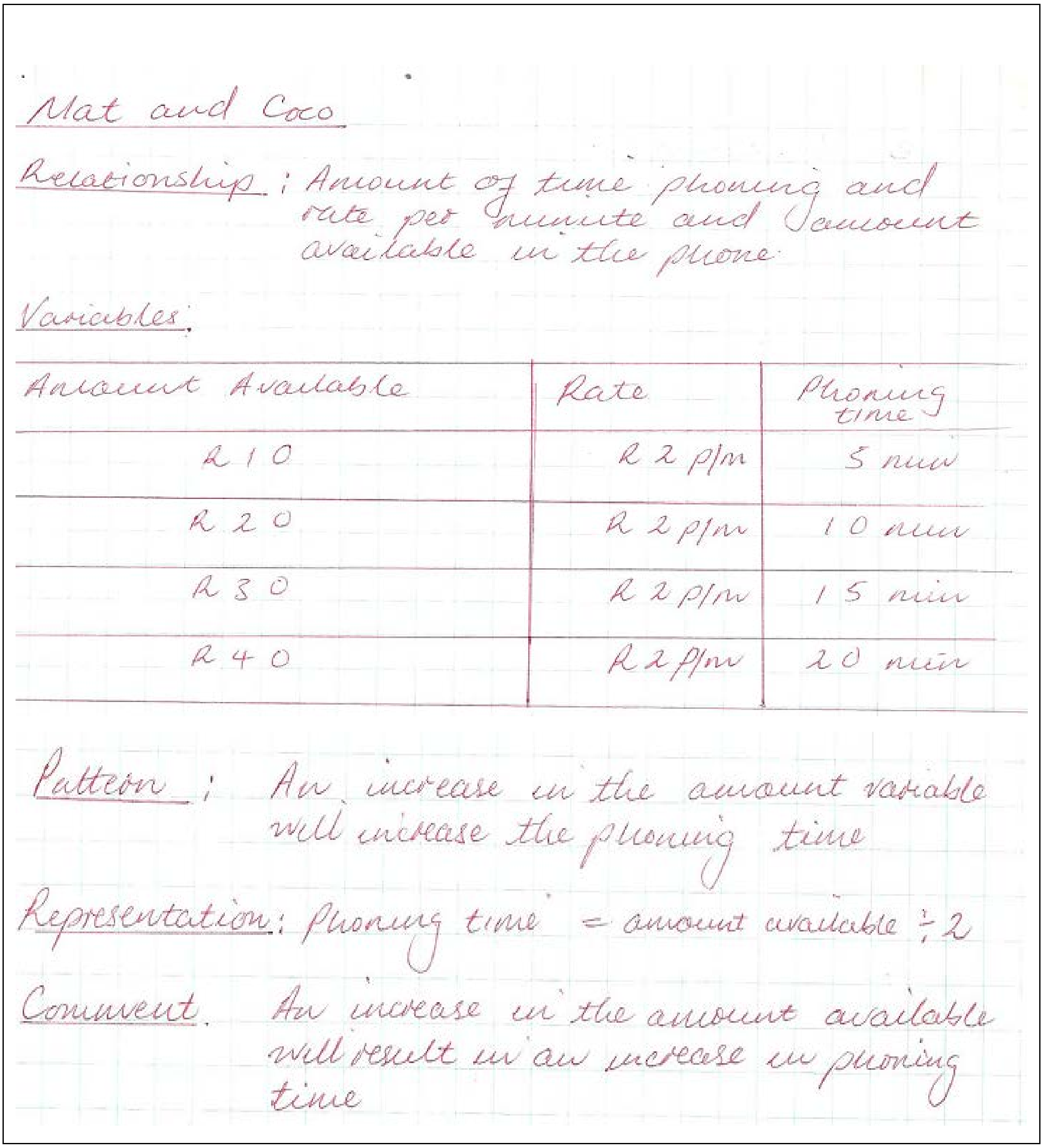
Pattern : An increase in either length or breadth will result in an increase in area

Area = length x Breadth

Comments : Area depends on the values of the length and breadth.

Figure 28: Monga and Edy's responses on activity 2

Monga and Edy could formulate a relationship and identify the dependent and independent variable. They were able to identify and explain the connection between the variables and use that connection to derive a formula connecting the variables. This indicates that the two learners had moved from the process level to the object level (Dubinsky & Harel, 1992) which was an improvement from where they started.



Mat and Coco

Relationship: Amount of time phoning and rate per minute and amount available in the phone.

Variables:

Amount Available	Rate	Phoning time
R 1 0	R 2 p/m	5 min
R 2 0	R 2 p/m	1 0 min
R 3 0	R 2 p/m	1 5 min
R 4 0	R 2 p/m	2 0 min

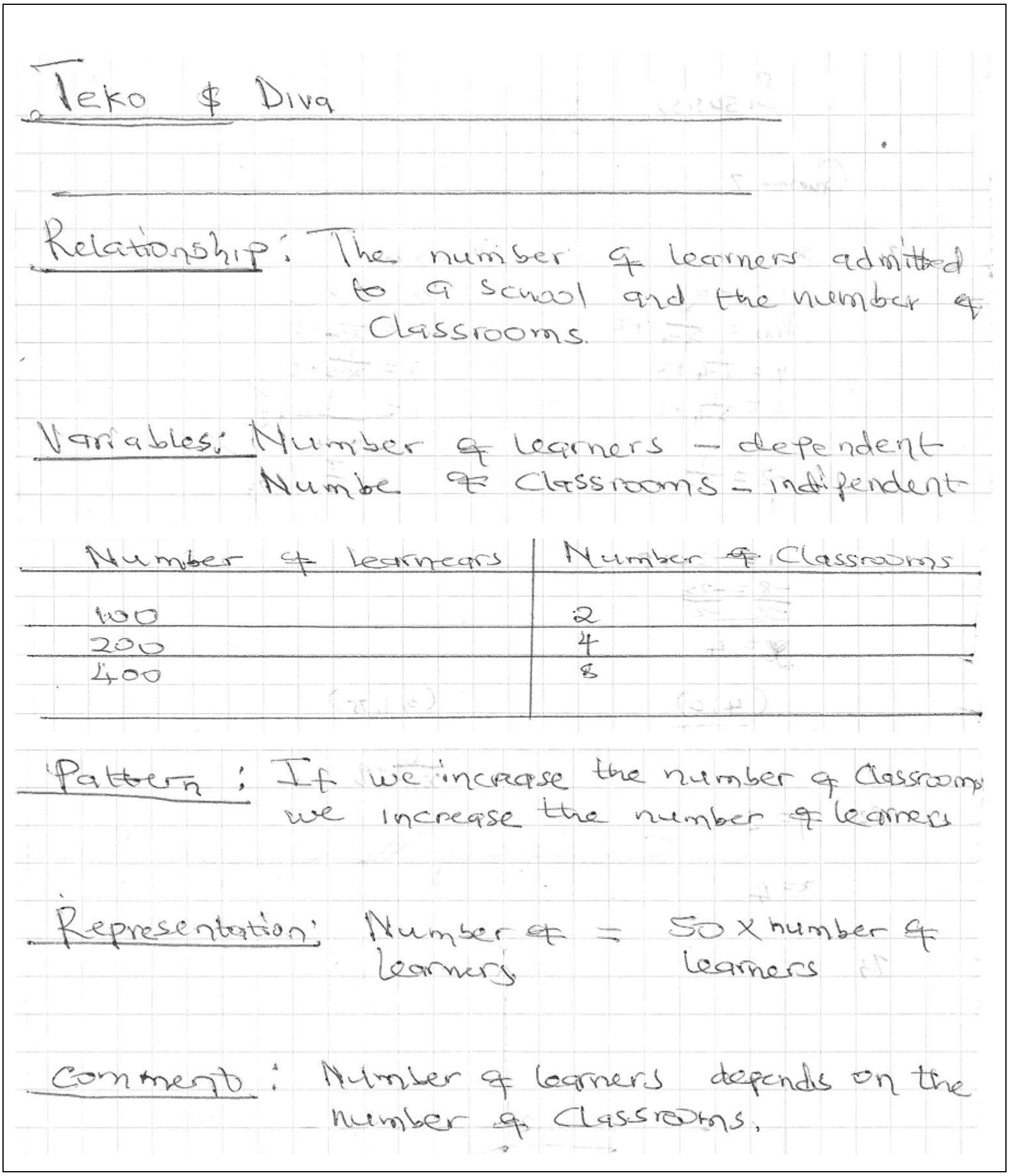
Pattern: An increase in the amount variable will increase the phoning time

Representation: Phoning time = amount available ÷ 2

Comment. An increase in the amount available will result in an increase in phoning time

Figure 29: Mat and Coco's responses on activity 2

Mat and Coco applied their experience on phoning to come out with a practical example of a relationship in which the amount of phoning time depends on the available amount of money recharged. They could identify the pattern in the relationship and give a formula for the relationship. Their ability to formulate own examples of functional relationships which they failed to do in previous tasks indicate that they had progressed from the action level to the process and object levels (Dubinsky & Harel, 1992).



Teko $ Diva

Relationship: The number of learners admitted to a school and the number of Classrooms.

Variables: Number of learners – dependent
Numbe of Classrooms – indifendent

Number of learnears	Number of Classrooms
100	2
200	4
400	8

Pattern: If we increase the number of Classroom we increase the number of learners

Representation: Number of learners = 50 x number of learners

Comment: Number of learners depends on the number of Classrooms.

Figure 30: Teko and Diva's responses to activity 2

Teko and Diva's example though superficial shows their understanding of relationships and the variables involved. They showed an improvement in their understanding of the connection between the dependent and independent variables in a relationship. This indicates they were operating at the action level (Dubinsky & Harel, 1992).

5.4.3.2 Retrospective analysis

Activity 1 worked well in developing all the six learners' ability to identify and explain the dependent and independent variables which was an indication that they were operating between the process and the object level (Dubinsky & Harel, 1992). For example, Monga and Edy explained the independent variable as the fixed variable and the one being changed by the fixed variable as the dependent variable. Teko and Diva regarded the independent variable as a variable which can affect the other variable, and the affected variable as the dependent variable. Similarly, Mat and Coco talked of a variable which depends on another variable as the dependent variable and the one which is depended on as the independent variable. Activity 2 worked well for four learners except for Teko and Diva who formulated a superficial relationship of classrooms and number of learners (action level). Although it may sound obvious that the number of learners at a school depend on the number of available classrooms, it can be argued that the classrooms can be overcrowded to accommodate all the available learners and if learners are few, they can still be distributed into all the classes. The two activities above enabled learners to take subject matter from reality and organize it according to mathematical patterns in order to solve problems from reality (Gravemeijer, 1994). As a result, learners were able to recognize mutual dependence between variables or varying quantities, determine the nature of the dependence relationship between variable quantities and to express and interpret quantitative relationships. These results indicate that learners had managed to move from the action to the process and the object level. However, learners could not use the key idea of the function concept to determine whether a given relation is a function or non-function. They could not use the proper functional language of domain and range in defining the function concept and in formulating examples and non-examples (Hitt, 1998). These results led to the developing of the HLT in the next phase.

5.4.4 Phase 4: Product and theory refinement

Product and theory refinement entails redesign and refinement of the problem, solutions (created in phase 3) and method are done based on the feedback received in phase 3 (interplay between theory and practice). Formative evaluation is used along with the reflection of the feedback, to do the redesign and refinement. This is an iterative stage where, the refinement is done for achieving successive approximation of theory, and refinement of design theory.

5.4.4.1 Teaching experiment 2 (Prototype 2)

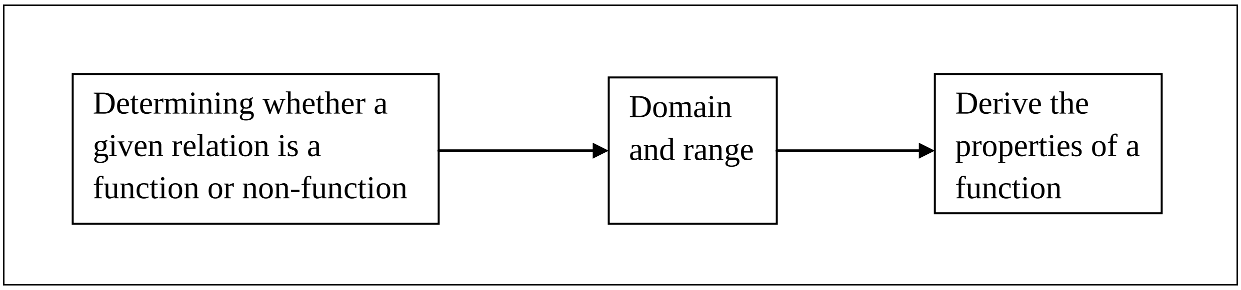


Figure 31: HLT for developing an understanding of the function concept

I designed the above HLT and the activities (instructional material) based on the principles of Realistic Mathematics Education (RME) to help shift learners from the process level to the object level and then to the schema level which is the main thrust of this study.

Learners should be able to:

i. distinguish a function from a non-function;

ii. identify the domain and range sets for a given function; and

iii. state and explain properties of a function.

In this lesson I wanted to help learners define the function concept in an informal way and at the same time introduce new terms, domain and range. To do so I made use of the principles of RME by taking subject matter from reality and organizing it according to mathematical patterns in order to solve problems from reality (Gravemeijer, 1994). I did this by allowing learners to act out the function concept and link the dependent and independent variables to the domain and range through acting. To achieve this I designed the following activity.

Activity 3: Table allocation game (determining whether a given relation is a function)

The purpose of this activity was to help learners understand the genetical and logical definitions of the function concept by having them act it out. My aim was to apply the RME principle of situating the function definition in reality by using an activity that is experientially real to learners. Acting out a function and a non-function would help learners understand how to determine whether a given relation is a function or not. This is linked to the definition of a function as a set of ordered pairs which Insook (1999) described as a logical definition. This will move learners from the process level to the object level (Dubinsky & McDonald, 2001).

Materials: Tables labelled with numbers on them for learners to move to. Learners are also labelled with letters A, B, C, D, E, and F.

Step 1: I wrote a relation that is a function in that every learner moves to a different table. Define the learners as the x-coordinates and the tables as the y-coordinates, for example, {(A, 1), (B, 2), (C, 3), (D, 4), (E, 5), (F, 6)}.

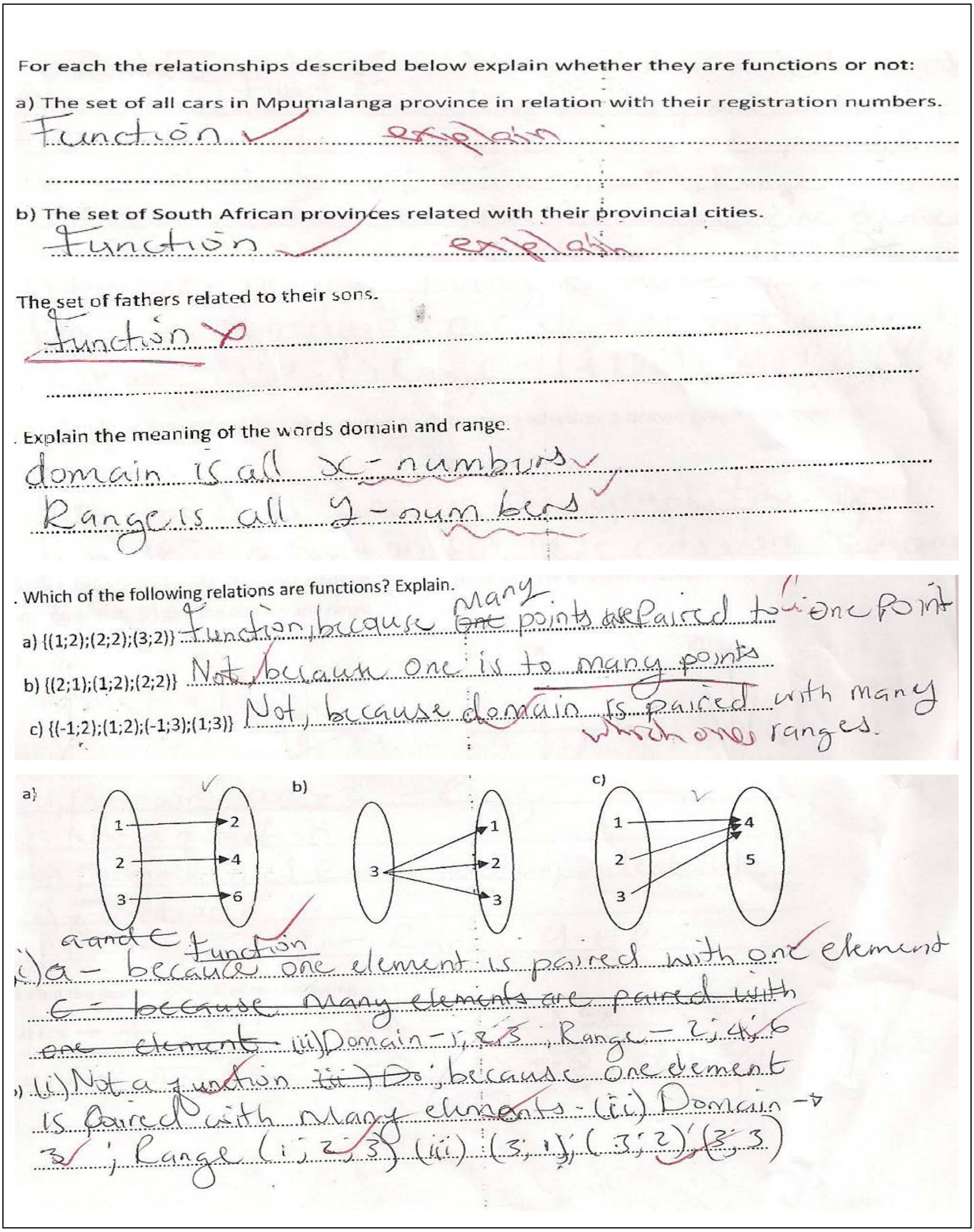
Step 2: I let learners act out the function by moving to the allocated table, for example, A moves to table 1, B moves to table 2, C moves to table 3 and so on. I told the learners that this relation is a function (I asked them to explain why) - answer: one learner is linked to a unique table.

Step 3: I wrote another relation that is a function. This time I wrote the relation so that more than one learner moves to one table, for example, (A, 1), (B, 2), (C, 1), (D, 3), (E, 4) and ((F, 5). I told the learners that this is a function as well. I checked to see if any of their guesses change. At this stage I introduced the aspects of domain (the learners) and range (where the learners are going to).

Step 4: I wrote a relation that is not a function, for example, (A, 1), (B, 2), (A, 3), (C, 4), (D, 5) and (E, 6). I asked them to explain why this relation was not a function. Answer: This is not a function because A will not be able to move to table 1 and table 3 at the same time. This helped learners to understand that "each element in the domain set is paired to only one element in the image set" (Dede & Soybas, 2011, p. 95).

Step 5: I then asked learners to derive the properties of the function concept from what they were acting out.

To assess learners' understanding of the aspects covered in teaching experiment 2, I gave them a worksheet (Task 1) in which they worked in pairs to determine which ones are functions and which ones are not. In developing questions for Task 1, I used the RME principle of using everyday experiences for items 1a, 1b and 1c. For question 2, I wanted learners to derive the meanings of domain and range from activity 3 a situation though imaginary allowed learners to imagine the meanings of these terms. In question 3 my use of set notation and arrow diagrams was meant to employ the RME principle of making the definition of the function concept real in the mind of the learner. This task intended to compel learners to use the properties of the function concept they had derived in step 5 above to identify a function and a non-function. The following are learners' written responses to the task:


For each the relationships described below explain whether they are functions or not:

a) The set of all cars in Mpumalanga province in relation with their registration numbers.

Function ✓ explain

b) The set of South African provinces related with their provincial cities.

Function explain

The set of fathers related to their sons.

Function ✗

. Explain the meaning of the words domain and range.

domain is all x-numbers ✓

Range is all y-numbers ✓

. Which of the following relations are functions? Explain.

a) {(1;2);(2;2);(3;2)} Function, because ~~One~~ many points are Paired to one point

b) {(2;1);(1;2);(2;2)} Not, because one is to many points

c) {(-1;2);(1;2);(-1;3);(1;3)} Not, because domain is paired with many which ones ranges.

i) a – because one element is paired with one element
a and c Function
~~c – because many elements are paired with one element~~ (ii) Domain – 1; 2; 3 ; Range – 2; 4; 6

b) (i) Not a function ~~(ii) Do~~; because one element is paired with many elements. (ii) Domain – 3 ; Range (1; 2; 3) (iii) (3; 1); (3; 2); (3; 3)

Figure 32: Monga and Edy’s written work on Task 1

Monga and Edy to a larger extent could identify functions and non-functions but could not explain why, indicating some signs of operating at the process level.

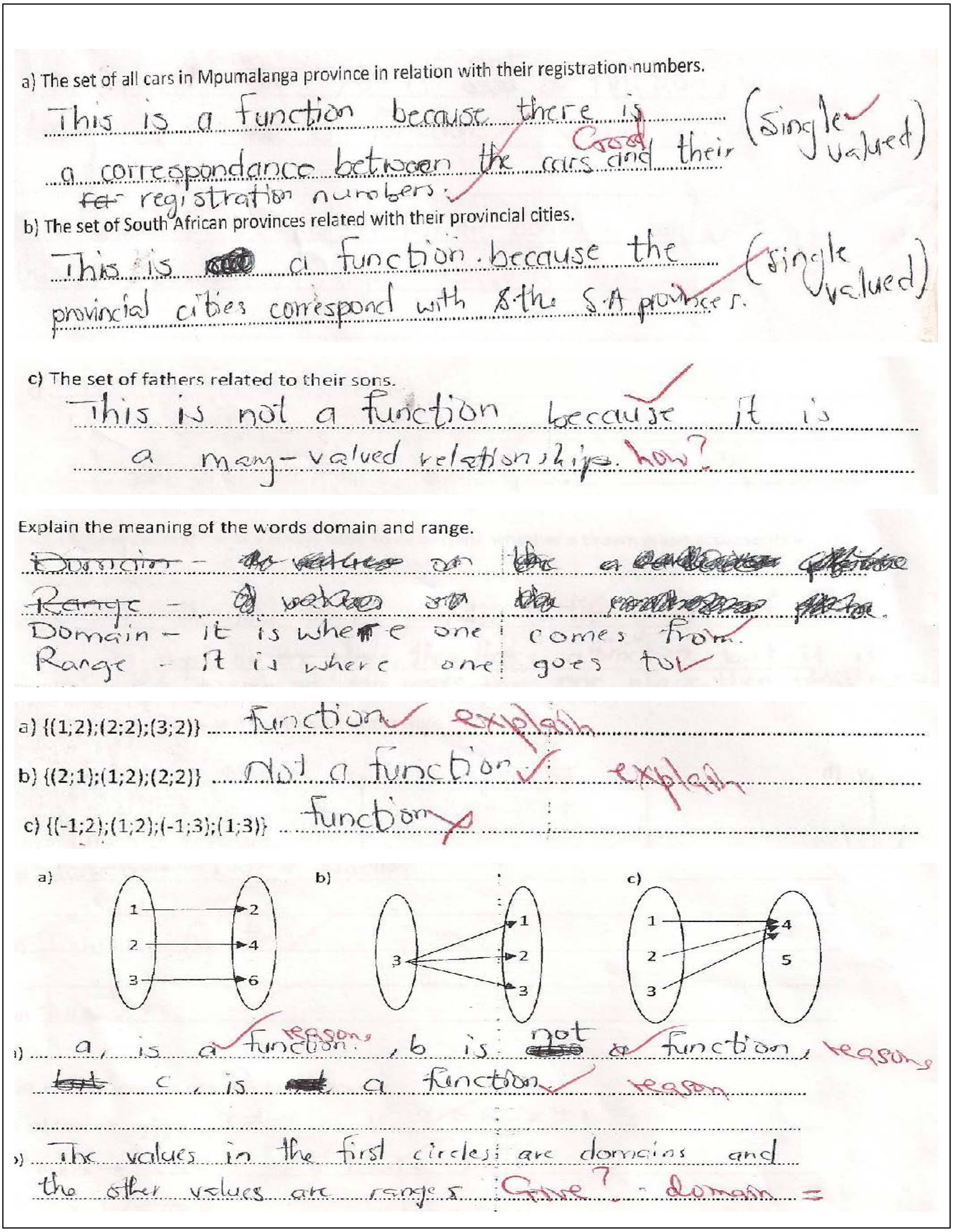


Figure 33: Teko and Diva's written work on Task 1

Teko and Diva showed that they had progressed from the action level to the process level though they could not explain why a given relation was a function or a non-function.

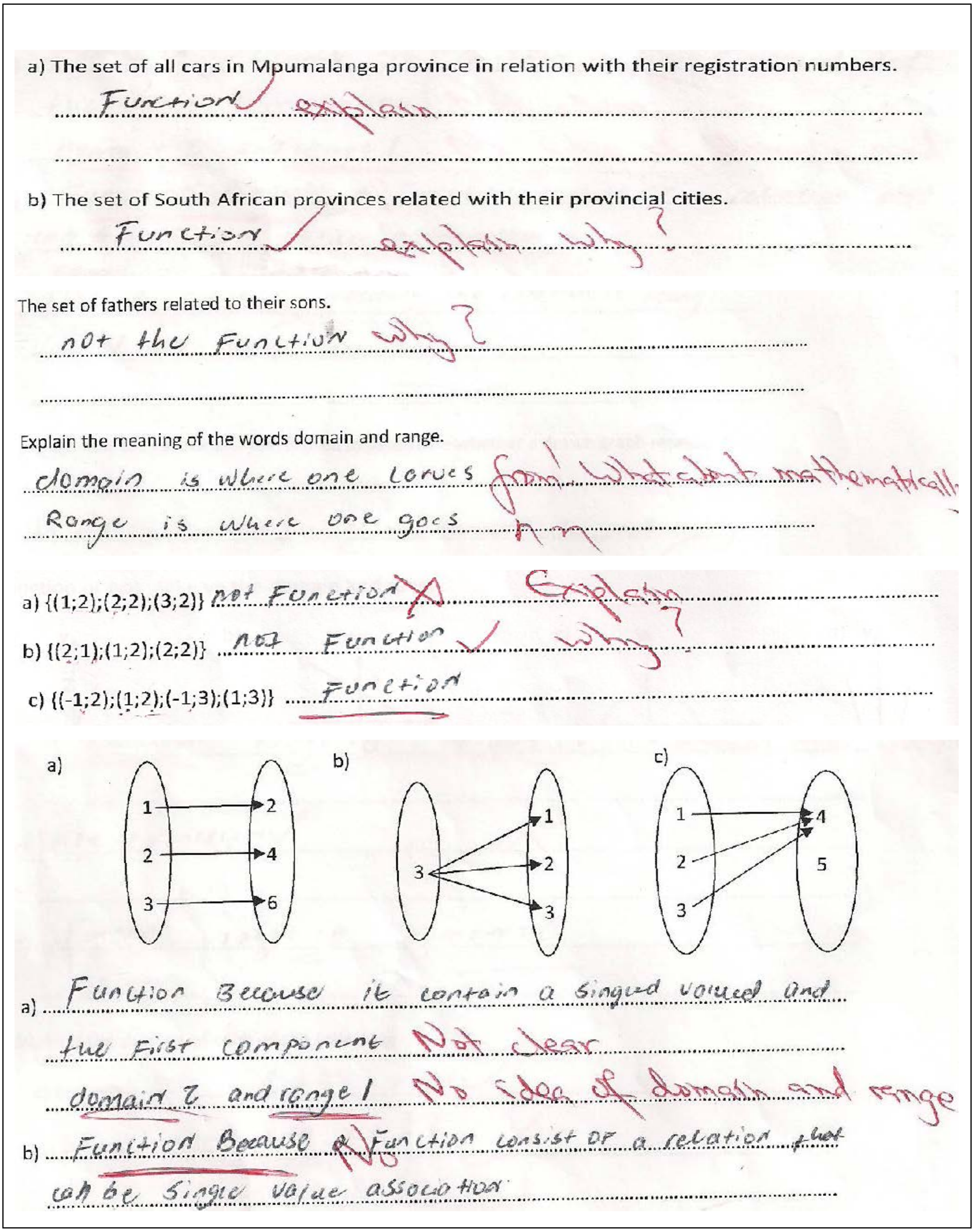


Figure 34: Mat and Coco's written work on Task 1

Mat and Coco could identify functions and non-functions but could not explain why, indicating some signs of operating at the process level.

5.4.4.2 Retrospective analysis

The purpose of Task 1 was to develop learners' ability to identify a function in its different representations. Evidence from learners' written work above indicates that Teko and Diva could identify a function and explain why the relationship is a function using the properties of a function for the first question on Task 1. They used the idea that every element of the domain set must have its own corresponding element in the range set. However, they could not give reasons for the second question indicating that they had progressed to the process level. However, Monga, Edy, Mat and Coco could not explain why a given relationship was a function or a non-function indicating that they were operating at the process level. They also showed a limited understanding of the concepts of domain and range. The one-to-one property was loosely used without an explanation of how it works and its limitations indicating that learners had a limited understanding of this property of the function concept. This was also documented in a study by Breidenbach et al. (1992). This led to the formulation of the following HLT:

5.4.4.3 Teaching experiment 3 (Prototype 3)

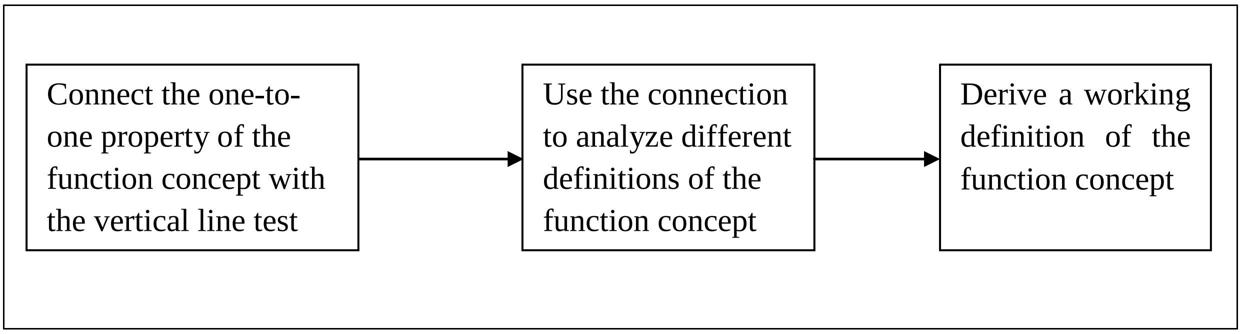


Figure 35: HLT for the working definition of the function concept

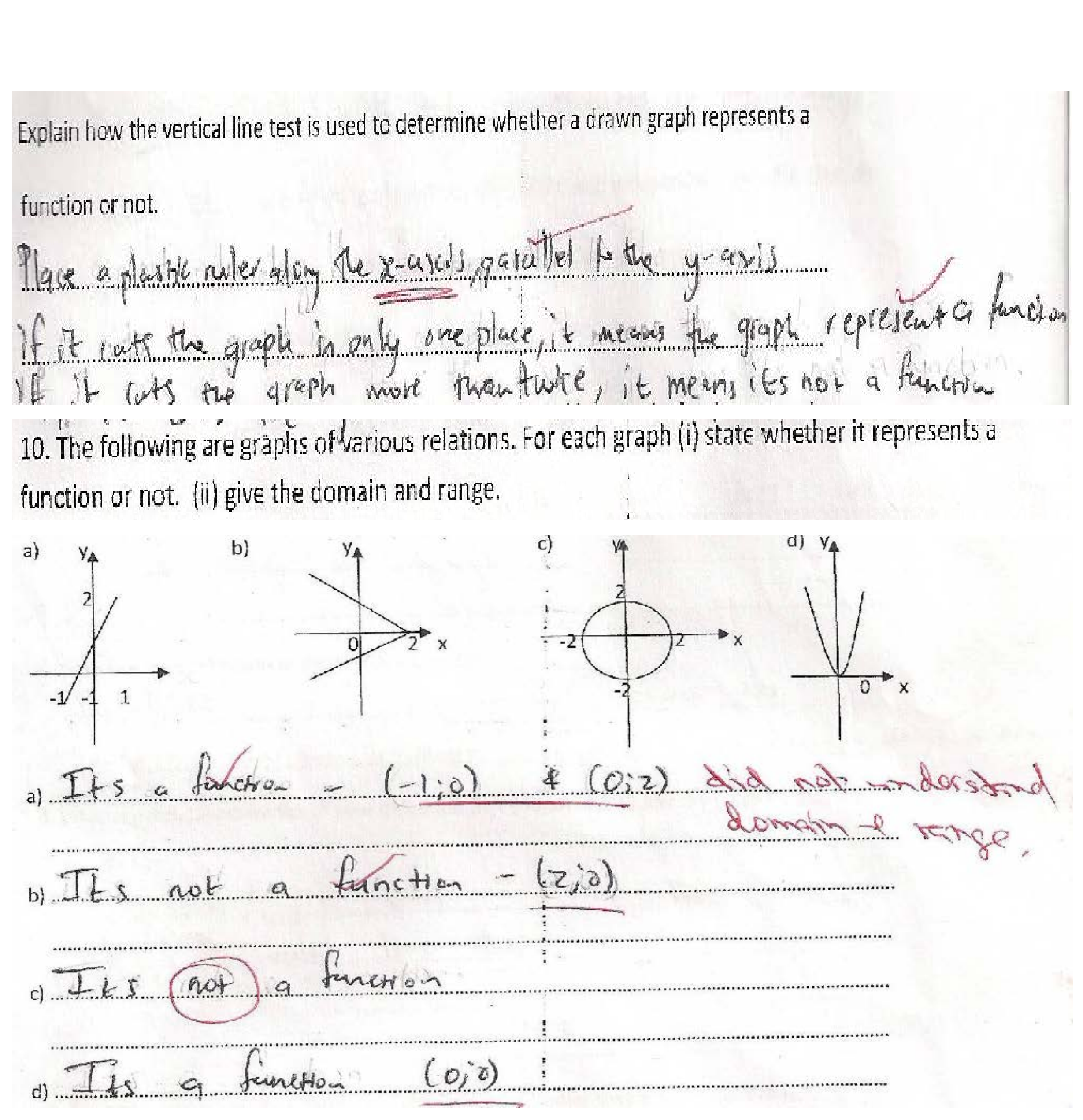
This HLT was intended to help learners understand the one-to-one property of the function concept and its limitations, that is, not every function is one-to-one, for example $y = x^2$ is not one-to-one on the domain of real numbers; and to identify relations that are functions and those that are non-functions. I referred learners back to the activity where they were acting out functions and non-functions in-order for them to understand the one-to-one property and its limitations. I then designed Task 2 to assess their ability to use the vertical line test. I then designed activity 4 to assess learners' ability to use the knowledge they had gained up to this point to evaluate different definitions that I took from various textbooks used locally and internationally. Discussions from this task were meant to lead learners to derive a working definition for the function concept. A definition that brings out the key idea of the function concept, helps learners to formulate examples of functions and non-functions, and enables learners to determine whether a given relation is a function or not.

By the end of this lesson learners should be able to:

i. explain the one-to-one property, its limitations and the use of the vertical line test;
ii. analyze the different definitions of the function concept; and
iii. derive a working definition of the function concept.

I revisited the activity where learners were acting out functions and non-functions and derived the one-to-one property. However, I explained to learners that not all functions are one-to-one using the example of $y = x^2$. I found it appropriate to analyze different definitions in learners' textbooks together with the learners using their knowledge from the previous activities to derive a working definition of the function concept. I used the definitions by Insook (1999); Laridon et al. (1987 & 2007); Foerster (1984) and Young (2011) discussed in Section 2.5 of Chapter 2 and the vertical line test to help learners identify functions and non-functions in graphical representations.

Task 2 assessed learners' conceptual understanding and ability to use the vertical line test to determine whether a drawn graph represents a function. Explaining how and why the vertical line test works requires learners to be operating at the object level while using the vertical line test is at the action level. This approach was supported by earlier studies reviewed in my literature (see Akkoc, 2006). Learners' responses to this task are shown in what follows:


Explain how the vertical line test is used to determine whether a drawn graph represents a function or not.

Place a plastic ruler along the x-axis, parallel to the y-axis

If it cuts the graph in only one place, it means the graph represent a function

If it cuts the graph more than twice, it means its not a function

10. The following are graphs of various relations. For each graph (i) state whether it represents a function or not. (ii) give the domain and range.

a) Its a function - (-1;0) & (0;2) did not understand domain & range.

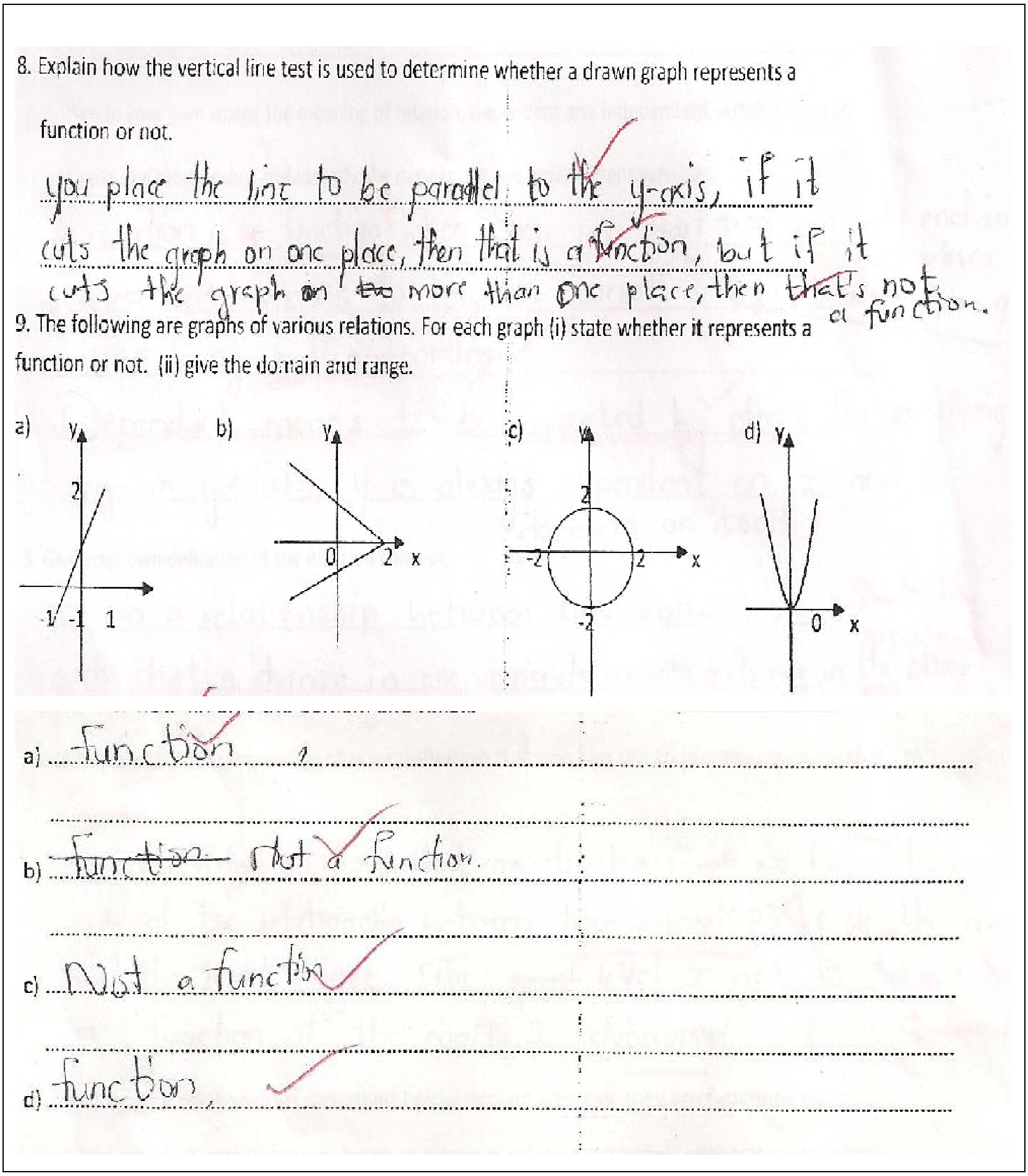
b) Its not a function - (2;0)

c) Its not a function

d) Its a function (0;0)

Figure 36: Monga and Teko's responses on Task 2

Monga and Teko could use the vertical line test to determine whether the drawn graphs represent functions. Item c) was wrongly ringed but is correct and I informed the learners about the mistake. However, they could not determine the domain and range from the drawn graphs. This problem was also reported in a recent study (Dede & Soybas, 2011). Their ability to explain how and why the vertical line test works indicates that they are operating at the object level of APOS theory but its use only places them at the process level.



8. Explain how the vertical line test is used to determine whether a drawn graph represents a function or not.

you place the line to be parallel to the y-axis, if it cuts the graph on one place, then that is a function, but if it cuts the graph on ~~two~~ more than one place, then that's not a function.

9. The following are graphs of various relations. For each graph (i) state whether it represents a function or not. (ii) give the domain and range.

a) function ?

b) ~~function~~ not a function.

c) Not a function

d) function

Figure 37: Mat and Diva's responses on Task 2

Mat and Diva were also able use the vertical line test and explain how it works but failed to find the domain and range for each graph. This indicate that they oscillating between the process and object levels of APOS theory.

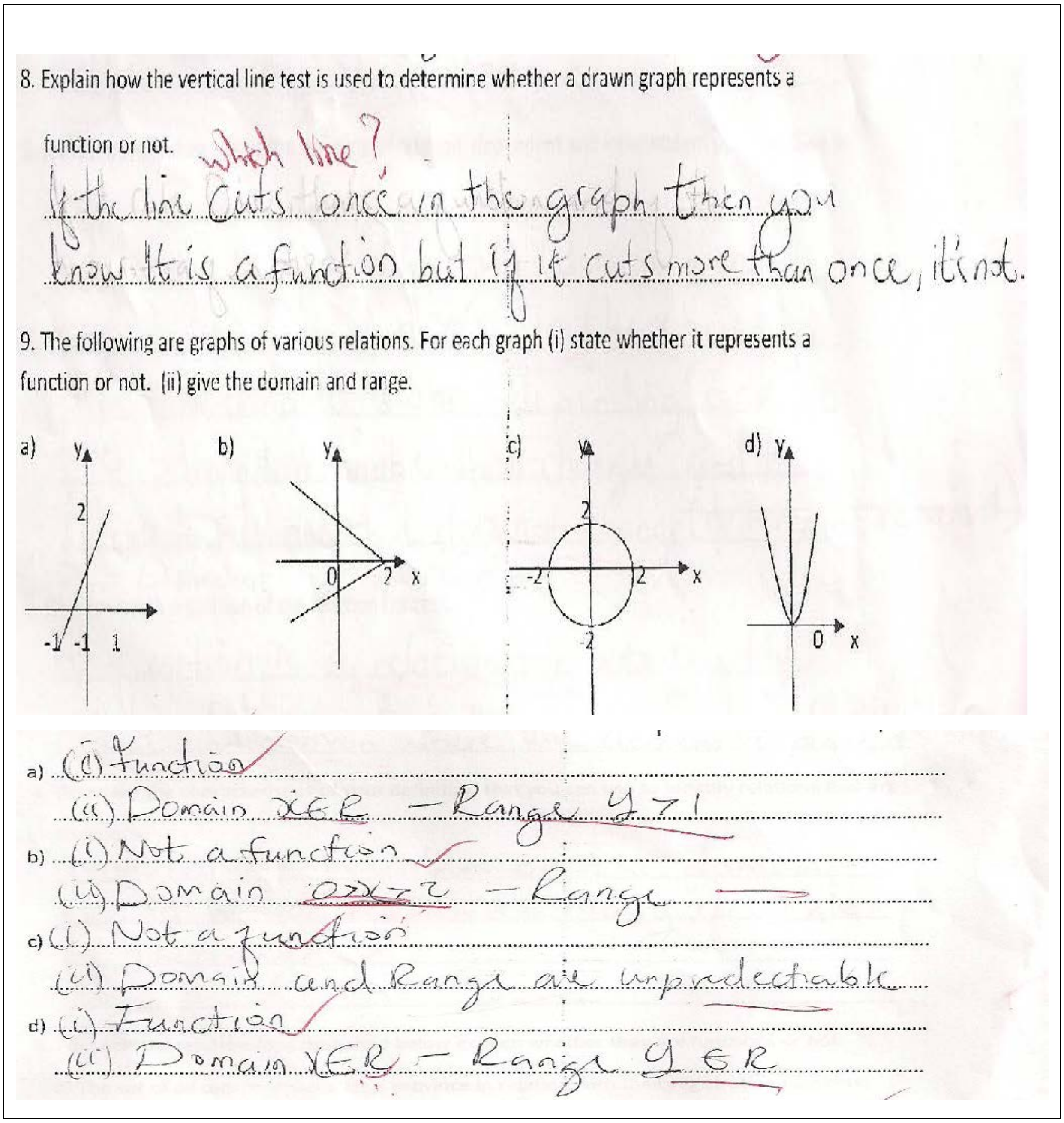


Figure 38: Coco and Edy's responses on Task 2

Again Coco and Edy could explain the use of the vertical line test but also failed to determine the domain and range of these graphs. Their failure to link a graph to its domain and range indicates that they are still operating at the process level.

After task 2 I asked learners to explain why the vertical line test works and how is it connected to the definition of the function concept. The following are the responses that I got from the learners:

Monga: If the vertical line crosses the graph once it means I will have one *x*-value corresponding to one *y*-value which is what the definition of a function wants. But if it cuts twice or more then one *x*-value will correspond to two or more *y*-values. It cannot be a function.

Mat: To me for a function it crosses once and for a non-function it crosses twice.

Edy: Where the vertical line crosses the graph is a point where a value of *x* corresponds with a *y* value. If the vertical line crosses the graph twice it won't be a function because one value of *x* will be having two values of *y*. A car cannot have two registration numbers.

Coco: What I know is that for a function the line crosses once and if it crosses twice or more it's not a function because one value of *x* will be having two values of *y*. A learner cannot go to tables at the same time

Diva: The vertical line works because it makes sure that one value of *x* is matched with one *y*-value.

Teko: It helps us to check that all the points on the graph have one value for *x* corresponding to one value of *y* for a graph to represent a function.

Learners' responses lead to the conclusion that a function is a "single valued relation in which the first value (*x*) should not be repeated". They added that each value of *x* has only one corresponding value of *y* saying "A child can only have one mother but a mother may have more than one child."

5.4.4.4 Retrospective analysis

Task 2 was intended to develop learners' understanding of the use of the vertical line test. From the learners' responses above I can conclude that they now understand why the vertical line test works and that learners can now use the vertical line test with understanding. Though learners' responses do not show indicators of the schema level, learners are operating above the process level towards the object level.

5.4.5 Phase 5: Final product and contribution to theory

5.4.5.1 Activity 4: Deriving the working definition of the function concept

A working definition of the function concept is a definition that captures most of the attributes (characteristics and aspects) of the function concept. I chose to use a working definition because learners had provided evidence that they cannot use the different definitions found in their textbooks. As such, a working definition will enable learners to appreciate the nature and origin of the function concept, identify functions and non-functions, formulate own examples of the function concept and identify dependent and independent variables in a given relationship. These attributes are found in the different definitions that span across most mathematics textbooks. I asked learners in pairs to come up with their own working definitions of the function concept. The discussion starts when learners were asked to compare the different working definitions they had formulated:

Monga: We must come up with a definition that will help us in answering questions like checking whether a given relation is a function or not. Edy can you read out our definition.

Edy: **(**reading) "A function is a relation associating the elements of one set with elements of another set."

Diva: According to me Monga and Edy's definition lacks the dependency of the elements; elements can be associated without necessarily depending on each other like my marks in maths and physical science.

Coco: I agree with Diva, let me read our definition (she reads) "A function is a relationship in which quantities depend on other quantities in such a way that if the latter are changed the former undergoes change."

Mat: Coco is right because the nature of the relationship must be clear that it is a dependency relationship between variable quantities. But we agreed with Teko that there must be a correspondence between the varying quantities in the relationship. Let Teko read our definition.

Teko: (reading): "A function is a dependence relationship between two sets of variable quantities in which each element from the first variable (domain set) has only one corresponding element in the second variable (range or image set)."

I gave learners the opportunity to comment and improve each pairs' working definition together as a group. Learners could identify weaknesses in each definition and they debated trying to convince one another to accept a particular definition. In the end learners in the sample seem to have agreed with Teko and Mat's definition. This definition is similar to the set-theoretic definition though expressed in simple terms but differs in that it captures the key idea of the function concept. However, it is a misconception to think that every function should be one-to-one or onto. Learners forgot that they had previously formulated multivariable functions like the area of a rectangle which is expressed in terms of two variables (length and breadth).

5.4.5.2 Retrospective analysis

Learners managed to identify the differences among their definitions and weaknesses in each definition. It became easy for the learners to use these definitions to formulate examples of functions and non-functions. Their working definitions included the core idea, the one-to-one onto properties. It was pleasing to note that learners could use their working definitions to formulate functions and non-functions and to determine whether given relations were functions or non-functions. I discovered that though learners had followed the HLTs as I had intended, only two learners were exhibiting some indicators of the schema level while the rest were oscillating between the process and object levels.

5.5 Summary of results on problem area 1

Learners were able to:

- explain and use the basic idea (that of a dependence relationship) behind the function concept in formulating function examples;
- use the uniqueness, the one-to-one correspondence and onto properties of the function concept to determine whether a given relation is a function or not;
- identify dependent and independent variables in a given function; and
- derive their own definition of the function concept which they used to formulate examples of functions and non-functions.

From the activities carried out during the teaching experiments, individual and group interviews and written work, learners seemed to have progressed from their initial action level to the current object and schema levels summarized in Table 9.

Table 9: Results of prototype 3

LEARNER	A	P	O	S	INDICATORS OF APOS THEORY CONCEPTION LEVELS
Monga			X		• see a function as something that's being acted on. • look at the graphical representation of a function and see that it is not a function because it does not pass the vertical line test.
Coco			X		• Can derive own definition of a function and use it to distinguish a function from a non-function. • can create own example of a function or non-function. • can link the vertical line test with the one-to-one property of the function concept
Mat				X	• can derive own and use own definition of the function concept in formulating examples and non-examples. • can use own definition to determine whether a given relation is a function or non-function.
Diva			X		• see a function as something that's being acted on. • look at the graphical representation of a function and see that it is not a function because it does not pass the vertical line test.
Edy			X		• see a function as something that's being acted on. • look at the graphical representation of a function and see that it is not a function because it does not pass the vertical line test.
Teko				X	• can use own definition of the function concept in formulating examples and non-examples. • can use own definition and vertical line test to determine whether a given relation is a function or non-function.

The next section focuses on learners' understanding of the representations of the function concept and how learners' understanding was used to design instructional sequences and activities to help them overcome their weaknesses and difficulties in translating from one representation to the other and move up the conceptual ladders of APOS theory.

5.6 Problem area 2: Representations of the function concept

5.6.1 Phase 1: Problem identification

Eisenberg (1992) stresses that an ability to make connections between graphical and analytical representations of the function concept is the main component of a robust understanding of the function concept. In the South African National Senior Certificate/ CAPS Mathematics paper 1 examination questions on functions learners are either asked to sketch the graph of a given function and other sub-questions relate to that sketch or to answer questions using a sketched graph. As such, learners need to understand the different types of representations in order for them to successfully answer questions involving these representations. Knuth (2000) added that since different representations emphasize different features of the function concept, the ability to move flexibly among representations is critical for learners to be able to choose the representation that will facilitate their ability to most efficiently solve a functional problem. According to Goldenberg (1988) many of the learners' misconceptions and difficulties may be traceable to the use of one or another representational setting of a function in isolation. He added that in most cases learners experience difficulties translating among these parallel representations because the translation process is overlooked, as a consequence, learners exhibit almost no flexibility whatsoever.

Recognizing the important role that multiple representations of the function concept play in learners' mathematical development, the National Council of Teachers of Mathematics emphasizes that learners should be able to "translate among tabular, symbolic, and graphical representations of functions" (NCTM 1989, p. 154). However, many learners leave high school without understanding the connections among these representations (Blume & Heckman, 1997). Flexibility to move between representations is an important mathematical development, however, the connectivity between these multiple representations is often absent (Knuth, 2000). This is because learners often fail to recognise the underlying equivalence between the graph, the equation of the graph, the verbal context or application and the table of values that the equation and graph represent. Knuth (2000) believes one must see these representations as 'informationally equivalent'. When this equivalence is evident it demonstrates a deeper understanding of the relationship between these representations.

Research question 1: What do learners understand by the representation of the function concept?

This research question was answered by analysing learners' responses with respect to what they said a representation of a function was, how they explained and calculated the aspects of a function representation (intercepts, asymptotes, turning points) and how they switched from one representation to the other. The interview excerpts in Sections 5.3.1 to 5.3.3 and the interview excerpts in this section provide the evidence of learners' level of understanding of the representation of the function concept. During session 4 (sessions 1 and 2 were interviews on questions 7 and 8 of the June 2011 examination while session 3 was the focus group interview) of the individual interviews I asked learners in the sample to explain how they represent a function and they gave me the following responses:

Diva: Representing a function. I can do it by an expression.

Edy: Sort of an equation.

Mat: From what I know a function can be represented as an equation, as you can use the table to get the function, and then 'mina' (myself) I favour the one which we use equation, because when you use equation, it becomes more easier and then straight forward.

Teko: Draw a graph by using an equation of a function. You can also write a table to represent your values.

Monga: You can write an equation.

Coco: Okay when you are given a function you represent it in a table and on a graph by using an equation and it's easy to get these using an equation. There are also other ways in which you could represent it but I usually do because sometimes you are asked to refer to the table and the graph.

Similar to Eisenberg's (1991) findings, learners in my sample also prefer the equation as a representation of the function concept. This may be because their teacher had taught this representation in isolation without linking it to other representations. This ability to repeat what was done in class verbatim indicates that the learners were operating at the action level of APOS theory.

Representation of the function concept included types of representations that learners were aware of; representations prescribed by the CAPS curriculum; concepts related to the graphical and symbolic representations (intercepts, asymptotes, turning points) and the process of translating from the graphical to symbolic representation and vice-versa. Learners' concept images of the representation of the function concept that emanated from the above excerpts and those in Section 5.3.1 were the graph, equation and table of values. Their reasoning about the representation was limited to seeing these representations as separate entities as indicated by their inability to translate from graph to the equation. Learners could not realise that these representations can actually represent the same function. In addition, learners' understanding of the important aspects of these representations (intercepts, turning points and asymptotes) was also weak. I repeat excerpts in Section 5.3.1 here for the reader's convenience.

Learners' understanding of the intercept

Interviewer: How do you explain an intercept?

Diva: Intercepts are the points you are supposed to plot so you can get your graph.

Interviewer: How do you calculate these intercepts?

Diva: I want to find the y-intercept first by putting $x = 0$ and then I'm going to find the x-intercept by putting $y = 0$ and calculate.

Interviewer: Why put $x = 0$ on y-intercept and $y = 0$ on x-intercept?

Diva : Sir because our teacher said when you calculate intercepts you must always let y or x be equal to zero.

Interviewer: Question 7.1 asks you to calculate the intercepts, what is your understanding of an intercept?

Coco: The point at which the graph will cut the axes.

Interviewer: How do you calculate the coordinates of the intercepts?

Coco: I'll change the x into zero for the y-intercept and y into zero for the x-intercept. That is what I will do.

Interviewer: Why change x and y into zero at the intercepts?

Coco: This is what we were told. I don't know why.

Interviewer: What does an intercept mean to you?

Monga: The y and the x-axis.

Interviewer: Tell me about your calculations here (pointing at his written solutions).

Monga: I just remember that at first I substitute x with 0 and then y with 0 in the equation but I don't know exactly what will be happening here I just substituted.

Interviewer: Can you explain to me the meaning of an intercept?

Teko: I know it as I calculate it here, but I don't know the meaning of it? I think the intercepts are the points of *y*.

Interviewer: How do you explain an intercept?

Mat: Is a line in a Cartesian plane where there are *x*-axis and *y*-axis, as we are told.

Interviewer: How do you calculate the coordinates of these intercepts?

Mat: Where there is *x*, you have to put 0 to calculate for a *y*-intercept, *x* has to be equal to 0.

Interviewer: Why put $x = 0$ at *y*-intercept?

Mat: I think it is because I didn't try to understand, but then I cram.

Interviewer: In your own words how do you describe an intercept?

Edy: The point where the *y* or the *x* is, that is the intercept. In the *y*-intercept, I think is the point y and in the *x*-intercept, I think is the point *x*.

Interviewer: How do you calculate the coordinates of the intercepts in 7.1?

Edy: I will put $y = \frac{1}{x-4} + 2$, so *y*-intercept, $x = 0$, then $y = \frac{1}{0-4} + 2$

Interviewer: Why is $x = 0$ at the *y*-intercept?

Edy: 'Oh' I don't know, the only thing I know is that, when you are calculating the *y* -intercept, $x = 0$, and for the *x*-intercept, $y = 0$.

Learners' responses in the excerpts above indicate that learners cannot give a precise and complete explanation of intercepts. This shows that learners had an incomplete conception of the concept of intercept and were operating at the action level. On the other hand, learners could calculate the intercepts but did not understand the procedure as evidenced by their inability to explain why $x = 0$ on *y*-intercept and $y = 0$ on the x-intercept indicating that they were at the process level.

Learners' understanding of an asymptote

Interviewer: You managed to find the asymptotes; can you explain to me what these are?

Diva: An asymptote, I think is the line where …, which shows us that the graph can only approach, not mean to touch or cross.

Interviewer: Can you explain how you obtained these asymptotes?

Diva: I don't know how to explain to someone how to find it. I say zero is equated to the denominator umm..., I forgot how I calculate like that....$f(x) = \frac{1}{x-4} + 2$, ok, for *x*, I will take this one, and I say $x - 4 = 0$.

Interviewer: Why do you equate *x* - 4 to 0?

Diva: This is what we were told!

Interviewer: Tell me, how did you sketch this graph?

Diva: I want to show you my axis before writing a …, is like this as I have plotted, and then it will be *q*, this is *y*-asymptote. Here I put *x*, *x* - axis and the *y* - axis, then I look for *y* = 0 is here, and *x* = 1.75, I'm going to put it here and for *x* = 0, *y* = 3.5, I think is here, so I check my asymptote, for *y* is 2, I write a dotted line, for *x* = 4, and for *y*, so then I join my points.

Interviewer: You also calculated the asymptotes correctly. Can you explain to me what an asymptote is?

Coco: Is the point in the graph that does not have to touch the line of the asymptote.

Interviewer: How do you calculate the asymptotes?

Coco: The constant 2 is my *y*-asymptote and I equate the denominator *x*-4 to zero to get the *x*-asymptote

Interviewer: Can you explain to me why your procedure works?

Coco: I am not sure but this is what we were told by our teacher.

Interviewer: How did you sketch the graph?

Coco: Okay for *y*, okay I am going to use 4 for *y* because 3.5 is where, I am looking for the asymptote, it is the point where it can't touch or pass or go beyond. So 3.5 cannot go beyond 4. So I don't know what happened here. This *y* is supposed to be 4 and then *x* is supposed to be 3 on this one. Then I am going to plot 3.5, then the 2.25 and then I was told if you have this graph on this third quadrant you have to have the same graph on the first quadrant and when you have it on the second you have to have the same graph on the fourth quadrant. So this means I will have another one here. 'Ah' I'm not sure whether I use the asymptote where the graph that is to cut and then the intercept.

Interviewer: Ok. How do you explain an asymptote?

Monga: I don't know!

Interviewer: How did you find these values you wrote?

Monga: Ok! The y-asymptote is this 2 standing on its own. I made a mistake to write 4 here, now I remember. This is what we were taught. For an *x*-asymptote I'm not sure but we were taught in class. For the *x*-asymptote you have to look for the number below, Ok! Like here (pointing at $\frac{1}{x-4}$), the first fraction, we have the number below, for a denominator , we look at the denominator... make it become 0, that is 4, this is my *x* asymptote. I confused my *x* and *y* here but I was told like that. Even my graph is wrong I was using wrong things, eish I made a mistake!

Interviewer: What is an asymptote?

Teko: The asymptote I should find a point where the intercept cannot touch or go beyond.

Interviewer: How do you calculate an asymptote?

Teko: It's going to be 4 ..... (pause) because..... (pause) I don't know.

Interviewer: I can see three graphs here, how did you draw these graphs?

Teko: I will start by drawing the asymptote of *y*, the dotted line *y* = 4. I was told to use a dotted line, so I can easily plot the graph.

Interviewer: What do you understand by an asymptote?

Mat: That is when there is *x*, when I meet the intercept, asymptote of *y*, I can close this one $\rightarrow \boxed{\frac{1}{x-4}}+2$

$\uparrow$

and that one is the asymptote of *y* (pointing at +2), when I need the asymptote for *x*, I close this one in box $\frac{1}{x-4}\boxed{+2}$ and take the denominator of $\frac{1}{x-4}$ which *x*-4 and equate it to zero then I solve for *x*.

Interviewer: Can you explain to me why you equated *x* - 4 to zero?

Mat: I just know how to write them down from the equation.

Interviewer: Show me how did you draw your graph?

Mat: I make sure that I always put the asymptotes first and make sure the asymptotes and the intercepts don't touch each other....Cartesian plane, they only need *x* and *y*, then when *x* is 4, *y* is here, 'oh', I first have to show the asymptote, *y* is 2, *y* = 0, then *x* = 4.

Interviewer: Ok. Explain to me what an asymptote is, as you understand it?

Edy: The asymptote is the marginal line where the graph will be, as are those ones we have drawn, usually plotted with dotted lines.

Interviewer: How do you calculate the asymptotes?

Edy: I don't know how to find the asymptote but I just think of cramming.

Interviewer: Tell me what did you cram?

Edy: I forgot sir!

Interviewer: Ok. How did you draw the sketch then?

Edy: When I plot my graph, it must not touch the asymptote points. 'Oh' the critical points, ok! we 'gonna' label the graph, here is the x -axis and here is the y -axis, ok! first I 'gonna' put my asymptote before plotting the other points on the y -axis, my asymptote is +2 and then, plot a broken line and then on the x -axis, my asymptote is 4, which I plot a broken line and then, already this one say: y = 3 and here x = 4.

Interviewer: But you wrote x = 3 and y =4. Are you changing it now?

Edy: I made a mistake when I was writing.

The responses on the asymptotes in above excerpts showed that learners did not have a clear understanding of what an asymptote is. Their explanations were limited to "a line that a graph does not touch or can only approach" without understanding why the graph does not touch the asymptote. Learners pointed out that they just memorised the procedure of finding the asymptote indicating little understanding of both the concept and the procedure, thus showing that they were operating at the action level.

Learners' understanding of a turning point

Interviewer: Looking at your solutions to question 8, I can see that you calculated the coordinates of the turning point correctly. What is your meaning of a turning point of a graph?

Diva: Turning point is where my graph turns, goes back where it comes from or same direction where it originates.

Interviewer: Explain to me how you calculated the coordinates of a turning point.

Diva: To calculate the turning point, I used the formula $x = \frac{-b}{2a}$.

Interviewer: Where does this formula come from and what does it say?

Diva: Yes, from the quadratic formula but I'm not sure but this is the x-coordinate.

Interviewer: In your own words how do you explain a turning point?

Coco: A turning point is where the graph is cutting, like in a parabola graph.

Interviewer: How do you calculate its coordinates?

Coco: So, before I get the value of b and the value a, I have to simplify this equation. Let's say I should calculate the turning point of A, I'm going to use this equation, h(x) = - $2x^2$ + 4x + 6, because, minus and minus is positive, this is the new equation, for the turning point, I have to use this formula $x = \frac{-b}{2a}$, y, then -2, this one is our a and this 4 is our b, it will be, 'ja' it will be 4, is $\frac{-b}{2a}$, all this, is equal to $\frac{-4}{-4}$, which gives me 1, x = 1, then from there, I have to get the y equation, substitute all the x values to get y. Because I want the y values. It will be $y = -2x^2 + 4x + 6$, this is equal to $-2(1)^2 + 4(1) + 6$, and this will give me -2 + 4 + 6 = 8 and then this gives me +8, then my turning point is (1; 8)

Interviewer: Correct. But tell me, how does this formula $x = \frac{-b}{2a}$ work?

Coco: We were just told that this is the x-coordinate of the turning point.

Interviewer: How do you explain a turning point?

Monga: A turning point of a parabola is a point whereby your graph turns, whether negative or positive.

Interviewer: Tell me how you calculated the coordinates of the turning point.

Monga: I will be using the turning point formula which is: $x = \frac{-b}{2a}$, first of all …, It originates from the quadratic formula. First of all before I go any further, I collect my data, where I have my a; b; and my c, but here I will be using my b and my a. My b is 4,then I will write the 4,all over the 2 from an original formula multiply by a, my a is -2, then -4 all over -4, then, this will cancel out, I will remain with 1. Using my x, where x = 1, I will use my original formula to find my y, $y = -2x^2 + 4x + 6$ where there is x, I will replace x by 1, $-2(1)^2 + 4(2) + 6$, then I will multiply out, I mean I add … because here, they stated they will allocate me 3 marks, no need to go step by step when calculating this, because my marks will be 3marks, unless if it was 5 marks, I will go step by step, therefore the turning point is (1;8), this 1 is for x, this 8 for y-axis.

Interviewer: Well done for getting most of the solutions correct this time. Ok, how do you explain a turning point?

Teko: The turning point is the maximum point that the graph can reach. It is the maximum and the minimum points that the graph can reach.

Interviewer: How did you calculate the coordinates of this turning point?

Teko: Ok! Then I say this -2 is a, from the original formula. This -2 will be my a and the 4 my b and the 6 will be my c. Ok! $\frac{-b}{2a}$ is $\frac{-4}{2}$ = -2, this is equal to $\frac{-4}{-4}$ then

$x = 1$, ok! Now I'm calculating the y. Ok! I must calculate here, $-2(1)^2 + 4(1) + 6 = 8$, therefore, ok! the turning point of A (1;8).

Interviewer: What is your meaning of a turning point?

Mat: The turning point is where the graph curves, that means where it changes directions.

Interviewer: How do you calculate the coordinates of the turning point?

Mat: What are we going to do here is that we 'gonna' multiply the things in brackets and then, 'ja', let's do it! We 'gonna' take the -2 as it is and put it down here, then open brackets, then I will say *x* times *x* is x^2, then I will say, ok! This is positive, then I will say *x* times 1 is *x*, then -3 times *x* is-3*x*, then -3 times 1 is -3 close [brackets], then next step, the -2 will remain as it is, I will put it down here, oh! Here there are like terms, the x^2 then, $x - 3x$ is $-2x$, then the -3 close [brackets]. Now we 'gonna' multiply, ok! The -2, (-2*x* times *x* is $-2x^2$), (-2 times -2*x* is +4*x*), (-2 times -3 is +6), so, this will be our original formula, ok! We 'gonna' calculate the turning point, we 'gonna'say $x = \frac{-b}{2a}$

Interviewer: Where is this formula coming from?

Mat: I just memorized it!

Interviewer: You did not attempt the other questions, why?

Mat: …eeh I can't proceed.

Interviewer: Please go through your solution to 8.1 with me I want to understand how you calculated the coordinates of the turning point.

Edy: I will use the formula $x = \frac{-b}{2a}$ where b = 4 and a = -2 from $h(x) = -2x^2 + 4x + 6$, we were told that *x* is the *x*-coordinate of the turning point then we substitute this value in the original equation to find the corresponding value of *y*.

Based on the responses on the turning point in the excerpts above, learners had an idea of what a turning point is, though their ideas are not complete which thus needed to be refined. Their responses indicate that learners just memorised that the *x*-coordinate of the turning point is $\frac{-b}{2a}$ without understanding where this formula is coming from and how it works indicating that they were operating at the action level. Though the learners could calculate to some extent the coordinates of the turning point they could not clearly explain how they were calculating and why they were calculating it in the way that they did.

Learners could explain how they sketched the graph using the asymptotes they had calculated though some of them were not sure as to where to start as indicated by their responses in the excerpts above. Most of the learners wanted to do a point by point plotting of the graph instead of only a few points (critical points) resulting in them taking longer to decide on the horizontal and vertical scales. They did not know what determines these scales as a result it took them long to draw the sketch (from my observation).

Learners' responses in the excerpts below revealed that learners had difficulties in answering questions that refer to the drawn graph.

Interviewer: What is Q on the diagram and how did you calculate its coordinates?

Diva: Q is a point on the *x*-axis, we know that on the *y*-axis, the value of *x* is 0 'umm' so we substitute this value into one of these equations, on this, on coordinates of *x* to find the value of *y*, so the coordinate of Q is (0; 6).

Interviewer: Correct. In 8.3 what is this m and b? How did you find them?

Diva: 8.3 numerical value of m and b....m is the gradient of this line b is on this point and then … the formula of the gradient m = $\frac{y_2-y_1}{x_2-x_1}$, the value of b which is 6, the value of m is -2.

Interviewer: Now how do you find the equation of *g(x)*?

Diva: I already calculated the gradient which is -2, so, usually this point b for *y*, it will be 2 for *x*, to find the value of c is 4, which means g (*x*) = -2*x* + 4.

Interviewer: How did you calculate the coordinates of Q?

Coco: I will say at Q, let *x* = 0 ...I'm not quite sure why *x* = 0. I want the value of *y*, and if you check the point at Q, *x* is already there, which is 0, so you need to calculate the *y*-value, so it will be like, $y = -2(0)^2 + 4(0) + 6$ then -2×0 it will give me 0 because any number ×0 is 0, then q = (0;6) then, the other point you determine the numerical value of m and b, which is, If I want to determine the numerical value, we use the point Q, because they are given and they are on the straight line, which means m is the gradient, so I need to calculate *x*, so m = $\frac{y_2-y_1}{x_2-x_1}$; and then, Q (0; 6), then P(3; 0), it will be …, I substituted *x* from the given coordinates will be y_2 is 0, y_1 is 6, x_2 is 3, then x_1 which is equal to 0, then $\frac{-6}{3}$, because we cannot say this minus the total of this, so m will be 2, will be -2, then the value of b, 'eish' this one is tough, I'm not quite sure how to calculate this, because there is this thing that r is the coordinate of (2; b), so b is the *y* -axis?

Interviewer: How did you calculate the coordinates of Q?

Monga: These last questions I was just writing what I had memorized I cannot actually explain how I got them.

Interviewer: Good! Take me through your solution of 8.2.

Teko: Ok! 8.2 is saying calculate the coordinate of Q. Before I calculate, I'm given $x = 0$, so I used the original formula again, then say $-2x^2 + 4x + 6$, ok! I will say where there is *x* I will put 0, ok!

$-2(0)^2 + 4(0) + 6$, then I punch this on the calculator, ok! I don't have to use the calculator because anything times by 0 is 0, so, I will say 'eh' Q(0;6). 'Q' is on the *y* -axis.

Interviewer: Why is $x = 0$ at Q?

Teko: We were told it's always zero there!

Interviewer: Ok. How did you determine the values of m and b?

Teko: This was simple! I used the coordinates of Q I got in 8.2 and coordinates P given and I used the formula for the gradient. For b, eish this one, I don't know. But for the equation of g(*x*) I used m above and for c I used the *y*-value for Q since it is the *y*-intercept.

Interviewer: How did you calculate the coordinates of Q?

Edy: I will replace the *x* by 0. This is obvious that this will be equal to 0 because we are on the y-axis and then my answer is 6, therefore I will answer the question, therefore Q is equal to (0;6). So my c the *y*-intercept is 6 already I will write the original formula $g(x) = mx + c$, then I will choose one point , is either P or Q, then, I have decided to take Q, whereby (0;6) and I will write *x* and write *y* there, and where there is *y*, 'oh' where there is g(*x*) I will put 6 =, my m, is -2, my *x* is 0, now I want the value of c, 6 = c, which means c = 6, then write $g(x) = -2x + 6$, then, this has been proven.

Interviewer: But you did not find the value of b. Why?

Edy: That one I don't even know where to start sir. I thought b was c at first but eh its not.

This was also documented in literature (Sierpinska, 1992). It was the most difficult part for most of the learners as they could not deduce the critical points from the sketched graph and use them to determine the required equations. Learners also had difficulties in determining the equation of the function represented by the drawn graph using the indicated critical points. This indicated that learners were not aware that the graph and the equation are just different representations of the same function that are well connected by critical points. These difficulties support findings in a study carried out by Knuth (2000).

Learners' responses both written and oral in the initial tasks and interviews on the representation of the function concept indicate that on the average, 5 learners were operating at the action level and Coco was operating at the process level as indicated by their explanations of meanings and procedures in calculating the critical points which are shown in the next table.

Table 10: Summary of learners' initial APOS theory conception levels on the representation of the function concept before teaching experiment

	Explanations on aspects of a representation of a function									
Learner	Preferred representation	Intercepts	Calculation of intercepts	Asymptotes	Determination of asymptotes	Turning point	Determination of coordinates of turning point	Procedure of sketching graph	Translation from graph to equation	APOS conception level in which a learner is generally operating in
Mat	Equation	A	A	A	P	A	P	P	A	ACTION
Teko	Table	A	A	A	A	A	P	A	A	ACTION
Coco	Table	P	P	A	P	A	P	P	A	PROCESS
Diva	Expression	A	P	A	P	A	P	A	A	ACTION
Monga	Equation	A	A	A	A	A	P	A	A	ACTION
Edy	Equation	A	A	A	A	A	P	A	A	ACTION

Key: A- Action, **P-** Process, **O-** Object and **S-** Schema

What are the weaknesses in learners' understanding of the representation of the function concept?

Learners' concept images, reasoning and difficulties indicate that they do not understand the meanings, procedures of calculating the critical points of a graph and the process of translating from the graphical representation to the symbolic representation. These critical points which I termed "connectors" in later sections are important to learners' ability to translate from one representation to another form. I was also able to detect that there are weaknesses in learners' understanding because their answers showed that they do not have indicators to show the schema level. Analysis of learners' interview responses in Sections 5.3.1 to 5.3.3 enabled me to find problem situations (difficulties), misconceptions and weaknesses in translating from one representation to the other. Learners' problem situations are summarised below:

- Learners had difficulties in answering questions that refer to a drawn graph. They could not deduce the critical points from the sketched graph and use them to determine the required equations.
- Learners also had difficulties in determining the equation of the function represented by the drawn graph using the indicated critical points. This indicated that learners did not understand the equivalence between the algebraic and graphical representation of the function concept.
- Learners could calculate the intercepts but did not know why, at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$. They could just follow the procedure without understanding it.
- They memorised $x = \frac{-b}{2a}$ for the turning point without knowing what it means and where this formula came from.
- They could not tell when a function had an asymptote and what an asymptote means.
- Learners could easily move from equation to graph but could not use a drawn graph to find the critical points and to determine the equation. I think this was because to move from an equation to a graph needs more of procedural understanding while to move from graph to equation requires more of conceptual understanding.

Markovits et al. (1986) and Zaslavsky (1997) each indicated that translation of functions from graphical to algebraic was more difficult than vice versa for learners. That was the case for participants in this study. They could not use the definitions of the critical points to identify and extract them from the drawn graph. Similarly, the data concurred with Eisenberg and Dreyfus (1994) who reported that the translation of functions from algebraic to graphical representation was easier for learners than the translation from graph to equation. Participants demonstrated difficulty translating from graph to equation. A question was raised by this finding: Why did grade 11 learners have difficulty with translation from graphical to algebraic representations? Analysis of the data led me to believe that the learners did not see the role of the critical points and their definitions in the translation process.

Learners indicated that they prefer to translate from equation to graph and that they find it difficult to know what to do when the graph is given for them to refer to (Eisenberg, 1991). This indicated that learners were taught these representations separately without highlighting the connections between them. The nature and prevalence of learners' difficulties and misconceptions indicated that there was a need for an intervention to help learners to appreciate that representations of the function concept (equation and graph in this particular case) are equivalent and well connected by the critical points.

Learners' difficulties with the connections among the different representations of the function concept were also prevalent in the literature review. Similar to the above difficulties, Knuth (2000) reported that "learners experienced difficulties in translating between the symbolic and the graph and as a result failed to identify the link or connection between these representations in a given problem situation" (p. 34). Ainsworth (1999) adds that "learners are faced with the complex task of not only understanding how each representation encodes and present information but also of understanding how these representations relate to the function concept" (p. 12).

5.6.2 Phase 2: Development of interventions

In this phase I referred to the RME's learning and teaching principles summarized in Section 5.4.2 in developing instructional activities that helped learners to improve their understanding of the representation of the function concept. Phase 2 addresses research question 2:

How can instruction be designed to improve learners' understanding of the representation of the function concept?

According to the theoretical framework for this study I brought in RME and related actual examples in my lessons to the learners' needs which I identified above in the form of weaknesses, difficulties, misconceptions and their APOS conception level. These weaknesses, difficulties, misconceptions and their APOS level indicated that there was need for intervention in order to help learners reduce their identified difficulties and misconceptions and help them to move up the conceptual ladders of APOS theory from the current process level to the object level then to the schema level. The inability of learners to translate flexibly from the graph to the equation indicated a conceptual gap that kept learners from understanding the equivalence between these representations. To bridge this gap I had to identify the HLT for overcoming this conceptual gap and plan activities that were likely to reduce this conceptual gap.

Being aware of learners' current understanding of translating from the equation to the graph and vice-versa; their weaknesses (difficulties), misconceptions and the APOS conception level which they were operating at, gave me the opportunity to formulate and choose activities that could help the learners to overcome their weaknesses and misconceptions. I designed instructional activities appealing to the principles of Realistic Mathematics Education (RME) and constructivism to help learners appreciate the connections between the equation and its graph as representations of the function concept and to enhance their ability to translate from one representation to the other. These activities were also intended to help shift learners from the process conception level to the object and then to the schema.

5.6.3 Phase 3: Tentative products

5.6.3.1 Teaching experiment 4

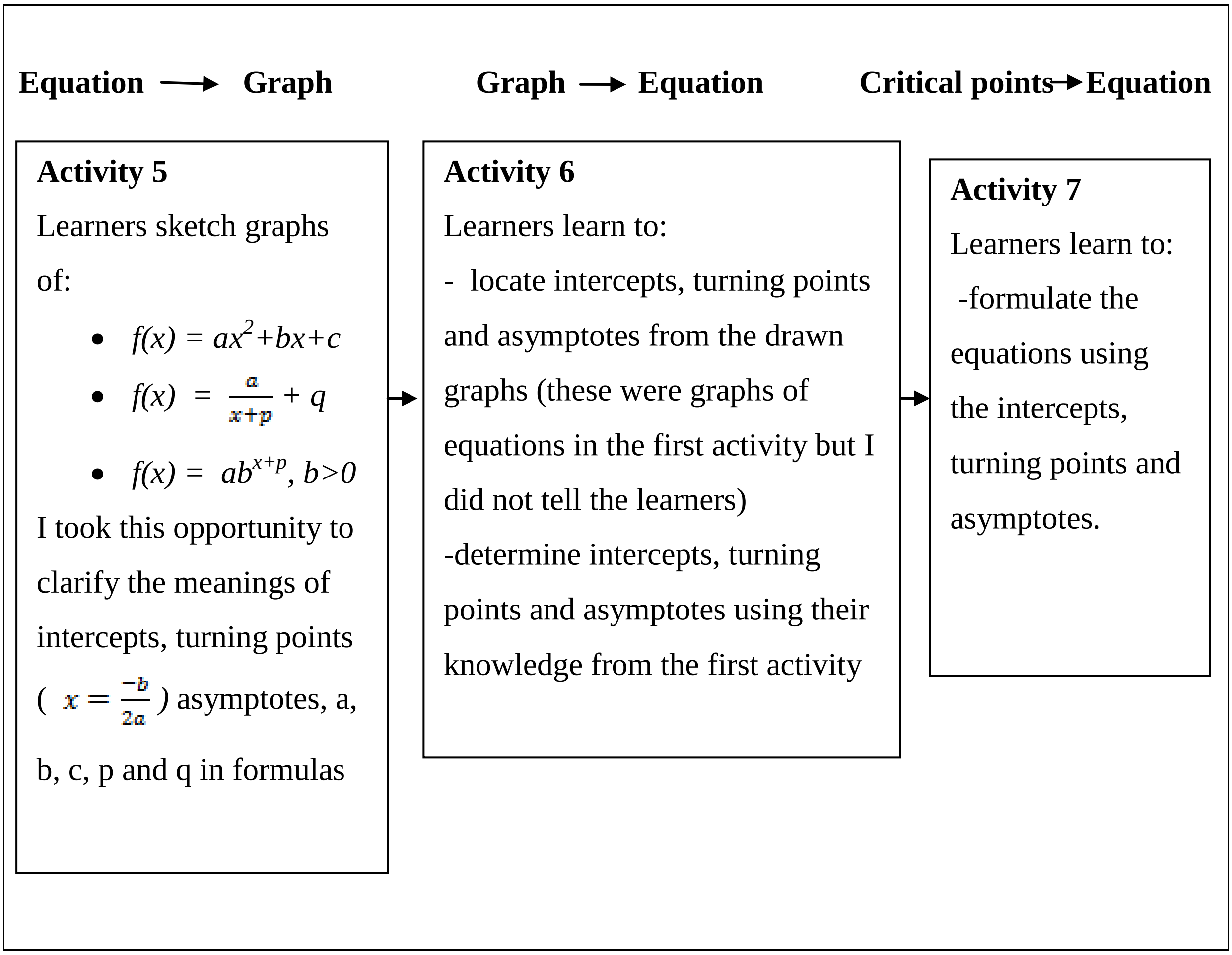


Figure 39: HLT for developing an understanding of the representation of the function

This HLT was intended to help learners realise that the critical points (intercepts, turning points and asymptotes) connect the equation and the graph. As such I hoped that if learners understand these critical points they will be in a better position to understand the connection between the equation and the graph, even with other representations like the table of values, verbal and the ordered pairs. The implication was that for learners' thinking to be adaptable, they need to readily move from one representation to the other and vice versa, requiring fluid translation among the representations and to view its representations as different views of the same construct. The term 'realistic' in this case refers not just to the connection with the real world, but also refers to problem situations which are real in learners' minds. Bottle (2005) also points out that '*realistic'* does not just mean real-life situations but can also mean 'to realise' or 'to imagine'. It therefore means that contexts that are realistic to learners (although imaginary) are included.

Teaching experiment 4 was designed to help learners appreciate the connections between the equation and its graph as representations of the function concept and enhance their ability to translate from one representation to the other. This will help learners to construct the process and object conceptions of the function concept as they could be able to imagine the process of translating from one representation to the other.

By the end of this teaching experiment learners should be able to:

i. sketch graphs of $f(x) = ax^2 + bx + c$, $f(x) = \frac{a}{x+p} + q$ and $f(x) = ab^{x+p}$; $b > 0$;

ii. use drawn graph to explain the location of the intercepts, asymptotes and turning points; and

iii. use the intercepts, asymptotes and turning points to formulate the equations.

The CAPS objectives require learners to be able to convert flexibly between these representations (tables, graphs, words and formulae). Types of functions included are linear and quadratic polynomial functions, exponential functions, and some rational functions.

The teaching experiment comprised three lessons.

Lesson 1

Learners should be able to:

Sketch graphs of $f(x) = ax^2 + bx + c$, $f(x) = \frac{a}{x+p} + q$ and $f(x) = ab^{x+p}$; $b > 0$.

The lesson was structured along the HLT. First I explained what it means to draw a sketch of a graph with point-by-point plotting. I then took this opportunity to clarify the meanings of intercepts, turning points and the use of $x = \frac{-b}{2a}$, asymptotes, and a, b, c, p and q. These clarifications also included the calculations of these critical points and the process of plotting them on the Cartesian plane and sketching the graphs. These explanations were meant to make these concepts imaginable to a learner which is a characteristic of RME. At the end of this lesson learners were expected to sketch the graphs of the above functions using the following characteristics: domain and range; intercepts with the axes; turning points; asymptotes; shape and symmetry. To assess learners' progress I designed the activities that follow, which they did in pairs taking advantage of social constructivism.

Activity 5: From equation to graph

Sketch the graphs of the following functions indicating all asymptotes, turning points and intercepts with the axes.

1. $f(x) = x^2 - 6x + 8$

2. $f(x) = -x^2 - 5x - 6$

3. $g(x) = \frac{3}{x-2} + 3$

4. $g(x) = -2.2^{x-1} + 1$

Activity 5 aimed to develop learners' ability to determine the critical points from the given equation of the function concept and to plot these points before joining them to draw the graph. Learners' written responses are shown and their analysis follows at the end.

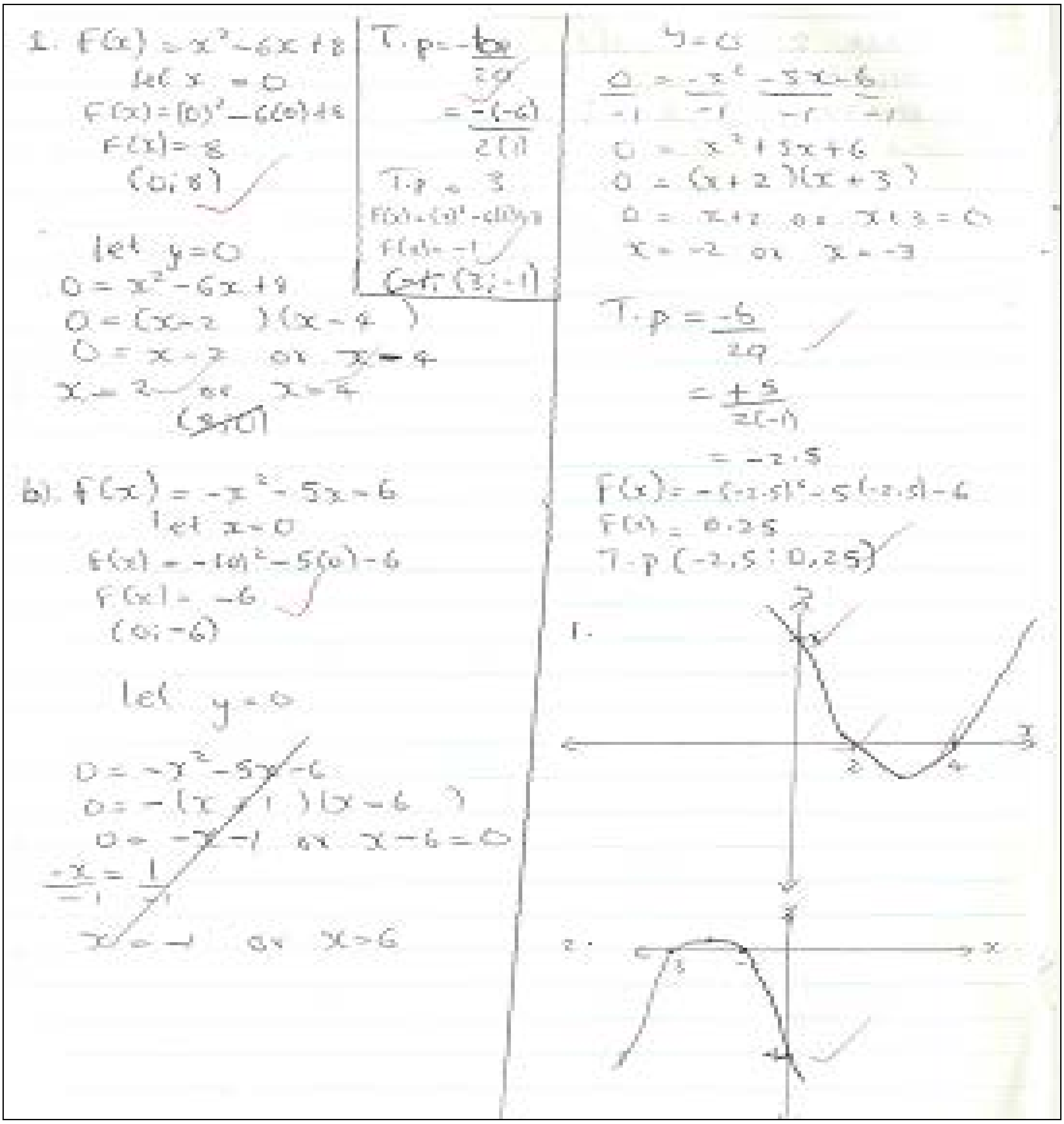


Figure 40: Monga and Teko's responses on activity 5

Monga and Teko showed that they had understood the process of translating from an equation to the graph by first calculating the critical points and plotting them. This indicates that they are now operating at the process level.

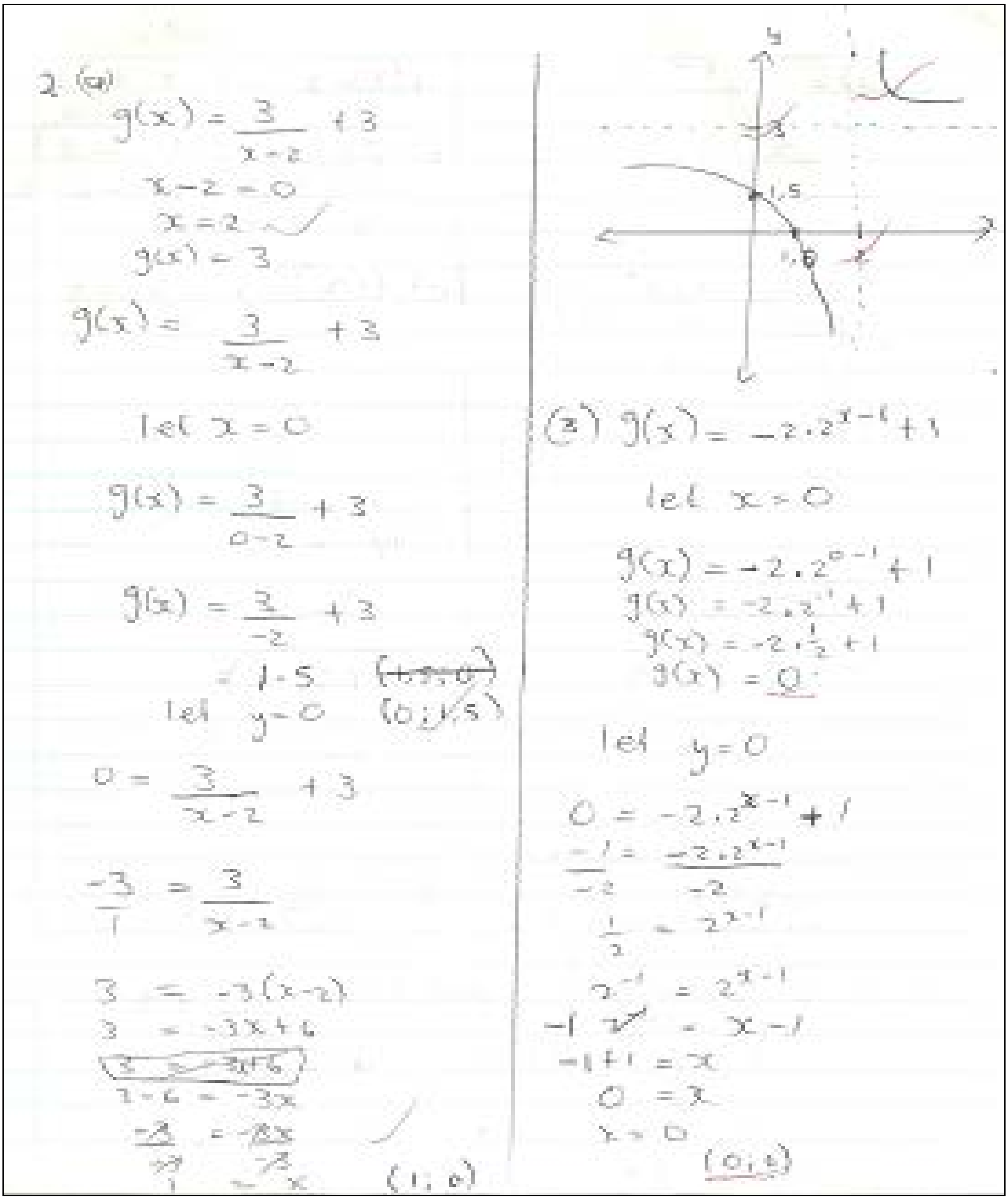


Figure 41: Mat and Coco's responses on activity 5

Mat and Coco showed that they had understood the process of translating from an equation to the graph by first calculating the critical points and plotting them. This indicates that they are now operating at the process level.

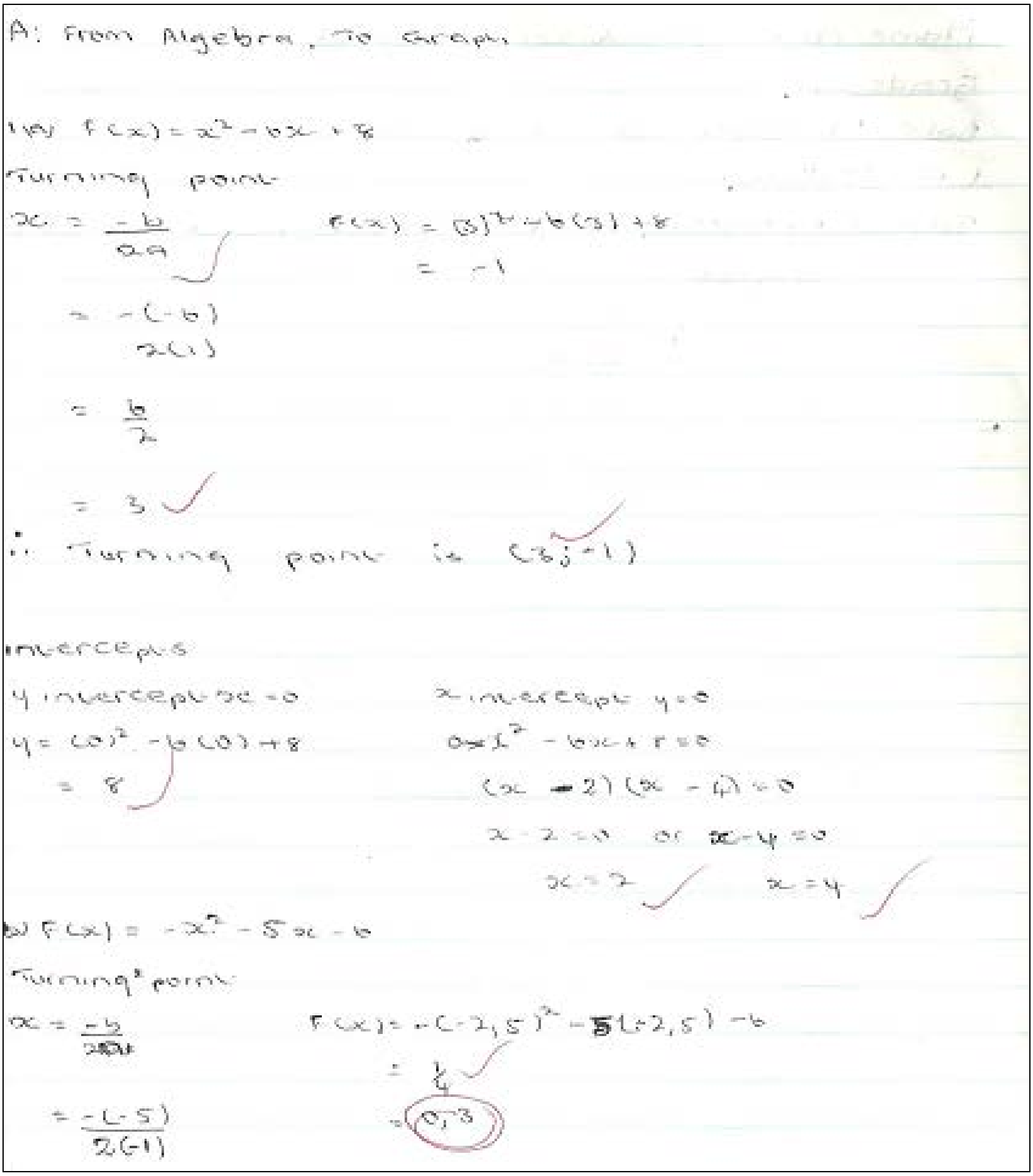


Figure 42: Diva and Edy's responses on activity 5

Diva and Edy could determine all the critical points from the given equation showing that they had understood this critical process though they failed to produce the required sketches. This indicates that they were operating between the action and process levels.

Lesson 2

By the end of this lesson learners should be able to:

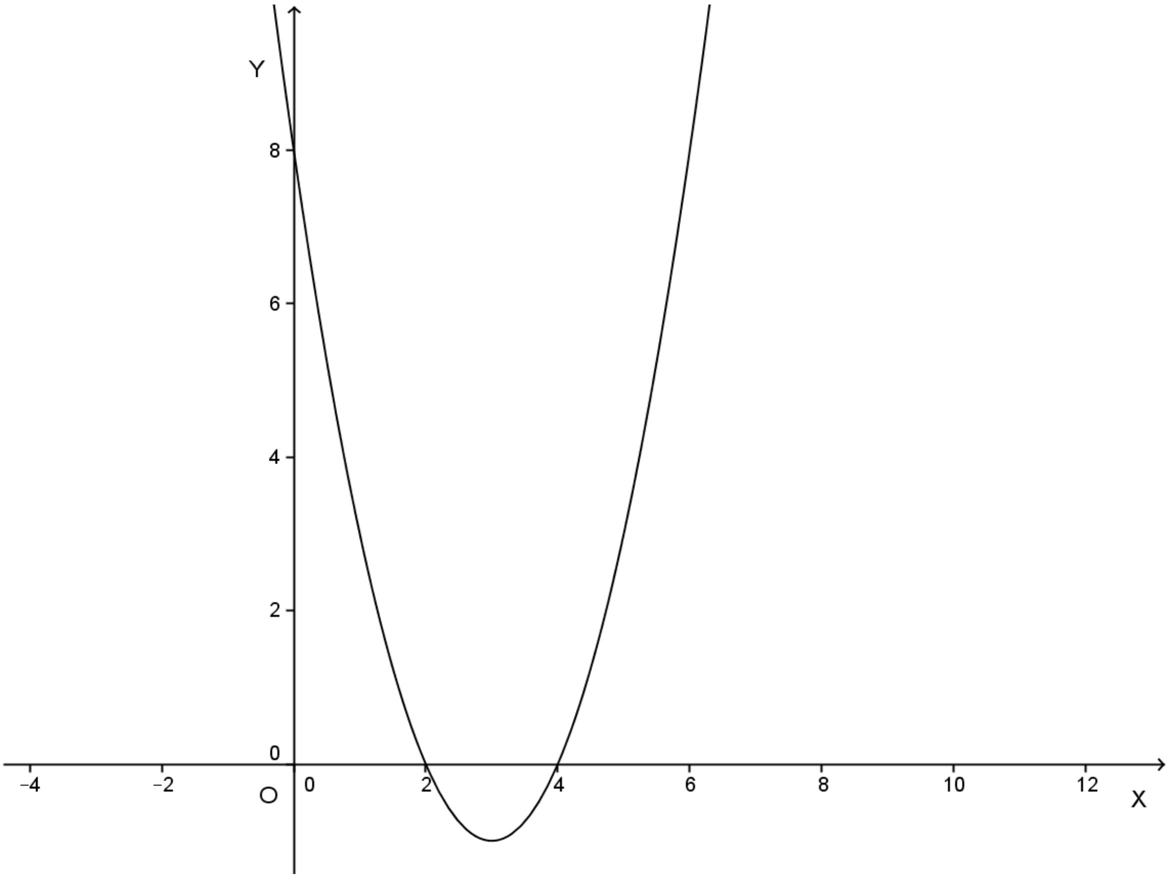
i. identify the intercepts, asymptotes and turning points from drawn graphs; and

ii. calculate the intercepts, asymptotes and turning points using the methods they used in activity 5.

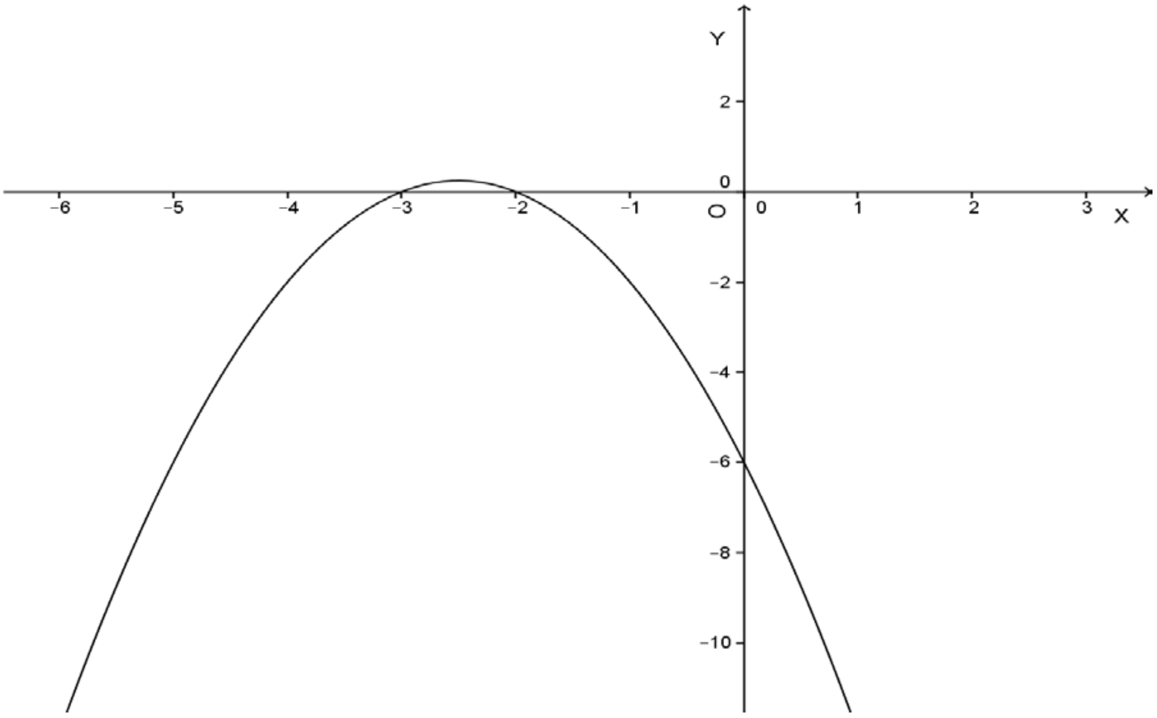
I provided learners with sheets of papers with drawn graphs. These were the graphs learners had drawn in activity 5 but I did not inform them of this. The intercepts, asymptotes and turning points for each graph were clearly shown and learners were asked to identify them. After identifying them learners would then calculate these critical points using the methods they used in activity 5. Since this activity was a bit challenging learners also worked in two groups of three members each.

Activity 6: From graph to equation

1.



2.



The graphs above represents the functions in the form of $f(x) = ax^2 + bx + c$. For each graph above:

a) Determine the values of a, b and c.

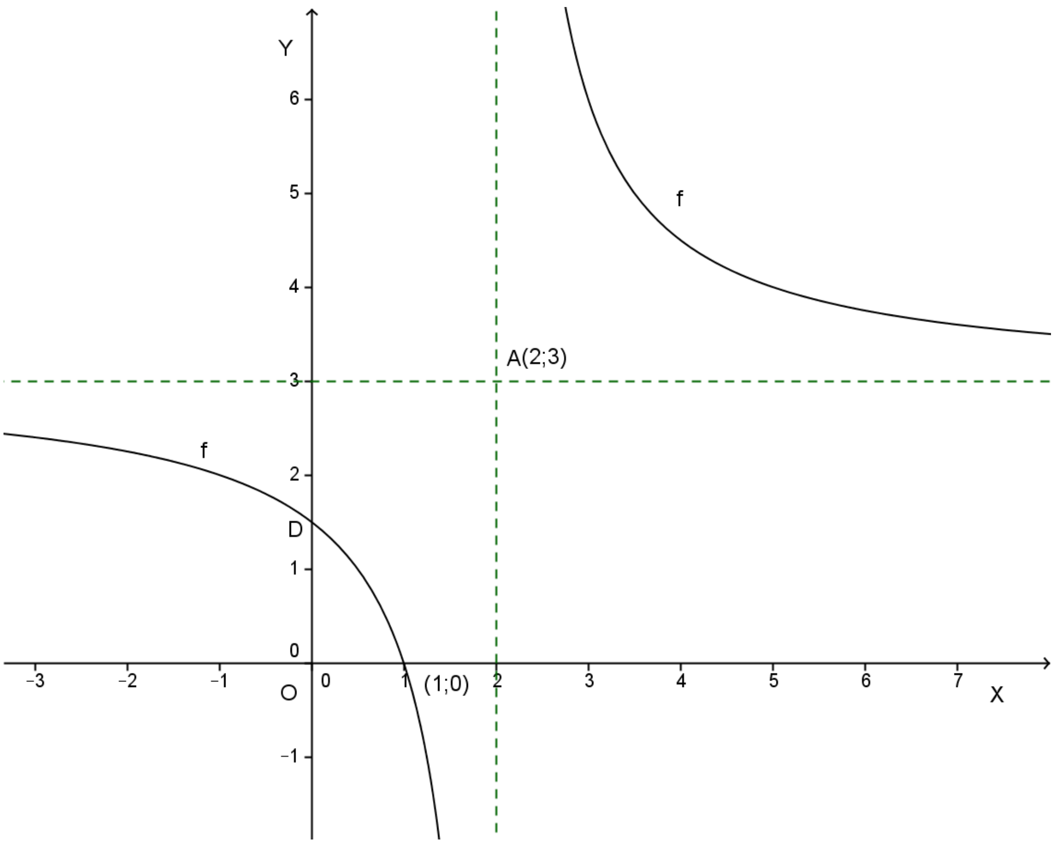
b) Determine the values of x for $f(x) = 0$.

c) Determine the coordinates of P, the turning point of $f(x)$.

d) Hence, determine the range of $f(x)$.

3. Given $f(x) = \frac{a}{x-p} + q$. The point A (2; 3) is the point of intersection of the asymptotes of *f*.

The graph of *f* intersects the *x*-axis at (1; 0). D is the *y*-intercept of *f*.

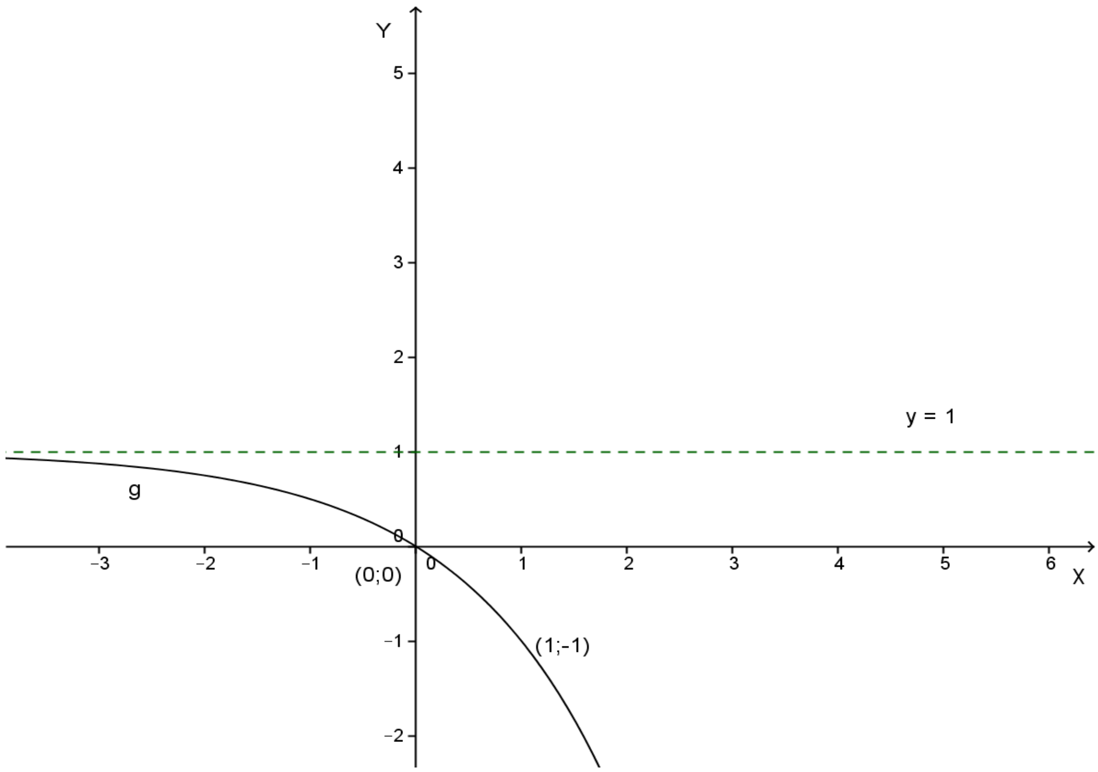


a) Write down the equations of the asymptotes of *f*.

b) Determine the equation of *f*.

c) Write down the coordinates of D.

4. The diagram below shows the graph of: $g(x) = ab^{x-1} + q$. A (1;-1) is a point on g and $y = 1$ is the asymptote of g.



a) Determine the values of a, b and q.

b) Determine the y-intercept of g.

c) Write down the range of g.

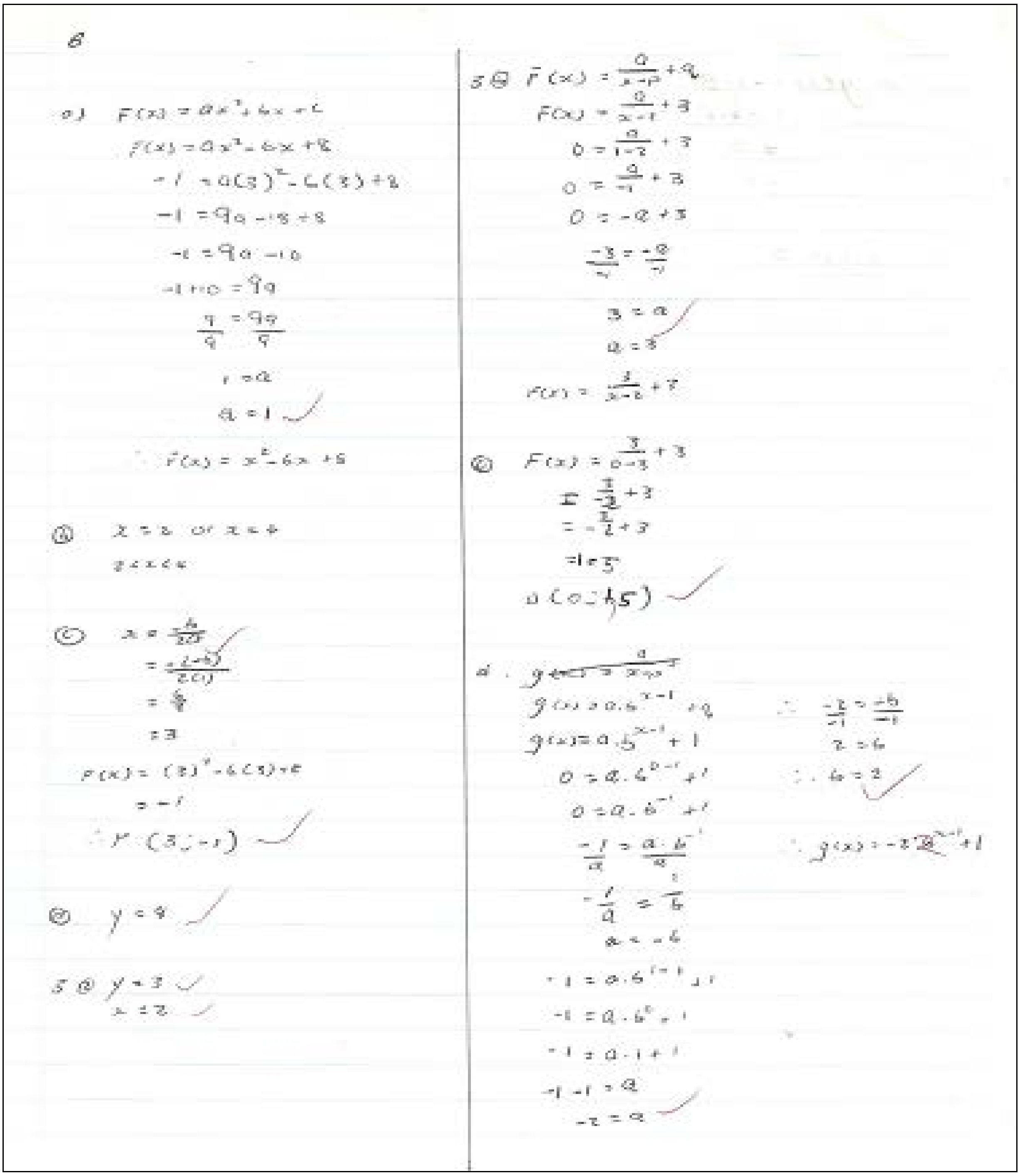


Figure 43: Coco, Mat and Teko's responses on activity 6

Though learners in this group got the correct answers they were still struggling with this process of translating from the graph to the equation. They could not agree on their understanding which was at the process level.

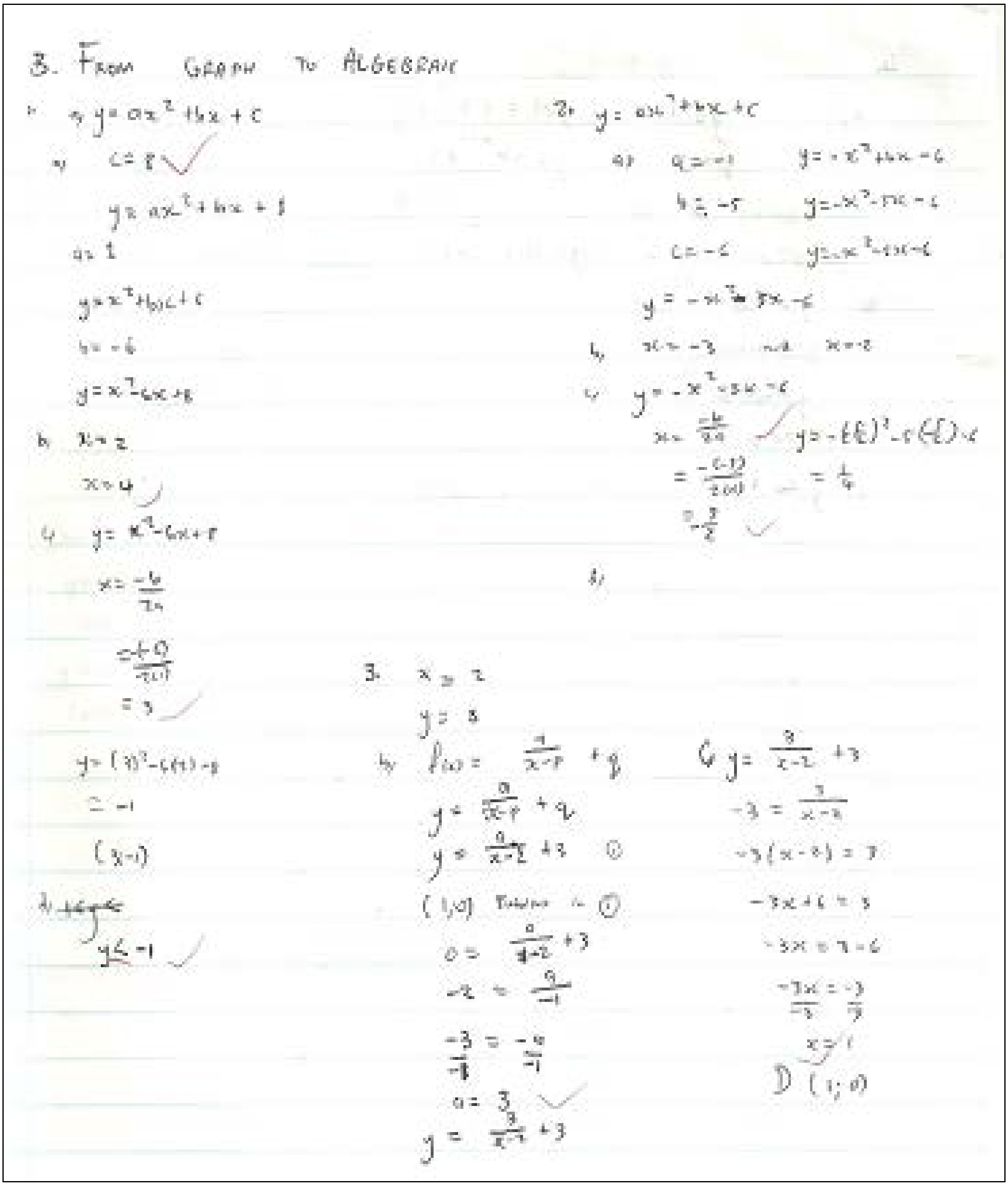


Figure 44: Monga, Edy and Diva's responses on activity 6

Learners in this group had similar challenges as the other group in that they also failed to agree on their understanding of translating from a graph to an equation. They could follow procedures and obtain correct answers without understanding them. This shows that they were operating at the process level.

Lesson 3

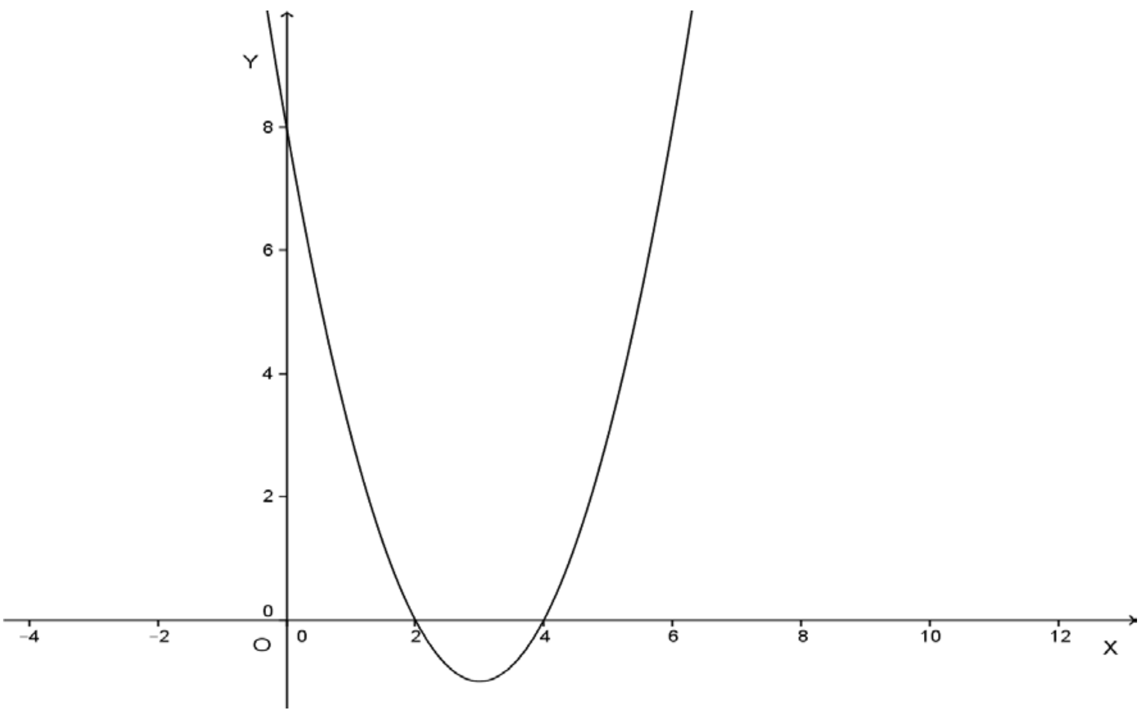
By the end of the lesson learners should be able to:

Use the intercepts, asymptotes and turning points to formulate the equations.

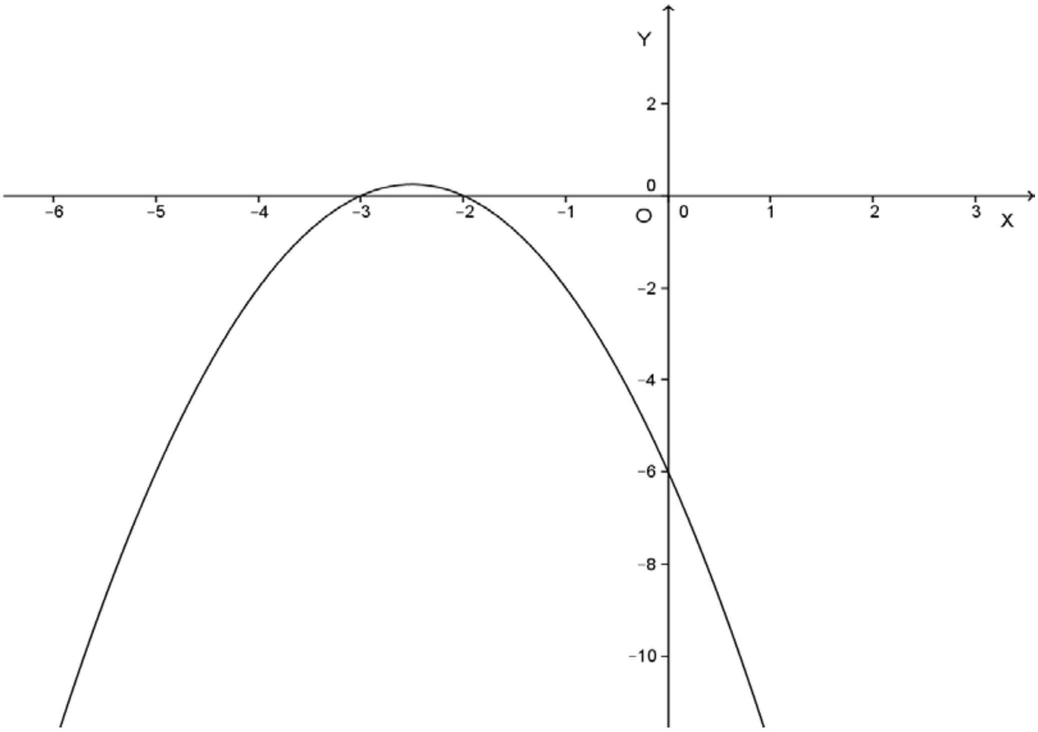
In this lesson the connection between the equation and the graph is unpacked by demonstrating that the critical points link these two representations of the function concept. The first two activities are revisited and learners were asked to make some observations from these two activities. A few learners realised that the graphs they had drawn in activity 5 were the same as the ones I provided them in activity 6. I showed and explained how the critical points connect the equation and the graph. After developing the links I explained the process of formulating equations using the identified critical points. This process was challenging to learners as it took long for them to realise and understand the formulation of equations. I was not surprised because this was their original difficulty which prohibited them from translating from the graph to the equation.

Activity 7: Identifying and using critical points from graphs to formulate equations

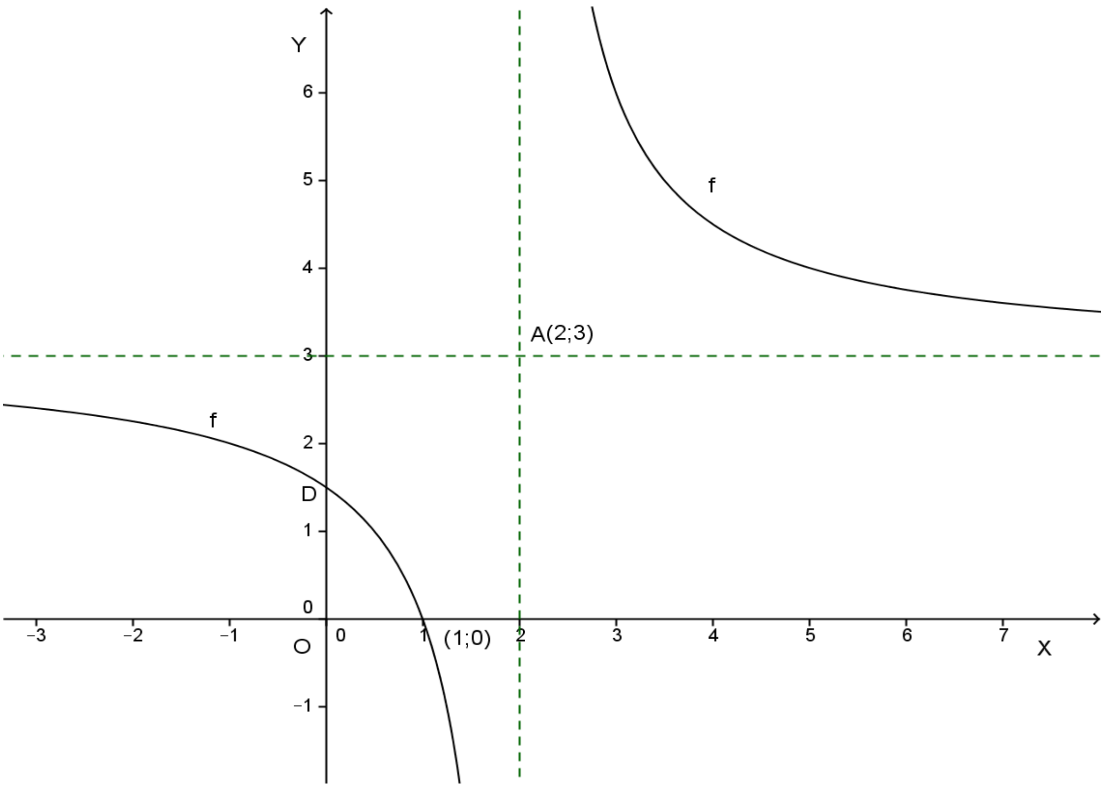
1.



2.


Y
2
0
−6
−5
−4
−3
−2
−1
O
0
1
2
3
X
−2
−4
−6
−8
−10

3.


Y
6
5
4
f
A(2;3)
3
f
2
D
1
0
−3
−2
−1
O
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(1;0)
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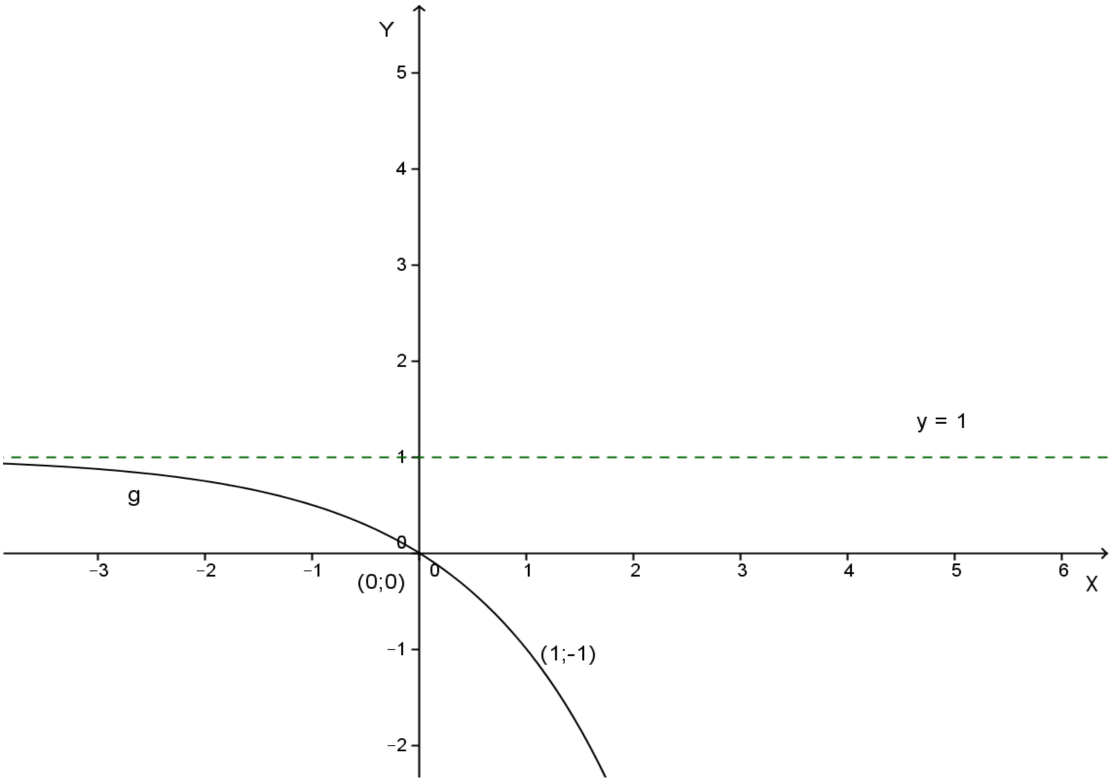
6
7
X
−1

4.



i. For each of the graphs above identify their respective critical points and use these critical points to formulate their equations.

ii. How do you know whether the equation is correct or not?

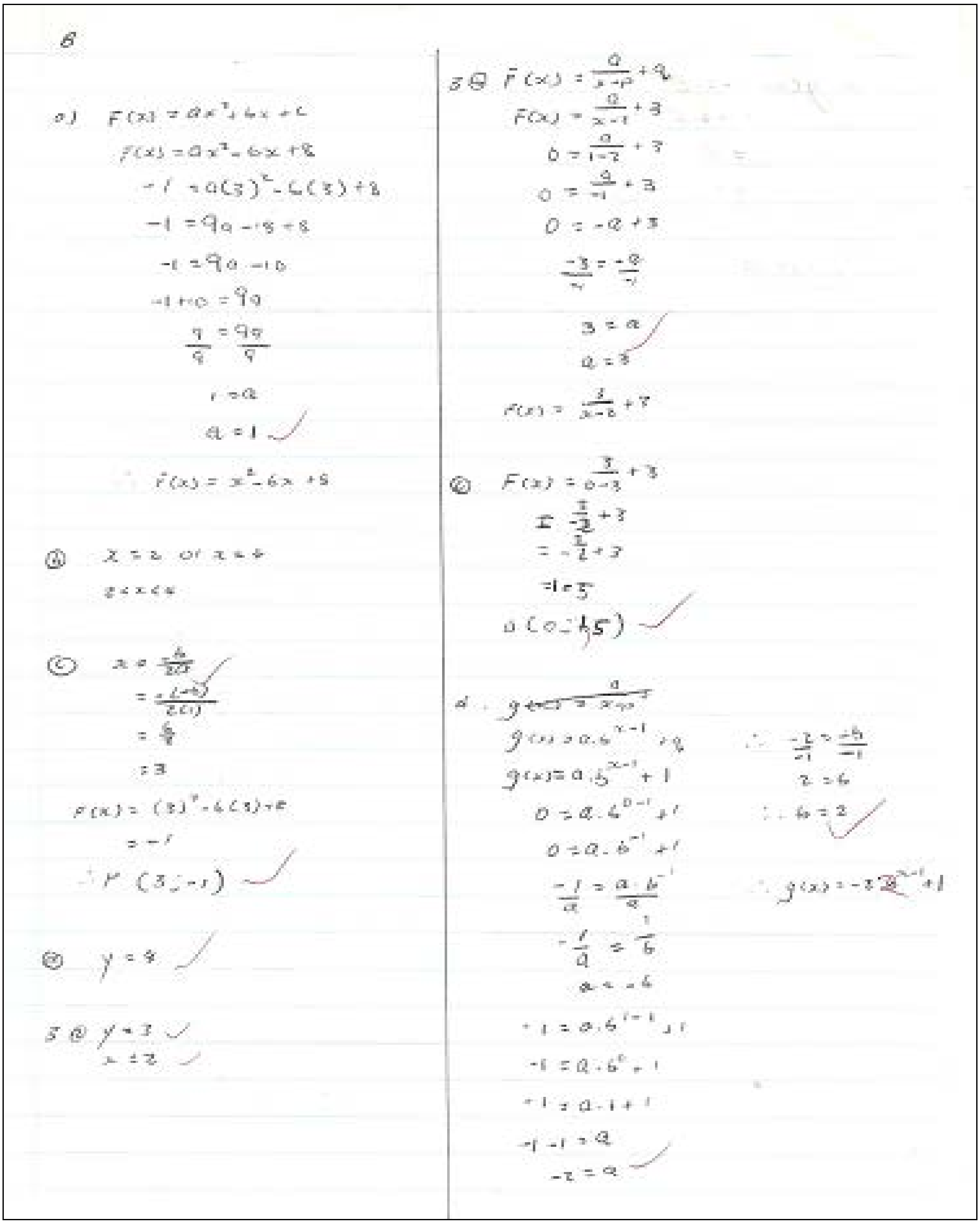


Figure 45: Learners' responses (they worked as a group of 6)

5.6.3.2 Retrospective analysis

The purpose of activity 5 was to develop learners' understanding of the process of translating from the symbolic to the graphical representation. Translation from an equation to the graph proved to be within the learners' comprehension as indicated by their written responses above. They were able to calculate the intercepts, asymptotes and turning points of different functions prescribed in the South African CAPS curriculum. As such, the first objective of my HLT was attained as intended.

Translation from graph to equation in activity 6 was still difficult for learners as I observed them arguing and at times agreeing to disagree in their groups. It seemed that previously learners were routinely given tasks that require translations in the equation-to-graph direction. As a consequence, learners were now having difficulties with tasks in which they had to proceed in the graph-to-equation direction. Learners' actual understanding of the translation process from graph to equation was superficial at best and mechanical in some instances. They also seemed to perceive that the graph is only the culmination of their ritualistic equation-to-graph procedure.

Activity 7 was well done maybe because learners were working as a whole group. The learners showed indicators of the object level though they were still not comfortable with the way critical points connect the graph and its equation. As a result I had to refine my previous HLT.

5.6.4 Phase 4: Product and theory refinement

5.6.4.1 Teaching experiment 5 (Prototype 5)

I had to adjust the previous HLT by putting more emphasis on basing the procedure on the definitions of the connection factors (critical points) as the bridge between procedural and conceptual understanding of the translation process between the equation and the graph. As a result, each step of the procedure begins with the definition of the critical point involved as follows:

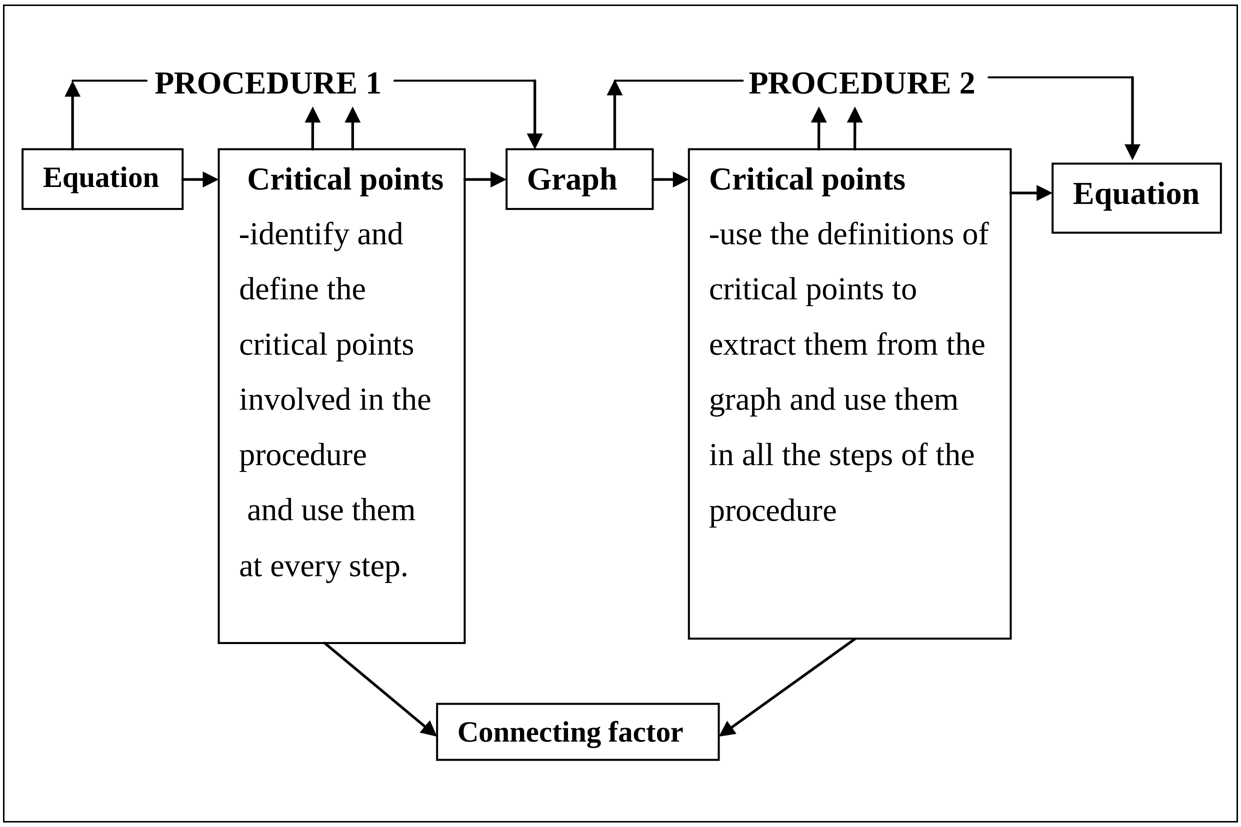


Figure 46: HLT for using a definition-based procedure

This was a consolidation lesson. The goal was to help learners to grasp the importance of the definitions of the critical points when translating from the graph to the equation. By the end of this lesson learners should be able to:

i. locate the intercepts, asymptotes and turning points on a drawn graph using their definitions;

ii. use the intercepts, asymptotes and turning points to formulate the equation; and

iii. translate from a graph to an equation.

It was assumed that the learners already knew how to calculate the intercepts, asymptotes and turning points. First I explained the adjusted HLT and illustrated it with examples for each type of graph.

5.6.4.2 Illustrative example for using the definition-based procedure

1. Equation $y = x^2 - 6x + 8$ **EQUATION**

↓

2. Critical points: i. Intercepts: *x-intercept*: $y = 0$ **CRITICAL POINTS**

$x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x = 2 \text{ or } x = 4$

(2; 0); (4; 0)

y –intercept: $x = 0$ …… $y = 8$

(0; 8)

ii. *Turning point*: $\frac{dy}{dx} = 0$ from $y = ax^2 + bx + c$

$2ax + b = 0$

$x = \frac{-b}{2a}$

From $y = x^2 - 6x + 8 \rightarrow a = 1 \text{ and } b = -6$

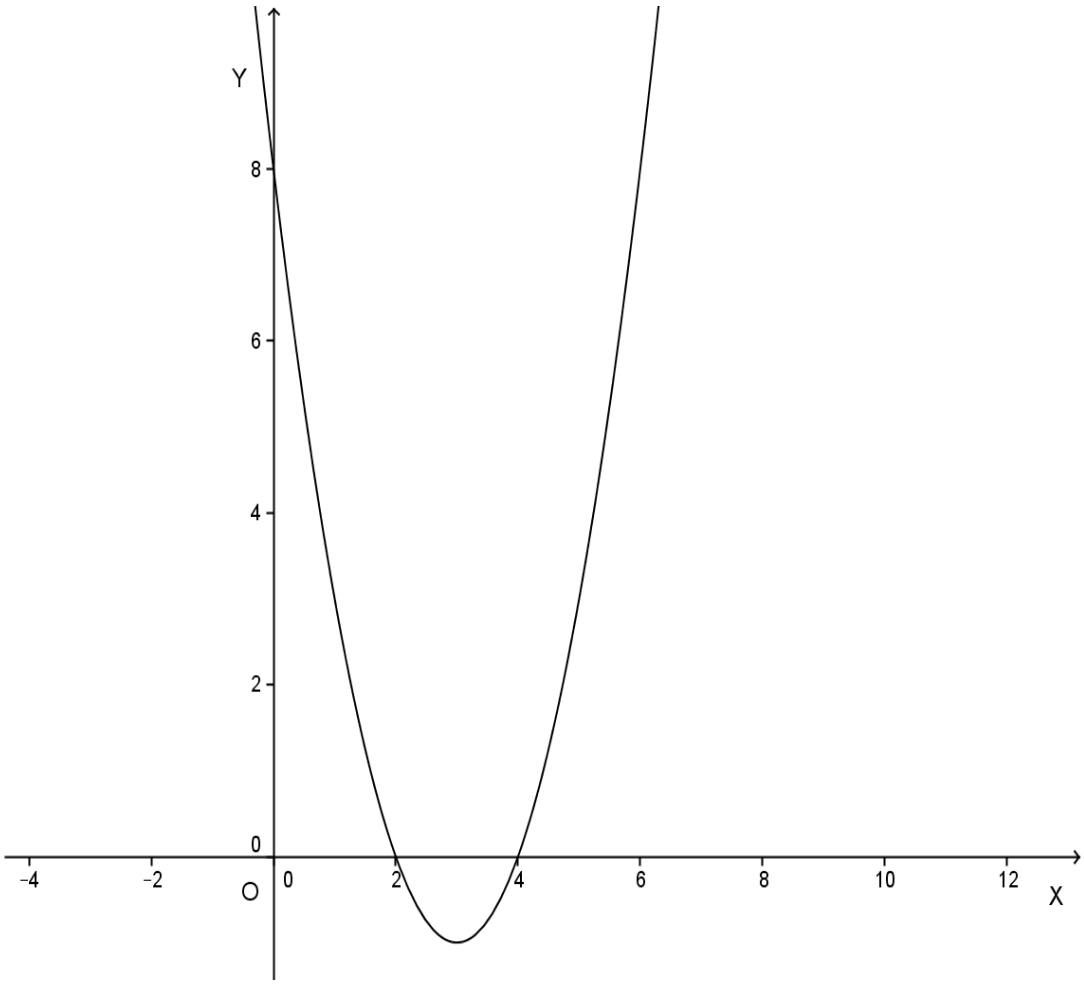
$x = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$

Corresponding *y*-value = $3^2 - 6(3) + 8 = -1$

(3; - 1)

↓

3. Plot all the critical points on the Cartesian plane and sketch the graph **GRAPH**



4. Critical points: *x*- intercepts: (2 ; 0) and (4 ; 0) **CRITICAL POINTS**

y – intercept (0 ; 8)

Turning point (3 ; -1)

5. Equation : general equation of a quadratic function: $f(x) = ax^2 + bx + c$ **EQUATION**

C is the y- intercept, so $c = 8$, so we have $y = ax^2 + bx + 8$

Since the intercepts lie on the curve they satisfy the equation

At (2 ;0) $4a + 2b + 8 = 0$............1

At (4 ; 0) $16a + 4b + 8 = 0$..........2

Multiplying .....1 by 2 $8a + 4b + 16 = 0$ ....3 in order to solve the equations

Subtracting .....3 from ....2 we get $8a – 8 = 0$

$a = 1$

Substituting the value of a in ....1 we get the value of $b = -6$

Then we substitute the values of a, b and c in the general equation $f(x) = ax^2 + bx + c$ to get $f(x) = x^2 – 6x + 8$ which is what we started with.

5.6.5 Phase 5: Final product and contribution to theory

5.6.5.1 Activity 8: Using definition-based procedures (DBPs) in the translation process

A definition-based procedure is one in which every step in it is based on the definitions of key concepts (intercepts, asymptotes and turning point) that are involved. This was to allow learners to develop a conceptual understanding of the procedures involved in translating from equation to graph and vice versa. This compelled learners to understand the meanings of all the critical points before they could calculate them or extract them from a drawn graph. Critical points are key concepts in the translation process and once they are understood it is easier for learners to translate from equation to graph and from graph to equation. The activities in which I engaged learners, were meant to compel learners to understand the importance of these key concepts in the procedure of calculating these critical points which they have been doing with little or no understanding. In this study I introduced the concept of Definition-Based Procedures (DBPs) to reduce this gap.

In activity 8 I asked learners to repeat question 7 of the June 2011 examination which learners had done for illustration purposes.

QUESTION 7

Given $f(x) = \frac{1}{x-4} + 2$

7.1	Calculate the coordinates of the x and y intercepts of f.	(4)
7.2	Determine the equations of the asymptotes of $f(x)$.	(2)
7.3	Sketch the graph of $f(x)$ showing all the critical points.	(4)
		[10]

This activity was done in pairs following the steps in the illustrative example in Section 5.6.4.2 but basing these steps on the definitions of the critical points. From my theoretical framework the term 'realistic' in RME also means to realise or imagine. This means that imaginary situations are also included. In this activity I first discussed the definitions of intercepts, turning point and asymptotes to make the definition-based procedure imaginable to learners. Together with learners we agreed on the following definitions of critical points that they were going to use:

Intercepts: *x-intercept* is the point where the graph crosses the x-axis and on the x-axis $y = 0$.

y-intercept is the point where the graph crosses the y-axis and on the y-axis $x = 0$.

Turning point: Is a point on a graph where the graph has just stopped increasing or decreasing and is about to change direction and start decreasing or increasing. A turning point is a minimum if the graph stops decreasing and is about to start increasing and is a maximum if it stops increasing and is about to start decreasing. At a turning point the graph is neither increasing nor decreasing resulting in the rate of change being zero ($\frac{dy}{dx} = 0$). This fact is used in deriving the x-coordinate of the turning point from the quadratic function $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b = 0$$

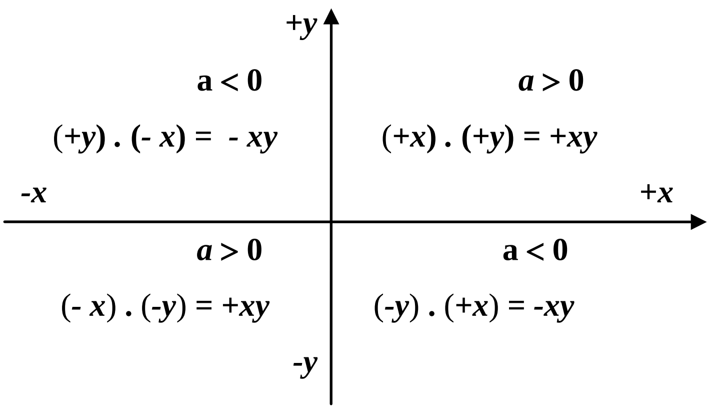
$$x = \frac{-b}{2a}$$

Asymptotes: For the hyperbola y $= \frac{a}{x-p} + q$.

x-asymptote (vertical asymptote) is the value of p which makes the function undefined. And for the function to be undefined *x – p = 0* because $\frac{a}{0}$ is undefined. This means the graph of the function can only approach this line but not touch it because once it touches this line it becomes undefined. p is not in the domain of the function and the graph is discontinuous at *x - p = 0*.

y-asymptote (horizontal asymptote): we first make *x* the subject in $y = \frac{a}{x-p} + q$ to obtain $x = \frac{a}{y-q} + p$. In this case, the *y-asymptote* is the value of q which makes the function undefined. And for the function to be undefined *y – q = 0* because $\frac{a}{0}$ is undefined. Thus, from $y = \frac{a}{x-p} + q$ it is shown that p is the *x*-asymptote and q is the *y*-asymptote

Effects of a: It should also be noted that ***a*** in the equation $y = \frac{a}{x-p} + q$, positions the two graphs in their respective quadrants. We start from $y = \frac{a}{x}$, the basic equation of the hyperbolic function which can be simplified to ***xy = a***.



If ***a* > 0** it means the product ***xy*** is positive. This is the case in the first and third quadrant. So, if **a > 0**, the graph lies in the first and third quadrants. If **a < 0** it means the product ***xy*** is negative. This is the case in the second and fourth quadrants. So, if **a < 0**, the graph lies in the second and fourth quadrants.

Learners' responses on using the definition-based procedure

Each pair was tasked to do and present one part of question 7 and their responses are shown in the figures below:

7.1 *x-intercept* is the point where the graph crosses the *x*-axis and on the *x*-axis $y = 0$

$f(x) = y = \frac{1}{x-4} + 2$

$0 = \frac{1}{x-4} + 2$, substitute y by 0

$-2 = \frac{1}{x-4}$

$-2(x - 4) = 1$, cross multiplying

$-2x + 8 = 1$, removing brackets

$-2x = -7$

$x = 3.5$. The coordinates of the *x-intercept* (3.5; 0)

y-intercept is the point where the graph crosses the *y*-axis and on the *y*-axis $x = 0$

$f(x) = y = \frac{1}{x-4} + 2$

$y = \frac{1}{0-4} + 2$, substitute x by 0 and simplify

$y = 1.75$. The coordinates of the *y*-intercept (0; 1.75)

Figure 47: Monga and Diva's response

Monga and Diva managed to use the definition of intercepts in the procedure of calculating the coordinates of the *x* and *y* intercepts. This indicates that they are operating at the process level of APOS theory.

7.2 *x-asymptote:* $f(x)$ is undefined if $x - 4 = 0$ $\longrightarrow$ $x = 4$ *(vertical asymptote)*

y-asymptote: $f(y)$ is undefined if $y - 2 = 0$ $\longrightarrow$ $y = 2$ *(horizontal asymptote)*, after making x the subject of the formula.

Figure 48: Coco and Edy's response

Coco and Edy demonstrated that they understood the concepts of horizontal and vertical asymptotes which made it easier for them to calculate these. This indicates that they are operating at the process level.

7.3. We first draw the two asymptotes which also form four quadrants. In this case a **= +1** so our graphs lie in the first and third quadrants. When drawing the graphs we make sure that they do not touch the asymptotes.

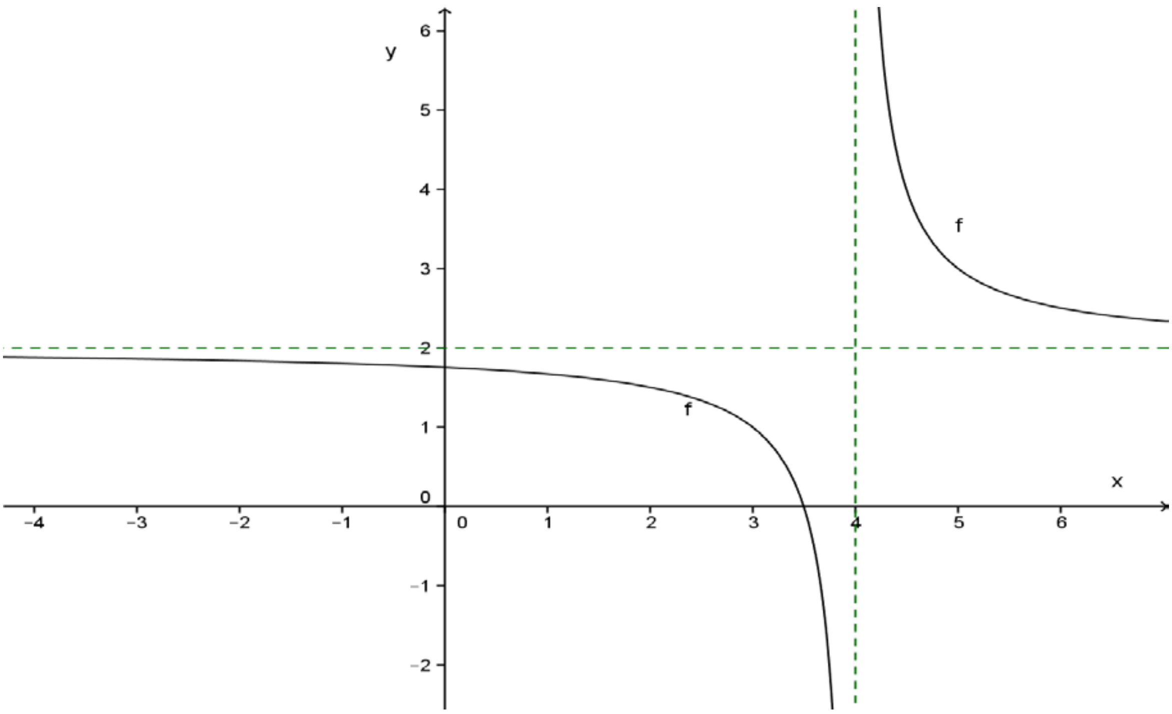


Figure 49: Mat and Teko's response

Mat and Teko managed to draw the asymptotes first and identify the quadrants in which the graphs lie. This ability to calculate the asymptote and to draw the graphs in their correct quadrants indicates that they are now operating at the object and schema levels.

5.6.5.2 Retrospective analysis

After the pair work I interviewed the learners again and their responses indicated that they had to a large extent overcome their difficulties and could now incorporate the definitions of the critical points into their procedures to make them meaningful instead of rote memorization of the steps involved in the procedures. In some instances it was not clear whether they had reached the schema level or not because they were showing more indicators of the object level and some were still at the process level. The table below summarizes learners' conception levels after the teaching experiment on the representation of the function concept:

5.6.5.3 Summary of results on problem area 2

Learners were able to:

- define the critical points and use these definitions in calculating intercepts, asymptotes and turning points;
- use the definitions of critical points to identify and extract critical points from a drawn graph; and
- use definition-based procedures in translating from a graph to an equation and vice versa.

Table 11: Summary of learners' APOS theory conception levels after the teaching experiment

Student	Intercepts	Calculation of intercepts	Asymptotes	Determination of asymptotes	Turning point	Determination of coordinates of turning point	Procedure of sketching graph	Translation from graph to equation	APOS conception level in which a learner is generally operating in
Mat	P	P	O	O	O	P	O	P	OBJECT
Teko	P	P	O	O	P	P	O	P	PROCESS
Coco	P	P	O	O	O	P	O	O	OBJECT
Diva	P	P	P	P	P	P	O	P	PROCESS
Monga	P	P	P	P	P	P	O	O	PROCESS
Edy	P	P	O	O	P	P	O	S	OBJECT

Reflections on the two *problem areas* and their teaching experiments above indicate that learners had not reached the schema level though some indicators of the schema level were beginning to show. It seemed as if learners were more comfortable operating at the process level than the object level. To ensure that learners' understanding of the function concept is extended to reach the schema level I included the inverse of a function which learners had not been taught in class because it is part of the grade 12 work and requires an understanding of the procedures and processes.

5.7 Problem area 3: The inverse of a function

5.7.1 Phase 1: Problem identification

What are the weaknesses in learners' understanding of the inverse of a function?

Since learners in the sample were not taught the inverse of a function no initial interviews were conducted and learners were working as a group so there were no individual activities, interviews and analysis of responses. I had to use learners' difficulties in understanding of the inverse of a function found in the literature and my experience as a mathematics teacher to come up with a hypothetical learning trajectory for learning the inverse of a function. Many learners in high school have difficulties with the concept of the inverse of a function: what it is, when it exists, how it is used, how to calculate it, and how to graph it (Marcus, 1999). In addition, Dubinsky and Harel (1992) believed that the inverse of a function was made difficult for many learners by the peculiarity of the representations and the absence of an algebraic formula. Ronda (2009) says that the most difficult part in teaching the concept of the inverse of a function is to make it make sense to learners and not so much in making the learners understand its definition or teaching them the process of finding the inverse function of a given function (by a graph or by a formula) or to "verify algebraically" that the functions are inverses.

From my personal experience I discovered that learners are normally taught the inverse of a function using the rule, 'interchange the positions of x and y and make y the subject of the formula'. The implication of this approach is that, to learners the inverse does not exist if there is no formula connecting variables. In most cases learners can easily follow this procedure correctly but without the slightest understanding of why they interchange the positions of x and y. Moreover, many teachers use this approach because it is easy for them to use but in most cases they do not explain to learners why they interchange the positions of x and y. This approach is also common in most of our textbooks and without the complete explanation it does not make any sense to the learners. The properties of 'one-to-one, onto and many-to-one' are the basic criterion that a function must meet to be reversed is also often misunderstood by learners. Teachers need to have an informal conception as well as more formal knowledge. It is important that learners understand what an inverse is (definition); conditions for its existence; how it is used (application); how it is calculated; relationship between the graph of a function and its inverse. My intention is to help grade 11 learners to make sense of the procedure of finding the inverse of a function by interchanging x and y and of the relationship between the graphs of a function and of its inverse.

5.7.2 Phase 2: Development of interventions

Phase 2 addresses research question 2:

How can instruction be designed to improve learners' understanding of the inverse of a function?

I used the RME's learning and teaching principles in Section 5.4.2 when I was designing HLTs and activities to assist learners to understand the concept of inverse of a function both procedurally and conceptually. Bayazit and Gray (2003) investigated learner learning of function inverses from two teachers, Ahmet and Mehmet. Ahmet focused his instruction on the idea of inverse "undoing" operations, whereas Mehmet on algorithmic and procedural skills (Bayazit & Gray, 2003). Learners were given a pre test and post test to evaluate their understanding of the inverse of a function before and after the classroom instruction. The authors concluded that in order to grasp the concept of the inverse of a function, learners should be given the opportunities to experience conceptually focused tasks (Bayazit & Gray, 2003, p. 109). This gave me an insight to introduce the inverse in this study by using real-life contexts modelling the inverse of a function. This could help learners to realise the existence of the inverse in their daily life and the conditions under which it exists, which will necessitate the construction of the definition of the inverse. Though learners might not have problems in calculating the inverse there was need to explain the procedure that they were using. I designed an activity of 'undoing' operations for learners before they could algebraically find the inverse. Lastly, I had to assist learners to discover the relationship between a function and its inverse.

5.7.2.1 Teaching experiment 6

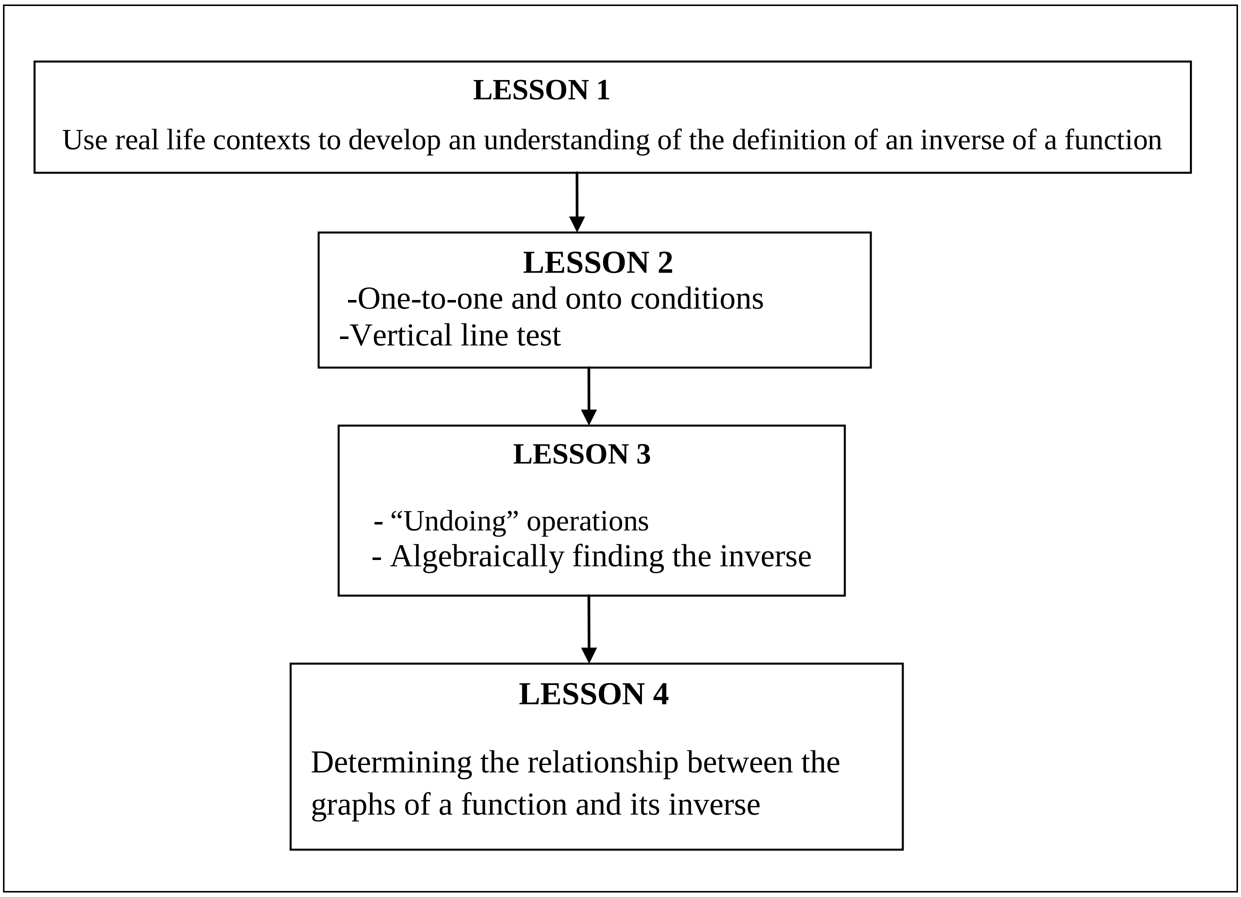


Figure 50: HLT for developing an understanding of the inverse of a function

The intention of the HLT was to enable learners to move more easily to the schema level. The teaching experiment comprised four lessons. Lesson 1 was on the introduction of the inverse in context, lesson 2 was on the conditions for the existence of an inverse, lesson 3 was on 'undoing' operations and calculating the inverse, and finally lesson 4 was on determining the relationship between the graph of a function and its inverse.

By the end of the teaching experiment learners should be able to:

i. define an inverse function;

ii. determine the inverse of a given function algebraically; and

iii. use the relationship between the graphs of a function and its inverse.

I designed two daily life situations modelling a function and how the idea of inverse is derived and discussed them with learners in my sample. This discussion enabled learners to formulate a function and then derive the inverse of that function without knowing that they were dealing with the inverse function. These two daily life situations were meant to demonstrate the idea of the inverse of a function at the layman/general and the mathematical levels.

a) At a layman or general level:

- If I need to call someone I am asking for her/his name on the list of my phone contacts.
- If someone of my contacts is calling me "my phone shows who is calling". This is the job of an inverse function: "finding the name corresponding to the number"

b) At a mathematical level:

- If Bheki makes R100/day. I know how to answer the question "After 7 days, how much money has he made?" I use the function W(t) = 100t.

But suppose I want to ask the reverse question:

- "If Bheki has made R700, how many days has he worked?" I use the reverse function t(W)=W/100. Given any amount of money, I divide it by 100 to find how many days he has worked. This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables.

5.7.3 Phase 3: Using tentative products and theories

Using daily life situations to teach the inverse of a function (Prototype 6)

Lesson 1

Activity 9: Using contexts to understand the definition and the purpose of inverses

On the market day Nonjabulo and friends are selling ice cream and yoghurt. Towards the end of the day they put their yoghurt on sale. They reduced the price of the yoghurt to R1,80 from R2,40. Use the new price for all the calculations that follow.

a. Copy and complete the table below:

Number of yoghurt cans	2	3	4	8	10
Price in rands (R)					

b. Determine the formula which they were using to find the price of any number of yoghurt cans.

c. There were 6 cans in each tray. How much will it cost to buy 3 trays? Is the cost of 18 cans three times the price of one tray?

d. You will notice that with each number of yoghurt cans there is an associated price. We can write these numbers as an ordered pair (number of yoghurt cans; price). Use the values in the table and write the values as a set of ordered pairs.

e. In the above relationship, identify the dependent variable and the independent variable and give reasons for you answer.

g. Thus, for every number of yoghurt cans bought, there is an associated price. This association resembles a function in which the number of yoghurt cans forms the domain while the price forms the range. Using this understanding, explain in your own words the meaning of the terms domain and range

2. Some customers who were coming to Nonjabulo and friends' table were saying that they want yoghurt for a specified amount of money e.g for R20 . This implies that at times Nonjabulo had to find out the number of yoghurt cans that can be bought with a specified amount of money.

a. Complete the following table:

Price in rands (R)	3, 60	5, 40	7, 20	14, 40	18, 00
Number of yoghurt cans					

b. Determine the formula which Nonjabulo and friends will use to find out the number of yoghurt cans that can be bought by any given amount of money. How do you it?

c. Would you consider this new relationship to be a function?

d. What is the domain and range of this new relationship?

e. In this new relationship, which is the dependent variable and which is the independent variable? Give reasons to support your answer.

f. Use the above table to write down the ordered pairs for this relationship and compare them with the ordered pairs in 1d. What do you notice? Explain in your own words.

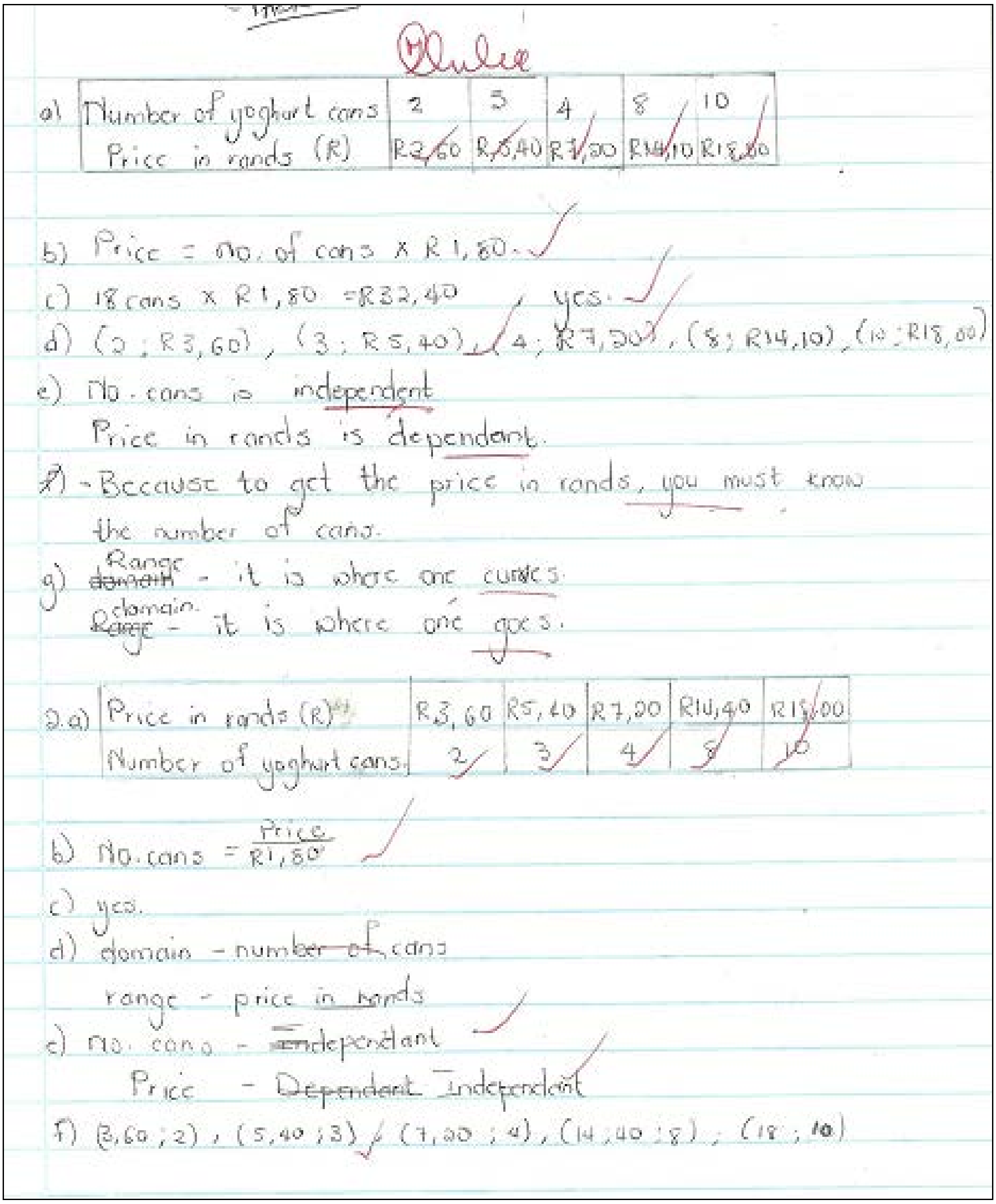


Figure 51: Learners' group responses

Learners' written work was fairly well done though they confused the dependent (range) and independent (domain). Learners managed to follow the procedures of the activity but failed to answer the questions that required them to make sense from the activity for example 2e and 2f were partially answered indicating that learners were operating at the process level of APOS theory. After their written work I interviewed learners as a group to elicit further information that I did not obtain from their written work:

Interviewer: What did you notice when you compared the ordered pairs in 1d and 2f?

Coco: The domain of 1d is the range of 2f and the domain of 2f is the range of 1d.

Monga: The domains and ranges interchange positions.

Diva: *x* and *y* coordinates swap positions

Teko: Domain of 1d becomes range of 2f.

Edy: *x*'s take the positions of *y*'s.

Mat: The *x*-axis becomes the *y*-axis.

Interviewer: Explain to me the connection between the two formulas in 1b and 2b?

Edy: In 1b we were multiplying the number of cans by R1.80 to get the total price while in 2b we were dividing the price by R1.80 to get the number of cans.

Coco: In 2b we were opposing what we did in 1b.

Diva: In 1b we multiplied by R1.80 and in 2b we divided by R1.80.

Monga: In 2b we reversed what we did in 1b.

Mat: In 1b we were given number of cans and unit price to find the price of a given number of cans. In 2b we were given the price of a certain number of cans to find that number.

Teko: The price and number of cans are connected by R1.80 the price of one can.

I then asked the learners to generate a definition of an inverse of a function as a group from what they had done in activity 9. In the end of their discussion they agreed on the following definition of the inverse of a function: '*The inverse of a function is a function which does the reverse of the original function and contains the same domain and range elements as the original function*'. Learners highlighted that the domain and range are switched in such a way that the domain of the function is the range of the inverse of a function, and vice versa. From their completion of the two tables in activity 9, learners emphasized that for every ordered pair (*x*; *y*) belonging to the original function, there is a corresponding ordered pair (*y*; *x*) that belongs to the inverse of a function. It was from this understanding that learners were able to demonstrate how to derive an inverse of the following function, *f*: $\{(-2;0),(1;3),(5;9)\}$ by swapping *x* and *y* to obtain its inverse f^{-1}: $\{(0;-2),(3;1),(9;5)\}$. This shows that learners were operating at the process level for the definition of the inverse of a function. Activity 9 prepared learners for understanding the one-to-one property as a condition that functions should possess for them to have inverses in lesson 2.

Lesson 2

I first reminded learners about 'one-to-one' functions using the previous real life examples: 'the set of all cars in Mpumalanga province is one-to-one with their registration numbers' and 'the set of South African provinces is one-to-one with their provincial cities'. These examples helped learners to derive their own definition of a one-to-one function as 'a *function in which every element of the range of the function corresponds to exactly one element of the domain'*. However, I reminded them that not all functions are one-to-one for example $f(x) = x^2$.

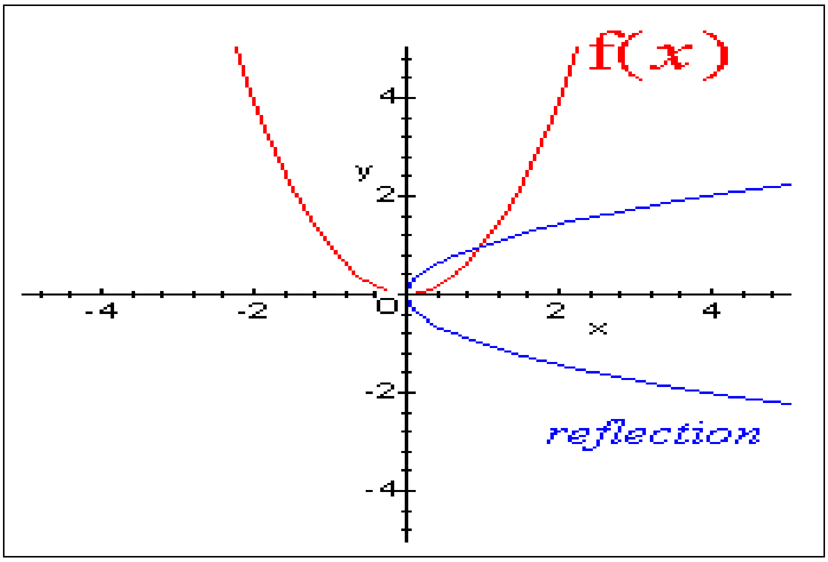
Activity 10

i. Show why the inverse of $f(x) = x^2$ is not a function.

ii. Draw the graph of $f(x) = x^2$ and its inverse.

The activity was done on the chalkboard. The learners directed the discussion as they indicated what should be written. *For* $f(x) = x^2$ there are two values of x that give the same $f(x)$. This is because both $f(x) = x^2$ and also $f(-x) = x^2$. There are two numbers that f takes to 4, $f(2) = 4$ and $f(-2) = 4$. If f had an inverse, then the fact that $f(2) = 4$ would imply that the inverse of f takes 4 back to 2. On the other hand, since $f(-2) = 4$, the inverse of f would have to take 4 to -2, hence the inverse is not a function. We cannot define f^{-1} of something to be two different things.

Looking at the same problem in terms of graphs. If f had an inverse, then its graph would be the reflection of the graph of f about the line $y = x$. The graph of f and its reflection about $y = x$ are drawn below.



Note that the reflected graph does not pass the *vertical line test*, so it is not the graph of a function. In the case of the inverse function, a display of conceptual knowledge would be the capacity of defining the inverse function, accompanied in the process of calculating the inverse by the questioning or testing of the existence of the inverse function. Lesson 2 helped learners to understand the conditions for an inverse function to exist and this led us to lesson 3 in which I taught learners how to use the inversion of operations that make up the function to find its inverse.

Lesson 3

I made use of the learners' knowledge of 'inverse operations' and attempted to expand their understanding of the inverse of a function as 'undoing' what a function does through conceptually focused tasks. The Mathematics Modlin Dictionary (2006) agrees with the learners' definition of an inverse of a function they derived in Lesson 2 and defines an inverse of a function as a function which 'does the reverse' of a given function, for example, if we have the function $f: x \rightarrow 2x + 1$, its inverse is found by reversing the operations that constructed it (reverse of multiplication is division and reverse of addition is subtraction) and the rules of precedence (BODMAS) are also reversed. The reversal of operations and rules of precedence gives the inverse denoted by $f^{-1}: x \rightarrow \frac{(x-1)}{2}$

On the other hand if we consider again the function $f(x) = 2x + 1$:

We can evaluate f at 1 by substituting x by 1 in the function: $f(1) = 2(1) + 1 = 3$ and it helps to think of f as transforming a 1 into 3. So if we think of f as 'acting on' numbers and transforming them, we define the inverse of *f* as the function that 'undoes' what *f* did. In other words, the inverse of *f* needs to take 3 back to 1. This was also demonstrated using the function machine. According to Benson and Buerman (2007) the inverse function should be understood as a way of breaking something down by reversing the operations performed in the relationship. They claim, for example, that students will have a greater understanding of inverse if they discuss reciprocals as multiplicative inverses and negatives as additive inverses.

Activity 11 (Oral): Reciprocals as additive and multiplicative inverses

Complete the following table

Function Rule	Inverse Rule
$x + 3$	$x - 3$
$\frac{x}{3}$	
	$4x$
	$\frac{x-1}{4}$
$\sqrt{x}$	

Activity 12: Algebraically finding the inverse of a given function

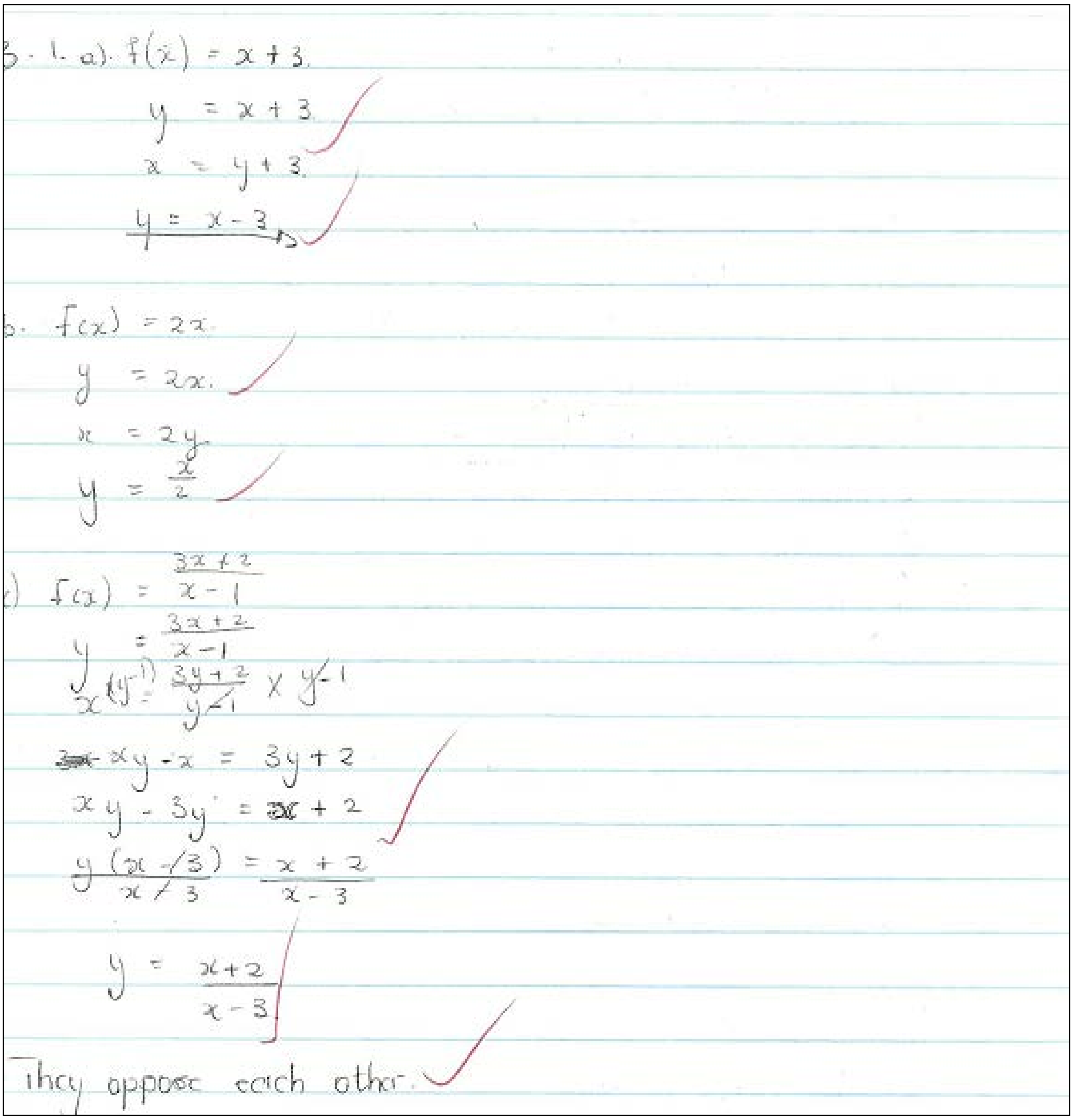
The purpose of this activity was to help learners to understand the algorithmic steps of switching *x* and *y* in the process of finding the inverse. Learners carried out the procedures internally showing that they were operating at the process level.

Find the inverses of the following functions

1. a) $f(x) = x + 3$ b) $f(x) = x - 3$ 2. a) $f(x) = 2x$ b) $f(x) = \frac{x}{2}$

3. a) $f(x) = \frac{3x+2}{x-1}$ b) $f(x) = \frac{x+2}{x-3}$

What did you notice about these inverses?


3. 1. a). $f(x) = x + 3$.
$y = x + 3$
$x = y + 3$
$y = x - 3$

b. $f(x) = 2x$
$y = 2x$
$x = 2y$
$y = \frac{x}{2}$

c) $f(x) = \frac{3x+2}{x-1}$
$y = \frac{3x+2}{x-1}$
$x(y-1) = \frac{3y+2}{y-1} \times y-1$
$xy - x = 3y + 2$
$xy - 3y = x + 2$
$\frac{y(x-3)}{x-3} = \frac{x+2}{x-3}$
$y = \frac{x+2}{x-3}$

They oppose each other.

Figure 52: Learners' group responses on activity 12

Lesson 4:

Activity 13: Understanding what it means to have an inverse graphically

The purpose of this activity was to develop learners' understanding of the relationship between the graphs of a function and its inverse and its links to the switching of the domain and range.

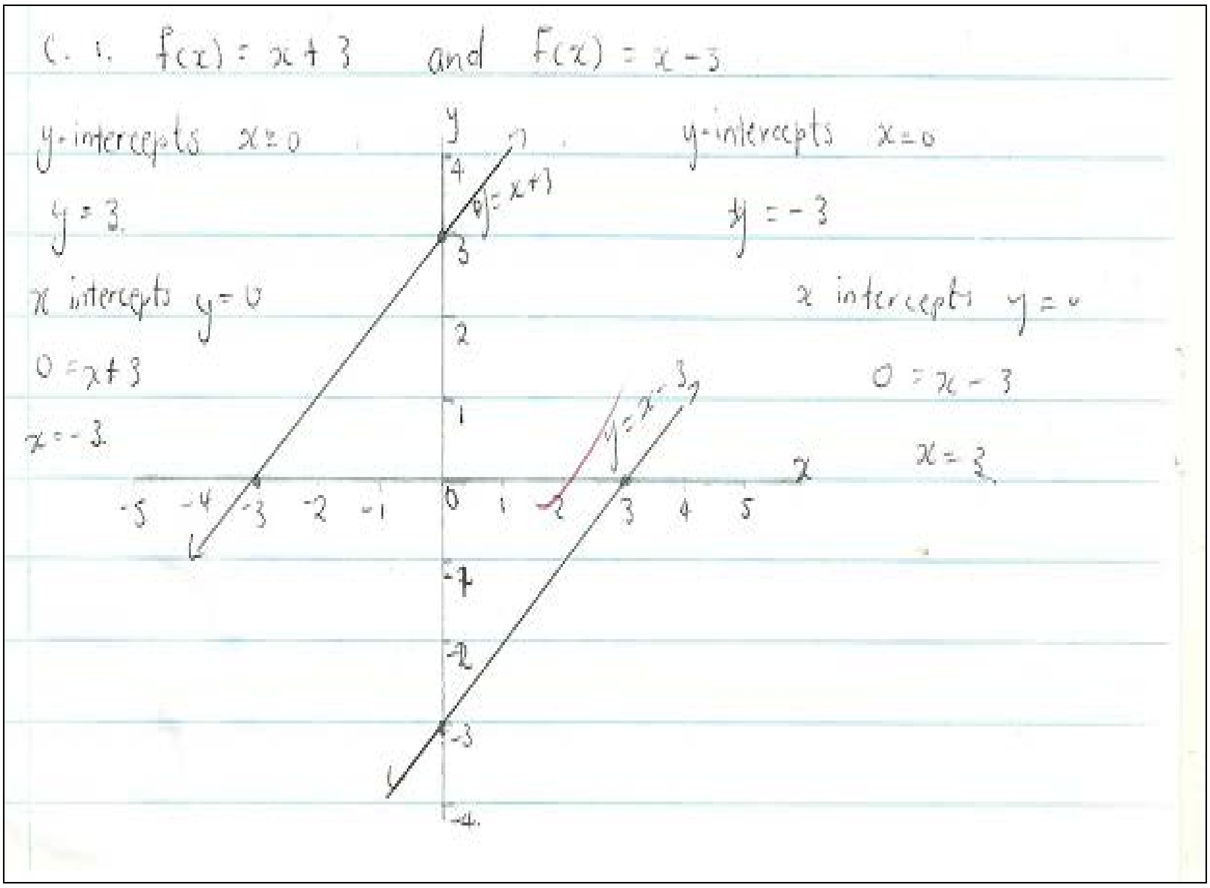
Sketch the graphs of the following functions on the same axes:

1. $f(x) = x + 3$ and $f(x) = x - 3$

2. $f(x) = 2x$ and $f(x) = \frac{x}{2}$

3. $f(x) = \frac{3x+2}{x-1}$ and $f(x) = \frac{x+2}{x-3}$

This activity was again done in one group to enable learners to interact and share their ideas and learn from each other. I wanted to avoid teaching by telling but to employ the principles of constructivism in which learners are guided to construct their own knowledge.



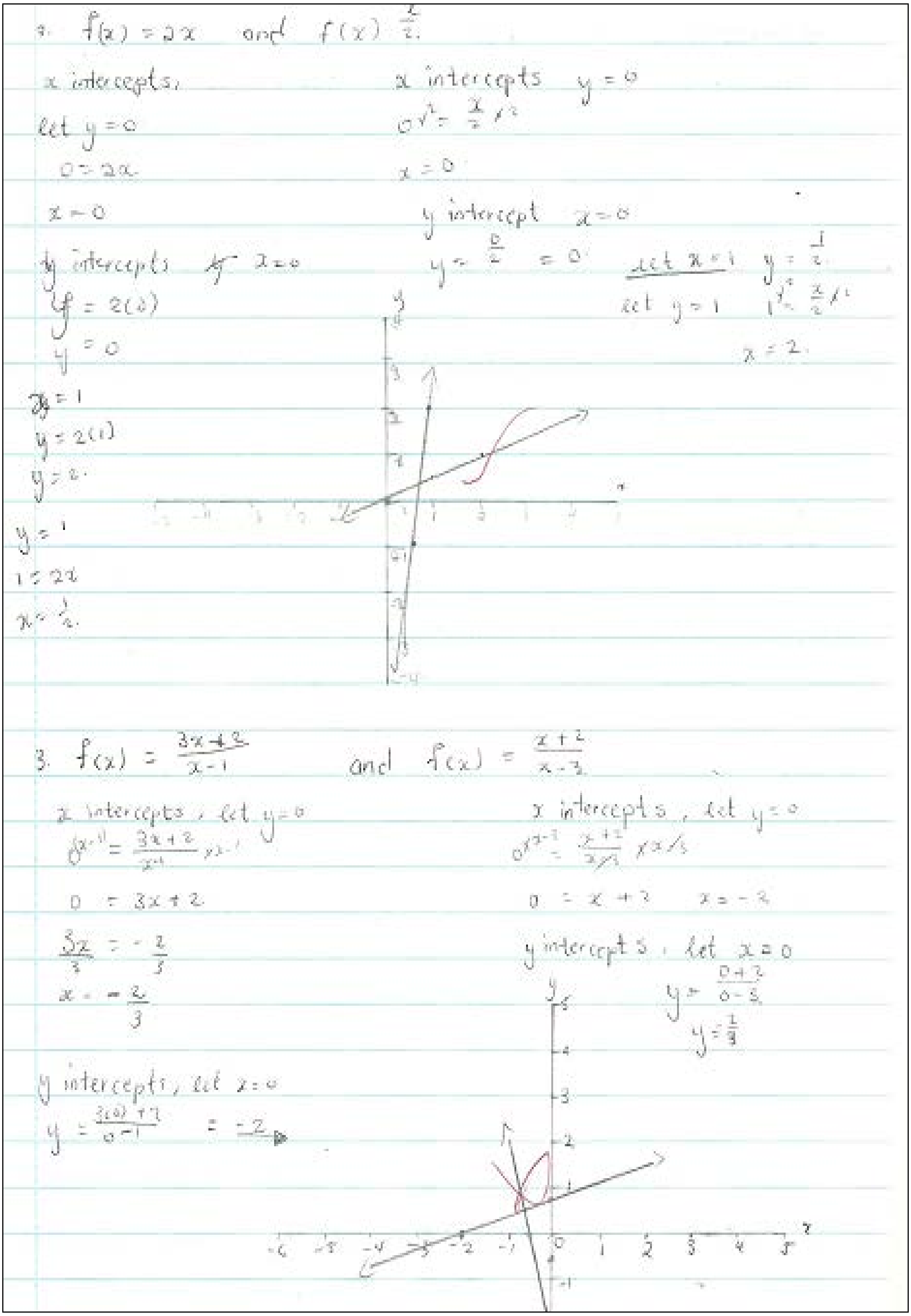


Figure 53: Learners' group responses on activity 13

Learners drew the first and second pairs of graphs correctly but the last pair was wrong and could not bring out the picture which I wanted them to discover. As a result they could not establish the relationship between the graphs of a function and its inverse as indicated by their interview responses below:

Interviewer: What did you notice about each pair of graphs you have drawn?

Monga: They are parallel to each other.

Edy: I don't agree what of the other two pairs they are not parallel.

Coco: They oppose each other like in a mirror.

Diva: I agree with Coco there is some reflection between these graphs.

Teko: The last two graphs confused me because they are crossing each other.

Mat: I can see that there is a relationship between our graphs but I can't explain it.

5.7.3.1 Retrospective analysis

Learners did activity 9 as a single group and their written and oral responses indicate that they understood what was happening in the activity. They realised the swapping of positions between the domain and range and that the second formula opposes what was done by the first formula. Even (1993) also found that many students conceptualized a function inverse using the notion of 'undoing'. "Undoing' is an informal meaning of inverse function which captures the essence of the definition" (Even, 1993, p. 557). The RME activities allowed the learners to experience this informal meaning and develop an understanding of the inverse. Using indicators of APOS theory for the inverse I can conclude from their responses that they are operating between the object level and the schema level.

Activity 10 showed that learners understood the one-to-one property of a function and used it to determine whether the inverse of $f(x) = x^2$ is a function or not. Activity 10 also helped learners to understand and use the vertical line test to determine whether a drawn graph is a function or not. Activity 11 was easily done by the learners as they used opposites of algebraic operations. Learners demonstrated their understanding of using the inverse of the function operations. When given the function equation, they knew how to reverse the calculations to arrive at the inverse equation. Their understanding of inverse operations progressed from the action to the process conception. These results confirmed the claims of Breidenbach et al. (1992) that learners built and then transformed mental representations. The procedural knowledge implies the mechanical computation of the inverse, following the algorithmic steps of switching x and y, and then solving for y.

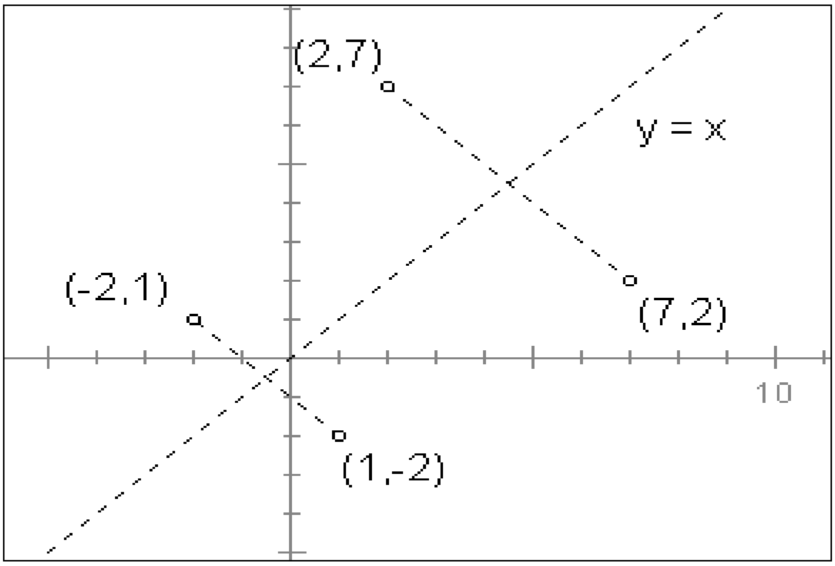
Learners could easily follow the procedure of switching the variables in activity 11 with an understanding that the domain becomes the range of an inverse function and the range becomes the domain. Learners also could realise that a function and its inverse oppose each other in terms of the rules of precedence (BODMAS). This indicates that learners are operating at the process level of APOS theory. However, activity 12 was not well done as learners failed to establish the relationship between the graphs of a function and its inverse. As a result I modified the HLT for the relationship between the graphs of a function and its inverse which led to phase 4.

5.7.3.2 Phase 4: Product and theory refinement

Take a function and its inverse from activity 12 and draw a table of values for each using the same domain and range. Then reflect the points in the table of values for the function in the line $y = x$ and compare your answers with the corresponding points in the table of values for the inverse.

Figure 54: HLT for improving learners' understanding the relationship between the graphs of a function and its inverse

The reflection of the point (a,b) about the line y = x is the point (b,a).



Taking for example question 3 in activity 12: The first function is $f(x) = \frac{3x+2}{x-1}$. Then its table of values is:

X	2	3	4
$f(x) = \frac{3x+2}{x-1}$	8	$\frac{11}{2}$	$\frac{14}{3}$

The second function is $f(x) = \frac{x+2}{x-3}$ and its table of values is as follows:

X	8	$\frac{11}{2}$	$\frac{14}{3}$
$f(x) = \frac{x+2}{x-3}$	2	3	4

From the two tables above $f(2) = 8$ and the point (2; 8) is on the graph of *f*. The inverse of *f* must take 8 back to 2, that is $f^{-1}(8) = 2$, so the point (8; 2) is on the graph of f^{-1}. This means that $f(x) = \frac{x+2}{x-3}$ is the inverse of $f(x) = \frac{3x+2}{x-1}$. The point (8; 2) is the reflection in the line $y = x$ of the point (2; 8). The same argument can be made for all points on the graphs of *f* and f^{-1}.

5.7.4 Interview questions on the inverse of a function (group interview)

1. In your own words explain what an inverse function is?
2. What symbol is used to represent an inverse of a function and what does that symbol mean to you?
3. If you are given the equation of a function explain how you find the inverse of that function?
4. What is the relationship between the domain and range (co-domain) of a function and that of its inverse?
5. What is the relationship between a function and its inverse?

The following is only a part of some of the responses from these interviews that I thought I could share with readers of this thesis:

Edy: The domain of the function will change to be the range of the inverse because the two interchange.

Coco: the domain changes to be the range and the range changes to be the domain.

Monga: I think the relationship between the function and the inverse is that they oppose each other.

Mat: An inverse undo a function, they are opposing each other.

Teko: The inverse oppose the function. A function ‘undoes’ what an inverse does.

Diva: To get the graph of the inverse I reflect the graph of the original function in the line $y = x$.

The inverse of a function is a reflection of the original function about the line $y = x$. The function and the inverse function are mirror images with the line $y = x$ acting as the mirror. The standard notation is $f(x)$ for the function and $f^{-1}(x)$ for the inverse function.

Retrospective analyses of learners’ responses indicated that the use of real life contexts and situations imaginable in learners’ minds were effective in developing learners’ conceptual understanding of the concept of inverse of a function which culminated into phase 5.

5.7.5 Phase 5: Final product and contribution to theory

Real life context-based development of the inverse of a function

This approach was supported by the following principles of RME:

- use of everyday activities that are experientially real to learners;
- making the idea of inverse real in the mind of the learner by enabling the learner to imagine the concept in his/her mind; and
- stressing understanding of processes (procedures) rather than memorizing that to find the inverse interchange x and y and make y the subject without understanding why x and y are swapped.

First I reviewed learners’ *working definition of a function* “A function is a dependence relationship between two sets of variable quantities in which each element from the first variable (domain set) has only one corresponding element in the second variable (range or image set).”

I asked learners to identify their own example of a function with restrictions and they gave me the following:

Diva (reporting): A taxi company charges R5 per kilometre within the 40 km radius. So our formula for calculating how much a passenger pays is: Fare = R5 times the number of kilometres travelled. In this situation distance is restricted to $0 \leq$ distance $\leq$ 40km and the fare is also restricted to $0 \leq$ fare $\leq$ R200

Learners' example indicates that they can formulate a function with some restrictions which is a sign that they are now operating between the object and the schema level of APOS theory. Since learners were able to formulate a real life example of a function it made it easier for me to develop the concept of inverse of a function using their example.

Developing the inverse of a function using the learners' real life example of a function

The inverse function is a kind of undoing function. Suppose you we are told that a passenger paid R150 on taxi fare, could we work backwards and find the number of kilometres the passenger travelled? The inverse relationship would be Fare / 5. If we divide the fare of R150 by 5, we would know that the passenger travelled 30km. The first relationship of Fare = R5 ×Number of kilometres travelled is a function. The undo rule of Number of km travelled = Fare / 5 is the inverse function. This led us to the graphs of the function and its inverse.

5.7.6 Summary of problem area 3

Learners were able to:

- derive own definition of an inverse of a function from real life contexts with which they were presented;
- derive the meaning of the one-to-one property of functions of having one domain element corresponding to exactly one element in the range set '*the domain of the function is equal to the range of the inverse and the range of the function is equal to the domain of the inverse'* from previous real life examples of functions;
- formulate own example of a function with a restricted domain;
- use the inversion of operations that constructed a function in finding its inverse; and
- use the relationship between the graphs of a function and its inverse '*the inverse of a function is a reflection of the original function about the line $y = x$'* to understand why they swap x and y in the procedure of finding the inverse algebraically.

Table 12: Summary of activities, tasks and interview sessions and when they happened

What happened	**When it happened**
1. Learners were asked to attempt question7 and 8 from the June 2011 examination paper as initial tasks	Starting point
2. Learners were interviewed individually on the initial tasks and their oral responses were taped	Interview session 1 on question 7 Interview session 2 on question 8
3. APOS analysis of learners' written and oral responses on the initial tasks revealed their difficulties and weaknesses	After transcribing learners' oral responses
4. Classification of learners' difficulties and weaknesses into problem area 1 and problem area 2	After identifying and listing learners' identified difficulties and weaknesses
Intervention for problem area 1 5. Learners worked in pairs on Activity 1 and were interviewed (APOS analysis of learners' responses) 6. Learners worked in pairs on Activity 2 and were interviewed (APOS analysis of learners' responses)	Teaching experiment 1 Interview session 3 on Activity 1
7. Learners were involved engaged in the Table allocation game in Activity 3 8. Learners wrote Task 1	Teaching experiment 2
9. Learners did Task 2 in pairs and were interviewed (APOS analysis of learners' responses) 10. Learners did Activity 4 as a group	Teaching experiment 3
Intervention for problem area 2 11. Learners did the following activities Activity 5: Translating from equation to graph Activity 6: Translating from graph to equation Activity 7: Critical points as connection factor	Teaching experiment 4
12. Learners did Activity 8 on using definition-based procedures	Teaching experiment 5
Intervention for problem area 3 13. Learners did the following activities: Activity 9: using real-life contexts to develop learners' understanding of the inverse Activity 10: relationship between function and its inverse Activity 11: reciprocals as additive and multiplicative inverses Activity 12: finding the inverse algebraically Activity 13: relationship between graphs of a function and its Inverse	Teaching experiment 6

5.8 Summary of chapter

In this Chapter I used APOS and Piaget's developmental theory levels to determine where learners were and developed HLTs that moved them towards the schema level. The tools to implement the HLTs were embedded in RME activities and based on constructivism. Research (Dubinsky & McDonald, 2001; Gray, 2007; Van de Walle, 2007) indicates that Piaget's levels may be age related but this differs for different cultures and for learners like the ones in my sample who have to engage in a second language. This could be the reason why my learners did not all reach the schema level. Though I wanted them to be at the formal operational level two out of the six learners were at the concrete operational level in some of the activities. The sequences of lessons and their activities for the definition, representation and inverse of the function concept which are the products of this study can be used in classroom situations to improve grade 11 learners' understanding of functions. Because timetables and scheduling vary from school to school, the amount of material per lesson will also vary depending on the available class time. I recommend that it be taught as a whole and in the sequence suggested. The next chapter presents a synthesis of the foregoing chapters from which I draw conclusions and recommendations for this study.

Chapter 6

Summary, conclusions and recommendations

6.1 Introduction

This chapter gives a summary of the study and conclusions based on the qualitative results presented in Chapter 5 to provide answers to the following research questions:

1. How do learners understand the function concept?
2. How can instruction be designed to improve learners' understanding of the function concept?

The structure of this chapter is guided by these research questions. I begin by summarizing the different perspectives on the understanding of functions including the one emanating from the current study. This is followed by a discussion of how the material improved learners' understanding of functions and a graphical display of trends of improvement in individual learners' understanding of the functions. A description of the final products and their contribution to theory and reflections on my theoretical framework and methodology follows. Conclusions drawn from this study are then summarized, followed by a discussion of the recommendations that can be adopted and implemented. Finally, a discussion of the limitations of this study and suggestions for further research make up the last section of this chapter.

6.2 What is an understanding of functions?

According to the CAPS grades 10-12 curriculum, learners have a full understanding of functions if they "demonstrate knowledge of the formal definition of the function concept and the ability to work with various types of representations by converting flexibly between numerical, graphical, verbal and symbolic representations" (DoBE, 2011, p. 12). As a teacher my initial interpretation of the statement was that learners are expected to understand the formal definition of the function concept first before understanding the different representations and the process of converting from one representation to the other. The teachers of participants in the study also believed the CAPS statement means that we should start with the formal definition. This is in contrast with the development of understanding of the function concept illustrated in this study. Literature has also shown that starting with the definition of the function concept does not guarantee understanding of the concept (Kwari, 2007). Literature further shows that while some believe that understanding of the function concept entails having knowledge of the function concept (Sajka, 2005) others see it as a learner's ability to use or demonstrate knowledge acquired about the function concept

(Markovits, Eylon & Bruckheimer, 1986). However, the findings from the current study contradict these two perspectives and regard understanding of the function concept as enabling the learners to:

- explain the key idea or core concept of the function concept which is a dependent relationship in which the value of one variable is dependent on the value of another;
- derive a working definition of the function concept based on the key idea;
- explain and use the basic properties of the function concept which are univalence and arbitrariness;
- classify relations into functions and non-functions;
- give examples which are functions, and relations which are not;
- translate from one representation to another;
- identify key features in graphs and tables including intercepts, asymptotes, symmetries, maximum and minimum, intervals where the function is increasing, decreasing, positive or negative;
- calculate or determine key features from given equations;
- sketch the graph using the given key features;
- derive the definition of an inverse of a function;
- use the property of one-to-one in determining whether a function has an inverse or not and to connect the graphs of a function and its inverse; and
- use the procedure of determining an inverse with an understanding of why x and y are swapped.

6.2.1 Grade 11 learners' understanding of the function definition and their weaknesses

Task-based clinical interviews were used to determine learners' understanding of functions in terms of its definition and representation which revealed their difficulties, misconceptions and weaknesses about the function concept. Findings from this study indicate that learners initially understood a function as any relationship (Diva, Edy, Mat and Monga), correspondence (Coco) or connection between inputs and outputs (Teko). The weakness in learners' understanding was that there was no mention of a dependence relationship between two sets of dependent and independent variables. This incomplete understanding of a function resulted in four of the learners (Teko, Monga, Mat and Diva) failing to formulate correct examples of functions and non-functions. Only two learners (Coco and Edy) managed to formulate correct examples of functions but also failed to give examples of non-functions. Learners' concept images of a function could not enable them to explain and identify the dependent and independent variables when given a function. I then used APOS theory indicators to inform my analysis of learners' responses to determine the level at which each learner was operating in terms of understanding the function definition. Five out of the six learners were found to be operating at the action level while only Coco was operating at the process level for the function definition.

6.2.2 Grade 11 learners' understanding of the function representation and their weaknesses

Learners in this study were more familiar with two representations of the function concept namely the graph (Teko, Coco) and the equation or expression (Diva, Edy, Mat and Monga). However, from the initial task-based clinical interviews all six learners could use a given equation to calculate the intercepts, asymptotes and turning points and to draw the graph but found it difficult to use a drawn graph to determine its equation. The main reason why learners had difficulties in translating from a graph to an equation was that they were calculating the critical points following memorized procedures without understanding how and why these procedures work. Moreover, learners did not understand the meanings of intercepts, asymptotes and turning points which are the important connectors of the graph and equation. The use of procedures by learners without understanding how and why they work indicates that they were operating at the process level of APOS theory.

6.3 Improving understanding of functions

Improving learners' understanding of functions entails helping them to overcome their conceptual difficulties and to move from their initial conception level to the next higher level. In the current study I used design research to improve the teaching and learning of functions. My focus was to enhance knowledge in the field of teaching and learning of functions and to enhance practice in the field by enabling better teaching and learning of functions. This was achieved by developing empirically grounded instructional sequences (HLTs) and activities for learning and understanding the function concept in terms of its definition, representation and inverse. This answers research question 2: *How can instruction be designed to improve learners' understanding of the function concept?*

6.3.1 Using design research in the classroom to improve learners' understanding of functions

This was achieved in two stages: firstly, by determining what to teach and secondly how to teach it. Firstly, what to teach can only be determined if the teacher knows what the learner already knows, which allowed me to apply the theory of constructivism which starts at the level of the learner. To determine what the learners in my sample already knew I conducted task-based clinical interviews before and after each intervention. I compared what learners knew and what they should understand at their grade level regarding the function-related concepts from the CAPS curriculum document. This helped me to establish conceptual and pedagogical gaps in learners' understanding of functions.

Secondly, after determining what to teach I had to find ways of teaching it which were determined by the following: the structure of the function-related concepts to be covered; forms or ways of understanding that exist for each concept; how learners should understand these function-related concepts; how these function-related concepts are developed; and approaches that help learners to understand these concepts. I used this knowledge to develop HLTs and activities that are imaginable and real to learners to be implemented in actual lessons in the classroom. In the current study I used the RME's teaching and learning principles to make the content real and to scaffold the learning to allow learners to "climb the ladder" of APOS theory conception levels of understanding functions. I also brought in Piaget's theory because I needed to work from what teachers know. Teachers are familiar with Piaget's theory which I related to APOS theory which is a more refined theory because of its four levels. APOS theory points to possible pedagogical strategies. After the teaching and learning activities I then designed and administered formative assessment tasks structured to enable learners to present and defend their solutions to task problems. These assessment tasks led to phase 1 of the five phases of design research explained in this study.

6.3.2 A qualitative summary of how learners improved their understanding of functions

Learners improved their understanding of the function concept in two stages corresponding to the definition and representation respectively. Learners' understanding of the function concept was extended by learning the inverse. In the first stage learners' initial understanding of the definition of the function concept was weak because they viewed a function as any relationship, correspondence or connection between things. This understanding of the definition of the function concept limited learners' ability to achieve the following: to formulate examples and non-examples of functions; to identify the dependent and independent variables in a given example of a function; and to determine whether a given relation is a function or not. This indicated that learners were operating at the action conception level of APOS theory. Learners improved from the action level to approximate the schema level when they were engaged in RME based activities which helped them to grasp the key idea of the function concept. Learners used the core idea of the function concept to derive a working definition which they were able to use to formulate examples and non-examples of functions and to determine whether a given relation is a function or not.

In the second stage involving representations of the function concept learners in the sample initially carried out calculations to determine intercepts, asymptotes and turning points without any understanding of the concepts indicating that they were operating at the process level of APOS theory. This caused learners to have difficulties in translating from a graph to an equation. Learners improved their ability to translate from a graph to an equation when they understood the key concepts which they used in every step of the procedure in translating from a graph to an equation. The use of a definition-based procedure helped learners to develop a conceptual understanding of the procedures involved in translating from equation to graph and vice versa. Though learners did not fully reach the schema level there was evidence of some indicators of the schema level.

Learners in this study extended their knowledge of deriving a working definition and using procedures with understanding when they learnt about the inverse of a function which they had not been taught. Learners were able to formulate a function and then derive its inverse through their engagement with activities based on daily life situations about selling ice cream cups which were modeling a function and how the idea of inverse is derived. Learners' engagement in these activities helped them to discover that for every ordered pair (x; y) belonging to the original function, there is a corresponding ordered pair (y; x) that belongs to the inverse of a function. It was from this understanding that learners were able to demonstrate how to derive an inverse of a given function by swapping the domain (x) and range (y) to obtain its inverse. Indications were that in the end, learners were operating at the schema level for the inverse of a function.

6.4 How the material improved learners' understanding of functions

To improve learners' understanding of the function concept which was initially at the action level I engaged them in constructive activities that, for them, were experientially real and imaginable. After the constructivist activities I administered problem solving tasks to help learners to move through various levels of abstraction. I paid attention to each learner's own written and interview responses and gave each the opportunity to reflect on their written and oral responses by explaining their solutions and their methods. Learners were given the opportunity to work in groups where they could exchange ideas and arguments and learn from each other. I used real world examples of functions to make the concept as applicable and understandable as possible. The use of real life relationships helped the learners to correctly identify the variables in the given relationships and to formulate their own relationships which they could manipulate to see how change in one variable would affect the other. For example, the table allocation activity in which learners acted out a function and a non-function helped them to understand how to determine whether a given relation is a function or not. This helped learners to derive a working definition of a function which they could use to formulate their own examples of both functions and non-functions. They could also determine whether a given relation was a function or not, an exercise which took them to the schema level for the definition of the function concept.

Activities 1 and 2 enabled learners to take subject matter from the real world and to organize it according to mathematical patterns. As a result, learners were able to recognize mutual dependence between variables or varying quantities, determine the nature of the dependence relationship between variable quantities and to express and interpret quantitative relationships. These two activities also developed the learners' ability to identify and explain the dependent and independent variables and to formulate examples and non-examples of functions. The learners managed to move from the action and process level to the object level. Initially, learners could not use the key idea of the function concept to determine whether a given relation is a function or non-function and could also not use the proper functional language of domain and range in defining the function concept and in formulating examples and non-examples. As a result I designed task 1 to develop learners' ability to identify a function in its different representations. Evidence from learners' written work indicated that this task helped them to identify a function and to explain why the relationship is a function using the properties of a function. Learners could use the idea that every element of the domain set must have its own corresponding element in the range set which indicated that they had progressed to the schema level.

I designed task 2 to develop learners' understanding of the link between the one-to-one property of the function concept and the use of the vertical line test. From the learners' responses I was able to conclude that they had understood why the vertical line test worked and they could use it intelligently. Though learners' responses could not show indicators of the schema level, learners were operating above the object level towards the schema level. I stopped at the object level because learners were now comfortable with working with these function-related concepts. Their written work also showed that they had overcome their difficulties.

I then asked learners to compare their definitions in order to come up with a working definition built on the key idea and the one-to-one property of the function concept. Learners examined their individual definitions as a group and had consensus on a working definition which they used easily to formulate examples of functions and non-functions and to determine whether given relations were functions or non-functions. Though learners had followed the HLTs as I had intended, only two learners were exhibiting some indicators of the schema level while the other four were oscillating between the process and object levels of APOS theory. However, their written work showed that they had overcome major difficulties that hinder such learners from reaching the schema level.

I designed activity 5 to develop learners' understanding of the process of translating from the symbolic to the graphical representation. Learners were able to calculate the intercepts, asymptotes and turning points of different functions. However, translation from graph to equation in activity 6 was difficult for learners. To help learners translate easily from graph to equation I designed activity 7 in which they worked as a group. This activity helped learners to appreciate the use of definition-based procedures which made the previously memorized procedures meaningful. Learners did not reach the schema level but they were exhibiting a greater number of indicators of the object level and the process level. Considering learners' initial low levels of conception, reaching the process and object level was an achievement. It appeared to be also easier for learners to move from the object level to the desired schema level.

I engaged learners with activity 9 to help them realise why the domain and the range swap their positions in the process of finding the inverse. Activity 10 was done to help learners establish the relationship between a function and its inverse, and also to use the vertical line test with an understanding. Activity 11 was done to help the learners to understand the opposites of algebraic operations which shifted their focus of attention from the notion of swapping x and y to the idea of an 'inverse operation'. This entailed the inversion of a sequence of algorithms in the process of a function by going from the end to the beginning. Learners demonstrated their understanding of using the inverse of the function operations and how to reverse the calculations to arrive at the inverse equation. The RME activities allowed the learners to experience this informal meaning and to develop an understanding of the inverse. Learners could easily follow the procedure of switching the variables in activity 11 with an understanding that the domain becomes the range of an inverse function and the range becomes the domain. Learners also could realise that a function and its inverse oppose each other in terms of the rules of precedence (BODMAS). This indicated that learners were operating at the schema level of APOS theory as I intended.

The findings of the study:

- *Learners initially carried out procedures to calculate intercepts, asymptotes and turning points without any understanding of the concepts.* This tendency limits learners' use of these procedures since they can only perform them in one direction. For example, learners in the current study were able to successfully use the procedures of determining the intercepts, asymptotes and turning points from a given equation but found it difficult to extract the same critical points from a drawn graph representing the same equation and then to formulate the equation. To assist learners in the current study to develop a conceptual understanding of the procedures of calculating the intercepts, asymptotes and turning points I introduced the concept of definition-based procedures (DBPs). The use of DBPs made it easier for learners to translate from equation to graph and *vice versa* and also compelled learners to understand the meanings of all the key concepts involved in carrying out these procedures. Once these key concepts are understood it is easier for learners to translate from equation to graph and from graph to equation.
- *Learners find it difficult to understand and use a prescribed definition of the function concept.* To help overcome this challenge I provided learners with opportunities to

develop a working definition of the function concept which they could easily use to identify functions and non-functions, formulate their own examples of the function concept, and identify dependent and independent variables in a given relationship. This definition also helped learners to appreciate the nature and origin of the function concept, distinguish a function from a non-function and apply it in problem solving.

- *The use of real life contexts formed the basis for learners' understanding of the inverse of a function.* In this study I introduced the inverse of a function by using real-life contexts and activities modelling the inverse of a function. This helped learners to realise the existence of the inverse in their daily life, the conditions under which it exists, and the relationship between a function and its inverse which necessitated the construction of the definition of the inverse. Real life activities also focussed learners' attention on understanding procedures and the reasons why procedures work rather than merely memorising them.
- *Design research improved learners' understanding of the function concept in terms of its definition, representation and inverse.* Design research together with APOS theory, RME and constructivism to a large extent succeeded in improving learners' conceptual understanding of the function concept through iterative design and development of instructional sequences (HLTs) and level appropriate RME based activities. In addition I provided for an enabling environment where learners shared their ideas and worked in groups which improved their understanding of the function concepts. Though not all the learners managed to reach the schema level, design research assisted learners to overcome their identified difficulties on functions and to move from their initial lower levels of APOS theory to the process and object levels and two learners reached the schema level as intended.

6.5 Trends of improvement in individual learners' understanding of the functions

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
Monga	Schema						
	Object		*	*		*	*
	Process				*		
	Action	*					
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
Monga	Schema						
	Object				*		*
	Process		*	*			
	Action	*					
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE							
		AFTER ACTIVITY					
LEARNER	CONCEPTION LEVEL	8	9	10	11	12	AVERAGE
Monga	Schema		*			*	*
	Object						
	Process	*		*	*		
	Action						
	Pre-function						

Figure 55: Monga's trends of improvement in understanding of the function concept

Initially Monga was operating at the action level but improved his understanding with each intervention. On average he was finally operating at the object-schema level, which means that the interventions had a positive effect on his understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
Coco	Schema						
	Object		*	*		*	*
	Process	*			*		
	Action						
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
Coco	Schema						
	Object				*		*
	Process	*	*	*			
	Action						
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE							
		AFTER ACTIVITY					
LEARNER	CONCEPTION LEVEL	8	9	10	11	12	AVERAGE
Coco	Schema		*				
	Object					*	*
	Process	*		*	*		
	Action						
	Pre-function						

Figure 56: Coco's trends of improvement in understanding of the function concept

Initially Coco was operating at the process level but improved her understanding with each intervention. On average she was finally operating at the object-schema level, which means that the interventions had a positive effect on her understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
	Schema						
	Object		*	*		*	*
Mat	Process				*		
	Action	*					
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
	Schema						
	Object				*		*
Mat	Process		*	*			
	Action	*					
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE						
		AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	8	9	10	11	AVERAGE LEVEL
	Schema	*	*	*	*	*
	Object					
Mat	Process					
	Action					
	Pre-function					

Figure 57: Mat's trends of improvement in understanding of the function concept

Initially Mat was operating at the action level but improved his understanding with each intervention. On average he was finally operating between the object and schema level, which means that the interventions had a positive effect on his understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
	Schema				*		
	Object		*	*		*	*
Diva	Process						
	Action	*					
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
	Schema				*		
	Object		*				*
Diva	Process			*			
	Action	*					
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE							
		AFTER ACTIVITY					
LEARNER	CONCEPTION LEVEL	8	9	10	11	12	AVERAGE
	Schema		*				
	Object					*	*
Diva	Process	*		*	*		
	Action						
	Pre-function						

Figure 58: Diva's trends of improvement in understanding of the function concept

Initially Diva was operating at the action level but improved his understanding with each intervention. On average he was finally operating at the object level, which means that the interventions had a positive effect on his understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
Edy	Schema						
	Object		*	*			*
	Process				*	*	
	Action	*					
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
Edy	Schema						
	Object				*		*
	Process		*	*			
	Action	*					
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE							
		AFTER ACTIVITY					
LEARNER	CONCEPTION LEVEL	8	9	10	11	12	AVERAGE
Edy	Schema		*				
	Object					*	*
	Process	*		*	*		
	Action						
	Pre-function						

Figure 59: Edy's trends of improvement in understanding of the function concept

Initially Edy was operating at the action level but improved his understanding with each intervention. On average he was finally operating at the object level, which means that the interventions had a positive effect on his understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF DEFINITION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	1	2	TASK 1	TASK 2	AVERAGE LEVEL
Teko	Schema						
	Object		*			*	*
	Process				*		
	Action	*		*			
	Pre-function						

EFFECTS OF EACH INTERVENTION ON LEARNER'S UNDERSTANDING OF REPRESENTATION							
			AFTER ACTIVITY				
LEARNER	CONCEPTION LEVEL	INITIAL LEVEL	5	6	7		AVERAGE LEVEL
Teko	Schema						
	Object				*		*
	Process		*	*			
	Action	*					
	Pre-function						

LEARNER'S UNDERSTANDING OF INVERSE							
		AFTER ACTIVITY					
LEARNER	CONCEPTION LEVEL	8	9	10	11	12	AVERAGE
Teko	Schema		*				
	Object	*				*	*
	Process			*	*		
	Action						
	Pre-function						

Figure 60: Teko's trends of improvement in understanding of the function concept

Initially Teko was operating at the action level but improved his understanding with each intervention. On average he was finally operating at the object level, which means that the interventions had a positive effect on his understanding of functions. This also indicates that the instructional material can improve learners' understanding of functions.

The learners displayed varying conception levels for the different function-related concepts. A learner could operate at the schema level for one concept but be on a lower level in another. The implication is that, the teacher may need to determine the learner's APOS conception level before teaching these function-related concepts in order to develop activities that help the learner to maintain the schema level or if operating at a lower level, to provide scaffolding to allow the learner to progress to a higher conception level.

6.6 Discussion of the final products and their contribution to theory

Teaching experiments conducted in the current study followed the five phases of design research and culminated in final products in their fifth phase which improved learners' understanding of the function concept. Phase 5 of the first problem area on the definition of the function concept resulted in learners deriving a '*working definition of the function concept'*. This working definition proved to be useful to the learners in this study as they could use it to formulate functions and non-functions and to determine whether given relations were functions or non-functions. The working definition is a contribution to theory because in the current mathematics textbooks for example, Laridon *et al., (*2007) definitions of a function are prescribed and teachers rely on them to develop an understanding of the function concept, an approach which does not guarantee an understanding of the function concept (Kwari, 2007). This study demonstrates that teachers should not rely on textbook definitions of a function but should develop a progression to allow learners to come up with a "working definition" of the function concept. Deriving a "working definition" adds to the dimensions of understanding (Usiskin, 2012) the function concept. The ability to use the "working definition" as a component of understanding the function concept concurs with Markovits, Eylon and Bruckheimer's (1986) view of learning and teaching the function concept. Results from this study suggest that the teacher should create an enabling environment in which learners are provided with real life situations modelling the function concept and activities to enable them to derive their own working definitions and to grasp the core idea of the function concept.

The concept of a '*definition-based procedure*' (DBP) was also introduced in phase 5 for the second problem area on representations of the function concept. In a definition-based procedure every step is based on the definitions of key concepts (intercepts, asymptotes, and turning points) that are involved. It was a different way of handling the application of these concepts integral to the use of functions. It compelled learners to understand the meanings of all the critical points before they could calculate them or extract them from a drawn graph. This is a contribution to the pedagogy of functions which previously focused on following procedures without understanding them which forced learners to memorize them. Learners should be assisted to understand the procedures of calculating intercepts, asymptotes, turning points and inverses instead of encouraging them to memorize them without any understanding.

From the literature reviewed in this study and my personal experiences with the teaching and learning of the inverse I have observed that teachers theorize this concept and do not show and explain to their learners the existence and use of an inverse in real life and why x and y are swapped in the procedure of determining the inverse. In the current study I developed the concept of inverse of a function using '*real life contexts*'. This practical approach is a contribution to the literature on the teaching and learning of the inverse of a function because it helped learners to realise the existence of the inverse in their daily lives and the conditions under which it exists which necessitated the construction of the definition of the inverse. Real life activities also focussed learners' attention on understanding procedures and the reasons why the procedures work in the way that they do rather than memorizing them. Engaging learners in real life activities modelling the inverse of a function helped them to understand the procedure of algebraically finding the inverse by swapping x and y and the relationship between the graphs of a function and its inverse. This also helped learners to make sense of the condition of one-to-one property of a function and a condition for a function to have an inverse.

The nature of questions in the June examination paper, namely, questions 7 and 8 compelled me to generate specific indicators for the action, process, object and schema level of APOS theory to complement the ones in Section 3.5. These indicators were also used in this study to determine the conception levels at which each learner was operating. They are outlined as follows:

Action level of a function

The following are indicators of a learner who is operating at the action level that I created:

- ties a function to a specific rule or formula;
- can only identify changes in a pattern but may fail to come up with a formula or rule

generating the pattern;

- gives only examples of functions done in the classroom;
- sees a function as a machine and understands that some value is put into the machine and the machine churns out a value;
- draws arrow diagrams and uses arrows to show relationships;
- can repeat the explanation of critical points just as they were given in class;
- uses rules without reason, for instance, they say “this is what we do or what we were told”; and
- can repeat the explanation of the procedures just as they were given in class.

A learner at the action level *cannot do the following*:

- create or formulate own example of a function or non-function;
- distinguish a function from a non-function;
- explain how and why the procedure they use works;
- explain why at the x-intercept $y = 0$ and at the y-intercept $x = 0$;
- explain why at the turning point the x-coordinate = $\frac{-b}{2a}$ and do not know how this formula came about;
- explain how and why the procedure they use works;
- explain why they interchange x and y in inverses; and
- understand the conditions that must be satisfied by a function to have an inverse.

Process level of a function

A learner operating at the process level *can do the following*:

- explains the meaning of the intercepts, turning points and asymptotes;
- easily calculates the critical points;
- gives partial explanations of their procedures;
- sketches the graph of a given equation by first calculating the critical points and plotting them;
- thinks of graph or curve sketching as an entire activity and internalizes the procedure;
- looks at the word “inverse function” as a verb and sees an inverse function as undoing something;
- gives a definition of inverse function that looks at the procedure as a whole with inputs, a process, and outputs;

- explains correctly the meaning of the inverse function;
- easily calculates the inverse function; and
- looks at an equation of a function and sees the procedure of finding its inverse as a whole without having to calculate it.

A learner at the process level *cannot do the following*:

- use the set-theoretical definition of a function to distinguish a function from a non-function;
- create own example of a function or non-function;
- link the vertical line test with the one-to-one property of the function concept;
- explain why they take the steps of the procedures that they use, such as finding the intercepts, turning points and asymptotes;
- attempt questions which require the use of a sketched graph;
- use the critical points they identify on a sketched graph to determine the equation of that graph;
- see that critical points connect the graph and its equation;
- explain why they take the steps of the procedures that they use like interchanging x and y;
- see the relationship between a function and its inverse;
- use the conditions for an inverse to exist; and
- link the vertical line test with the one-to-one property of the function concept and the condition that is required for an inverse function to exist.

Object level of a function

A learner operating at the object level *can do the following*:

- looks at the graphical representation of a function and verify whether it is a function or not by using the vertical line test;
- can explain why a graph which passes the vertical line test represents a function;
- is able to interpret and relate parts of algebraic expressions or equations representing functions;
- has knowledge of the rules and properties that enables the learner to describe how s/he transforms functions and predicts how functions are transformed by looking at the

graphs of transformed functions and to arrive at conclusions of the properties of graphs relating to different equations;

- can identify critical points on a drawn graph and write down their coordinates;
- is aware of the reasons why they follow the steps of procedures that they use, such as interchanging *x* and *y* in calculating the inverse;
- can see the relationship between a function and its inverse only in terms of the domain and range;
- is partially aware of the conditions for an inverse to exist; and
- is aware of the graphical relationship between a function and its inverse.

A learner at the object level *cannot do the following*:

- use the logical definition of the function concept in formulating examples and non-examples;
- use the logical definition to determine whether a given relation is a function or non-function;
- easily switch from graph to equation;
- link the critical points located on the drawn graph and the ones they calculated using the equation;
- see that the critical points connect the graph and its equation;
- link the vertical line test with the one-to-one property of the function concept and the condition that is required for an inverse function to exist; and
- explain the graphical relationship between a function and its inverse.

Schema level of a function

A learner at the schema level *can do the following*:

- use the logical definition of the function concept in formulating examples and non-examples;
- use the logical definition to determine whether a given relation is a function or non-function;
- switch from graph to equation and from equation back to graph;
- link the critical points located on the drawn graph and the ones calculated using the equation;
- see that the critical points connect the graph and its equation;

- link the vertical line test with the one-to-one property of the function concept and the condition that is required for an inverse function to exist; and
- explain the graphical relationship between a function and its inverse.

These indicators are a contribution to the literature on functions since they worked well in this study and they can be improved through their use in classrooms to teach the function concept.

6.7 Reflections on the theoretical framework

The theoretical framework was compiled by bringing together the theories of Piaget, APOS and RME within the constructivist paradigm in order to take advantage of their complementary themes and characteristics. The RME teaching and learning principles helped me to design the HLTs and activities that helped learners to move faster through APOS theory conception levels. APOS and Piaget's theory complemented RME by informing the analysis. I used APOS theory conception levels to determine the level at which a particular learner was operating using the learner's responses to task questions and clinical interview questions. APOS theory was also used as a tool to explain objectively learners' difficulties with a broad range of function-related concepts and to suggest ways that learners can learn these concepts. APOS theory pointed me towards pedagogical strategies that led to marked improvements in learners' understanding of the function concept. However, the explanations offered by a APOS analysis were limited to descriptions of the thinking processes of which a learner was capable, rather than what "really" happened in a learner's mind, since this is probably unknowable. In practice, the mental constructions specified by APOS theory rarely occur in such a simple logical sequence. Moreover, transitions between levels were not always clear especially at the schema level which is not well defined.

I believe that the framework I used in this study served its purpose well as it allowed me to classify learners in APOS theory conception levels and to use the RME's teaching and learning principles to design stage appropriate and realistic activities that moved the learners from one conception level to the next. For my teaching, understanding how a learner moves through this developmental process enhanced my notion of how learners learn and therefore provided me with opportunities to help them to understand the function concept. As such, teachers need to determine the various stages at which their learners are and plan stage-appropriate activities relative to learners in a particular stage.

However, my reflections on the theoretical framework indicated that more time needs to be devoted to helping learners develop the mental structures at the process, object and schema levels. Deriving and using a working definition should facilitate the development of mental structures at the process and object levels, while using definition-based procedures in translating from a graph to an equation should aid object conceptions. Organising and linking the working definitions and definition-based procedures could also link the relevant actions, processes and objects to form organised schemas.

6.8 Reflections on my research methodology

The aim of this study was to use design research to develop instructional material that could improve the learning of functions. In order for me to achieve this aim I followed a qualitative paradigm through a case study which enabled me to use task-based clinical interviews, follow-up group interviews and observation. It was important for me to conduct both individual and group interviews because they enabled me to discover learners' thinking, processes and patterns which allowed me to evaluate their level of understanding in terms of APOS theory conception levels. By listening to learners' interpretations, explanations, justifications and observing them interacting and responding to each other's thought processes I was able to learn about their understanding of the function concept, concept images, difficulties and misconceptions. Group interviews generated a wider range of responses from learners and enabled them to challenge each other.

The use of a case study provided me with a wealth of detail on learners' understanding of functions which gave credibility to classroom situations and problems concerning their learning of functions. The actual results from this case study also furnished me with real or concrete solutions to the problems being experienced in the teaching and learning of functions. In general, findings from a case study may only be applicable to similar cases but in this study I was not seeking to generalize my findings but interested in discovering "learners' understanding of functions" so that I could use that to design instruction.

There were times when I felt the need to adjust the HLT or instructional activity for the next lesson. I made minor changes in the HLT because of incidents in the classroom such as my anticipations of failure, strategies I had not foreseen, activities that were too difficult, and so on. In such cases, I had a micro-cycle of design, experiment, and analysis within a macro-cycle of design research. Such micro-cycles in the HLT created optimal conditions and I regarded them as elements of the data corpus. Hence, I had to reflect well on these changes. In addition, the information gained strength when I supported changes with theoretical considerations. It is a result of the methodology used in this study that I was able to elicit learners' interpretations, difficulties and misconceptions about the function concept and then to use these to design HLTs and RME-based activities that improved learners' understanding of function-related concepts.

6.9 Conclusions

In drawing conclusions from the results presented in chapter 5, I was mainly interested in learners' overall improvement in their understanding of functions spelt out in Section 2.4 of Chapter 2. The following conclusions were made from this study.

- *Learners find it difficult to understand and use a prescribed definition of the function concept.*

To help overcome this challenge I provided learners with opportunities to develop the following working definition of the function concept:

> "A function is a dependence relationship between two sets of variable quantities in which each element from the first variable (where we are coming from/source) has only one corresponding element in the second variable (where we are going to/destination)."

This definition captures important attributes of the function concept, that is, the key concept or dependence relationship and the domain and range. Learners in this study could easily use it to identify functions and non-functions, formulate their own examples of the function concept, and identify dependent and independent variables in a given relationship. This definition also helped learners to appreciate the nature and origin of the function concept, distinguish a function from a non-function and apply it in problem solving. However, this working definition did not cater for constant functions. I was satisfied by their use of this working definition at their level of cognitive development. Figure 61 summarizes the HLTs that I designed and implemented which resulted in improvements in learners' conceptual understanding of the definition of the function concept.

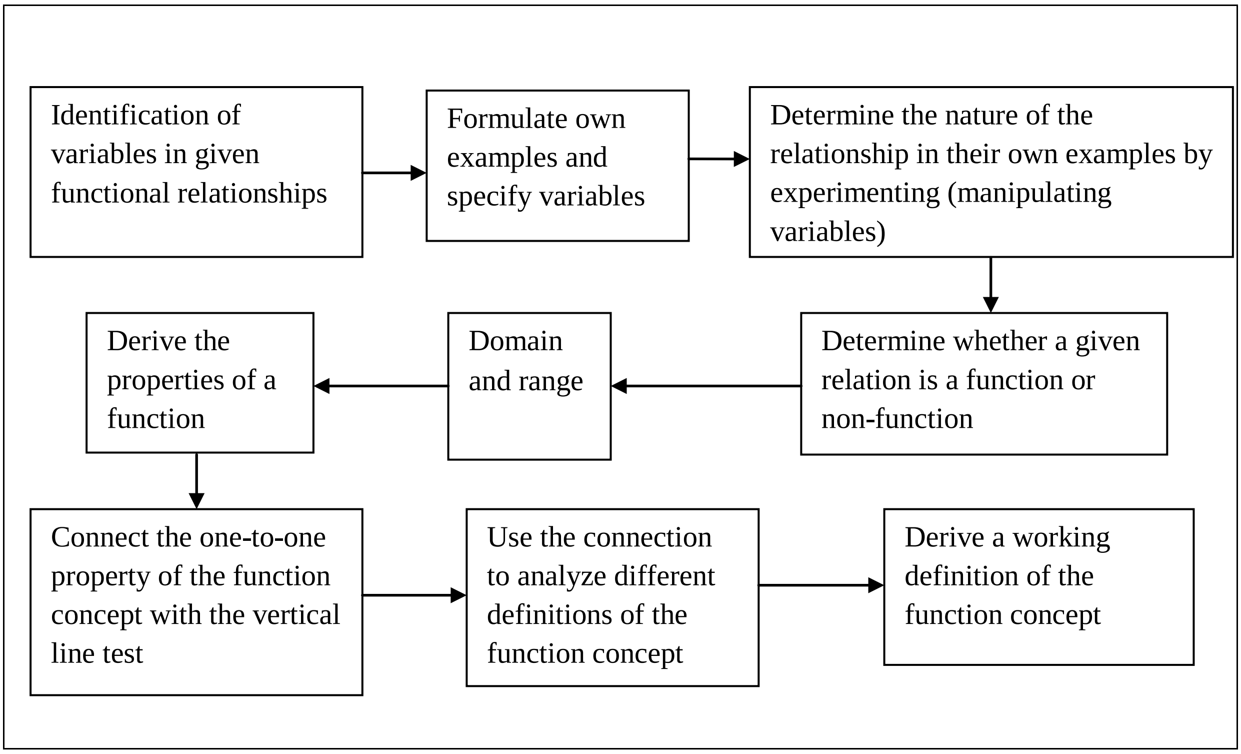


Figure 61: Summary of HLTs for the definition of the function concept

- *Learners' have a tendency of memorising and carrying out procedures/algorithms without understanding them.*

This tendency limits learners' use of these procedures since they can only perform them in one direction. For example, learners in the current study were able to successfully use the procedures of determining the intercepts, asymptotes and turning points from a given equation but found it difficult to extract the same critical points from a drawn graph representing the same equation and then to formulate the equation. Schwarz and Hershkowitz (1999) attribute the difficulties in translating from one representational form to another to the fact that different representations of a function have different properties for mathematical work with functions. For example, the critical points can be read from a graph but can only be calculated from a given equation. As such learners should be helped to read or extract these critical points from graphs and calculate them with understanding. Moreover, it is important for teachers to help learners understand the mathematical procedures before applying them in problem solving. To assist learners in the current study to develop a conceptual understanding of the procedures of calculating the intercepts, asymptotes and turning points I introduced the concept of definition-based procedures (DBPs). The use of

DBPs made it easier for learners to translate from equation to graph and *vice versa* and also compelled learners to understand the meanings of all the key concepts involved in carrying out these procedures. Once these key concepts are understood it is easier for learners to translate from equation to graph and from graph to equation. Figure 62 gives a summary of the HLTs I designed to help learners understand the procedures of translating from an equation to a graph and *vice versa*.

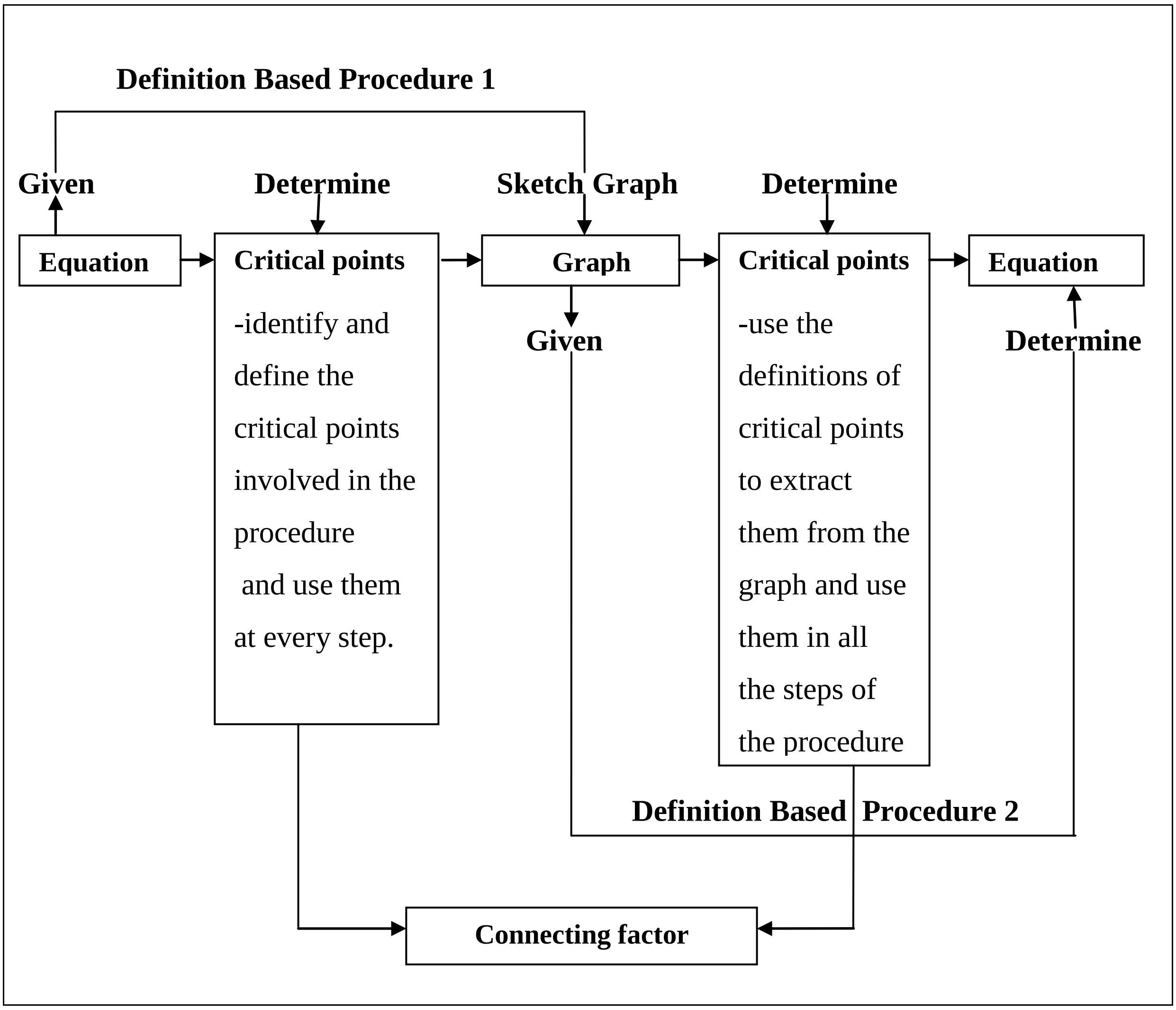


Figure 62: Summary of HLTs for DBPs for translating from equation to graph and *vice versa*

- *The use of real life contexts formed the basis for their understanding of the inverse of a function.*

In this study I introduced the inverse of a function by using real-life contexts and activities modelling the inverse of a function. This helped learners to realise the existence of the inverse in their daily life, the conditions under which it exists, and the relationship between a function and its inverse which necessitated the construction of the definition of the inverse. Though learners did not have difficulty in calculating the inverse there was need to explain the procedure they were using. Real life activities also focussed learners' attention on understanding procedures and the reasons why procedures work rather than merely memorising them. Figure 63 summarizes the HLTs that were designed with the intention of enabling learners to move more quickly to the schema level.

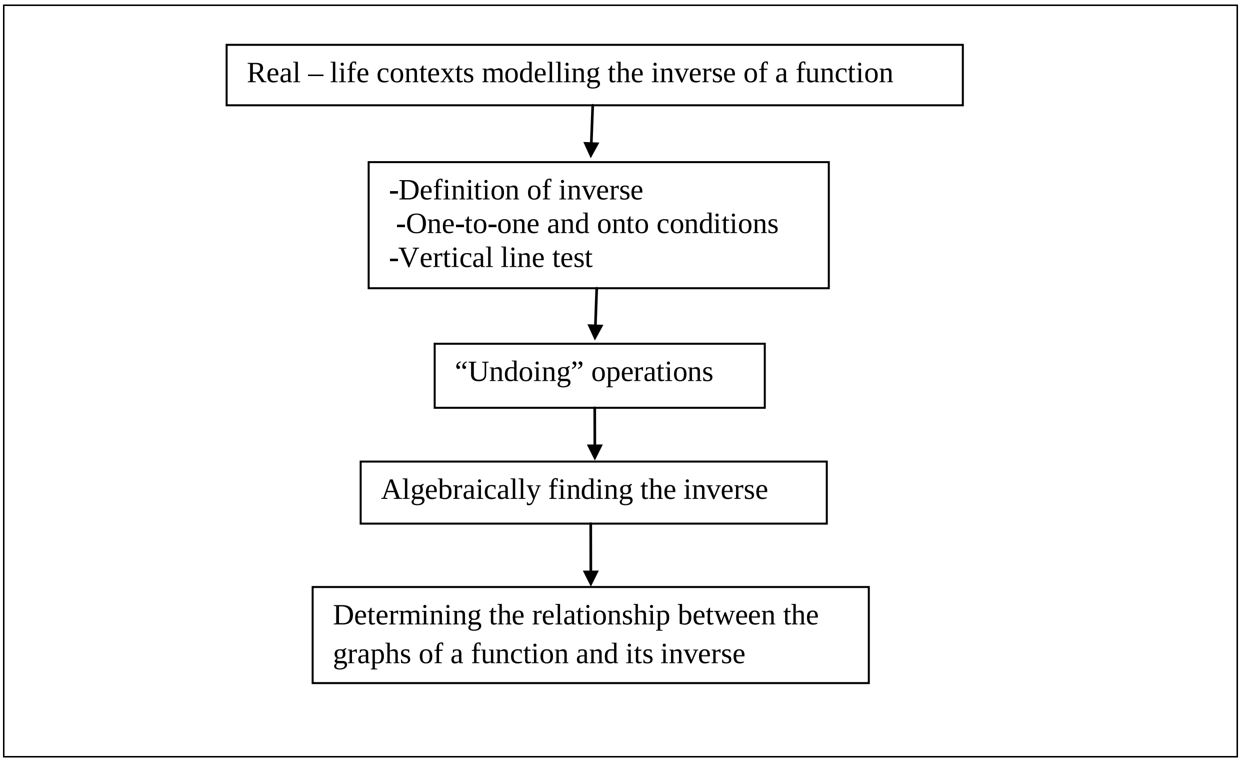


Figure 63: Summary of HLTs for the inverse of a function

- *Design research improved learners' understanding of the function concept in terms of its definition, representation and inverse.*

Design research together with APOS theory, RME and constructivism to a large extent succeeded in improving learners' conceptual understanding of the function concept through iterative design and development of instructional sequences (HLTs) and level appropriate RME based activities.

Though not all the learners managed to reach the schema level, the research assisted learners to overcome their identified difficulties on functions. Learners also managed to move from their initial lower levels of APOS theory to the process and object levels and two learners reached the schema level as intended.

6.10 Recommendations

- *Use of a design research model in designing, developing and implementing instruction on functions*

A design research model in teaching the function concept is recommended as planning of lessons starts with the learner unlike the traditional lesson planning which focuses on the content to be taught and how it can be taught.

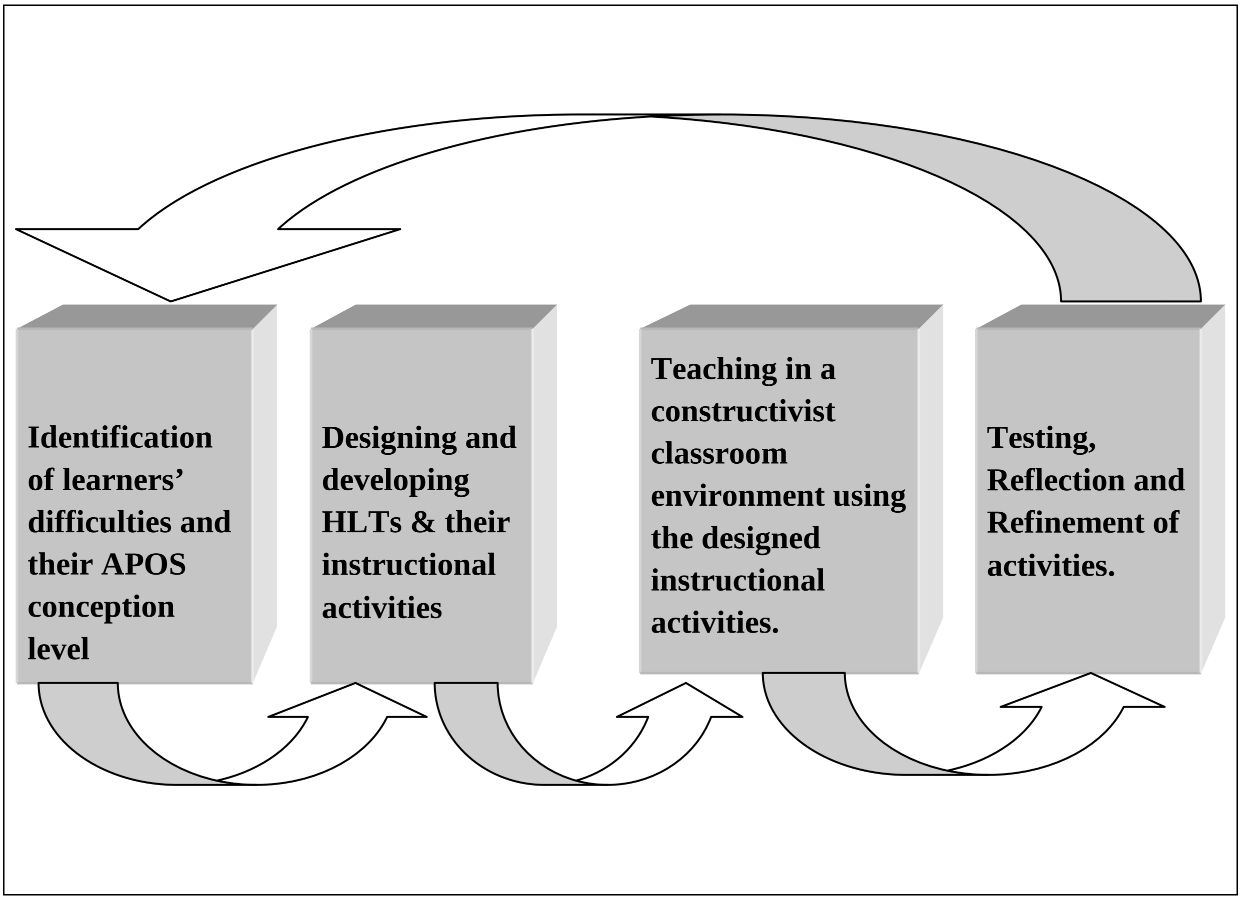


Figure 64: Simplified illustration of Wademan's (2005) adapted Generic Model

- *Combining constructivist theories of APOS, Piaget, and RME in design research*

Teachers can use APOS and Piaget's theory to inform the analysis of their learners' written or verbal responses which point to possible pedagogical strategies. On the other hand the RME's learning and teaching principles can be used by teachers to inform their instruction, that is, the planning and implementation. In this study design research together with APOS, Piaget's theory, RME and constructivism to a large extent succeeded in improving learners' conceptual understanding of the function concept in terms of its definition, representations and inverse. This occurred through iterative design and development of instructional sequences (HLTs) and level appropriate RME based activities. In this study I also used constructivism, RME and APOS theory conception levels on the recommendation of the literature I reviewed. However, in South African schools, for different reasons these ideas of teaching the function concept are presently not being used. The results of this study suggest the use of these approaches facilitated meaningful learning. Teachers should use constructivism as their theoretical framework in teaching mathematics. A constructivist classroom environment enables learners to actively participate in the construction and development of their own understanding. It also promotes more learner to learner interaction which helps learners to compare their solution strategies and learn from each other unlike the teacher-centred approach which is commonly being used in our classrooms.

- *Use of APOS theory conception levels as a diagnostic tool for learners' understanding of the function concept and as a pointer to possible pedagogical strategies*

I recommend the use of APOS theory conception levels as a diagnostic tool for learners' understanding of the function concept and as a pointer to possible pedagogical strategies. Teachers can use conception levels of APOS theory in determining the level at which learners are operating, using the available evidence in the form of the learners' responses to task questions and clinical interview questions. This will help teachers design instructional materials to move their learners from one conception level to the other. The ultimate aim is to reach the schema level.

The theory of RME which proved to be very effective in this study is also recommended to drive the design and implementation of the instructional sequences and activities for teaching and learning the function concept. RME places learners' mathematical reasoning at the centre of the design process while simultaneously proposing the specific means by which the development of their reasoning could be systematically supported (Cobb, Zhao & Visnovska, 2008). RME allows the teacher to initially present the definition, representation and inverse of the function concept within a concrete context allowing learners to develop informal strategies, but gradually through the process of guided 'mathematisation', allow learners to progress to more formal, abstract, standard strategies. It compels the teacher to use realistic context situations as the starting point or as the source for learning the function concept and speaks to the need to locate instructional starting points that are experientially real to learners and that take into account learners' current mathematical ways of knowing. Teachers need to examine their learners' informal solution strategies and interpretations that might suggest pathways by which more formal mathematical practices might be developed. RME also places emphasis on understanding processes, rather than learning algorithms and the focus is on the growth of the learners' knowledge and understanding of mathematical concepts. There is a need to establish a link between learners' own understandings and the correct mathematical ideas.

In order to combine these theories into a workable plan for the practicing teacher, workshops are required. Teachers who trained in the seventies are familiar with Piaget's theories while those who trained in the eighties onwards are more familiar with constructivism. There needs to be a focus on connecting Piaget's theories with the APOS theory. The ideas surrounding constructivism should be considered in the context of RME.

6.11 Limitations of the study

A teaching experiment required clinical interviews to probe further and deeper into each of the learners' thinking. My sample was reduced from twelve to six because of two learners who just dropped out after the first round of interviews and four others whose attendance at the sessions was erratic making the size of the sample smaller than what I had planned. A larger sample could have yielded more concept images about the function concept. To begin with, the learners were not free to express themselves after I had told them that I was tape recording our conversations. I tried to reduce the effect of this reluctance by being more informal in our discussions and diverting the learner's attention from the tape recorder. Since I was observing learners alone I might also have missed some important points learners were making in their group discussions. Constraints of time did not allow me to take each learner to the schema level but I noted that they escalated from the level where they originally were located. In addition I had assumed that since learners had been taught this topic they would begin at the process level of APOS theory, but I found that some of them were at a lower level. So I had to design more RME activities which required more time to accomplish. This could be the reason that some of the learners were operating below the schema level.

As a result of the above limitations I suggest further longitudinal research that involves more types of functions and a larger sample of learners, more researchers and possibly video-recording even the facial expressions of learners when they start a problem. In a normal class of 30-35 learners I would advise teachers to allow their learners to conduct these activities in groups and then to present their solutions to the whole class. This should then be followed by a whole class discussion where learners explain and justify their solutions which will enable learners to share ideas and learn from each other.

6.12 Closing remarks

This study sought to answer the following research questions in terms of the definition, representation and inverse of the function concept

1 How do learners understand the function concept?

2 How can instruction be designed to improve learners' understanding of the function concept?

Learners regarded a function as any relationship. This narrow view of a function resulted in learners failing to formulate examples and non-examples of functions. To help overcome this challenge I provided learners with opportunities to develop a working definition of the function concept. Learners in the current study were able to use the procedures of determining the intercepts, asymptotes and turning points from a given equation without any understanding of the concepts. As a result they found it difficult to extract the same critical points from a drawn graph representing the same equation and then to formulate the equation. To assist learners develop a conceptual understanding of the procedures of calculating the intercepts, asymptotes and turning points I introduced the concept of definition-based procedures (DBPs) which made it easier for learners to translate from equation to graph and *vice versa*. In this study I introduced the inverse of a function by using real-life contexts and activities modelling the inverse of a function. This helped learners to realise the existence of the inverse in their daily life, the conditions under which it exists, and the relationship between a function and its inverse which necessitated the construction of the definition of the inverse. Real life activities also focussed learners' attention on understanding procedures and the reasons why procedures work rather than merely memorising them.

The context of the classroom informed the instructional sequences to teach grade 11 functions which are the product of this study. Seeing that timetables and scheduling vary from school to school, the amount of material per lesson will also vary depending on the available class time. I recommend that the lessons be taught in the sequence of this study.

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Appendices

Appendix 1: Interview Schedule for the June 2011 Examination

Introduction

"My name is Tinoda Chimhande and I am doing a research study to determine how grade 11 learners understand the function concept in terms of its definition, representation and inverse. I also want to understand and develop the connections that learners make between the definition, representation and inverse of the function concept. Are you aware that this interview will be audio taped and I will make a transcript, but that your identity will be kept confidential?

Questions

1. You wrote the June mathematics examination Paper 1 that had questions on the function concept. Can you explain what thoughts came to your mind when you read the problems on the function concept in the examination?
2. Did the problems look familiar to you in any way?
3. Did you attempt to solve such problems previously? If so, when?
4. Could you please take me, step by step through your solutions? You are going to solve each part of the questions on functions on a separate page and while you are solving it I want you to tell me what you are doing. I want to get a better idea of your thinking behind each solution strategy on all parts of the questions on the function concept. There is no right or wrong solution and you are free to refer to your marked script; I just want to get a better understanding of your thinking as you solve the problems on functions. I am also going to ask you questions as we go along to help clarify things that I may not understand".
5. Do you think that there are other strategies to solve the same problems?
6. If you were given similar problems, would you be able to answer them?
7. How would you explain such solutions to your colleagues?
8. Was it difficult to determine the answers?
9. Why did you choose these particular solutions?
10. Did you check your answers to see if they were correct?

Appendix 2

Task 1: Definition of the function concept

1. For each the relationships described below explain whether they are functions or not:

a) The set of all cars in Mpumalanga province in relation with their registration numbers.

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b) The set of South African provinces related with their provincial cities.

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c) The set of fathers related to their sons

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2. Explain the meaning of the words domain and range.

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3. Which of the following relations are functions? Explain.

a) {(1; 2) ;(2; 2) ;(3; 2)}

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b) {(2; 1) ;(1; 2) ;(2; 2)}

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c) {(-1; 2) ;(1; 2) ;(-1; 3) ;(1; 3)}

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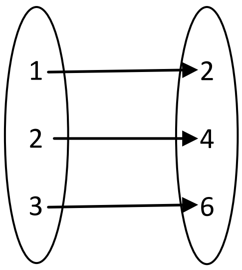
4. For each of the following

(i) state, with reasons, which are functions

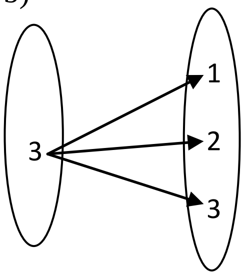
(ii) give the domain and range in each case

(iii) represent the relation as a set of ordered pairs.

a)



b)



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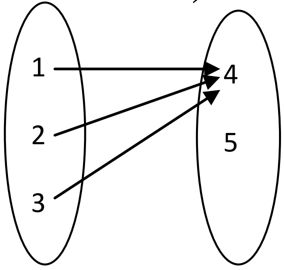

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Task 2: Definition of the function concept

1. Explain how the vertical line test is used to determine whether a drawn graph represents a function or not.

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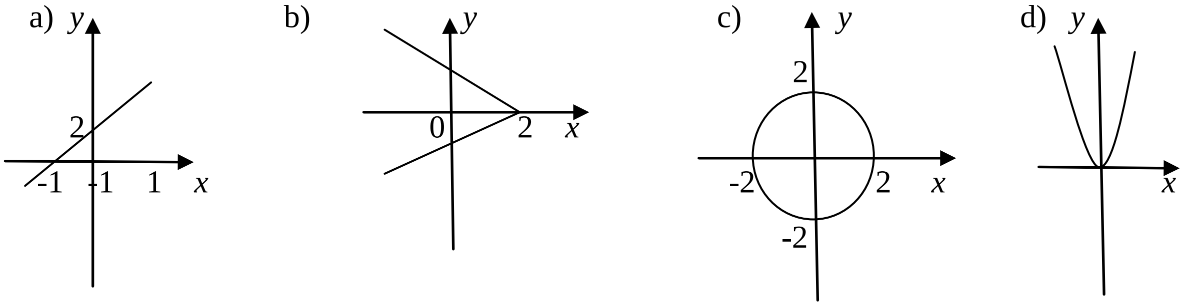
2. The following are graphs of various relations. For each graph (i) state whether it represents a function or not. (ii) give the domain and range.



a)…………………………………………………………………………………………………

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b)…………………………………………………………………………………………………

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d)…………………………………………………………………………………………………

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Appendix 3: Interview Schedule for the definition of the function concept

1. What do you think of when you hear the word function in mathematics?
2. Using your own words and any diagrams you need to express your ideas, explain the meaning of the word function.
3. List and explain any special properties of a function that you can recall and explain how you would illustrate them.
4. Give me two examples of a function and two examples of non-functions.
5. How do you distinguish a function from a non-function?
6. A function represents a relationship between an independent and a dependent variable. Explain in your own words what you understand by an independent variable and a dependent variable?
7. How do you identify the independent and dependent variables in a given functional relationship?
8. A function has a domain and a range (co-domain). Explain in your own words the meaning of a domain and range.
9. Where do you use functions in real life? You can use an example to explain the application of functions in real life.

Appendix 4: Interview Schedule for the representation of the function concept

1. Explain how you might represent the function and of these representations which one do you favour most and why?

(a) **Equations**

(i). When you are given the equation of a function how do you draw the graph of the function?

(ii). What are the main features (critical points) of the graph?

a. What are the intercepts of a graph and how do you calculate them?

b. What is an asymptote of a graph? How do you know from a given equation that there is an asymptote and how do you determine the equation of that asymptote?

c. Some graphs have turning points. What do you understand by a turning point of a graph and how do you determine this turning point?

(b). **Graphs**

(i). What are the critical points of a graph?

(ii). How do you determine the equation of a function from a drawn graph?

(c). Tables.

(i). When given a table of values how do you draw the graph represented by that table of values?

(ii). When is it possible to determine the equation of the function from the table? How do you determine the equation of a function from the table of values?

(iii). What other critical points of the function can be determined from the table of values?

(iv). If the table is not given but you want to use it, what information will you need to complete the table?. Which other representations can you use to fill in the table?

Appendix 5: Interview Schedule for inverse of a function

1. In your own words explain what an inverse function is?
2. What symbol is used to represent an inverse of a function and what does that symbol mean to you?
3. If you are given the equation of a function explain how you find the inverse of that function?
4. What is the relationship between the domain and range (co-domain) of a function and that of its inverse?
5. What is the relationship between a function and its inverse? The form of these interviews will change from student to student, depending on the answers the students will give. The changes will occur only in the shape of the interview and the order of the questions, not in the substance of the tasks.

Appendix 6: Initial tasks and activities used in this study

Initial Tasks: From the June 2011 Examination

QUESTION 7

Given $f(x) = \frac{1}{x-4} + 2$

7.1 Calculate the co-ordinates of the x and y intercepts of f. (4)

7.2 Determine the equations of the asymptotes of $f(x)$. (2)

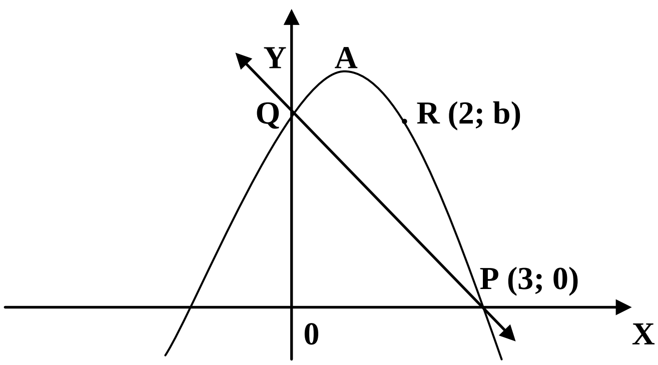
7.3 Sketch the graph of $f(x)$ showing all the critical points. (4)

[10]

QUESTION 8

The sketch below, not drawn to scale, shows the graphs of the functions defined by: $h(x) = -2(x-3)(x+1)$ and $g(x) = mx+c$.

where A is the turning point of $h(x)$ and R(2;b) is a point on $h(x)$



8.1 Calculate the coordinates of the turning point A. (3)

8.2 Calculate the coordinates of Q. (1)

8.3 Determine the numerical values of m and b. (4)

8.4 Write down the equation of $g(x)$. (1)

[9]

Activity 1: Identifying and explaining variables

Work in pairs and identify the variables

i. The amount of hours someone works to the amount they get paid.

ii. The amount of petrol put in the tank and the amount of money paid by the motorist.

iii. The number of loaves of bread and the amount paid by the customer.

iv. The number of nights spent at a hotel and the amount of money the guest has to pay..

v. The number of patients admitted in a hospital and the number of beds in the hospital.

Activity 2: Applying the knowledge

Procedure for activity 2:

1. Learners formulate their own relationship which they will be able to manipulate in an activity by varying the quantities.
2. Identify quantities that vary in the course of the activity and focus on the relationship between those variables.
3. Create a record of the corresponding values of the varying quantities by using a table or graph.
4. Identify patterns in the records created.
5. Create a representation of the identified pattern in the relationship.
6. What can you say about the representation you created in 5?

Activity 3: Table allocation game (determining whether a given relation is a function)

Materials: Tables labelled with numbers on them for learners to move to. Learners are also labelled with letters A, B, C, D, E, and F.

Step 1: I wrote a relation that is a function in that every learner moves to a different table. Define the learners as the x-coordinates and the tables as the y-coordinates, for example, {(A, 1), (B, 2), (C, 3), (D, 4), (E, 5), (F, 6)}.

Step 2: I let learners act out the function by moving to the allocated table, for example, A moves to table 1, B moves to table 2, C moves to table 3 and so on. I told the learners that this relation is a function (I asked them to explain why) - answer: one learner is linked to a unique table.

Step 3: I wrote another relation that is a function. This time I wrote the relation so that more than one learner moves to one table, for example, (A, 1), (B, 2), (C, 1), (D, 3), (E, 4) and ((F, 5). I told the learners that this is a function as well. I checked to see if any of their guesses change. At this stage I introduced the aspects of domain (the learners) and range (where the learners are going to).

Step 4: I wrote a relation that is not a function, for example, (A, 1), (B, 2), (A, 3), (C, 4), (D, 5) and (E, 6). I asked them to explain why this relation was not a function. Answer: This is not a function because A will not be able to move to table 1 and table 3 at the same time.

Step 5: I then asked learners to derive the properties of the function concept from what they were acting out.

Activity 4: Deriving the working definition of the function concept

I asked learners in pairs to come up with their own working definitions of the function concept. The discussion starts when learners were asked to compare the different working definitions they had formulated

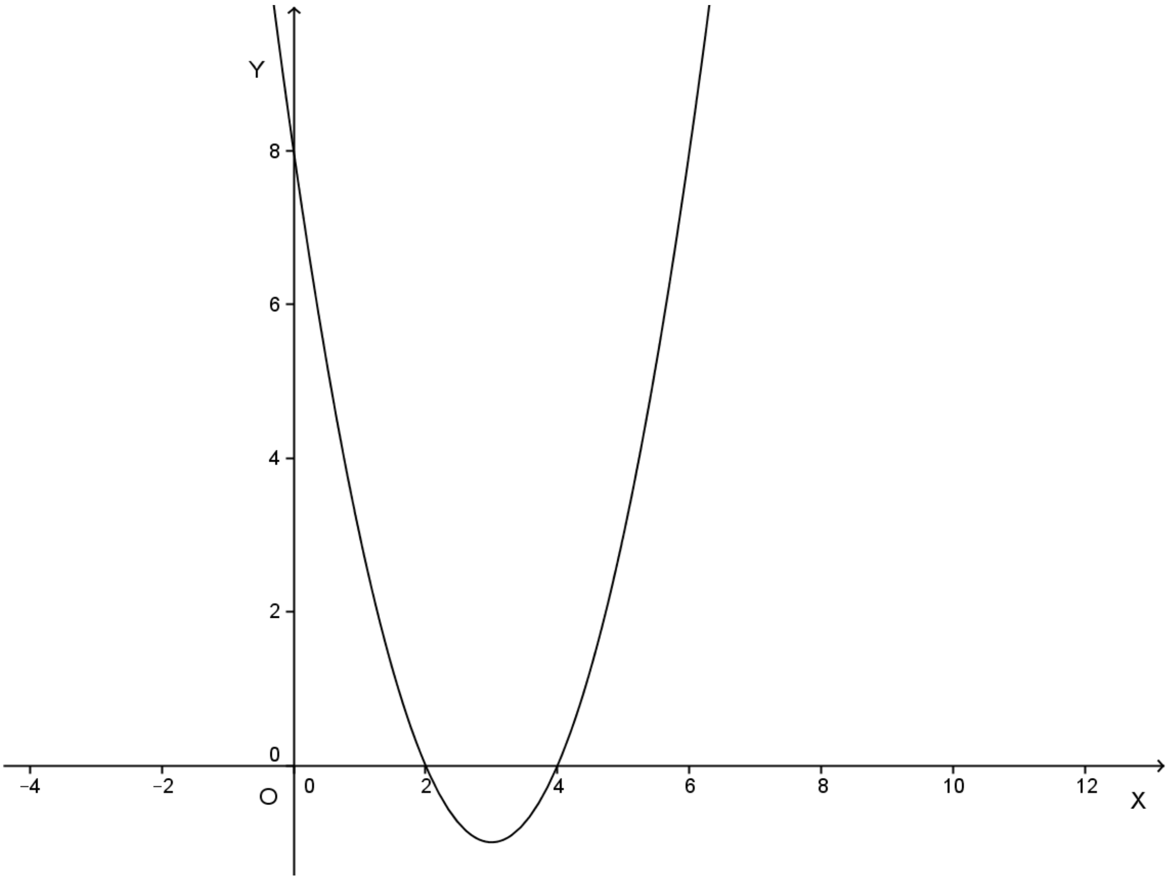
Activity 5: From equation to graph

Sketch the graphs of the following functions indicating all asymptotes, turning points and intercepts with the axes.

1. $f(x) = x^2 - 6x + 8$
2. $f(x) = -x^2 - 5x - 6$
3. $g(x) = \frac{3}{x-2} + 3$
4. $g(x) = -2.2^{x-1} + 1$

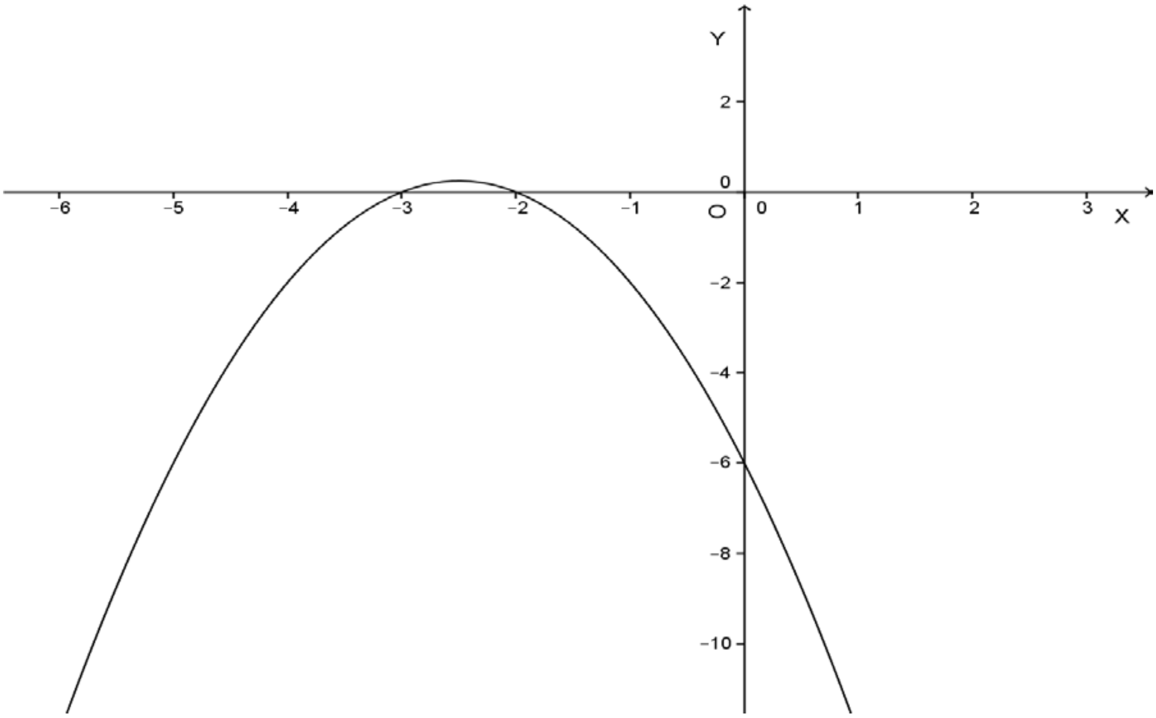
Activity 6: From graph to equation

1.



Activity 6: From graph to equation continued

2.



The graphs above represents the functions in the form of $f(x) = ax^2 + bx + c$. For each graph above:

a) Determine the values of a, b and c.

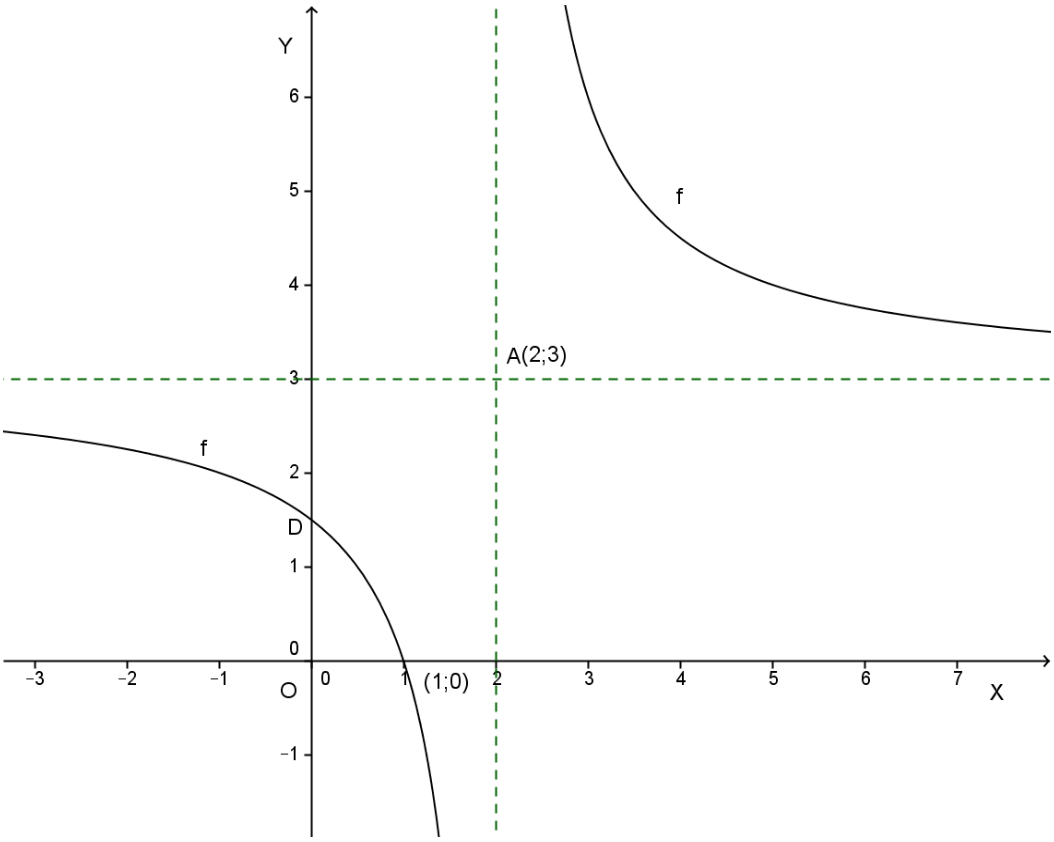
b) Determine the values of x for $f(x) = 0$.

c) Determine the coordinates of P, the turning point of $f(x)$.

d) Hence, determine the range of $f(x)$.

Activity 6: From graph to equation continued

3. Given $f(x) = \frac{a}{x-p} + q$. The point A (2; 3) is the point of intersection of the asymptotes of *f*. The graph of *f* intersects the *x*-axis at (1; 0). D is the *y*-intercept of *f*.



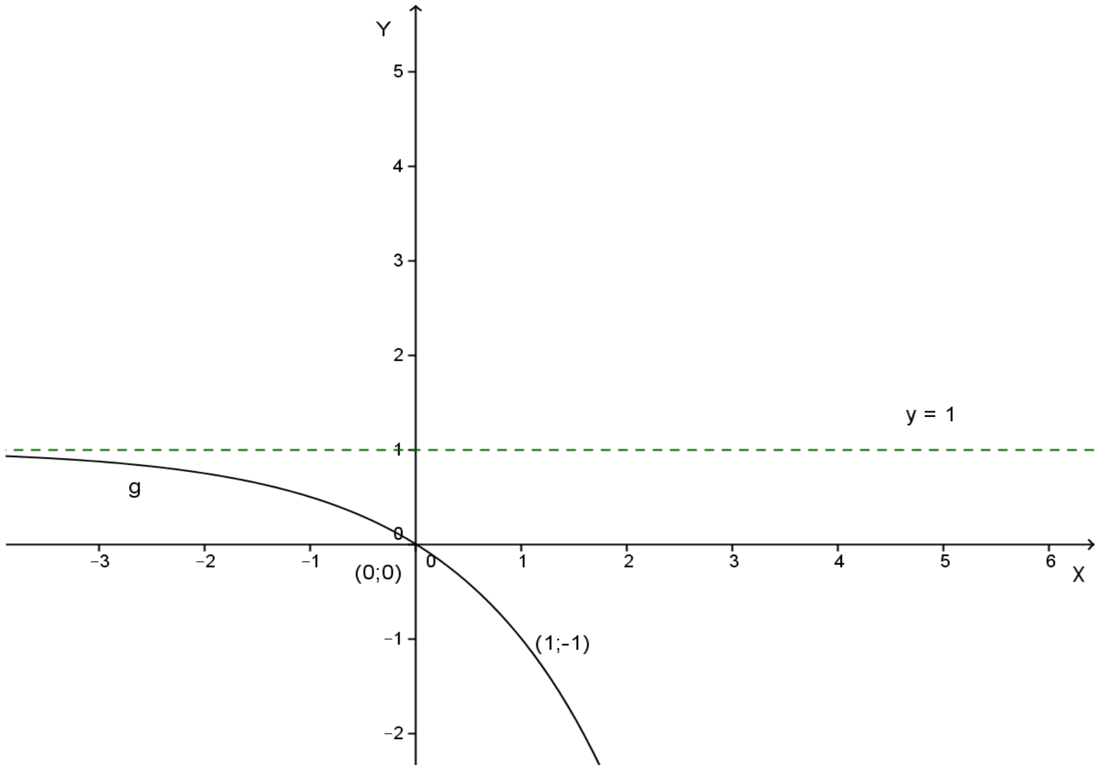
a) Write down the equations of the asymptotes of *f*.

b) Determine the equation of *f*.

c) Write down the coordinates of D.

Activity 6: From graph to equation continued

4. The diagram below shows the graph of: $g(x) = ab^{x-1} + q$. A (1;-1) is a point on g and $y = 1$ is the asymptote of g.



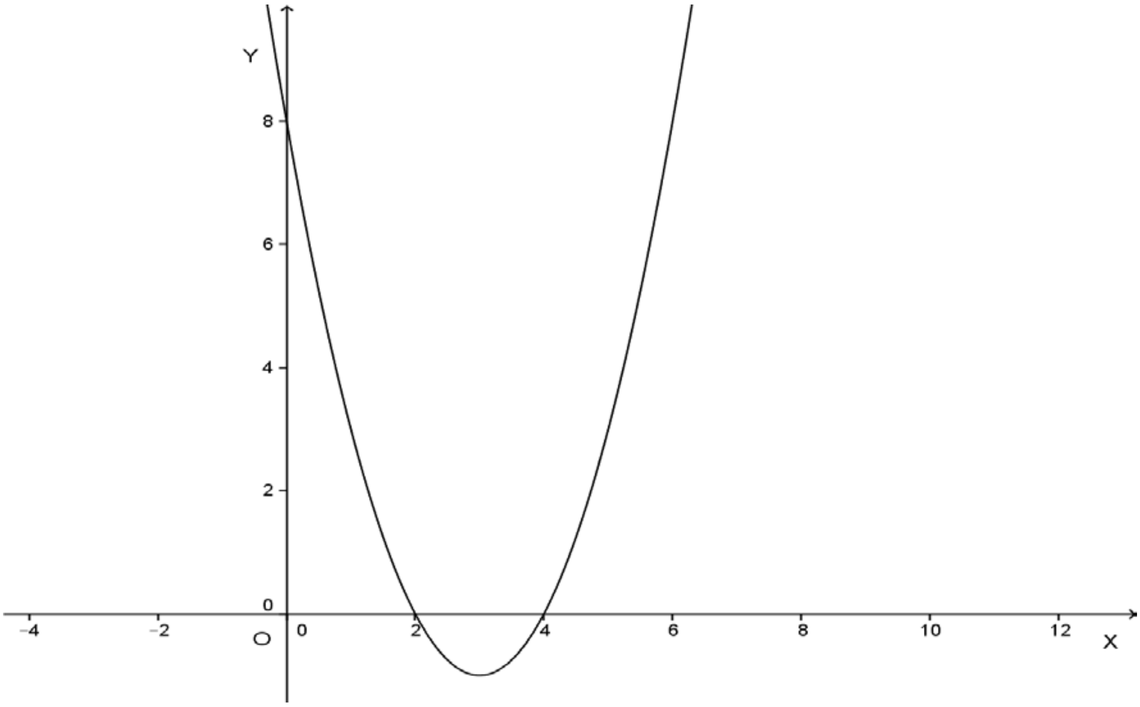
a) Determine the values of a, b and q.

b) Determine the y-intercept of g.

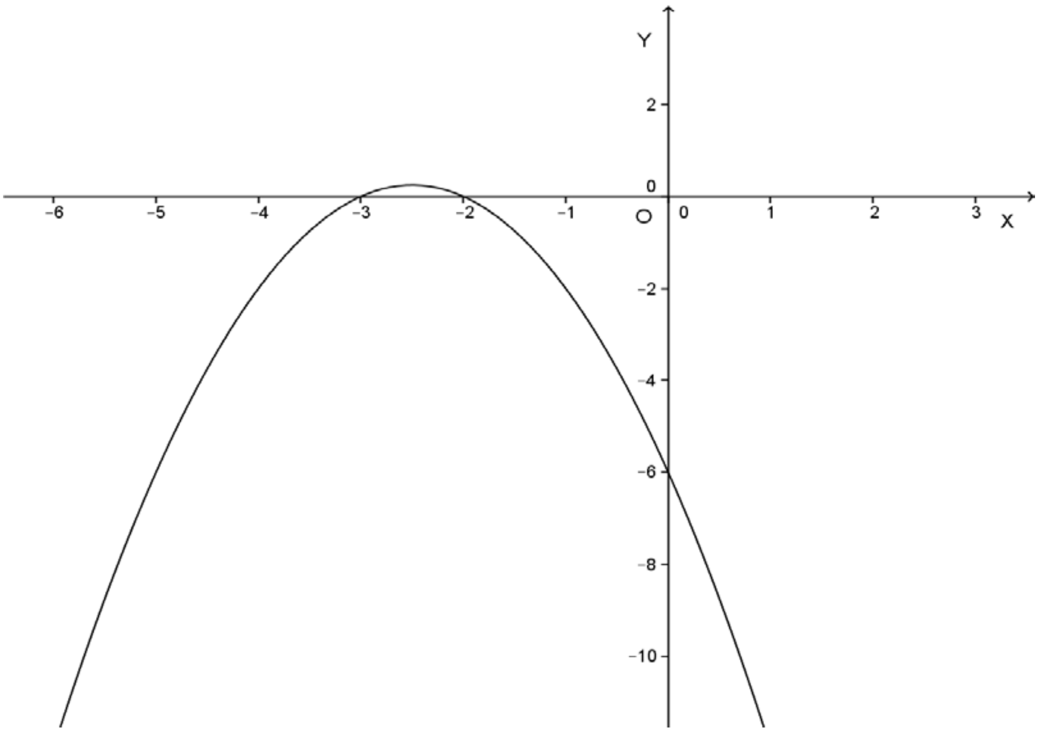
c) Write down the range of g.

Activity 7: Identifying and using critical points from graphs to formulate equations

1.

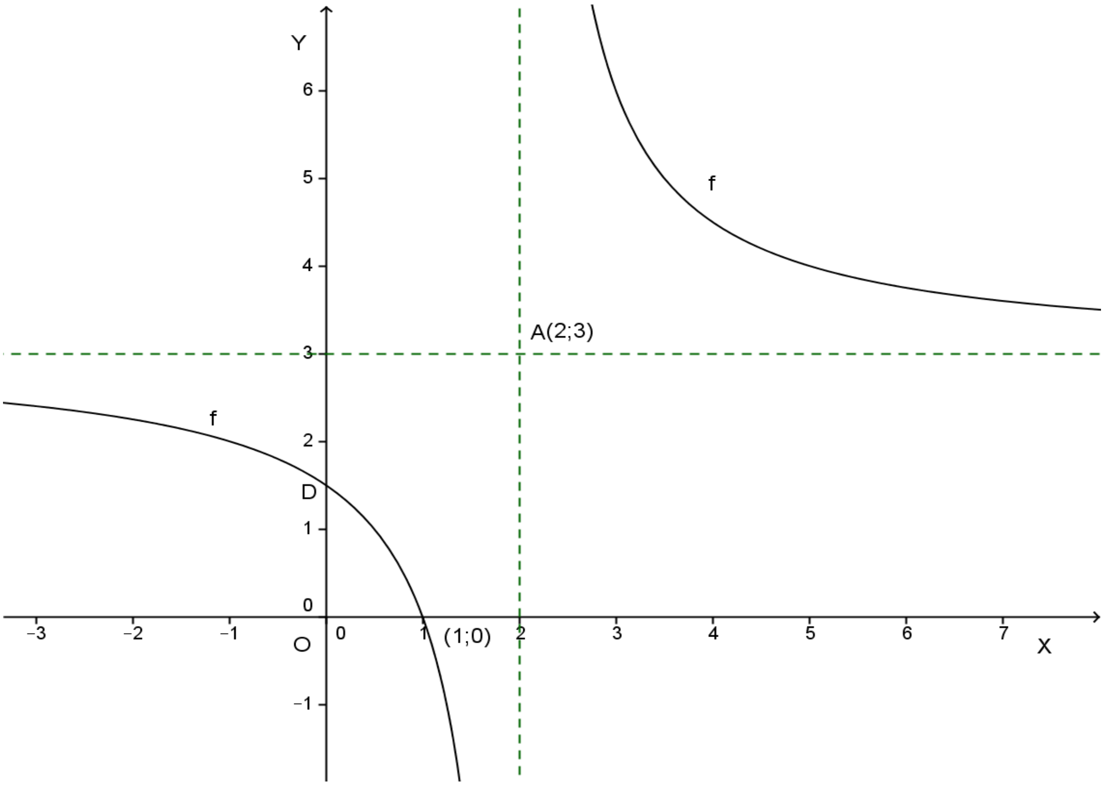


2.



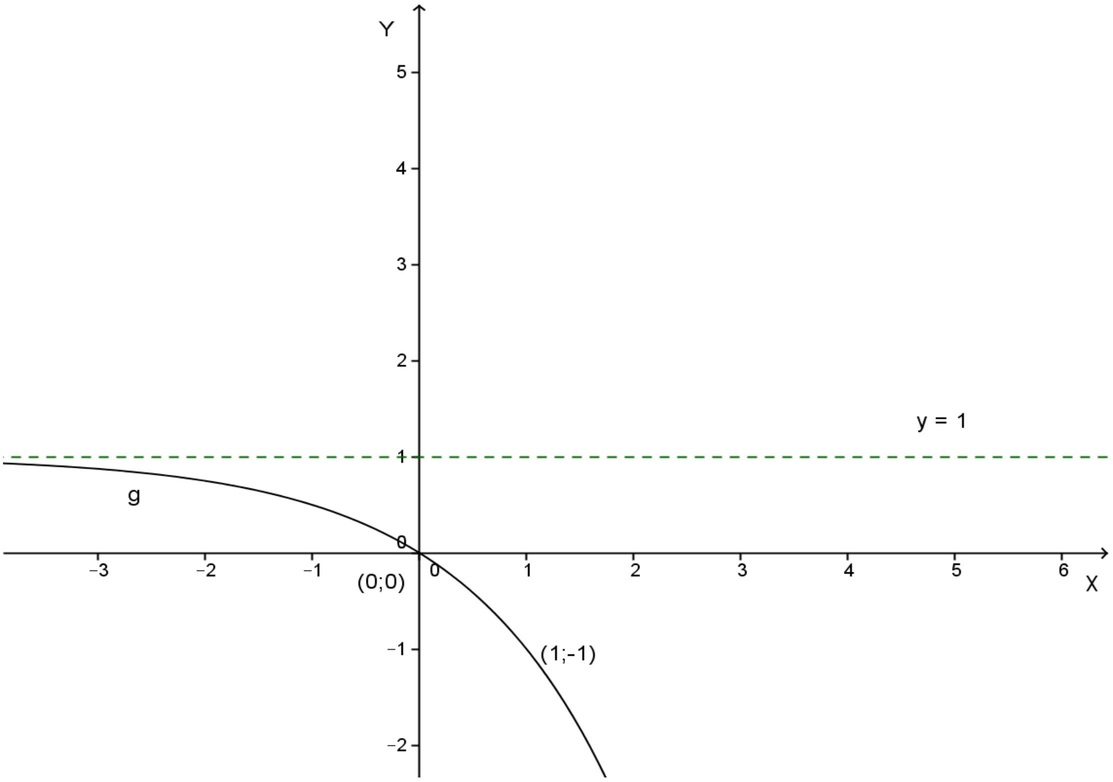
Activity 7: Identifying and using critical points from graphs to formulate equations (contd)

3.



Activity 7: Identifying and using critical points from graphs to formulate equations (continued)

4.



i. For each of the graphs above identify their respective critical points and use these critical points to formulate their equations.

ii. How do you know whether the equation is correct or not?

Activity 8: Using definition-based procedures (DBPs) in the translation process

Given $f(x) = \frac{1}{x-4} + 2$

7.1 Calculate the coordinates of the x and y intercepts of f. (4)

7.2 Determine the equations of the asymptotes of $f(x)$. (2)

7.3 Sketch the graph of $f(x)$ showing all the critical points. (4)

[10]

Activity 9: Using contexts to understand the definition and the purpose of inverses

On the market day Nonjabulo and friends are selling ice cream and yoghurt. Towards the end of the day they put their yoghurt on sale. They reduced the price of the yoghurt to R1,80 from R2,40. Use the new price for all the calculations that follow.

a. Copy and complete the table below:

Number of yoghurt cans	2	3	4	8	10
Price in rands (R)					

b. Determine the formula which they were using to find the price of any number of yoghurt cans.

c. There were 6 cans in each tray. How much will it cost to buy 3 trays? Is the cost of 18 cans three times the price of one tray?

d. You will notice that with each number of yoghurt cans there is an associated price. We can write these numbers as an ordered pair (number of yoghurt cans; price). Use the values in the table and write the values as a set of ordered pairs.

e. In the above relationship, identify the dependent variable and the independent variable and give reasons for you answer.

g. Thus, for every number of yoghurt cans bought, there is an associated price. This association resembles a function in which the number of yoghurt cans forms the domain while the price forms the range. Using this understanding, explain in your own words the meaning of the terms domain and range

Activity 9: Using contexts to understand the definition and the purpose of inverses (continued)

2. Some customers who were coming to Nonjabulo and friends' table were saying that they want yoghurt for a specified amount of money e.g for R20 . This implies that at times Nonjabulo had to find out the number of yoghurt cans that can be bought with a specified amount of money.

a. Complete the following table:

Price in rands (R)	3, 60	5, 40	7, 20	14, 40	18, 00
Number of yoghurt cans					

b. Determine the formula which Nonjabulo and friends will use to find out the number of yoghurt cans that can be bought by any given amount of money. How do you it?

c. Would you consider this new relationship to be a function?

d. What is the domain and range of this new relationship?

e. In this new relationship, which is the dependent variable and which is the independent variable? Give reasons to support your answer.

f. Use the above table to write down the ordered pairs for this relationship and compare them with the ordered pairs in 1d. What do you notice? Explain in your own words.

Activity 10

i. Show why the inverse of $f(x) = x^2$ is not a function.

ii. Draw the graph of $f(x) = x^2$ and its inverse.

Activity 11 (Oral): Reciprocals as additive and multiplicative inverses

Complete the following table

Function Rule	Inverse Rule
$x + 3$	$x - 3$
$\frac{x}{3}$	
	$4x$
	$\frac{x-1}{4}$
$\sqrt{x}$	

Activity 12: Algebraically finding the inverse of a given function

Find the inverses of the following functions

1. a) $f(x) = x + 3$ b) $f(x) = x - 3$ 2. a) $f(x) = 2x$ b) $f(x) = \frac{x}{2}$

3. a) $f(x) = \frac{3x+2}{x-1}$ b) $f(x) = \frac{x+2}{x-3}$

What did you notice about these inverses?

Activity 13: Understanding what it means to have an inverse graphically

Sketch the graphs of the following functions on the same axes:

1. $f(x) = x + 3$ and $f(x) = x - 3$

2. $f(x) = 2x$ *and* $f(x) = \frac{x}{2}$ **3**. $f(x) = \frac{3x+2}{x-1}$ *and* $f(x) = \frac{x+2}{x-3}$

Appendix 7: Ethical clearance certificate

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

RESEARCH ETHICS COMMITTEE

CLEARANCE CERTIFICATE	CLEARANCE NUMBER : SM 11/05/02
DEGREE AND PROJECT	PhD Design research towards improving understanding of functions: a South African case study
INVESTIGATOR(S)	Tinoda Chimhande
DEPARTMENT	Science, Mathematics and Technology Education
DATE CONSIDERED	21 August 2013
DECISION OF THE COMMITTEE	APPROVED

Please note:
For Masters applications, ethical clearance is valid for 2 years
For PhD applications, ethical clearance is valid for 3 years.

CHAIRPERSON OF ETHICS COMMITTEE — Prof Liesel Ebersöhn

DATE — 21 August 2013

CC — Jeannie Beukes
Liesel Ebersöhn
Prof A Naidoo
Dr GH Stols

This ethical clearance certificate is issued subject to the following conditions:

1. A signed personal declaration of responsibility
2. If the research question changes significantly so as to alter the nature of the study, a new application for ethical clearance must be submitted
3. It remains the students' responsibility to ensure that all the necessary forms for informed consent are kept for future queries.

Please quote the clearance number in all enquiries.

Appendix 8: Permission letters

Principal's letter

TO: THE PRINCIPAL
XXX HIGH SCHOOL
PRIVATE BAG X4036
KWALUGEDLANE
1341

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT XXX HIGH SCHOOL

Dear Sir/ Madam

My name is Tinoda Chimhande and I am presently studying for my PhD degree in Mathematics Education through the University of Pretoria, and as such I am required to carry out research to write up a thesis. The title of my research is, "**Design research towards improving understanding of functions: a South African case study**". The aim of this study is to use learners' cognition when learning the function concept to design and develop instructional material to improve learners' understanding of the function concept. The second aim is to reduce the difficulties that characterise the teaching and learning of functions at high school.

I hereby request permission to carry out my research at XXX high school. The study will be carried out when the school is in session but all research activities will be done outside school hours and during weekends. Research activities will be done within and around the school environment but with the knowledge and permission of the H.O.D and class teachers. Initially learners in the sample will write the 2011 June Mathematics examination paper and I will analyse their solution strategies and then interview them on how and why they answered the questions in the way they will do. This will assist me to understand learners' cognitive processes when solving functional problems which will enable me to design and develop instructional material to observe where the students are in their thinking and assist them accordingly.

Criteria for participation include: willingness to participate voluntarily, ability to express oneself clearly and being grade 11 at that particular high school. Since participation is purely voluntary, participants are at liberty to withdraw from the study at any time if they so wish without being penalized. Participants' anonymity and confidentiality throughout the project, as well as in the reporting of the findings, is assured by the use of pseudonyms. I will not name the school at any stage. All the information gathered from a learner will be used solely for research purposes. For further information you can contact my supervisor (Professor Ana Naidoo at 0124205686 or ana.naidoo@up.ac.za). I trust that my request is acceptable.

Yours in Education

Tinoda Chimhande (Student Number: 10573462)

....

The Director's letter

TO: THE DIRECTOR
MPUMALANGA DEPARTMENT OF EDUCATION
PRIVATE BAG X 1014
KANYAMAZANE 1214

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT XXX HIGH SCHOOL IN YYY DISTRICT

Dear Sir/Madam

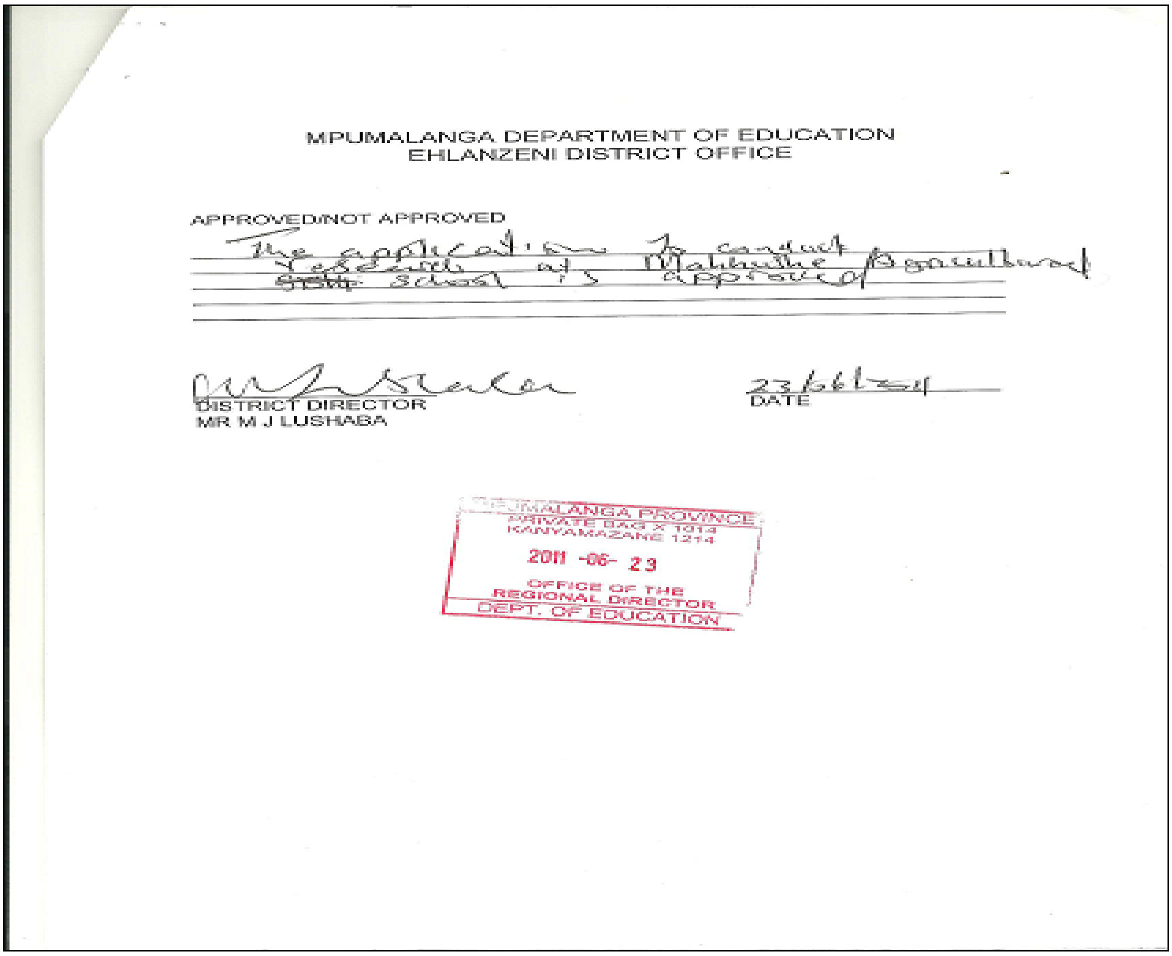
My name is Tinoda Chimhande and I am presently studying for my PhD degree in Mathematics Education through the University of Pretoria, and as such I am required to carry out research to write up a thesis. The title of my research is, "**Design research towards improving understanding of functions: a South African case study**". The aim of this study is to use learners' cognition when learning the function concept to design and develop instructional material to improve learners' understanding of the function concept. The second aim is to reduce the difficulties that characterise the teaching and learning of functions at high school.

I hereby request permission to carry out my research at XXX high school in YYY district. The study will be carried out when the school is in session but all research activities will be done outside school hours and during weekends. Research activities will be done within and around the school environment but with the knowledge and permission of the principal, H.O.D and class teachers. Initially learners in the sample will write the 2011 June Mathematics examination paper and I will analyse their solution strategies and then interview them on how and why they answered the questions in the way they will do. This will assist me to understand learners' cognitive processes when solving functional problems which will enable me to design and develop instructional material to observe where the students are in their thinking and assist them accordingly.

Criteria for participation include: willingness to participate voluntarily, ability to express oneself clearly and being grade 11 at that particular high school. Since participation is purely voluntary, participants are at liberty to withdraw from the study at any time if they so wish without being penalized. Participants' anonymity and confidentiality throughout the project, as well as in the reporting of the findings, is assured by the use of pseudonyms. I will not name the school at any stage. All the information gathered from a learner will be used solely for research purposes. For further information you can contact my supervisor (Professor Ana Naidoo at 0124205686 or ana.naidoo@up.ac.za). I trust that my request is acceptable.

Yours in Education

Tinoda Chimhande (Student Number: 10573462)



MPUMALANGA DEPARTMENT OF EDUCATION
EHLANZENI DISTRICT OFFICE

APPROVED/NOT APPROVED

The application to conduct research at Mahlhuthe Agricultural ~~High~~ School is approved

DISTRICT DIRECTOR
MR M J LUSHABA

23/06/2011
DATE

MPUMALANGA PROVINCE
PRIVATE BAG X 1014
KANYAMAZANE 1214
2011 -06- 23
OFFICE OF THE
REGIONAL DIRECTOR
DEPT. OF EDUCATION

Informed Assent Letter

Dear Student

My name is Tinoda Chimhande and I am presently studying for my PhD degree in Mathematics Education through the University of Pretoria, and as such I am required to carry out research to write up a thesis. The title of my research is, "**Design research towards improving understanding of functions: a South African case study**". The aim of this study is to use learners' cognition when learning the function concept to design and develop instructional material to improve learners' understanding of the function concept. The second aim is to reduce the difficulties that characterise the teaching and learning of functions at high school.

I hereby invite you to participate in the research. The study will be carried out when the school is in session but all research activities will be done outside school hours and during weekends. I will make transport arrangements and refreshments for you during weekend sessions. Research activities will be done within and around the school environment but with the knowledge and permission of the principal, H.O.D and class teachers. Initially you will write the 2011 June Mathematics examination paper and I will analyse your solution strategies on the questions on the function concept and then interview you on how and why you answered the questions in the way you will do. This will assist me to understand your thinking processes when solving functional problems which will enable me to design and develop instructional material to observe where you are in your thinking and assist you accordingly.

Participation in this study is voluntary, you have the right to withdraw or discontinue your participation at any point during the course of the study with no negative consequences. Participants' anonymity and confidentiality throughout the project, as well as in the reporting of the findings, is assured by the use of pseudonyms. All the information gathered from you will be used solely for research purposes.

The interviews will be audio taped. Access to the tapes will be restricted to the researcher. The tape will be stored in a secure area (e.g., locked filing cabinet) and the tapes will be destroyed one year after the completion of the study. The tapes will be transcribed, and your words may be quoted. If so, a pseudonym will be used to ensure that you cannot be identified in any way.

If you agree to participate in this study, please sign and return the attached consent slip to your teacher as soon as possible. If you have any questions about any aspect of the study, please do not hesitate to contact me for further information or clarification at 0794874936.

Yours sincerely

Tinoda Chimhande

Student Consent Slip

I --------------------------------------- agree to participate in the research and I understand that this participation is entirely voluntary; I can withdraw consent at any time without penalty and have the results of this participation (up to the date of withdrawing), to the extent that it can be identified as mine, returned to me, removed from the research records, or destroyed. The results of this participation will be confidential, and will not be released in any individually identifiable form without the prior consent of myself and my parent/guardian, unless otherwise required by law. The interviews will be audio taped. The tapes will be transcribed, and my words may be quoted. If so, a pseudonym will be used to ensure that I cannot be identified in any way.

Signature of Researcher	Signature of Student
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Date: ---------------------	Date: -----------------

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Parental Permission Letter

Dear Parent

My name is Tinoda Chimhande and I am presently studying for my PhD degree in Mathematics Education through the University of Pretoria, and as such I am required to carry out research to write up a thesis. The title of my research is, "**Design research towards**

improving understanding of functions: a South African case study". The aim of this study is to use learners' cognition when learning the function concept to design and develop instructional sequences and activities to improve learners' understanding of the function concept. The second aim is to reduce the difficulties that characterise the teaching and learning of functions at high school.

I hereby request your permission to allow your child to participate in the research. The study will be carried out when the school is in session but all research activities will be done outside school hours and during weekends. I will make transport arrangements and refreshments for the students during weekend sessions. Research activities will be done within and around the school environment but with the knowledge and permission of the principal, H.O.D and class teachers. Initially learners in the sample will write the 2011 June Mathematics examination paper and I will analyse their solution strategies and then interview them on how and why they answered the questions in the way they will do. This will assist me to understand learners' cognitive processes when solving functional problems which will enable me to design and develop instructional material to observe where the students are in their thinking and assist them accordingly.

Participation in this study is voluntary, you have the right to withdraw your consent at any time without consequences, and your child can discontinue his or her participation at any point during the course of the study with no negative consequences. Your permission in no way obligates your child to participate in the study if s/he is unwilling. Participants' anonymity and confidentiality throughout the project, as well as in the reporting of the findings, is assured by the use of pseudonyms. All the information gathered from your child will be used solely for research purposes.

The interviews will be audio taped. Access to the tapes will be restricted to the researcher. The tape will be stored in a secure area (e.g., locked filing cabinet) and the tapes will be destroyed one year after the completion of the study. The tapes will be transcribed, and the words of your child may be quoted. If so, a pseudonym will be used to ensure that your child cannot be identified in any way.

If you agree to allow your child to participate in this study, please sign and have your child return the attached permission slip to his or her teacher as soon as possible. If you have any questions about any aspect of the study, please do not hesitate to contact me for further information or clarification at 0794874936. I would be happy to talk with you! You may also contact my supervisor Professor Ana Naidoo, ana.naidoo@up.ac.za), 0124205686. Thank you for your time and consideration.

Yours sincerely

Tinoda Chimhande

Parental Permission slip

I give my consent for my child ____________________________ to participate in the research and I understand that this participation is entirely voluntary; I or my child can withdraw consent at any time without penalty and have the results of the participation, to the extent that it can be identified as my child's, returned to me, removed from the research records, or destroyed. The results of this participation will be confidential, and will not be released in any individually identifiable form without the prior consent of myself and my child, unless otherwise required by law.

Name of Student: ---

Printed Name of Parent/Guardian: ---

Signature of Parent /Guardian: ---

Date: -------------------------------------

Signature of Researcher: --

Date: -----------------------