

## Natural convection/isothermal viscous incompressible flows by the velocity-vorticity formulation

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### ABSTRACT

2D natural convection/isothermal viscous incompressible fluid flows are presented from the unsteady Boussinesq approximation and the Navier-Stokes equations in its velocity-vorticity formulation. These flows are obtained with a simple numerical method based on a fixed point iterative process to solve the nonlinear elliptic system that results once a time discretization is performed.

### Introduction

2D natural convection/isothermal viscous incompressible flows from the unsteady Boussinesq approximation and the Navier-Stokes equations in its velocity-vorticity formulation are presented; they are obtained with a numerical procedure based on a fixed point iterative process to solve the nonlinear elliptic system that results once a convenient second order time discretization is made. The iterative process leads to the solution of uncoupled, well-conditioned, symmetric linear elliptic problems for which efficient solvers exist either by finite differences or finite elements as far as rectangular domains are considered. On the lid driven cavity problem thermal flows up to Rayleigh numbers  $Ra = 10^5$  and isothermal ones up to Reynolds numbers  $Re = 1000$  are presented; both cases with aspect ratio (ratio of the height to the width)  $A = 1$  and 2.

We would like to point out that with the velocity-vorticity formulation is more difficult to

solve these flows, at least with a numerical procedure similar to the one applied in the stream function-vorticity formulation using a fixed point iterative process to solve an analogous nonlinear elliptic system, [1]. Moreover, works dealing with the velocity-vorticity formulation are scarce, even more for thermal flows. [2] uses a control-volume finite difference approach to discretize the problem and then a direct solution procedure to solve the algebraic system via a block tridiagonal matrix algorithm; they present isothermal results for the driven cavity problem, and heat transfer results for natural and mixed convection. [3] presents 3D isothermal results for the driven cavity problem up to  $Re \leq 2000$  using finite differences combined with an ADI procedure for the parabolic velocity Poisson equations and the continuity equation to solve the resultant algebraic system by a diagonally dominant tridiagonal matrix algorithm. [4] through a Parallel multi-block method reports isothermal results also for 2D and 3D for the driven cavity

problem.

### Continuous problem and numerical method

Let  $\Omega \subset R^N (N = 2, 3)$  be the region of the flow of an unsteady viscous incompressible thermal fluid, and  $\Gamma$  its boundary. This kind of flows is governed by the non-dimensional system, in  $\Omega \times (0, T)$ ,  $T > 0$ ,

$$\begin{cases} \mathbf{u}_t - \frac{1}{Re} \nabla^2 \mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{f} & (a) \\ \nabla \cdot \mathbf{u} = 0 & (b) \\ \theta_t - \frac{1}{RePr} \nabla^2 \theta + \mathbf{u} \cdot \nabla \theta = 0, & (c) \end{cases} \quad (1)$$

known as the Boussinesq approximation in primitive variables if  $\mathbf{f} = \frac{Ra}{PrRe^2} \theta \mathbf{e}$ , where  $\mathbf{u}$ ,  $p$ , and  $\theta$  are the velocity, pressure, and temperature of the flow, and  $\mathbf{e}$  is the unitary vector in the gravitational direction. The dimensionless parameters  $Re$ ,  $Ra$  and  $Pr$  are the Reynolds, Rayleigh and Prandtl numbers, given respectively by  $Re = UL/\nu$ , with  $\nu = \frac{UL}{Re}$  = kinematic viscosity,  $Ra = \frac{\beta l^3 \kappa g \rho_0^2}{\mu^3 c_p} (T_l - T_0)$ ,  $Pr = \kappa / \mu c_p$ , involving thermal coefficients, characteristic quantities, and the gravitational constant  $g$ . If the flow does not depend on the temperature the coupling with (1c) is eliminated and  $\mathbf{f}$  does not depend on  $\theta$ , then (1a–b) give the Navier-Stokes equations for isothermal flows.

The system must be supplemented with initial and boundary conditions. For instance  $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0$  and  $\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x})$  in  $\Omega$ ; and  $\mathbf{u} = \mathbf{f}$  and  $B\theta = 0$  on  $\Gamma$ ,  $t \geq 0$ , where  $B$  is a temperature boundary operator which can involve Dirichlet, Neumann or mixed boundary conditions.

Taking the curl in both sides of equation (1a), one obtains the non-dimensional form of the vorticity  $\boldsymbol{\omega}$  transport equation in  $\Omega \times (0, T)$

$$\boldsymbol{\omega}_t - \frac{1}{Re} \nabla^2 \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \mathbf{f}, \quad (2)$$

where the new  $\mathbf{f}$  is the curl of the old one and the vorticity vector  $\boldsymbol{\omega}$  is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \quad (3)$$

Taking the curl in (3), from the identity  $\nabla \times \nabla \times \mathbf{a} = -\nabla^2 \mathbf{a} + \nabla(\nabla \cdot \mathbf{a})$  and using (1b), the following velocity Poisson equation is obtained

$$\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega} \quad (4)$$

Hence, equations (2), with the corresponding  $\mathbf{f}$  depending on  $\theta$  as stated before, and (4) coupled to (1c) give the Boussinesq approximation

in velocity-vorticity formulation. It can be easily verified that the vorticity, scalar,  $\omega$  transport equation in  $\Omega \times (0, T)$ ,  $\Omega \subset R^2$ , is given by

$$\omega_t - \frac{1}{Re} \nabla^2 \omega + \mathbf{u} \cdot \nabla \omega = \frac{Ra}{PrRe^2} \frac{\partial \theta}{\partial x} \quad (5)$$

where, from the 2D restriction in (3),

$$\boldsymbol{\omega} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \quad (6)$$

and, from (4), the two Poisson equations for the velocity components are expressed as

$$\begin{cases} \nabla^2 u_1 = -\frac{\partial \omega}{\partial y} & (a) \\ \nabla^2 u_2 = \frac{\partial \omega}{\partial x} & (b) \end{cases} \quad (7)$$

Then, the vector Boussinesq approximation system (2) and (4) coupled to (1c) is reduced to a scalar system of four equations in 2D: one given by (5) and two by (7), coupled to (1c); (5) and (7) are related through (6) from which the boundary condition for  $\omega$  in (5) should be obtained from that of  $\mathbf{u} = (u_1, u_2)$ . A description of the numerical method follows.

For the time derivatives appearing in the vorticity equation (5) and temperature equation (1c) the following well known second-order approximation is used

$$f_t(\mathbf{x}, (n+1)\Delta t) = \frac{3f^{n+1} - 4f^n + f^{n-1}}{2\Delta t}, \quad (8)$$

where  $\mathbf{x} \in \Omega$ ,  $n \geq 1$ ,  $\Delta t$  denotes the time step, and  $f^r \equiv f(\mathbf{x}, r\Delta t)$ , assuming  $f$  is smooth enough.

Then, from (5), (7), and (1c), a fully implicit time-discretization gives rise to a nonlinear system of elliptic equations, at each time level  $(n+1)\Delta t$ , of the following form, in  $\Omega$ ,

$$\begin{cases} \nabla^2 u_1 = -\frac{\partial \omega}{\partial y} \\ \nabla^2 u_2 = \frac{\partial \omega}{\partial x}, \\ \mathbf{u} = \mathbf{u}_{bc} \text{ on } \Gamma, \\ \alpha \omega - \nu \nabla^2 \omega + \mathbf{u} \cdot \nabla \omega = \frac{Ra}{PrRe^2} \frac{\partial \theta}{\partial x} + f_\omega, \\ \omega = \omega_{bc} \text{ on } \Gamma, \\ \alpha \theta - \gamma \Delta \theta + \mathbf{u} \cdot \nabla \theta = f_\theta, \\ B\theta|_\Gamma = 0. \end{cases} \quad (9)$$

where  $\alpha = \frac{3}{2\Delta t}$ ,  $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$ ,  $f_\theta = \frac{4\theta^n - \theta^{n-1}}{2\Delta t}$ ,  $\frac{1}{Re}$  has been replaced by the kinematic viscosity  $\nu$  considering  $U = L = 1$ , and  $\gamma = \frac{1}{PrRe^2}$ ;  $\mathbf{u}_{bc}$  and  $\omega_{bc}$  denote the boundary condition for  $\mathbf{u}$  and  $\omega$ , and  $B$  is still the temperature boundary operator. This work involves isothermal flows, governed by the Navier-Stokes equations and natural convection flows under the Boussinesq approximation; then, for the latter case,  $Re = 1 = \nu$ ,

[5]. Since system (9) is of non-potential type, a fixed point iterative process may be used, which can be seen either as an adaptation to natural convection of the one for mixed convection in the stream function-vorticity formulation, [1], or as an extension to natural convection of the one for isothermal flows in velocity-vorticity formulation, [6].

If we denote

$$R_\omega(\omega, \mathbf{u}) \equiv \alpha\omega - \Delta\omega + \mathbf{u} \cdot \nabla\omega - \frac{Ra}{Pr} \frac{\partial\theta}{\partial x} - f_\omega \quad \text{in } \Omega$$

$$R_\theta(\theta, \mathbf{u}) \equiv \alpha\theta - \gamma\Delta\theta + \mathbf{u} \cdot \nabla\theta - f_\theta \quad \text{in } \Omega,$$

system (9) is equivalent to, in  $\Omega$ ,

$$\begin{cases} \nabla^2 u_1 & = -\frac{\partial\omega}{\partial y} \\ \nabla^2 u_2 & = \frac{\partial\omega}{\partial x}, & \mathbf{u}|_\Gamma = \mathbf{u}_{bc} \\ R_\theta(\theta, \mathbf{u}) & = 0, & B\theta|_\Gamma = 0 \\ R_\omega(\omega, \mathbf{u}, \theta) & = 0, & \omega|_\Gamma = \omega_{bc}. \end{cases} \quad (10)$$

Then, (10) is solved by the fixed point iterative process:

With  $\omega^0 = \omega^n$  and  $\theta^0 = \theta^n$  given, solve until convergence on  $\theta$  and  $\omega$ , in  $\Omega$ ,

$$\begin{cases} \nabla^2 u_1^{m+1} = -\frac{\partial\omega^m}{\partial y} \\ \nabla^2 u_2^{m+1} = \frac{\partial\omega^m}{\partial x}, \\ \mathbf{u}^{m+1}|_\Gamma = \mathbf{u}_{bc} \\ \theta^{m+1} = \theta^m - \\ \rho_\theta(\alpha I - \gamma\Delta)^{-1} R_\theta(\theta^m, \mathbf{u}^{m+1}); \\ B\theta^{m+1}|_\Gamma = 0, \quad \rho_\theta > 0, \\ \omega^{m+1} = \omega^m - \\ \rho_\omega(\alpha I - \Delta)^{-1} R_\omega(\omega^m, \mathbf{u}^{m+1}, \theta^{m+1}), \\ \omega^{m+1}|_\Gamma = \omega_{bc}^m, \quad \rho_\omega > 0, \end{cases} \quad (11)$$

then take  $(u_1^{n+1}, u_2^{n+1}, \theta^{n+1}, \omega^{n+1}) = (u_1^{m+1}, u_2^{m+1}, \theta^{m+1}, \omega^{m+1})$ .

Therefore, once the third and fourth equalities for  $\theta^{m+1}$  and  $\omega^{m+1}$  in (11) are multiplied by the operators  $\alpha I - \gamma\Delta$  and  $\alpha I - \Delta$  respectively, it turns out that at each iteration *four* uncoupled, symmetric linear elliptic problems, two associated with these operators and two with  $\nabla^2$ , have to be solved.

For the space discretization of elliptic problems either finite differences or finite elements may be used, as far as rectangular domains are concerned; in either case efficient solvers exist.

For the finite element case, variational formulations have to be chosen and then restrict them to finite dimensional spaces, like those in [5] and [7]. For the results in the next Section, the second order approximation of the Fishpack solver in rectangular domains, [8], is used. Then, such second order approximation in space combined with the second order approximation in (8) for the first derivatives in time, the approximation with second order central differences at the interior points, and with (8) on the boundary, for all the first derivatives of including those that appear in the local Nusselt number  $Nu(x)$  and de second order trapezoidal rule to calculate the global Nusselt number  $\overline{Nu}$ ,  $Nu(x)$  and  $\overline{Nu}$  defined for instance in [9], imply that the discrete problem relies on second order discretizations.

Like for the isothermal case, [6], contrary to what it was thought not all the results can be obtained with second order discretizations, a fourth order one is required for some of them. The fourth order discretization is accomplished with the fourth order option of Fishpack to approximate elliptic problems and with the one in [10] to approximate the first derivatives.

## Numerical experiments

The numerical experiments take place in rectangular cavities, then  $\Omega = (0, a) \times (0, b)$  with  $a, b > 0$ . By viscosity and since for natural convection all the walls of the cavity are solid and fixed, the boundary condition for  $\mathbf{u}$  is  $\mathbf{0}$  everywhere on  $\Gamma$  whereas that for  $\theta$  is given by

$$\begin{cases} \theta = 1 \text{ on } \Gamma|_{x=0}, \quad \theta = 0 \text{ on } \Gamma|_{x=a}; \\ \frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma|_{y=0,b}, \end{cases}$$

hence the horizontal walls are insulated and the left vertical wall is the hot one. The Prandtl number is given by  $Pr = 0.72$  because it is assumed that the cavity is filled with air. For isothermal flows, the well known driven cavity problem is considered, then  $\mathbf{u}(x, b) = (1, 0)$  on the moving wall  $y = b$  and  $\mathbf{0}$  elsewhere. The initial conditions for velocity, vorticity, and temperature are given by  $\mathbf{u}(\mathbf{x}, 0) = (0, 0)$ ,  $\omega(\mathbf{x}, 0) = 0$ , and  $\theta(\mathbf{x}, 0) = 0$  in  $\Omega$ . Results that converge to the asymptotic steady state are reported at moderate Rayleigh and Reynolds numbers. Concerning the iterative process,  $\rho_\theta = \rho_\omega = 0.7$  and the convergence is achieved with tolerance of  $10^{-5}$ . The mesh sizes are denoted by  $h_x$  and  $h_y$ , and the time step by  $\Delta t$ ; they will be specified in each case under study. The results are reported through the streamlines (left) and

isotherms (right) for natural convection flows and through the streamlines (left) and the isovorticity contours (right) for isothermal flows; moreover, the results reported here were obtained with 4<sup>th</sup> order approximations.

For natural convection flows  $Ra = 10^4$  and  $Ra = 10^5$  are considered whereas  $Re = 400$  and  $1000$  for isothermal ones; for both, the effect on the aspect ratio  $A$  is analyzed. Figures 1 and 2 picture the natural convection flows for  $Ra = 10^5$  with  $A = 1$  (square cavity) and  $2$  respectively; the respective meshes are  $(h_x, h_y) = (1/64, 1/64)$  and  $(h_x, h_y) = (1/64, 2/128)$ , and the time steps  $\Delta t = 0.0001$  and  $\Delta t = 0.00001$ . Table 1 shows the activity of the flow through the minimum and maximum of the stream function,  $\psi_{min}$  and  $\psi_{max}$  ( $\psi_{max} = 0$ ), and the corresponding global Nusselt number  $\overline{Nu}$ , denoted with the subscript  $v - w$ , compared with the ones of the stream function-vorticity formulation, denoted  $sf - w$ ; even though they show some noticeable discrepancy the agreement with the physics is good: the flow is more active if  $A$  increases ( $|\psi_{min}|$  is bigger) and the heat transfer is lower. The case for  $Ra = 10^4$  does not give any trouble at all; however, to reinforce the validation something will be pointed out next about it with results for  $A = 1$  and  $2$  obtained with meshes  $(h_x, h_y) = (1/64, 1/64)$  and  $(h_x, h_y) = (1/64, 2/128)$ , and time steps  $\Delta t = 0.0001$  and  $\Delta t = 0.00001$ . On this regard, Tables 2 and 3 show results for  $Ra = 10^5$  and for  $Ra = 10^4$  respectively to compare with [14] whose results are obtained through a false transient. The comparison is made with  $|\psi_{mid}| = |\psi_{value}|$  at the center of the cavity and with the global Nusselt number  $\overline{Nu}$ . It can be observed that they are close for  $A = 1$ , in [14] results for  $A = 2$  are not reported which is indicated on the Tables by \*'s, despite the fact the methods are different. For  $Ra = 10^5$ , as has already been mentioned, the heat transfer is lower if  $A$  increases, which agrees also with the stream function formulation, Table 1, as it happens for higher  $Ra$ ; however, as Table 3 shows, for  $Ra = 10^4$  this situation is the opposite.

Finally, Figures 3 and 4 show the isothermal flows for  $Re = 400$  with  $A = 1$  and  $2$  respectively, the latter rotated  $-90^\circ$ ; the respective meshes are  $(h_x, h_y) = (1/80, 1/80)$  and  $(h_x, h_y) = (1/120, 2/240)$ , and the time steps  $\Delta t = 0.01$  and  $\Delta t = 0.001$ ; the one with  $A = 1$  agrees with that in [6] with the contour values given by [11] while here it is reported with the contour values of [12], the one with  $A = 2$  is

compared with that obtained from a scheme of the stream function-vorticity formulation, which has been validated with higher Reynolds numbers, [1]. Table 4 shows the activity of the flow through  $\psi_{min}$  and  $\psi_{max}$  compared with the ones of the stream function-vorticity formulation; even though they show some noticeable discrepancy the agreement with the physics is also good: the behavior of the flow is almost the same if  $A$  or  $Re$  increase. Figure 5 pictures the local Nusselt number  $Nu(x)$  for  $Ra = 10^5$  and  $A = 1$  and  $2$ . It shows that the heat transfer is greater as the aspect ratio increases; the maximum value of  $Nu(x)$  occurs close to  $y=0$ , which agrees with other formulations.

## Conclusions

We have reported numerical results of 2D natural convection/isothermal flows of viscous incompressible fluids from the unsteady Boussinesq approximation/Navier-Stokes equations in the velocity-vorticity formulation. The numerical procedure shows to be good to capturing asymptotic steady states for moderate Rayleigh and Reynolds numbers with aspect ratios  $A \geq 1$ . Despite the numerical procedure applied to this formulation is not as good as for the stream function-vorticity, [1], or the primitive variables, [13], formulation the way it behaves, through the discretization parameters (the meshes are coarser and the time step bigger) and through the discretization order (in general, fourth order is required), gives us another point of view of the behavior of the flows under different numerical methods and different formulations of the problem, showing, nevertheless, the agreement with the physics of the flows. Moreover, from the natural convection flows obtained here with the velocity-vorticity formulation further investigation can be done, for instance with a mesh size and time step independence studies, to find out if for  $Ra \leq 10^4$  the heat transfer is really bigger as  $A$  increases, and if that were the case which would the critical value  $Ra_c < 10^5$  be to get the opposite situation?

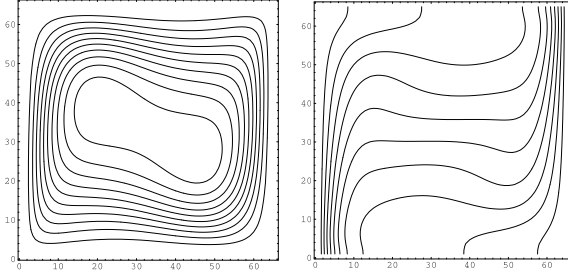


Figure 1:  $Ra=100000$ ,  $h_x=1/64$ ,  $h_y=1/64$

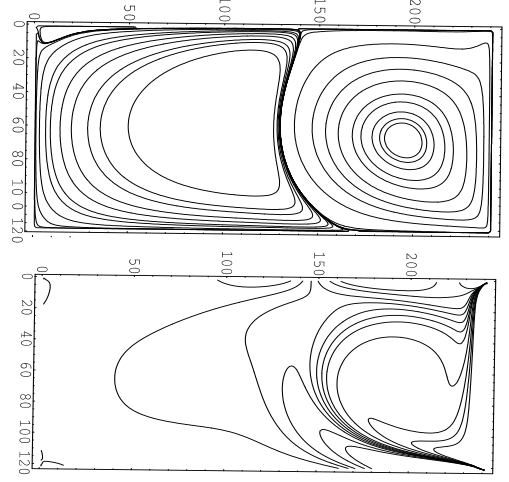


Figure 4:  $Re=400$ ,  $h_x=1/120$ ,  $h_y=2/240$

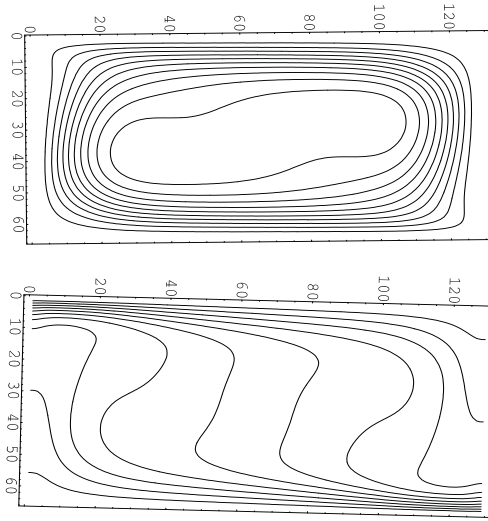


Figure 2:  $Ra=100000$ ,  $h_x=1/64$ ,  $h_y=2/128$

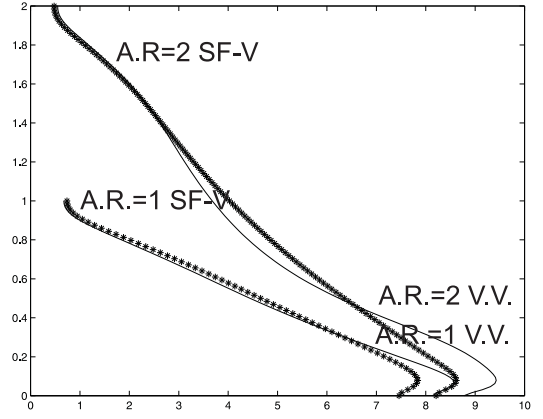


Figure 5: Local Nusselt numbers for the stream function vorticity (SF-V) and Velocity-Vorticity (VV) formulations with different aspect ratios

$Ra$	$A$	$\psi_{v-w}^{min}$	$\psi_{sf-w}^{min}$	$\overline{Nu}_{v-w}$	$\overline{Nu}_{sf-w}$
$10^5$	1	-15.655	-13.514	4.574	4.547
$10^5$	2	-31.414	-21.675	4.350	4.320

Table 1.  $\psi_{min}$  and global Nusselt number for  $Ra=10^5$  with different aspect ratios

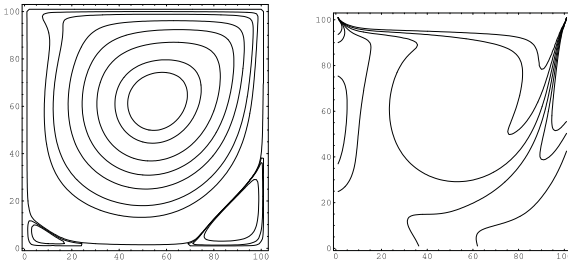


Figure 3:  $Re=400$ ,  $h_x=1/100$ ,  $h_y=1/100$

$A$	$ \psi ^{mid}$	$ \psi _{Vd}^{mid}$	$\overline{Nu}$	$\overline{Nu}_{Vd}$
1	15.111	11.97	4.577	4.454
2	31.414	***	4.350	***

Table 2.  $Ra = 10^5$ :  $\psi^{min}$  Vs  $\psi_{Vd}^{min}$ ,  $\overline{Nu}$  Vs  $\overline{Nu}_{Vd}$

$A$	$ \psi ^{mid}$	$ \psi _{V_d}^{mid}$	$\overline{Nu}$	$\overline{Nu}_{V_d}$
1	7.515	5.176	2.259	2.212
2	10.160	***	2.363	***

Table 3.  $Ra = 10^4$ :  $\psi^{min}$  Vs  $\psi_{V_d}^{min}$ ,  $\overline{Nu}$  Vs  $\overline{Nu}_{V_d}$

$Re$	$A$	$\psi_{v-w}^{min}$	$\psi_{s_f-w}^{min}$	$\psi_{v-w}^{max}$	$\psi_{s_f-w}^{max}$
400	1	-1.214	-1.113	.00081	.00061
400	2	-1.218	-1.116	.0103	.0088
1000	1	-1.117	-1.156	.0022	.0017
1000	2	-1.159	-1.164	.0217	.0130

Table 4.  $\psi_{min}$  and  $\psi_{max}$  for  $Re$  400 and 1000 with different aspect ratios

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