All is Number

Jan C. A. Boeyens and Demetrius C. Levendis

Abstract Rational numbers, which correctly describe many recognizable patterns in the physical world, are often seen to converge in the process to irrational limits or even singularities. As a common example, atomic numbers are well known as fundamental parameters in chemistry, but by demonstrating that the periodicity of atomic matter is simulated by the convergence of rational fractions, from unity to the golden ratio, the importance of limiting processes and irrational limits in the modelling of chemical systems and of phenomena such as superconduction is emphasized. Other limiting formulae feature in atomic spectral series, radioactive decay, circular measure, absolute temperature, the speed of light, structure of the solar system and gravitational collapse. In virtually all cases the convergence involves the irrational golden ratio and the golden spiral, the essential properties of which are briefly reviewed in summary of the arguments developed in this volume. The suspicion that molecular shape should have a related number basis could not be substantiated. Only in the double-helical base pairing of DNA could any correlation between molecular structure and number theory be demonstrated. It is tempting to conjecture that the ubiquitous appearance of irrational limits signals the inadequacy of the $\mathbb{R}^3$ number system to provide a detailed account of the four-dimensional world.

Keywords: discrete set, golden section, limiting law, periodicity, wave mechanics

Corresponding author: Jan C. A. Boeyens
Unit for Advanced Scholarship, University of Pretoria, South Africa.
e-mail: jan.boeyens@up.ac.za

Demetrius C. Levendis
Molecular Sciences Institute, School of Chemistry, University of the Witwatersrand, Johannesburg,
e-mail: demetrius.levendis@wits.ac.za

1 Introduction

The statement, which inspired the title of this paper, is traditionally credited to Pythagoras in recognition of the fundamental discovery that the pitch produced by the stretched string on a musical instrument is numerically related to the fractional length of its vibrating part. It was repeated many centuries later in somewhat different terms by Émile de Chancourtois, co-discoverer of elemental periodicity, who claimed [1] that

> ... the properties of the elements are the properties of numbers.

At some stage most scientists benefit by ″the unreasonable effectiveness″ of mathematical models in science, but too few look for the cause behind this happenstance. There is a school of thought that detects a parallel between the subjective choice of topics for analysis and the formulation of matching formulae. At the other extreme numerical systems are considered to have independent existence in the same way as their physical counterparts. It is, for instance, an undisputed fact that the sophistication of cosmological models faithfully follows developments in number theory. While the concept of infinity remains mathematically unresolved the cosmos remains infinite. However, whether number theories are invented or discovered does not decide their utility in science. If nobody understands the remarkable similarity between Farey sequences, Ford circles and the periodicity of matter, it is no excuse not to exploit this consilience to develop powerful new number-theoretic models for chemistry.

With this prospect in mind some unexpected parallels between numbers and material behaviour will be examined as simplified models of complicated natural phenomena.

2 Numbers and Nature

Most historians trace the rise of modern physics back to Newton and Leibniz who developed the mathematical tools to describe continuous systems and processes. For this reason the appearance of quantum mechanics, with its emphasis on discrete events, came as a major surprise, still not completely digested. The joke is that Newtonian mechanics concerns the continuous motion of discrete particles whereas quantum theory deals with discrete rearrangements of continuous wave structures. We see a need to differentiate between discrete and continuous systems and to enquire into the feasibility of what happens to be an uneasy compromise, epitomized by the notion of wave-particle duality.

The appearance of numbers in chemistry has an equally interesting history.

The modern view of matter developed from John Dalton's theory which, for the first time, combined the abstractions of atom and element into a single practically useful concept, albeit at the cost of unwanted diversity. Resolution of the dilemma, by postulating hydrogen as the common building block of more complex atoms, was

resisted so fiercely that the alleged author, William Prout, had to publish his proposal anonymously. The importance of number was at the heart of his hypothesis. Despite experimental evidence which contradicted the notion, Prout's hypothesis was not without support and remained alive until its final vindication in the discovery of isotopes and atomic number, which ironically also signalled the demise of atomic particle theory.

The next fundamantal advance in the theory of matter, like that of Prout, failed to gain recognition and the name of the author, Hantaro Nagaoka, is now all but forgotten. In line with the old tradition of cosmic symmetry between macrocosm and microcosm Nagaoka conjectured [2] that, like the planet Saturn with its rings, an atom consisted of a solid positively charged core surrounded by a striated electron cloud with regularly spaced local maxima at characteristic distances from the core. In order to avoid electrodynamic instability of the system the electron density was modelled as standing waves which introduced the set of integers known to characterize atomic spectra. Although never acknowledged by his imitators, Rutherford and Bohr, it is to his credit that he appreciated the defects of a particle model and developed his theory without reference to quantum mechanics.

It is remarkable how the work of William Harkins on the periodicity of matter, dating from the same era, also remained unnoticed and dormant for decades. Whereas Mendeleéff was acclaimed for postulating the properties of undiscovered elements on the basis of his periodic law, Harkins, who characterized seven unknown nuclides on the basis of a more fundamental periodic law, has been forgotten. He was the first to notice, before discovery of the neutron, that the ratio $(A-Z)/A$ for known nuclides never exceeds 0.62 [3], later identified more accurately as the golden ratio, $\tau = 0.61803\ldots$ [4].

The most serious setback for a modern theory of matter was the deliberate suppression of Erwin Schrödinger's demonstration that the behaviour of electrons in an atom cannot be described correctly by a particle model and quantum jumps [5]. A beautiful theory, based on a wave model of matter, was buried through professional rivalry to be replaced by incomprehensible concepts such as particles with wavelike properties — even Zitterbewegung, infinite self-energy, probability density, non-Boolean algebra of observables and other weird properties. Remember how Newton described particles as

solid, massy, hard, impenetrable, movable and indestructable.

Niels Bohr, one of Schrödinger's adversaries, despite his rejection of the wave model, referred on occasion [6] to

... one of the boldest dreams of natural science, namely to build up an understanding of the regularities of nature upon the consideration of pure numbers.

Only waves can generate these numbers.

2.1 Discrete Sets

The Pythagorean theory of music, for lack of a more convincing explanation, became firmly associated with celestial events and structures, to such an extent that initiates like Johannes Kepler could actually hear the "music of the spheres". Although this sound is less audible today, the original theory has survived in the form of the modern standing-wave model of harmonic vibrations. Many other number patterns of the same type have also survived into the modern era as the basis of erudite new theories.

2.1.1 The Bode-Titius Law

This ancient law, which relates the planetary orbits of the solar system to natural numbers, was recently shown [7] to apply to all natural satellite systems in the solar system, subject to a general optimization model which also governs the electronic structure of atoms and the hierarchy of bond orders that characterize covalent interactions in molecules. More than an entertaining exercise in numerology, this law, with the prediction of the asteroid belt to its credit, now emerges as a convincing manifestation of self-similarity on a cosmic scale. An amazing consequence of this symmetry in operation is that the *quantum* effects, hitherto postulated as the distinctive characteristic of microphysical systems, are predicted to be also observable on a cosmic scale. Although not generally endorsed, observations that support this prediction sporadically surface in the recent astronomical literature. It begs a rethink of quantum theory.

2.1.2 Atomic Number

The discoverers of unexpected number patterns are traditionally given a hard time by establishment critics. Nobody more so than William Prout who spotted the tendency of many atomic weights, known in the early 19th century, to be improbably close to whole numbers. He was ridiculed and maligned almost as badly as Alfred Wegener, who first mooted the idea of continental drift. Wegener never received the recognition that he deserved, arguably for overlooking the "true" basis of the phenomenon. The same happened to Prout and his handful of supporters. In order to account for exceptional atomic weights, like the 35.5 of chlorine, there are several accounts on record, proposing the possibility of such elements being mixtures of atoms that differ in mass, but not in chemical properties. William Harkins who actually implicated the existence of a *neutron*, many years before the discovery of such a particle, also failed to be taken seriously. What this history shows, is that the idea of number led the way for about a century. It ceased to be a joke only when imagination was no longer needed in order to appreciate the obvious.

This saga has still not ended. Once the penny had dropped and the many-headed finally accepted the notion of isotopes, their thinking stagnated again into a parti-

cle count to explain the appearance of integers. This is the brick wall that stopped Schrödinger when he proposed a wave model as the natural origin of quantum integers.

2.1.3 The Balmer Formula

The quantum theory of matter developed from an innovative suggestion by Johann Balmer to account for the optical spectrum of atomic hydrogen, observed as a series of emission lines at narrowly defined wavelengths. He related the measured wavelengths to a series of integers by the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right), n = 3,4,5\ldots,$$

that existed as an unexplained oddity outside of mainstream physics for many years, until a matching relationship with the extranuclear distribution of electrons on an atom was conjectured by Nagaoka and later incorporated into the Bohr formula, which linked it to the quantum theory of radiation. It is interesting to note that both Nagaoka and Bohr based their proposals on a perceived likeness between atoms and celestial structures.

The self-similarity between atoms and planetary systems goes even further. Whereas an atomic nucleus is surrounded by an electronic core and a valence shell, the solar system has an inner core of rocky planets with major planets beyond the asteroid belt. A major planet like Saturn has a core of rings and minor shepherding moons, surrounded by regularly spaced major moons. In the case of the solar system all planets have characteristic quantun numbers that converge from both ends to the golden ratio at the chaotic asteroid belt. The boundary between planetary rings and major moons is likewise defined by τ.

By reference to Fig. 2, which follows later on, a chaotic atomic core is readily identified as the regions labeled s, d and f. It is precisely for the regular p-block elements that the covalence formulae based on the golden ratio work best.

2.2 Irrational Limits

The Balmer formula defines an infinite series that converges to the limit in which $\lambda = 4/R$. On approaching this limit the spectral lines are bunched together more closely, such that the discrete line spectrum turns into a continuum. The spectrum becomes blurred, but the limit is accurately measurable.

On closer examination the atomic numbers of stable nuclides, consisting of Z protons and N neutrons, are also seen to converge to $Z = 102$ as the ratio of Z/N approaches a constant value, now recognized as the golden ratio, τ [12]. Remarkably, the same constant characterizes the logarithmic spiral that generates the Bode-Titius

law. It is therefore of interest to examine other well-known limiting processes which could reveal a more general common pattern.

2.2.1 Radioactivity

Radioactive decay is one example of a natural process, said to proceed *exponentially* as a function of time. In practice it is observed that the measured activity A at time t has diminished to half its initially measured value of A_0 after a characteristic time interval known as the half-life of the process. This result is valid for any A_0. In explanation the rate process of radioactive decay has been formulated as

$$A = A_0 e^{-\lambda t},$$

according to which

$$\frac{A_0}{2} = A_0 e^{-\lambda T_{\frac{1}{2}}}$$

for half-life $T_{\frac{1}{2}}$ and decay constant λ. In logarithmic notation the expression

$$2e^{-\lambda T_{\frac{1}{2}}} = 1$$

reduces to $\ln 2 = \lambda T_{\frac{1}{2}}$, which defines constant $T_{\frac{1}{2}}$.

This result demonstrates the important principle that, in theory, the decay process can continue for ever, albeit with diminishing activity. At any given time another period of $T_{\frac{1}{2}}$ is required to half the currently measured activity. Theoretically a nearly infinite number of half-lives expire before all activity ceases. The decay process may therefore be considered as approaching a final unreachable state ever more closely.

Mathematically the process is described precisely by the exponential function, which is based on e, the basis of natural logarithms. This constant is presented to the first 3441 decimal places on a coffee mug for sale in Germany. It also defines e by the formulae

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!} \quad , \quad e^{i\pi} = -1 \quad \text{and} \quad e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

None of these can ever specify the ultimate value of e and any refinement to improve the accuracy can be continued indefinitely, without the possibility of ever reaching finality.

The remarkable fact is that the exponential function had been known for centuries before the discovery of radioactivity and its formulation could therefore not have been inspired by the natural process. One may ask, what led Euler to this invention – or discovery?

2.2.2 Circular Measure

Another well-known transcendental number, called π, measures the ratio between the diameter and circumference of a circle in the Euclidean (or complex) plane. Like e it is associated with a well-defined, but unreachable limit – in this case the circumference of a regular polygon with infinitely many sides. For regular polygons with an even number of sides it is easy to demonstrate that the ratio between longest diameter and total circumference is calculated as $\pi_N = N \sin(180°/N)$. This value increases steadily with N from $\pi_4 = 2.83$, $\pi_6 = 3$, $\pi_8 = 3.06$, $\pi_{64} = 3.1403$ to $\pi_{1024} = 3.1415877$ and beyond. It approaches, but never reaches the value of π, which, like e, is an irrational number that keeps many computers occupied to specify its value with improving accuracy. A second coffee mug defines

$$\pi = 4 \times (1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{15} + \tfrac{1}{17} - \tfrac{1}{19} + \ldots),$$

which oscillates covergently around $\pi = 3.141592....$

There is an intriguing philosophical issue related to such a pointless search for infinity. The elusive absolute value of π, and perhaps of e as well, has a different value in curved space and a completely different meaning in curved space-time. The concept of a linear infinite measure has, for example, no meaning in topologically closed spaces. The limiting point that quantifies π and e only exists in hypothetical Euclidean space, which provides a convenient frame of reference for projection from the globally curved underlying space-time into three-dimensional local tangent space. Humans are conditioned to analyze all observations in familiar three-dimensional tangent space as a function of universal time.

2.2.3 Zero Temperature

Probably the most familiar example of an unreachable limit in physical science is the absolute zero of temperature. Rather than a mathematical impossibility, 0K implies the cessation of all motion, which renders it experimentally inaccessible. A logical explanation of the effect may well be another futile search for infinity. Energy, matter and motion can only occur in curved space-time. An eventless situation is, by definition, restricted to flat, infinite Euclidean space, which occurs nowhere in the real world. In this sense the third law of thermodynamics and phenomena such as superfluidity, that occurs near 0K, provide experimental evidence of space-time curvature.

2.2.4 The Speed of Light

The most famous limitation in physical science is the limiting speed c, imposed by the special theory of relativity and known as the speed of light. Again, this is a limitation imposed by the mathematical properties of Euclidean space on the per-

ception of unaccelerated relative speed as measured in different frames of reference. The inability to measure the relative speed of electromagnetic signals is perceived as distortion of space-time geometry in the form of spacial contraction, time dilation and increasing mass. All of these perceptions attest to the non-Euclidean nature of four-dimensional events. The familiar dynamical concepts of classical mechanics, as mere projections, have no meaning in four-dimensional space-time. Classical dynamics refers to the analysis of observations as a function of three-dimensional Euclidean space and one-dimensional universal time. The projection of four-dimensional phenomena into this tangent space creates the impression of magic at work.

2.2.5 The Black Hole

The mother of all irrational limits is the singularity at the centre of a black hole. It epitomizes the infinite gravitational field at $r = 0$, predicted by the inverse-square law of force, $F = -GMm/r^2$, and its effect on a run-away accumulation of matter. Despite many erudite theories it remains the foremost enigma of astronomy and the harshest test for the infinity concept. A proposed resolution of the problem [8] seeks an outlet for the infinitely accumulated matter through an Einstein-Rosen bridge across the interface that separates antipodal regions of the double cover of the projective space-time model. We consider the conjecture that the universe originated in an initial singularity to be completely over the top.

2.2.6 The Common Limit

Many limiting processes in Nature appear to be associated with the irrational numbers π, e and τ, each of which is defined geometrically in three-dimensional tangent space. The geometry of a circle defines π, a regular pentagon defines τ and e relates to the complex plane.

These definitions are no longer valid in 4D curved space-time. It has been known since 1603 by the theorem of Thomas Harriot [9] that for a spherical triangle Δ, with angles α, β and γ, the angular excess

$$\delta = \alpha + \beta + \gamma - \pi$$

is proportional to the area(Δ), and hence non-zero.

Alternatively, the radius of a great circle, constructed in the surface of a unit sphere, $r = 1$, define an elliptic $\pi_e = 2$. Without further proof it becomes obvious that, although the excess in curved space-time may not be measurable in local tangent space, it necessitates serious correction over astronomical distances. Extrapolation to the sub-atomic scale is equally hazardous, with the effect of 4D curvature no longer a negligible factor. In a sense, each irrational limit in tangent space amounts to a three-dimensional singularity, which disappears in curved space-time.

The Pythagorean cult disintegrated for a similar reason following the discovery of irrational numbers, a concept at variance with their dogma. If modern scientists had the same courage of their convictions singularities and infinities in physical theory would be anathema. Instead we disguise our ignorance as subtle theory.

Singularities and infinities are only unreachable on analysis in an inadequate number system. The irrational limits that emerge in three-dimensional space are regular points in four-dimensional space-time. By following natural numbers as they appear in three-dimensional theory of chemical systems invariably leads to irrational numbers, such as the golden ratio, which signals progression to four dimensions.

Resolution of the singularity problem in four dimensions can only be guessed at. A possible guide towards a mathematical elucidation is provided by the mapping of a spiral inside a golden rectangle. It amounts to inscribing semi-circles within the squares that divide the rectangle into smaller copies of itself. The process continues as an endless approach towards the unreachable focus of the spiral at the intersection of rectangle diameters. The same process unfolds in the opposite direction on addition of larger squares to the growing golden rectangles. Instead of the infinitely small, the spiral now extends to the infinitely large. In four-dimensional projective space the two points of convergence may be identified as a single point at infinity.

On this basis we conjecture that the golden spiral exists in 4D as two conjugate parts of a closed construct, which also models the natural number system[1]. The imaginary half represents extension of reality into a parallel antiworld. Singular points in gravity, temperature, the composition of atomic nuclei and the number system, are now identified as the 4D points of contact between conjugate worlds. There is no closer approach to 4D reality and no better device to model this reality. It is no longer surprising to find that chemical systems are amenable to golden spiral optimization and easier to understand why chemical interactions are accurately modelled by the resulting wave structure.

2.3 The Golden Section

The irrational number, known as the golden ratio, is said to be the most irrational of them all. Like other irrationals it also occurs as the limit of a regular series of rational fractions; in this case the Fibonacci fractions. In Nature it occurs as the convergence limit of the mass fractions of stable nuclides, $Z/(A - Z)$. As a clue to its physical meaning it is noted that the stability of nuclides depends on their space-time environment [4]. In regions where space-time curvature approaches infinity the mass ratio $Z/(A - Z) \to 1$. In the hypothetical situation of zero curvature matter does not exist. It is inferred that in an intermediate situation of curvature, condusive to the development of biological life, the mass ratio $Z/(A - Z) \to \tau$.

---

[1] The geometrical definition of both π and τ depends on space-time curvature and since both e and τ represent converging number sequences, the identity $e^{i\pi} = -1$ cannot hold universally, unless the number system also adapts

Some of the remarkable properties of the golden section, with relevance to chemistry, are summarized in the introductory paper of this volume. Perhaps most surprising is the close relationship between golden ratio and the Fibonacci series, which was first formulated to model the population growth in a rabbit colony.

Not only the golden ratio itself, but also its integral powers, τ^n, are embedded as convergent series in the Fibonacci sequence. Comparison of the first few of these reveals a striking pattern:

$$\tau : \tfrac{1}{1}\ \tfrac{1}{2}\ \tfrac{2}{3}\ \tfrac{3}{5}\ \tfrac{5}{8}\ \tfrac{8}{13}\ \tfrac{13}{21}\ \tfrac{21}{34} \to 0.6180\ldots\ \ 1$$

$$\tau^2 : \tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{2}{5}\ \tfrac{3}{8}\ \tfrac{5}{13}\ \tfrac{8}{21}\ \tfrac{13}{34}\ \tfrac{21}{55} \to 0.3819\ldots\ \ 1$$

$$\tau^3 : \tfrac{1}{3}\ \tfrac{1}{5}\ \tfrac{2}{8}\ \tfrac{3}{13}\ \tfrac{5}{21}\ \tfrac{8}{34}\ \tfrac{13}{55}\ \tfrac{21}{89} \to 0.2360\ldots\ \ 2$$

$$\tau^4 : \tfrac{1}{5}\ \tfrac{1}{8}\ \tfrac{2}{13}\ \tfrac{3}{21}\ \tfrac{5}{34}\ \tfrac{8}{55}\ \tfrac{13}{89}\ \tfrac{21}{144} \to 0.1458\ldots\ \ 3$$

$$\tau^5 : \tfrac{1}{8}\ \tfrac{1}{13}\ \tfrac{2}{21}\ \tfrac{3}{34}\ \tfrac{5}{55}\ \tfrac{8}{89}\ \tfrac{13}{144}\ \tfrac{21}{233} \to 0.0901\ldots\ \ 5$$

$$\tau^6 : \tfrac{1}{13}\ \tfrac{1}{21}\ \tfrac{2}{34}\ \tfrac{3}{55}\ \tfrac{5}{89}\ \tfrac{8}{144}\ \tfrac{13}{233}\ \tfrac{21}{377} \to 0.0557\ldots\ \ 8$$

$$\tau^n = \lim_{i\to\infty} \frac{F_i}{F_{i+n}}$$

The modularity of each sequence, defined by the matrix of successive fractions, for the series τ^n,

$$m = \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = \pm F_n,$$

the nth Fibonacci number.

The unique properties of the golden section show that the τ-sequence

$$F_\tau = \ldots, \tau^n, \tau^{n-1}, \ldots, \tau^0, \tau^{-1}, \tau^{-2}, \ldots, \tau^{-n} \ldots$$

is not only a double geometric sequence but also a Fibonacci sequence. The generalized Fibonacci series, $G^{8,2}$, also known as the Funk-Hellet sequence [10], with seed values $S_0 = 8$, $S_1 = 2$, reduces to $1000F_\tau$, rounded off, on aligning G_0 with τ^{12}:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_τ	τ^{11}	τ^{10}	.	.	.	.	.	.	.	.	.	τ^0	τ^{-1}	τ^{-2}
$1000F_\tau$	5	8	13	21	34	56	90	146	236	382	618	1000	1618	2618
$G^{8,2}$	2	10	12	22	34	56	-	-	-	-	-	-	-	-

Note the special properties of $G_{14} = 2 \times 7 \times 11 \times 17$, $12G_{14} = 10^4\pi$.

The construction of a golden spiral is also readily reduced to Fibonacci numbers. A dilative rotation of 2π transforms any point on the spiral into a homothetic point, which is similarly placed and directed. The spiral, said to be homothetic to itself, therefore has the property of self-similarity at all scales. This means that the spiral is endless in both directions. When seen to converge towards a focal point, where

the diagonals of the golden rectangles intersect, it means that the spiral continues in more dimensions within a cone. The diverging segments meet at a point at infinity to create a single closed curve in projective space. This could be the property referred to by Johannes Kepler in stating that,

> ...the divine proportion served the Creator as an idea when He introduced the continuous generation of similar objects from similar objects.

Logarithmic spirals are widely used, especially in the engineering sciences and for botanical phyllotaxis to solve optimization problems. In the case of a distribution with multiple sub-maxima the procedure consists in the variation of a divergence angle between adjacent maxima on the spiral, until the optimal distribution is realized.

2.3.1 Superconduction

Metallic superconductors that operate at temperatures below 30K were discovered a hundred years ago, followed by high-temperature superconductors ($T_c \sim 90K$) 75 years later. A centennial survey [11] concluded:

> Exactly how such high-temperature superconductors (HTSS) work remains a mystery, but that has not stopped engineers from trying to exploit them.

However, a plausible mechanism, based on the golden ratio and which applies to all superconductors, exists [12] as the property of atomic nuclei rather than the interaction of electrons with lattice phonons.

The observation that the convergence of $Z/N \to \tau$ promotes nuclear stability was interpreted to imply a packing mode for nucleons in three-dimensional analogy of botanical phyllotaxis. The general property, $N > Z$, of stable nuclides then results in a golden excess of $x = Z - \tau N$ protons, which form an electrically positive layer in the nuclear surface, according to Fig. 1. The analysis predicts enhanced superconduction of isotropically pure materials in favourable allotropic modification, such as low-temperature ^{117}Sn with spin of 11/2. These predictions await technological exploitation.

The existence of high-spin surface layers suggests an intriguing new interpretation of magnetic phenomena associated with superconduction. To account for the anisotropy and anomalous signs of Hall coefficients it is noted that an external magnetic field simply aligns the excess spin system to generate an oppositely directed internal magnetic field. Alternatively the internally generated magnetic field of a superconductor is sufficient for the diagnostic levitation of a small magnet over the system, in what is known as the Meisner effect.

It is important to note that whereas the BCS theory only applies to special low T_c materials, the golden-excess model covers all superconductors. The disadvantage of currently known HTSS is the brittleness of the ceramic materials that complicates the manufacture of flexible wires made up of well-aligned crystals. Raising the critical temperature, T_c, of cheap industrially important metals by isotropic enrichment offers a more manageable alternative technology.
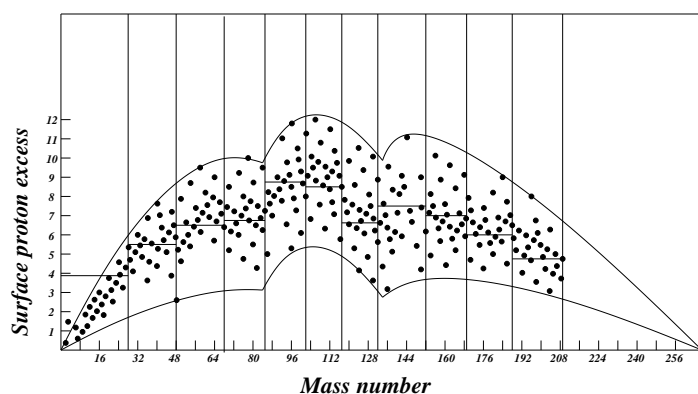
Fig. 1 Proton surface excess, $x = Z - \tau N$, as a function of mass number. Nuclide periodicity predicts maximal surface spin to occor in the regions as marked, in general agreement with the measured spin and elemental superconductivity of odd mass number nuclides.

3 Chemistry by Number

The most important contribution of chemistry to science, the periodic table of the elements, can be derived directly by elementary number theory. The fundamental relationship between Farey sequences and Ford circles is readily demonstrated.
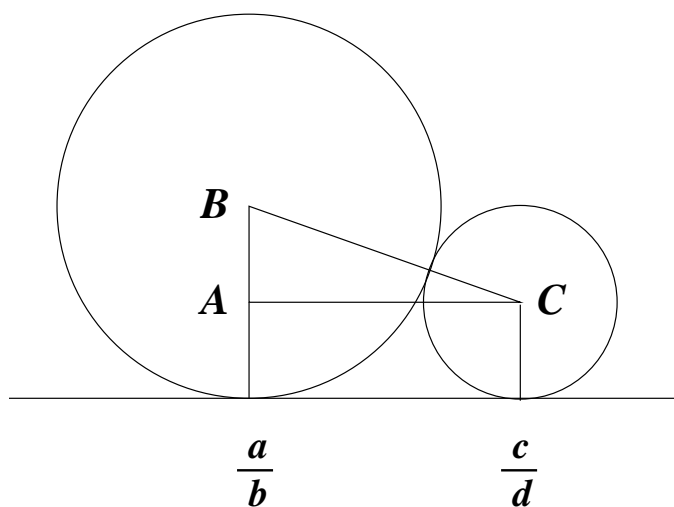
Consider two touching circles with diameters of $1/b^2$ and $1/d^2$, centred at respective points a/b and c/d as shown in the diagram, and such that $b < d$ and $a/b < c/d$. By construction, in the right triangle ABC:

$$AB = \frac{1}{2b^2} - \frac{1}{2d^2}$$

$$AC = \frac{c}{d} - \frac{a}{b}$$

$$BC = \frac{1}{2b^2} + \frac{1}{2d^2} \quad \text{(sum of radii)}$$

By Pythagoras:

$$(BC)^2 - (AB)^2 = \frac{1}{2(bd)^2} + \frac{1}{2(bd)^2} = \left(\frac{1}{bd}\right)^2 = (AC)^2 = \left(\frac{bc - ad}{db}\right)^2$$

Hence

$$bc - ad = \begin{vmatrix} c & a \\ d & b \end{vmatrix} = 1$$

This is the unimodular condition that characterizes neighbouring fractions in a Farey sequence.
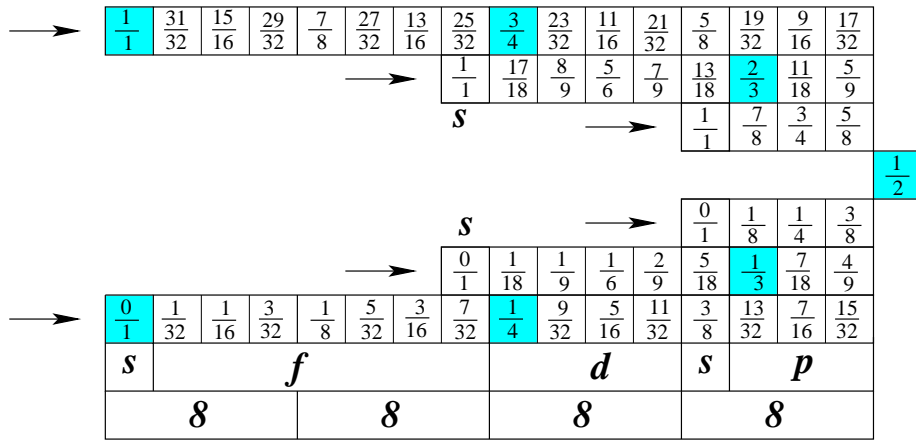
| $\frac{1}{1}$ | $\frac{31}{32}$ | $\frac{15}{16}$ | $\frac{29}{32}$ | $\frac{7}{8}$ | $\frac{27}{32}$ | $\frac{13}{16}$ | $\frac{25}{32}$ | $\frac{3}{4}$ | $\frac{23}{32}$ | $\frac{11}{16}$ | $\frac{21}{32}$ | $\frac{5}{8}$ | $\frac{19}{32}$ | $\frac{9}{16}$ | $\frac{17}{32}$ |

| | $\frac{1}{1}$ | $\frac{17}{18}$ | $\frac{8}{9}$ | $\frac{5}{6}$ | $\frac{7}{9}$ | $\frac{13}{18}$ | $\frac{2}{3}$ | $\frac{11}{18}$ | $\frac{5}{9}$ |
| | s | | | $\frac{1}{1}$ | $\frac{7}{8}$ | $\frac{3}{4}$ | $\frac{5}{8}$ |

$\frac{1}{2}$

| | s | | $\frac{0}{1}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{3}{8}$ |
| | $\frac{0}{1}$ | $\frac{1}{18}$ | $\frac{1}{9}$ | $\frac{1}{6}$ | $\frac{2}{9}$ | $\frac{5}{18}$ | $\frac{1}{3}$ | $\frac{7}{18}$ | $\frac{4}{9}$ |

| $\frac{0}{1}$ | $\frac{1}{32}$ | $\frac{1}{16}$ | $\frac{3}{32}$ | $\frac{1}{8}$ | $\frac{5}{32}$ | $\frac{3}{16}$ | $\frac{7}{32}$ | $\frac{1}{4}$ | $\frac{9}{32}$ | $\frac{5}{16}$ | $\frac{11}{32}$ | $\frac{3}{8}$ | $\frac{13}{32}$ | $\frac{7}{16}$ | $\frac{15}{32}$ |

| s | f | | | | d | | | s | p |
| 8 | 8 | | | | 8 | | | 8 | |

Fig. 2 Periodic table of the elements, defined directly by the Farey sequence $\mathscr{F}_4$. Shaded blocks show how the Farey fractions subdivide the periods into regions that correspond to the s, p, d, f blocks of chemical elements. Blocks of 2 and 8 elements define the compact form of the Appended periodic table.

Ford-circle mapping of the periodic table [13], based on this idea, consists of a central circle flanked on both sides by three satellites, which together represent the $\mathscr{F}_4$ Farey sequence:

$$\mathscr{F}_4 = \left\{ \frac{0}{1} \frac{1}{4} \frac{1}{3} \frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{1}{1} \right\}$$

In each fraction, h/k, the denominator is interpreted to distinguish between periodic systems, $k = 1 \rightarrow 4$, of size $2k^2$. The different fractions in $\mathscr{F}_4$ represent subsets with $j = 2\sum_{i=1}^{k}(2i-1)$, i.e. $2(1+3+\ldots \leq k)$, elements. Each subset with fixed k expands into a sequence of smaller fractions, $n/j, n = 0 \rightarrow j$. On arranging these

fractions symmetrically around $\frac{1}{2}$ the layout of the periodic table [13] is reproduced, as shown in Fig. 2, providing each block represents two chemical elements in the order defined by the arrows.

Without assuming an exclusion principle, the Farey fractions subdivide the periodic sets into subsets that correspond to the traditional s, p, d, f energy levels, derived from atomic spectra. The resulting sublevel structure is highlighted by shading in the following diagram. It is necessary to emphasize that these fractions do not refer to Z/N ratios.

Apart from the relative abundance of isotopes all other basic data, usually shown on a periodic table of the elements are predicted by elementary number theory. Based on the periodic table shown as an Appendix, the chemically important parameters of ionization radius and electronegativity have been used, together with the golden section and golden spirals, to derive all essential parameters pertaining to covalent interaction and the optimization of molecular structure by MM. The detailed results are described in the preceding papers in this volume.

3.1 Molecular Structure

Chemical science is based on more than atomic structure. More precisely, it concerns the properties of molecules, and this is where all current theories fall short. Efforts to simulate molecular structure by elementary number theory have been equally unsuccessful, with the exception of one partial success pertaining to the structure of DNA base pairing.
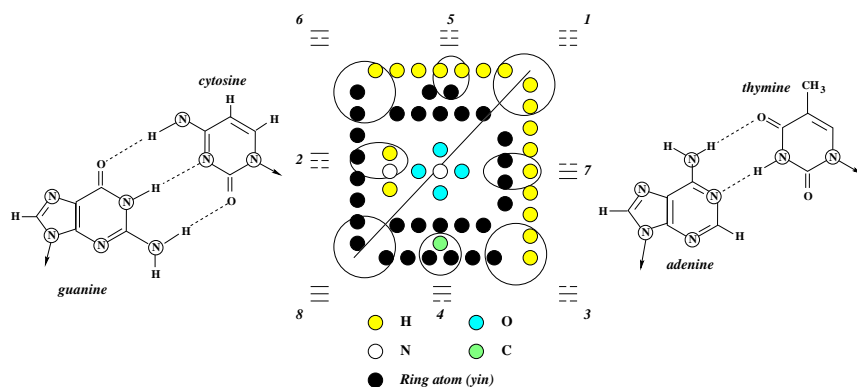

Fig. 3 Ho Tu simulation of DNA base pairing

A mysterious number code, known as *Ho Tu* [15] emerged in China during the fourth millenium BCE, in the form of 55 dots arranged as in Fig. 3. The circled clusters represent *yin-yang* trigrams in the eight radial compass directions, arranged

in an 8×8 matrix, known as *I Ching*, generate the numbers 0 to 63 as hexagrams in binary code.

Not only do the I Ching hexagrams correspond to the DNA genetic code [16], but the Ho Tu dots have also been shown [15] to symbolize the molecular structure of the two H-bonded base pairs. The two base pairs are made up of 55 atoms, just like the Ho Tu consists of 55 dots. The dots above the diagonal in Fig. 3 model the cytosine–guanine pair and those below the diagonal represent adenine–thymine, as drawn.

In each Ho Tu segment there are 15 black (yin) dots, just as each base pair has 15 central ring atoms. These 15 atoms define a (5+4)- and a 6-membered cyclic system. The five central yang dots allude to the five hydrogen bonds in groups of 3 and 2. The extracyclic H, N, O and C atoms are colour coded to emphasize the correlation. The match between Ho Tu and the base pairing is complete.

The equivalence of I Ching and the binary number system is beyond dispute. Its correspondence with the 64 codon DNA genetic code [16] is equally convincing. Derivation of the 8-trigram compass mandala, that generates the I Ching from the Ho Tu pattern, is readily demonstrated. Unless the homology of the Ho Tu and DNA base-pairing is a massive coincidence there can be little doubt that it presents a detailed code of biological replication, still not fully understood. The origin of the code is shrouded in the mists of time. Legend has it that the code was delivered to the emperor Fu Hsi by a "dragon horse", which has been likened [15] to a rocket from outer space.

The cross-sectional profile of the most common form of crystalline DNA, with a 10_1 screw axis, has an enclosing form scaled by the limiting factor [17] of:

$$\mu(10/3) = \frac{\cos 3\pi/10}{\cos \pi/10} = \tau,$$

the golden ratio. This form of DNA double helix also represents the most efficient space-filling arrangement.

The degree of success in relating the structure of the most intricate of molecules to pure numbers raises the haunting suspicion that the inability to formulate a valid theory of molecular structure may be traced back to an inadequate understanding of cosmic self-similarity. With a little insight the accumulated empirical wisdom on molecular conformation could perhaps fall into place to solve the mystery of molecular shape.

4 Conclusion

In all cases where the golden section or the golden spiral correlates with chemical phenomena convergence to some singularity is observed. The most striking example, shown in Fig. 4, occurs as the composition of stable nuclides, measured as Z/N, converges to the golden ratio as $Z \to 102$. At the same time the hem lines, which
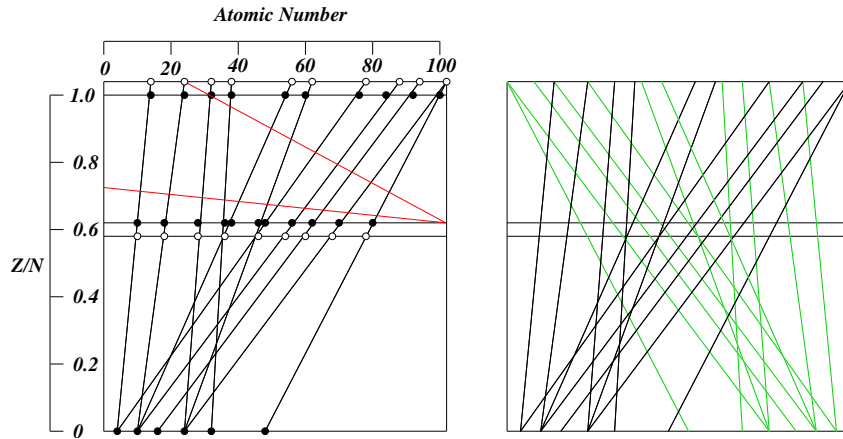
Fig. 4 Variability of the periodic table of the elements, a subset of the 24-member periodicity of stable nuclides, depends on space-time curvature and is mapped by 11 hem lines in the frame on the left. The field of stability is defined by the limiting lines that converge to τ at $Z = 102$. The symmetrical version on the right is only resolved in 4D projective space.

define nuclide periodicity of 24, map out the observed periodic table of the elements at $Z/N = \tau$.

Extrapolation of the hem lines from 0 to 1.04 defines special periodic functions at $Z/N = 0$, 0.58, τ, 1 and 1.04, with $0 \leq Z \leq 102$. The periodic function defined at $Z/N = 0$ refers to neutron periodicity; at $Z/N = 0.58$ the sequence of points corresponds to the Schrödinger solution for H; the periodic functions at $Z/N = 1$ and 1.04 are inverse to those at τ and 0.58 respectively. The logic behind this is that the Schrödinger model, which ignores all but the Coulomb attraction between proton and electron, refers to an empty (Euclidean flat) universe, inverse to the infinitely curved geometry at $Z/N = 1.04$. As the general curvature of space-time follows a golden spiral the observed periodicity occurs at $Z/N = \tau$ and nuclear synthesis by $\alpha-$particle addition at $Z/N = 1$. The structure inversion is due to the known effect of pressure on electronic energy levels in atoms.

By the same argument the ratio of $Z/N = \varepsilon \simeq 0.58$ should be interpreted as characteristic of free space. It is of interest that also the ubiquitous mysterious ratio [18] of $0.1115 \simeq 2\varepsilon$. Maybe it is no accident that the approximate formula for a Fibonacci spiral [19]

$$r \simeq 2\varepsilon \exp(\theta \cdot \tau/2)$$

contains both ε and τ, in line with the conjecture that the golden spiral embodies the relationship between Euclidean tangent space and the underlying four-dimensional curved space-time manifold.

The points at $Z/N = 1.04$ are arranged symmetrically around a central point. On reflection of the entire pattern around the line at $Z = 51$ a set of closed lines, which cannot be embedded in two dimensions, is generated. Drawn on the surface of a
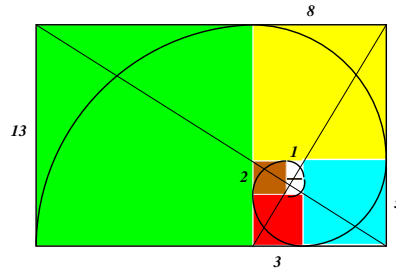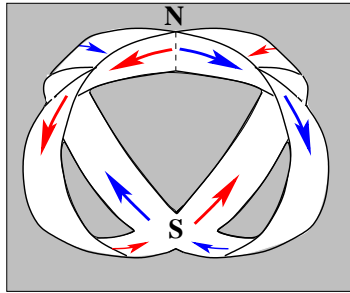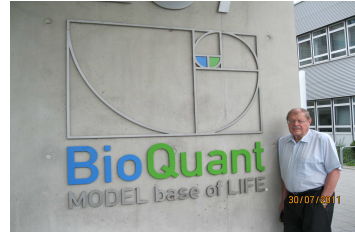
Fig. 5 Projective space is visualized by the space-filling intersection of an infinitesimal Möbius strips, like the two bands, shown in the lower left frame [8]. A golden spiral results from inscribing circular segments into the gnomons defined by Fibonacci numbers in a golden rectangle.

Möbius band the double pattern is readily identified as representing both matter and antimatter and requires the points at 0 and 102 to coincide. In order to achieve this the single edge of the Möbius band must be glued to itself to create a closed space which can no longer be embedded in three dimensions and is known as projective space.

We conclude that all of the results described in this volume can be generated from two related constructs – the golden spiral and projective geometry. Remarkably, models of exactly these two things have recently appeared on the Neuenheimer campus of the Ruprecht-Karls University of Heidelberg where a large part of this work was done. These are pictured with one of us in Fig. 5.

Appendix

PERIODIC TABLE OF THE ELEMENTS

Legend cell: Z, Sym / r_0 (Å) / χ — Closed s shell | Closed p shell | Closed d shell | Closed f shell | Radio–active

Period								
2	1 H 0.98 / 6.25	2 He						
8	3 Li 2.36 / 2.60	4 Be 2.20 / 2.79	5 B 1.88 / 3.26	6 C 1.78 / 3.45	7 N 1.69 / 3.63	8 O 1.60 / 3.83	9 F 1.52 / 4.03	10 Ne 1.44 / 4.26
8	11 Na 3.09 / 1.98	12 Mg 2.87 / 2.14	13 Al 2.74 / 2.24	14 Si 2.62 / 2.34	15 P 2.51 / 2.44	16 S 2.47 / 2.48	17 Cl 2.30 / 2.67	18 Ar 2.21 / 2.78
18	19 K 3.50 / 1.75	20 Ca 3.08 / 1.99						
18	21 Sc 3.02 / 2.03	22 Ti 2.99 / 2.05	23 V 2.96 / 2.07	24 Cr 3.02 / 2.03	25 Mn 2.90 / 2.11	26 Fe 2.87 / 2.14	27 Co 2.85 / 2.15	28 Ni 2.81 / 2.18
	29 Cu 2.88 / 2.13	30 Zn 2.76 / 2.22	31 Ga 3.00 / 2.04	32 Ge 2.89 / 2.12	33 As 2.80 / 2.19	34 Se 2.71 / 2.26	35 Br 2.61 / 2.35	36 Kr 2.52 / 2.43
18	37 Rb 3.81 / 1.61	38 Sr 3.54 / 1.73						
	39 Y 3.27 / 1.88	40 Zr 3.24 / 1.89	41 Nb 3.30 / 1.86	42 Mo 3.26 / 1.88	43 Tc 3.14 / 1.95	44 Ru 3.20 / 1.92	45 Rh 3.16 / 1.94	46 Pd 2.29 / 2.68
	47 Ag 3.11 / 1.97	48 Cd 3.00 / 2.04	49 In 3.31 / 1.85	50 Sn 3.19 / 1.92	51 Sb 3.09 / 1.98	52 Te 2.98 / 2.06	53 I 2.88 / 2.13	54 Xe 2.75 / 2.23
32	55 Cs 4.03 / 1.52	56 Ba 3.75 / 1.64	57 La 4.62 / 1.33	58 Ce 4.69 / 1.31	59 Pr 4.68 / 1.31	60 Nd 4.66 / 1.32	61 Pm 4.64 / 1.32	62 Sm 4.63 / 1.32
	63 Eu 4.61 / 1.33	64 Gd 4.51 / 1.36	65 Tb 4.58 / 1.34	66 Dy 4.57 / 1.34	67 Ho 4.55 / 1.35	68 Er 4.54 / 1.35	69 Tm 4.52 / 1.36	70 Yb 4.50 / 1.36
	71 Lu 3.54 / 1.73	72 Hf 3.50 / 1.75	73 Ta 3.47 / 1.77	74 W 3.44 / 1.78	75 Re 3.40 / 1.80	76 Os 3.37 / 1.82	77 Ir 3.34 / 1.84	78 Pt 3.41 / 1.80
	79 Au 3.38 / 1.81	80 Hg 3.24 / 1.89	81 Tl 3.43 / 1.79	82 Pb 3.32 / 1.85	83 Bi 3.22 / 1.90	84 Po 3.12 / 1.97	85 At 3.03 / 2.02	86 Rn 2.93 / 2.09

References

1. de Chancourtois AEB (1889) Nature 41:186.
2. Nagaoka H (1904) Nature 69:392.
3. W.D. Harkins WD (1931) Phys Rev 38:1270.
4. Boeyens JCA (2003) J Radioanal Nucl Chem 259:33.
5. Schrödinger E (1926) Naturwiss 28:664.
   – (1952) Brit J Phil Sci 3:109.
6. Bohr N, quoted by [20].
7. Boeyens JCA (2009) Physics Essays, 22:493.
8. Boeyens JCA (2010) Chemical Cosmology. Springer, Heidelberg.
9. Stillwell J (1992) Geometry of Surfaces. Springer-Verlag, Heidelberg.
10. Funk-Hellet C. (1956) La Bible et la Grand Pyramide d'Egypte. Hellet Vincent, Montreal.
11. The Economist Technology Quarterly, December 3rd 2011: Superconductors 100 years on.
12. Boeyens JCA, Levendis DC (2008) Number Theory and the Periodicity of Matter. Springer.com.
13. Boeyens JCA, Comba P (2012) This volume.
14. Boeyens JCA (2008) Chemistry from First Principles, Springer.com
15. Walter K (1994) Tao of Chaos. Kairos Center, Austin.
16. Boeyens JCA (2003) Crystal Engin 6:167.
17. Janner A (2001) Crystal Engin 4:119.
18. Maddox J (1983) Nature 304:11.
19. Boeyens JCA (2012) Atomic Structure. This volume.
20. Holton G (1952) Introduction to Concepts and Theories in Physical Science. Addison-Wesley, Reading MA.