Lattices of properties of countable graphs and the Hedetniemi Conjecture

by

Moroli D. V. Matsoha
(Student number 25332750)

Submitted in partial fulfillment of the requirements for the degree

Magister Scientiae in Mathematics

in the Department of Mathematics and Applied Mathematics

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

November 2013

DECLARATION

I, the undersigned declare that the dissertation, which I hereby submit for the degree **Magister Scientiae in Mathematics** at the University of Pretoria, is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other university.

SIGNATURE: ....

NAME: Moroli D. V. Matsoha

DATE: November 2013

Lattices of properties of countable graphs and the Hedetniemi Conjecture

by

Moroli D. V. Matsoha

supervised by

Prof Izak Broere

and co-supervised by

Dr Tomáš Vetrìk

in the

Department of Mathematics and Applied Mathematics

for the degree

MSc Mathematics

Abstract

Lattices of hereditary properties of finite graphs have been extensively studied. We investigate the lattice $\mathbb{L}$ of induced-hereditary properties of countable graphs. Of interest to us will be some of the members of $\mathbb{L}$. Much of our focus will be on hom-properties $\to G$. We analyse their behaviour and consider their link to solving the long standing Hedetniemi Conjecture. We then discuss universal graphs and construct a universal graph U_n for the property $\to K_n$. Then we use the structure of U_n to prove a theorem by Szekeres and Wilf. Lastly we offer a new proof of a theorem by Duffus, Sands and Woodrow.

Contents

1 Preliminaries

1.1 Graph theoretic definitions and notation

The following definitions and notation are predominantly those of [4]. In those instances when the definition is from elsewhere we shall reference the source then.

A **graph** G is a pair $(V(G), E(G))$, where the first, $V(G)$, referred to as the **vertex set** of G, is a non-empty set, and the second, $E(G)$, referred to as the **edge set** of G, is a possibly empty set of 2-element subsets of $V(G)$. The **order** of G refers to $|V(G)|$, the cardinal number of $V(G)$.

The elements of $V(G)$ are the **vertices** of G, while the elements of $E(G)$ are the **edges** of G. For all $\{x, y\} \in E(G)$, we shall write xy instead of $\{x, y\}$. Furthermore, we make no distinction between xy and yx.

For a graph G we call x and y **adjacent** vertices or say x is **adjacent to** y if $xy \in E(G)$. If $xy \notin E(G)$, then we say x and y are **non-adjacent** vertices. Given $x \in V(G)$ the **neighbours** of x are those vertices adjacent to x.

A graph G' is a **subgraph** of a graph G, denoted $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. We shall also refer to G' as an **internal subgraph** of G. When $V(G')$ is a proper subset of $V(G)$ or $E(G')$ is a proper subset of $E(G)$, then we call G' a **proper subgraph** of G. A subgraph G' of G, is an **internal induced subgraph** of G, written $G' \leq G$, if, for all $x, y \in V(G')$, $xy \in E(G)$ implies $xy \in E(G')$. Given any non-empty subset A of $V(G)$, an induced subgraph of G whose vertex set is A will be called the **subgraph of G induced by A**, written $G[A]$. For a graph G, a vertex u of G and an edge e of G, we write $G - u$ to denote the subgraph of G induced by the set $V(G) \setminus \{u\}$ and write $G - e$ to denote the graph obtained from G after the removal of e from the edge set of G.

Two graphs G and H are said to be **isomorphic** if there exists a bijective mapping $\phi : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. Such a map we call an **isomorphism**. We say G and H are **non-isomorphic** if they are not isomorphic. If G'' is isomorphic to G', an internal subgraph of G, we shall also refer to G'' as a subgraph of G, similarly for induced subgraphs.

A **path** P in a graph G, is a sequence of vertices of G such that consecutive vertices are adjacent in G and no vertex in this sequence is repeated. The first and last members of this sequence, when they exist, are called the **end vertices** of P. Any vertex of P that is not an end vertex is called an **internal vertex** of P. If u and v are end vertices of a path P then we on occasion refer to P as a $u - v$ path. The **length** of P is $|V(P)| - 1$. A **cycle** in G is a path in G with adjacent end vertices. The **length** of a cycle is the order of the cycle. An **even cycle** is a cycle of even order, while an **odd cycle** is one of odd order.

A graph G is **connected** if, for all $u, v \in V(G)$, there exists a $u - v$ path. It is **disconnected** if it is not connected. A **component** of a graph G is a connected subgraph of G that is not a proper subgraph of any connected subgraph of G. Given a component G' of a graph G we use $G - G'$ to represent the graph $G[V(G) \setminus V(G')]$, provided $V(G) \setminus V(G') \neq \emptyset$.

A complete graph is one in which any two distinct vertices are adjacent. For a positive integer n we shall use K_n and C_n to represent a complete graph of order n and a cycle of order n, respectively. Here C_n is seen as an autonomous graph, a "cycle in itself". We use the notation $K_{\aleph_0}$ to represent the complete graph whose vertex set is the set $\{1, 2 \ldots\}$.

The **disjoint union** $G \sqcup H$ of graphs G and H with disjoint vertex sets is that graph with vertex set $V(G \sqcup H) = V(G) \cup V(H)$ and edge set $E(G \sqcup H) = E(G) \cup E(H)$. The disjoint union of more than two graphs is defined similarly. The **join** $G \vee H$ of graphs G and H with disjoint vertex sets is that graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}.$$

The **cross product** $G \times H$ of graphs G and H is a graph whose vertex set $V(G \times H) = V(G) \times V(H)$ and whose edge set is such that two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if $x_1y_1 \in E(G)$ and $x_2y_2 \in E(H)$. Since this is the only graph product we deal with we shall refer to it as the **product**. The **Mycielski construct** of a graph G, denoted by $M(G)$, is the graph obtained by first introducing a new vertex w and then introducing, for each $x \in V(G)$, a new vertex x' and making it adjacent to w and to all the neighbours of x in G. The **Hajós construction** [10] on graphs G and H is implemented by deleting an edge xy in G and an edge uv in H, identifying the vertices x and u, and adding the edge yv. A graph obtained from this construction we call a **Hajós construct**.

For a positive integer n and a graph G, an n**-colouring** of G is an assignment of colours to the vertices of G, where one colour from n colours is given to each vertex. A **proper** n**-colouring** of G is an n-colouring of G in a manner such that no two adjacent vertices are coloured the same. A graph G is n**-colourable** if there exists a proper n-colouring of G. Given a proper n-colouring of a graph G using colours $1, 2, \ldots, n$ the set V_i $(1 \leq i \leq n)$ of vertices of G that are coloured i is called a **colour class**. The **chromatic number** of a graph G which admits a proper n-colouring for some finite n, $\chi(G)$, is the least integer m such that G has a proper m-colouring. A graph G with $\chi(G) = n$ is an n**-chromatic graph** . A graph G is n**-critical**, $n \geq 2$, if, for all edges $e \in E(G)$, $\chi(G - e) < \chi(G)$.

The **clique number** $\omega(G)$ of a graph G is the largest order of all complete subgraphs of G.

1.2 Lattice theoretic definitions and notation

The following definitions and notation are consistent with [5].

By **partial order** we mean a binary relation, $\leq$, on some set P, that is **reflexive, antisymmetric** and **transitive**. Where, for $x, y, z \in P$, reflexive means $x \leq x$, antisymmetric means if $x \leq y$ and $y \leq x$ then $x = y$, and transitive means that if $x \leq y$ and $y \leq z$ then $x \leq z$. By **partially ordered set**(or **poset**) we mean a pair $\langle P, \leq \rangle$, where $\leq$ is a partial order on P. On occasion we write P for the partially ordered set $\langle P, \leq \rangle$. This we do when no ambiguity can arise regarding the partial order on P. In a partially ordered set $\langle P, \leq \rangle$, a **minimal element** $x \in P$ is one such that if $y \leq x$, for some $y \in P$, then $y = x$. A **maximal element** $x \in P$ is one such that if $y \geq x$, for some $y \in P$, then $y = x$. If P has one minimal element then we call it the **least element** in P. Similarly, if P has one maximal element we call it the **greatest element** in P. Two elements x and y in P are **incomparable** if $x \nleq y$ and $y \nleq x$.

Given any subset S of a poset $\langle P, \leq \rangle$, we say $x \in P$ is an **upper bound** of S if $x \geq y$ for all $y \in S$. We say $x \in P$ is a **lower bound** of S if $x \leq y$ for all $y \in S$. The **join** of S, $\bigvee S$, also known as the **least upper bound** of S, if it exists, is the least element in the set of upper bounds of S. The **meet** of S, $\bigwedge S$, also known as the **greatest lower bound** of S, if it exists, is the greatest element in the set of lower bounds of S. For any $x, y \in P$, we write $x \vee y$ and $x \wedge y$ for the join and meet, respectively, of the set $\{x, y\}$, when these exist.

A poset $\langle P, \leq \rangle$ is called a **lattice** if the join and meet of any two elements in P exist. At times we shall use the notation $\langle P, \vee, \wedge \rangle$ to mean that a poset P is a lattice whose joins and meets are defined by $\vee$ and $\wedge$, respectively. A poset P is said to be a **complete lattice** if the join and meet of all subsets of P exist. A lattice L is **modular** if for all $x, y, z \in L$ such that $x \leq y$ then $x \vee (z \wedge y) = (x \vee z) \wedge y$. A lattice L is **distributive** if $x \vee (z \wedge y) = (x \vee z) \wedge (x \vee y)$

for all $x, y, z \in L$.

A subset M of a lattice L, together with the partial order $\leq$ inherited from L, is a **sublattice** of L if the join and meet of any two elements in $\langle M, \leq \rangle$ exist and are the join and meet of said elements in L. That is, for all $x, y \in M$ we have

$$x \vee_M y = x \vee_L y \quad \text{and} \quad x \wedge_M y = x \wedge_L y,$$

where $\vee_M, \wedge_M$ and $\vee_L, \wedge_L$ represent the joins and meets in M and L, respectively. We call M a **join-semi-sublattice** of L if,

$$x \vee_M y = x \vee_L y \qquad \text{for all } x, y \in M$$

and a **meet-semi-sublattice** of L if,

$$x \wedge_M y = x \wedge_L y \qquad \text{for all } x, y \in M.$$

An element x in a lattice L is **join-reducible** if there exist distinct elements $y, z \in L$ such that $x = y \vee z$ and $y \neq x \neq z$. We say x is **join-irreducible** if x is not join-reducible and x is not the least element in L. Similarly, x is a **meet-reducible** element in L if there exist distinct elements $y, z \in L$ such that $x = y \wedge z$ and $y \neq x \neq z$. If x is not meet-reducible and it is not the greatest element in L then we call x a **meet-irreducible** element of L.

A closure operator is a mapping $c\colon \langle P, \leq \rangle \longrightarrow \langle P, \leq \rangle$ such that, for all $x, y \in P$,

1. $x \leq c(x)$ (extensive),
2. $x \leq y$ implies $c(x) \leq c(y)$ (monotone),
3. $c(c(x)) = c(x)$ (idempotent).

Let L_1 and L_2 be lattices, then a mapping $f\colon L_1 \longrightarrow L_2$ is a **lattice homomorphism** if, for all $a, b \in L_1$,

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

1.3 Summary

We begin by introducing the lattice $\mathbb{L}$ of induced-hereditary properties of countable graphs, followed by a brief discussion of graph homomorphisms and their properties. This we do in preparation for an investigation of the distributive lattice Hom, which is a sublattice of $\mathbb{L}$. Continuing with the theme of graph homomorphisms, we take a look at the properties of the core of a graph, and proceed to describe the lattice of hom-equivalence classes of graphs. After this we discuss properties of finite character and compact elements in $\mathbb{L}$. This is followed by the construction of universal graphs for hom-properties, achieved with the aid of the *Rado* graph. We then study the well-known conjecture by Hedetniemi, which states that, for all finite graphs G and H, $\chi(G \times H) = min\{\chi(G), \chi(H)\}$. We give equivalent formulations of this conjecture, one of which we make in terms of the meet-irreducibility of the hom-property $\rightarrow K_n$. Duffus, Sands and Woodrow verified the Hedetniemi Conjecture for some special cases. Using the Hajós construction we give a new proof of their theorem.

2 Lattices

2.1 The lattice $\mathbb{L}$

Let $\mathcal{I}$ be the set of all countable unlabelled simple graphs. By 'simple' we mean loop-less, undirected graphs without parallel edges. In addition we make no distinction between isomorphic graphs. Although we are dealing with unlabelled graphs we will on occasion assign labels to their vertices. This we will do only better to describe the vertex set and edge set of the graph in question. A **property** is any subset $\mathcal{P}$ of $\mathcal{I}$.

Definition 1. [2] *A property $\mathcal{P}$ is an **induced-hereditary property** or **i-h property**, for short, if $G \in \mathcal{P}$ implies $H \in \mathcal{P}$ for all $H \leq G$.*

Definition 2. [2] *A property $\mathcal{P}$ is **additive** if whenever $G, H \in \mathcal{P}$ then $G \sqcup H \in \mathcal{P}$.*

Let $\mathbb{L}$ be the set of all induced-hereditary properties. Below are a few examples of additive induced-hereditary properties taken from [2]. From here on, by $\mathbb{N}$ we shall mean the set $\{1, 2, 3, \ldots\}$.

Examples

1. $\mathcal{I}$
2. $\mathcal{O} = \{G \in \mathcal{I} \mid E(G) = \emptyset\}$
3. $\mathcal{K} = \{G \in \mathcal{I} \mid$ all components of G are complete graphs $\}$
4. $\mathcal{O}_k = \{G \in \mathcal{I} \mid$ each component of G has at most k vertices $\}$, $k \in \mathbb{N}$.
5. $\mathcal{I}_k = \{G \in \mathcal{I} \mid G$ does not contain $K_{k+1}\}$, $k \in \mathbb{N}$.

Proposition 1. *The pair $\langle \mathbb{L}, \subseteq \rangle$, where $\subseteq$ means 'to be a subset of', is a poset.*

Proof. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbb{L}$. Clearly $\mathcal{P} \subseteq \mathcal{P}$, and if $\mathcal{P} \subseteq \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$ then $\mathcal{P} = \mathcal{Q}$. Also, if $\mathcal{P} \subseteq \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{R}$ then $\mathcal{P} \subseteq \mathcal{R}$. All these are a result of set theoretic properties. $\square$

Proposition 2. *The poset $\langle \mathbb{L}, \subseteq \rangle$ is a complete lattice, where joins and meets are defined by set union and intersection, respectively. That is, the poset $\langle \mathbb{L}, \subseteq \rangle$ is the lattice $\langle \mathbb{L}, \cup, \cap \rangle$.*

Proof. Let $\mathbb{S} \subseteq \mathbb{L}$, then $\bigcup \mathbb{S}, \bigcap \mathbb{S} \in \mathbb{L}$. For all properties $\mathcal{Q} \in \mathbb{S}$, $\mathcal{Q} \subseteq \bigcup \mathbb{S}$ and $\bigcap \mathbb{S} \subseteq \mathcal{Q}$. Therefore $\bigcup \mathbb{S}$ is an upper bound of $\mathbb{S}$, and $\bigcap \mathbb{S}$ is a lower bound of $\mathbb{S}$.

Suppose $\mathcal{P} \in \mathbb{L}$ is an upper bound of $\mathbb{S}$. Since $\mathcal{Q} \subseteq \mathcal{P}$ for all $\mathcal{Q} \in \mathbb{S}$ it follows that $\bigcup \mathbb{S} \subseteq \mathcal{P}$, therefore $\bigcup \mathbb{S} = \bigvee \mathbb{S}$.

Now suppose $\mathcal{P} \in \mathbb{L}$ is a lower bound of $\mathbb{S}$. Then $\mathcal{P} \subseteq \mathcal{Q}$ for all $\mathcal{Q} \in \mathbb{S}$. From this it follows that $\mathcal{P} \subseteq \bigcap \mathbb{S}$, therefore $\bigwedge \mathbb{S} = \bigcap \mathbb{S}$. $\square$

In keeping with [2] we obtain the following.

Proposition 3. *The lattice $\mathbb{L}$ is distributive.*

Proof. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbb{L}$. Then

$$\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) = \mathcal{P} \cup (\mathcal{Q} \cap \mathcal{R}) = (\mathcal{P} \cup \mathcal{Q}) \cap (\mathcal{P} \cup \mathcal{R}) = (\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$$

and

$$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) = \mathcal{P} \cap (\mathcal{Q} \cup \mathcal{R}) = (\mathcal{P} \cap \mathcal{Q}) \cup (\mathcal{P} \cap \mathcal{R}) = (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R}).$$

Thus $\mathbb{L}$ is a distributive lattice. $\square$

Definition 3. *For all $G, H \in \mathcal{I}$, a mapping $f \colon V(G) \longrightarrow V(H)$* ***preserves edges*** *if f is such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. The mapping f* ***preserves non-edges*** *if $uv \notin E(G)$ implies $f(u)f(v) \notin E(H)$.*

Definition 4. [2] *For all $G, H \in \mathcal{I}$, a* ***homomorphism*** *from G to H is a mapping $f : V(G) \longrightarrow V(H)$ that preserves edges. We write $G \longrightarrow H$ to represent the existence of a homomorphism from G to H. On occasion, instead of $G \longrightarrow H$, we will say G is* ***homomorphic*** *to H.*

Lemma 1. *Let φ be a homomorphism from G to H. The restriction of φ to any subset A of $V(G)$ is a homomorphism from $G[A]$ to H.*

Proof. Let φ be a homomorphism from G to H, and let A be any subset $V(G)$. Then the mapping $\gamma\colon A \longrightarrow V(H)$ defined, for all $x \in A$, by $\gamma(x) = \varphi(x)$ is the restriction of φ to A. Given any adjacent vertices u and v of $G[A]$ it follows that $\gamma(u)\gamma(v) = \varphi(u)\varphi(v) \in E(H)$. Thus γ is a homomorphism. $\square$

Lemma 2. *If $G \longrightarrow H$ then $G' \longrightarrow H$ for all subgraphs G' of G.*

Proof. Let $G \longrightarrow H$ for some $G, H \in \mathcal{I}$. Then there exists a homomorphism φ from G to H. Let $G' \subseteq G$ then $V(G') \subseteq V(G)$. It follows by Lemma 1 that the restriction of φ to $V(G')$, call it f, is a homomorphism from $G[V(G')]$ to H. Therefore f preserves the edges of G' as well, thus it is also a homomorphism from G' to H. $\square$

Lemma 3. *If $G \longrightarrow H$ and $H \longrightarrow F$ then $G \longrightarrow F$*

Proof. Let $G, H, F \in \mathcal{I}$ be such that $G \longrightarrow H$ and $H \longrightarrow F$. Then there exist homomorphisms φ and γ from G to H and from H to F, respectively. We claim that the composite function $\gamma \circ \varphi$ is a homomorphism from G to F. Let $uv \in E(G)$, then $\varphi(u)\varphi(v) \in E(H)$ and consequently $\gamma(\varphi(u))\gamma(\varphi(v)) \in E(F)$. Thus $\gamma \circ \varphi$ is a homomorphism from G to F. $\square$

Lemma 4. *If $G \in \mathcal{I}$ is such that each of its components is homomorphic to a graph $H \in \mathcal{I}$ then $G \longrightarrow H$.*

Proof. Let G and H satisfy the above conditions. Then the mapping from $V(G)$ to $V(H)$ whose restriction to each component of G is a homomorphism from said component to H is a homomorphism from G to H. $\square$

Lemma 5. *For all $G \in \mathcal{I}$ we have $\chi(G) \leq n$, where $n \in \mathbb{N}$, if and only if $G \longrightarrow K_n$.*

Proof. Select any graph $G \in \mathcal{I}$ such that $\chi(G) \leq n$ for some $n \in \mathbb{N}$, and let $v_1, \ldots, v_n$ be the n vertices of K_n. Since $\chi(G) \leq n$ it follows that there exist an integer $j \leq n$ such that $V(G)$ can be partitioned into j colour classes $V_1, \ldots, V_j$. Then the mapping $\varphi\colon V(G) \longrightarrow V(K_n)$ defined by $\varphi(u) = v_i$ if and only if $u \in V_i$, where $1 \leq i \leq j$, is a homomorphism from G to K_n.

Let $G \in \mathcal{I}$ be such that $G \longrightarrow K_n$ for some $n \in \mathbb{N}$. Then there exists a homomorphism φ from G to K_n. Allow $v_1, \ldots, v_n$ to be the n vertices of K_n. In addition, for each $1 \leq i \leq n$, let $\varphi^{-1}(v_i) = \{u \in V(G) \mid \varphi(u) = v_i\}$. Then the set $\{\varphi^{-1}(v_i) \mid 1 \leq i \leq n \text{ and } \varphi^{-1}(v_i) \neq \emptyset\}$ is a partition of $V(G)$ into at most n colour classes. Therefore $\chi(G) \leq n$. $\square$

Lemma 6. *If $G \longrightarrow H$ for some $G, H \in \mathcal{I}$ then $\chi(G) \leq \chi(H)$*

Proof. Let $G \longrightarrow H$. If $\chi(H)$ is not finite we are done. So, assume $\chi(H)$ is finite. Then $H \longrightarrow K_{\chi(H)}$ by Lemma 5. From this and Lemma 3 we obtain $G \longrightarrow K_{\chi(H)}$. Therefore, by Lemma 5, we obtain $\chi(G) \leq \chi(H)$. $\square$

Notice that $\chi(G) \leq \chi(H)$ does not imply $G \longrightarrow H$. As an example the Mycielski construct of C_5, $M(C_5)$, also known as the Grötsch graph [4], has chromatic number 4, yet K_3, which is 3-chromatic, is not homomorphic to it.

Lemma 7. *If $\omega(G) > \omega(H)$ then $G \not\longrightarrow H$.*

Proof. Let G and H be graphs in $\mathcal{I}$ such that $\omega(G) > \omega(H)$. Then G contains a complete subgraph of order $\omega(G)$. Call this subgraph G'. Suppose, to the contrary, that $G \longrightarrow H$. Let γ be a homomorphism from G to H. Then, for all distinct vertices $u, v \in V(G')$, $\gamma(u)\gamma(v) \in E(H)$ since $uv \in E(G')$. Therefore $H[\{\gamma(u) \mid u \in V(G')\}]$ is a complete graph of order $\omega(G)$. Which implies $\omega(H) \geq \omega(G)$, clearly a contradiction. $\square$

Lemma 8. *If φ is a homomorphism from a countable connected graph G to a countable graph H, then φ maps G to one component of H.*

Proof. Let $G, H \in \mathcal{I}$ be such that G is connected and $G \longrightarrow H$. If H is connected then we are done, so assume H is not connected. Suppose there exists a homomorphism φ that maps G to at least 2 components of H. Then there exist vertices u and v in G such that $\varphi(u)$ and $\varphi(v)$ belong to different components of H. G being connected, there exists a $u - v$ path P in G. As a result there exist adjacent vertices x and y in P which are such that $\varphi(x)$ and $\varphi(y)$ belong to different components of H. Therefore $\varphi(x)\varphi(y) \neq E(H)$ and thus φ does not preserve the edge xy. Which implies that φ is not a homomorphism. Clearly this is a contradiction. □

2.2 The lattice Hom

Definition 5. [2] *A property $\mathcal{P} \in \mathbb{L}$ is a **hom-property** if there exists a graph $G \in \mathcal{I}$ such that, for all graphs $H \in \mathcal{I}$,*

$$H \in \mathcal{P} \text{ if and only if } H \longrightarrow G.$$

Whence we shall write $\to G$ for $\mathcal{P}$.

It follows that $\to G$ is a Hom-property for all $G \in \mathcal{I}$. Let $Hom = \{\to G \mid G \in \mathcal{I}\}$.

Lemma 9. *All hom-properties are additive.*

Proof. Let $G, H \in \mathcal{I}$ be such that all components of G belong to $\to H$. Then $G \longrightarrow H$ by Lemma 4. Therefore $G \in \to H$, and thus $\to H$ is additive. □

Lemma 10. *For all graphs $G, H \in \mathcal{I}$*

$$\to(G \sqcup H) = (\to G) \cup (\to H) \cup \mathcal{D},$$

where $\mathcal{D} = \{F \in \mathcal{I} \mid$ each component of F belongs to $\to G$ or $\to H\}$.

Proof. Let $F \in \to(G \sqcup H)$, then there exists a homomorphism φ from F to $G \sqcup H$. If F is connected then φ maps F into some component of G or H. This follows by Lemma 8. Therefore $F \in \to G$ or $F \in \to H$, thus $F \in (\to G) \cup (\to H) \cup \mathcal{D}$. If F is not connected then the restriction of φ to each component of F is a homomorphism to a component of G or H. This follows by the application of Lemma 1 and Lemma 8. Then $F \in \mathcal{D}$, hence $F \in (\to G) \cup (\to H) \cup \mathcal{D}$. Thus we obtain $\to(G \sqcup H) \subseteq (\to G) \cup (\to H) \cup \mathcal{D}$.

Now supose $F \in (\to G) \cup (\to H) \cup \mathcal{D}$. If $F \in \to G$ then $F \longrightarrow G$, therefore $F \longrightarrow (G \sqcup H)$ by Lemma 4. We obtain the same result for $F \in \to H$. So assume $F \in \mathcal{D}$, then each component of F is homomorphic to G or H. Therefore each component of F is homomorphic to $G \sqcup H$. By Lemma 4 we obtain $F \longrightarrow (G \sqcup H)$, thus $F \in \to(G \sqcup H)$. As a result $(\to G) \cup (\to H) \cup \mathcal{D} \subseteq \to(G \sqcup H)$. Therefore $(\to G) \cup (\to H) \cup \mathcal{D} = \to(G \sqcup H)$. □

Proposition 4. *The pair $\langle Hom, \subseteq\rangle$ is a lattice, where joins and meets are described as follow,*

$$\to G \vee \to H = \to(G \sqcup H)$$

and

$$\to G \wedge \to H = \to G \cap \to H = \to(G \times H).$$

Proof. Clearly $(\to G) \cup (\to H) \cup \mathcal{D}$, where $\mathcal{D}$ is as described above, is an upper bound of $\to G$ and $\to H$ in Hom. Suppose $\to F$ is an upper bound of $\to G$ and $\to H$. Since $\to F$ is additive it follows that $\mathcal{D} \subseteq \to F$, hence $(\to G) \cup (\to H) \cup \mathcal{D} \subseteq \to F$, therefore

$$\to(G \sqcup H) = (\to G) \cup (\to H) \cup \mathcal{D} = \to G \vee \to H.$$

What remains now is to show that

$$\to G \wedge \to H = \to G \cap \to H = \to(G \times H).$$

Since the meet of two properties in Hom as in $\mathbb{L}$ is defined by their intersection we are only required to show that $\to G \cap \to H = \to(G \times H)$. Let $F \in \to(G \times H)$ then $F \longrightarrow (G \times H)$. Now let $\varphi_1 : V(G \times H) \longrightarrow V(G)$ and $\varphi_2 : V(G \times H) \longrightarrow V(H)$ be mappings defined as follows. For all $(u, v) \in V(G \times H)$,

$$\varphi_1((u, v)) = u \text{ and } \varphi_2((u, v)) = v.$$

Then φ_1 and φ_2 are homomorphisms, so $(G \times H) \longrightarrow G$ and $(G \times H) \longrightarrow H$. By Lemma 3 we obtain $F \longrightarrow G$ and $F \longrightarrow H$, therefore $F \in (\to G \cap \to H)$.

Assume $F \in (\to G \cap \to H)$, then there exist homomorphisms γ_1 and γ_2 from F to G and from F to H, respectively. The mapping $\gamma : V(F) \longrightarrow V(G \times H)$ defined by $\gamma(u) = (\gamma_1(u), \gamma_2(u))$ is a homomorphism from F to $G \times H$, therefore $F \in \to(G \times H)$, completing our proof. □

Next we prove that Hom is in fact a distributive lattice. For this we will employ the theorem below, taken from [5].

Theorem 1. *A lattice L is distributive if and only if, for all $a, b, c \in L$,*

$$a \vee b = c \vee b \text{ and } a \wedge b = c \wedge b \text{ implies } a = c.$$

Theorem 2. *The lattice Hom is distributive.*

Proof. Let $\to G_1, \to G_2, \to G_3$ be properties in Hom satisfying the following,

$$\to G_1 \vee \to G_2 = \to G_3 \vee \to G_2$$

and

$$\to G_1 \wedge \to G_2 = \to G_3 \wedge \to G_2.$$

We endeavour to show that $\to G_1 = \to G_3$. Our proof shall be by contradiction. So assume $\to G_1 \neq \to G_3$, then there exists a graph G belonging to only one of these properties. Without loss of generality let $G \in \to G_1$. Now assume $G \in \to G_2$, then

$$G \in \ (\to G_1 \cap \to G_2) = (\to G_3 \cap \to G_2).$$

This, of course, implies $G \in \to G_3$, which is a contradiction. Therefore G does not belong to $\to G_2$. Since $G \in \to G_1$ it follows, by Proposition 4, that $G \in \ (\to G_1 \vee \to G_2)$, and therefore $G \in \ (\to G_3 \vee \to G_2)$. By Lemma 10 we have

$$\to G_3 \vee \to G_2 = \ (\to G_3) \cup (\to G_2) \cup \mathcal{D},$$

where $\mathcal{D}$ is described in a similar fashion as earlier. Since $G \notin \to G_3$ and $G \notin \to G_2$ it follows that $G \in \mathcal{D}$ and at least one component of G does not belong to $\to G_3$. Let G_α be such a component. Then $G_\alpha \in \to G_2$ and $G_\alpha \notin \to G_3$. In addition $G_\alpha \in \to G_1$ since $G_\alpha \leq G \in \to G_1$. As a result

$$G_\alpha \in \ (\to G_1 \cap \to G_2) = (\to G_3 \cap \to G_2),$$

which implies $G_\alpha \in \rightarrow G_3$, a contradiction. Thus our initial assumption is false. It follows that Hom is a distributive lattice. □

The following two lemmas were obtained from Lemma 5.11 of [5].

Lemma 11. *An element x from a distributive lattice L is meet-irreducible in L if and only if whenever $x \geq b \wedge c$ for some elements b and c from L it follows that $x \geq b$ or $x \geq c$.*

Proof. Suppose x is meet-irreducible and $x \geq b \wedge c$. Then $x = x \vee (b \wedge c) = (x \vee b) \wedge (x \vee c)$ so that $x = x \vee b$ or $x = x \vee c$ by the meet-irreducibility of x. But then it follows that $x \geq b$ or $x \geq c$.

Suppose for the converse that from $x \geq b \wedge c$ it follows that $x \geq b$ or $x \geq c$ and that $x = b \wedge c$. Then $x \leq b$ and $x \leq c$ follow from this equation while $x \geq b$ or $x \geq c$ follows from the given condition, hence $x = b$ or $x = c$ as required. □

Lemma 12. *An element x from a distributive lattice L is join-irreducible in L if and only if whenever $x \leq b \vee c$ for some elements b and c from L it follows that $x \leq b$ or $x \leq c$.*

Proof. Suppose x is join-irreducible and $x \leq b \vee c$. Then $x = x \wedge (b \vee c) = (x \wedge b) \vee (x \wedge c)$, therefore $x = x \wedge b$ or $x = x \wedge c$, giving us $x \leq b$ or $x \leq c$.

Suppose for the converse that from $x \leq b \vee c$ it follows that $x \leq b$ or $x \leq c$ and that $x = b \vee c$. Then $x \geq b$ and $x \geq c$ hence $x = b$ or $x = c$, completing our proof. □

Next we identify join-irreducible elements in Hom and save the discussion of meet-irreducible elements in Hom for later.

Proposition 5. *The join-irreducible elements in Hom are those hom-properties $\rightarrow G$ such that G is connected.*

Proof. We first show that if G is disconnected then $\rightarrow G$ is join-reducible. Let G be a disconnected graph. If G_1 and G_2 are two components of G such that $G_1 \longrightarrow G_2$, then $G \longrightarrow (G - G_1)$ by Lemma 4. Since $(G - G_1) \longrightarrow G$ it follows that $\rightarrow G = \rightarrow(G - G_1)$. Therefore it is sufficient to consider only those disconnected graphs G whose components are mutually non-homomorphic.

Let G be a disconnected graph whose components are mutually non-homomorphic and let G_1 and G_2 be any two components of G. Now consider $\rightarrow(G - G_1)$ and $\rightarrow(G - G_2)$. The first does not contain the graph G_1 while the second does and the second does not contain the graph G_2 while the first does. Therefore these two properties are incomparable. Furthermore each is a proper subset of the property $\rightarrow G$ and

$$\rightarrow(G - G_1) \vee \rightarrow(G - G_2) = \rightarrow((G - G_1) \sqcup (G - G_2)) = \rightarrow G,$$

which, of course, implies that $\rightarrow G$ is join-reducible.

Now we prove that if G is connected, then $\rightarrow G$ is join-irreducible. Assume, for a proof by contradiction, that $\rightarrow G$ is join-reducible, then there exist graphs H and F such that $\rightarrow H, \rightarrow F \subset \rightarrow G$ and

$$\rightarrow H \vee \rightarrow F = \rightarrow G,$$

which implies $\rightarrow(H \sqcup F) = \rightarrow G$ by Proposition 4. Which, in turn, implies that $G \in \rightarrow(H \sqcup F)$. Therefore G is homomorphic to $H \sqcup F$. Since G is connected it follows by Lemma 8 that G is homomorphic to H or F. Without loss of generality let G be homomorphic to H then $G \in \rightarrow H$. Therefore $\rightarrow G \subseteq \rightarrow H$, which is a contradiction. $\square$

2.3 The core of a graph

Definition 6. [9, 2] *For a finite graph $G \in \mathcal{I}$, the* **core** *of G, denoted $C(G)$, is a subgraph H of G such that $G \longrightarrow H$ and $G \not\longrightarrow H'$ for all proper subgraphs H' of H. When $C(G) = G$ we simply say G is a* **core**.

From this definition a finite graph is not a core if and only if it is homomorphic to a proper subgraph of itself. Next we define the core of an infinite graph in $\mathcal{I}$.

Definition 7. *For a infinite graph $G \in \mathcal{I}$, the* **core** *of G, denoted $C(G)$, is a subgraph H of G that is non-isomorphic to G and is such that $G \longrightarrow H$ and $G \not\longrightarrow H'$ for all proper subgraphs H' of H.*

This definition does not allow an infinite graph $G \in \mathcal{I}$ to be a core of itself. By Lemma 3 any graph G that has a core satisfies $G' \longrightarrow C(G)$ for all subgraphs G' of G.

Examples of Cores

1. All finite complete graphs.
2. All $(k+1)$-critical graphs, $k \in \mathbb{N}$.
3. The Mycielski construct of an odd cycle.
4. The graph $K_1 \vee G$, where G is a $(k+1)$-critical graph, $k \in \mathbb{N}$.
5. The Petersen graph.

The following results were obtained from [9].

Lemma 13. *Every graph $G \in \mathcal{I}$ that is homomorphic to a finite subgraph of itself has a core.*

Proof. Let $G \in \mathcal{I}$ have a finite subgraph H such that $G \longrightarrow H$. Define $\mathcal{S}$ to be the set of all subgraphs of H that G is homomorphic to. That is $\mathcal{S} = \{H' \subseteq H \mid G \longrightarrow H'\}$. Since $H \longrightarrow H$ it follows that $\mathcal{S}$ is non-empty. The relation $\subseteq$, to be a subgraph of, is a partial order on $\mathcal{S}$, hence $\mathcal{S}$ is a poset. Since $\mathcal{S}$ is finite it follows that it has a minimal element, thus G has a core. $\square$

Not all graphs in $\mathcal{I}$ have a core. The graph $G = (K_1 \sqcup K_2 \sqcup K_3 \sqcup \ldots)$ does not have a core [9]. This is because the set of subgraphs of G that G is homomorphic to, call this set $\mathcal{S}$, has elements of the form $K_i \sqcup K_j \sqcup \ldots$, where $j > i \geq 1$. Therefore given any member $K_i \sqcup K_j \sqcup \ldots$ of $\mathcal{S}$, there exists a homomorphism from G to $K_j \sqcup \ldots$, a proper subgraph of $K_i \sqcup K_j \sqcup \ldots$. Thus $\mathcal{S}$ has no minimal element. Consequently G has no core.

There are however infinite graphs in $\mathcal{I}$ that do have cores. The graph, $K_3 \sqcup C_5 \sqcup C_7 \sqcup \ldots$, constructed by taking the disjoint union of a single copy of each odd cycle, has K_3 as it's core. The one-way infinite path $P_{\mathbb{N}}$ which has vertex set $\mathbb{N}$ and edge set $\{ij \mid j = i + 1\}$ has a core. Any subgraph of $P_{\mathbb{N}}$ that is isomorphic to K_2 is a core of $P_{\mathbb{N}}$.

Lemma 14. *If G is a finite graph then G has a core.*

Proof. Let G be a finite graph. Since $G \longrightarrow G$ it follows by Lemma 13 that $C(G)$ exists. $\square$

Lemma 15. *Let $G \in \mathcal{I}$ be such that $\omega(G) = \chi(G) = n$ for some positive integer n. Then $C(G) = K_n$.*

Proof. Let $G \in \mathcal{I}$ be a graph with $\omega(G) = \chi(G) = n$ for some positive integer n. Then G has a complete subgraph H of order n that is isomorphic to K_n. Therefore $G \longrightarrow H$ and $G \not\longrightarrow H'$, for all $H' < H$. Thus $H = C(G)$ and so $C(G) = K_n$. $\square$

The graph $K_3 \sqcup C_5 \sqcup C_7 \sqcup \ldots$, whose components have chromatic number 3, has chromatic number 3 and clique number 3. By Lemma 15, we are able to quickly arrive at the conclusion that K_3 is its core. Lemma 15 can also be used to show that all finite complete graphs are cores.

Lemma 16. *Let G be a countable graph with a finite core. If F_1 and F_2 are cores of G then they are isomorphic.*

Proof. Let G be a countable graph with a finite core and let F_1 and F_2 be cores of G.

First, we show that F_1 and F_2 are finite. Let F be a finite core of G and assume F_1 is not finite. Because F is a subgraph of G and $G \longrightarrow F_1$, it follows that $F \longrightarrow F_1$. Since F is finite it follows that all homomorphisms from F to F_1 map F onto a proper subgraph H of F_1. From $G \longrightarrow F$ and $F \longrightarrow H$ we obtain $G \longrightarrow H$. Which implies F_1 is not a core of G. This is a contradiction, thus F_1 is finite. The same argument can be applied to F_2 to show that it is finite.

Now we prove uniqueness. Since both F_1 and F_2 are subgraphs of G, there exist homomorphisms ϕ_{12} and ϕ_{21} from F_1 to F_2 and from F_2 to F_1, respectively.

We claim that both homomorphisms are surjective. Suppose, without loss of generality, that ϕ_{12} is not surjective, then F_1 is mapped onto a proper subgraph H of F_2, that is $F_1 \longrightarrow H$. Thus $G \longrightarrow H$, implying F_2 is not a core, a contradiction.

By this result and the finiteness of said cores, we obtain $|V(F_1)| = |V(F_2)|$. Therefore ϕ_{12} and ϕ_{21} are injective, and are thus bijective. But any homomorphism from a core of G to a core of G must preserve not only adjacency but also non-adjacency. Consequently F_1 and F_2 are isomorphic, proving uniqueness. $\square$

Lemma 17. *If $G \in \mathcal{I}$ has a finite core, then $C(G)$ is an induced subgraph of G.*

Proof. Suppose this is not so, then there exists a graph $G \in \mathcal{I}$ with a finite core that is not an induced subgraph of G. Then $G[V(C(G))]$ and $C(G)$ have a finite number of edges. In addition $G[V(C(G))]$ has more edges than $C(G)$ and $G[V(C(G))] \longrightarrow C(G)$. Let ϕ be a homomorphism from $G[V(C(G))]$ to $C(G)$, then ϕ preserves all edges of $G[V(C(G))]$, therefore at least two edges in $G[V(C(G))]$ are preserved by one edge in $C(G)$. This means two vertices of $G[V(C(G))]$ are mapped to the same vertex in $C(G)$. Therefore ϕ maps $G[V(C(G))]$ to a proper subgraph of $C(G)$. Let H be this proper subgraph, then from $C(G) \longrightarrow G[V(C(G))]$ and $G[V(C(G))] \longrightarrow H$ we obtain $C(G) \longrightarrow H$, which implies $G \longrightarrow H$. Clearly this is a contradiction. $\square$

The core of a graph need not be connected. As an example, let G be the disconnected graph whose one component is K_3 and whose other component is the Mycielski construct of C_5. Then both components of G are cores and none is homomorphic to the other. Therefore $C(G) = G$.

Lemma 18. *Let $G, H \in \mathcal{I}$ be such that $C(G)$ and $C(H)$ exist. Then $G \longrightarrow H$ if and only if $C(G) \longrightarrow C(H)$.*

Proof. Let $G, H \in \mathcal{I}$ be such that $C(G)$ and $C(H)$ exist.

Suppose $G \longrightarrow H$, then $C(G) \longrightarrow H$, since $C(G) \subseteq G$. This, together with $H \longrightarrow C(H)$, yields $C(G) \longrightarrow C(H)$.

Now assume $C(G) \longrightarrow C(H)$. From $G \longrightarrow C(G)$ we have $G \longrightarrow C(H)$. Since $C(H) \longrightarrow H$ it follows that $G \longrightarrow H$. $\square$

2.4 Equivalence classes

Definition 8. [1] *Call two graphs $G, H \in \mathcal{I}$ equivalent, denoted $G \sim H$, if $G \longrightarrow H$ and $H \longrightarrow G$.*

We quickly show that $\sim$ is an equivalence relation on $\mathcal{I}$. Let $G \in \mathcal{I}$, then $G \longrightarrow G$ therefore $G \sim G$, thus $\sim$ is a reflexive relation. Given $G, H \in \mathcal{I}$ such that $G \sim H$, it follows that $G \longrightarrow H$ and $H \longrightarrow G$ therefore $H \sim G$, hence $\sim$ is a symmetric relation. Now let $G_1, G_2, G_3 \in \mathcal{I}$ be such that $G_1 \sim G_2$ and $G_2 \sim G_3$. Then $G_1 \longrightarrow G_2$, $G_2 \longrightarrow G_3$, $G_3 \longrightarrow G_2$ and $G_2 \longrightarrow G_1$. By Lemma 3 we have $G_1 \longrightarrow G_3$ and $G_3 \longrightarrow G_1$, thus $G_1 \sim G_3$. Therefore $\sim$ is a transitive relation.

It follows that, for all $G \in \mathcal{I}$, the set $[G] = \{H \in \mathcal{I} \mid H \sim G\}$ is the equivalence class of G. Let $\mathbb{E} = \{[G] \mid G \in \mathcal{I}\}$. The following are some of the members of $\mathbb{E}$.

1 $[K_1] = \{G \in \mathcal{I} \mid E(G) = \emptyset\}$,

2 $[K_2] = \{G \in \mathcal{I} \mid \chi(G) = 2\}$,

3 $[K_n] = \{G \in \mathcal{I} \mid \chi(G) = n$ and $\omega(G) = n\}$ for integers $n \geq 3$.

Definition 9. *For any $G, H \in \mathcal{I}$, $[G] \longrightarrow [H]$ if and only if $G \longrightarrow H$.*

We show that this definition yields a well-defined relation, independent of the choice of representatives. Let $G_1, G_2, H_1, H_2 \in \mathcal{I}$ and suppose that $[G_1] = [G_2]$ and $[H_1] = [H_2]$. We intend to show that $G_1 \longrightarrow H_1 \iff G_2 \longrightarrow H_2$. Without loss of generality let $G_1 \longrightarrow H_1$. Since $[G_1] = [G_2]$ and $[H_1] = [H_2]$ it follows that $G_1 \sim G_2$ and $H_1 \sim H_2$. Therefore $G_2 \longrightarrow G_1$ and $H_1 \longrightarrow H_2$. By Lemma 3 we obtain $G_2 \longrightarrow H_2$.

Lemma 19. *The pair $\langle \mathbb{E}, \longrightarrow \rangle$ is a partially ordered set.*

Proof. Let $[G], [H], [F] \in \mathbb{E}$. Since $G \longrightarrow G$ it follows that $[G] \longrightarrow [G]$. If $[G] \longrightarrow [H]$ and $[H] \longrightarrow [G]$ then $G \longrightarrow H$ and $H \longrightarrow G$, therefore $G \sim H$ hence $[G] = [H]$. Now suppose $[G] \longrightarrow [H]$ and $[H] \longrightarrow [F]$, then $G \longrightarrow H$ and $H \longrightarrow F$, therefore, by Lemma 3, $G \longrightarrow F$, giving us $[G] \longrightarrow [F]$. □

Theorem 3. *The pair* $\langle \mathbb{E}, \longrightarrow \rangle$ *is a lattice, where*

$$[G] \vee [H] = [G \sqcup H]$$

and

$$[G] \wedge [H] = [G \times H]$$

for all $[G], [H] \in \mathbb{E}$.

Proof. Let $[G], [H] \in \mathbb{E}$, then $G \longrightarrow (G \sqcup H)$ and $H \longrightarrow (G \sqcup H)$. From this we get $[G] \longrightarrow [G \sqcup H]$ and $[H] \longrightarrow [G \sqcup H]$, thus $[G \sqcup H]$ is an upper bound of $\{[G], [H]\}$. Suppose $[F]$ is an upper bound of $\{[G], [H]\}$. Since $[G] \longrightarrow [F]$ and $[H] \longrightarrow [F]$ it follows that there exist homomorphisms γ_1 and γ_2 from G to F, and from H to F, respectively. Let $\gamma \colon V(G \sqcup H) \longrightarrow V(F)$ be the mapping defined, for all $x \in V(G \sqcup H)$, by

$$\gamma(x) = \begin{cases} \gamma_1(x) & \text{if} \quad x \in V(G); \\ \gamma_2(x) & \text{if} \quad x \in V(H). \end{cases}$$

Then γ is a homomorphism from $G \sqcup H$ to F, hence $[G \sqcup H] \longrightarrow [F]$. Proving that $[G \sqcup H]$ is the join of $[G]$ and $[H]$.

From $(G \times H) \longrightarrow G$ and $(G \times H) \longrightarrow H$ we arrive at $[G \times H] \longrightarrow [G]$ and $[G \times H] \longrightarrow [H]$. Therefore $[G \times H]$ is a lower bound of $\{[G], [H]\}$. Suppose $[F]$ is a lower bound of $\{[G], [H]\}$, then $[F] \longrightarrow [G]$ and $[F] \longrightarrow [H]$, therefore there exist homomorphisms φ_G and φ_H from F to G and from F to H, respectively. Then the mapping $\varphi \colon V(F) \longrightarrow V(G \times H)$ defined, for all $x \in V(F)$, by

$$\varphi(x) = (\varphi_G(x), \varphi_H(x)),$$

is a homomorphism from F to $G \times H$, therefore $[F] \longrightarrow [G \times H]$, proving that $[G \times H]$ is the meet of $[G]$ and $[H]$. $\square$

Let the mapping $\phi : Hom \longrightarrow \mathbb{E}$ be defined as $\phi(\to G) = [G]$.

Theorem 4. *ϕ is a lattice isomorphism from $\langle Hom, \subseteq\rangle$ onto $\langle E, \longrightarrow\rangle$*

Proof. Let $\to G, \to H \in Hom$, then

$$\phi(\to G \vee \to H) = \phi(\to(G \sqcup H)) = [G \sqcup H] = [G] \vee [H] = \phi(\to G) \vee \phi(\to H)$$

and

$$\phi(\to G \wedge \to H) = \phi(\to(G \times H)) = [G \times H] = [G] \wedge [H] = \phi(\to G) \wedge \phi(\to H),$$

therefore ϕ preserves joins and meets, thus it is a lattice homomorphism.

It is easy to see that ϕ is onto. To see that it is one-to-one, assume $[G] = [H]$ then it follows that $G \sim H$, therefore $G \in \to H$ and $H \in \to G$, and consequently $\to G = \to H$. $\square$

Theorem 5. *$\langle \mathbb{E}, \longrightarrow\rangle$ is a distributive lattice*

Proof. We have already established that Hom is a distributive lattice. Since ϕ is a lattice homomorphism and thus preserves distributivity, it follows that $\mathbb{E}$ is a distributive lattice. $\square$

2.5 Properties of finite character

Definition 10. *A property $\mathcal{P} \in \mathbb{L}$ is of finite character (written f-c) if*

$$G \in \mathcal{P} \iff \text{all finite induced subgraphs of } G \text{ are in } \mathcal{P}.$$

Let $\mathbb{F} = \{\mathcal{P} \in \mathbb{L} \mid \mathcal{P} \text{ is of } f\text{-}c\}$. We aim to prove that $\mathbb{F}$ is sublattice of $\mathbb{L}$. To achieve this we have to prove that the intersection and the union of any two elements in $\mathbb{F}$ belong to $\mathbb{F}$. First we introduce some notation: For any graph $G \in \mathcal{I}$, let $\mathcal{F}(G) = \{F \in \mathcal{I} \mid F \leq G \text{ and } F \text{ is finite }\}$. In terms of this notation we have: $\mathcal{P}$ is of *f-c* if $G \in \mathcal{P} \iff \mathcal{F}(G) \subseteq \mathcal{P}$.

Lemma 20. *For all graphs $G \in \mathcal{I}$, $\mathcal{F}(G) \in \mathbb{L}$.*

Proof. Given any graph $H \in \mathcal{F}(G)$ and any induced subgraph F of H it follows that F is a finite induced subgraph of G, therefore $F \in \mathcal{F}(G)$ and thus $\mathcal{F}(G) \in \mathbb{L}$, completing our proof. □

Now we show that $\mathbb{F}$ is a sublattice of $\mathbb{L}$, that is, it is closed under finite intersections and finite unions. In the case of intersections we can show more than is needed.

Theorem 6. *Let $\mathbb{S}$ be any subset of $\mathbb{F}$ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$, where $k \in \mathbb{N}$, be any properties in $\mathbb{F}$. Then*
(a) $\bigcap \mathbb{S} \in \mathbb{F}$ and
(b) $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k \in \mathbb{F}$.

Proof. (a) Given any $G \in \mathcal{I}$ such that $G \in \bigcap \mathbb{S}$ it follows that $G \in \mathcal{P}$ for all $\mathcal{P} \in \mathbb{S}$. By definition of *f-c*, $\mathcal{F}(G) \subseteq \mathcal{P}$ for all $\mathcal{P} \in \mathbb{S}$, therefore $\mathcal{F}(G) \subseteq \bigcap \mathbb{S}$. Now assume $\mathcal{F}(G) \subseteq \bigcap \mathbb{S}$ for some $G \in \mathcal{I}$, then $\mathcal{F}(G) \subseteq \mathcal{P}$ for all $\mathcal{P} \in \mathbb{S}$. By definition of *f-c*, we have $G \in \mathcal{P}$ for all $\mathcal{P} \in \mathbb{S}$ therefore $G \in \bigcap \mathbb{S}$. Thus $\bigcap \mathbb{S} \in \mathbb{F}$.

(b) Let us assume that there are properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ in $\mathbb{F}$ for which $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k \notin \mathbb{F}$. Then there exists a graph H whose finite induced subgraphs are in $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k$ yet $H \notin \mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k$. But then $\mathcal{F}(H) \nsubseteq \mathcal{P}_i$ for each i, otherwise $\mathcal{F}(H) \subseteq \mathcal{P}_j$ for some j, which would imply that $H \in \mathcal{P}_j$ since $\mathcal{P}_j \in \mathbb{F}$, and in turn imply that $H \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k$.

From $\mathcal{F}(H) \nsubseteq \mathcal{P}_i$ (for each i) now follows that there exists, for each i, a graph $G_i \in \mathcal{F}(H)$ with $G_i \notin \mathcal{P}_i$. Now let $G = H[V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)]$, i.e., G is the subgraph of H induced by the union of the vertex sets of the G_i. Clearly G is finite since each $V(G_i)$ is finite, hence $G \in \mathcal{F}(H)$. Furthermore, each G_i is an induced subgraph of G.

Finally, since $G \in \mathcal{F}(H)$, we have that $G \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_k$ and hence that $G \in \mathcal{P}_j$ for some j. But then, each G_i, being an induced subgraph of G, is in this i-h property $\mathcal{P}_j$. In particular we then have, for this j, that $G_j \in \mathcal{P}_j$, a contradiction. $\square$

Given $\mathcal{P} \in \mathbb{L}$ let $c(\mathcal{P}) = \{G \in \mathcal{I} \mid \mathcal{F}(G) \subseteq \mathcal{P}\}$, that is, $c(\mathcal{P})$ is the set of all graphs whose finite induced subgraphs belong to $\mathcal{P}$.

Lemma 21. *For all $\mathcal{P} \in \mathbb{L}$, $\mathcal{P} \subseteq c(\mathcal{P})$.*

Proof. Given any $G \in \mathcal{P}$ and any finite $F \leq G$ it follows that $F \in \mathcal{P}$ since $\mathcal{P}$ is an induced-hereditary property. Therefore $\mathcal{F}(G) \subseteq \mathcal{P}$. As a consequence $G \in c(\mathcal{P})$. $\square$

Lemma 22. *For all $\mathcal{P} \in \mathbb{L}$, $c(\mathcal{P}) \in \mathbb{L}$.*

Proof. Let $G \in c(\mathcal{P})$ and $H \leq G$. Assume, to the contrary, that $H \notin c(\mathcal{P})$, then $\mathcal{F}(H) \nsubseteq \mathcal{P}$, which follows by the definition of $c(\mathcal{P})$. Since $\mathcal{F}(H) \subseteq \mathcal{F}(G)$ it follows that $\mathcal{F}(G) \nsubseteq \mathcal{P}$, which implies $G \notin c(\mathcal{P})$, a contradiction. $\square$

Lemma 23. *$\mathcal{P} \in \mathbb{F}$ if and only if $\mathcal{P} = c(\mathcal{P})$.*

Proof. Let $\mathcal{P} \in \mathbb{F}$. If $G \in c(\mathcal{P})$ then $\mathcal{F}(G) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is of f-c we have $G \in \mathcal{P}$. Therefore $c(\mathcal{P}) \subseteq \mathcal{P}$. By Lemma 21 we have $\mathcal{P} \subseteq c(\mathcal{P})$, thus $\mathcal{P} = c(\mathcal{P})$.

Now suppose, for some $\mathcal{P} \in \mathbb{L}$, that $\mathcal{P} = c(\mathcal{P})$. Let $G \in \mathcal{I}$ be such that $\mathcal{F}(G) \subseteq \mathcal{P}$ then $G \in c(\mathcal{P})$. Since $\mathcal{P} \in \mathbb{L}$ it follows that $\mathcal{F}(G) \subseteq \mathcal{P}$ for all $G \in \mathcal{P}$. Therefore $\mathcal{P} \in \mathbb{F}$. $\square$

Clearly c is a mapping from $\mathbb{L}$ into $\mathbb{L}$.

Proposition 6. *The mapping c is a closure operator on $\mathbb{L}$*

Proof. To prove that c is a closure operator we need to show that c is extensive, monotone and idempotent.

By Lemma 21, c is extensive. Suppose, for some $\mathcal{P}, \mathcal{Q} \in \mathbb{L}$, that $\mathcal{P} \subseteq \mathcal{Q}$. Select any graph $G \in c(\mathcal{P})$, then $\mathcal{F}(G) \subseteq \mathcal{P}$. Therefore $\mathcal{F}(G) \subseteq \mathcal{Q}$. By the definition of $c(\mathcal{Q})$ it follows that $G \in c(\mathcal{Q})$. Hence $c(\mathcal{P}) \subseteq c(\mathcal{Q})$ and thus c is monotone.

By the monotonicity of c we have $c(\mathcal{P}) \subseteq c(c(\mathcal{P}))$ for all $\mathcal{P} \in \mathbb{L}$. Now, if $G \in c(c(\mathcal{P}))$ then $\mathcal{F}(G) \subseteq c(\mathcal{P})$. By Lemma 23, $c(\mathcal{P}) \in \mathbb{F}$ therefore $G \in c(\mathcal{P})$, hence $c(c(\mathcal{P})) \subseteq c(\mathcal{P})$. Thus $c(c(\mathcal{P})) = c(\mathcal{P})$, proving that c is idempotent. $\square$

The following results were obtained using Proposition 7.2 in [5].

Proposition 7. *For all $\mathcal{P} \in \mathbb{L}$ we have $c(\mathcal{P}) = \bigcap\{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\}$.*

Proof. Let $\mathcal{P} \in \mathbb{L}$. Since $\mathcal{P} \subseteq c(\mathcal{R})$ for all $c(\mathcal{R}) \in \{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\}$ it follows that $c(\mathcal{P}) \subseteq c(c(\mathcal{R})) = c(\mathcal{R})$, therefore $c(\mathcal{P}) \subseteq \bigcap\{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\}$.

From $\mathcal{P} \subseteq c(\mathcal{P})$ it follows that $c(\mathcal{P}) \in \{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\}$, therefore $\bigcap\{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\} \subseteq c(\mathcal{P})$, yielding $c(\mathcal{P}) = \bigcap\{c(\mathcal{Q}) \mid \mathcal{P} \subseteq c(\mathcal{Q})\}$. $\square$

Proposition 8. *The sublattice $\mathbb{F}$ of $\mathbb{L}$ is a complete lattice with meets and joins defined as follows. For every subset $\mathbb{S}$ of $\mathbb{F}$,*

$$\bigwedge \mathbb{S} = \bigcap \mathbb{S} \text{ and } \bigvee \mathbb{S} = c(\bigcup \mathbb{S})$$

Proof. Let $\mathbb{S}$ be a subset of $\mathbb{L}$. Since $\bigcap \mathbb{S}$ is the meet of $\mathbb{S}$ in $\mathbb{L}$ it follows that to prove $\bigwedge \mathbb{S} = \bigcap \mathbb{S}$ in $\mathbb{F}$ we need only prove that $\bigcap \mathbb{S} \in \mathbb{F}$. Theorem 6 offers exactly this, thus completing the first part of our proof.

For all $\mathcal{P} \in \mathbb{S}$ we have $\mathcal{P} \subseteq \bigcup \mathbb{S} \subseteq c(\bigcup \mathbb{S})$, therefore $c(\bigcup \mathbb{S})$ is an upper bound of $\mathbb{S}$ in $\mathbb{F}$. Let $\mathcal{Q} \in \mathbb{F}$ be an upper bound of $\mathbb{S}$, then $\bigcup \mathbb{S} \subseteq \mathcal{Q}$. With the help of Proposition 6 and Lemma 23 we obtain $c(\bigcup \mathbb{S}) \subseteq c(\mathcal{Q}) = \mathcal{Q}$. Which provides us with $\bigvee \mathbb{S} = c(\bigcup \mathbb{S})$, completing our proof. $\square$

2.6 Compact properties

Definition 11. *A property $\mathcal{P} \in \mathbb{L}$ is* ***compact*** *if, for every $\mathbb{S} \subseteq \mathbb{L}$,*

$$\mathcal{P} \subseteq \bigvee \mathbb{S} \Longrightarrow \mathcal{P} \subseteq \bigvee \mathbb{T} \text{ for some finite } \mathbb{T} \subseteq \mathbb{S}$$

Lemma 24. *All finite properties in $\mathbb{L}$ are compact.*

Proof. Let $\mathcal{P} \in \mathbb{L}$ be finite and allow $\mathbb{S} \subseteq \mathbb{L}$ to be such that $\mathcal{P} \subseteq \bigvee \mathbb{S}$, then $H \in \bigvee \mathbb{S}$ for all $H \in \mathcal{P}$. Since $\bigvee \mathbb{S} = \bigcup \mathbb{S}$ it follows, for all $H \in \mathcal{P}$, that $H \in \mathcal{Q}$ for some $\mathcal{Q} \in \mathbb{S}$. From this and the finiteness of $\mathcal{P}$ there exist a finite number of properties $\mathcal{Q}$ in $\mathbb{S}$ such that $\mathcal{P}$ is a subset of the union of said properties. □

Definition 12. *Given any $\mathcal{F} \subseteq \mathcal{I}$, let*

$$(\mathcal{F}) = \{H \in \mathcal{I} \mid H \leq G \text{ for some } G \in \mathcal{F}\}.$$

We call $(\mathcal{F})$ the ***i-h property generated*** *by $\mathcal{F}$. For a graph $G \in \mathcal{I}$ we write (G) for the property $(\{G\})$, sometimes written $\leq G$.*

For all $\mathcal{F} \subseteq \mathcal{I}$, the property $(\mathcal{F})$ belongs to $\mathbb{L}$. This follows immediately from the definition. For all $\mathcal{F} \subseteq \mathcal{I}$ we have $\mathcal{F} \subseteq (\mathcal{F})$, and $\mathcal{F} = (\mathcal{F})$ if and only if $\mathcal{F} \in \mathbb{L}$. In fact, if $\wp(\mathcal{I})$ is the set of all properties then $(.) : \wp(\mathcal{I}) \longrightarrow \wp(\mathcal{I})$ is a closure operator, taking $\wp(\mathcal{I})$ into $\mathbb{L}$. The property, (G), generated by a finite graph G is a finite property and is thus a compact element in $\mathbb{L}$. Later we will show that all properties $(\mathcal{F})$ generated by a finite subset $\mathcal{F}$ of $\mathcal{I}$ are compact elements in $\mathbb{L}$. However not all elements in $\mathbb{L}$ are compact. For example, the property, $\mathcal{P} = \{K_n \mid n \in \mathbb{N}\}$, which is the set of all finite complete graphs, belongs to $\mathbb{L}$ but is not compact. To see this consider the set $\mathbb{S}$ of all properties generated by a finite complete graph, that is, $\mathbb{S} = \{(K_n) \mid n \in \mathbb{N}\}$. Since $\bigvee \mathbb{S} = \bigcup_{n \in \mathbb{N}} (K_n)$ it follows that $\mathcal{P} \subseteq \bigvee \mathbb{S}$. But clearly $\mathcal{P}$ is not a subset of the union of any finite number of elements in $\mathbb{S}$. Thus $\mathcal{P}$ is not compact.

Lemma 25. *If* $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{I}$ *then* $(\mathcal{F}_1) \subseteq (\mathcal{F}_2)$.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be subsets of $\mathcal{I}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let H be any graph in $(\mathcal{F}_1)$. Then $H \leq G$ for some $G \in \mathcal{F}_1$. By the initial hypothesis we obtain $G \in \mathcal{F}_2$, from which we arrive at $H \in (\mathcal{F}_2)$ by means of Definition 12. $\square$

Lemma 26. *If* $\mathcal{F}_1, \ldots, \mathcal{F}_n$ *are any subsets of* $\mathcal{I}$, *for some* $n \in \mathbb{N}$, *then*

$$(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n) = (\mathcal{F}_1) \cup \ldots \cup (\mathcal{F}_n)$$

Proof. Let $H \in (\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n)$, then $H \leq G$ for some $G \in \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$. Therefore $G \in \mathcal{F}_i$, for some $1 \leq i \leq n$, and thus $H \in (\mathcal{F}_i)$. As a consequence $H \in (\mathcal{F}_1) \cup \ldots \cup (\mathcal{F}_n)$, therefore $(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n) \subseteq (\mathcal{F}_1) \cup \ldots \cup (\mathcal{F}_n)$.

By Lemma 25 we obtain $(\mathcal{F}_i) \subseteq (\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n)$ for all $1 \leq i \leq n$, which gives $(\mathcal{F}_1) \cup \ldots \cup (\mathcal{F}_n) \subseteq (\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n)$. $\square$

Theorem 7. $\mathcal{P}$ *is a compact element in* $\mathbb{L}$ *if and only if* $\mathcal{P} = (\mathcal{F})$ *for some finite* $\mathcal{F} \subseteq \mathcal{I}$.

Proof. Suppose $\mathcal{P}$ is a compact property in $\mathbb{L}$. Let $\mathbb{Q} = \{(G) \,|\, G \in \mathcal{P}\}$ then $\mathcal{P} \subseteq \bigvee \mathbb{Q}$. By our initial hypothesis it follows that there exists a finite subset $\mathbb{T}$ of $\mathbb{Q}$ such that $\mathcal{P} \subseteq \bigvee \mathbb{T}$. Thus, for some positive integer n, we have $\mathbb{T} = \{(G_1), \ldots, (G_n)\}$, where $(G_i) \in \mathbb{Q}$ for all $1 \leq i \leq n$. Consequently we have

$$\mathcal{P} \subseteq \bigvee \mathbb{T} = (G_1) \vee \ldots \vee (G_n) = (G_1) \cup \ldots \cup (G_n).$$

Now let $\mathcal{F} = \{G_1, \ldots, G_n\}$ then $(\mathcal{F}) = (G_1) \cup \ldots \cup (G_n)$ by Lemma 26, therefore $\mathcal{P} \subseteq (\mathcal{F})$. Since $\mathcal{P}$ is an induced-hereditary property and $G_i \in \mathcal{P}$ for all $1 \leq i \leq n$, we obtain $(G_i) \subseteq \mathcal{P}$ for all $1 \leq i \leq n$. Therefore $(G_1) \cup \ldots \cup (G_n) \subseteq \mathcal{P}$, and thus $\mathcal{P} = (\mathcal{F})$.

Now suppose $\mathcal{P} = (\mathcal{F})$ for some finite $\mathcal{F} \subseteq \mathcal{I}$. Then, for some positive integer n, $\mathcal{F} = \{F_1, \ldots, F_n\}$, where $F_i \in \mathcal{I}$ for all $1 \leq i \leq n$. Furthermore

$\mathcal{P} = (F_1) \cup \ldots \cup (F_n)$. Select $\mathbb{S} \subseteq \mathbb{L}$ such that $\mathcal{P} \subseteq \bigvee \mathbb{S}$, then $(F_1) \cup \ldots \cup (F_n) \subseteq \bigvee \mathbb{S}$. Therefore, for all $1 \leq i \leq n$, there exists a property $\mathcal{Q}_i \in \mathbb{S}$ such that $F_i \in \mathcal{Q}_i$. Which implies $(F_i) \subseteq \mathcal{Q}_i$ for all $1 \leq i \leq n$. Therefore $\mathcal{P} = (F_1) \cup \ldots \cup (F_n) \subseteq \bigcup_{i=1}^{n} \mathcal{Q}_i$, hence $\mathcal{P}$ is compact. $\square$

Let $\mathbb{C} = \{\mathcal{P} \in \mathbb{L} \mid \mathcal{P} \text{ is compact }\}$.

Proposition 9. *The join of any two elements in $\mathbb{C}$ is compact.*

Proof. Let $\mathcal{P}, \mathcal{Q} \in \mathbb{C}$, then by Theorem 7 there exist finite $\mathcal{F}, \mathcal{G} \subseteq \mathcal{I}$ such that $\mathcal{P} = (\mathcal{F})$ and $\mathcal{Q} = (\mathcal{G})$. Therefore $\mathcal{P} \cup \mathcal{Q} = (\mathcal{F}) \cup (\mathcal{G}) = (\mathcal{F} \cup \mathcal{G})$ by Lemma 26. Since $\mathcal{F} \cup \mathcal{G}$ is a finite subset of $\mathcal{I}$ it follows by Theorem 7 that $\mathcal{P} \cup \mathcal{Q} \in \mathbb{C}$, therefore $\mathcal{P} \vee \mathcal{Q} = \mathcal{P} \cup \mathcal{Q} \in \mathbb{C}$. $\square$

As we shall see in the following proposition, $\mathbb{C}$ is not a sublattice of $\mathbb{L}$ but is instead a join-semi-sublattice of $\mathbb{L}$.

Proposition 10. *There are compact elements in $\mathbb{C}$ whose intersection is not compact.*

Proof. We offer two compact elements in $\mathbb{C}$ whose intersection is not in $\mathbb{C}$. Let G be the disjoint union of all finite complete graphs, that is $G = K_1 \sqcup K_2 \sqcup K_3 \sqcup \ldots$. Then G belongs to $\mathcal{I}$.

Now let $\mathcal{P} = (G)$ and $\mathcal{Q} = (K_{\aleph_0})$. By Theorem 7 the properties $\mathcal{P}$ and $\mathcal{Q}$ are compact. We claim that $\mathcal{P} \cap \mathcal{Q} = \{K_n \mid n \in \mathbb{N}\}$. For all $n \in \mathbb{N}$, $K_n \leq G$ and $K_n \leq K_{\aleph_0}$, therefore $\{K_n \mid n \in \mathbb{N}\} \subseteq \mathcal{P} \cap \mathcal{Q}$. Let $H \in \mathcal{P} \cap \mathcal{Q}$, then $H \in \mathcal{Q}$, therefore H is an induced subgraph of $K_{\aleph_0}$, which implies H is a complete graph. Since H is a complete graph that also belongs to $\mathcal{P}$ it follows that H is finite, therefore $H = K_n$ for some $n \in \mathbb{N}$. Which gives us $\mathcal{P} \cap \mathcal{Q} \subseteq \{K_n \mid n \in \mathbb{N}\}$. From which we obtain $\mathcal{P} \cap \mathcal{Q} = \{K_n \mid n \in \mathbb{N}\}$.

Earlier, in this section, we showed that the property $\{K_n \mid n \in \mathbb{N}\}$ is not compact, thus $\mathcal{P} \cap \mathcal{Q}$ is not compact. This completes our proof. $\square$

2.7 Universal Graphs

Definition 13. [11]

We say a graph G is a ***weakly universal graph of*** *a property $\mathcal{P} \in \mathbb{L}$ if, for all $H \in \mathcal{P}$, there exists an injective homomorphism from H to G. If $G \in \mathcal{P}$ then we say G is a* ***weakly universal graph in*** *$\mathcal{P}$.*

Since we are dealing with unlabelled graphs it follows that a graph G is a weakly universal graph of a property $\mathcal{P}$ if and only if, for all $H \in \mathcal{P}$, H is a subgraph of G.

Definition 14. [11]

We say a graph G is a ***universal graph of*** *a property $\mathcal{P} \in \mathbb{L}$ if, for all $H \in \mathcal{P}$, there exists an injective homomorphism φ from H to G such that, for all $u, v \in V(H)$, if $uv \notin E(H)$ then $\varphi(u)\varphi(v) \notin E(H)$. If $G \in \mathcal{P}$ then we say G is a* ***universal graph in*** *$\mathcal{P}$.*

The homomorphism φ described in Definition 14 preserves non-edges. Therefore a graph G is a universal graph of a property $P \in \mathbb{L}$ if and only if, for all $H \in \mathcal{P}$, H is an induced subgraph of G.

Lemma 27. [3] *Every property $\mathcal{P} \in \mathbb{L}$ has a universal graph.*

Proof. Let $\mathcal{P} \in \mathbb{L}$ and let G be the disjoint union of all graphs in $\mathcal{P}$. Then $H \leq G$ for all $H \in \mathcal{P}$, hence G is a universal graph of $\mathcal{P}$. □

Lemma 27 does not guarantee that a property $\mathcal{P} \in \mathbb{L}$ has a countable universal graph. Since our concern lies with countable graphs we are interested in a countable universal graph for a property $\mathcal{P} \in \mathbb{L}$. In addition we desire this universal graph to be in $\mathcal{P}$.

Lemma 28. *A compact property $\mathcal{P} \in \mathbb{L}$ has a countable universal graph.*

Proof. Let $\mathcal{P} \in \mathbb{L}$ be compact. Then there exists a finite subset $\mathcal{F}$ of $\mathcal{I}$ such that $\mathcal{P} = (\mathcal{F})$. Let $G = \bigsqcup\{G' \mid G' \in \mathcal{F}\}$, then $G \in \mathcal{I}$ and $H \leq G$ for all $H \in \mathcal{P}$. $\square$

Before pursuing universal graphs of properties in $\mathbb{L}$ we introduce the Rado graph R [11], which will come in handy in our construction of universal graphs. The Rado graph is that countable graph whose vertex set $V(R) = \{1, 2, \ldots\}$ and whose edge set $E(R)$ is described as follows. For any two vertices x and y in $V(R)$ where $x < y$, x is adjacent to y if and only if y has a 1 in the xth position of its binary expansion. Rado showed that this graph is a universal graph in $\mathcal{I}$.

For each graph $G \in \mathcal{I}$ let U'_G be the graph with $V(U'_G) = V(G) \times V(K_{\aleph_0})$ and edge set

$$E(U'_G) = \{(x, u)(y, v) \mid xy \in E(G) \text{ and } (x, u), (y, v) \in V(U'_G)\}.$$

For each $x \in V(G)$ we call the set $\{(x, v) \mid v \in V(K_{\aleph_0})\}$ the **tower of U'_G generated by** x. For each $i \in V(K_{\aleph_0})$ we call the set $\{(y, i) \mid y \in V(G)\}$ the **ith level** of U'_G. Then each tower of U'_G is an independent set of vertices and two vertices $p, q \in V(U'_G)$ are adjacent if and only if p and q belong to towers generated by adjacent vertices of G.

Lemma 29. *For all graphs $G \in \mathcal{I}$, U'_G is a weakly universal graph in the property $\to G$.*

Proof. Select any $G \in \mathcal{I}$. Our first order of business will be to prove that $U'_G \in \to G$. Since a countable union of countable sets is countable we have $U'_G \in \mathcal{I}$. Let φ be the mapping that delivers every tower of U'_G to the vertex in G that generated it. Given any adjacent vertices in U'_G it follows that these vertices belong to towers of U'_G generated by adjacent vertices in G, therefore φ maps said vertices to adjacent vertices in G. Thus φ is a homomorphism from U'_G to G, which gives us $U'_G \in \to G$. Next we prove that $H \subseteq U'_G$ for all

$H \in \to G$. Let $H \in \to G$ then there exists a homomorphism, φ, from H to G. For each $x \in V(G)$ let $W_x = \{z \in V(H) \mid \varphi(z) = x\}$. Then $\{W_x \mid x \in V(G)\}$ is a partition of $V(H)$ into independent sets. Since, for all $x \in V(G)$, W_x is countable there exists a one-to-one mapping from W_x into the tower of U'_G generated by x. For each $x \in V(G)$, let f_x be such a mapping. Now let $\gamma\colon V(H) \longrightarrow V(U'_G)$ be the mapping defined, for all $z \in V(H)$, by

$$\gamma(z) = f_x(z) \text{ if } z \in W_x.$$

Then γ is one-to-one. In addition, given any adjacent vertices z_1 and z_2 in H it follows that $\varphi(z_1) \neq \varphi(z_2)$ and $\varphi(z_1)\varphi(z_2) \in E(G)$. Therefore $\gamma(z_1)$ and $\gamma(z_2)$ belong to towers of U'_G generated by adjacent vertices. As a consequence $\gamma(z_1)$ is adjacent to $\gamma(z_2)$ in U'_G. Thus γ is a one-to-one homomorphism from H to U'_G, which completes our proof. □

For $n \in \mathbb{N}$ let $U_n = K_n \times R$, the product of K_n and R, where $V(K_n) = \{1, \ldots, n\}$. For each $1 \leq i \leq n$ we call the set $\{(i, v) \mid v \in V(R)\}$ the ***i*th tower** of U_n, denoted T_i. For each $i \in V(R)$ we call the set $\{(x, i) \mid x \in V(K_n)\}$ the ***i*th level** of U_n. Since the vertex set of U_n is a finite union of countable sets it follows that $U_n \in \mathcal{I}$. We aim to show that, for all $n \in \mathbb{N}$, U_n is a universal graph in $\to K_n$.

Lemma 30. *For all $n \in \mathbb{N}$, $U_n \in \to K_n$.*

Proof. Let $n \in \mathbb{N}$. We have already remarked that U_n is countable. What remains is to show that $U_n \longrightarrow K_n$. Let φ be the mapping which, for each $1 \leq i \leq n$, delivers the ith tower of U_n to the vertex i of K_n. Then φ is a homomorphism from U_n to K_n, yielding $U_n \in \to K_n$. □

Proposition 11. *For all $n \in \mathbb{N}$, U_n is a universal graph in $\to K_n$.*

Proof. Fix a positive integer n. By Lemma 30, we have $U_n \in \to K_n$. We are required to show that, for all $H \in \to K_n$, there exists a one-to-one homomorphism, from H to U_n, that preserves non-edges. Allow H to be a graph in $\rightarrow K_n$. By Theorem 1 of [11] the exists a one-to-one homomorphism, φ, from H to R that preserves non-edges. Since $H \longrightarrow K_n$, there exists a partition $W_1, \ldots, W_n$ of $V(H)$ into n colour classes. Let the mapping $\gamma\colon V(H) \longrightarrow V(U_n)$ be defined, for all $z \in V(H)$, by

$$\gamma(z) = (i, \varphi(z)) \text{ if } z \in W_i,$$

where $1 \leq i \leq n$. Given any two $x, y \in V(H)$, $\gamma(x) \neq \gamma(y)$ since $\varphi(x) \neq \varphi(y)$. Thus γ is one-to-one. Let x and y be any two vertices in H, then there exist integers $1 \leq i, j \leq n$ such that $x \in W_i$ and $y \in W_j$. In addition $\varphi(x) \neq \varphi(y)$. If $xy \in E(H)$ then $\varphi(x)\varphi(y) \in E(R)$ and $i \neq j$, therefore $\gamma(x) = (i, \varphi(x))$ is adjacent to $\gamma(y) = (j, \varphi(y))$ in U_n. If $xy \notin E(H)$ then $\varphi(x)\varphi(y) \notin E(R)$ and therefore $\gamma(x) = (i, \varphi(x))$ is not adjacent to $\gamma(y) = (j, \varphi(y))$ in U_n. Thus γ is a one-to-one homomorphism that preserves non-edges, giving us the desired result. □

We state two useful features of the graph U_n, both of which are reminiscent of the *extension property* of the Rado graph.

Theorem 8. *(a) For every two finite, disjoint subsets U and V of $V(U_n)$ of which no two elements, one from U and the other from V, are on the same level of the towers of U_n and for which $U \cap T_i = \emptyset$ for some $1 \leq i \leq n$, there is a vertex in the ith tower that is adjacent to all members of U and non-adjacent to all members of V.*

(b) For every finite subset W of $V(U_n)$ for which $W \cap T_i = \emptyset$ for some $1 \leq i \leq n$, there is a vertex in the ith tower if U_n that is adjacent to every vertex of W.

Proof. First we prove (a). Assume the subsets U and V of $V(U_n)$ satisfy the conditions stated above. Let $U' = \{u \mid (j, u) \in U\}$ and $V' = \{v \mid (j, v) \in V\}$. Let (i, w) be the vertex of U_n where w is a positive integer whose binary expansion has a 1 in every position $u \in U'$, a 0 in every position $v \in V$ and a 1 in some position $x > y$ for every $y \in U' \cup V'$. Then (i, w) has all the required properties. The proof of (b) is similar to that of (a). $\square$

Theorem 9. *For a positive integer n and a graph $G \in \mathcal{I}$ the following statements are equivalent.*

(a) $G \leq U_n$

(b) $G \longrightarrow U_n$

(c) $G \longrightarrow K_n$

(d) *G is n-colourable*

Proof. Clearly (d) implies (c). By Proposition 11 we obtain (a) and (b) from (c). Both imply $G \in \rightarrow K_n$ which in turn implies (d). $\square$

We remark that condition (a) of the above theorem is seemingly more demanding than condition (b). This theorem provides us with an inducing supergraph for all n-colourable graphs, allowing us to study n-colourable graphs by limiting our investigation to U_n.

For each graph $G \in \mathcal{I}$ let $U_G = G \times R$. Then, for each $x \in V(G)$, we call the set $\{(x, v) \mid v \in V(R)\}$ the **tower of U_G generated by** x. For each $i \in V(R)$ we call the set $\{(y, i) \mid y \in V(G)\}$ the **ith level** of U_G. Then every tower and every level of U_G is an independent set of vertices.

Lemma 31. *For all $G \in \mathcal{I}$, $U_G \in \rightarrow G$.*

Proof. Let $G \in \mathcal{I}$, then U_G is countable. Let φ be the mapping which delivers each tower of U_G to the vertex in G that generated it. Then φ is a homomorphism from U_G to G, thus $U_G \in \rightarrow G$. $\square$

Proposition 12. *For all $G \in \mathcal{I}$, U_G is a universal graph in $\rightarrow G$.*

Proof. The proof of this is similar to that of the proof of Proposition 11. □

Next we provide a new proof of a known result by Szekeres and Wilf. One that utilises the structure of the universal graph U_n. For a finite graph H we shall write $\delta(H)$ for the minimum degree of the vertices of H.

Theorem 10 (Szekeres and Wilf, [12]). *For a finite graph G we have*

$$\chi(G) \leq 1 + max_{H \leq G}\delta(H),$$

where the maximum is taken over all induced subgraphs H of G.

Proof. Let G be a graph of order $p \in \mathbb{N}$ and let $n = 1 + max_{H \leq G}\delta(H)$. Then every induced subgraph of G has a vertex of degree at most $n-1$. We now label, in reverse order, the p vertices of G as $v_p, v_{p-1}, \ldots, v_1$: Let v_p be any vertex in $V(G)$ with degree in G at most $n-1$; let v_{p-1} be any vertex in $V(G) \setminus \{v_p\}$ with degree in $G[V(G) \setminus \{v_p\}]$ at most $n-1$; let, in general, for any integer j with $1 \leq j \leq p-1$, v_{p-j} be any vertex in $V(G) \setminus \{v_p, v_{p-1}, \ldots, v_{p-j+1}\}$ with degree in $G[V(G) \setminus \{v_p, v_{p-1}, \ldots, v_{p-j+1}\}]$ at most $n-1$; until v_1 is the single vertex left in $V(G) \setminus \{v_p, v_{p-1}, \ldots, v_2\}$ with degree in $G[\{v_1\}]$ at most $n-1$, in fact, sadly, only zero. We construct a homomorphism $f\colon G \longrightarrow U_n$ by applying (b) of Theorem 8 repeatedly to define $f(v_1), \ldots, f(v_p)$ recursively. Let $f(v_1)$ be any vertex of U_n. Now suppose that for some integer $k < p$ that $f(v_1), \ldots, f(v_k)$ have been defined in such a manner that edges are preserved, that is, for every $1 \leq i, j \leq k$, $v_iv_j \in E(G)$ implies $f(v_i)f(v_j) \in E(U_n)$.
Let $W = \{f(v_i) \mid i \leq k \text{ and } v_iv_{k+1} \in E(G)\}$. Then, by the manner in which we chose $v_1, \ldots, v_p$, $|W| < n-1$. Thus there is a tower T_i of U_n such that $W \cap T_i = \emptyset$. Applying (b) of Theorem 8 we can find a vertex w of T_i which is adjacent to each vertex in W. By defining $f(v_{k+1}) = w$ we have a vertex of U_n with desired properties. Using this recursive procedure, the proof can be completed. □

3 The Hedetniemi Conjecture

3.1 The Hedetniemi Conjecture and meet irreducible elements in the lattice Hom

In this section we state the Hedetniemi Conjecture and reveal it's link to meet-irreducible elements in Hom.

Conjecture 1 (Hedetniemi, [7]). *For all finite graphs G and H we have*

$$\chi(G \times H) = min\{\chi(G), \chi(H)\}.$$

We have already established that $(G \times H) \longrightarrow G$ and $(G \times H) \longrightarrow H$ for all $G, H \in \mathcal{I}$, hence, if G and H are of finite chromatic number, it follows that $\chi(G \times H) \leq \chi(G)$ and $\chi(G \times H) \leq \chi(H)$. Therefore $\chi(G \times H) \leq min\{\chi(G), \chi(H)\}$ for all countable graphs G and H with finite chromatic number.

Let $Hom^* = \{\to G \mid \chi(G) \text{ is finite }\}$, then $Hom^* \subseteq Hom$. Let $\to G$ and $\to H$ be properties in Hom^*, then $G \sqcup H$ and $G \times H$ are of finite chromatic number, therefore $\to(G \sqcup H)$ and $\to(G \times H)$ belong to Hom^*. From this it follows that Hom^* is a sublattice of Hom.

Lemma 32. *The following are equivalent:*

(a) For all finite graphs G and H we have $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.

(b) For all $G, H \in \mathcal{I}$ with finite chromatic number we have $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.

Proof. It is easy to see that (b) implies (a), so we prove that (a) implies (b). Suppose (a) and let G and H be countable graphs with finite chromatic number. By the Compactness Theorem [6] there exist finite induced subgraphs G' and H' of G and H, respectively, with $\chi(G') = \chi(G)$ and $\chi(H') = \chi(H)$.

Our assumption guarantees that $\chi(G' \times H') = min\{\chi(G'), \chi(H')\}$. Hence $min\{\chi(G), \chi(H)\} = min\{\chi(G'), \chi(H')\} = \chi(G' \times H') \leq \chi(G \times H)$ since $G' \times H' \leq G \times H$. Since the reverse inequality is true for all G and H, the required equation follows. $\square$

We shift our focus to meet-irreducible elements in Hom. The following are known meet-irreducible elements in Hom: $\rightarrow K_1, \rightarrow K_2, \rightarrow K_3$ and $\rightarrow K_4$. Having not managed to characterize the meet-irreducible elements in Hom, we offer a theorem that we hope brings us a step closer to identifying these elements.

Theorem 11. *The Hedetniemi Conjecture is true if and only if* $\rightarrow K_n$ *is meet-irreducible in* Hom^* *for all* $n \in \mathbb{N}$.

Proof. Let the Hedetniemi Conjecture be true and assume that, for some n, $\rightarrow K_n$ is meet-reducible. Then there exist countable graphs G and H, of finite chromatic number, such that $\rightarrow K_n \subset \rightarrow G, \rightarrow H$ and
$\rightarrow(G \times H) = \rightarrow G \wedge \rightarrow H = \rightarrow K_n$. Which implies $(G \times H) \longrightarrow K_n$ and $K_n \longrightarrow (G \times H)$. Thus $n = \chi(G \times H) = \min\{\chi(G), \chi(H)\}$, which suggests $\chi(G) = n$ or $\chi(H) = n$. Without loss of generality let $\chi(G) = n$, then G is homomorphic to K_n and consequently $\rightarrow G \subseteq \rightarrow K_n$, a contradiction.

For all $n \in \mathbb{N}$, let $\rightarrow K_n$ be meet-irreducible in Hom^*, and assume that, for some n, there exist graphs G and H such that
$n = \chi(G \times H) < min\{\chi(G), \chi(H)\}$. Then

$$\rightarrow K_n \subset \rightarrow(G \sqcup K_n) \text{ and } \rightarrow K_n \subset \rightarrow(H \sqcup K_n).$$

Now

$$(G \sqcup K_n) \times (H \sqcup K_n) = (G \times H) \sqcup (G \times K_n) \sqcup (K_n \times H) \sqcup (K_n \times K_n)$$

where

$$\chi(G \times H) = \chi(G \times K_n) = \chi(K_n \times H) = \chi(K_n \times K_n) = n,$$

since $\chi(K_n \times F) = n$ for all graphs F with $\chi(F) \geq n$. From this it follows that $\chi((G \sqcup K_n) \times (H \sqcup K_n)) = n$, therefore
$\to((G \sqcup K_n) \times (H \sqcup K_n)) \subseteq \to K_n$. Since $K_n \leq K_n \times K_n \leq (G \sqcup K_n) \times (H \sqcup K_n)$, it follows that K_n is homomorphic to $(G \sqcup K_n) \times (H \sqcup K_n)$.
Thus $\to K_n \subseteq \to((G \sqcup K_n) \times (H \sqcup K_n))$ and as a consequence

$$\to(G \sqcup K_n) \wedge \to(H \sqcup K_n) = \to((G \sqcup K_n) \times (H \sqcup K_n)) = \to K_n,$$

which implies that $\to K_n$ is meet-reducible, a contradiction. $\square$

3.2 More equivalent forms of the Hedetniemi Conjecture

Theorem 12. *The following statements are equivalent:*

(a) *For all finite graphs $G, H \in \mathcal{I}$ we have $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.*

(b) *For all $G, H \in \mathcal{I}$ with finite chromatic number we have $\chi(G \times H) = min\{\chi(G), \chi(H)\}$*

(c) *For all $n \in \mathbb{N}$ the hom-property $\rightarrow K_n$ is meet-irreducible in $\langle Hom^*, \wedge, \vee\rangle$.*

(d) *For every two hom-properties $\rightarrow G$ and $\rightarrow H$ in Hom^* if $\rightarrow K_n = \rightarrow G \cap \rightarrow H$ then $\rightarrow K_n = \rightarrow G$ or $\rightarrow K_n = \rightarrow H$.*

(e) *For all $n \in \mathbb{N}$ if $\rightarrow G \cap \rightarrow H \subseteq \rightarrow K_n$ then $\rightarrow G \subseteq \rightarrow K_n$ or $\rightarrow H \subseteq \rightarrow K_n$.*

(f) *For all $n \in \mathbb{N}$ and every two n-critical graphs G and H there exists an n-critical graph F such that $F \longrightarrow G$ and $F \longrightarrow H$.*

(g) *Given $n \in \mathbb{N}$ and any graphs G and H with $\chi(G) = \chi(H) = n + 1$ and $\omega(G) = \omega(H) = n$, there exists a graph F of chromatic number $n + 1$ that is homomorphic to G and H.*

Proof. By Lemma 32 we have established that (a) and (b) are equivalent. By Theorem 11 we have (b) and (c) are equivalent, therefore (a) and (c) are equivalent. The equivalence of (c) and (d) follows by the definition of meet-irreducibility in a lattice.

We show that (b) and (e) are equivalent. Assume (b), then $\rightarrow G \cap \rightarrow H \subseteq \rightarrow K_n$ implies $\chi(G \times H) \leq n$. By our assumption $\chi(G) \leq n$ or $\chi(H) \leq n$. Therefore $G \longrightarrow K_n$ or $H \longrightarrow K_n$. Thus $\rightarrow G \subseteq \rightarrow K_n$ or $\rightarrow H \subseteq \rightarrow K_n$. Now assume (e). Given $G, H \in \mathcal{I}$ we have $\rightarrow (G \times H) \subseteq \rightarrow K_{\chi(G \times H)}$. By (e) we obtain $G \longrightarrow K_{\chi(G \times H)}$ or $H \longrightarrow K_{\chi(G \times H)}$. From which we obtain $\chi(G) \leq \chi(G \times H)$ or $\chi(H) \leq \chi(G \times H)$. This, of course, implies $\chi(G \times H) \geq min\{\chi(G), \chi(H)\}$. Therefore $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.

Now we show (b) and (f) are equivalent. Let $G, H \in \mathcal{I}$ be n-critical and assume (b). Then $\chi(G \times H) = n$, therefore $G \times H$ has an n-critical subgraph F. Since $F \longrightarrow (G \times H) \longrightarrow G$ and $(G \times H) \longrightarrow H$, it follows by Lemma 3 that $F \longrightarrow G$ and $F \longrightarrow H$. Now assume (f) and let $G, H \in \mathcal{I}$. Without loss of generality let $\chi(G) \leq \chi(H)$. Then, for all positive integers $n \leq \chi(G)$, there exist n-critical subgraphs of G and H. By (f) there exists, for each n, an n-critical graph F that is homomorphic to G and H, and as a result is homomorphic to $G \times H$. Thus $\chi(G \times H) \geq n$ for all positive integers $n \leq \chi(G)$. As a consequence $\chi(G \times H) = \chi(G) = min\{\chi(G), \chi(H)\}$.

To show that (c) and (g) are equivalent we begin by showing that if (c) does not hold then (g) does not hold. So assume, for some $n \in \mathbb{N}$, that $\rightarrow K_n$ is meet-reducible. Then there exist graphs G and H such that $\rightarrow K_n = \rightarrow G \wedge \rightarrow H$ with $\rightarrow K_n \subset \rightarrow G$ and $\rightarrow K_n \subset \rightarrow H$. Let G' and H' be subgraphs of G and H, respectively, with chromatic number $n+1$. We claim $\omega(G'), \omega(H') < n+1$. Without loss of generality assume $\omega(G') = n+1$, then we have $H' \longrightarrow K_{n+1} \longrightarrow G' \longrightarrow G$ and $H' \longrightarrow H$. Which implies $H' \in \rightarrow G \wedge \rightarrow H = \rightarrow K_n$, and in turn implies $H' \longrightarrow K_n$. Surely, this is a contradiction since $\chi(H') = n+1$. Let $G^* = G' \sqcup K_n$ and $H^* = H' \sqcup K_n$. Then G^* and H^* satisfy the conditions in (g). We argue that by our initial hypothesis there can not exist a graph F of chromatic number $n+1$ that is homomorphic to G^* and H^*. Suppose such a graph F existed, then F would be homomorphic to G and H since $G^* \longrightarrow G$ and $H^* \longrightarrow H$. Which would imply $F \in (\rightarrow G \wedge \rightarrow H) = \rightarrow K_n$, and thus imply $F \longrightarrow K_n$. Which again is a contradiction since $\chi(F) = n+1$. Now we show that (c) implies (g). Assume $\rightarrow K_n$ is meet-irreducible for all $n \in \mathbb{N}$ and let G and H be any graphs in $\mathcal{I}$ with $\chi(G) = \chi(H) = n+1$ and $\omega(G) = \omega(H) = n$. Then $K_n \longrightarrow G$ and $K_n \longrightarrow H$ therefore $K_n \in \rightarrow(G \times H)$. Which produces

$\rightarrow K_n \subseteq \rightarrow(G \times H) = \rightarrow G \wedge \rightarrow H$. Therefore $\rightarrow K_n \subset \rightarrow(G \times H)$ since $\rightarrow K_n$ is meet-irreducible. Thus there exists a graph $F \in \rightarrow(G \times H)$ which does not belong to $\rightarrow K_n$. Therefore $\chi(F) = n + 1$ and F is homomorphic to G and H. This completes our proof. $\square$

3.3 A new proof of the Duffus-Sands-Woodrow Theorem

The following result was obtained by Duffus, Sands and Woodrow [7].

Theorem 13. *Given any $n \in \mathbb{N}$ and any two connected graphs, G and H, with $\chi(G), \chi(H) > n$ and $\omega(G) = \omega(H) = n$, then $\chi(G \times H) > n$.*

We present an alternative proof to this theorem. Unlike the proof by said authors, which employs colouring functions or that of El-Zaher and Sauer [8], which utilizes the exponential graph method, ours employs the construction of a graph that is homomorphic to G and H.

From the lattice Hom we observed that given $n \in \mathbb{N}$ and graphs G and H,

$$\chi(G \times H) > n \iff \exists\, F \in \rightarrow G \cap \rightarrow H \text{ such that } \chi(F) > n,$$

where $F \in \rightarrow G \cap \rightarrow H$ simply means F is homomorphic to G and H.

The theorem below is therefore equivalent to Theorem 13.

Theorem 14. *Given any $n \in \mathbb{N}$ and any two connected graphs G and H with $\chi(G) = \chi(H) = n + 1$ and $\omega(G) = \omega(H) = n$, there exists a graph F of chromatic number $n + 1$ that is homomorphic to both G and H.*

Proof. Our proof is as follows. Let G and H be as specified. We isolate subgraphs of G and H that will enable us to find two graphs which on application of the Hajós construction will yield the graph F. The biggest challenge will be in proving that our construct is homomorphic to G and H. To ease this exercise the two subgraphs we are in search of shall be obtained by 'trimming' the graphs G and H of as many vertices and edges as we can whilst preserving the characteristics mentioned above.

Let G' and H' be $(n+1)$-critical internal subgraphs of G and H, respectively. Furthermore, let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be the vertices of a complete internal subgraph of G and H, respectively. Then let G'' be the internal subgraph of G with vertex set $V(G'') = V(G') \cup \{v_1, \ldots, v_n\}$ and edge set $E(G'') = E(G') \cup E(G[\{v_1, \ldots, v_n\}])$. Similarly, let H'' be the internal subgraph of H with vertex set $V(H'') = V(H') \cup \{w_1, \ldots, w_n\}$ and edge set $E(H'') = E(H') \cup E(H[\{w_1, \ldots, w_n\}])$.

We make two distinctions. If $V(G') \cap \{v_1, \ldots, v_n\} \neq \emptyset$, we say G'' is of ***Type 1***. If $V(G') \cap \{v_1, \ldots, v_n\} = \emptyset$, we say G'' is of ***Type 2***.

If G'' is of type 1 then it is a connected subgraph of G with chromatic number $n+1$ and clique number n. We rename G''. Let $G^* = G''$. On the other hand, if G'' is of type 2 then it is a disconnected subgraph of G with chromatic number $n+1$ and clique number n. We require a connected subgraph of G, so we find a suitable replacement for G''. Since G is connected it follows that there exists a path in G connecting a vertex in $V(G')$ to a vertex in $\{v_1, \ldots, v_n\}$. Let P be a path of minimum length that achieves this connection. Then only the end vertices of P belong to $V(G'')$. Let G^* be the graph obtained after including P into the graph G''. Then G^* is a connected $(n+1)$-chromatic subgraph of G with clique number n, and is of type 2.

We do the same thing for H'', giving us either a graph H^* of type 1 or of type 2.

We are almost ready to implement Hajós's construction. But first we note the following. Either (i) both G^* and H^* are of type 1, (ii) G^* and H^* are of different types or (iii) both G^* and H^* are of type 2. We tackle each separately.

Case (i). Since both G^* and H^* are of type 1, there exist $1 \leq i, j \leq n$ such that v_i is in G' and w_j is in H'. There also exist $x \in V(G')$ and $y \in V(H')$ such that $xv_i \in E(G')$ and $yw_j \in E(H')$, otherwise this would imply v_i and w_j are isolated vertices of an $(n+1)$-critical graph, which cannot be. We apply Hajós's construction on G' and H' in order to obtain F. We do this by first deleting the edges xv_i and yw_j, then we identify the vertices v_i and w_j

to form the vertex z, and finally we introduce the the edge xy. The graph F is $(n+1)$-chromatic since the Hajós construct of two $(n+1)$-critical graphs is $(n+1)$-critical.

Now we prove that F is homomorphic to G^* and H^*. Observe that the induced subgraph $F[(V(G') \setminus \{v_i\}) \cup \{z\}]$ of F, call it Q, is isomorphic to $G' - xv_i$. Similarly, $F[(V(H') \setminus \{w_j\}) \cup \{z\}]$, the induced subgraph of F, which we will call R, is isomorphic to $H' - yw_j$. Therefore Q and R are n-chromatic and are thus homomorphic to $H^*[\{w_1 \dots w_n\}]$ and $G^*[\{v_1 \dots v_n\}]$. In addition all homomorphisms from Q to $H^*[\{w_1 \dots w_n\}]$ map the vertices x and z to the same vertex in $H^*[\{w_1 \dots w_n\}]$. Similarly all homomorphisms from R to $G^*[\{v_1 \dots v_n\}]$ map the vertices y and z to the same vertex in $G^*[\{v_1 \dots v_n\}]$.

Let γ_1 be any homomorphism from Q to $H^*[\{w_1 \dots w_n\}]$, such that $\gamma_1(z) = w_j$. Then $\gamma_1(x) = w_j$. Now let γ_2 be the mapping $\gamma_2 \colon V(F) \setminus V(Q) \longrightarrow V(H^*)$ defined, for all $a \in V(F) \setminus V(Q)$, by $\gamma_2(a) = a$. Then γ_2 preserves all edges of the subgraph of F induced by $V(F) \setminus V(Q)$. Finally let γ be the mapping $\gamma \colon V(F) \longrightarrow V(H^*)$ defined by

$$\gamma(a) = \begin{cases} \gamma_1(a) & \text{if } a \in V(Q)\ ; \\ \gamma_2(a) & \text{if } a \in V(F) \setminus V(Q)\ . \end{cases}$$

Then γ is a homomorphism, where the edge xy in F is preserved by the edge yw_j in H^*.

We can replicate the above construction to obtain a homomorphism from F to G^*.

Case (ii). Without loss of generality, assume H^* is of type 2. We construct an $(n+1)$-chromatic graph, H^{***}, which on application of Hajós's construction with G' will yield a graph F that is homomorphic to G^* and H^*. To achieve this we shall, first, need to revisit the path P that was instrumental in constructing H^*. Already, we have ascertained that P connects a vertex

$z \in V(H')$ to w_j, for some integer $1 \le j \le n$, in such a manner that all internal vertices of P do not belong to H''. Therefore $P = (z, z_1, \ldots, z_{k-1}, w_j)$ for some $k \in \mathbb{N}$. Consider H', the $(n+1)$-critical subgraph of H^*, whose vertex set can be partitioned into $n+1$ colour classes $V_1, \ldots, V_{n+1}$ where $V_{n+1} = \{z\}$. Let H^{**} be the graph with vertex set

$$V(H^{**}) = V(H') \setminus \{z\} \;\cup\; \{u_1, \ldots, u_n\}$$

and edge set

$$E(H^{**}) = E(H' - z) \cup \{xu_i \mid xz \in E(H') \text{ and } x \notin V_i\}.$$

Clearly $V_1 \cup \{u_1\}, \ldots, V_n \cup \{u_n\}$ is a partition of $V(H^{**})$ into n colour classes, therefore H^{**} is n-chromatic. A valuable property of H^{**} is that every successful proper n-colouring of H^{**} requires any two elements of the set $\{u_1, \ldots, u_n\}$ to be assigned different colours. Suppose, to the contrary, that the vertex set of H^{**} can in fact be partitioned into n colour classes, $W_1, \ldots, W_n$, where W_1 contains at least two elements of $\{u_1, \ldots, u_n\}$. Without loss of generality let $u_1, u_2 \in W_1$. Then every vertex that is adjacent to z in H' is adjacent to u_1 or u_2 in H^{**} and thus all such vertices do not belong to W_1. For all integers $1 \le i \le n$, let W_i^* be the set W_i after the removal of all members of $\{u_1, \ldots, u_n\}$, then $W_1^* \cup \{z\}, W_2^* \ldots, W_n^*$ is a partition of $V(H')$ into n colour classes. This, of course, is a contradiction.

Now we utilise H^{**} to construct another graph. Let H^{***} be the graph with vertex set

$$V(H^{***}) = V(H^{**}) \;\cup\; \{u\} \;\cup \Big(\bigcup_{1 \le i \le k-1} \{u_{i,1}, \ldots, u_{i,n}\} \Big),$$

where u and the $u_{i,j}$ are completely new vertices, and edge set

$$E(H^{***}) = E(H^{**}) \cup \bigcup_{i,j,s} \left[\{u_i u_{1,j}\} \cup \{u_{s,i} u_{s+1,j}\} \cup \{u_{k-1,j} u\} \right],$$

where $1 \leq i \neq j \leq n$ and $1 \leq s \leq k-2$. The graph H^{***} is $(n+1)$-chromatic and is homomorphic to H^*. In addition $H^{***} - uu_{k-1,1}$ is n-chromatic.

We quickly prove these claims. Suppose, to the contrary, that H^{***} is n-chromatic. Then we can partition $V(H^{***})$ into n colour classes $C_1, \ldots, C_n$. By an earlier mentioned property of H^{**} no two elements of the set $\{u_1, \ldots u_n\}$ belong to the same colour class. We can therefore assume without loss of generality that $u_i \in C_i$ for all $1 \leq i \leq n$. Thus $u_{1,i} \in C_i$ for all $1 \leq i \leq n$. As a consequence $u_{s,i} \in C_i$ for all $1 \leq i \leq n$ and $2 \leq s \leq k-1$. But $u \in C_j$ for some $1 \leq j \leq n$, thus C_j contains adjacent vertices u and $u_{k-1,j}$. This is a contradiction. Hence H^{***} is $(n+1)$-chromatic.

By the aforementioned properties of H^{**} we can partition $V(H^{**})$ into n colour classes $D_1, \ldots, D_n$ such that $u_i \in D_i$ for all $1 \leq i \leq n$.
Let $D_1^* = D_1 \cup \{u_{1,1}, u_{2,1}, \ldots, u_{k-1,1}\} \cup \{u\}$ and
$D_i^* = D_i \cup \{u_{1,i}, u_{2,i}, \ldots, u_{k-1,i}\}$ for all $2 \leq i \leq n$ then $D_1^*, \ldots, D_n^*$ is a partition of $V(H^{***} - uu_{k-1,1})$ into n colour classes. Therefore $H^{***} - uu_{k-1,1}$ is n-chromatic. Furthermore $H^{***} - uu_{k-1,1} \longrightarrow G^*[\{v_1 \ldots v_n\}]$ and all homomorphisms γ from $H^{***} - uu_{k-1,1}$ to $G^*[\{v_1 \ldots v_n\}]$ are such that $\gamma(u_{k-1,1}) = \gamma(u)$.

The mapping $\phi\colon V(H^{***}) \longrightarrow V(H^*)$ defined by

$$\phi(a) = \begin{cases} a & \text{if } a \in H^{***}[V(H') \setminus \{z\}]\ ; \\ z & \text{if } a = u_i \text{ for some } 1 \leq i \leq n; \\ z_t & \text{if } a = u_{t,s} \text{ for some } 1 \leq s \leq n; \\ w_j & \text{if } a = u\ . \end{cases}$$

where $1 \leq t \leq k-1$, is a homomorphism from H^{***} to H^*.

Since G^* is of type 1 there exists an integer $1 \leq i \leq n$ such that v_i is in G'. In addition there exist $x \in V(G')$ such that $xv_i \in E(G')$. We implement Hajós's construction on G' and H^{***}. We delete the edges xv_i of G' and $uu_{k-1,1}$ of H^{***}, then identify the vertices u and v_i to form a new vertex y. Lastly we make x and $u_{k-1,1}$ adjacent. This new graph we call F.

Next we show that $F \longrightarrow G^*$ and $F \longrightarrow H^*$. The subgraph of F induced by the vertex set $V(H^{***}) \setminus \{u\} \cup \{y\}$, call it R, is isomorphic to $H^{***} - uu_{k-1,1}$. It follows that $R \longrightarrow G^*[\{v_1 \dots v_n\}]$ and every homomorphism, γ, from R to $G^*[\{v_1 \dots v_n\}]$ is such that $\gamma(u_{k-1,1}) = \gamma(y)$.

Let γ_1 be a homomorphism from R to $G^*[\{v_1 \dots v_n\}]$ such that $\gamma_1(y) = v_i$. Then $\gamma_1(u_{k-1,1}) = v_i$. Let γ_2 be the mapping $\gamma_2\colon V(F) \setminus V(R) \longrightarrow V(G^*)$ defined as follows. For all $a \in V(F) \setminus V(R)$, $\gamma_2(a) = a$. Then, let γ be the mapping $\gamma\colon V(F) \longrightarrow V(G^*)$ defined, for all $a \in V(F)$, by

$$\gamma(a) = \begin{cases} \gamma_1(a) & \text{if } a \in V(R) \ ; \\ \gamma_2(a) & \text{if } a \in V(F) \setminus V(R) \ . \end{cases}$$

Then γ is a homomorphism, where the edge $xu_{k-1,1}$ in F is preserved by the edge xv_i in G^*.

Similarly the subgraph of F induced by the vertex set $V(G') \setminus \{v_i\} \cup \{y\}$, call it Q, is isomorphic to $G' - xv_i$. Therefore $Q \longrightarrow H^*[\{w_1 \dots w_n\}]$ and every homomorphism, φ, from Q to $H^*[\{w_1 \dots w_n\}]$ is such that $\varphi(x) = \varphi(y)$.

Let φ_1 be a homomorphism from Q to $H^*[\{w_1 \dots w_n\}]$ such that $\varphi_1(x) = w_j$. Then $\varphi_1(y) = w_j$. Let φ_2 be the mapping $\varphi_2\colon V(F) \setminus V(Q) \longrightarrow V(H^*)$ defined, for all $a \in V(F) \setminus V(Q)$, by $\varphi_2(a) = \phi(a)$, where previously $\phi\colon V(H^{***}) \longrightarrow V(H^*)$ and $V(F) \setminus V(Q) \subseteq V(H^{***})$. Finally let φ be the mapping $\varphi\colon V(F) \longrightarrow V(H^*)$ defined, for all $a \in V(F)$, by

$$\varphi(a) = \begin{cases} \varphi_1(a) & \text{if } a \in V(Q) \ ; \\ \varphi_2(a) & \text{if } a \in V(F) \setminus V(Q) \ . \end{cases}$$

Then φ is a homomorphism, where the edge $xu_{k-1,1}$ in F is preserved by the edge $w_j z_{k-1}$ in H^*.

Case (iii). Similar to the situation for H^* in Case (ii), there exist paths P_1 and P_2 belonging to G^* and H^*, respectively, with P_1 of length k, and P_2 of length ℓ. In addition P_1 connects a vertex e of G' to a vertex v_i in a manner such that the internal vertices of P_1 do not belong to G' or $G^*[\{v_1, \dots, v_n\}]$.

Similarly P_2 connects a vertex f of $H^*[V(H')]$ to a vertex w_j in a manner that all internal vertices of P_2 do not belong to H' or to $H^*[\{w_1,\ldots,w_n\}]$. Let $P_1=(e,z_1,\ldots,z_{k-1},v_i)$ and $P_2=(f,z'_1,\ldots,z'_{\ell-1},w_j)$. Construct graphs G^{**} and H^{**} in a fashion similar to that of H^{**} in Case (ii) with

$$V(G^{**})=V(G')\setminus\{e\}\ \cup\ \{u_1,\ldots,u_n\}$$

and

$$V(H^{**})=V(H')\setminus\{f\}\ \cup\ \{u'_1,\ldots,u'_n\}.$$

Finally, construct graphs G^{***} and H^{***} with vertex sets

$$V(G^{***})=V(G^{**})\ \cup\ \{u\}\ \cup\Big(\bigcup_{1\le i\le k-1}\{u_{i,1},\ldots,u_{i,n}\}\Big)$$

and

$$V(H^{***})=V(H^{**})\ \cup\ \{u'\}\ \cup\Big(\bigcup_{1\le i\le \ell-1}\{u'_{i,1},\ldots,u'_{i,n}\}\Big),$$

in the same manner that H^{***} of Case (ii) was constructed. Then G^{***} and H^{***} are $(n+1)$-chromatic graphs that are homomorphic to G^* and H^*, respectively. Furthermore $G^{***}-\{uu_{k-1,1}\}$ and $H^{***}-\{u'u'_{\ell-1,1}\}$ are n-chromatic.

Delete the edges $uu_{k-1,1}$ of G^{***} and $u'u'_{\ell-1,1}$ of H^{***} and identify the vertices u and u' to form the vertex y. Lastly make $u_{k-1,1}$ and $u'_{\ell-1,1}$ adjacent. This new graph we call F. The mapping $\phi_1\colon V(G^{***}-u)\longrightarrow V(G^*)$ defined by

$$\phi_1(x)=\begin{cases} x & \text{if } x\in G^{***}[V(G')\setminus\{e\}]\ ;\\ e & \text{if } x=u_i \text{ for some } 1\le i\le n;\\ z_t & \text{if } x=u_{t,s} \text{ for some } 1\le s\le n.\end{cases}$$

where $1\le t\le k-1$, is a homomorphism from $G^{***}-u$ to G^*. Whilst the mapping $\phi_2\colon V(H^{***}-u')\longrightarrow V(H^*)$ defined by

$$\phi_2(x)=\begin{cases} x & \text{if } x\in H^{***}[V(H')\setminus\{f\}]\ ;\\ f & \text{if } x=u'_i \text{ for some } 1\le i\le n;\\ z'_t & \text{if } x=u'_{t,s} \text{ for some } 1\le s\le n.\end{cases}$$

where $1 \leq t \leq \ell - 1$, is a homomorphism from $H^{***} - u'$ to H^*. We shall use these two homomorphisms to prove that $F \longrightarrow G^*$ and $F \longrightarrow H^*$.

The subgraph of F induced by the vertex set $V(H^{***}) \setminus \{u'\} \cup \{y\}$, call it R, is isomorphic to $H^{***} - u'u'_{\ell-1,1}$. Thus $R \longrightarrow G^*[\{v_1 \dots v_n\}]$ and every homomorphism, γ, from R to $G^*[\{v_1 \dots v_n\}]$ is such that $\gamma(u'_{\ell-1,1}) = \gamma(y)$.

Let γ_1 be a homomorphism from R to $G^*[\{v_1 \dots v_n\}]$ such that $\gamma_1(y) = v_i$. Then $\gamma_1(u'_{\ell-1,1}) = v_i$. Now, let γ be the mapping $\gamma \colon V(F) \longrightarrow V(G^*)$ defined, for all $a \in V(F)$, by

$$\gamma(a) = \begin{cases} \gamma_1(a) & \text{if } a \in V(R) \; ; \\ \phi_1(a) & \text{if } a \in V(F) \setminus V(R) \; . \end{cases}$$

Then γ is a homomorphism, where the edges $yu_{k-1,i}$, $2 \leq i \leq n$, and the edge $u'_{\ell-1,1}u_{k-1,1}$ of F are preserved by the edge $v_i z_{k-1}$ in G^*.

The subgraph of F induced by the vertex set $V(G^{***}) \setminus \{u\} \cup \{y\}$, call it Q, is isomorphic to $G^{***} - uu_{k-1,1}$. Therefore $Q \longrightarrow H^*[\{w_1 \dots w_n\}]$ and every homomorphism, φ, from Q to $H^*[\{w_1 \dots w_n\}]$ is such that $\varphi(u_{k-1,1}) = \varphi(y)$.

Let φ_1 be a homomorphism from Q to $H^*[\{w_1 \dots w_n\}]$ such that $\varphi_1(y) = w_j$. Then $\varphi_1(u_{k-1,1}) = w_j$. Let φ be the mapping $\varphi \colon V(F) \longrightarrow V(H^*)$ defined, for all $a \in V(F)$, by

$$\varphi(a) = \begin{cases} \varphi_1(a) & \text{if } a \in V(Q) \; ; \\ \phi_2(a) & \text{if } a \in V(F) \setminus V(Q) \; . \end{cases}$$

Then φ is a homomorphism, where the edges $yu'_{\ell-1,i}$, $2 \leq i \leq n$, and the edge $u_{k-1,1}u'_{\ell-1,1}$ of F are preserved by the edge $w_j z'_{\ell-1}$ in H^*. $\square$

Lemma 33. *The following statements are equivalent:*

(a) *For all $n \in \mathbb{N}$ and every two n-critical graphs G and H there exists an n-critical graph F such that $F \longrightarrow G$ and $F \longrightarrow H$.*

(b) *For all $n \in \mathbb{N}$ and every two connected n-critical graphs G and H there exists an n-critical graph F such that $F \longrightarrow G$ and $F \longrightarrow H$.*

Proof. Clearly (a) implies (b). Thus our task is to prove that (b) implies (a). So assume (b) and allow G and H to be n-critical graphs where at least one is disconnected. Then G and H have n-critical components G' and H', respectively. Therefore, by (b), there exists an n-critical graph F such that $F \longrightarrow G'$ and $F \longrightarrow H'$. Since $G' \longrightarrow G$ and $H' \longrightarrow H$ it follows by Lemma 3 that $F \longrightarrow G$ and $F \longrightarrow H$. This completes our proof. $\square$

With the aid of Theorem 12 we have that (b) of Lemma 33 is equivalent to the Hedetniemi Conjecture. Which implies that we need not consider disconnected graphs when proving the Hedetniemi Conjecture. Thus Theorem 14 is a special case of the Hedetniemi Conjecture. Therefore what has been proven so far is that for all positive integers $n > 1$ and every two connected n-critical graphs G and H with $\omega(G) = \omega(H) = n - 1$ there exists an n-critical graph F that is homomorphic to G and H. What remains, in order to prove the Hedetniemi Conjecture, is to show that there exists such a graph F for the following two scenarios:

(a) $\omega(G) = n - 1$ and $\omega(H) < n - 1$

(b) $\omega(G) < n - 1$ and $\omega(H) < n - 1$.

Our hope is that the technique employed in the proof of Theorem 14 could be applied, with possibly some modification, in proving the result for the two above mentioned scenarios. We should mention that Duffus, Sands and Woodrow, in [7], pondered an approach to proving the Hedetniemi Conjecture through a technique that utilises Hajós's construction. Various note-worthy techniques have been used in proving special cases of Hedetniemi's conjecture and one more does not hurt.

References

[1] Bruce L. Bauslaugh, List-Compactness of Directed Graphs, Graphs and Combinatorics, **17** 2001 17-38

[2] M. Borowiecki, I. Broere, M. Frick, G. Semanišin and P. Mihók, A survey of hereditary properties of graphs, Discussiones Mathematicae Graph Theory, **17** 1997 5–50

[3] I. Broere and J. Heidema and P. Mihok, Constructing Universal Graphs for Induced-Hereditary Graph Properties, Mathematica Slovaca, **63** 2013 191-200

[4] G. Chartrand, L. Lesniak and P. Zhang, Graphs and digraphs, Fifth Edition, CRC Press, Boca Raton, 2011

[5] B.A Davey and H.A Priestley, Introduction to Lattices and Order, Second Edition, Cambridge University Press, Cambridge, 2002

[6] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations, Indag. Math., **13** 1951 371-373

[7] D. Duffus and B. Sands and R.E Woodrow, On the Chromatic Number of the Product of Graphs, Journal of Graph Theory, **9** 1985 487-495

[8] M. El-Zahar and N.W Sauer, The Chromatic Number of the Product of Two 4-Chromatic Graphs is 4, Combinatorica, **5** 1985 121-126

[9] P. Hell and J. Nešetřil, The core of a graph, Discrete Mathematics, **109** 1992 117-126

[10] Tommy R. Jensen and Gordon F. Royle, Hajós Constructions of Critical Graphs, Journal of Graph Theory, **30** 1998 37-50

[11] R. Rado, Universal graphs and universal functions, Acta Arithmetica, **9** 1964 331-340

[12] G. Szekeres and H. S. Wilf, An inequality for the chromatic number of a graph, Journal of Combinatorial Theory, **4** 1968 1-3