Extension of results about p-summing operators to Lipschitz p-summing maps and their respective relatives

by

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DECLARATION

I, the undersigned declare that the dissertation, which I hereby submit for the degree **Magister Scientiae in Mathematics** at the University of Pretoria is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other university.

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DATE: July 2013.

SUMMARY

In this dissertation, we study about the extension of results of p-summing operators to Lipschitz p-summing maps and their respective relatives for $1 \leq p < \infty$.

Lipschitz p-summing and Lipschitz p-integral maps are the non-linear version of (absolutely) p-summing and p-integral operators respectively. The p-summing operators were first introduced in the paper [13] by Pietsch in 1967 for $1 < p < \infty$ and for $p = 1$ go back to Grothendieck which he introduced in his paper [9] in 1956. They were subsequently taken on with applications in 1968 by Lindenstrauss and Pelczynski as contained in [12] and these early developments of the subject are meticulously presented in [6] by Diestel et al.

While the absolutely summing operators (and their relatives, the integral operators) constitute important ideals of operators used in the study of the geometric structure theory of Banach spaces and their applications to other areas such as Harmonic analysis, their confinement to linear theory has been found to be too limiting. The paper [8] by Farmer and Johnson is an attempt by the authors to extend known useful results to the non-linear theory and their first interface in this case has appealed to the uniform theory, and in particular to the theory of Lipschitz functions between Banach spaces. We find analogues for p-summing and p-integral operators for $1 \leq p < \infty$. This then divides the dissertation into two parts.

In the first part, we consider results on Lipschitz p-summing maps. An application of Bourgain's result as found in [2] proves that a map from a metric space X into $\ell_1^{2^X}$ with $|X| = n$ is Lipschitz 1-summing. We also apply the non-linear form of Grothendieck's Theorem to prove that a map from the space of continuous real-valued functions on $[0, 1]$ into a Hilbert space is Lipschitz p-summing for some $1 \leq p < \infty$. We also prove an analogue of the 2-Summing Extension Theorem in the non-linear setting as found in [6] by showing that every Lipschiz 2-summing map admits a Lipschiz 2-summing extension. When X is a separable Banach space which has a subspace isomorphic to ℓ_1, we show that there is a Lipschitz p-summing map from X into $\mathbb{R}^2$ for $2 \leq p < \infty$ whose range contains a closed set with empty interior. Finally, we prove that if a finite metric space X of cardinality 2^k is of supremal metric type 1, then every Lipschitz map from X into a Hilbert space is Lipschitz p-summing for some $1 \leq p < \infty$.

In the second part, we look at results on Lipschitz p-integral maps. The main result is that the natural inclusion map from ℓ_1 into ℓ_2 is Lipschitz 1-summing but not Lipschitz 1-integral.

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Part II: LIPSCHITZ p-INTEGRAL MAPS RESULTS

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INTRODUCTION

The aim of this dissertation as said in the summary is to study the extension of results about p-summing operators to Lipschitz p-summing maps and their respective relatives for $1 \leq p < \infty$. This study is basically a question of general nature first posed by the authors of the paper [8] as an open problem 6, namely

Problem 6. What results about p-summing operators have analogues for Lipschitz p-summing operators?

Several special cases of Problem 6 are isolated in the same paper as problems 1 to 5, namely:

1. (Problem 1) Is there a composition formula for Lipschitz p-summing operators? That is, do we have $\pi_p^L(TS) \leq \pi_r^L(T)\pi_s^L(S)$, where $\frac{1}{p} \leq (\frac{1}{r} + \frac{1}{s}) \wedge 1$?

2. (Problem 2) Is every Lipschitz 2-summing operator Lipschitz 2-integral?

3. (Problem 3) When Y is a Banach space and X is a finite metric space, what is the dual of $\Pi_p^L(X, Y)$?

4. (Problem 4) Is every Lipschitz mapping from an L_1 space to a Hilbert space Lipschitz 1-summing? Is every Lipschitz mapping from $C(K)$ space to a Hilbert space Lipschitz 2-summing?

5. (Problem 5) If $T : X \to Y$ is Lipschitz, is $\pi_p^L(T)$ the supremum of $\pi_p^L(TS)$ as S ranges over all mappings from finite subsets of $\ell_{p'}$ into X having Lipschitz constant at most one?

The paper [8] by Farmer and Johnson is an attempt by the authors to extend known useful results to the non-linear theory and their first interface in this case has appealed to the uniform theory, and in particular to the theory of Lipschitz functions between Banach spaces.

The study of ideals of absolutely p-summing operators and their relatives touches on quite a broad spectrum of related issues in the geometric, analytic and measure-theoretic underpinnings related to various properties of Banach spaces and including the Randon-Nikodym property of Banach spaces and the stability questions. It is not intended that all the problems stated above be addressed at the same time or in their order of appearance.

NOTATION

Throughout this dissertation, we will assume all our metric spaces to be pointed metric spaces, that is, each one has a special point designated by 0. We will use the usual convention, $\mathbb{R}$ and $\mathbb{C}$ will denote the real or complex numbers respectively. We shall use $\mathbb{K}$ to represent $\mathbb{R}$ or $\mathbb{C}$. B_X will denote the closed unit ball of our space X (metric or Banach space). Therefore, we summarize some definitions from [6], [15], and [3].

Bounded linear maps between Banach spaces are referred to as *operators*. Otherwise, the word *map* or *mapping* will have a possibly unbounded non-linear connotation. The collection $\mathcal{L}(X, Y)$ of all operators $u : X \to Y$ is a Banach space with respect to the norm

$$\|u\| = \sup_{x \in B_X} \|u(x)\|.$$

A Banach space operator $u : X \to Y$ is said to be *isometric* if $\|u(x)\| = \|x\|$ for all $x \in X$; it is an *isometry* if it is also onto.

The dual of a Banach space X will be denoted by $X^* := \mathcal{L}(X, \mathbb{K})$; its typical element will be denoted by x^*, and for $x \in X$, we shall write $\langle x^*, x\rangle = x^*(x)$ (or $\langle x, x^*\rangle$) for the action of x^* on x.

A subset K of X^* is said to be a *norming subset* if it has the property that $\|x\| = \sup\{|f(x)| : f \in K\}$ for all $x \in X$.

A sequence (x_n) in X is *norm-null* if $\lim_n \|x_n\| = 0$. Similarly, a sequence (x_n) in X is *weakly-null* if $\lim_n x^*(x_n) = 0$ for all $x^* \in X^*$.

If x_n is a sequence in X, then (x_n) is *norm Cauchy* if and only if given strictly increasing sequences (j_n) and (k_n) of positive integers, the sequence $(x_{k_n} - x_{j_n})$ is norm null. Similarly, if x_n is a sequence in X, then (x_n) is *weakly Cauchy* if and only if given strictly increasing sequences (j_n) and (k_n) of positive integers, the sequence $(x_{k_n} - x_{j_n})$ is weakly null.

Denote by $\ell_p^{\text{strong}}(X)$ the set of all sequences (x_n) in X such that $(\|x_n\|) \in \ell_p$, a vector space under pointwise operations with a natural norm given by

$$\|(x_n)\|_p^{\text{strong}} := \Big(\sum_n \|(x_n)\|^p\Big)^{\frac{1}{p}}.$$

Denote by $\ell_p^{\text{weak}}(X)$ the set of all sequences (x_n) in X such that $(x^*(x_n)) \in \ell_p$ for every $x^* \in X^*$, a vector space under pointwise operations with a norm given by

$$\|(x_n)\|_p^{\text{weak}} := \sup\Big\{\Big(\sum_n |x^*(x_n)|^p\Big)^{\frac{1}{p}} : x^* \in B_{X^*}\Big\}.$$

A Banach space Z is *injective* if whenever Y_0 is a subspace of a Banach space Y, any $u \in \mathcal{L}(Y_0, Z)$ has an extension $\tilde{u} \in \mathcal{L}(Y, Z)$ with $\|u\| = \|\tilde{u}\|$.

Let (X,δ) and (Y,d) be metric spaces. A map $f : X \to Y$ is said to be *Lipschitz* if there exists a constant $C \geq 0$ (we use the convention, $\frac{0}{0} = 0$) such that

$$d(f(x), f(y)) \leq C \cdot \delta(x,y)$$

for all $x, y \in X$ where the notation $d(f(x), f(y)) = \|f(x) - f(y)\|$ is the distance from $f(x)$ to $f(y)$ in Y and also $\delta(x,y) = \|x - y\|$ is the distance from x to y in X. The least such C is called the *Lipschitz constant* of f and is denoted by $\mathrm{Lip}(f)$. We will also write

$$\mathrm{Lip}(f) = \sup_{x,y \in X} \left\{ \frac{d(f(x), f(y))}{\delta(x,y)} : \delta(x,y) \leq 1, \text{ and } x \neq y \right\}.$$

Sometimes reference will be made to a metric space X without specifying the metric. We will denote by $\mathcal{L}ip(X,Y)$ the set of all Lipischitz functions from X to Y.

Two finite metric spaces X and Y are called *C-isomorphic* if there is a map $\varphi : X \to Y$ such that $\mathrm{Lip}(\varphi) \cdot \mathrm{Lip}(\varphi^{-1}) \leq C$.

Let $C_2^n = \{(x_1, \ldots, x_n) | x_i \in \{0,1\}\} = \{0,1\}^n$. For every pair $x = (x_j)_{j=1}^n$, $x' = (x'_j)_{j=1}^n$ in C_2^n, the *Hamming metric* is defined by $h(x,x') = \sharp\{i | x_i \neq x'_i\}$. Sometimes, $C_2^n = \{-1,1\}^n$ is more convenient to use.
Whenever $C_2^n = \{0,1\}^n$, the Hamming metric coincides with the standard ℓ_1 metric $d(x,x') = \sum_{i=1}^n |x_i - x'_i|$.

Let $1 \leq p \leq \infty$. Call the metric space (C_2^n, δ_p) the *ℓ_p n-cube* (or ℓ_p^n -cube) if $C_2^n = \{0,1\}^n$ and

$$\delta_p(x,x') = \left(\sum_{i=1}^n |x_i - x'_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i - x'_i| \right)^{\frac{1}{p}}$$

for any pair x, x' in C_2^n. The ℓ_1 n-cube is often called the Hamming cube.

Let (X,δ) be a metric space with cardinality 2^n ($|X| = 2^n$) and consider a one to one map $\varphi : C_2^n \to X$. The map φ give some way of ordering the elements of X by n-dimensional binary vectors. Let $x = (x_j)_{j=1}^n$ be a point in C_2^n and denote the opposite point on the given cube by $x^c = (1 - x_j)^n$. A *diagonal* in X is defined as the unordered pair $(\varphi(x), \varphi(x^c))$ and its *length* is defined by

$$\mathrm{diag}_\varphi(x) = \delta(\varphi(x), \varphi(x^c)).$$

Denoted by D the set of all diagonals. Then, $|D| = 2^{n-1}$. The same notation will be used for a diagonal and its length.

Every unordered pair of points (x,x') in C_2^n differing in one binary coordinate only defines an *edge* in X. The *length* of the edge (usually just called 'edge') is defined by

$$\mathrm{edge}_\varphi(x,x') = \delta(\varphi(x), \varphi(x')).$$

Denote by E the set of all edges. Then, $|E| = n2^{n-1}$.

If two edges $(\varphi(x_1), \varphi(x_2))$ and $(\varphi(x'_1), \varphi(x'_2))$ share one point, they are said to be *connected*. For example, if $\varphi(x_2) = \varphi(x'_1)$.

A set of n-connected edges from $\varphi(x)$ to $\varphi(x^c)$ defines a *path* belonging to $\mathrm{diag}_\varphi(x)$.

1 PRELIMINARIES ON LIPSCHITZ p-SUMMING MAPS

1.1 DEFINITIONS AND ELEMENTARY PROPERTIES

In this chapter, we use the standard notation as can be found in [6], and [8].

Definition 1.1.1
Suppose X and Y are Banach spaces. A linear operator $u : X \to Y$ is *p-summing* for $1 \leq p < \infty$ precisely when there is a constant $C \geq 0$ such that for $x_1, \cdots, x_n \in X$ and regardless of the natural number n, we have

$$\left(\sum_{i=1}^{n} \|ux_i\|^p\right)^{\frac{1}{p}} \leq C \cdot \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}}.$$

The least C for which the above inequality holds will be denoted by $\pi_p(u)$. We denote by $\Pi_p(X, Y)$, the set of all p-summing operators from X into Y.

Inspired by this useful concept, Farmer and Johnson introduced in [8] the following definition;

Definition 1.1.2
Suppose X and Y are metric spaces and $1 \leq p < \infty$. A Lipschitz map $T : X \to Y$ is *Lipschitz p-summing* if there is a constant C such that for all (x_i), (y_i) in X and all positive reals a_i, we have

$$\sum a_i \|Tx_i - Ty_j\|^p \leq C^p \cdot \sup_{f \in B_{X^\sharp}} \sum a_i |f(x_i) - f(y_i)|^p, \tag{1.1}$$

where $B_{X^\sharp}$ is the unit ball of $X^\sharp$, and $X^\sharp$ is the Lipschitz dual of $X :=$ $\{f : X \to \mathbb{R} : |f(x) - f(y)| \leq C \cdot |x - y|,\ \text{for some } C > 0, f(0) = 0\}$; that is, $X^\sharp$ is the Banach space of all real valued Lipschitz functions under the (semi)-norm Lip(.) and $\|Tx - Ty\|$ is the distance from Tx to Ty in Y. X is a pointed metric space, that is, $0 \in X$, and $0 \in Y$. $B_{X^\sharp}$ is a compact Hausdorff space in the topology of pointwise convergence on X.

If we restrict to $a_i = 1$ because of density of numbers, the definition is the same. We denoted by $\pi_p^L(T)$ the least C for which the above inequality (1.1) holds and by $\Pi_p^L(X, Y)$ the set of all Lipschitz p-summing mappings from X into Y.

The proof of the following proposition can be found in ([15], Proposition 1.2.3).

Proposition 1.1.3 ([15], Proposition 1.2.3)
Let X and Y be metric spaces, and $\hat{X}$, $\hat{Y}$ be their completions. If $T : X \to Y$ is a Lipschitz map, then T has an extension $\hat{T} : \hat{X} \to \hat{Y}$ such that

$$\mathrm{Lip}(T) = \mathrm{Lip}(\hat{T}).$$

An application of Proposition 1.1.3 gives the following proposition.

Proposition 1.1.4
Suppose $1 \leq p < \infty$. Let X and Y be metric spaces, and $\hat{X}$, $\hat{Y}$ be their completions. If $T : X \to Y$ is Lipschitz p-summing, then $\hat{T} : \hat{X} \to \hat{Y}$ is also Lipschitz p-summing with

$$\pi_p^L(T) = \pi_p^L(\hat{T})$$

where $\hat{T}$ is the extension of T by density.

Proof.
For every $(x_i), (y_i) \subset X$, and all $a_i > 0$,

$$\sum_i a_i \|Tx_i - Ty_i\|^p \quad \leq \quad \big(\pi_p(T)\big)^p \cdot \sup_{f \in B_{X^\sharp}} \left(\sum_i a_i |f(x_i) - f(y_i)|^p \right).$$

Let $(\hat{x}_i), (\hat{y}_i)$ be sequences in $\hat{X}$. Then, there exists $({x_i}^{(n)}), ({y_i}^{(n)})$ with $\lim_n \|\hat{x}_i - {x_i}^{(n)}\| = 0$ and $\lim_n \|\hat{y}_i - {y_i}^{(n)}\| = 0$.
By definition, $\hat{T}(\hat{x}_i) = \hat{T}(\lim_n {x_i}^{(n)}) = \lim_n Tx_i^{(n)}$. Similarly, $\hat{T}(\hat{y}_i) = \lim_n Ty_i^{(n)}$. Hence, we have

$$\begin{aligned}
\sum a_i \|\hat{T}\hat{x}_i - \hat{T}\hat{y}_i\|^p &= \sum a_i \| \lim_n [Tx_i^{(n)} - Ty_i^{(n)}]\|^p \\
&= \lim_n \sum a_i \|Tx_i^{(n)} - Ty_i^{(n)}\|^p \\
&\leq \pi_p^L(T)^p \cdot \sup_{f \in B_{X^\sharp}} \big(\sum_i \lim_n a_i |f(x_i^{(n)}) - f(y_i^{(n)})|^p \big) \\
&= \pi_p^L(T)^p \cdot \sup_{f \in B_{X^\sharp}} \big(\sum_i a_i |\hat{f}(\hat{x}_i) - \hat{f}(\hat{y}_i)|^p \big) \\
&\leq \pi_p^L(T)^p \cdot \sup_{\hat{g} \in B_{\hat{X}^\sharp}} \big(\sum_i a_i |\hat{g}(\hat{x}_i) - \hat{g}(\hat{y}_i)|^p \big).
\end{aligned}$$

Thus, $\pi_p^L(\hat{T}) \leq \pi_p^L(T)$ and by extension, $\pi_p^L(T) = \pi_p^L(\hat{T} \mid_X) \leq \pi_p^L(\hat{T})$. Therefore, $\hat{T}$ is Lipschitz p-summing with $\pi_p^L(\hat{T}) = \pi_p^L(T)$ and this concludes the proof of the proposition. ■

1.2 IDEAL PROPERTY

Before we state and prove the Ideal Property for Lipschitz p-summing maps, we have the following result which can be found in ([15], Proposition 1.2.2).

Proposition 1.2.1 ([15], Proposition 1.2.2)
Let X, Y, and Z be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are Lipschitz maps, then $g \circ f : X \to Z$ is also a Lipschitz map and $\mathrm{Lip}(f \circ g) \leq \mathrm{Lip}(f) \cdot \mathrm{Lip}(g)$.

Proof.
Let $f : X \to Y$ and $g : Y \to Z$ be Lipschitz maps. Then for all $x, y \in X$, we have the following set of inequalities

$$\begin{aligned} \|(f \circ g)(x) - (f \circ g)(y)\| &= \|f(g(x)) - f(g(y))\| \\ &\leq \mathrm{Lip}(f) \cdot \|g(x) - g(y)\| \\ &\leq \mathrm{Lip}(f) \cdot \mathrm{Lip}(g) \cdot \|x - y\|. \end{aligned}$$

Hence, $f \circ g$ is a Lipschitz map and

$$\mathrm{Lip}(f \circ g) \leq \mathrm{Lip}(f) \cdot \mathrm{Lip}(g). \qquad \blacksquare$$

In the linear theory, a consequence of Definition 1.1.1 is that Π_p satisfies the Ideal Property for p-summing operators. This property states that if $v : X \to Y$ is a p-summing operator between Banach spaces X and Y, then for any Banach spaces X_0 and Y_0, and any $u \in \mathcal{L}(Y, Y_0)$ and $w \in \mathcal{L}(X_0, X)$, the operator $uvw : X_0 \to Y_0$ is p-summing with

$$\pi_p(uvw) \leq \|u\| \cdot \pi_p(v) \cdot \|w\|.$$

This property with its proof can be found in [6] by Diestel et al as Ideal Property of p-Summing Operators. In the non-linear setting, there is also a version of the Ideal Property. This was observed by Farmer and Johnson in their paper [8] as an immediate consequence of Definition 1.1.2 which we now state and prove.

Proposition 1.2.2 (Non-linear Ideal Property)
Let $T : X \to Y$ be Lipschitz p-summing, $A : W \to X$ and $B : Y \to Z$ be Lipschitz mappings. Then, $BTA : W \to Z$ is Lipschitz p-summing and

$$\pi_p^L(BTA) \leq \mathrm{Lip}(A) \cdot \pi_p^L(T) \cdot \mathrm{Lip}(B).$$

Proof.

Let (x_i) and (y_i) be in W and a_i be positive scalars such that $a_i = 1$ for each i. The fact that T is Lipschitz p-summing and B is a Lipschitz map implies that

$$\begin{aligned}\Delta &= \sum \|BTAx_i - BTAy_i\|_Z^p \leq \mathrm{Lip}(B)^p \cdot \sum \|TAx_i - TAy_i\|_Y^p \\ &\leq \mathrm{Lip}(B)^p \cdot \pi_p^L(T)^p \cdot \sup_{f \in B_{X^\sharp}} \sum |f(Ax_i) - f(Ay_i)|^p \\ &= \mathrm{Lip}(B)^p \cdot \pi_p^L(T)^p \cdot \sup_{f \in B_{X^\sharp}} \sum |f \circ A(x_i) - f \circ A(y_i)|^p.\end{aligned}$$

Since by definition, both $f : X \to \mathbb{R}$ and $A : W \to X$ are Lipschitz maps, then $f \circ A$ is a Lipschitz map by Proposition 1.2.1, with $\mathrm{Lip}(f \circ A) \leq \mathrm{Lip}(f) \cdot \mathrm{Lip}(A)$ and $f \circ A(0) = 0$. So, $f \circ A \in W^\sharp$. Since $\mathrm{Lip}(f) \leq 1$, it follows by normalizing that

$$\Delta \leq \mathrm{Lip}(B)^p \pi_p^L(T)^p \sup_{f \in B_{X^\sharp}} \left((\mathrm{Lip}(f \circ A))^p \sum \left| \frac{f \circ A(x_i)}{\mathrm{Lip}(f \circ A)} - \frac{f \circ A(y_i)}{\mathrm{Lip}(f \circ A)} \right|^p \right)$$

$$\leq \mathrm{Lip}(B)^p \pi_p^L(T)^p \sup_{f \in B_{X^\sharp}} \left(\mathrm{Lip}(f)^p \mathrm{Lip}(A)^p \sum \left| \frac{f \circ A(x_i)}{\mathrm{Lip}(f \circ A)} - \frac{f \circ A(y_i)}{\mathrm{Lip}(f \circ A)} \right|^p \right)$$

$$\leq \mathrm{Lip}(B)^p \cdot \pi_p^L(T)^p \cdot \mathrm{Lip}(A)^p \cdot \sup_{g \in B_{W^\sharp}} \sum |g(x_i) - g(y_i)|^p.$$

Therefore, BTA is Lipschitz p-summing, and

$$\pi_p^L(BTA) \leq \mathrm{Lip}(B) \cdot \pi_p^L(T) \cdot \mathrm{Lip}(A). \qquad \blacksquare$$

Farmer and Johnson in their paper [8] showed that Definition 1.1.2 of Lipschitz p-summing maps is the 'precise' analogue of Definition 1.1.1 due to the following Proposition 1.2.3. This proposition was proved by Farmer and Johnson in ([8], Theorem 2).

Proposition 1.2.3 ([8], Theorem 2)
Suppose $1 \leq p < \infty$. Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y. Then, T is p-summing if and only if it is Lipschitz p-summing and

$$\pi_p^L(T) = \pi_p(T).$$

Proof.
Suppose T is p-summing. Then, regardless of the natural number m and the choice of $x_1, \ldots, x_m$ in X, we have

$$\sum_{i=1}^{m} \|Tx_i\|^p \leq \pi_p(T)^p \cdot \sup_{f \in B_{X^*}} \left(\sum_{i=1}^{m} |f(x_i)|^p \right).$$

Since T is linear, then for $\{z_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ in X with $x_i = z_i - y_i$ and taking $a_i = 1$ for each i, we have

$$\begin{aligned}
\sum_{i=1}^m \|Tz_i - Ty_i\|^p &= \sum_{i=1}^m \|T(z_i - y_i)\|^p \\
&\leq (\pi_p(T))^p \cdot \sup_{f\in B_{X^*}} \left(\sum_{i=1}^m |f(z_i - y_i)|^p\right) \\
&= (\pi_p(T))^p \cdot \sup_{f\in B_{X^*}} \left(\sum_{i=1}^m |f(z_i) - f(y_i)|^p\right) \\
&\leq (\pi_p(T))^p \cdot \sup_{f\in B_{X^\sharp}} \left(\sum_{i=1}^m |f(z_i) - f(y_i)|^p\right).
\end{aligned}$$

Hence

$$\sum_{i=1}^m \|Tz_i - Ty_i\|^p \leq (\pi_p(T))^p \cdot \sup_{f\in B_{X^\sharp}} \left(\sum_{i=1}^m |f(z_i) - f(y_i)|^p\right).$$

This shows that T is Lipschitz p-summing and

$$\pi_p^L(T) \leq \pi_p(T). \tag{1.2}$$

Conversely, if T is Lipschitz p-summing, then it is p-summing by Theorem 2 in [8] with

$$\pi_p(T) \leq \pi_p^L(T). \tag{1.3}$$

Combining (1.2) and (1.3), we have

$$\pi_p^L(T) = \pi_p(T). \qquad \blacksquare$$

Remark 1.2.4
We also observe that since the operators in ([6], Examples 2.9(a)-(e)) are bounded linear operators and p-summing, then by Proposition 1.2.3, all these operators are also Lipschitz p-summing. We also observe from Proposition 1.2.3 that the multiplication operators as found in ([6], Example 2.9(a)); $M_\varphi : C(K) \to L_p(\mu) : f \mapsto f\cdot\varphi$ and Example 2.9(c); $M_\varphi : L_\infty(\mu) \to L_p(\mu) : f \mapsto f\cdot\varphi$ also satisfies $\pi_p^L(M_\varphi) = \|\varphi\|_p$. Similarly, for the formal inclusion operators as found in ([6], Example 2.9(b)); $j_p : C(K) \to L_p(\mu)$ where K is a compact Hausdorff space and μ is a positive Borel measure on K, and Example 2.9(d); $i_p : L_\infty(\mu) \to L_p(\mu)$ for any finite measure space (Ω, Σ, μ) also satisfies $\pi_p^L(j_p) = \mu(K)^{\frac{1}{p}}$ and $\pi_p^L(i_p) = \mu(\Omega)^{\frac{1}{p}}$ respectively. Finally, for the diagonal operators as found in ([6], Example 2.9(e)); $D_\lambda : \ell_\infty \to \ell_p : (a_n) \mapsto (\lambda_n a_n)$ where λ_n is any member of ℓ_p also satisfy $\pi_p^L(D_\lambda) = \|\lambda\|_p$.

1.3 INJECTIVITY PROPERTY

In the linear setting, an application of the Ideal Property for p-summing operators gives rise to the Injectivity Property of p-summing operators. This property states that the operator $v : X \to Y$ is p-summing if and only if the operator $uv : X \to Z$ is p-summing where the operator $u : Y \to Z$ is isometric, and X, Y and Z are Banach spaces. We even have $\pi_p(uv) = \pi_p(v)$ in such a case. This result of the Injectivity Property for p-summing operators and its proof can be found in ([6], Injectivity of Π_p). An analogue of this property also exists in the non-linear setting. Before stating and proving the non-linear version of the Injectivity Property, we have the following definitions and results. Theorem 1.3.3 is an analogue of the result obtained in chapter 2 of [6] in the non-linear setting.

Definition 1.3.1
Let (X, δ) be a complete metrizable topological vector space and $1 \leq p < \infty$. A sequence (x_n) in X is *strongly p-summable* if the scalar sequence $(\delta(x_n, 0)_n)$ is in ℓ_p. We denote by $\ell_p^{\text{strong}}(X)$ the set of all such sequences in X and the metric by

$$\|(x_n) - (y_n)\|_p^{\text{strong}} = \Big(\sum_n \|x_n - y_n\|^p\Big)^{\frac{1}{p}}.$$

Definition 1.3.2
Let (X, δ) be a complete metrizable topological vector space and $1 \leq p < \infty$. A sequence (x_n) in X is *weakly p-summable* if the scalar sequences $((fx_n)_n)$ are in ℓ_p for every $f \in X^\sharp$. We denote by $\ell_p^{\text{weak}}(X)$ the set of all such sequences in X and the metric by

$$\|(x_n) - (y_n)\|_p^{\text{weak}} = \sup_{f \in B_{X^\sharp}} \Big(\sum |f(x_n) - f(y_n)|^p\Big)^{\frac{1}{p}}.$$

Suppose $u : X \to Y$ is a Lipschitz map between complete metrizable topological vector spaces X and Y, the correspondence

$$\hat{u} : (x_n)_n \mapsto (ux_n)_n$$

induces Lipschitz maps $\ell_p^{\text{weak}}(X) \to \ell_p^{\text{weak}}(Y)$ and $\ell_p^{\text{strong}}(X) \to \ell_p^{\text{strong}}(Y)$ respectively. In both cases, the norm is clearly $\text{Lip}(u)$. Indeed, we show that $u : X \to Y$ is Lipschitz if and only if $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{weak}}(Y)$ is Lipschitz. For this purpose, suppose u is Lipschitz and let (x_n) and (y_n) be in $\ell_p^{\text{weak}}(X)$. We then have

$$\begin{aligned}
\|\hat{u}(x_n)_n - \hat{u}(y_n)_n\| &= \|\hat{u}(x_n) - \hat{u}(y_n)\|_p^{\text{weak}} = \|(ux_n)_n - (uy_n)_n\|_p^{\text{weak}} \\
&= \sup_{g \in B_{Y^\sharp}} \Big(\sum_n |g(ux_n) - g(uy_n)|^p\Big)^{\frac{1}{p}} \\
&\leq \text{Lip}(g \circ u) \cdot \sup_{f \in B_{X^\sharp}} \Big(\sum_n |f(x_n) - f(y_n)|^p\Big)^{\frac{1}{p}} \\
&\leq \text{Lip}(u) \cdot \|(x_n) - (y_n)\|_p^{\text{weak}}.
\end{aligned}$$

Hence, $\hat{u}$ is Lipschitz and

$$\mathrm{Lip}(\hat{u}) \leq \mathrm{Lip}(u). \tag{1.4}$$

Conversely, suppose $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{weak}}(Y)$ is a Lipschitz map. Let (x_n) and (y_n) be in $\ell_p^{\text{weak}}(X)$. We then have

$$\|(ux_n)_n - (uy_n)_n\|_p^{\text{weak}} = \|\hat{u}(x_n) - \hat{u}(y_n)\|_p^{\text{weak}} \leq \mathrm{Lip}(\hat{u}) \cdot \|(x_n) - (y_n)\|_p^{\text{weak}}.$$

In partcular, put $x_n = x$, $\forall n$, $y_n = y$, $\forall n$ and consider length-1 sequences. Then

$$\sup_{g \in B_{Y^\sharp}} |gux - guy| \leq \sup_{f \in B_{X^\sharp}} |fx - fy|$$

if and only if

$$\|ux - uy\| \leq \mathrm{Lip}(u) \cdot \|x - y\|$$

and so, u is Lipschitz with

$$\mathrm{Lip}(u) \leq \mathrm{Lip}(\hat{u}). \tag{1.5}$$

Combining (1.4) and (1.5), we have $\mathrm{Lip}(u) = \mathrm{Lip}(\hat{u})$.

A similar argument shows that $u : X \to Y$ is Lipschitz if and only if $\hat{u} : \ell_p^{\text{strong}}(X) \to \ell_p^{\text{strong}}(Y)$ is Lipschitz with $\mathrm{Lip}(u) = \mathrm{Lip}(\hat{u})$. This process may even produce a Lipschitz map $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{strong}}(Y)$. This happens precisely when u is Lipschitz p-summing.

Theorem 1.3.3
Suppose $1 \leq p < \infty$ and let X and Y be complete metrizable topological vector spaces. A mapping $u : X \to Y$ is Lipschitz p-summing if and only if $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{strong}}(Y)$ is Lipschitz. In this case, $\pi_p^L(u) = \mathrm{Lip}(\hat{u})$.

Proof.
Suppose first that u is Lipschitz p-summing. Then, for all (x_n), (y_n) in X and all positive reals a_i such that $a_i = 1$ for each i, we have

$$\sum_n \|ux_n - uy_n\|^p \leq \left(\pi_p^L(u)\right)^p \cdot \sup_{f \in B_{X^\sharp}} \left(\sum_n |f(x_n) - f(y_n)|^p\right).$$

Let (x_n), (y_n) be in $\ell_p^{\text{weak}}(X)$, then we have

$$\begin{aligned}
\|\hat{u}(x_n) - \hat{u}(y_n)\|_p^{\text{strong}} &= \left(\sum_n \|ux_n - uy_n\|^p\right)^{\frac{1}{p}} \\
&\leq \pi_p^L(u) \cdot \sup_{f \in B_{X^\sharp}} \left(\sum_n |f(x_n) - f(y_n)|^p\right)^{\frac{1}{p}} \\
&= \pi_p^L(u) \cdot \|(x_n) - (y_n)\|_p^{\text{weak}}.
\end{aligned}$$

Consequently, $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{strong}}(Y)$ is Lipschitz and

$$\mathrm{Lip}(\hat{u}) \leq \pi_p^L(u). \tag{1.6}$$

Conversely, suppose $\hat{u} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{strong}}(Y)$ is Lipschitz. Then for finite sequences (x_n), (y_n) in X, we have

$$\begin{aligned} \Big(\sum_n \|ux_n - uy_n\|^p\Big)^{\frac{1}{p}} &= \|\hat{u}(x_n) - \hat{u}(y_n)\|_p^{\text{strong}} \\ &\leq \text{Lip}(\hat{u}) \cdot \|(x_n) - (y_n)\|_p^{\text{weak}} \\ &= \text{Lip}(\hat{u}) \cdot \sup_{f \in B_{X^\sharp}} \Big(\sum_n |f(x_n) - f(y_n)|^p\Big)^{\frac{1}{p}}. \end{aligned}$$

Hence, u is Lipschitz p-summing and

$$\pi_p^L(u) \leq \text{Lip}(\hat{u}). \tag{1.7}$$

Combining (1.6) and (1.7), we have

$$\pi_p^L(u) = \text{Lip}(\hat{u}). \qquad \blacksquare$$

From the above theorem, we have shown that
$\Pi_p^L(X,Y) = \mathcal{L}ip\big(\ell_p^{\text{weak}}(X), \ell_p^{\text{strong}}(Y)\big)$ isometrically isomorphically.

We now have the the following injectivity property of Lipschitz p-summing maps.

Theorem 1.3.4 (Injectivity of Π_p^L)
Let X, Y and Y_0 be complete metrizable topological vector spaces. If $u : Y \to Y_0$ is an isometric map, then $v \in \Pi_p^L(X,Y)$ if and only if $uv \in \Pi_p^L(X,Y_0)$. In such a case, we also have

$$\pi_p^L(uv) = \pi_p^L(v).$$

Proof.
Suppose $v \in \Pi_p^L(X,Y)$. Since $u : Y \to Y_0$ is isometric, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "uv"] \arrow[dr, "v"'] & & Y_0 \\ & Y \arrow[ur, "u"'] & \end{tikzcd}$$

Let $\widehat{uv} : \ell_p^{\text{weak}}(X) \to \ell_p^{\text{strong}}(Y_0)$ be a map. In order to apply Theorem 1.3.3, we show that $\widehat{uv}$ is Lipschitz. For this purpose, let $(x_n)_n, (y_n)_n \in \ell_p^{\text{weak}}(X)$ with $x = (x_n)_n$ and $y = (y_n)_n$. Since u is isometric and v is Lipschitz

p-summing, we have

$$\begin{aligned}
\|\widehat{uv}(x) - \widehat{uv}(y)\|_p^{\text{strong}} &= \Big(\sum_n \|uvx_n - uvy_n\|_{Y_0}^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_n \|vx_n - vy_n\|_Y^p\Big)^{\frac{1}{p}} \\
&\leq \pi_p^L(v) \cdot \sup_{f \in B_{X^\sharp}} \Big(\sum_n |f(x_n) - f(y_n)|^p\Big)^{\frac{1}{p}} \\
&= \pi_p^L(v) \cdot \|x - y\|_p^{\text{weak}}.
\end{aligned}$$

Hence, $\widehat{uv}$ is Lipschitz and $\mathrm{Lip}(\widehat{uv}) \leq \pi_p^L(v)$.
By Theorem 1.3.3, we conclude that uv is Lipschitz p-summing and

$$\pi_p^L(uv) = \mathrm{Lip}(\widehat{uv}) \leq \pi_p^L(v) \tag{1.8}$$

Conversely, suppose $uv \in \Pi_p^L(X, Y_0)$, we show that $v \in \Pi_p^L(X, Y)$. Let (w_n) and (z_n) be in X, and since u is isometric, we have

$$\begin{aligned}
\Big(\sum_n \|vw_n - vz_n\|_Y^p\Big)^{\frac{1}{p}} &= \Big(\sum_n \|uvw_n - uvz_n\|_{Y_0}^p\Big)^{\frac{1}{p}} \\
&\leq \pi_p^L(uv) \cdot \sup_{f \in B_{X^\sharp}} \Big(\sum_n |f(w_n) - f(z_n)|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

Hence, v is Lipschitz p-summing and

$$\pi_p^L(v) \leq \pi_p^L(uv) \tag{1.9}$$

By (1.8) and (1.9), we have

$$\pi_p^L(v) = \pi_p^L(uv) \qquad \blacksquare$$

2 PIETSCH THEOREMS

2.1 DOMINATION AND FACTORIZATION

The proofs of the following two theorems, that is, the Pietsch Domination Theorem 2.1.1 and the Pietsch Factorization Theorem 2.1.2 can be found in [6].

Pietsch Domination Theorem 2.1.1 (Linear version) ([6], Pietsch Domination Theorem)
Suppose that $1 \leq p < \infty$. Let $u : X \to Y$ be a Banach space operator, and K be a weak*-compact norming subset of B_{X^*}. Then u is p-summing if and only if there exists a constant C and a regular probability measure μ on K such that for each $x \in X$

$$\|ux\| \leq C \cdot \left(\int_K |x^*(x)|^p d\mu(x^*) \right)^{\frac{1}{p}}.$$

In such a case, $\pi_p(u)$ is the least of all the constants C for which such a measure exists.

Pietsch Factorization Theorem 2.1.2 (Linear version) ([6], Pietsch Factorization Theorem)
Suppose $1 \leq p < \infty$. Let X and Y be Banach spaces, and K be a weak*-compact norming subset of B_{X^*}. Let B be B_{Y^*} (or a norming subset thereof). For every operator $u : X \to Y$, the following are equivalent

(1) u is p-summing.

(2) There exist a regular Borel probabilty measure μ on K, a (closed) subspace X_p of $L_p(\mu)$ and an operator $\tilde{u} : X_p \to Y$ such that
(a) $j_p i_X(X) \subset X_p$ and (b) $\tilde{u} j_p i_X(x) = ux$ for all $x \in X$.
In other words, if j_p^X is the map $i_X(X) \to X_p$ induced by j_p, then the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "i_X"'] & Y \\
i_X(X) \arrow[r, "j_p^X"] \arrow[d, hook] & X_p \arrow[u, "\tilde{u}"'] \arrow[d, hook] \\
C(K) \arrow[r, "j_p"] & L_p(\mu)
\end{tikzcd}$$

(3) There exist a regular probability measure μ on K and an operator $\hat{u} : L_p(\mu) \to \ell_\infty^B$ such that the following diagram commutes:

$$\begin{tikzcd}
C(K) \arrow[rr, "j_p"] & & L_p(\mu) \arrow[d, "\hat{u}"] \\
X \arrow[u, "i_X"] \arrow[r, "u"] & Y \arrow[r, "i_Y"] & \ell_\infty^B
\end{tikzcd}$$

(4) There exists a probability space (Ω, Σ, μ) and operators $\hat{u} : L_p(\mu) \to \ell_\infty^B$ and $v : X \to L_\infty(\mu)$ such that the following diagram commutes

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "i_p"] & & L_p(\mu) \arrow[d, "\hat{u}"] \\ X \arrow[u, "v"] \arrow[r, "u"] & Y \arrow[r, "i_Y"] & \ell_\infty^B \end{tikzcd}$$

Moreover, we may choose μ and $\tilde{u}$ in (2) or μ and $\hat{u}$ in (3) so that $\|\tilde{u}\| = \|\hat{u}\| = \pi_p(u)$; in (4) we may arrange that $\|v\| = 1$ and $\|\hat{u}\| = \pi_p(\mu)$.

Inspired by the above two theorems, Farmer and Johnson introduced in [8] the following theorem;

2.2 NON-LINEAR PIETSCH THEOREM

Non-Linear Pietsch Theorem 2.2.1 ([8], Theorem 1)
Suppose $1 \leq p < \infty$. Let $T : X \to Y$ be a mapping between metric spaces X and Y. Then the following are equivalent for the mapping T and $C \geq 0$.

(1) $\pi_p^L(T) \leq C$. (that is, T is Lipschitz p-summing)

(2) There is a probability μ on $B_{X^\sharp}$ such that

$$\|Tx - Ty\|^p \leq C^p \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f)$$

(Pietsch Domination).

(3) For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\ X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z \end{tikzcd}$$

with μ a probability and Lip(A) $\cdot$ Lip(B)$\leq C$ (Pietsch Factorization).

(4) There is a probability μ on K, the closure in the topology of pointwise convergence on X of the extreme points of $B_{X^\sharp}$ such that

$$\|Tx - Ty\|^p \leq C^p \int_K |f(x) - f(y)|^p d\mu(f)$$

We shall also rely on the following result to prove the Non-linear Pietsch Theorem.

Remark 2.2.2
Suppose X is a set, (Y, d) is a metric space and $f : X \to Y$ is an injective function. The injective function f and d induce a metric on X, namely $(a, b) \mapsto d(f(a), f(b))$. This metric makes X an isometric copy of the metric subspace $(f(X), d)$ of (Y, d) and f is an isometry. This result is an example as can be found in ([14], Example 1.4.4).

Proof of the Non-linear Pietsch Theorem.

(2) implies (3): Since every space X embeds in a $C(K)$ space, we have the following commutative diagram by taking $K = B_{X^\sharp}$;

$$\begin{tikzcd}
C(B_{X^\sharp}) \arrow[r, "J_\infty"] & L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & I_{\infty,p}AX \arrow[r, hook] \arrow[d, "B_1"] & L_p(\mu) \arrow[dl, "B"] \\
& X \arrow[ul, "i_X"] \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z &
\end{tikzcd}$$

with $B_1 \circ I_{\infty,p} \circ A = J \circ T$. We are able to extend B_1 to B because Z is 1-injective with $\text{Lip}(B_1) = \text{Lip}(B)$. Thus, $B_1 = B_{|_{I_{\infty,p}AX}}$. Since i_X and J_∞ are Lipschitz maps and a composition of Lipschitz maps is Lipschitz by Proposition 1.2.1, then A is also a Lipschitz map. Also, since J is an isometry and by condition (2), we have for any $x, y \in X$

$$\begin{aligned}
\|B_1 I_{\infty,p}Ax - B_1 I_{\infty,p}Ay\|_Z^p &= \|JTx - JTy\|_Z^p = \|Tx - Ty\|^p \\
&\leq C^p \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f).
\end{aligned}$$

Since the functions f are Lipschitz, $f \in B_{X^\sharp}$ so that $\text{Lip}(f) \leq 1$, and $I_{\infty,p} \circ A$ is injective, then by Remark 2.2.2, we have

$$\begin{aligned}
\|B_1 I_{\infty,p}Ax - B_1 I_{\infty,p}Ay\|_Z^p &\leq C^p \cdot \text{Lip}(f)^p \cdot \|x - y\|_X^p \\
&\leq C^p \cdot \|x - y\|_X^p \\
&= C^p \cdot \|I_{\infty,p}Ax - I_{\infty,p}Ay\|_{L_p(\mu)}^p.
\end{aligned}$$

Therefore, B_1 is Lipschitz, and $\text{Lip}(B_1) \leq C$. Also,

$$\text{Lip}(A) = \text{Lip}(J_\infty i_X) \leq \text{Lip}(J_\infty)\text{Lip}(i_X) \leq 1.$$

Since B_1 has an extension, i.e., B_1 extends to B with $B_1 = B_{|_{I_{\infty,p}AX}}$ and $\text{Lip}(B_1) = \text{Lip}(B)$, we have

$$\text{Lip}(A) \cdot \text{Lip}(B) \leq C.$$

(3) implies (1): Since $I_{\infty,p}$ is Lipschitz p-summing with $\pi_p^L(I_{\infty,p}) = 1$ by Remark 1.2.4, and J is an isometry, then by condition (3) and the Non-

linear Ideal Property, we have

$$\begin{aligned}\pi_p^L(T) = \pi_p^L(JT) &= \pi_p^L(B_1 \circ I_{\infty,p} \circ A) \\ &\leq \text{Lip}(B_1) \cdot \pi_p^L(I_{\infty,p}) \cdot \text{Lip}(A) \\ &= \text{Lip}(B_1) \cdot \pi_p(I_{\infty,p}) \cdot \text{Lip}(A) \\ &= \text{Lip}(B_1) \cdot \text{Lip}(A) \\ &= \text{Lip}(B) \cdot \text{Lip}(A) \leq C.\end{aligned}$$

(1) implies (2): Suppose $\pi_p^L(T) = 1$. Let Q be a convex cone in $C(B_{X^\sharp})$ consisting of all positive linear combinations of the functions of the form $\|Tx - Ty\| - C^p|f(x) - f(y)|^p$, as x and y range over X. Now condition (1) says that Q is disjoint from the positive cone $P = \{F \in C(B_{X^\sharp}) : F(f) > 0 \quad \forall f \in B_{X^\sharp}\}$. P is clearly an open and convex subset of $C(B_{X^\sharp})$. Indeed, P is open since $P = \bigcup_F F^{-1}(0, \infty)$ where $F \in C(B_{X^\sharp})$. P is convex since it is a cone, and so $Q \cap P = \emptyset$, otherwise, $g_M \in Q$ for some finite set $M \subset X$ and $g_M(f) > 0$ for all $f \in B_{X^\sharp}$ where

$$g_M(f) = \sum_{x,y \in M} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p.$$

Indeed on the contrary, if $g_M \in Q$ for some finite set $M \subset X$ and $g_M(f) > 0$ for all $f \in B_{X^\sharp}$, then

$$\sum_{x,y \in M} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p > 0$$

so that

$$\sum_{x,y \in M} \|Tx - Ty\|^p > C^p \cdot \sum_{x,y \in M} |f(x) - f(y)|^p.$$

Hence

$$\sum_{x,y \in M} \|Tx - Ty\|^p > C^p \cdot \sup_{f \in B_{X^\sharp}} \sum_{x,y \in M} |f(x) - f(y)|^p$$

contrary to T being Lipschitz p-summing. Therefore, $Q \cap P = \emptyset$.
Hence, by the Separation Theorem and the Riesz Representation Theorem, there is a finite signed Baire measure μ on $B_{X^\sharp}$ and a real number c so that for all $G \in Q$ and $F \in P$,

$$\int_{B_{X^\sharp}} G d\mu \leq c < \int_{B_{X^\sharp}} F d\mu.$$

Since $0 \in Q$ then $c \geq 0$. Also, since all positive constant functions belong to P, then $c < 0$ so that $c = 0$. Since $\int_{B_{X^\sharp}} .d\mu$ is positive on the positive cone P of $C(B_{X^\sharp})$, the signed measure μ is positive which we can assume by rescaling is a probability measure. Hence

$$\int_{B_{X^\sharp}} G d\mu \leq 0 < \int_{B_{X^\sharp}} F d\mu$$

so that

$$\int_{B_{X^\sharp}} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p d\mu(f) \leq 0.$$

Therefore
$$\|Tx - Ty\|^p \leq C^p \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f).$$
(1) implies (4): The proof that (1) implies (4) is like the proof that (1) implies (2) since the supremum on the right hand side of (1.1), the definition of the Lipschitz p-summing norm, is the same as
$$\sup_{f \in K} \sum a_i |f(x_i) - f(y_i)|^p. \qquad \blacksquare$$

The strong form of the Pietsch Domination Theorem says that if X is a subspace of a $C(K)$ space for some compact Hausdorff space K, and if T is a p-summing linear operator with domain X, then there is a probability measure μ on K so that for all $x \in X$,
$$\|Tx\|^p leq \pi_p(T)^p \int_K |x(t)|^p d\mu(t).$$
Unfortunately, a non-linear version for this result does not hold.

Indeed, let D_n be the discrete metric space with n-points so that the distance between any two distinct points is one. We now embed D_n into $C(\{-1,1\}^n)$ in two different ways. First, define $D_n = \{x_1, \cdots, x_n\}$ with $|D_n| = n$. Let

$$d(x_i, y_j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

and $f(x_k) = \frac{1}{2} r_k$, where r_k is the projection onto the k^{th} coordinate (i.e., $r_k(x) = k^{\text{th}}$-coordinate of x).
If $x \in \{-1,1\}^n$ then $x = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ and $|\{-1,1\}^n| = 2^n$. Let $E_1 = \{x \in \{-1,1\}^n : |\frac{r_i}{2}(x) - \frac{r_j}{2}(x)|^p = 1\}$ and $E_2 = \{x \in \{-1,1\}^n : |\frac{r_i}{2}(x) - \frac{r_j}{2}(x)|^p = 0\}$. Then
$$d(x_i, y_j) = \left\| \frac{r_i}{2} - \frac{r_j}{2} \right\|_\infty = 1.$$
Let j_p be the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \mu)$ where μ is the uniform probability on $\{-1,1\}^n$. We then have the following commutative diagram;

$$\begin{tikzcd}
C(\{-1,1\}^n) \arrow[r, "j_p"] & L_p(\{-1,1\}^n, \mu) \\
D_n \arrow[u, "f"] \arrow[r, "I_{D_n}"] & j_p f(D_n) \arrow[u]
\end{tikzcd}$$

Let $\mu(E_1) = \mu(E_2) = \frac{1}{2}$. Then
$$\begin{aligned}
\|j_p f(x_i) - j_p f(x_j)\|_p^p &= \left\| j_p\left(\frac{r_i}{2}\right) - j_p\left(\frac{r_j}{2}\right) \right\|_p^p \\
&= \int_{\{-1,1\}^n} \left| \frac{r_i}{2}(x) - \frac{r_j}{2}(x) \right|^p d\mu \\
&= \int_{E_1} \left| \frac{r_i}{2}(x) - \frac{r_j}{2}(x) \right|^p d\mu + \int_{E_2} \left| \frac{r_i}{2}(x) - \frac{r_j}{2}(x) \right|^p d\mu \\
&= 1 \cdot \mu(E_1) + 0 \cdot \mu(E_2) = \frac{1}{2}.
\end{aligned}$$

Since f is an isometry, we have

$$\begin{aligned}\int_{\{-1,1\}^n} \Big|x_i(x) - x_j(x)\Big|^p d\mu &= \int_{\{-1,1\}^n} \Big|f(x_i)(x) - f(x_j)(x)\Big|^p d\mu \\ &= \int_{\{-1,1\}^n} \Big|\frac{r_i}{2}(x) - \frac{r_j}{2}(x)\Big|^p d\mu \quad = \quad \frac{1}{2}.\end{aligned}$$

Therefore

$$\Big\|j_p\Big(\frac{r_i}{2}\Big) - j_p\Big(\frac{r_j}{2}\Big)\Big\|_p^p = \frac{1}{2} \leq 1 = 2 \cdot \int_{\{-1,1\}^n} \Big|x_i(x) - x_j(x)\Big|^p d\mu.$$

Hence

$\pi_p^L(I_{D_n}) \leq 2^{\frac{1}{p}}$ so that $\pi_1^L(I_{D_n}) \leq 2$. We also have $\pi_2^L(I_{D_n}) \leq \sqrt{2}$ and so on. Therefore, $\pi_1^L(I_{D_n}) \leq 2$.

Secondly, Let $D_n = \{g_1, \cdots, g_n\} = \{g_k : 1 \leq k \leq n\}$ where the g_k are the unit vectors with disjoint support. i.e.,

$$\{x \in \{-1,1\}^n : g_i(x) \neq 0\} \cap \{x \in \{-1,1\}^n : g_j(x) \neq 0\} = \emptyset.$$

Let $E_i = \{x \in \{-1,1\}^n : g_i(x) \neq 0\}$ and $E_j = \{x \in \{-1,1\}^n : g_j(x) \neq 0\}$. Then, $\|g_i - g_j\|_\infty = 1$.
Let j_p be the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \nu)$ where ν is any probability measure on $\{-1,1\}^n$ with $\nu(E_i) = \nu(E_j) = \frac{1}{n}$. Hence

$$\begin{aligned}\|j_p g_i - j_p g_j\|_p^p &= \int_{\{-1,1\}^n} |j_p g_i(x) - j_p g_j(x)|^p d\nu \\ &= \int_{\{-1,1\}^n} |g_i(x) - g_j(x)|^p d\nu \\ &= \int_{E_i \cup E_j} |g_i(x) - g_j(x)|^p d\nu \\ &= \int_{E_i} |g_i(x) - g_j(x)|^p d\nu + \int_{E_j} |g_i(x) - g_j(x)|^p d\nu \\ &= \int_{E_i} |g_i(x)|^p d\nu + \int_{E_j} |g_j(x)|^p d\nu \\ &\leq 1 \cdot \nu(E_i) + 1 \cdot \nu(E_j) = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.\end{aligned}$$

Therefore

$$
\begin{aligned}
\|j_p g_i - j_p g_j\|_p &\le \left(\frac{2}{n}\right)^{\frac{1}{p}} \cdot 1 = \left(\frac{2}{n}\right)^{\frac{1}{p}} \cdot \left(\int_{\{-1,1\}^n} |(g_i)(x) - (g_j)(x)|^p d\nu\right)^{\frac{1}{p}} \\
&= \left(\sum_{x\in\{-1,1\}^n} \int_{[x]} |(g_i)(x) - (g_j)(x)|^p d\nu\right)^{\frac{1}{p}} \\
&= \left(\sum_{x\in\{-1,1\}^n} \int_{[x]} |(g_k)(x)|^p d\nu\right)^{\frac{1}{p}} \\
&= \left(2^n \cdot \|g_k\|^p \cdot \frac{1}{2^n}\right)^{\frac{1}{p}} \\
&= 1 \qquad (g_k \quad \text{unit vectors})
\end{aligned}
$$

where $k = i$ or j depending on $x \in E_i$ or E_j. We therefore have $\pi_p^L(j_p) \le \left(\frac{2}{n}\right)^{\frac{1}{p}}$. Hence, $\pi_1^L(I_{D_n}) = \pi_1^L(j_p \mid_{I_{D_n}}) \le \left(\frac{2}{n}\right)^{\frac{1}{p}}$. As $n \to \infty$, $\pi_p^L(I_{D_n}) \le 0$. Hence, $\pi_p^L(I_{D_n}) = 0$.

But as shown in [8], $\pi_p^L(I_{D_n}) \to 2^{\frac{1}{p}}$ as $n \to \infty$. The extreme points K_n of $B^\sharp_{D_n}$ are of the form $\pm\chi_A$ where A is a non-empty subset of $D_n\backslash\{0\}$. We now calculate $\pi_p^L(I_{D_n})$ in the case that n is even. Let μ be the uniform measure on $J_{\frac{n}{2}} := \{\chi_A : |A| = \frac{n}{2}, A \subset D_n\backslash\{0\}\}$ (so that $\mu(e) = 0$ for $e \in K_n\backslash J_{\frac{n}{2}}$). Then μ is a probability measure on K_n and for each pair of distinct points x and y in D_n, we have

$$\|I_{D_n}(x) - I_{D_n}(y)\|^p = \big(d(x,y)\big)^p = 1 = \frac{2(n-1)}{n} \cdot \int_{K_n} |f(x) - f(y)|^p d\mu(f).$$

Hence, $\pi_p^L(I_{D_n}) = \left(2 - \frac{2}{n}\right)^p$ since μ is a Pietsch measure. As $n \to \infty$, $\pi_p^L(I_{D_n}) \to 2^{\frac{1}{p}}$. Since $I_{D_n} = j_p|_{D_n}$, we would expect $0 = \pi_p^L(j_p) \ge \pi_p^L(j_p|_{D_n}) \ne 0$ and this is a contradiction. Therefore, the strong form of the Pietsch domination theorem does not hold in the non-linear theory.

3 APPLICATION OF THE NON-LINEAR PIETSCH THEOREM

3.1 INCLUSION THEOREM

In their paper [8], Farmer and Johnson noted that one immediate consequence of the Non-Linear Pietsch Theorem 2.2.1 is that π_p^L is a monotonely decreasing function of p. This is the non-linear version of the Inclusion Theorem. Suppose X and Y are Banach spaces and $1 \leq p \leq q < \infty$. If $u : X \to Y$ is p-summing, then it is q-summing and

$$\pi_q(u) \leq \pi_p(u).$$

This result is the linear version of the Inclusion Theorem and its proof can be found in ([6], Inclusion Theorem 2.8). We now state and prove the Non-linear Inclusion Theorem.

Non-linear Inclusion Theorem 3.1.1
Let $1 \leq p \leq q < \infty$. If $T : X \to Y$ is a Lipschitz p-summing map between metric spaces X and Y, then it is Lipschitz q-summing and

$$\pi_q^L(T) \leq \pi_p^L(T).$$

Proof.
Let $\pi_p^L(T) < \infty$, then by the Non-Linear Pietsch Theorem 2.2.1, for some probability measure μ on $B_{X^\sharp}$, we have

$$\|Tx - Ty\|^p \leq \left(\pi_p^L(T)\right)^p \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f).$$

By monotonicity of the L_p-metrics, we therefore have

$$\begin{aligned} \|Tx - Ty\| &\leq \pi_p^L(T) \cdot \left(\int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f)\right)^{\frac{1}{p}} \\ &\leq \pi_p^L(T) \cdot \left(\int_{B_{X^\sharp}} |f(x) - f(y)|^q d\mu(f)\right)^{\frac{1}{q}}. \end{aligned}$$

Hence

$$\|Tx - Ty\|^q \leq \left(\pi_p^L(T)\right)^q \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^q d\mu(f).$$

This shows that T is Lipschitz q-summing by the Non-Linear Pietsch Theorem 2.2.1, and

$$\pi_q^L(T) \leq \pi_p^L(T). \qquad \blacksquare$$

An immediate consequence of the Non-linear Inclusion Theorem is that whenever a map $T : X \to Y$ between metric spaces X and Y is Lipschitz 1-summing for $1 \leq p < \infty$, then it is Lipschitz p-summing and

$$\pi_p^L(T) \leq \pi_1^L(T).$$

Another consequence of the Non-Linear Pietsch Theorem 2.2.1 is that there is a version of the Grothendieck's Theorem (linear version).

3.2 GROTHENDIECK'S THEOREM

The proof of the following theorem, that is, Grothendieck's Theorem 3.2.1 (linear version) can be found in ([6], Theorem 3.4).

Grothendieck's Theorem 3.2.1 (Linear version) ([6], Theorem 3.4) Regardless of the measures μ and ν, every operator $u : L_1(\mu) \to L_2(\nu)$ is 1-summing with $\pi_1(u) \leq \mathcal{K}_G \cdot \|u\|$, where $\mathcal{K}_G$ is Grothendieck's constant.

Diestel et al in ([6], Remark 2.20) have noted a particular case, that is, the natural inclusion map $i : \ell_1 \to \ell_2$ is 1-summing. Applying Proposition 1.2.3, we also note from the above theorem (Grothendieck Theorem 3.2.1) that a linear operator T from an L_1 space into a Hilbert space is Lipschitz 1-summing with $\pi_1^L(T) \leq \mathcal{K}_G \cdot \mathrm{Lip}(T)$.

In the non-linear setting of Grothendieck's Theorem, weighted trees play a role analogous to that of an L_1 space in the linear theory. This observation is due to Farmer and Johnson in their paper [8]. The reason is that every finite subset of a weighted tree has the lifting property, which is to say that if X is a finite weighted tree, $T : X \to Y$ is a Lipschitz map from X into a metric space Y, and $Q : Z \to Y$ is a 1-Lipschitz quotient mapping, then for each $\epsilon > 0$, there is a mapping $S : X \to Z$ such that $\mathrm{Lip}(S) \leq \mathrm{Lip}(T) + \epsilon$ and $T = Q \circ S$.

The following formal definitions of which are given in Definition 3.2.2 and Definition 3.2.3 can be found in ([7], Definition 3.1) and in [11] respectively.

Definition 3.2.2
A metric space X is a *finite metric tree* if it is a finite connected graph T that has no cycles endowed with an edge weighted path metric.

Definition 3.2.3
A Lipschitz map f between the metric spaces X and Y is called a *Lipschitz quotient map* if there is a $C > 0$ (the smallest such C, the co-Lipschitz constant, is denoted by co-Lip(f)) so that for every $x \in X$ and $r > 0$, $f\big(B_X(x,r)\big) \supset B_Y(f(x), \frac{r}{C})$.
If there is a Lipschitz quotient mapping from X onto Y with $\mathrm{Lip}(f) \cdot$

co-Lip$(f) \leq C$, we say that Y is a *C-Lipschitz quotient* of X.

The following version of the Non-linear Grothendieck's Theorem is due to Chen and Zheng and its proof can be found in their paper ([4], Corollary 2.3). We will call this version of Chen and Zheng as Theorem 3.2.4.

Theorem 3.2.4 ([4], Corollary 2.3)
Suppose X is a metric tree and T is a Lipschitz map from X into ℓ_2, then for any $1 < p \leq 2$,

$$\pi_p^L(T) \leq B_q \cdot A_1^{-1} \cdot \mathrm{Lip}(T),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and B_q, A_1^{-1} are the respective Khintchine constants.

Another slightly different proof of the Non-linear Grothendieck's Theorem to that of Chen and Zheng is given in the following theorem which we now prove.

Non-Linear Grothendieck's Theorem 3.2.5
Let $T : X \to Y$ be a Lipschitz mapping where X is a finite weighted tree and Y a Hilbert space. Then, T is Lipschitz 1-summing and

$$\pi_1^L(T) \leq \mathcal{K}_G \cdot \mathrm{Lip}(T).$$

Proof.
By the lifting property of X and Y being a quotient of an L_1 space, there exists a map $S : X \to L_1$ such that $T = Q \circ S$

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "S"'] & & Y \\ & L_1 \arrow[ur, "Q"'] & \end{tikzcd}$$

and for every $\epsilon > 0$

$$\mathrm{Lip}(S) \leq \mathrm{Lip}(T) + \epsilon. \tag{3.1}$$

By the notes after Grothendieck's Theorem 3.2.1, Q is Lipschitz 1-summing and

$$\pi_1^L(Q) \leq \mathcal{K}_G \cdot \mathrm{Lip}(Q). \tag{3.2}$$

By the Non-linear Ideal Property, $T = Q \circ S$ is also Lipschitz 1-summing and

$$\pi_1^L(T) \leq \mathrm{Lip}(S) \cdot \pi_1^L(Q). \tag{3.3}$$

By (3.1) and (3.2) and letting $\epsilon \to 0$, (3.3) becomes

$$\pi_1^L(T) \leq \mathcal{K}_G \cdot \mathrm{Lip}(Q) \cdot \mathrm{Lip}(T).$$

Since Q is a 1-Lipschitz quotient mapping, then co-Lip$(Q) = 1$ and Lip$(Q) \cdot$ co-Lip$(Q) \leq 1$, so that

$$\begin{aligned}
\pi_1^L(T) &\leq \mathcal{K}_G \cdot \mathrm{Lip}(Q) \cdot \mathrm{Lip}(T) \\
&= \mathcal{K}_G \cdot \mathrm{Lip}(Q) \cdot 1 \cdot \mathrm{Lip}(T) \\
&= \mathcal{K}_G \cdot \mathrm{Lip}(Q) \cdot \text{co-}\mathrm{Lip}(Q) \cdot \mathrm{Lip}(T) \\
&\leq \mathcal{K}_G \cdot \mathrm{Lip}(T).
\end{aligned}$$

Therefore

$$\pi_1^L(T) \leq \mathcal{K}_G \cdot \mathrm{Lip}(T). \qquad \blacksquare$$

4 FURTHER RESULTS ON LIPSCHITZ p-SUMMING MAPS

4.1 BOURGAIN'S THEOREM

Before stating and proving Bourgain's Theorem, we have the following definitions and results as can be found in [2] and [6].

Definition 4.1.1
Let (X, δ) and (Y, d) be finite metric spaces. Suppose $f : X \to Y$ is an injective mapping, we denote the *distortion of a map* f by

$$\operatorname{dist}(f) := \inf \left\{ \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(f(x), f(y))}{\delta(x, y)} \cdot \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\delta(x, y)}{d(f(x), f(y))} \right\}.$$

Definition 4.1.2
Let $1 \leq p < \infty$. Suppose X and Y are Banach spaces. An operator $u : X \to Y$ is said to be p-nuclear if there exists operators $w \in \mathcal{L}(\ell_p, Y)$, $v \in \mathcal{L}(X, \ell_\infty)$ and a sequence $\lambda \in \ell_p$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "v"'] & Y \\
\ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[u, "w"']
\end{tikzcd}$$

We denoted by $\mathcal{N}_p(X, Y)$ the collection of all p-nuclear operators from X into Y. With each $u \in \mathcal{N}_p(X, Y)$, we define its p-nuclear norm by

$$\nu_p = \inf \|v\| \cdot \|M_\lambda\| \cdot \|w\|$$

where the infimum is taken over all operators v and w as in the above diagram.

Suppose X is a finite metric space. We denote for each positive number s by $\mathcal{P}_s$ the set of all subsets of X with cardinal $[s]$ and by 2^X the set of all subsets of X where $[s]$ denotes the greatest integer less that or equal to s. Let $\Lambda : \ell_\infty^{2^X} \to \ell_1^{2^X}$ be the diagonal operator defined by $\Lambda_{(A)} = s^{-1}|\mathcal{P}_s|^{-1}$ where $A \subset X$ and $|A| = s$. Define also the map $u : X \to \ell_\infty^{2^X}$ such that for each $x \in X$, $u(x) = \{d(x, A)\}_{A \subset X}$. The map u satisfies the condition that $\operatorname{Lip}(u) \leq 1$. Indeed, for each $x, y \in X$ with $x \neq y$ and every $A \subset X$, we have

$$|u(x)(A) - u(y)(A)| = |d(x, A) - d(y, A)| \leq d(x, y) = \|(x) - (y)\|$$

so that

$$\|u(x) - u(y)\| = \sup_{A \subset X} |u(x)(A) - u(y)(A)| \leq d(x, y) = \|x - y\|.$$

Therefore, $\mathrm{Lip}(u) \leq 1$.

In his paper [2], Bourgain proved that every diagonal operator $\Lambda : \ell_\infty^{2^X} \to \ell_1^{2^X}$ is Lipschitz 1-summing. The proof of Bourgain's result will be given in the following theorem which we will call as Bourgain's Theorem.

Bourgain's Theorem 4.1.3
The diagonal operator $\Lambda : \ell_\infty^{2^X} \to \ell_1^{2^X}$ defined above is Lipschitz 1-summing and

$$\pi_1^L(\Lambda) \leq \log |X|.$$

Proof.
Clearly, Λ is a linear operator. Indeed for every $x, y \in X$, let $\Delta = \Lambda\Big\{\{d(x,A)\}_{A\subset X} + \{d(y,A)\}_{A\subset X}\Big\}$ so that

$$\begin{aligned}
\Delta &= \Lambda\{d(x,A) + d(y,A)\}_{A\subset X} \\
&= \left\{\frac{d(x,A) + d(y,A)}{s|\mathcal{P}_s|}\right\}_{A\subset X} \\
&= \left\{\frac{d(x,A)}{s|\mathcal{P}_s|}\right\}_{A\subset X} + \Lambda\left\{\frac{d(y,A)}{s|\mathcal{P}_s|}\right\}_{A\subset X} \\
&= \Lambda\{d(x,A)\}_{A\subset X} + \Lambda\{d(y,A)\}_{A\subset X}.
\end{aligned}$$

Also

$$\begin{aligned}
\Lambda\Big\{\alpha \cdot \{d(x,A)\}_{A\subset X}\Big\} &= \Lambda\{\alpha \cdot d(x,A)\}_{A\subset X} \\
&= \left\{\frac{\alpha \cdot d(x,A)}{s|\mathcal{P}_s|}\right\}_{A\subset X} = \alpha \cdot \left\{\frac{d(x,A)}{s|\mathcal{P}_s|}\right\}_{A\subset X} \\
&= \alpha \cdot \Lambda\{d(x,A)\}_{A\subset X}.
\end{aligned}$$

We now factor Λ through $\ell_1^{2^X}$ as follows

$$\begin{tikzcd}
\ell_\infty^{2^X} \arrow[r, "\Lambda"] \arrow[d, "v"'] & \ell_1^{2^X} \arrow[d, "w"] \\
\ell_\infty^{2^X} \arrow[r, "\Lambda_1"] & \ell_1^{2^X}
\end{tikzcd}$$

with $v = \mathrm{id}_{\ell_\infty^{2^X}}$ and $w = \mathrm{id}_{\ell_1^{2^X}}$.
Therefore, Λ is 1-nuclear by the definition of the nuclear operator, and

$$\begin{aligned}
\nu_1(\Lambda)_{\ell_\infty^{2^X} \to \ell_1^{2^X}} &= \inf \|v\| \cdot \|\Lambda_1\| \cdot \|w\| = \|\Lambda_1\| \\
&= \|\Lambda\|_{\ell_\infty^{2^X} \to \ell_1^{2^X}} = \int_1^{|X|} \frac{ds}{s} = \log |X|.
\end{aligned}$$

By Remark 1.2.4, Λ is Lipschitz 1-summing and

$$\pi_1^L(\Lambda) = \pi_1(\Lambda) = \left\| \left(\frac{1}{s|\mathcal{P}_s|} \right) \right\|_{\ell_1} \leq \nu_1(\Lambda) = \log |X|. \qquad \blacksquare$$

We also observe by the Non-linear Inclusion Theorem for $1 \leq p < \infty$ that Λ is also Lipschitz p-summing and $\pi_p^L(\Lambda) \leq \log |X|$.

4.2 APPLICATION OF BOURGAIN'S THEOREM

If $|X| = n$, Bourgain in his paper [2] showed that X embeds in ℓ_1 (that is, X admits a Lipschitz embedding into ℓ_1 with distortion at most $C \cdot \log n$ where C is an absolute constant). As a consequence of this known result and an application of Bourgain's Theorem, we have the following theorem.

Theorem 4.2.1
Let $T : X \to \ell_1^{2^X}$ be the embedding with $|X| = n$. Then T is Lipschitz 1-summing and

$$\pi_1^L(T) \leq \log n.$$

Proof.
Since T factors through $\ell_\infty^{2^X}$ as follows

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "u"'] & & \ell_1^{2^X} \\ & \ell_\infty^{2^X} \arrow[ur, "\Lambda"'] & \end{tikzcd}$$

and Λ is Lipschitz 1-summing by Bourgain's Theorem, it follows by the Non-linear Ideal Property that $T = \Lambda \circ u$ is also Lipschitz 1-summing. Furthermore,

$$\pi_1^L(T) = \pi_1^L(\Lambda \circ u) \leq \mathrm{Lip}(u) \cdot \pi_1^L(\Lambda) \leq \pi_1^L(\Lambda) \leq \log |X| = \log n. \qquad \blacksquare$$

We also observe by the Non-linear Inclusion Theorem for $1 \leq p < \infty$ that T is also Lipschitz p-summing and $\pi_p^L(T) \leq \log n$.

4.3 FURTHER STRUCTURAL RESULTS

In their paper [11], Johnson et al noted that if X is any separable Banach space containing ℓ_1, then there is a Lipschitz quotient map from X onto any separable Banach space Y. They went on further by noting a particular case, that is, there is a Lipschitz quotient map from $C[0,1]$ onto ℓ_1. In fact, known results in the linear theory reduce the general theorem to this special case. This provides also the first known examples of pairs of separable Banach spaces X and Y so that there is a Lipschitz quotient map from X to Y but no such linear quotient map.

Bates et al in ([1], Proposition 4.2) proved that if Y is any separable Banach space, then there is a Lipschtz quotient T of ℓ_1 onto Y.

These results by Johnson et al and Bates et al lead us to the following theorem.

Theorem 4.3.1
Let $1 \leq p < \infty$ and H be a Hilbert space. Then the map $u : C[0,1] \to H$ is Lipschitz p-summing.

Proof.
Let $q_1 : C[0,1] \to \ell_1$ and $q_2 : \ell_1 \to H$ be the quotient maps as explained above. Then the map u factors through ℓ_1 as follows

$$\begin{tikzcd} C[0,1] \arrow[rr, "u"] \arrow[dr, "q_1"'] & & H \\ & \ell_1 \arrow[ur, "q_2"'] & \end{tikzcd}$$

By the Non-linear Grothendieck's Theorem, q_2 is Lipschitz 1-summing and

$$\pi_1^L(q_2) \leq \mathcal{K}_G \cdot \mathrm{Lip}(q_2).$$

By the Non-linear Ideal Property, $u = q_2 \circ q_1$ is also Lipschitz 1-summing and

$$\pi_1^L(u) = \pi_1^L(q_2 \circ q_1) \leq \mathrm{Lip}(q_1) \cdot \pi_1^L(q_2) \leq \mathcal{K}_G \cdot \mathrm{Lip}(q_1) \cdot \mathrm{Lip}(q_2).$$

Since $1 \leq p < \infty$, then by the Non-linear Inclusion Theorem, u is Lipschitz p-summing and

$$\pi_p^L(u) \leq \pi_1^L(u) \leq \mathcal{K}_G \cdot \mathrm{Lip}(q_1) \cdot \mathrm{Lip}(q_2). \qquad \blacksquare$$

In the linear theory, we have the Π_2-Extension Theorem as can be found in ([6], Π_2-Extension Theorem 2.15) which states that if X, Y, and W are Banach spaces with X a subspace of W, then each 2-summing operator $u : X \to Y$ admits a 2-summing extension $\hat{u} : W \to Y$ with

$$\pi_2(u) = \pi_2(\hat{u}).$$

An application of Proposition 1.1.3 gives us an analogue of Π_2-Extension Theorem in the non-linear setting.

Theorem 4.3.2 (Π_2^L-Extension Theorem)
If X, Y and W are metric spaces with X a metric subspace of W, then each Lipschitz 2-summing map $u : X \to Y$ admits a Lipschitz 2-summing extension $\hat{u} : W \to Y$ with

$$\pi_2^L(u) = \pi_2^L(\hat{u}).$$

Proof.
Any Π_2^L-summing map $u : X \to Y$ has a factorization

$$\begin{tikzcd}
L_2(\mu) \arrow[d, "B"] & & L_\infty(\mu) \arrow[ll, "I_{\infty,2}"'] & \\
Z & Y \arrow[l, "J"'] & X \arrow[l, "u"'] \arrow[u, "A"] \arrow[r, hook] & W \arrow[ul, "\hat{A}"']
\end{tikzcd}$$

for some isometric embedding J of Y into a 1-injective space Z, where μ is a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \leq C$. By $L_\infty(\mu)$'s injectivity, A has an extension $\hat{A} \in \mathcal{L}ip(W, L_\infty(\mu))$ with $\mathrm{Lip}(\hat{A}) = \mathrm{Lip}(A)$. Hence the factorization

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,2}"] & & L_2(\mu) \arrow[d, "B"] \\
W \arrow[u, "\hat{A}"] \arrow[r, "\hat{u}"] & Y \arrow[r, "J"] & Z
\end{tikzcd}$$

defines the map $\hat{u}$ by $\hat{u} \mid_X = u$ such that $\hat{u}$ is Π_2^L-summing with

$$\begin{aligned}
\pi_2^L(\hat{u}) &\leq \mathrm{Lip}(\hat{A}) \cdot \mathrm{Lip}(B) \\
&= \mathrm{Lip}(A) \cdot \mathrm{Lip}(B).
\end{aligned}$$

It follows that

$$\pi_2^L(\hat{u}) \leq \pi_2^L(u).$$

On the other hand, since $\hat{u}$ extends u

$$\pi_2^L(u) = \pi_2^L(\hat{u} \mid_X) \leq \pi_2^L(\hat{u})$$

and so

$$\pi_2^L(u) = \pi_2^L(\hat{u}). \qquad \blacksquare$$

The proof of the following proposition can be found in ([1], Proposition 4.2).

Proposition 4.3.3 ([1], Proposition 4.2)
For any separable Y, there is a Lipschitz quotient u of ℓ_1 onto Y which maps a hyperplane to zero.

An application of Proposition 4.3.3 and the Π_2^L-Extension Theorem is the following corollary.

Corollary 4.3.4
Suppose that X is a Banach space such that ℓ_1 is Lipschitz embeddable in X, then, X admits a Lipschitz quotient which is Lipschitz equivalent to ℓ_2 where the Lipschitz quotient is Lipschitz 2-summing.

Proof.
By Proposition 4.3.3, if Y is a subspace of X which is Lipschitz equivalent to ℓ_1, we are assured of a Lipschitz quotient map u into $\mathcal{L}ip(Y, \ell_2)$. By the Non-linear Grothendieck's Theorem, u is Lipschitz 2-summing. By the Π_2^L-Extension Theorem, u admits a Π_2^L-extension $\hat{u} : X \to \ell_2$ which is surjective with

$$\pi_2^L(u) = \pi_2^L(\hat{u}). \qquad \blacksquare$$

Recall [11] that a Lipschitz map f from a metric space X into a metric space Y is said to be *ball non-collapsing* provided there is a $c > 0$ so that for every $r > 0$, the image under f of any ball of radius r contains a ball of radius cr. If f is a Lipschitz map from a metric space X into $\mathbb{R}^n$, it is said that f is *measure non-collapsing* provided there is a $c > 0$ so that for every $r > 0$, the image under f of any ball of radius r has Lebesgue measure at least Cr^n.

It has been shown that if X is finite dimensional and $f : X \to \mathbb{R}^n$ is measure non-collapsing, then f is ball non-collapsing. This concept was noted as a weakening of the notion of Lipschitz quotient maps by G. David and S. Semmes in [5]. In ([11], Theorem 2.4), we have the proof of the following theorem.

Theorem 4.3.5 ([11], Theorem 2.4)
Let X be a separable Banach space which has a subspace isomorphic to ℓ_1. Then there is a measure non-collapsing mapping f from X into $\mathbb{R}^2$ whose range is a closed set with empty interior.

The following theorem is an application of Theorem 4.3.5.

Theorem 4.3.6
Let X be a separable Banach space which has a subspace isomorphic to ℓ_1. Then, there is a Lipschitz p-summing map from X into $\mathbb{R}^2$ for $2 \leq p < \infty$ whose range contains a closed set with empty interior.

Proof.
If $\mathbb{R}^2$ is given the usual norm, it becomes a Hilbert space. Let Y be the isomorphic copy of ℓ_1 in X. The restriction to the isomorphic copy of ℓ_1 in X of the map f alleged to in Theorem 4.3.5 is Lipschitz 1-summing by the Non-linear Grothendieck's Theorem. Applying the Non-linear Inclusion Theorem, $f\,|_Y$ is also Lipschitz 2-summing. By the Π_2^L-Extension Theorem, $f\,|_Y$ has a Π_2^L-extension $F : X \to \mathbb{R}^2$. Applying the Non-linear Inclusion Theorem once more, F is Lipschitz p-summing and its range contains a closed set with empty interior. $\blacksquare$

4.4 METRIC TYPE

Bourgain et al introduced in their paper [3] the metric version of type p. But first, suppose Y is an infinite dimensional Banach space and $p \geq 1$, then Y is said to be of *Rademacher type p* if there exists a constant $C > 0$ such that for sequences $(y_1, \cdots, y_n) \in Y$, we have

$$\left(\frac{1}{2^n}\sum_x \left\|\sum_{i=1}^n x_i y_i\right\|^p\right)^{\frac{1}{p}} \leq C \cdot \left(\sum_{i=1}^n \|y_i\|^p\right)^{\frac{1}{p}}$$

where the summing is over all $x \in \{-1,1\}^n$. Bourgain observed in [3] that this definition of Rademacher type p is valid for $1 \leq p \leq 2$.

The introduction of the theory of metric type by Bourgain et al in [3] has implications on the geometric structure of Banach spaces. On the other hand, it enables one to look for results in metric spaces which are extensions of well known results. The metric type p is the analogue of the Rademacher type p for metric spaces and is given in the following definition.

Definition 4.4.1
Suppose $p \geq 1$. An (infinite) dimensional metric space (X, δ) has *metric type p* if there is a constant α such that for every k and any k-cube defined by any map $\varphi : C_2^k \to X$, we have the following inequality

$$\left(\sum_D \mathrm{diag}^2\right)^{\frac{1}{2}} \leq \alpha \cdot k^{\frac{1}{p}-\frac{1}{2}}\left(\sum_E \mathrm{edge}^2\right)^{\frac{1}{2}}$$

where the summing is over all the diagonals and all the edges of the k-cube. The least α for which the above inequality holds will be denoted by $\alpha_p(X)$ and is called the metric p-type constant of (X, δ). For a given k, the smallest constant is denoted by $\alpha_p(k, X)$.

In a similar manner, the metric type constant can also be defined for a family of finite metric spaces (X_N, δ_N) where $|X_N| = N$ $(N \uparrow \infty)$ as the best constant obtained simultaneously for all the spaces.

Suppose (X_N, δ_N) is a family of finite metric spaces where $|X_N| = N$ $(N \uparrow \infty)$. Then such a family of finite metric spaces is atleast of type 1 with the 1-type constant atmost 1, that is, for every k-cube

$$\left(\sum_D \mathrm{diag}^2\right)^{\frac{1}{2}} \leq k^{\frac{1}{2}}\left(\sum_E \mathrm{edge}^2\right)^{\frac{1}{2}}.$$

Indeed, let $\mathrm{diag}(x)$ be any diagonal and we take the identity path $E_x(\mathrm{Id})$. Applying the triangle and Cauchy-Schwartz inequalities, we have

$$\mathrm{diag}^2(x) \leq \left(\sum_{E_x(\mathrm{Id})} \mathrm{edge}\right)^{\frac{1}{2}} \leq k\left(\sum_{E_x(\mathrm{Id})} \mathrm{edge}^2\right). \tag{4.1}$$

Summing over all diagonals, we have

$$\sum_{D} \mathrm{diag}^2(x) \leq k \sum_{D} \Big(\sum_{E_x(\mathrm{Id})} \mathrm{edge}^2 \Big).$$

Therefore

$$\Big(\sum_{D} \mathrm{diag}^2 \Big)^{\frac{1}{2}} \leq k^{\frac{1}{2}} \Big(\sum_{E} \mathrm{edge}^2 \Big)^{\frac{1}{2}}$$

since the edges belonging to the identity paths are all distinct.

Proposition 4.4.2 ([3], Proposition 2.2)
Suppose (X, δ) is a metric space of cardinality 2^k such that there exists a map $\varphi : C_2^k \to X$ for which the following inequality holds

$$\Big(\sum_{D} \mathrm{diag}^2 \Big)^{\frac{1}{2}} = k^{\frac{1}{2}} \Big(\sum_{E} \mathrm{edge}^2 \Big)^{\frac{1}{2}}.$$

Then X is isometric to the C_2^k cube with the usual ℓ_1 (Hamming) metric.

Proof.
The above equality in the proposition means equality in (4.1) for every diagonal. For any x, we take the diagonal $\mathrm{diag}(x)$. Now the equality in the Cauchy-Schwartz inequality (4.1) implies that all the edges in $E_x(\mathrm{Id})$ are mutually equal.
In a similar manner, for any given permutation λ, and every x, all the edges of $E_x(\lambda)$ are mutually equal. Therefore, all $k \cdot 2^{n-1}$ edges are equal. We denote this length by β. The equality in the triangle inequality in (4.1) implies that the length of every diagonal is $k\beta$. We take any x, x' in C_2^k such that the Hamming distance $h(x, x') = h$. Then

$$h\beta = \mathrm{diag}(x) - (k-h)\beta \leq \delta(\varphi(x), \varphi(x')) \leq h\beta.$$

Hence $\delta(\varphi(x), \varphi(x')) = h\beta$, that is $\beta h(x, x') = \delta(\varphi(x), \varphi(x'))$. This shows that X is isometric to the C_2^k cube with the usual ℓ_1 (Hamming) metric.
■

The following theorem is the main result on metric type which is an application of Proposition 4.4.2.

Theorem 4.4.3
Let $1 \leq p < \infty$. If a finite metric space X of cardinality 2^k is of supremal metric type 1, then every Lipschitz mapping from X into a Hilbert space is Lipschitz p-summing.

Proof.
Let T be the Lipschitz mapping from X into a Hilbert space H. By Proposition 4.4.2, X is isometric to the C_2^k with the usual ℓ_1 (Hamming) metric. Let $u : X \to C_2^k$ be the isometric mapping. By the lifting property of ℓ_1 and H being a quotient of an ℓ_1 space, there exists a mapping $\varphi : C_2^k \to \ell_1$

such that $v = Q \circ \varphi$ where Q is a Lipschitz quotient mapping from ℓ_1 into H. We therefore have the following factorization diagram;

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "u"'] & H \\ C_2^k \arrow[ur, "v"] \arrow[r, "\varphi"] & \ell_1 \arrow[u, "Q"'] \end{tikzcd}$$

By the Non-linear Grothendieck's Theorem, Q is Lipschitz 1-summing and

$$\pi_1^L(Q) \leq \mathcal{K}_G \cdot \mathrm{Lip}(Q). \tag{4.2}$$

where $\mathcal{K}_G$ is Grothendieck's constant. Applying the Non-linear Inclusion Theorem for $1 \leq p < \infty$, Q is Lipschitz p-summing and by (4.2)

$$\pi_q^L(Q) \leq \pi_1^L(Q) \leq \mathcal{K}_G \cdot \mathrm{Lip}(Q). \tag{4.3}$$

By the Non-linear Ideal Property, $v = Q \circ \varphi$ is Lipschitz p-summing and by (4.3) we have

$$\pi_p^L(v) \leq \mathrm{Lip}(\varphi) \cdot \pi_p^L(Q) \leq \mathcal{K}_G \cdot \mathrm{Lip}(Q). \tag{4.4}$$

Applying the Non-linear Ideal Property once more, $T = v \circ u$ is Lipschitz p-summing and

$$\pi_p^L(T) \leq \mathrm{Lip}(u) \cdot \pi_p^L(v). \tag{4.5}$$

By (4.4) and (4.5), we have

$$\pi_p^L(T) \leq \mathrm{Lip}(u) \cdot \pi_p^L(v) \leq \mathcal{K}_G \cdot \mathrm{Lip}(\varphi) \cdot \mathrm{Lip}(Q) \cdot \mathrm{Lip}(u). \qquad \blacksquare$$

5 FUNDAMENTALS ON LIPSCHITZ p-INTEGRAL MAPS

5.1 DEFINITIONS

The main purpose of this chapter is to present analogues of p-integral operators in the non-linear setting. The main result in this chapter is that the natural inclusion map $i : \ell_1 \to \ell_2$ is 1-Lipschitz summing but not 1-Lipschitz integral. We now summarize some definitions and results as can be found in [6] and [8].

Definition 5.1.1
Suppose $1 \leq p \leq \infty$. A linear mapping $u : X \to Y$ between Banach spaces X and Y is a *p-integral operator* if there are a probability measure μ and (bounded linear) operators $B : L_p(\mu) \to Y^{**}$ and $A : X \to L_\infty(\mu)$ giving rise to the following commutative diagram

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "u"] & Y \arrow[r, "K_Y"] & Y^{**}
\end{tikzcd}$$

where $I_{\infty,p} : L_\infty(\mu) \to L_p(\mu)$ is the formal identity and $K_Y : Y \to Y^{**}$ is the canonical isometric embedding.

The collection of all p-integral operators from X to Y will be denoted by

$$\mathcal{I}_p(X, Y).$$

With each $u \in \mathcal{I}_p(X, Y)$, its p-integral norm is defined by,

$$\iota_p(u) = \inf \|A\| \cdot \|B\|,$$

where the infimum is taken over all measures μ and operators A and B as in the above diagram.

Inspired by this useful concept, Farmer and Johnson introduced in [8] the following definition;

Definition 5.1.2
Suppose $1 \leq p \leq \infty$. A Lipschitz mapping $u : X \to Y$ between metric spaces is *Lipschitz p-integral* if there are a probability measure μ and Lipschitz mappings $B : L_p \to (Y^\sharp)^*$ and $A : X \to L_\infty(\mu)$ satisfying the following commutative diagram

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & (Y^\sharp)^*
\end{tikzcd}$$

where $I_{\infty,p} : L_\infty(\mu) \to L_p(\mu)$ is the formal identity and $J : Y \to (Y^\sharp)^*$ is the canonical isometry.

The collection of all Lipschitz p-integral mappings from X to Y will be denoted by

$$\mathcal{I}_p^L(X, Y).$$

With each $u \in \mathcal{I}_p^L(X, Y)$, its Lipschitz p-integral semi-norm is given by

$$\iota_p^L(u) = \inf \big(\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\big).$$

5.2 STRUCTURAL RESULTS

Farmer and Johnson in their paper [8] noted the following important non-linear results. Proposition 6.2.1 is the non-linear version of the Inclusion Theorem for p-integral operators which is a consequence of Definition 5.1.1 as noted in ([6], Proposition 5.1). It states that if $u : X \to Y$ is a p-integral operator between Banach spaces X and Y and $1 \leq p < q < \infty$, then u is also q-integral and $\iota_q(u) \leq \iota_p(u)$. That is, $\mathcal{I}_p(X, Y) \subset \mathcal{I}_q(X, Y)$. There is also a version of the Ideal property for p-integral operators in the non-linear setting which is Proposition 5.2.3 in this dissertation. This result was also noted by Farmer and Johnson in their paper [8]. The proof of the Ideal property for p-integral operators in the linear setting can be found in ([6], Theorem 5.2(b)) which states that if $v : X \to Y$ is a p-integral operator between Banach spaces X and Y and for $w \in \mathcal{L}(X_0, X)$ and $u \in \mathcal{L}(Y, Y_0)$, the operator $uvw : X_0 \to Y_0$ is also p-integral where X_0 and Y_0 are also Banach spaces. Furthermore, $\iota_p(uvw) \leq \|u\| \cdot \iota_p(v) \cdot \|w\|$. Proposition 5.2.1 and Proposition 5.2.2 in this dissertation were noted by Farmer and Johnson in their paper [8] which we now state and prove.

Proposition 5.2.1
Let $1 < p \leq q \leq \infty$. If $T : X \to Y$ is a Lipschitz p-integral map between metric spaces X and Y, then it is Lipschitz q-integral and

$$\iota_q^L(T) \leq \iota_p^L(T).$$

Proof.
Suppose T is Lipschitz p-integral, then we have the following factorization diagram

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & (Y^\sharp)^*
\end{tikzcd}$$

with $\iota_p^L(T) = \inf \big(\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\big)$.
Note that $I_{\infty,p}$ can be obtained by composing $I_{q,p}$ with $I_{\infty,q}$, that is,

$I_{\infty,p} = I_{q,p} \circ I_{\infty,q}$, so that we have the following commutative diagram

$$\begin{tikzcd}
L_q(\mu) \arrow[dr, "I_{q,p}"] & \\
L_\infty(\mu) \arrow[u, "I_{\infty,q}"] \arrow[r, "I_{\infty,p}"] & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & (Y^\sharp)^*
\end{tikzcd}$$

The above diagram now becomes

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,q}"] & & L_q(\mu) \arrow[d, "\widetilde{B}"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & (Y^\sharp)^*
\end{tikzcd}$$

with $\widetilde{B} = B \circ I_{q,p}$.
Hence, T is Lipschitz q-integral and

$$\begin{aligned}
\iota_q^L(T) \leq \mathrm{Lip}(A) \cdot \mathrm{Lip}(\widetilde{B}) &= \mathrm{Lip}(A) \cdot \mathrm{Lip}(B \circ I_{q,p}) \\
&\leq \mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \cdot \pi_p^L(I_{q,p}) \\
&= \mathrm{Lip}(A) \cdot \mathrm{Lip}(B).
\end{aligned}$$

Passing to the infimum, we have

$$\iota_q^L(T) \leq \inf \big(\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\big) = \iota_p^L(T). \qquad \blacksquare$$

Proposition 5.2.2
Let $1 \leq p < \infty$. If $T : X \to Y$ is a p-integral linear operator between Banach spaces X and Y, then, it is Lipschitz p-integral and

$$\iota_p^L(T) \leq \iota_p(T).$$

Proof.
Suppose T is p-integral. Then, we have a factorization

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "K_Y"] & Y^{**}
\end{tikzcd}$$

with

$$\iota_p(T) = \inf \|A\| \cdot \|B\|.$$

Since Y^{**} is norm one complemented in $(Y^\sharp)^*$ via a projection P, then we have a factorization

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \arrow[dr, "\widetilde{B}"] & \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "K_Y"] & Y^{**} \arrow[r, "P_{|Y^{**}}"] & (Y^\sharp)^*
\end{tikzcd}$$

with $P_{|_{Y^{**}}} = \mathrm{id}_{Y^{**}}$.
The above diagram now becomes

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,q}"] & & L_p(\mu) \arrow[d, "\widetilde{B}"] \\ X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & (Y^\sharp)^* \end{tikzcd}$$

with $\widetilde{B} = P_{|_{Y^{**}}} \circ B$ and $J = P_{|_{Y^{**}}} \circ K_Y$.
Hence, T is Lipschitz p-integral and

$$\begin{aligned} \iota_p^L(T) &\leq \|A\| \cdot \|\widetilde{B}\| \\ &= \|A\| \cdot \|P_{|_{Y^{**}}} \circ B\| \\ &\leq \|A\| \cdot \|P_{|_{Y^{**}}}\| \cdot \|B\| \\ &= \|A\| \cdot \|\mathrm{id}_{Y^{**}}\| \cdot \|B\| \\ &= \|A\| \cdot \|B\|. \end{aligned}$$

Passing to the infimum, we have

$$\iota_p^L(T) \leq \inf \|A\| \cdot \|B\| = \iota_p(T). \qquad \blacksquare$$

Proposition 5.2.3 (Ideal Property of Lipschitz p-Integral maps)
Let $v : X \to Y$ be Lipschitz p-integral, $w : X_0 \to X$ and $u : Y \to Y_0$ be Lipschitz mappings between metric spaces. Then, $uvw : X_0 \to Y_0$ is Lipschitz p-integral and

$$\iota_p^L(uvw) \leq \mathrm{Lip}(w) \cdot \iota_p^L(v) \cdot \mathrm{Lip}(u).$$

Proof.
We first consider a composition of Lipschitz mappings

$$X_0 \xrightarrow{w} X \xrightarrow{v} Y \xrightarrow{u} Y_0.$$

Since v is Lipschitz p-integral, then we have a factorization

$$\begin{tikzcd} X_0 \arrow[r, "A_0"] \arrow[dr, "w"'] & L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\ & X \arrow[u, "A"] \arrow[r, "v"] & Y \arrow[r, "J_Y"] & (Y^\sharp)^* \end{tikzcd}$$

with $A_0 = A \circ w$ and $\iota_p^L(v) = \inf \big(\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\big)$.

Taking note of the fact that we also have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "J_Y"] \arrow[d, "u"'] & (Y^\sharp)^* \arrow[d, "(u^\sharp)^*"] \\ Y_0 \arrow[r, "J_{Y_0}"] & (Y_0^\sharp)^* \end{tikzcd}$$

with $J_{Y_0} \circ u = (u^\sharp)^* \circ J_Y$, we arrive at the following factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "\widetilde{B}"] \\ X_0 \arrow[u, "A_0"] \arrow[r, "uvw"] & Y_0 \arrow[r, "J_{Y_0}"] & (Y_0^\sharp)^* \end{tikzcd}$$

with $A_0 = A \circ w$ and $\widetilde{B} = (u^\sharp)^* \circ B$.
Hence, uvw is Lipschitz p-integral and

$$\begin{aligned} \iota_p^L(uvw) &\leq \mathrm{Lip}(A_0) \cdot \mathrm{Lip}(\widetilde{B}) \\ &= \mathrm{Lip}(A \circ w) \cdot \mathrm{Lip}\big((u^\sharp)^* \circ B\big) \\ &\leq \mathrm{Lip}(A) \cdot \mathrm{Lip}(w) \cdot \mathrm{Lip}((u^\sharp)^*) \cdot \mathrm{Lip}(B) \\ &= \mathrm{Lip}(A) \cdot \mathrm{Lip}(w) \cdot \mathrm{Lip}(u) \cdot \mathrm{Lip}(B). \end{aligned}$$

Taking the infimum, we have

$$\begin{aligned} \iota_p^L(uvw) &\leq \inf\Big(\big(\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\big) \cdot \mathrm{Lip}(u) \cdot \mathrm{Lip}(w)\Big) \\ &= \mathrm{Lip}(u) \cdot \iota_p^L(v) \cdot \mathrm{Lip}(w). \end{aligned}$$

■

Diestel et al proved in their book ([6], Proposition 5.1) that if an operator $v : X \to Y$ between Banach spaces X and Y is p-integral for $1 \leq p < \infty$, then, it is p-summing and

$$\pi_p(v) \leq \iota_p(v).$$

There is also an analogue of this proposition in the non-linear setting which is Proposition 5.2.4 in this dissertation.

Proposition 5.2.4
Let $1 \leq p < \infty$. Suppose X and Y are metric spaces. If $u : X \to Y$ is Lipschitz p-integral, then it is Lipschitz p-summing and

$$\pi_p^L(u) \leq \iota_p^L(u).$$

Proof.
Since u is Lipschitz p-integral, then we have the following Lipschitz p-integral factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\ X \arrow[u, "A"] \arrow[r, "u"] & Y \arrow[r, "J"] & (Y^\sharp)^* \end{tikzcd}$$

Since $I_{\infty,p}$ is Lipschitz p-summing by Remark 1.2.4 with $\pi_p^L(I_{\infty,p}) = 1$, then by the Non-linear Ideal Property, Ju is also Lipschitz p-summing. Since J

is an Isometry, then by the Injectivity of Π_p^L, u is also Lipschitz p-summing and

$$\begin{aligned}\pi_p^L(u) = \pi_p^L(Ju) &= \pi_p^L(B \circ I_{\infty,p} \circ A)\\ &\leq \mathrm{Lip}(B) \cdot \pi_p^L(I_{\infty,p}) \cdot \mathrm{Lip}(A)\\ &= \mathrm{Lip}(A) \cdot \mathrm{Lip}(B).\end{aligned}$$

Passing to the infimum, we have

$$\pi_p^L(u) \leq \inf\{\mathrm{Lip}(A) \cdot \mathrm{Lip}(B)\} = \iota_p^L(u). \qquad \blacksquare$$

We recall [6] that a linear map $u : X \to Y$ between Banach spaces X and Y is p-summing if and only if it takes weakly p-summable sequences in X to strongly p-summable sequences in Y. The class of linear maps $u : X \to Y$ which take weakly null sequences in X to norm null sequences in Y are the natural companion of p-summing operators and these maps are said to be *completely continuous.*

If we take limits, the linear map $u : X \to Y$ is completely continuous precisely when it takes weakly convergent sequences to norm convergent sequences and this happens when u takes weakly compact sets into norm compact sets. In fact, if $u : X \to Y$ is completely continuous, then u takes each weakly Cauchy sequence into a norm Cauchy (and therefore convergent) sequence.

Proposition 5.2.5 ([6], Example 5.11.)
The natural inclusion map $i : \ell_1 \to \ell_2$ is 1-summing but not 1-integral.

Proof.
By a remark after Grothendieck's Theorem 3.2.1, the natural inclusion map $i : \ell_1 \to \ell_2$ is 1-summing. Suppose on the contrary that it is also 1-integral. Then, we have the following commutative diagram

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,1}"] & & L_1(\mu) \arrow[d, "B"] \\ \ell_1 \arrow[u, "A"] \arrow[r, "i"] & \ell_2 \arrow[r, "K_{\ell_2}"] & (\ell_2^*)^* \end{tikzcd}$$

with $(\ell_2^*)^* = \ell_2$ and $\iota_1(i) = \inf\big(\|A\| \cdot \|B\|\big)$.

We therefore have the following commutative diagram

$$\begin{tikzcd} L_\infty(\mu) \arrow[r, "I_{\infty,1}"] & L_1(\mu) \arrow[d, "B"] \\ \ell_1 \arrow[u, "A"] \arrow[r, "i"] & \ell_2 \end{tikzcd}$$

By Remark 1.2.4, $I_{\infty,1} \circ A$ is 1-summing. Applying Gronthendieck's Theorem 3.2.1, B is also 1-summing. Therefore, $i = B \circ I_{\infty,1} \circ A$ would be a compact operator by a remark after Proposition 5.2.4. Since for all $(e_n) \in \ell_1$, $(ie_n) \in \ell_2$, there exists a convergent subsequence e_{n_k} of (e_n) which must then be a Cauchy sequence. But $\|e_{n_k} - e_{n_l}\| = \sqrt{2} \nrightarrow 0$ as $k, l \to \infty$. Hence, we have a contradiction and therefore, i cannot be 1-integral. ■

The above proposition leads us to the main result of this chapter which is an application of the p-summing and p-integral operators for $1 \leq p < \infty$ in the non-linear theory.

Proposition 5.2.6
The natural inclusion map $i : \ell_1 \to \ell_2$ is 1-Lipschitz summing but not 1-Lipschitz integral.

Proof.
By Proposition 5.2.5, i is 1-summing so that it is 1-Lipschitz summing by Proposition 1.2.3. Since it is not 1-integral by Proposition 5.2.5, we see from Proposition 5.2.2 that it cannot be 1-Lipschitz integral and this completes the proof of the proposition. ■

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