A BIVARIATE GENERALISATION OF GAMMA DISTRIBUTION

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Abstract: In this paper a bivariate generalisation of the gamma distribution is proposed by using an unsymmetrical bivariate characteristic function; an extension to the noncentral case also receives attention. The probability density functions of the product and ratio of the correlated components of this distribution are also derived. The benefits of introducing this generalised bivariate gamma distribution and the distributions of the product and the ratio of its components will be demonstrated by graphical representations of their density functions. An example of this generalised bivariate gamma distribution to rainfall data for two specific districts in the North West province is also given to illustrate the greater versatility of the new distribution.

Mathematical subject classification: 62E20

Key words: bivariate characteristic function; Laguerre polynomial; noncentral; product; ratio; rainfall data.

1 Introduction

A bivariate generalisation of the gamma distribution can be obtained by considering the joint distribution with bivariate characteristic function

$$\phi(t_1,t_2) = (1-it_1)^{-n_1}(1-it_2)^{-n_2}(1+\frac{\lambda t_1 t_2}{(1-it_1)(1-it_2)})^{-n_3}. \tag{1}$$

The parameter λ in (1) is related to the density correlation ρ by the expression: $\rho = \dfrac{n_3\lambda}{\sqrt{n_1 n_1}}$. This bivariate characteristic function (1) produces the marginals of characteristic functions of gamma variables. The first major focus in this paper is to derive the corresponding bivariate distribution of (1) by using the inverse Fourier transform; it uniquely determines the distribution. This generalised bivariate gamma distribution will be extended to the noncentral case by introducing marginals of characteristic functions of noncentral gamma variables.

The distributions of products and ratios of correlated gamma variables are of interest in many areas of science for example in hydrology, rainfall and water quality (see e.g. Nadarajah and Kotz (2007a), (2007b) and Loáiciga and Leipnik (2005)). The proposed generalised bivariate gamma distribution can be helpful in constructing plausible models for such data. Therefore, in this paper we also study the product and the ratio of the correlated components of this generalised bivariate gamma distribution.

The paper is organized as follows: The key idea is presented in Section 2, that is a generalised bivariate gamma distribution by deriving the probability density function (p.d.f.) from (1). In section 3 and 4, the exact expressions for the distributions of the product X_1X_2 and the ratios $\frac{X_1}{X_2}$ and $\frac{X_1}{X_1+X_2}$ are derived where (X_1,X_2) has this generalised bivariate gamma distribution. In section 4 some percentage points of $\frac{X_1}{X_2}$ are also included - since the measure $P(X_1 < X_2)$ is of interest in the study of the stress-strength model that describes the lifetime of a component with random strength X_2 subjected to a random stress X_1. In section 5 we extend the generalised bivariate gamma distribution to the noncentral case. The benefits of introducing the generalised bivariate gamma distribution and the distributions of the product and the ratio of its components will be demonstrated by graphical representations of their density functions. An example to rainfall data is used to compare this model with other bivariate gamma models.

2 A generalised bivariate gamma distribution

In this section the exact expression for the joint p.d.f. of the generalised bivariate gamma distribution is derived, utilizing the inverse Fourier transform. Some properties of this distribution are also given.

2.1 Genesis of the distribution

Theorem 1

Given the bivariate characteristic function in (1) where X_1 and X_2 are standard gamma variables with parameters n_1 and n_2, the joint p.d.f. is given by

$$f_{X_1,X_2}\left(x_1,x_2\right) = x_1^{n_1-1} x_2^{n_2-1} e^{-x_1-x_2} \sum_{k=0}^{\infty} \frac{\left(n_3\right)_k \, k! \, \lambda^k}{\Gamma(n_1+k)\Gamma(n_2+k)} L_k^{n_1-1}[x_1] \, L_k^{n_2-1}[x_2] \tag{2}$$

for $x_1, x_2 > 0, n_1, n_2, n_3 > 0, 0 \leq \lambda \leq \frac{\sqrt{n_1 n_2}}{n_3}$ and where $L_k^{\alpha}[z] = \sum_{j=0}^{k} (-1)^j \binom{k+\alpha}{k-j} \frac{z^j}{j!}$ is the Laguerre polynomial of order k (see Gradshteyn and Ryzhik (2007), pp. 1000).

Proof:

The p.d.f. can be calculated by taking the inverse Fourier exponential transform of the bivariate characteristic function in (1):

$$f_{X_1,X_2}\left(x_1,x_2\right) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x_1-it_2x_2} (1-it_1)^{-n_1} (1-it_2)^{-n_2} \left(1 - \frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)}\right)^{-n_3} dt_1 dt_2$$

Replacing $\left(1 - \frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)}\right)^{-n_3}$ with the binomial series $\sum_{k=0}^{\infty} (n_3)_k \frac{\left(\frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)}\right)^k}{k!}$, and using term wise integration, the p.d.f. can be written as

$$f_{X_1,X_2}(x_1,x_2) = \sum_{k=0}^{\infty} \frac{(n_3)_k \, \lambda^k}{k!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it_1x_1} (it_1)^k (1-it_1)^{-n_1-k} \, dt_1 \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it_2x_2} (it_2)^k (1-it_2)^{-n_2-k} \, dt_2$$

The integrals are solved by using Gurland (1955, pp.124). After simplifying, we establish the result given by equation (2). ∎

Remarks 1.1

1. Using the characteristic function given in (1), various moments about the origin are calculated:

$$E(X_i) = \left.\frac{-i\,\partial\phi(t_1,t_2)}{\partial t_1}\right|_{t_1=t_2=0} = n_i \text{ , for } i=1,2$$

$$E(X_i^2) = \left.\frac{-\,\partial^2\phi(t_1,t_2)}{\partial t_i^2}\right|_{t_1=t_2=0} = n_i(n_i+1) \text{ , for } i=1,2$$

$$E(X_1X_2) = \left.\frac{-\,\partial^2\phi(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} = n_1n_2 + n_3\lambda$$

Therefore we have $var(X_i) = n_i$ for $i = 1, 2$ and $cov(X_1, X_2) = n_3\lambda$. Pearson's product moment correlation coefficient is then given by $\frac{n_3\lambda}{\sqrt{n_1 n_2}}$. Consequently we have $0 \leq \lambda \leq \frac{\sqrt{n_1 n_2}}{n_3}$. This shows that if two correlated gamma variables are jointly distributed according to equation (2), then no negative value of the correlation coefficient exists between the two variables.

2. To illustrate the dependence character of this generalised bivariate gamma distribution, note that (2) can be expressed as follows:

$$f_{X_1,X_2}(x_1,x_2) = f_{n_1}(x_1)\, f_{n_2}(x_2) \sum_{k=0}^{\infty} \frac{(n_3)_k\, k!\, \lambda^k}{(n_1)_k\,(n_2)_k} L_k^{n_1-1}[x_1]\, L_k^{n_2-1}[x_2]$$

$(x_1, x_2 > 0, n_1, n_2, n_3 > 0{,}0 \leq \lambda \leq \frac{\sqrt{n_1 n_2}}{n_3})$ where $f_{n_i}(x_i) = \dfrac{1}{\Gamma(n_i)} x_i^{n_i-1} e^{-x_i}, i = 1, 2.$

3. Provided the variables X_1 and X_2 are transposed to $X_1 + \xi_1$ and $X_2 + \xi_2$, similarly to the transposed bivariate gamma distribution discussed by Loáiciga and Leipnik (2007), the joint p.d.f. of X_1 and X_2 can be calculated using (1) and is given by

$$\begin{aligned} &f_{X_1,X_2}(x_1,x_2) \\ &= (x_1-\xi_1)^{n_1-1}(x_2-\xi_2)^{n_2-1}e^{-(x_1-\xi_1)-(x_2-\xi_2)} \textstyle\sum_{k=0}^{\infty} \frac{(n_3)_k\, k!\, \lambda^k}{\Gamma(n_1+k)\Gamma(n_2+k)} L_k^{n_1-1}[x_1-\xi_1]\, L_k^{n_2-1}[x_2-\xi_2] \end{aligned}$$

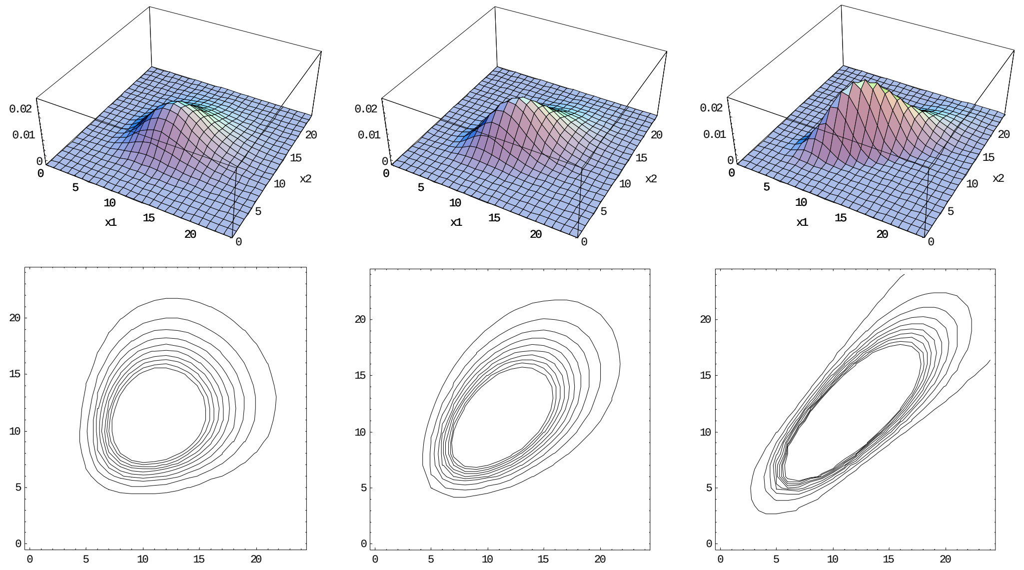
for $x_i > \xi_i$, $i = 1, 2$ (see van den Berg, 2010).

4. The bivariate characteristic function in (1) can easily be extended for two parameter gamma variables X_1 and X_2.

5. Note that $\int_0^\infty \int_0^\infty f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2 = 1$ for $f_{X_1,X_2}(x_1,x_2)$ defined in (2) by using equation (8) of Gradshteyn and Ryzhik (2007 pp. 809).

6. For $n_3 = n_1$,(1) reduces to the equicorrelated case of the bivariate characteristic function proposed by Jensen (1970) (we refer to the latter model as the standardised Jensen model); and if $n_3 = n_2 = n_1$ then (1) is the bivariate characteristic function proposed by Kibble (1941) (see Balakrishnan and Lai (2009) pp. 306; 315).

7. Figure 1 illustrates the effect of parameter n_3 on the generalised bivariate gamma distribution with p.d.f. (2). In figure 1 the presentations are for $n_3 = 4$, 12 and 20 respectively with $n_1 = n_2 = 12$ and λ= 0.5. The parameter n_3 influences the shape of the plot where higher values of n_3 result in higher peaks. The correlation of each example is also given. The programming was done by making use of built-in routines of the package Mathematica.



(a)$n_3 = 4$ $\text{corr}(X_1, X_2) = 0.166$ (b)$n_3 = 12$ $\text{corr}(X_1, X_2) = 0.5$ (c)$n_3 = 20$ $\text{corr}(X_1, X_2) = 0.833$

Figure 1: The generalised bivariate gamma distribution (see (2)) **for** $n_1 = n_2 = 12$ **and** $\lambda = 0.5$ **and different values of** n_3.

Remarks 1.2

The bivariate characteristic function:

$$\phi(t_1, t_2) = e^{i(\zeta_1 t_1 + \zeta_2 t_2)}((1 - it_1 b_1)^{-\alpha_1}(1 - it_2 b_2)^{-\alpha_2} + \beta t_1 t_2)^{-\gamma} \tag{3}$$

was proposed by Loáiciga and Leipnik (2005, page 332, equation(15)) and is also a generalization of Kibble's model. For $b_1 = b_2 = 1, \zeta_1 = \zeta_2 = 0$, it will be compared to our model in section 6.

3 Distribution of the product

In this section, the exact expression for the distribution of the product X_1X_2 is derived when (X_1, X_2) follows this generalised bivariate gamma distribution (2).

Theorem 2

If X_1 and X_2 are jointly distributed according to equation (2), the p.d.f. of $Z = X_1X_2$ is given by

$$f_Z(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} \frac{2K_{n_2-n_1-j+l}(2\sqrt{z})\, z^{\frac{1}{2}(n_1+n_2+j+l)-1}(n_3)_k \binom{k}{j}\binom{k}{l}\lambda^k(-1)^{j+l}}{k!\,\Gamma(n_1+j)\,\Gamma(n_2+l)} \tag{4}$$

for $z > 0$,$n_1, n_2, n_3 > 0$,$0 \le \lambda \le \frac{\sqrt{n_1 n_2}}{n_3}$and where $K_m(.)$ is the modified Bessel function of the second kind (see section 8.4 of Gradshteyn and Ryzhik (2007)).

Proof:

Consider the transformation defined by $Z = X_1X_2$. The p.d.f. of Z and X_2 is given by

$$
\begin{aligned}
f_{Z,X_2}(z,x_2) &= \frac{1}{x_2} f_{X_1,X_2}\left(\frac{z}{x_2},x_2\right) \\
&= \frac{1}{x_2}\left(\frac{z}{x_2}\right)^{n_1-1} x_2^{n_2-1} e^{-\frac{z}{x_2}-x_2} \sum_{k=0}^{\infty} \frac{(n_3)_k\, k!\, \lambda^k}{\Gamma(n_1+k)\Gamma(n_2+k)} L_k^{n_1-1}[\frac{z}{x_2}] L_k^{n_2-1}[x_2]
\end{aligned}
$$

Using the series expansion for the Laguerre polynomials $L_k^{n_1-1}[\frac{z}{x_2}]$ and $L_k^{n_2-1}[x_2]$ it follows that

$$
f_{Z,X_2}(z,x_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} z^{n_1+j-1} x_2^{n_2-n_1-j+l-1} e^{-\frac{z}{x_2}-x_2} \frac{(n_3)_k \binom{k}{j}\binom{k}{l} \lambda^k (-1)^{j+l}}{k!\,\Gamma(n_1+j)\Gamma(n_2+l)}
$$

Now, the p.d.f. of Z is given by

$$
\begin{aligned}
f_Z(z) &= \int_0^{\infty} f_{Z,X_2}(z,x_2)\, dx_2 \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} \frac{z^{n_1+j-1}(n_3)_k \binom{k}{j}\binom{k}{l} \lambda^k (-1)^{j+l}}{k!\,\Gamma(n_1+j)\Gamma(n_2+l)} \int_0^{\infty} x_2^{n_2-n_1-j+l-1} e^{-\frac{z}{x_2}-x_2}\, dx_2
\end{aligned}
$$

The integral is solved by equation (9) from Gradshteyn and Ryzhik (2007 pp. 368). After simplifying, we establish the result given by equation (4). ∎

Remark

Figure 2 displays the p.d.f. of $Z = X_1X_2$ given by equation (4) for $n_3 = 4, 12$ and 20 respectively with $n_1 = n_2 = 12$ and $\lambda = 0.5$. For fixed n_1, n_2 and λ, the p.d.f. of Z is stochastically increasing with respect to n_3.

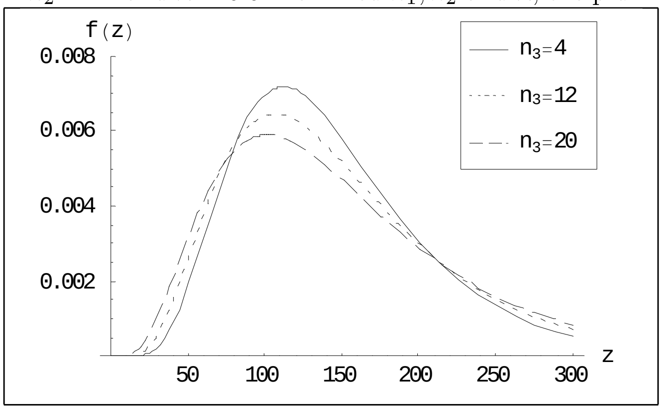


Figure 2: The p.d.f. of Z=X_1X_2 (see (4)) **for** $n_1 = n_2 = 12$ **and** $\lambda = 0.5$ **and different values of** n_3.

4 Distribution of the ratios

In this section the exact expressions for the distributions of the ratios $\frac{X_1}{X_2}$ and $\frac{X_1}{X_1+X_2}$ are derived when (X_1, X_2) follows the generalised bivariate gamma distribution (2).

4.1 The distribution of $\frac{X_1}{X_2}$

Theorem 3

If X_1 and X_2 are jointly distributed according to equation (2), the p.d.f. of $W = \frac{X_1}{X_2}$ is given by

$$
f_W(w) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} \frac{w^{n_1+j-1}(n_3)_k \binom{k}{j}\binom{k}{l} \lambda^k (-1)^{j+l}\Gamma(n_1+n_2+j+l)}{(w+1)^{n_1+n_2+j+l}\, k!\,\Gamma(n_1+j)\,\Gamma(n_2+l)} \tag{5}
$$

for $w > 0, n_1, n_2, n_3 > 0, 0 \le \lambda \le \frac{\sqrt{n_1 n_2}}{n_3}$.

Proof:

Consider the transformation defined by $W = \frac{X_1}{X_2}$. The p.d.f. of W and X_2 is given by

$$\begin{aligned} f_{W,X_2}(w, x_2) &= x_2 f_{X_1,X_2}(w x_2, x_2) \\ &= x_2 (w x_2)^{n_1-1} x_2^{n_2-1} e^{-w x_2 - x_2} \sum_{k=0}^{\infty} \frac{(n_3)_k\, k!\, \lambda^k}{\Gamma(n_1+k)\Gamma(n_2+k)} L_k^{n_1-1}[w x_2] L_k^{n_2-1}[x_2] \end{aligned}$$

Using the series expansion for the Laguerre polynomials $L_k^{n_1-1}[w x_2]$ and $L_k^{n_2-1}[x_2]$ we have

$$f_{W,X_2}(w, x_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{k} w^{n_1+j-1} x_2^{n_1+n_2+j+l-1} e^{-(w+1)x_2} \frac{(n_3)_k \binom{k}{j} \binom{k}{l} \lambda^k (-1)^{j+l}}{k!\, \Gamma(n_1+j)\, \Gamma(n_2+l)}$$

Now, the p.d.f. of W is given by

$$\begin{aligned} f_W(w) &= \int_0^{\infty} f_{W,X_2}(w, x_2)\, dx_2 \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{k} \frac{w^{n_1+j-1} (n_3)_k \binom{k}{j} \binom{k}{l} \lambda^k (-1)^{j+l}}{k!\, \Gamma(n_1+j)\, \Gamma(n_2+l)} \int_0^{\infty} x_2^{n_1+n_2+j+l-1} e^{-(w+1)x_2}\, dx_2 \end{aligned}$$

The integral is solved by equation (3) from Gradshteyn and Ryzhik (2007) pp 340. After simplifying, we establish the result given by equation (5). ∎

Remarks

1. Figure 3 displays the p.d.f. of $W = \frac{X_1}{X_2}$ given by equation (5) for $n_3 = 4, 12$ and 20 respectively with $n_1 = n_2 = 12$ and λ= 0.5. For fixed n_1, n_2 and λ, the p.d.f. of W is stochastically decreasing with respect to n_3.

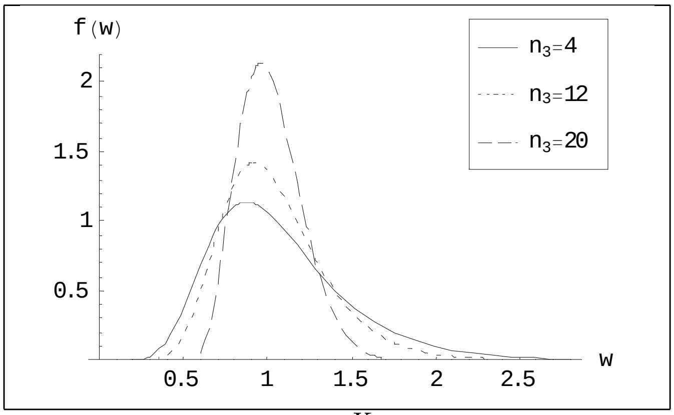


Figure 3: The p.d.f. of $W = \dfrac{X_1}{X_2}$ (see (5)) **for** $n_1 = n_2 = 12$ **and** $\lambda = 0.5$ **and different values of** n_3.

2. The stress-strength model describes the lifetime of a component with random strength X_2 subjected to a random stress X_1. The measure $P(X_1 < X_2)$ is of interest, therefore we give some percentage points of $W = \dfrac{X_1}{X_2}$ in Table 1, where (X_1, X_2) has the generalised bivariate gamma distribution (2). The percentage points w_α of W are obtained numerically by solving the equation $\int_0^{w_\alpha} f(w)dw = \alpha$ for $n_1 = n_2 = 12$ and $\lambda = 0.5$. The programming was done by making use of built-in routines of the package Mathematica. Note the percentage points in Table 1 confirm the shapes for the densities in figure 3.

Table 1: The percentage points w_α of W (see (5)).

n_3	$\alpha = 0.01$	0.1	0.25	0.5	0.75	0.99
4	0.406698	0.615181	0.775664	1	1.28922	2.45882
12	0.494215	0.685124	0.820873	1	1.12821	2.02341
20	0.675878	0.787844	0.879561	1	1.13693	1.47956

4.2 The distribution of $\frac{X_1}{X_1+X_2}$

Theorem 4

If X_1 and X_2 are jointly distributed according to equation (2), the p.d.f. of $U = \frac{X_1}{X_1+X_2}$ is given by

$$f_U(u) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} \frac{u^{n_1+j-1}(1-u)^{n_2+l-1}(n_3)_k\binom{k}{j}\binom{k}{l}\lambda^k(-1)^{j+l}\Gamma(n_1+n_2+j+l)}{k!\,\Gamma(n_1+j)\,\Gamma(n_2+l)} \tag{6}$$

for $0 < u < 1$,$n_1, n_2, n_3 > 0$,$0 \le \lambda \le \frac{\sqrt{n_1 n_2}}{n_3}$.

Proof:

If $U = \frac{X_1}{X_1+X_2}$ and $W = \frac{X_1}{X_2}$, then $W = \frac{U}{1-U}$. Because $f_U(u) = f_W\left(\frac{u}{1-u}\right)\frac{dw}{du}$ and $\frac{dw}{du} = \frac{1}{(1-u)^2}$, we have

$$f_U(u) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k} \frac{\left(\frac{u}{1-u}\right)^{n_1+j-1}(n_3)_k\binom{k}{j}\binom{k}{l}\lambda^k(-1)^{j+l}\Gamma(n_1+n_2+j+l)}{(\frac{u}{1-u}+1)^{n_1+n_2+j+l}\,k!\,\Gamma(n_1+j)\,\Gamma(n_2+l)}\frac{1}{(1-u)^2}$$

After simplifying, we establish the result given by equation (6). ∎

Remark

Figure 4 illustrates the p.d.f. of $U = \frac{X_1}{X_1+X_2}$ (see (6)) for n_3 =4, 12 and 20 respectively for $n_1 = n_2 = 12$ and λ= 0.5. Note that each p.d.f. is symmetric about its mean.

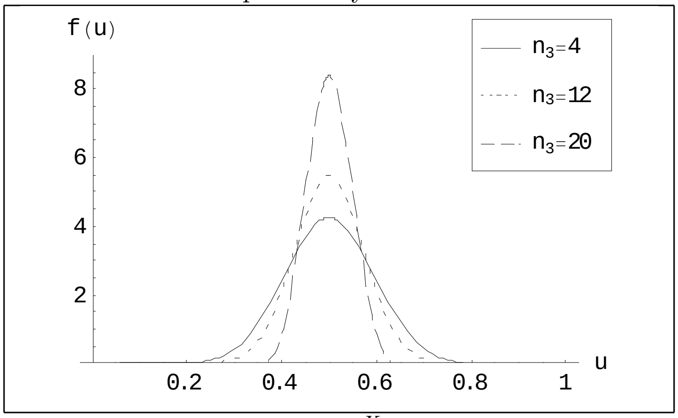


Figure 4: The p.d.f. of $U = \frac{X_1}{X_1+X_2}$ (see (6)) for $n_1 = n_2 = 12$ and $\lambda = 0.5$ and different values of n_3.

5 The noncentral case

In this section the noncentral case of the bivariate generalisation of the gamma distribution (2) is studied of which the marginal distributions are noncentral gamma with p.d.f. given by

$$f_{X_i}(x_i) = \sum_{k=0}^{\infty}\frac{e^{-\delta_i}\delta_i^k}{k!}\frac{x_i^{n_i+k-1}e^{-x_i}}{\Gamma(n_i+k)}$$

for $x_i, n_i, \delta_i > 0$ for $i = 1, 2$.

We propose the bivariate characteristic function with correlated noncentral gamma marginals X_1 and X_2 given by

$$\phi(t_1,t_2) = e^{i\delta_1 t_1(1-it_1)^{-1}}(1-it_1)^{-n_1} e^{i\delta_2 t_2(1-it_2)^{-1}}(1-it_2)^{-n_2}(1+\frac{\lambda t_1 t_2}{(1-it_1)(1-it_2)})^{-n_3} \tag{7}$$

Theorem 5

Given the bivariate characteristic function in (7) where X_1 and X_2 are noncentral gamma variables with parameters (n_1, δ_1) and (n_2, δ_2), the joint p.d.f. is given by

$$f_{X_1,X_2}(x_1,x_2) = x_1^{n_1-1}x_2^{n_2-1}e^{-x_1-x_2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n_3)_k\lambda^k\delta_1^j\delta_2^l(k+j)!(k+l)!(-1)^{j+l}}{k!j!l!\,\Gamma(n_1+k+j)\Gamma(n_2+k+l)}L_{k+j}^{n_1-1}[x_1]L_{k+l}^{n_2-1}[x_2] \tag{8}$$

for $x_1, x_2 > 0$, $n_1, n_2, n_3, \delta_1, \delta_2 > 0$, $0 \le \lambda \le \frac{\sqrt{(n_1+2\delta_1)(n_2+2\delta_2)}}{n_3}$.

Proof:

The p.d.f. can be calculated by taking the inverse Fourier exponential transform of the c.f. given in (7).

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x_1-it_2x_2}e^{i\delta_1 t_1(1-it_1)^{-1}}(1-it_1)^{-n_1} \\ &\quad \cdot e^{i\delta_2 t_2(1-it_2)^{-1}}(1-it_2)^{-n_2}(1-\frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)})^{-n_3}\, dt_1 dt_2 \end{aligned}$$

Replacing $(1-\frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)})^{-n_3}$ with the binomial series $\sum_{k=0}^{\infty}(n_3)_k\frac{(\frac{\lambda it_1 it_2}{(1-it_1)(1-it_2)})^k}{k!}$ and $e^{i\delta_1 t_1(1-it_1)^{-1}}$ and $e^{i\delta_2 t_2(1-it_2)^{-1}}$ with the exponential series $\sum_{j=0}^{\infty}\frac{\left(\frac{i\delta_1 t_1}{1-it_1}\right)^j}{j!}$ and $\sum_{l=0}^{\infty}\frac{\left(\frac{i\delta_2 t_2}{1-it_2}\right)^l}{l!}$, and using term wise integration, the p.d.f. can be written as

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(n_3)_k\,\lambda^k\delta_1^j\delta_2^l}{k!j!l!}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-it_1x_1}(1-it_1)^{-n_1}\left(\frac{it_1}{(1-it_1)}\right)^{k+j}dt_1 \\ &\quad \cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-it_2x_2}(1-it_2)^{-n_2}\left(\frac{it_2}{(1-it_2)}\right)^{k+l}dt_2 \end{aligned}$$

The integrals are solved using Gurland (1955, pp.124). After simplifying, we establish the result given by equation (8). ∎

Remarks

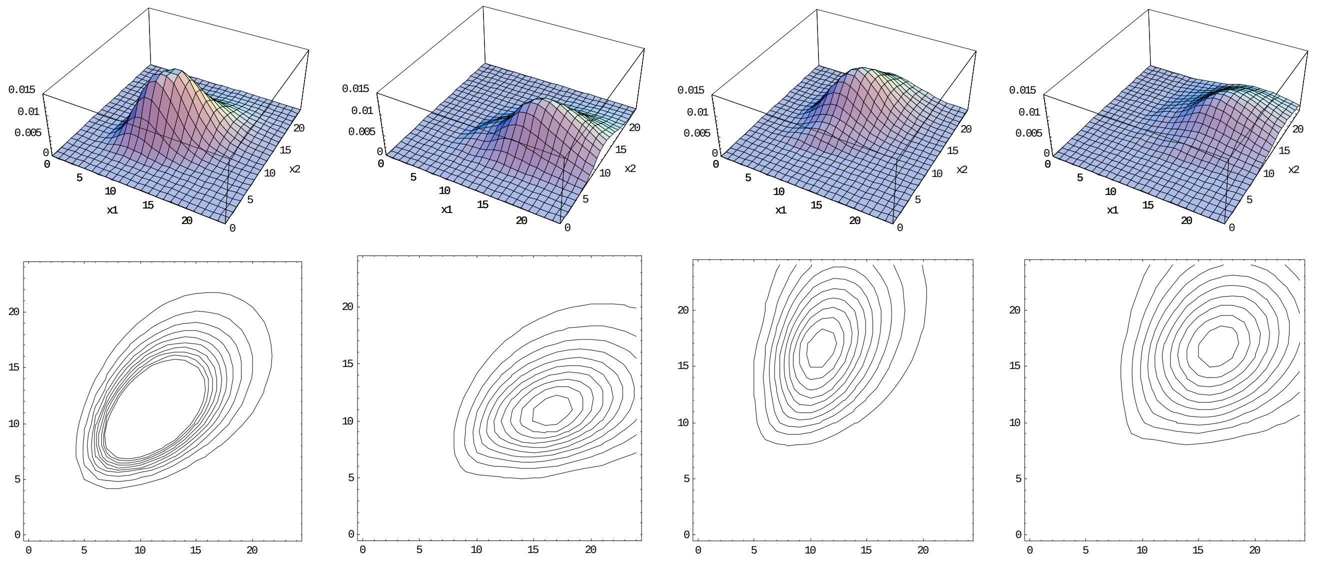
1. Using the bivariate characteristic function given in (8), the following moments about the origin are calculated:

$$\begin{aligned} E(X_i) = \frac{-i\,\partial\phi(t_1,t_2)}{\partial t_1}\Bigg|_{t_1=t_2=0} &= n_i+\delta_i \quad \text{for } i=1,2. \\ E(X_i^2) = \frac{-\,\partial^2\phi(t_1,t_2)}{\partial t_i^2}\Bigg|_{t_1=t_2=0} &= (n_i+\delta_i)^2+(n_i+2\delta_i) \quad \text{for } i=1,2. \\ E(X_1X_2) = \frac{-\,\partial^2\phi(t_1,t_2)}{\partial t_1\partial t_2}\Bigg|_{t_1=t_2=0} &= (n_1+\delta_1)(n_2+\delta_2)+n_3\lambda \end{aligned}$$

Therefore we have $var(X_i) = n_i + 2\delta_i$ for $i = 1, 2$ and $cov(X_1, X_2) = n_3\lambda$. Pearson's product moment correlation coefficient is then given by $\frac{n_3\lambda}{\sqrt{(n_1+2\delta_1)(n_2+2\delta_2)}}$

Consequently we have $0 \leq \lambda \leq \frac{\sqrt{(n_1+2\delta_1)(n_2+2\delta_2)}}{n_3}$. This shows that if two correlated gamma variables are jointly distributed according to equation (8), then no negative value of the correlation coefficient exists between the two variables.

2. Figure 5 illustrates the shape of the p.d.f. (see (8)) for the case $n_1 = n_2 = n_3 = 12$ and λ= 0.5 and for different values for the additional parameters (δ_1, δ_2). The plots for $(\delta_1, \delta_2) = (0, 0), (\delta_1, \delta_2) = (6, 0), (\delta_1, \delta_2) = (0, 6)$ and $(\delta_1, \delta_2) = (6, 6)$ are given respectively. The plots show that for a fixed n_1, n_2, n_3 and λ, X_1 and X_2 are stochastically increasing with respect to δ_1 and δ_2.



(a)(δ_1, δ_2)= $(0, 0)$	(b)$(\delta_1, \delta_2) = (6, 0)$	(c)$(\delta_1, \delta_2) = (0, 6)$	(d)$(\delta_1, \delta_2) = (6, 6)$
corr$(X_1, X_2) = 0.5$	corr$(X_1, X_2) = 0.354$	corr$(X_1, X_2) = 0.354$	corr$(X_1, X_2) = 0.25$

Figure 5: The noncentral generalised bivariate gamma distribution (see (8)) **for** $n_1 = n_2 = n_3 = 12$ **and** $\lambda = 0.5$ **and different values of** (δ_1, δ_2).

3. The derivations for the exact expressions for the p.d.f.'s for X_1X_2, $\frac{X_1}{X_2}$ and $\frac{X_1}{X_1+X_2}$ if (X_1, X_2) has the noncentral generalised bivariate gamma distribution (see (8)) are similar to these in sections 3 and 4. The graphical representations of the distributions of X_1X_2, $\frac{X_1}{X_2}$ and $\frac{X_1}{X_1+X_2}$ are given in figure 6 respectively for different values for the noncentral parameters.

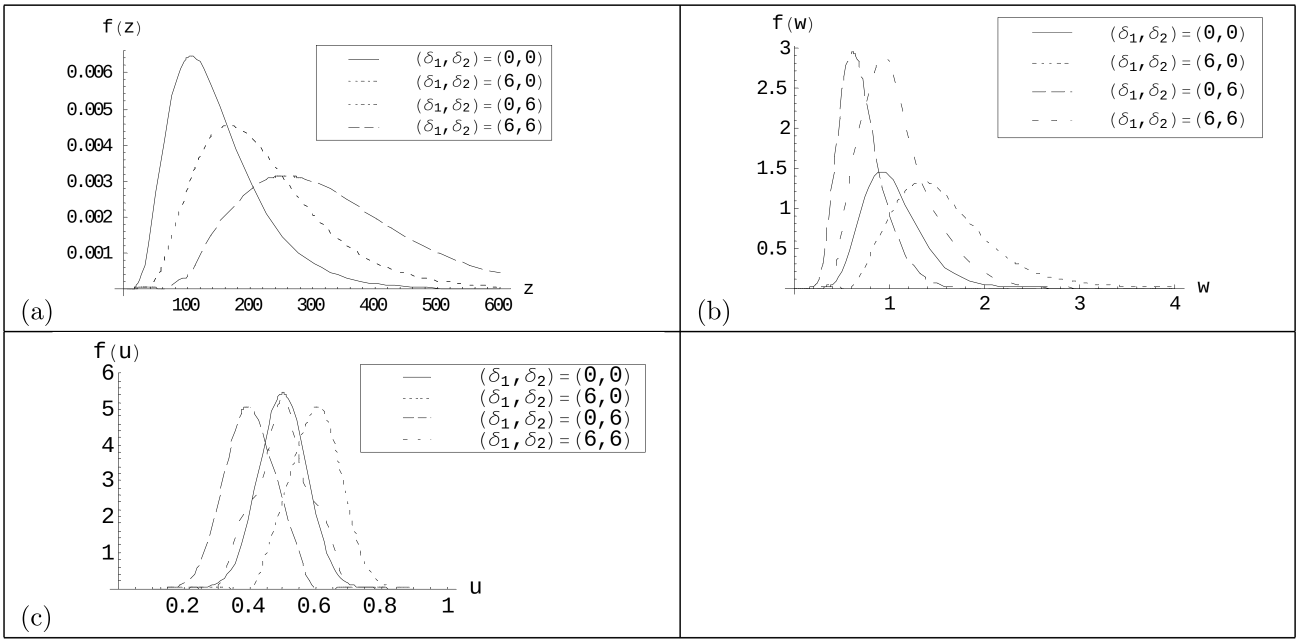


Figure 6: The p.d.f. of (a)$Z = X_1X_2$ **, (b)**$\frac{X_1}{X_2}$ **and (c)**$\frac{X_1}{X_1+X_2}$ **for** $n_1 = n_2 = n_3 = 12$**,** $\lambda = 0.5$ **and different values of** δ_1 **and** δ_2**.**

6 Example

The North West Province is a summer-rainfall region that lies in the northern part of South Africa next to the Botswana border. The landscape is largely flat regions of scattered trees and grassland. The province is an important food basket in South Africa. Maize is an important crop and the North West is the major producer of white maize in the country. In order for farmers to have a successful harvest, rain needs to be evenly distributed throughout the raining season. Not only is rain important when maize is planted (September to December), but also before it is harvested (January to April).
The example concerns the rainfall of two districts, Klerksdorp and Wolmaransstad, in the North West province. Variable X_1 represents the total amount of rain (in centimeters) that fell during the first four months of the raining season (September to December). Variable X_2 represents the total amount of rain (in centimeters) that fell during the four months that follow (January to April). Variable X_2 is often correlated to X_1. Our aim is to analyse this data as follows:

- to compare this bivariate model (2) with other Kibble related bivariate models (see remark 1.1(6) and remark 1.2);
- to investigate the distribution of $W = \frac{X_1}{X_2}$ for Klerksdorp and Wolmaransstad.

To compare the fitted models the Akaike information criterion (AIC) was used; the fit is better when AIC is lower. The maximum likelihood estimators were calculated numerically by using MATLAB. One of MATLAB's optimisation constraints is that it can only do minimisation. The likelihood function was therefore multiplied by -1 and the minimisation function *fminsearch* was used to find the maximum likelihood estimators. The infinite sum in the probability density function could not be programmed and a threshold for the number of summations was set. In this application, a threshold of 30 was chosen. Table 2 shows the parameter estimates and the values for AIC for Kibble related bivariate distributions. It can be noticed that the new model (2) is more appropriate to model this rainfall data set. However, the performance of our model and the case of Loáiciga and Leipnik is alike for this data set.

Table 2: Parameter estimates and AIC for the rainfall data for Klerksdorp and Wolmaransstad.

City	**Model**	**Parameter estimates**	**AIC**
Klerkdorp	Kibble	$\hat{n}_1$= 25.7031; $\hat{\lambda}$ = 0.7219	481.8145
Klerkdorp	Standardised Jensen	$\hat{n}_1$= 21.8329; $\hat{n}_2$= 30.5621; $\hat{\lambda}$ = 0.716 56	364.5002
Klerkdorp	Loáiciga and Leipnik bivariate gamma	$\hat{\alpha}_1$= 0.4691; $\hat{\alpha}_2$ = 0.6486; $\hat{\gamma}$ = 46.2049; $\hat{\beta}$ = 0.3479	354.1529
Klerkdorp	Generalised bivariate gamma	$\hat{n}_1$= 21.5762; $\hat{n}_2$= 30.2684; $\hat{n}_3$= 39.1165; $\hat{\lambda}$ =0.459 01	353.0156
Wolmaransstad	Kibble	$\hat{n}_1$= 23.8281; $\hat{\lambda}$ = 0.8170	459.8103
Wolmaransstad	Standardised Jensen	$\hat{n}_1$= 19.7264; $\hat{n}_2$= 29.1491; $\hat{\lambda}$ = 0.620 47	362.3551
Wolmaransstad	Loáiciga and Leipnik bivariate gamma	$\hat{\alpha}_1$= 0.5288; $\hat{\alpha}_2$ = 0.7851; $\hat{\gamma}$ = 37.2537; $\hat{\beta}$ = 0.2990	359.7768
Wolmaransstad	Generalised bivariate gamma	$\hat{n}_1$= 19.7274; $\hat{n}_2$= 29.1513; $\hat{n}_3$= 142.27; $\hat{\lambda}$ = 0.086025	359.4678

Figure 7 displays the distribution of $\frac{X_1}{X_2}$ for Klerksdorp and Wolmaransstad, using equation (5) with the parameter estimates as in table 2 above. The probability that the rainfall in the late rain season will be less than the rainfall in the early rain season, $P(\frac{X_1}{X_2} > 1)$, is 0.00453021 for Klerksdorp and 0.023692 for Wolmaransstad. The graph shows a bigger spread for Wolmaransstad, which indicates that the summer rainfall in Wolmaransstad is more evenly distributed throughout the raining season.

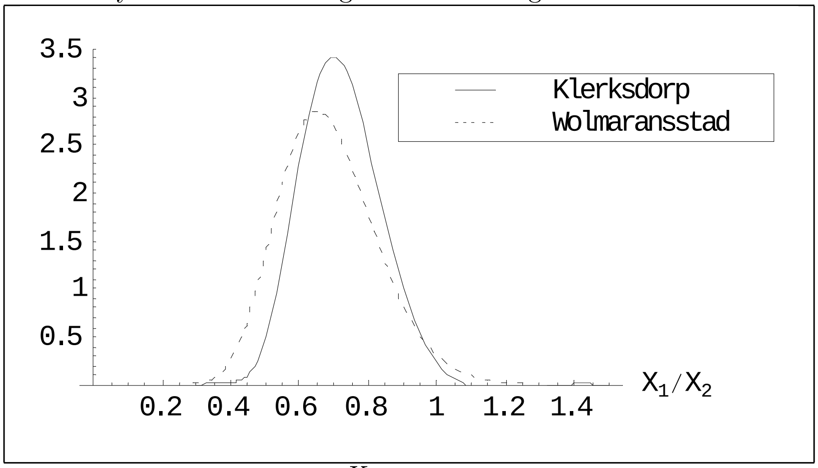


Figure 7: The p.d.f. of $W = \dfrac{X_1}{X_2}$ (see (5)) **using parameter estimates for rainfall data from Klerksdorp and Wolmaransstad.**

7 Conclusion

A generalised bivariate gamma distribution with correlated standard and noncentral gamma marginals were proposed where the correlation coefficient is a function of the parameters. Balakrishnan and Lai (2009) give various applications for bivariate gamma distributions which could possibly be extended to this generalised bivariate gamma distribution. Exact expressions for the p.d.f.'s of the product and ratios of the correlated components of (2) were also derived; some percentage points of $W = \dfrac{X_1}{X_2}$ were given as well. The effect of the parameter, n_3, on the shapes of this generalised bivariate gamma p.d.f., as well as the p.d.f.'s of the product and ratios were illustrated. An example to rainfall data two specific districts in the North West province illustrates the value which is added by proposing this bivariate model. Several applications for the distributions of product and ratio of components of bivariate gamma distributions exist in hydrology, rainfall and water quality. The applications could be used to analyse the effects and causes of different international problems facing the world today, such as global warming and water pollution. The proposed generalised bivariate gamma distribution can be helpful in constructing plausible models for such applications.

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