Solid-gas interaction in nonlinear acoustics

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Acknowledgements

iv

Abstract

This dissertation is concerned with aspects of the newly-proposed approach to nonlinear acoustics in which the Lagrangian description of gas motion is followed. It contains a systematic survey of the approach which leads to the so-called dynamic piston problem. Then new situations regarding the piston problem are studied. These situations cover cases of varying applied pressure and results concerning the formation of shock discontinuities are presented.

Contents

Chapter 1

Introduction

Classical acoustics as recorded in the published literature began with the question of determining the speed at which a sonic disturbance was propagated in air. The first reported experiment was by the French intellectual Pierre Gassendi (1592 – 1655) who, in 1635 measured the time taken to observe the sound of a canon shot fired a known distance away. The result was a "sound speed" of 475 m/s [6, Historical Introduction, p. xviii]. Shortly afterwards Newton had the experiment repeated in England and then proceeded to find a theoretical basis for calculating sound speed [5]. The theoretical predictions were, however, not in accordance with observations. Much later Pierre-Simon Laplace [3] realized that the compressibility of whatever gas transmitted sonic disturbances had to be taken into account and produced results still used today.

It was in 1858 that Earnshaw [7] returned to the matter of sound speed and suggested that the concept may not be as simple as it seemed. He argued that if a piston were pushed into a gas-filled pipe with the gas initially at rest the gas movement ahead of the piston would catch up with the gas further along in the pipe and a discontinuity of motion would result. He goes on to suggest that the speed of the disturbance generated in this way could be higher than the theoretical speed of sound. At the time this phenomenon was well described within the context of water waves [14] and was called a *bore*. Earnshaw [8] and Riemann [9] investigated the problem by using the Euler-description for gas/fluid motion. Their work gave rise to the so-called Earnshaw-Riemann invariants [1] which is intimately connected to the method of characteristics prominent in the investigation of hyperbolic partial differential equations. Somewhat later Lord Rayleigh [6] attempted a mathematical discussion of Earshaw's ideas but soon reverted back to the wave equation as a means of studying the nature of sonic disturbances. In this approach the Laplace sound speed (about 335 m/s for air) is central. The approach is still used by most acoustic engineers.

In 2010 the theoretical description of sonic disturbances was re-opened by Sauer [10] this time using the Lagrange-description of gas motion. The reason for this was that the notion of a "closed system" from thermodynamics was much more natural since the motion of a single or an identified collection of gas particles could be followed as they travelled in time. The result was a system of nonlinear partial differential equations that could not be easily related to linear wave motion. In addition, an inequality constraint that represents the physical fact that a body of gas cannot be compressed to zero volume, was identified. Computed results showed that sonic disturbances predicted from this approach could not be approximated by linear theory. The same paper also touched upon Earnshaw's image of a piston being moved onto a gas-filled pipe. The approach models the piston as a physical body interacting with the gas in the pipe and indicates that shock-discontinuities will result. This *dynamic piston problem* was taken further in a subsequent paper [11] where jump conditions were derived and applied to the case of a positive constant external force acting on the piston.

The aim of this thesis is to further investigate and analyse the interaction that occurs between the solid and a gas. In particular we investigate a class of cases in which the applied force is positive and decreasing. For the sake of completeness and coherence, we give a detailed presentation of the underlying theory which is only described in the briefest of ways in the Sauer papers.

In Chapter 2 we deal with classical thermo-statics. The chapter motivates the theory discussed in Chapter 3 where the Lagrange approach is followed. This is in line with the discussion in Chapter 2 where we also consider volumes consisting of the same gas particles. Here we introduce the notion of *compression* in a static/global sense. It is defined as a ratio of volumes which could be viewed as purely geometrical but later it joins with the relevant physical concepts. This leads to a constraint and a relationship between compression and pressure which may be very fruitful in the study of acoustic phenomena.

In Chapter 3 we discuss the basic ideas of the Lagrange approach. This will be the underlying framework for the derivation of equations. We also introduce the physical principle of conservation of mass which tells us that the mass of a body consisting of the same gas particles does not change as particles move from one configuration to another. The notion of compression as described in the previous chapter is now localized by letting the ratio of volumes consisting of the same gas particles shrink to a point (single particle). This leads to a localized version of the compression-pressure relation which we put forward as the basic acoustic postulate ([10]). The consideration of kinematics of volume then leads to our first dynamic equation for pressure and an inequality constraint on pressure which shows that there is a lower bound on the pressure that can develop which in turn implies that it is impossible to compress a body of gas until there is no volume.

In Chapter 4 the principle of balance of linear momentum is described and combined with the principle

of conservation of mass in a manner that suits our purpose. This yields the second dynamic equation for velocity which is then augmented with the dynamic equation for pressure obtained in Chapter 3. These equations are then scaled to dimensionless form and this shows that the acoustic hypothesis resembles the classic Boyle-Marriot law. The scaling will also enable us to "mix" variables at a later stage. After this we analyse the special case of one-dimensional motion in order to compare the consequences with the well known behaviour derived from the linear wave equation. One-dimensional consideration reduces the system of equations to two equations for two unknowns, velocity and pressure.

It is immediately seen that the linear system cannot possibly yield solutions for the nonlinear system. One reason is that the linear system is symmetric in velocity and pressure while the nonlinear system is not. Moreover, the constraint is not found in the linearized equations while the whole character of the nonlinear system is no longer hyperbolic ([2, p. 42]) and the wave-like behaviour of the solutions may be at risk if the constraint is not considered. This constraint is then utilized to transform the one-dimensional equations into a symmetric hyperbolic system which exhibits the true nonlinear character of the problem.

In Chapter 5 the scaled system of equations is applied to the piston problem. We formulate the dynamic piston problem and use the method of characteristics to reduce the equations to a single nonlinear transport equation with a dynamic condition at the boundary $x = 0$. A point overlooked in [11] is that the basic physical constraint is not sufficient to ensure that one of the families of characteristics gives a boundary condition on the axis $x = 0$. This is cleared up and it is shown that positive lower and upper bounds for the pressure are needed. Physically this means that a given volume of gas can only be compressed to a positive lower value and rarefied to a positive upper value. Under these circumstances the *dynamic boundary condition* at $x = 0$ reduces to a single nonlinear ordinary differential equation for the piston velocity that can be solved explicitly in terms of the externally applied force expressed as a pressure. We then proceed to find results that give bounds for the piston velocity and will ensure that the piston velocity is increasing/decreasing.

In Chapter 6 characteristics emanating from the t axis (x = 0) are studied and these characteristics will give the motion of the gas ahead of the piston. Intersection of characteristics and the point of intersection is analysed and it is shown that on the portion where the piston velocity is increasing discontinuities in the solution will develop. Finally the bounding curve (envelope) is discussed and it is shown how to compute it.

Chapter 7 is concerned with jump curves. These are curves along which jump discontinuities occur. Under such circumstances the basic differential equations are replaced by *jump conditions* that are derived from the fundamental balance and conservation laws. We derive the jump condition for the dynamic

piston problem from balance of linear momentum and conservation of mass. This condition is then used to derive properties of the jump curve involved and the calculation of the jump curve. This leads to the calculation of the gas motion under shock conditions.

In several instances we used numerical methods to calculate illustrative examples. In most cases the numerical methods are of an elementary nature with no doubt about the accuracy of the outcomes. In the case of the jump curve, the method employed is the same as that used in [11] where an ordinary differential equation with initial condition is first transformed into an integral equation of Volterra type which is then handled by means of numerical quadrature procedures. The quadrature used in this case is the trapezium rule and this leads to the numerical solution of a nonlinear equation by the *regula falsi* (secant) method. For this complex procedure the accuracy of the outcome is not established and we relied on stability obtained by decreasing time steps. It turned out that even fairly big steps yielded results that did not show remarkable differences on plots.

Chapter 2

Thermo-statics and classical acoustics

2.1 Elementary considerations

In order to motivate the theory described in Chapter 3 we consider the thermo-static situation as depicted in Figure 2.1 below. What we see is a closed container with gas in different states of compression and what we intend to do is to compare the states. For this purpose a fixed state, the "reference state", is chosen and other states in "general" are compared to it. The comparison is static and bears no reference to time. The dynamics of change is not an issue. A most important point is that the gas in the different states consists of the same material particles. We refer to the geometry of each state as the *configuration*.

States of configurations are described mathematically by the *thermostatic variables* Volume, Mass density and Pressure. For these the following notations are used respectively

Reference state: V_0, ρ, p_0.

Generic state: V, σ, p.

Suppose the gas experiences compression, possibly as the result of external pressure, so that its volume changes to V and density to σ. The *compression r* of the gas (relative to the reference state) is defined as the ratio of increase in volume to the "original" volume. That is,

$$r = \frac{V - V_0}{V_0} = \frac{V}{V_0} - 1.$$

(2.1)

7
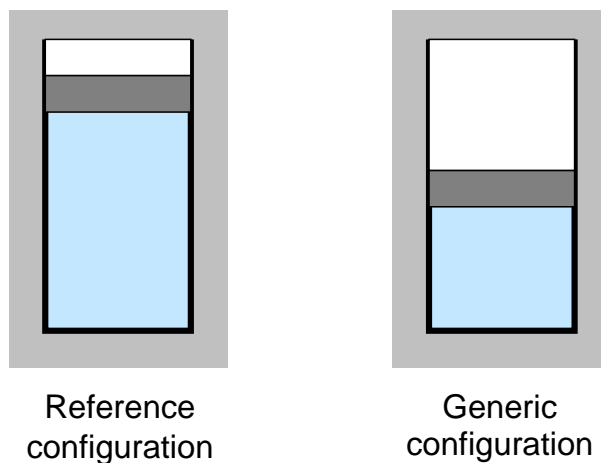
Figure 2.1: Static configurations and states

If it is assumed that gas volumes must always be positive, we have from (2.1) that there is a constraint on r namely

$$1 + r > 0. \tag{2.2}$$

This means that there is a lower bound to the compression. In fact, a given physical volume of gas V_0 cannot be compressed to zero.

Since the gas in the different states consists of the same physical material, we can invoke the principle of *conservation of mass* which proclaims that in both states the mass of gas in the container is the same. Mathematically this is expressed in the following way: $\rho V_0 = \sigma V$. Therefore, by (2.1),

$$\frac{V}{V_0} = \frac{\rho}{\sigma} = 1 + r. \tag{2.3}$$

The expression (2.3) gives the relationship between compression and density.

2.2 The acoustic assumption

The standard thermo-static assumption is that the density of the gas in a given state is determined by the pressure p in that state. That is

$$\sigma = f(p). \tag{2.4}$$

The *static acoustic assumption* is that for an isentropic gas the rate of change of density with pressure

is constant and positive:

$$\frac{d\sigma}{dp} = f'(p) = c^{-2}. \tag{2.5}$$

The constant c has the dimension of speed. It is called the *static sound speed*. The static sound speed is a property of the kind of gas under consideration.

Combination of (2.4) and (2.3) gives

$$f(p) = \frac{\rho}{1+r}, \tag{2.6}$$

which shows that the compression r is a function of pressure. Differentiation of (2.6) with respect to p and combination of the result with (2.5) leads to the differential equation

$$-\rho(1+r)^{-2}\frac{dr}{dp} = \frac{1}{c^2}. \tag{2.7}$$

The equation (2.7) can immediately be put into integrable form:

$$\frac{d}{dp}\left[(1+r)^{-1}\right] = \frac{1}{\rho c^2}. \tag{2.8}$$

By definition, $r(p_0) = 0$ (in the reference state there is no compression). Hence, integration of (2.8) from p_0 to p gives

$$\frac{1}{1+r} = \frac{p-p_0}{\rho c^2} + 1. \tag{2.9}$$

The identity (2.9) will become, in localized form, the acoustic assumption of Chapter 3.

Chapter 3

Classical acoustics

3.1 The Lagrange approach in hydrodynamics

In this approach we follow the motion of individual fluid particles and volumes consisting of the same particles as they move in space as opposed to the Euler approach whereby the state of the fluid motion at arbitrary fixed points $x = (x_1, x_2, x_3)$ in 3-dimensional Euclidean space is considered. This is in line with the discussion in Chapter 2 where we considered a volume consisting of the same gas particles.

For this reason, let $\Phi \subset \mathbb{R}^3$ be the reference configuration of fluid particles. Particles in the fluid are identified by their position $x \in \Phi$. The *motion* of the fluid is described by a function $\chi(., t) : x \in \Phi \mapsto X(t) := \chi(x, t)$ with $X(t)$ the position of x at time t. We shall not be concerned with situations in which different particles can be in the same place at the same time. Also the same particle cannot be in different places at the same time and therefore we may assume that χ is one-to-one.

For the fluid to have 'bulk' at all, we assume that the reference configuration Φ is an open subset of $\mathbb{R}^3$ and we consider only smooth motions. That is χ and its inverse χ^{-1} have continuous derivatives with respect to all variables. The derivatives will be denoted by $\partial_t := \frac{\partial}{\partial t}$ and $\partial_k := \frac{\partial}{\partial x_k}$. Since $\chi(x, t) = (\chi_1(x, t), \chi_2(x, t), \chi_3(x, t))$, $\partial_t \chi(x, t)$ is a vector quantity.
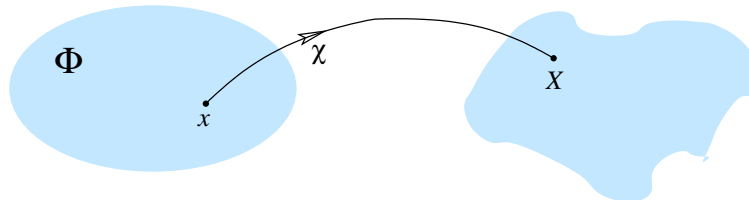
Figure 3.1: The motion of a volume of gas

Our first objective is to find, on the basis of classical physics, a system of partial differential equations that will enable us to determine the motion χ. We prepare for this by introducing the notion of the gradient of 3-vector fields. Let $f(x) = (f_1(x), f_2(x), f_3(x))$ be a differentiable 3 - vector field defined for $x \in \Omega \subset \mathbb{R}^3$, with Ω an open set. Then the *gradient* L_x of f is defined as the matrix function with components $L_{x,ij}(f)(x) = \partial_j f_i(x)$. The transpose matrix function $L_x^T f$ is defined as the transpose of $L_x f(x)$. The motion $\chi(.,t)$ is simply an invertible transformation from one coordinate system x to another X with Jacobian $J(x,t) := det[L_x \chi(x,t)]$. The Jacobian can never be zero because χ is invertible. By suitable choices of the directions of coordinate axes, it may be assumed that $J(x,t)$ is positive.

We need some calculus-identities given below. In order to make notations clear, we explain as follows:

If an $n \times n$ matrix A is invertible, its transpose A^T is also invertible and $[A^T]^{-1} = [A^{-1}]^T$. We shall use the notation $A^{-T} = [A^T]^{-1}$. If A and B are $n \times n$ matrices with components a_{ij} and b_{ij} we use notation $A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij}$, the so-called *Frobenius inner product*.

Suppose that $F(X,t)$ is a scalar field defined on the configuration of fluid particles at time t. Let $f(x,t) := F(\chi(x,t),t)$. Then it follows from the chain rule that the gradients of F and f are related in this way:

$$\nabla_X F(X,t) = L_x \chi(x,t)^{-T} \nabla_x f(x,t). \tag{3.1}$$

If $F(X,t)$ is a vector field defined on the configuration at time t and $f(x,t) = F(\chi(x,t),t)$ we have the following for the divergence of F:

$$\nabla_X \cdot F(X,t) = L_x \chi(x,t)^{-T} : L_x f(x,t). \tag{3.2}$$

3.2   Compression and Conservation of mass

As already remarked, the Lagrangian description is very much in line with the thermo-static considerations of Chapter 2. We proceed to follow that development within the present context, but localize the concepts by letting moving volumes "shrink to a point".

To begin with, we define compression at a point taking our cue from the previous chapter. The compression $r(x,t)$ experienced by particle x at time t is defined as

$$r(x,t) := \lim_{|\Upsilon| \to 0} \frac{|\Upsilon(t)| - |\Upsilon|}{|\Upsilon|} ; x \in \Upsilon. \tag{3.3}$$

Here the absolute value denotes volume. $|\Upsilon|$ represents an arbitrary volume of particles in the reference configuration/state and $|\Upsilon(t)|$ represents the volume of particles in the configuration at time t. The only

restriction is that $x \in \Upsilon$. We note that Υ and $\Upsilon(t) = \chi(\cdot, t)[\Upsilon]$ are representations of the same material. From the expression (3.3) we have the following:

$$1 + r(x, t) = \lim_{|\Upsilon| \to 0} \left[\frac{|\Upsilon(t)|}{|\Upsilon|}\right] = \lim_{|\Upsilon| \to 0} \left[\frac{\int_\Upsilon J dx}{\int_\Upsilon dx}\right] = J \tag{3.4}$$

in which the final step is a result of the Radon-Nikodým Theorem.

Next, the *principle of conservation of mass* states that the mass of an arbitrary collection of particles is constant as it moves along in time. In other words, the mass of a collection of particles will never change, regardless of how the constituent parts rearrange themselves. We can express this principle as follows: let Υ be an open set of Φ, and let $\Upsilon(t) := \chi(., t)[\Upsilon]$ be its configuration at time t. With this definition $\Upsilon(t)$ is the image of Υ under χ. Then the principle of conservation of mass tells us that the mass of Υ is the same as the mass of $\Upsilon(t)$.

To make the principle mathematically workable we use the notion of *mass density*. This is a function from which the mass of a given configuration of fluid particles may be computed. Since the fluid is compressible, the mass density (mass per unit volume) may change. Let $\rho(x)$ be the density function in the reference configuration and $\sigma(X, t)$ the density function in the configuration at time t. The conservation of mass then simply states that

$$\int_\Upsilon \rho(x) dx = \int_{\Upsilon(t)} \sigma(X, t) dX. \tag{3.5}$$

The integrals in this expression are volume integrals and the volume measures such as dx should be thought of as being $dx = dx_1 dx_2 dx_3$. The right of (3.5) may be transformed into an integral over the reference volume Υ by letting $X = \chi(x, t)$. By the substitution rule we then obtain

$$\int_\Upsilon \rho(x) dx = \int_\Upsilon \sigma(\chi(x, t), t) J(x, t) dx. \tag{3.6}$$

Since the expression (3.6) holds for arbitrary open subsets of Φ, a standard theorem on Lebesgue integration states that the integrands must be equal. We therefore have

$$\rho(x) = J(x, t) \sigma(\chi(x, t), t), \tag{3.7}$$

which is the principle of conservation of mass expressed in terms of mass densities.

We see that the compression r only involves the motion χ. It is, however, related to the physical principle of conservation of mass as expressed in (3.7). By eliminating the Jacobian J from (3.4) and

(3.7) we obtain

$$1 + r(x,t) = \frac{\rho(x)}{\sigma(\chi(x,t),t)}.$$ (3.8)

The expression (3.8) is the local version of (2.3). Once again, taking (3.7) as well as the positivity of J into account, we find the local version of the constraint (2.2):

$$1 + r(x,t) > 0.$$ (3.9)

3.3   The acoustic assumption in continuous form

We now introduce the *localized acoustic assumption* by postulating the relation

$$\frac{1}{J(x,t)} = \frac{1}{1 + r(x,t)} = 1 + \frac{p(x,t) - p_0}{\rho c^2}$$ (3.10)

which is simply is the result (2.9) localized to the point (x,t). It assumes that the static sound speed c and the reference pressure p_0 are constants. The relation (3.10) together with (3.9) leads to an equivalent form of the constraint namely

$$\rho c^2 + [p(x,t) - p_0] > 0.$$ (3.11)

3.4   The kinematics of volume

The kinematic property of compression is found in the balance of volume which states that the rate of change of volume of any piece of material is explained by the normal components of velocity at its boundary. In precise terms, this is

$$\frac{d}{dt} \int_{\Upsilon(t)} dX = \int_{\partial \Upsilon(t)} V(Y,t).n(Y)ds(Y) = \int_{\Upsilon(t)} \nabla_X \cdot V(X,t)dX$$ (3.12)

having made use of the divergence theorem.

If we transform back to the reference configuration and make use of (3.2) the result is

$$\frac{d}{dt} \int_{\Upsilon} J(x,t)dx = \int_{\Upsilon} J(x,t)L_x\chi(x,t)^{-T} : L_xv(x,t)dx$$ (3.13)

If we assume that the time derivative and the integral can be interchanged and that J has an integrable

time derivative, we obtain

$$\int_\Upsilon J_t(x,t)dx = \int_\Upsilon J(x,t)L_x\chi(x,t)^{-T} : L_xv(x,t)dx \tag{3.14}$$

Since Υ is arbitrary it follows from (3.14) that

$$J_t(x,t) = J(x,t)L_x\chi(x,t)^{-T} : L_xv(x,t) \tag{3.15}$$

which is the analogue of Euler's formula [13, p. 131] in the Lagrangian description.

From (3.10) we have

$$J(x,t) = \frac{\rho c^2}{\rho c^2 + p(x,t) - p_0}. \tag{3.16}$$

Differentiation of (3.16) with respect to t yields

$$J_t(x,t) = \frac{-p_t\rho c^2}{(\rho c^2 + p(x,t) - p_0)^2}. \tag{3.17}$$

If (3.16) and (3.17) are substituted into (3.15) the result is

$$\frac{-p_t\rho c^2}{(\rho c^2 + p(x,t) - p_0)^2} = [\frac{\rho c^2}{\rho c^2 + p(x,t) - p_0}]L_x\chi(x,t)^{-T} : L_xv(x,t)$$

This can be simplified to

$$p_t(x,t) + [\rho c^2 + p(x,t) - p_0]L_x\chi(x,t)^{-T} : L_xv(x,t) = 0 \tag{3.18}$$

which is a dynamic equation involving the rate of change of pressure.

It should be noted that, since ρc^2 has the unit of pressure, the expression (3.18) is dimensionally correct.

Chapter 4

Continuum mechanics

4.1 Balance of linear momentum

So far we have investigated only the notions of compression, density and pressure and the way they are inter-related. The dynamics of the underlying motion — the underlying causes of the motion — now come into focus. The basic principle here is *balance of linear momentum* which is the extension of Newtons second law of motion for a single particle.

Balance of linear momentum states that: The rate of change of total linear momentum in the considered control volume is equal to the total force acting on the mass in that volume. To formulate the principle of balance of linear momentum mathematically the notions of velocity and mass density as discussed in Chapter 3 are essential.

Since we deal with an ideal gas (no viscosity) the internal force observed at X at time t is due to pressure only. We assume that there are no external forces. If Υ is an arbitrary open set of the reference configuration and $\Upsilon(t) = \chi(.,t)[\Upsilon]$ its image at time t, the principle of balance linear momentum may be expressed by

$$\frac{d}{dt} \int_{\Upsilon(t)} \sigma(X,t)V(X,t)dX = -\int_{\partial\Upsilon(t)} P(X,t)n(X)dS(x) = -\int_{\Upsilon(t)} \nabla_X P(X,t)dX \qquad (4.1)$$

Here P(X,t) is the pressure at X at time t and $n(X)$ the unit exterior normal at X to the surface $\partial\Upsilon(t)$. The minus sign on the right is due to the use of the exterior normal in the expression. The final expression in (4.1) is obtained from the divergence theorem.

Balance of linear momentum can under certain assumptions be expressed as a partial differential equation. We can transform the expression (4.1) to the reference coordinates x by letting $X = \chi(x,t)$.

The result is

$$\frac{d}{dt}\int_{\Upsilon}\sigma(\chi(x,t),t)v(x,t)J(x,t)dX = -\int_{\Upsilon}L_x\chi(x,t)^{-T}\nabla_x p(x,t)J(x,t)dx, \tag{4.2}$$

having made use of (3.1). If conservation of mass as expressed in (3.7) is combined with (4.2) the result is

$$\frac{d}{dt}\int_{\Upsilon}\rho(x)v(x,t)dx = -\int_{\Upsilon}L_x\chi(x,t)^{-T}\nabla_x p(x,t)J(x,t)dx. \tag{4.3}$$

If we assume that the time derivative and the integral in (4.3) can be interchanged and that v has an integrable time derivative, we obtain

$$\int_{\Upsilon}[\rho(x)v_t(x,t) + J(x,t)L_x\chi(x,t)^{-T}\nabla_x p(x,t)]dx = 0. \tag{4.4}$$

Since Υ is an arbitrary open subset of Φ, it follows that the integrand in (4.4) is zero. Thus we obtain the dynamic equation

$$\rho(x)v_t(x,t) + J(x,t)L_x\chi(x,t)^{-T}\nabla_x p(x,t)dx = 0. \tag{4.5}$$

The acoustic postulate (3.16) enables us to eliminate the determinant J from (4.5) to obtain, after some simplification,

$$v_t(x,t) + \frac{c^2}{\rho c^2 + p(x,t) - p_0}L_x\chi(x,t)^{-T}\nabla_x p(x,t)dx = 0. \tag{4.6}$$

It should be noted that the equation (4.6) represents a combination of balance of linear momentum, conservation of mass and the acoustic postulate.

4.2   Scaling of the system

In our considerations so far we have found the partial differential equations (4.6), (3.18) and the constraint (3.11). This will be the system we shall work with. The next step is to scale the equations to dimensionless form. In scaled form the equations and the constraint will be stripped of constants so that they will be mathematically clearer. In addition dimensionless entities can be algebraically "mixed" without inadmissible consequences. All of this will come to fruit in Sections 4.3 and 4.6. We shall take the reference density ρ as a constant. That is to say that in the reference configuration the mass density is constant.

For the purpose of scaling we assume that there is some length parameter L in terms of which position will be scaled. This parameter is free at this stage. Later a specific choice will be made. Time is scaled in terms of $T = L/c$, velocity in terms of c and pressure in terms of ρc^2. The details of scaling are shown in Section 8.1. The following substitutions will be used:

Position: $x \longrightarrow \frac{x}{L}$;

Time: $t \longrightarrow \frac{t}{T}$;

Velocity: $v \longrightarrow \frac{v}{c}$;

Pressure: $p \longrightarrow \frac{p-p_0}{\rho c^2}$

This leads to

$$v_t(x,t) + [1 + p(x,t)]^{-1} L_x \chi(x,t)^{-T} \nabla_x p(x,t) = 0; \tag{4.7}$$

$$p_t(x,t) + [1 + p(x,t)] L_x \chi(x,t)^{-T} : L_x v(x,t) = 0. \tag{4.8}$$

In addition the constraint (3.11) now takes the form

$$1 + p(x,t) > 0. \tag{4.9}$$

Since the compression r, being a ratio of volumes, is already dimensionless and therefore insensitive to scaling, the scaled form of the acoustic assumption (3.10) is

$$[1 + p(x,t)][1 + r(x,t)] = 1, \tag{4.10}$$

which is analogous to the Boyle-Marriott law. Detailed calculations are shown in Section 8.1.

In the sequel we shall work with these equations, but they are still insufficient in number since the three components of the motion χ are unknown. In the remaining sections we study a particular form of the motion in which the problem of the determination of the motion itself is remedied. From the system (4.7), (4.8) of transport equations it is seen that there are propagation speeds which depend on pressure in a very specific way.

4.3 System in one-dimensional motion

The consideration of the one-dimensional motion results in the system of equations being reduced to two equations for two unknowns, velocity and pressure. The original studies of the propagation of sound

were very much focused on one-dimensional motions. Let us therefore consider motions of the form $\chi(x,t) = (\psi(x_1,t), x_2, x_3) = (\chi_1(x,t); \chi_2(x,t); \chi_3(x,t))$.

For this motion by definition $L_x\chi(x,t) = \frac{\partial \chi_i}{\partial x_j}$ so that we have

$$L_x\chi(x,t) = \begin{bmatrix} \frac{\partial \psi(x_1,t)}{\partial x_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}(\partial_1\psi, 1, 1)$$

Also, $v(x,t) = \chi_t(x,t) = (\psi_t(x_1,t), 0, 0)$. Since $1 + r(x,t) = J(x,t)$ and by (4.10) we have $1 + r(x,t) = (1 + p(x,t))^{-1} = \partial_1\psi(x_1,t)$, and

$$L_x v(x,t) = \begin{bmatrix} \partial_1\psi_t(x_1,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{diag}(\partial_1 v_1, 0, 0), \tag{4.11}$$

$$L_x\chi(x,t)^{-1} = \frac{1}{\partial_1\psi}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \partial_1\psi & 0 \\ 0 & 0 & \partial_1\psi \end{bmatrix} = \begin{bmatrix} 1+p(x,t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L_x\chi(x,t)^{-T}. \tag{4.12}$$

If we substitute back into (4.7) the result is

$$\begin{bmatrix} \partial_t v(x_1,t) \\ 0 \\ 0 \end{bmatrix} + \frac{1}{1+p(x,t)}\begin{bmatrix} 1+p(x,t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \partial_1 p \\ \partial_2 p \\ \partial_3 p \end{bmatrix} = 0$$

and if we work out this equation we obtain, since $1 + p(x,t) > 0$,

$$\partial_t v_1(x,t) + \partial_1 p(x,t) = 0, \tag{4.13}$$

and $\partial_2 p(x,t) = \partial_3 p(x,t) = 0$. Hence p does not depend on x_2 and x_3.

The equation (4.8) becomes

$$p_t(x,t) + [1+p][1+p]\begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+p(x,t))^{-1} & 0 \\ 0 & 0 & (1+p(x,t))^{-1} \end{bmatrix} : \begin{bmatrix} \partial_1 v_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

which turns out to be

$$p_t(x,t) + [1 + p(x,t)]^2 \begin{bmatrix} \partial_1 v_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

and this reduces to

$$\partial_t p(x_1, t) + [1 + p(x_1, t)]^2 \partial_1 v_1 = 0. \tag{4.14}$$

For the one dimensional case the notation can be simplified by writing x for x_1 and v for v_1. Also, in suitable notation for partial derivatives, the equations (4.13) and (4.14) are now written in the form

$$v_t(x,t) + p_x(x,t) = 0; \tag{4.15}$$

$$p_t(x,t) + [1 + p(x,t)]^2 v_x(x,t) = 0. \tag{4.16}$$

The assumption of one-dimensionality of the motion therefore leads to a very simple system of first order partial differential equations (4.15), (4.16) which is essentially nonlinear as will become clear in Section 4.5.

4.4 Linearization

The system (4.15), (4.16) can be linearized by a "plausibility" reasoning based on the assumption that the compression is "small". When it is assumed that the compression is "small" ($|r| \ll 1$), it follows from (4.10) that the same is true for pressure because

$$p(x,t) = \frac{-r(x,t)}{1 + r(x,t)}.$$

Therefore, if $0 < |r| < \varepsilon < 1$,

$$|p(x,t)| = \frac{|r(x,t)|}{1 + r(x,t)} \leq \frac{|r(x,t)|}{1 - \varepsilon} \ll 1.$$

The plausibility argument is that the term $(1 + p)^2$ in (4.16) can "safely" be replaced by 1 and this leads to the linearized system

$$v_t(x,t) + p_x(x,t) = 0; \tag{4.17}$$

$$p_t(x,t) + v_x(x,t) = 0. \tag{4.18}$$

This leads to the wave equation through elimination of either v or p by taking suitable derivatives. But it is not necessary to do that since as we know that by taking more derivatives one loses information. If we let $u_1 := [v + p]/2$ and $u_2 := [v - p]/2$, addition and subtraction of the two linear equations (4.17), (4.18) lead to the decoupled equations

$$u_{1,t}(x,t) + u_{1,x}(x,t) = 0; \tag{4.19}$$

$$u_{2,t}(x,t) - u_{2,x}(x,t) = 0. \tag{4.20}$$

For $-\infty < x < \infty$ and $t > 0$ the Cauchy problem, (4.19), (4.20) to which the initial conditions

$$\left. \begin{array}{l} u_1(x,0) = u_{10}(x); \\ u_2(x,0) = u_{20}(x), \end{array} \right\} \tag{4.21}$$

are added, is readily solved by the method of characteristics (see *e.g.* [12, Chapter 3]). The solutions are

$$\left. \begin{array}{l} u_1(x,t) = u_{10}(x - t); \\ u_2(x,t) = u_{20}(x + t). \end{array} \right\} \tag{4.22}$$

It is quite simple to obtain the initial states u_{10}, u_{20} from the initial states of v and p and equally simple to translate from the solutions u_1, u_2 back to v and p. Thus the linearized equations pose no formal difficulties and the solutions obtained are typical "sound waves" in a traditional sense. One difficulty however, would be obedience to the constraint (4.9).

Comparison of the system (4.15), (4.16) with the linearization (4.17), (4.18) shows another problem. In the nonlinear system the roles of v and p cannot be interchanged while in the linearized system they can.

4.5   The system plus the constraint

Let us incorporate the constraint $1 + p > 0$ into the equations by making the substitution

$$1 + p(x,t) = \exp\{q(x,t)\} > 0 \tag{4.23}$$

Since $p_t(x,t) = \exp\{q(x,t)\}q_t(x,t)$ and $p_x(x,t) = \exp\{q(x,t)\}q_x(x,t)$, the equations (4.15) and (4.16) now become

$$\left.\begin{array}{l} v_t(x,t) + \exp\{q(x,t)\}q_x(x,t) = 0 \\ q_t(x,t) + \exp\{q(x,t)\}v_x(x,t) = 0. \end{array}\right\} \tag{4.24}$$

The transformed system (4.24) has gained a remarkable symmetry even though the new equations are both nonlinear. The other advantage is that the substitutions

$$\left.\begin{array}{l} u_1(x,t) = \frac{1}{2}[v(x,t) + q(x,t)]; \\ u_2(x,t) = \frac{1}{2}[v(x,t) - q(x,t)], \end{array}\right\} \tag{4.25}$$

which works so well in the linear system, at least have some effect although the system is not decoupled. The resulting equations are

$$\left.\begin{array}{l} u_{1,t}(x,t) + \exp\{q(x,t)\}u_{1,x}(x,t) = 0; \\ u_{2,t}(x,t) - \exp\{q(x,t)\}u_{2,x}(x,t) = 0; \\ q(x,t) = u_1(x,t) - u_2(x,t). \end{array}\right\} \tag{4.26}$$

Still the roles of u_1 and u_2 cannot be interchanged. These are the equations we shall use to analyse the nonlinear acoustic problem.

4.6 Characteristics of the system

The characteristics of the system of Equations (4.26) are the level curves of the functions u_1 and u_2 (see *e.g.* Courant-Friedrichs [1], Jeffrey [2]). Let us denote the curves by C_1 and C_2.

Let the C_1-curve, on which u_1 is constant, be represented parametrically by

$$x = \Xi_1(\tau), \quad t = T(\tau),$$

and let $U_1(\tau) = u_1(\Xi_1(\tau), T(\tau))$. Then

$$\frac{dU_1}{d\tau} = u_{1,x}(\Xi, T)\Xi_1'(\tau) + u_{1,t}(\Xi, T)T'(\tau) = 0$$

If we compare this equation and the first equation in system (4.26) we see that

$$\Xi_1'(\tau) = \exp\{q_1(x,\tau)\} \text{ and } T'(\tau) = 1$$

From this it follows that $t = \tau$ so that

$$\Xi_1'(t) = \exp\{q_1(x,t)\}$$

with q_1 the evaluation of q on the C_1 curve.

For a C_2 curve, let

$$x = \Xi_2(\tau), \quad t = T(\tau).$$

Then, as above but comparison with the second equation in (4.26) yields

$$\Xi_2'(t) = -\exp\{q_2(x,t)\}$$

with q_2 the evaluation of q on C_2. Put together as a system we have

$$\left.\begin{aligned}
\Xi_1'(t) &= \exp\{q_1(x,t)\}, \\
\Xi_2'(t) &= -\exp\{q_2(x,t)\}
\end{aligned}\right\} \tag{4.27}$$

Chapter 5

The Dynamic Piston problem

5.1   Introduction

Within the context of Euler's description of fluid motion, the effects of the motion of a piston in a semi-infinite tube is well studied see *e.g.* Courant-Friedrichs [1]. The approach is to let the piston perform a prescribed motion. To realize such a motion may be impossible. We approach the problem differently by viewing the piston as a physical body with mass and surface area which is moved as the result of external forces. The applied external force will be in the form of an applied pressure. A schematic of the problem is shown in Figure 5.1 below.

AP(t) → ← *Ap(0,t)*
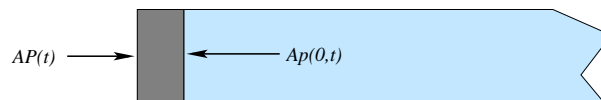
Figure 5.1: The dynamic piston problem

Let the reference configuration be the interval $(0, \infty)$ and assume the piston acts at the point $x = 0$. If M is the mass of the piston, A its surface area, $w(t)$ its velocity at time t and $P(t)$ the applied pressure, then the equation of motion for the piston is

$$Mw'(t) = AP(t) - Ap(0, t) \tag{5.1}$$

with p the unscaled internal pressure. The minus-sign on the right of (5.1) is in accordance with the pressure term in the formulation of balance of linear momentum as expressed in (4.1). This shows that internal pressure will oppose motion of the piston.

To this we add the contact condition

$$w(t) = v(0, t) \tag{5.2}$$

with v once again the unscaled internal velocity, ($w(t)$ is actually the limit as $x \to 0^+$ of the gas velocity). The condition (5.2) means that the velocity of the piston is the same as that of the particles directly ahead of it.

Scaling and combination of (5.1), (5.2) is in accordance with Section 4.2 but with the choice $L = \frac{M}{\rho A}$ gives the dimensionless equation

$$\frac{d}{dt}[v(0, t)] = P(t) - p(0, t). \tag{5.3}$$

This scaling is shown explicitly in Section 8.2. The problem we shall study in the following sections will be in terms of the functions u_1, u_2 introduced in Section 4.5. That is, the system (4.26) defined for $x > 0; t > 0$ together with the initial conditions $v(x, 0) = 0; p(x, 0) = 0$, and the dynamic boundary condition (5.3). Thus the system of differential equations in this case is:

$$\left. \begin{array}{l} u_{1,t}(x, t) + \exp\{q(x, t)\}u_{1,x}(x, t) = 0; \quad x > 0; t > 0; \\ u_{2,t}(x, t) - \exp\{q(x, t)\}u_{2,x}(x, t) = 0; \quad x > 0; t > 0; \\ q(x, t) = u_1(x, t) - u_2(x, t). \end{array} \right\} \tag{5.4}$$

The initial conditions can be found from (4.23) and (4.25). Indeed, by (4.23) $p(x, 0) = 0$ implies that $q(x, 0) = 0$. In addition, $v(x, 0) = 0$, and it follows from (4.25) that $u_1(x, 0)$ and $u_2(x, 0)$ are both zero. That is

$$u_1(x, 0) = u_2(x, 0) = 0 \text{ for } x > 0. \tag{5.5}$$

By the same token, since $v = u_1 + u_2$, $q = u_1 - u_2$ and $p = \exp\{q\} - 1$, the terms in the dynamic boundary condition (5.3) can all be expressed in terms of u_1 and u_2. With some re-arrangement this condition now takes the form

$$\frac{d}{dt}[u_1(0, t) + u_2(0, t)] + \exp\{u_1(0, t) - u_2(0, t)\} = 1 + P(t). \tag{5.6}$$

5.2   Reduction of equations by use of characteristics

Under special circumstances consideration of the characteristics of the system (5.4) as discussed in Section 4.6 yields information that reduces the present problem to a single nonlinear transport equation with a dynamic boundary condition.

In [10] and [11] it is assumed without proof that the C_2-characteristics emanating from the positive

x-axis cover the quadrant $\{x > 0;\ t > 0\}$. If that is the case, the fact that u_2 is constant along these characteristics and the homogeneous boundary conditions (5.5) shows that $u_2 = 0$ on the whole quadrant. If that is the case, the reduction is achieved with u_1 the only unknown function.

To find sufficient conditions for the C_2-characteristics to cover the quadrant in question, we revert to the very beginning in Chapter 2 where the notion of compression was first introduced. According to (2.1),

$$1 + r = \frac{V}{V_0}. \tag{5.7}$$

Physical considerations tell us that the volume of gas cannot be compressed to below a certain volume or the gas may liquefy. Also, it cannot be rarefied to a vacuum. All this is relative to the reference volume V_0. Thus we are lead to the *strict constraints*

$$0 < R_0 \le \frac{V}{V_0} \le R_1 < \infty, \tag{5.8}$$

with R_0 and R_1 physical constants for the particular gas. Combination of (5.7) and (5.8) leads to

$$0 < R_0 \le 1 + r \le R_1 < \infty. \tag{5.9}$$

We shall carry the inequalities (5.9) over to the continuum-situation. This is still in accordance with the constraint $1 + r > 0$. From the Boyle-Mariott analogy (4.10) it follows that $R_0(1 + p) \le 1 \le R_1(1 + p)$ and we obtain the inequalities

$$R_1^{-1} \le (1 + p) \le R_0^{-1}. \tag{5.10}$$

Theorem 5.2.1. *Under the strict constraints (5.10) the C_2-characteristics emanating from the positive x-axis cover the quadrant $\{x > 0,\ t > 0\}$. If this is the case, $u_2 = 0$ on the quadrant and $u_1(x,t) = 0$ in the region $\{t > 0,\ x > t\}$.*

Proof. From the definition of q (4.23) and the differential equations for the characteristics (4.27) it is seen that $-R_0^{-1} \le -(1 + p(\Xi_2(t), t) = \Xi_2'(t) \le -R_1^{-1}$. Thus, for a point $y > 0$ on the x-axis where $\Xi_2(0) = y$ we have $y - R_0^{-1}t \le \Xi_2(t) \le y - R_1^{-1}t$. It follows that $\Xi_2(t) = 0$ for some t such that $R_0 y \le t \le R_1 y$. If we let y run through all positive values, we see that at every point on the t-axis a C_2-characteristic emanating from the positive x-axis will meet. Thus, since C_2 is a decreasing curve, the covering of the quadrant by C_2-curves is proved which means that in the quadrant $u_2 = 0$ since $u_2(x, 0) = 0$ for $x > 0$.

Finally we consider a C_1-characteristic emanating from a point $y > 0$ on the x-axis. Along such a

characteristic, $u_1 = 0$. But we have already shown that $u_2 = 0$ everywhere. Hence $q = u_1 - u_2 = 0$ along the characteristic, and therefore, by (4.27), the slope of the characteristic is 1. In fact the equation of the characteristic is $x = y + t$. By letting y vary over the positive x-axis we see that the octant in question is covered by the C_1-characteristics. $\qquad\square$

Remark 5.2.2. The strict constraints (5.9) impose restrictions for initial conditions. Since the initial values of u_1 and u_2 are carried along the characteristics, care should be taken that these constraints are obeyed at time zero. This means that $R_1^{-1} \leq 1 + p(x,0) = \exp\{q(x,0)\} = \exp\{u_1(x) - u_2(x)\} = 1 \leq R_0^{-1}$. This will be satisfied if $R_0 \leq 1 \leq R_1$ which seems to be physically reasonable if the reference volume can be compressed.

In the following we shall assume that the C_2-characteristics coming from the positive x-axis cover the whole quadrant on which the problem is defined. If this is the case, the second of the equations in (5.4) is satisfied in a way consistent with the homogeneous initial state $u_2(x,0) = 0$. What remains to be solved is the first equation together with the dynamic boundary condition (5.6). The only unknowns are now $u_1(x,t)$ and the boundary value $w(t) = v(0,t) = u_1(0,t) + u_2(0,t) = u_1(0,t)$. Let us use the easier notation $u(x,t) = u_1(x,t)$. Then the problem to be solved is

$$\left.\begin{aligned}
&u_t(x,t) + \exp\{u(x,t)\}u_x(x,t) = 0, x > 0; t > 0, \\
&u(x,0) = 0, \\
&u(0,t) = w(t)
\end{aligned}\right\} \tag{5.11}$$

and

$$w'(t) + \exp\{w(t)\} = 1 + P(t). \tag{5.12}$$

5.3   Analysis of the piston velocity

The behaviour of the solutions of (5.11) is determined by the piston velocity w. The ordinary differential equation (5.12) can be solved explicitly by using the substitution

$$g(t) = \exp\{-w(t)\}. \tag{5.13}$$

Indeed we have, $g'(t) = -g(t)w'(t) = g(t)[1 + P(t) + 1/g(t)]$ by (5.12). This leads to the linear differential equation

$$g'(t) + [1 + P(t)]g(t) = 1. \tag{5.14}$$

If we assume, as we shall do from now on, that the function P is continuous, the equation (5.14) can be uniquely solved by integrating factor provided the initial value $g(0)$ is known. To find that, we let $w(0)$ be compatible with the initial state $v(x,0) = u(x,0)$. Thus we let $w(0) = \lim_{x \to 0^+} u(x,0) = 0$. From (5.13) it is seen that

$$g(0) = \exp\{-w(0)\} = 1. \tag{5.15}$$

With the integrating factor I defined by

$$I(t) := \exp\{t + \int_0^t P(\tau)d\tau\}, \tag{5.16}$$

the solution is

$$g(t) = \frac{1 + \int_0^t I(\tau)d\tau}{I(t)} > 0. \tag{5.17}$$

Thus the piston velocity $w(t) = -\ln\{g(t)\}$ is well defined. This is a general expression for g and we can see that g, and therefore w, depends on the history of the applied pressure. Let us continue to study this dependence to see in what way the applied pressure affects the piston velocity.

To begin with we note that from (5.14) and (5.15)

$$g'(0) = 1 - [1 + P_0] = -P_0, \tag{5.18}$$

where $P_0 = P(0)$. Hence, if P is initially positive, g will decrease for some time and w will therefore increase during the same time interval. If the applied pressure is not kept at a sufficiently high level, the piston may eventually run out of steam as we shall see later.

A first concern is to obtain upper bounds and positive lower bounds for g. This may appear to be easy since we already have an explicit expression embodied in (5.16) and (5.17), but this is not as easy as it may seem. Instead we take an indirect approach by comparing solutions corresponding to different applied pressures. Let $\gamma(t)$ be the solution of

$$\left.\begin{array}{l} \gamma'(t) + [1 + Q(t)]\gamma(t) = 1, \\ \gamma(0) = 1. \end{array}\right\} \tag{5.19}$$

We compare g and γ by letting $h(t) = g(t) - \gamma(t)$. Subtraction of (5.14) and (5.19) gives, after some manipulation,

$$\left.\begin{array}{l} h'(t) + [1 + Q(t)]h(t) = [Q(t) - P(t)]g(t), \\ h(0) = 0. \end{array}\right\} \tag{5.20}$$

We can now formulate the Comparison Theorem [11].

Theorem 5.3.1. *If for all $t > 0$, $Q(t) \geq P(t)$, then $\gamma(t) \leq g(t)$.*

Proof. Since $g(t) > 0$, it follows from the first equation in (5.20) that $h'(t) + [1 + Q(t)]h(t) \geq 0$. To resolve this inequality we consider the "integrating factor" $K(t)$ given by

$$K(t) = \exp\{\int_0^t (1 + Q(\tau))d\tau\} = \exp\{t + \int_0^t Q(\tau)d\tau\} > 0, \tag{5.21}$$

and note that

$$\frac{d}{dt}[K(t)h(t)] = K(t)[h'(t) + [1 + Q(t)]h(t)] \geq 0. \tag{5.22}$$

Integration of the inequality (5.22) from 0 to t and the initial condition in (5.20) leads to the conclusion that $K(t)h(t) \geq 0$. Since $K(t)$ is always positive, $h(t)$ cannot be negative. $\square$

Remark 5.3.2. It is readily seen from the proof of Theorem 5.3.1 that $Q(t) \leq P(t)$ implies that $\gamma(t) \geq g(t)$.

Corollary 5.3.1. *If for all $t > 0$, there are constants $m > -1$ and M such that $m \leq P(t) \leq M$, then*

$$\frac{1 + M \exp\{-(1 + M)t\}}{1 + M} \leq g(t) \leq \frac{1 + m \exp\{-(1 + m)t\}}{1 + m}.$$

Proof. For a constant applied pressure $P(t) = f > -1$ the expression (5.17) can readily be evaluated. The result is

$$g(t) = \frac{1 + f \exp\{-(1 + f)t\}}{1 + f} \tag{5.23}$$

Now apply Theorem 5.3.1 with $\gamma(t) = m$ and $\gamma(t) = M$ $\square$

Corollary 5.3.2. *Nonnegative applied pressures produce nonnegative piston velocities.*

Proof. For $m = 0$ it follows that $g(t) \leq 1$ and therefore $w(t) = -\ln g(t) \geq 0$. $\square$

The next dependency result [11] is

Theorem 5.3.3. *Suppose that P is continuously differentiable. If either $P_0 > 0$ and $P'(t) \geq 0$ or $P_0 = 0$ and $P'(t) > 0$, the piston velocity $w(t)$ will be strictly increasing. Likewise if either $P_0 < 0$ and $P'(t) \leq 0$ or $P_0 = 0$ and $P'(t) < 0$, the piston velocity will be strictly decreasing.*

Proof. Let $h(t) := g'(t)$. Differentiation of (5.14) and re-substitution in the result shows that

$$h'(t) + [1 + P(t) - \frac{P'(t)}{1 + P(t)}]h(t) = -\frac{P'(t)}{1 + P(t)}. \tag{5.24}$$

Also, from (5.18), $h(0) = -P_0$. The integrating factor for the (linear) equation (5.24) is found to be

$$J(t) = (\frac{1 + P_0}{1 + P(t)})I(t),\tag{5.25}$$

with $I(t)$ defined in (5.16). By following a standard procedure, we obtain

$$J(t)h(t) = -P_0 - \int_0^t (\frac{P'(t)}{1 + P(t)})J(\tau)d\tau.\tag{5.26}$$

It is seen from (5.25) that $J(t) > 0$ and it follows from (5.26) that $h(t) < 0$. Hence $g(t)$ is strictly decreasing. The second statement follows in precisely the same way from the calculations already made. □

The following result [11] gives a sufficient condition for the derivative of g to be increasing when g itself is decreasing.

Theorem 5.3.4. *If $g'(t) < 0$ and $P'(t) \le 0$ then $g''(t) > 0$*

Proof. By using (5.14) the expression (5.24) can be re-phrased as

$$g''(t) = P'(t)\frac{g'(t) - 1}{1 + P(t)} - [1 + P(t)] = -P'(t)g(t) - [1 + P(t)]g'(t).\tag{5.27}$$

Again, by (5.14), $g'(t) < 0$ if and only if $[1 + P(t)]g(t) > 1$. Thus, $1 + P(t) > 0$ and by (5.27) the proof is complete. □

Example 5.3.5. This example is crucial in the rest of the work. We consider the case of a decreasing positive applied pressure given by

$$P(t) = \frac{1}{1 + t}.\tag{5.28}$$

Since $P(t)$ is strictly decreasing $P'(t) < 0$ for all t. In this case the expressions involved in (5.17) can be obtained explicitly:

$$I(t) = \exp\{t + \ln(1 + t)\} = (1 + t)\exp\{t\};$$

$$\int_0^t I(\tau)d\tau = \int_0^t (1 + \tau)\exp\{\tau\}d\tau = t\exp\{t\},$$

which leads to

$$g(t) = \frac{1 + t\exp\{t\}}{(1 + t)\exp\{t\}} = \frac{t + \exp\{-t\}}{1 + t}.\tag{5.29}$$

The behaviour of the functions $g(t)$ and $w(t) = -\ln g(t)$ is shown in Figures 5.2 and 5.3 below.
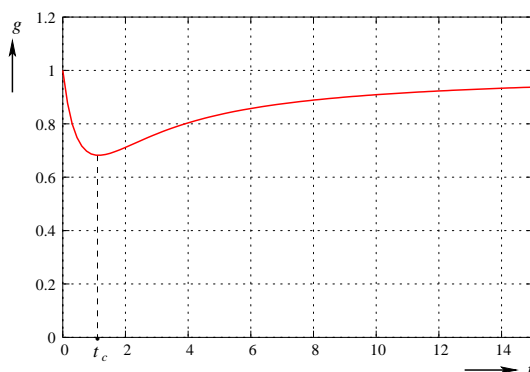
Figure 5.2: Plot of the function $g(t)$
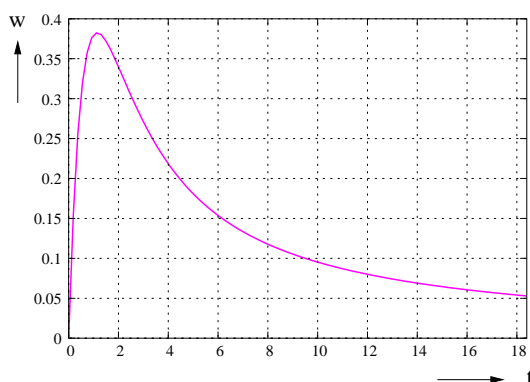
Figure 5.3: Piston velocity with positive decreasing input

This is evidently a case, as mentioned earlier, where the piston loses speed after a while because the driving force is not sufficiently sustained.

Figure 5.2 furthermore illustrates Theorem 5.3.4: Suppose $g'(t_c) = 0$. In the time interval $[0, t_c)$; $t_c \approx 1.1462$, $g'(t) < 0$ and $P'(t) < 0$ for all t. Therefore, in the mentioned time interval, $g''(t) > 0$. The "critical" time t_c is calculated by numerically solving the equation

$$g'(t) = \frac{1 - (2 + t)\exp\{t\}}{(1 + t)^2} = 0.$$

It is also seen that g' increases from -1 to a positive maximum then decreases to zero as t goes to infinity. Verification of this is by elementary calculus.

Chapter 6

Characteristics and the piston problem

6.1 Characteristics emanating from the t-axis

In Chapter 5 we saw that when the quadrant $Q = \{(x,t) : t > 0, \ x > 0\}$ is covered by C_2-characteristics the piston problem reduces to the single transport equation (5.11) for velocity. The boundary velocity $w(t)$ represents the velocity of the piston and is determined by the dynamic equation (5.12). This equation could be solved explicitly by the substitution $w(t) = \exp\{-g(t)\}$ under the initial condition $w(0) = 0$ which is the same as the initial condition $g(0) = 1$. We also analysed the piston velocity. In this chapter and the next we turn our attention to the motion of the gas ahead of the piston. Let us, by way of re-capitulation, state the dynamic problem under investigation:

$$\left.\begin{aligned}
&u_t(x,t) + \exp\{u(x,t)\}u_x(x,t) = 0, x > 0; t > 0;\\
&u(x,0) = 0;\\
&u(0,t) = w(t) = -\ln\{g(t)\}.
\end{aligned}\right\} \tag{6.1}$$

The problem (6.1) will be analysed by the method of characteristics. The characteristics emanating from the x-axis ($t = 0$) are, because of the homogeneous initial condition, found to be the family of straight lines $x = t + y; \ y > 0$ (see *e.g.* [4], [12]). They cover the upper octant $\{(x,t) : x > t > 0\}$ and there $u(x,t) = 0$.

The more intricate problem is to study the characteristics emanating from the t-axis ($x = 0$). They will give the motion of the gas ahead of the piston. Let us represent those characteristics (u is constant

along them) in the form $t = T(x)$. Along these curves $U(x) := u(x, T(x))$ is constant and it follows that

$$U'(x) = u_x(x, T(x)) + T'(x)u_t(x, T(x)) = 0. \tag{6.2}$$

Comparison of the partial differential equation in (6.1), evaluated on the curve, with (6.2) gives

$$T'(x) = \exp\{-U(x)\} = \exp\{-U_0\}, \tag{6.3}$$

where $U_0 = U(0)$. From (6.3) we deduce that the characteristics in question are straight lines given by $t = T(x) = s + x \exp\{-U_0\}$. On such a characteristic $U_0 = u(0, s) = w(s)$ and hence the equation of a characteristic coming from the $t-$ axis is, by the boundary condition in (6.1),

$$t = s + x \exp\{-w(s)\} = s + xg(s). \tag{6.4}$$

Thus, to calculate the solution of (5.11) for given (x, t) in the quadrant Q of the (xt) plane it is necessary to solve (6.4) for s. At such a point $u(x, t) = w(s)$. The point s on the line $x = 0$ depends on x and t. From (6.4) we see that

$$s_x(x, t) = -\frac{g(s)}{1 + xg'(s)}. \tag{6.5}$$

The spatial derivative of the velocity $u(x, t)$ may now be expressed entirely in terms of the known function g. Indeed, since $u(x, t) = w(s(x, t)) = -\ln(g(s(x, t)))$, we find that $u_x = -\frac{g'(s)}{g(s)}s_x$. This combined with (6.5) yields

$$u_x(x, t) = \frac{g'(s(x, t))}{1 + xg'(s(x, t))}. \tag{6.6}$$

In the same way, since $1 + p(x, t) = \exp\{w(s(x, t))\}$, we find that

$$p_x(x, t) = \frac{g'(s(x, t))}{g(s(x, t))[1 + xg'(s(x, t))]}. \tag{6.7}$$

It should be noted that, by (5.14), the derivative g' can be expressed in terms of the applied pressure P and g itself. Since, by (5.16) and (5.17), g itself depends entirely on P. Thus the expressions above depend solely on P.

6.2 Characteristics intersecting

Next we consider the case where the piston velocity is strictly increasing in some time interval. Then there must be $s_1 < s_2$ such that $w(s_1) < w(s_2)$. Since the function g is strictly decreasing in that

interval, the characteristic passing through $0, s_1$ will have a slope which is larger compared to the one passing through $0, s_2$ and consequently they will intersect at some point (x, t) with $x, t > 0$. Indeed, from (6.4), $t = s_1 + xg(s_1) = s_2 + xg(s_2)$. We can solve for x:

$$x = -\frac{s_2 - s_1}{g(s_2) - g(s_1)} = -\frac{1}{g'(s)} \geq 0; \quad s_1 \leq s \leq s_2, \tag{6.8}$$

where the last two inequalities follow from the mean value theorem. By (6.4) the $t-$ coordinate of the point of intersection is given by

$$t = s_i - \frac{g(s_i)}{g'(s_i)}; \quad i = 1, 2. \tag{6.9}$$

Thus the existence of an increasing part of the piston velocity $w(t)$ (or a decreasing part of the function $g(t)$) leads to the intersection of characteristics. This implies that the solution function $u(x, t)$ may become multi-valued or discontinuous. Also (6.8) $\Rightarrow xg'(s) + 1 = 0$ resulting in the non-existence of the spatial derivatives u_x and p_x from (6.6) and (6.7), respectively. Figure 6.1 depicts the situation described
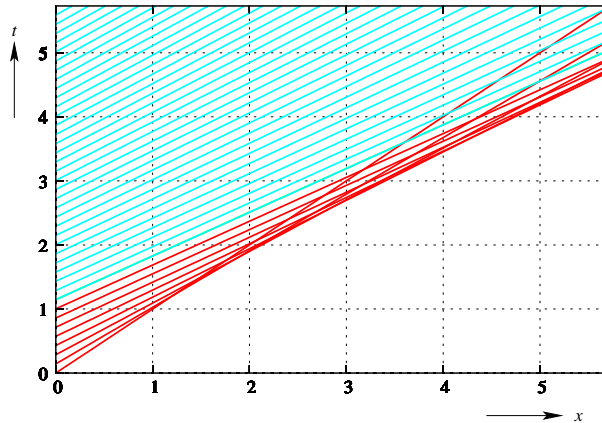
Figure 6.1: Characteristics emanating from the t- axis

in Example 5.3.5 (specifically Figures 5.2 and 5.3) where $g(t)$ is strictly decreasing in the time interval $[0, t_c)$ and therefore some of the characteristics will intersect. Because of the representation $t = T(x)$ for the characteristics we chose the t-axis vertically.

6.3   Where do characteristics intersect?

When viewing a plot of characteristics such as in Figure 6.1 the question of the location of the points of intersection comes up. We approach this question by investigating the equation

$$F(s; x, t) := s + xg(s) - t = 0; \quad (x, t) \in Q. \tag{6.10}$$

Here (x, t) is the point through which a characteristic passes. Thus x and t are parameters while s is the variable. The definition (6.10) is in accordance with the representation (6.4) of characteristics in general.

The analysis of the zero's of the function $F(s; x, t)$ as defined in (6.10) can yield much more because the number of zero's is precisely the number of characteristics through the point $(x, t) \in Q$ while the solution(s) s tell us where the characteristic(s) emanate from. The *index $i(x, t)$* of a point (x, t) is defined as the number of solutions of the equation (6.10).

The analysis for general $g(t)$ would be a tremendous task. We therefore restrict the investigation to a simpler class of cases, in line with Example 5.3.5. These restrictions will be as follows:

1. $P(t) \geq 0; \quad P_0 > 0$;

2. $P'(t) < 0$;

3. For some $t_c > 0$, $g'(t) < 0$ for $0 < t < t_c; \quad g'(t) > 0$ for $t > t_c$.

The analysis in [11] pertains to the much simpler case when $g' < 0$ and $g'' > 0$. In this analysis provision is made for the possibility that g'' can change sign. The following result is needed to be able to cope with the more complex situation at hand:

Lemma 6.3.1. *Under the restrictions above the following is true:*

$$g''(t) > 0; \quad for\ 0 < t < t_c;$$
$$g'(t_c) = 0;$$
$$g_\infty = \lim_{t \to \infty} g(t)\ exists,\ and \quad \frac{1}{1 + P_0} \leq g_\infty \leq 1;$$
$$\lim_{t \to \infty} g'(t) = 0;$$
$$Let\ P_\infty = \lim_{t \to \infty} P(t).\ Then\ g_\infty = \frac{1}{1 + P_\infty}.$$

Proof. Under the restrictions above $g'(t) < 0$ for $0 < t < t_c$ and $P'(t) < 0$ for all t. By Theorem 5.3.4, $g''(t) > 0$ for $0 < t < t_c$.

If $g'(t) < 0$ for $0 < t < t_c$ and $g'(t) > 0$ for $t > t_c$ it follows from continuity that $g'(t_c) = 0$

By Corollary 5.3.1 the function g is bounded. In fact, since $0 < P(t) \leq P_0$, $[1 + P_0]^{-1} \leq g(t) \leq 1$ for all t. Since $g(t)$ is increasing for $t > t_c$, it follows that the supremum $g_\infty = \sup_{t > t_c} g(t)$ exists. By the definition of supremum g_∞ is the limit we have in mind.

Now we can prove that $\lim_{t \to \infty} g'(t) = 0$ by using the mean-value theorem. Let $a > 0$ be fixed. Then $g(t + a) - g(t) = ag'(\xi)$ with $t \leq \xi \leq t + a$. Therefore, $g'(\xi) = \frac{g(t+a)-g(t)}{a} \to 0$ as $t \to \infty$.

The final expression is derived by noticing, as above, that the limit $P_\infty = \inf_{t > 0} P(t)$ exists. When the results we have already obtained are applied to the differential equation $g'(t) = 1 - [1 + P(t)]g(t)$ (5.14), the resulting expression is $1 - [1 + P_\infty]g_\infty = 0$ which is what we want.

$\square$

For further reference we record the following:

$$\left.\begin{aligned}
&F(0; x, t) = x - t; \\
&F_s(s; x, t) = 1 + xg'(s); \\
&F_s(0; x, t) = 1 - P_0 x; \\
&F_s(t_c; x, t) = 1; \\
&F_{ss}(s; x, t) = xg''(s); \\
&F(s; x, t) \sim s + \frac{x}{1+P_\infty} - t \ \text{ as } \ s \to \infty.
\end{aligned}\right\} \tag{6.11}$$

The asymptotic behaviour of F for large s follows from Lemma 6.3.1. Indeed, if $F_\infty(s; x, t) = s + xg_\infty - t$, then $|F_\infty(s; x, t) - F(s; x, t)| = x[g_\infty - g(s)]$ which tends to zero when $s \to \infty$. The next result is crucial:

Lemma 6.3.2. *If $P_0 x \geq 1$, there is precisely one point $s = s_m(x, t) \in [0, t_c)$ where $F_s(s; x, t) = 0$. If $P_0 x = 1$, $s_m = 0$. Otherwise $s_m > 0$. At $s = s_m$, $F(s; x, t)$ has a local minimum. In addition, $s_m \in [0, t_c)$.*

If $P_0 x < 1$ there is no such point as s_m.

There are no local maxima of F.

Proof. Since, by Lemma 6.3.1, $g'(s) \to 0$ as $s \to \infty$; $F_s = 1 + xg'(s) \to 1$ as $s \to \infty$. Therefore since $F_s(0; x, t) = 1 - P_0 x$, F_s has values in the interval $[1 - P_0 x, 1]$. Possibly some values are not in this interval. From the continuity of F_s as function of s it is seen that the possibility of zero's of F_s exists. We investigate the three possibilities pertaining to the signature of $1 - P_0 x$. With this in mind, we note that for $s \in [0, t_c]$, $1 - P_0 x \leq F_s(s; x, t) = 1 + xg'(s) \leq 1$. In addition, $g''(s) > 0$ in this interval, and $g'(s) > 0$ for $s > t_c$. We investigate the three possibilities pertaining to the signature of $1 - P_0 x$.

CASE 1: If $P_0 x > 1$ then $F_s(0; x, t) = 1 - P_0 x < 0$. Hence F_s increases from a negative to a positive value in the interval $[0, t_c]$ and remains positive thereafter. Therefore it can have only one zero

$s_m \in (0, t_c)$.

CASE2: If $P_0 x = 1$, $F_s(0; x, t) = 1 - P_0 x = 0$. Since $F_s(s; x, t) > 0$ for all other values of s, this is the only zero.

CASE3: If $P_0 x < 1$, F_s will start with a positive value and will remain positive. In this case there cannot be a zero.

Finally, since $F_{ss} > 0$, there cannot be local maxima.

□

We shall consider points $(x, t) \in Q$ in certain subsets, namely

$$S_+ = \{(x, t) \in Q : t > x > 0\};$$
$$S_0 = \{(x, t) \in Q : 0 < x < 1/P_0\};$$
$$S_- = \{(x, t) \in Q : x \geq t > 0\}.$$

These regions S_+ and S_- cover the whole quadrant Q. S_+ is the region above and S_- below the line $t = x$. The set S_0 plays a very definite role in Lemma 6.3.2. Indeed, for $(x, t) \in S_0$, F has a local minimum F_m at $s = s_m \in [0, t_c)$. The different regions are shown in Figure 6.2 below.

We will deal with points in S_+ and S_- separately.

THE REGION S_+: In this region, by (6.11) $F(0, x, t) = x - t < 0$ and $F(s, x, t) \to \infty$ as $s \to \infty$. This means there is at least one positive solution of $F(s, x, t) = 0$. In order to investigate the existence of multiple solutions, we consider the derivative $F_s = 1 + x g'(s)$. By Lemma 6.3.2 there can be at most one minimum-point of F in this region and at that point F can only have a negative value otherwise it must be positive everywhere, in conflict with $F(0, x, t) =< 0$. Thus, in this region $i(x, t) = 1$.

THE REGION S_-: In this region $F(0, x, t) = x - t \geq 0$ and $F_s(0, x, t) = 1 - P_0 x$ (from (6.11)). By Lemma 6.3.2 the function F can therefore have only one zero for $(x, t) \in S_- \cap S_0$ and that is on the line $t = x$. That zero is $s = 0$. In $S_- \cap S_0 \backslash \{t = x\}$ there can be no zeros.

For $(x, t) \in S_- \backslash S_0$ the function F has a minimum at $s = s_m \in (0, t_c)$. The number of zero's of F is therefore determined by the minimum-value $F_m = F(s_m; x, t)$. The conclusion is

$$i(x, t) = 0 \text{ if } F_m > 0;$$
$$i(x, t) = 1 \text{ if } F_m = 0;$$
$$i(x, t) = 2 \text{ if } F_m < 0.$$

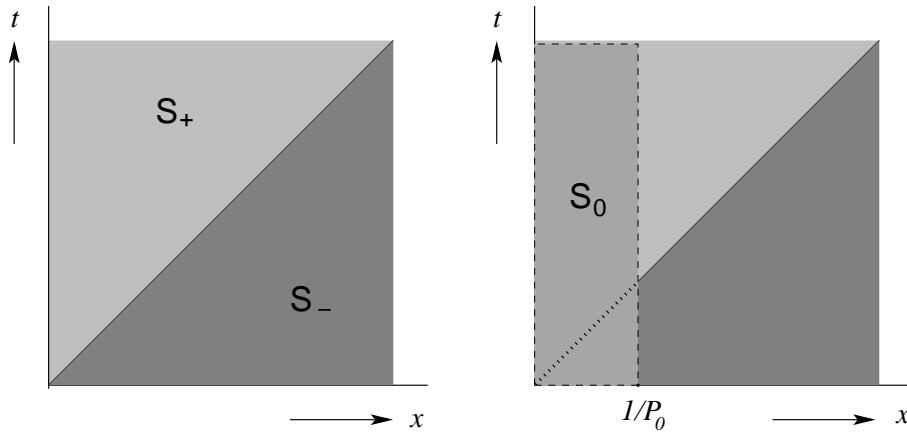The case $F_m = 0$ is of special interest as it defines a "bounding curve" in the quadrant Q. Indeed,

Figure 6.2: The regions in question

we have remarked earlier that s_m depends on the point (x, t). The curve we refer to is therefore of the form $E(x, t) = F(s_m(x, t); x, t) = 0$ and is represented as an implicit relation between t and x. From our discussion above it is seen that the bounding curve is defined for $x \geq x_0 = 1/P_0$ and $t \geq t_0 = 1/P_0$. Furthermore, the curve is below the line $t = x$, only touching the line at the point (x_0, t_0). The bounding curve therefore is the interface between the regions where $i = 0$ and $i = 2$.

6.4 The bounding curve

If the point (x, t) is on the bounding curve, it is seen from the previous section that $F(s_m(x, t); x, t) = 0$ and $F_s(s_m(x, t); x, t) = 0$. This means that the bounding curve is an envelope of characteristics emanating from the t-axis. We further investigate the envelope by calculating its slope. The following calculation is essential:

$$E_x = F_s(s_m)\frac{\partial s_m}{\partial x} + F_x(s_m) + F_t(s_m)\frac{dt}{dx} = F_x(s_m) + F_t(s_m)\frac{dt}{dx} = 0. \tag{6.12}$$

Thus we can solve for dt/dx from (6.12) to obtain

$$\frac{dt}{dx} = -\frac{F_x(s_m)}{F_t(s_m)} = g(s_m). \tag{6.13}$$

From (6.13) we immediately see that the envelope is an increasing curve since g is always positive. In addition, from Lemma 6.3.2 it is seen that at the point (x_0, t_0), $s_m = 0$ so that at this point $dt/dx = g(0) = 1$. This means that at the starting point the envelope is tangential to the line $t = x$. In turn this means that the envelope $E(x, t) = 0$ and the curve $t - x = 0$ form a cusp at (x_0, t_0).

We need to examine what happens below the envelope. Suppose that $E(x, t) = 0$ and $0 < t^* < t$. In

other words, the point (x, t^*) is below the envelope. Now, $F(0; t^*, x) = x - t* > x - t > 0$. Hence, the minimum value of F at this point will be positive, and no characteristic emanating from the t-axis will go through this point.

Our findings in this and the previous section can be summarized as follows:

Theorem 6.4.1.

$$If \quad t > x > 0, \quad i(x, t) = 1;$$
$$if \quad t = x > 0 \quad and \quad 0 < x \le \frac{1}{P_0}, \quad i(x, t) = 1;$$
$$if \quad 0 < t < x \quad and \quad 0 < x \le \frac{1}{P_0}, \quad i(x, t) = 0.$$

Suppose that $x > 1/P_0$ and $E(x, t^\dagger) = 0$, then

$$if \quad x \ge t > t^\dagger, \quad i(x, t) = 2;$$
$$i(x, t^\dagger) = 1;$$
$$if \quad t < t^\dagger, \quad i(x, t) = 0.$$

6.5   Computing an envelope of characteristics

The objective here is to find points $(x, t) \in Q$ for which $E(x, t) = 0$. From Sections 6.3 and 6.4 we see that it is only feasible for $x > x_0 = 1/P_0$. For $x = x_0$ we already know that $t = t_0 = 1/p_0$. The equations at hand for the task are

$$\left. \begin{array}{l} F(s, x, t) = s + xg(s) - t = 0; \\ F_s(s, x) = 1 + xg'(s) = 0. \end{array} \right\} \tag{6.14}$$

The task of finding the points (x, t) on the envelope will therefore entail the elimination of s from the equations (6.14). In the case treated in [11] it was possible to do this explicitly. In the situation of Example 5.3.5 explicit elimination of s does not seem to be possible. The envelope has to be found by a numerical algorithm which is based on the following:

1. Choose $x > x_0$.

2. Solve for s from $1 + xg'(s) = 0$.

3. Calculate t from $t = s + xg(s)$.

It is the second step of the algorithm that may be analytically difficult and will then require a numerical procedure. We illustrate by continuing with Example 5.3.5 where $P(t) = 1/(1+t)$. From the expression (5.29) for $g(t)$ we obtain

$$g'(s) = \frac{1 - (2+s)\exp(-s)}{(1+s)^2}.$$

(6.15)

Use of (5.12) in Step 2 of the algorithm leads to the equation

$$\frac{x[1 - (2+s)\exp(-s)] + (1+s)^2}{x(1+s)^2} = 0,$$

and this reduces to

$$f(s; x) = x[1 - (2+s)\exp(-s)] + (1+s)^2 = 0,$$

(6.16)

with s the unknown (x having been chosen in Step 1).

In order to find a suitable numerical method for solving the equation (6.16) we investigate the function f for $s \geq 0$. First we note that $f(0; x) = 1 - x$. Since $x > 1/P_0 = 1$ in this case, it is seen that $f(0; x) < 0$. Next, we consider f_s which turns out to be $f_s(s; x) = (1+s)[x\exp(-s) + 2] > 0$. We see that the function f increases from a negative value at $s = 0$ and will eventually become positive. Thus the Newton-Raphson algorithm

$$s_{n+1} = s_n - \frac{f(s_n; x)}{f_s(s_n; x)}$$

would be quite suitable.

Once a sufficiently accurate approximation for the solution of (6.16) is found, the value of t on the curve can be calculated by Step 3 above. The result of the calculation with added perspectives can be seen in Figure 6.3
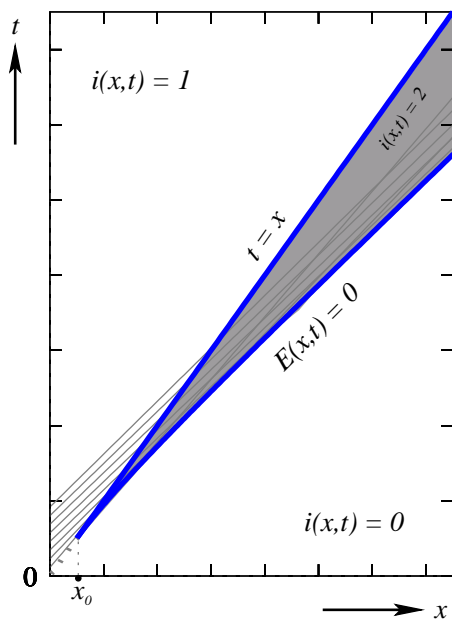
Figure 6.3: Characteristics emanating from the t-axis, their envelope and the index i in various regions. In this case $P(t) = 1/(1 + t)$.

Chapter 7

Jump curves

7.1 The jump condition

Under circumstances when the piston velocity increases, jump discontinuities in the solution $u(x,t)$ of the problem (5.11) will develop. The resulting discontinuous function cannot be a classical solution of a differential equation. In fact, the underlying differential equations were derived from the fundamental physical principles of conservation of mass and balance of linear momentum under the assumption that the entities involved were differentiable. These principles as formulated can hold even when the functions involved are not smooth. We can therefore revert to them and abandon the differential equations.

In the problem at hand, the function $u(x,t)$ is the scaled velocity of particle x at time t. We therefore deal with the case of a velocity function that can have a jump discontinuity. We first go back to the balance of linear momentum in unscaled form as set out in Section 4.1. In the one-dimensional case this has a particular form: Let $[a,b]$ be an arbitrary subinterval of the reference configuration $(0,\infty)$ and, as in Section 4.3, let $\psi(x,t)$ define the one-dimensional motion. The equation (4.1) should be re-interpreted. For this purpose we need to realize that if $A = \psi(a,t)$ and $B = \psi(b,t)$ then the unit exterior normals at A and B are -1 and 1 respectively. Thus the boundary term in (4.1) is replaced by $-[P(B,t)n(B) - P(A,t)n(A)] = P(A,t) - P(B,t) = p(a,t) - p(b,t)$. Here we still use the notation: if $F(X,t)$ is a function defined for X in the configuration at time t then $f(x,t) := F(\psi(x,t),t)$. Hence,

(4.1) for one dimensional motion, can be stated in the form

$$
\begin{aligned}
\frac{d}{dt} \int_{\psi(a,t)}^{\psi(b,t)} \sigma(X,t)V(X,t)dX &= \int_{\psi(a,t)}^{\psi(b,t)} P(X,t)n(X)dS \\
&= P(\psi(a,t),t) - P(\psi(b,t),t) \\
&= P(A,t) - P(B,t) \\
&= p(a,t) - p(b,t).
\end{aligned}
\tag{7.1}
$$

Let us assume that a jump discontinuity in the particle velocity occurs over the smooth curve $x = j(t)$ in the reference configuration. Suppose that $a < j(t) < b$ and let $J(t) = \psi(j(t),t)$. For $\varepsilon > 0$ but sufficiently small, we have that $a < j(t) - \varepsilon < j(t) + \varepsilon < b$. We deal with integrals such as on the left of (7.1) in the following way: Let $J_-(t) = \psi(j(t) - \varepsilon, t)$ and $J_+(t) = \psi(j(t) + \varepsilon, t)$. Then, since $\psi_x > 0$, it follows that $A = A(t) < J_-(t) < J(t) < J_+(t) < B - B(t)$. In general we resort to improper integrals and write

$$
\int_{A(t)}^{B(t)} F(X,t)dX = \lim_{\varepsilon \to 0} \left[\int_{A(t)}^{J_-(t)} F(X,t)dX + \int_{J_+(t)}^{\psi(b,t)} F(X,t)dX. \right]
\tag{7.2}
$$

If it is assumed that below and above the curve of discontinuity everything is fine, then transformation of (7.2) back to the reference configuration yields

$$
\int_{A(t)}^{B(t)} F(X,t)dX = \lim_{\varepsilon \to 0} \left[\int_a^{j(t)-\varepsilon} f(x,t)\psi_x(x,t)dx + \int_{j(t)+\varepsilon}^b f(x,t)\psi_x(x,t)dx \right].
\tag{7.3}
$$

If we apply (7.3) to the integrand $F(X,t) = \sigma(X,t)V(X,t)$ and take into account the conservation of mass (3.7) in the form $\rho(x) = \psi_x(x,t)\sigma(\psi(x,t),t)$, we obtain from (7.3) the expression

$$
\frac{d}{dt} \int_a^b \rho(x)v(x,t)dx = \frac{d}{dt} \left(\lim_{\varepsilon \to 0} \left[\int_a^{j(t)-\varepsilon} \rho(x)v(x,t)dx + \int_{j(t)+\varepsilon}^b \rho(x)v(x,t)dx \right] \right).
\tag{7.4}
$$

If it is assumed that in (7.4) the limit and time derivative can be interchanged, we can proceed to evaluate the time derivatives. From the chain rule it follows that

$$
\frac{d}{dt} \int_a^{j(t)-\varepsilon} \rho(x)v(x,t)dx = j'(t)\rho(j(t)-\varepsilon)v(j(t)-\varepsilon,t) + \int_a^{j(t)-\varepsilon} \rho(x)v_t(x,t)dx,
\tag{7.5}
$$

and

$$
\frac{d}{dt} \int_{j(t)+\varepsilon}^b \rho(x)v(x,t)dx = -j'(t)\rho(j(t)+\varepsilon)v(j(t)+\varepsilon,t) + \int_{j(t)+\varepsilon}^b \rho(x)v_t(x,t)dx.
\tag{7.6}
$$

By (4.5) (in one-dimensional form), $\rho(x)v_t(x,t) = -p_x(x,t)$ in $[a, j(t)-\varepsilon]$ and in $[j(t)+\varepsilon, b]$. This,

substituted into (7.5) and (7.6) and use of the fundamental theorem of the calculus yields

$$\frac{d}{dt} \int_a^{j(t)-\varepsilon} \rho v(x,t)dx = j'(t)\rho(j(t)-\varepsilon)v(j(t)-\varepsilon,t) - [p(j(t)-\varepsilon) - p(a,t)], \tag{7.7}$$

and

$$\frac{d}{dt} \int_{j(t)+\varepsilon}^b \rho v(x,t)dx = -j'(t)\rho(j(t)+\varepsilon)v(j(t)+\varepsilon,t) - [p(b,t) - p(j(t)+\varepsilon)]. \tag{7.8}$$

Substitution of (7.7) and (7.8) into (7.4) yields the following:

$$\frac{d}{dt} \int_a^b \rho(x)v(x,t)dx = \lim_{\varepsilon \to 0} -j'(t) \left[\rho(j(t)+\varepsilon)v(j(t)+\varepsilon,t) - \rho(j(t)-\varepsilon)v(j(t)-\varepsilon,t) \right]$$

$$+ \lim_{\varepsilon \to 0} \left[p(j(t)+\varepsilon) - p(j(t)-\varepsilon) \right] + \left[p(a,t) - p(b,t) \right]. \tag{7.9}$$

Next we define the *jump* of a function $f(x,t)$ over the curve $x = j(t)$ as follows:

$$[f](t) := \lim_{x \downarrow j(t)} f(x,t) - \lim_{x \uparrow j(t)} f(x,t).$$

If the function f is continuous, the jump is zero. With this notation and the additional assumption that the reference density $\rho(x)$ is a continuous function, it follows from (7.9) that

$$\frac{d}{dt} \int_a^b \rho(x)v(x,t)dx = -\rho(j(t))j'(t)[v](t) + [p](t) + [p(a,t) - p(b,t)], \tag{7.10}$$

see *e.g.* [4]. Comparison of (7.1), (7.4) and (7.10) leads to the unscaled *jump condition*

$$\rho(j(t))j'(t)[v](t) - [p](t) = 0. \tag{7.11}$$

Scaling of the equation (7.11) in accordance with Section 4.2, taking ρ to be constant, gives the scaled jump condition

$$[v](t)j'(t) - [p](t) = 0. \tag{7.12}$$

The scaling is explained in Section 8.3. It should be noted that the constant density ρ is absorbed into the scaling.

Remark 7.1.1. The jump condition (7.11) was derived under a number of assumptions such as the commutation of derivatives and limits, continuity of the density function and switching of integrals and derivatives. Thus the jump conditions above should be seen as other expressions of physical laws rather than consequences of the principles of conservation of mass and balance of linear momentum.

7.2   The jump condition for the piston problem

The jump condition (7.12) derived in the previous section is general and does not specifically pertain to the piston problem we began studying in Chapter 5. What we saw in Chapter 5 is that for the piston problem, characteristics emanating from the t-axis can intersect. Here we specifically refer to the situation described in Lemma 6.3.1 that formed the backdrop to our study further on. We also remind ourselves that the characteristics emanating from the x-axis also exist, are the straight lines of the form $x = y + t$ (or $t = x - y$), cover the whole first quadrant and carry with them zero-values of the function $u(x,t)$ (Theorem 5.2.1). This is depicted in Figure 7.1 below
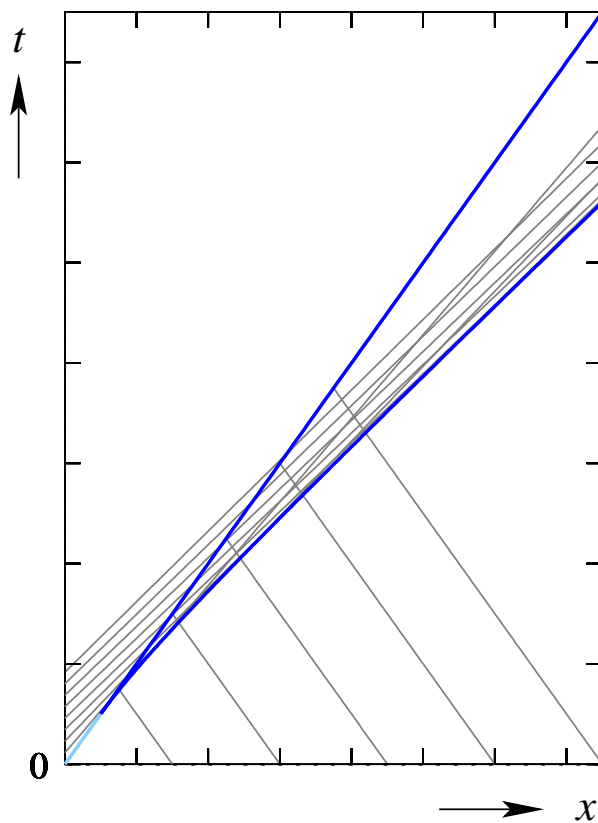

Figure 7.1: Characteristics emanating from the t and x-axes for the case $P(t) = 1/(1 + t)$. Refer also to Figure 6.3.

From Section 5.2 we have $v = u_1 + u_2 = u_1 = u$. So the jump in v is the same as the jump in u. (i.e. $[v](t) = [u](t)$). We also have that $q = u_1 - u_2 = u_1 = u$. Since, by (4.23), $p = \exp\{q\} - 1 = \exp\{u\} - 1$,

it follows that $[p](t) = [\exp(u)]$. Hence the jump condition (7.12) for the piston problem becomes

$$[u](t)j'(t) - [\exp\{u\}](t) = 0. \tag{7.13}$$

We shall use relationship (7.13) to calculate the jump curve from the initial condition $j(\frac{1}{P_0}) = \frac{1}{P_0}$ which means that the jump curve begins at the point $(1/P_0, 1/P_0)$ where the bounding curve starts.

For the piston problem the jumps in (7.13) can be further simplified. If we refer to Figure 6.3 and recall the arguments of Sections 6.2 and 6.3, it becomes clear that the jump curve can only occur in the shaded region in Figure 6.3. Figure 7.1, shows that on one side of the jump curve (the one reached by characteristics from the x-axis) $u = p = 0$. Thus the jump in u is determined by a characteristic from the t-axis. This means that $[u](t) = 0 - w(s) = \ln\{g(s)\}$ is determined by a characteristic through the point $(j(t), t)$ emanating from a point $0, s$ on the positive t-axis. Also, $[\exp\{u\}](t) = \exp\{0\} - \exp\{w(s)\} = 1 - \exp\{w(s)\} = 1 - 1/g(s)$. Thus (7.13) simplifies to

$$j'(t) = \frac{\exp\{w(s)\} - 1}{w(s)} = \frac{1 - g(s)}{-g(s)\ln\{g(s)\}}. \tag{7.14}$$

7.3 Properties of the jump curve

We are now in a position to derive some important properties of the jump curve $x = j(t)$.

Theorem 7.3.1. *Under the restrictions 1–3 of Section 6.3 the jump curve for the piston problem is defined for $t > t_0 = 1/P_0$ and is strictly increasing. Furthermore, $\lim_{t \to t_c} j(t) = x_0 = 1/P_0$ and $\lim_{t \to t_c} j'(t) = 1$.*

Proof. The jump curve is defined for $t > t_0$ because at t_0 we have the initial condition. From Lemma 6.3.1, $0 < g(s) < 1$. $\ln\{g(s)\} < 0$, as a result $-g(s)\ln\{g(s)\} > 0$ and therefore $j'(t) = \frac{1-g(s)}{-g(s)\ln\{g(s)\}} > 0$ which implies $j(t)$ is strictly increasing. At the point (x_0, t_0), $s = 0$. Hence,

$$\lim_{t \to t_0} j'(t) = \lim_{s \to 0} \frac{1 - g(s)}{-g(s)\ln\{g(s)\}}$$
$$= \lim_{x \to 1} \frac{1 - x}{-x\ln x}, \text{ because } g \to 1.$$

By L'Hospital's rule if $f(x) = 1 - x$ and $h(x) = -x\ln x$ then $\lim_{x \to 1} \frac{f'(x)}{h'(x)} = \lim_{x \to 1} \frac{-1}{-[1+\ln x]} = 1$, therefore $\lim_{x \to 1} \frac{1-x}{-x\ln x} = 1$. At the point (x_0, t_0) we have the initial condition that $j(t_0) = j(\frac{1}{P_0}) = x_0 = \frac{1}{P_0}$. $\square$

Theorem 7.3.2. *The bounding curves $x = t$, $E(x, t) = 0$ cannot be the jump curve.*

Proof. The only characteristic through a point on the bounding curve $E(x,t) = 0$ is tangent to the curve and is not intersected by any other characteristic. Therefore the envelope cannot be a jump curve.

If the line $t = x$ is the jump curve then at every point on the line two characteristics must intersect. We can use Theorem 6.4.1 which says that if $x = t$ and $x \le \frac{1}{P_0}$ then $i(x,t) = 1$ to show that the curve $x = t$ is not the jump curve.   □

Theorem 7.3.2 is important in subsequent considerations in the following sense: Through every point $(j(t), t)$ of the jump curve precisely two characteristics from the t-axis will pass. The question is, which one of the two should be used to calculate the value of u? This question can be re-phrased by observing that $u = w(s)$. There will be two s-values involved. From the theorem in question we deduce that the correct s-value must be such that $s > s_m$ where $F(s_m; x, t) = 0$ determines the point (x, t) on the bounding envelope curve at $t > t_0$.

7.4   Calculating the jump curve

For a point (x, t) on the jump curve, let $s(x, t)$ be the s-value from which $u = w(s)$ is determined according to the previous section. The ordinary differential equation (7.14) can then be re-formulated as

$$j'(t) = \frac{1 - g(s(x,t))}{-g(s(x,t))\ln\{g(s(x,t)\}} := G(s(j(t),t)).; \quad x = j(t). \tag{7.15}$$

An apparently accurate way of numerically solving (7.15) is to rewrite it as an integral equation and to approximate the integral with the trapezium rule. Thus

$$\int_t^{t+h} j'(\tau)d\tau = j(t+h) - j(t) = \int_t^{t+h} G(s(j(\tau)))d\tau \approx \frac{h}{2}[G(s(j(t),t)) + G(s(j(t+h), t+h))].$$

From here we obtain the approximate equation

$$x_2 - x_1 = \frac{h}{2}[G(s(x_1,t)) + G(s(x_2, t+h))] \tag{7.16}$$

with $x_1 = j(t)$ and $x_2 = j(t+h)$. Thus, if x_1 is known, x_2 can in principle be found by solving (7.16). If we start at $t = t_0 = 1/P_0$ where, by Theorem 7.3.1, $x_1 = j(t_0) = 1/P_0$, the value $x_2 = j(t_0 + h)$ can be calculated. There is a problem, though, because we need to find $s(x_2, t_0 + h)$. This is overcome by use of the *regula falsi* (secant) method which we describe next.

Suppose that $t > t_0$ is given, that x_1 is known and that $s_1 = s(x_1, t)$ has been calculated. Let

$G_1 = G(s_1)$ and set

$$H(x_2) = \frac{h}{2}[G_1 + G(s(x_2, t + h))] - [x_2 - x_1].$$

The problem is to find x_2 for which $H(x_2) = 0$. The algorithm is as follows:

1. Calculate the number x_m such that $E(x_m, t + h) = 0$. That is, the point $(x_m, t + h)$ lies on the envelope curve.

2. By stepping through values of $x \in (x_m, t + h)$ find x_- and x_+ for which $H_- = H(x_-) < 0$ and $H_+ = H(x_+) > 0$.

3. Apply the *regula falsi* estimate:
$$x = \frac{x_- H_+ - x_+ H_-}{H_+ - H_-}.$$

4. Evaluate $H = H(x)$. If $H < 0$, let $x_- = x$ and $H_- = H$. If $H > 0$, let $x_+ = x$ and $H_+ = H$.

5. Go back to Step 3 and repeat until $|H| < \delta$ for some pre-determined error limit $\delta > 0$. Let $x_2 = x$.

Remark 7.4.1. In performing Steps 2 and 4 above the most delicate point is to find the appropriate zero of the function $F(s; x, t) = s + xg(s) - t$ defined in (6.10). An example of this function for x and t in the appropriate range is shown in Figure 7.2 below. From the observations in Section 7.3, it is
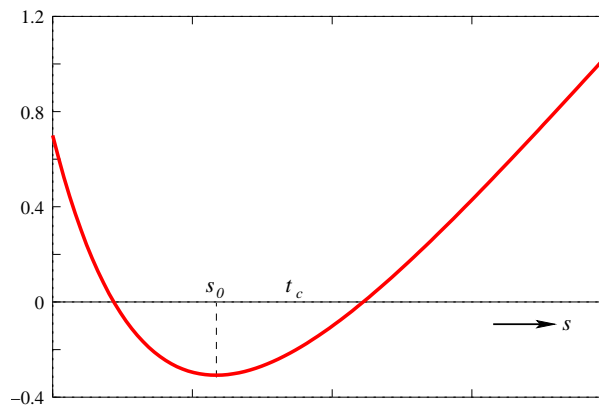
Figure 7.2: The function F

clear that we will be looking for the larger root s indicated in the sketch. Since $F_s > 0$ for $s > s_0$, the Newton-Raphson algorithm would be most efficient provided that the starting point is to the right of the minimum point s_0. To find the minimum point, in turn we note from (6.11) that it is found from $F_s(s_0; x, t) = 1 + xg'(s_0) = 0$. Evidently, s_0 is to the left of the critical point $s = t_c$ (Lemma 6.3.1) where $g' = 0$ for $g'(s_0) = -1/x < 0$. So the Newton-Raphson iterations could start to the right of $s = t_c$.

To calculate the jump curve numerically, a small time increment $h > 0$ is chosen. At times $t_0 = 1/P_0$ and $t_n = t_0 + nh$, $n = 1, 2 \ldots$ the x-coordinates x_n on the jump curve are calculated according to the algorithm described above. We begin at t_0 where $j(t_0) = 1/P_0$ and calculate x_1. With $j(t_1) = x_1$ we can proceed to calculate x_2 and so on. The jump curve for case $P(t) = 1/(1+t)$ of Example 5.3.5 calculated with $h = 0.05$ is shown in Figure 7.3 below. It lies, as expected, between the two bounding curves. C++ code for the procedure is given in Section 8.4.

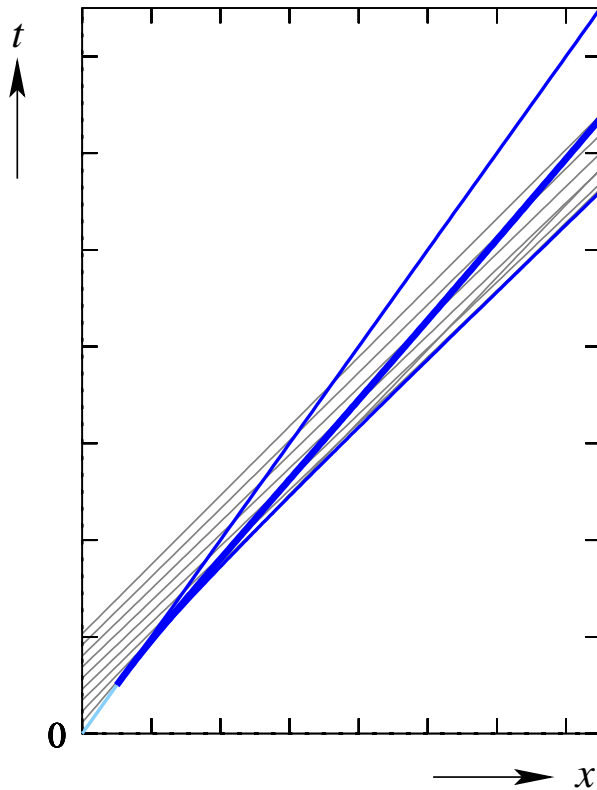Figure 7.3: The jump curve for the case $P(t) = 1/(1+t)$, the bounding curves and some characteristics. Refer also to Figures 6.3 and 7.1.

7.5   Motion of the gas

Once the jump curve $x = j(t)$ has been calculated, the gas velocity $u(x,t)$ at a given point (x,t) in the quadrant $\{x > 0,\ t > 0\}$ can be calculated.

Theorem 7.3.1 shows that the jump curve plays no role when $0 < x < x_0 = 1/P_0$. In this case we are

looking at the region S_0 (Figure 6.2). Below and on the line $t = x$ only the characteristics coming from the x-axis play a role and therefore $u = 0$ there. At points above this line the characteristics emanating from the t-axis come into play. For points (x, t) with $0 < x < x_0$ and $t > t_0 = x_0$, only a single such characteristic will pass through the point (Figure 6.3). If this characteristic emanates from the point $s(x, t)$ on the $t - axis$, then $u(x, t) = w(s(x, t))$.

From the considerations in Section 7.2 it is seen that for the case $x > x_0$ the region below the jump curve carries the value $u = 0$ (Figure 7.1). In the region above the jump curve a characteristic passing through the point (x, t) emanating from the point $s(x, t)$ on the t-axis was identified and once again $u(x, t) = w(s(x, t))$. Thus the velocity $u(x, t)$ can be calculated throughout the positive quadrant.

Once the velocity $u(x, t)$ is known, one can calculate the pressure. Indeed, from (4.26) it is seen that, since $u_2 = 0$, $q(x, t) = u(x, t)$. Hence, by (4.23), $p(x, t) = \exp\{u(x, t)\} - 1$. Thus, since the exponent is an increasing function, the pressure would follow a pattern similar to the velocity.

A graphic way of depicting the development of the velocity profile is to obtain the curves $x \to u(x, t)$ for a number of fixed values of t. For the special case of Example 5.3.5 the results are shown in Figure 7.2. Following [11] we shall refer to the developed discontinuity as *Earnshaw's bore*.
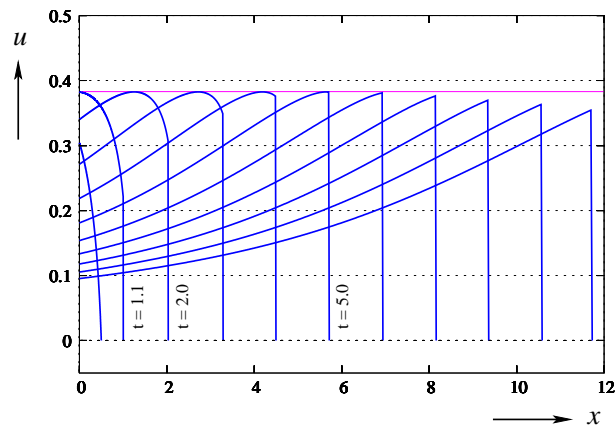
Figure 7.4: The development of Earnshaw's bore

It is seen that, for the particular example, the velocity profiles follow a pattern. The velocity (as a function of x) after some time reaches a local maximum before the discontinuity occurs. After some stage the discontinuity occurs before a maximum is reached and then the supremum seams to decay. This observation needs further clarification.

In the first place we notice that, by the contact condition (5.2) and by the notation adopted in Section 5.2, that at $x = 0$ we have

$$u(0, t) = v(0, t) = w(t). \tag{7.17}$$

Thus every curve (such as in Figure 7.2) begins at the value $w(t)$.

In the second place we note that if the characteristic through the point (x,t) emanates from the point $s(x,t)$ on the t-axis, then

$$u(x,t) = w(s(x,t)). \tag{7.18}$$

Also (6.6) states that

$$u_x(x,t) = \frac{g'(s(x,t))}{1 + xg'(s(x,t))}. \tag{7.19}$$

Thus $u_x(0,t) = g'(s(0,t)) = g'(t)$ and this implies that $u_x(0,t) < 0$ if $t < t_c$, $u_x(0,t) = 0$ if $t = t_c$ and $u_x(0,t) > 0$ when $t > t_c$. We see therefore that the velocity curves decrease initially if $t < t_c$. For $t > t_c$ they will increase.

Lastly we concentrate on the case $t > t_c$ and look for places where they will reach a local maximum. From (7.19) we see that this will be the case when $s(x,t) = t_c$ for $t = t_c$ is the unique time when $g'(t) = 0$. Suppose that x_{max} is the x-value where $u(x,t)$ reaches a local maximum. By (7.18) the maximum value of u will equal the maximum value w_{max} of w which is given by

$$u_{max} = w_{max} = -\ln(g(t_c)). \tag{7.20}$$

In the special case of Example 5.3.5, $u_{max} \approx 0.3825$ and is shown in Figure 7.4. Finally, the characteristic passing through the point (x_{max}, t) also passes through the point $s = t_c$ on the t-axis and, by (6.4),are of the form $t = t_c + x_{max}g(t_c)$. From this we can solve for x_{max} to obtain

$$x_{max} = \frac{t - t_c}{g(t_c)}. \tag{7.21}$$

From (7.21) we see that a local maximum cannot be reached if $t < t_c$. If $t > t_c$ a maximum is only achieved if $x_{max} < j(t)$ which explains why some of the curves in Figure 7.2 do not reach the maximum possible value.

It should be noted that, by Lemma 6.3.1, $g(t)$ decreases to the limit $g_\infty = 1/(1 + P_\infty)$ as $t \to \infty$. This means that $w(t)$ decreases to $w_\infty = \ln(1 + P_\infty)$ as $t \to \infty$. Thus is follows that the solution curves will decay to a constant value when $t \to \infty$. For the case of Example 5.3.5 where $P_\infty = 0$, the decay will be to zero. The decaying applied pressure will not be able to sustain any motion in the long run.

Chapter 8

Appendices

8.1 Scaling of the dynamic equations

Here we give details of the results that are given in Section 4.2 and for convenience, we recall some details/proofs.

For the purpose of scaling we assume that there is some length parameter ℓ in terms of which position will be scaled. We shall scale velocity in terms of the static sound velocity c. To find a measure of pressure, we note that (with units expressed in square brackets)

$$[\rho c^2] = \frac{kg}{m^3} \cdot \frac{m^2}{s^2} = kg.m^{-1}s^{-2} = \frac{kg.m.s^{-2}}{m^2} = \frac{\text{Newton}}{m^2} = \text{Pressure.}$$

We shall therefore scale pressure in terms of the quantity ρc^2. Time will be scaled in terms of $T = \frac{\ell}{c}$.

We now make the following substitutions

$x = \ell y$ where y is dimensionless ;

$t = T\tau$;

$v(x, t) = cV(y, \tau)$;

$p - p_0 = \rho c^2 P(y, \tau)$

$\chi = \ell \tilde{\chi}$

Now, by the chain rule,

$$v_t = \frac{\partial V}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{c}{T} V_\tau.$$

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}(p - p_0) = \rho c^2 \frac{\partial P}{\partial y_i} \frac{\partial y_i}{\partial x_i} = \frac{\rho c^2}{\ell} \frac{\partial P}{\partial y_i}$$

Thus

$$\nabla_x p = \frac{\rho c^2}{L} \nabla_y P$$

The vector quantity χ is more complicated:

$$L_x \chi = \left(\frac{\partial \chi_i}{\partial x_j} \right) = \ell \sum_{k=1}^{n} \frac{\partial \tilde{\chi}_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} = \ell \frac{1}{\ell} \frac{\partial \tilde{\chi}_i}{\partial y_j} = L_y \tilde{\chi},$$

because

$$\left(\frac{\partial y_k}{\partial x_j} \right) = \frac{1}{\ell} I,$$

with I the unit 3×3 matrix.

Substitution into the equation (4.6) now gives

$$\frac{c}{T} \frac{\partial V}{\partial \tau} + \frac{c^2}{\rho c^2 + \rho c^2 P} L_y^{-T} \tilde{\chi} (\frac{\rho c^2}{\ell}) \nabla_y P,$$

which upon re-arrangement becomes

$$V_\tau + \frac{cT}{\ell} (\frac{1}{1+P}) L_y^{-T} \tilde{\chi} \nabla_y P = 0.$$

By the definition of T, however, $cT/\ell = 1$, and we end up with the scaled equation

$$V_\tau + (\frac{1}{1+P}) L_y^{-T} \tilde{\chi} \nabla_y P = 0.$$

The equation (3.18) is scaled in precisely the same way and the outcome is

$$P_\tau + [1+P] L_y^{-T} \tilde{\chi} : V_y = 0.$$

We note at this stage that the unit ℓ of length is still free.

8.2   Scaling of the Dynamic Boundary Condition

If M is the mass of the piston, A its surface area, $w(t)$ its velocity at time t and $P(t)$ the applied pressure, then the equation of motion of the piston is

$$Mw'(t) = AP(t) - Ap(0, t)$$

With p the unscaled pressure. To this we add the contact condition

$$w(t) = v(0, t)$$

With v once again the unscaled velocity. The scaling is done as in the previous section. We let

$w(t) = cW(\tau);$

$t = T\tau$ where $T = \frac{\ell}{c};$

$p(0, t) - p_0 = \rho c^2 P(0, \tau);$

$P(t) - p_0 = \rho c^2 \Pi(\tau);$

As before we obtain

$$w'(t) = c\frac{dW}{d\tau}\frac{d\tau}{dt} = \frac{c}{T}W'(\tau).$$

Substitution into the equation of motion of the piston (5.1) yields

$$M\frac{c}{T}W'(\tau) = A\rho c^2[\Pi(\tau) - P(0, \tau)],$$

which simplifies to

$$\frac{M}{l\rho A}W'(\tau) = \Pi(\tau) - P(0, \tau).$$

If we now choose the length $\ell = \frac{M}{\rho A}$ and use the fact that $W(\tau) = V(0, \tau)$ the result is

$$\frac{d}{d\tau}[V(0, \tau)] = \Pi(\tau) - P(0, \tau).$$

8.3   Scaling of the jump condition

The scaling of the jump condition (7.11) is done in the same way as all the others if we notice that the scaling of the jump is simply

$$j(t) = \ell J(\tau),$$

that the jump in pressure is given by

$$[p](t) = \rho c^2[P(\tau),$$

and the jump in velocity should be

$$[v](t) = c[V](\tau).$$

The result, after some simplification, is

$$J'(\tau)[V](\tau) = [P](\tau).$$

8.4   C++ code: Jump Curve

The following is C++ code used to calculate points on the jump curve for the case

$$P(t) = \frac{1}{1+t}.$$

Part of the calculation yields the envelope of characteristics emanating from the t-axis. Inputs are t_{max}, the time span for the plot, and $t_{incr} = h$, the time step for the calculations. The output was used to produce Figure 7.3.

<div align="center">jump.cc</div>

```cpp
#include <complex>
#include <iostream>
#include <fstream>
#include <stdlib.h>

using namespace std;

// Identify defined functions here; e.g.

int sgn(double); /* Calculates the sign of x */
double gee(double); /* Calculates g(s) */
double gee_p(double); /* Calculates g'(s) */
double ef(double, double); /* Calculates f(s,t) = (s-t)g'(s) - g(s) */
double aitch(double, double, double); /* Calculates h(s,x,t) = s-xg(s)-t */
double envpt(double); /* Calculates envelope point for t>1 given.*/
double comp_s(double, double); /*Calculates s-value for x,t given */

main()
{
  double s, x, t, t_max, x_plus, h, h_2, h_fin, h_fin2, h_x;
  double G, X_0, gamma, x_1, x_2, F, F_1, F_2;
  double epsilon = 1e-9;

 // Edit Name.dat; e.g.
 ofstream outfile("env_jump.dat", ios::out);
 if (!outfile) {
   cerr << "Error: file could not be opened. \n";
   exit(1);}

// Dialogue (inputs)
  cout << "t_max= ";
  cin >> t_max;
  cout << "t_incr= ";
  cin >> h;
// End dialogue
```

```cpp
// Initialize
  h_2 = h/2;
  t = 1; x = 1; x_plus = 1, s = 0;
  outfile << t << "\t" << x << "\t" << x_plus << "\t" << s <<"\n";
  G = 1;
// Init done

// Main loop starts here
  while(t < t_max){
    X_0 = x + h_2*G;
    t += h;
    x_plus = envpt(t);
    h_x = (x_plus - t)/10;

    // Localize zero
    x_1 = x_plus -h_x;
    x_2 = x_1 - h_x;
    s = comp_s(t,x_1);
    gamma = 1/gee(s);
    G = (gamma-1)/log(gamma);
    F_1 = X_0 + h_2*G - x_1;
    s = comp_s(t,x_2);
    gamma = 1/gee(s);
    G = (gamma-1)/log(gamma);
    F_2 = X_0 + h_2*G - x_2;

    while(sgn(F_1) == sgn(F_2)){
        x_1 = x_2;
        x_2 = x_1 - h_x;
        F_1 = F_2;
        s = comp_s(t,x_2);
        gamma = 1/gee(s);
        G = (gamma-1)/log(gamma);
        F_2 = X_0 + h_2*G - x_2;}
    // Localization done

    // Regula-falsi
    F = F_1;
    while (abs(F) >= epsilon){
        x = (F_2*x_1 - F_1*x_2)/(F_2 - F_1);
        s = comp_s(t,x);
        gamma = 1/gee(s);
        G = (gamma-1)/log(gamma);
        F = X_0 + h_2*G - x;
        if(sgn(F) == sgn(F_1)){
          x_1 = x;
          F_1 = F;}
        else{
          x_2 = x;
          F_2 = F;}
    }

    outfile << t << "\t" << x << "\t" << x_plus << "\t" << s << "\n";}
// Main loop ends

  outfile.close();

 return 0;
}

/* Defined functions */
```

```
/* Sign function */
int sgn(double x)
{
   int sign;
   sign =1;
   if (x==0) sign = 0;
   else if (x<0) sign = -1;
return sign;
}

/* The function g(s)*/
double gee(double s)
{
  return (s+exp(-s))/(1+s);
}

/* The function g'(s)*/
double gee_p(double s)
{
  double sigma;
  sigma = 1+s;
  return (1-(1+sigma)*exp(-s))/(sigma*sigma);
}

/*The function f(s,t) = (s-t)g'(s) - g(s) */

double ef(double s, double t)
{
  return (s-t)*gee_p(s) - gee(s);
}

/* The function h(s,x,t) = s + xg(s) -t   */

double aitch(double s, double x, double t)
{
  return s + x*gee(s) -t;
}

/* To calculate envelope point (t given) */

double envpt(double t)
{
  double h, s_1, s_2, s, f_1, f_2, f, epsilon;
//   Prepare for Regula-falsi

// Init
  epsilon = 1e-9;
  h = 0.01;
  s_1 = 0;
  s_2 = h;
  f_1 = ef(s_1,t);
  f_2 = ef(s_2,t);

// Localize zero
  while (sgn(f_1) == sgn(f_2)){
    s_1 = s_2;
    f_1 = f_2;
    s_2 += h;
    f_2 = ef(s_2,t);}
// Localization done
```

```cpp
// Regula-falsi
  f = f_1;
  while (abs(f) >= epsilon){
    s = (f_2*s_1 - f_1*s_2)/(f_2 - f_1);
    f= ef(s,t);
    if(sgn(f) == sgn(f_1)){
        s_1 = s;
        f_1 = f;}
         else{
            s_2 = s;
            f_2 = f;}
         }
   return -1/gee_p(s);
}


/* To find s, given x and t */

double comp_s(double t, double x)
{
   double h, s_1, s_2, h_1, h_2, s, ah;
   double epsilon = 1e-9, s_c = 1.14619, g_c = 0.682156, g_pm = 0.0460764;
   double c = g_c - s_c*g_pm;

// Init
   s_1 = t - c*x;
   h = s_1/100;
   s_2 = s_1 - h;
   h_1 = aitch(s_1,x,t);
   h_2 = aitch(s_2,x,t);

// Localize zero
   while (sgn(h_1) == sgn(h_2)){
     s_1 = s_2;
     h_1 = h_2;
     s_2 -= h;
     h_2 = aitch(s_2,x,t);}
// Localization done

// Regula-falsi
   ah = h_1;
   while (abs(ah) >= epsilon){
     s = (h_2*s_1 - h_1*s_2)/(h_2 - h_1);
     ah= aitch(s,x,t);
     if(sgn(ah) == sgn(h_1)){
         s_1 = s;
         h_1 = ah;}
          else{
             s_2 = s;
             h_2 = ah;}
           }
   return s;
}
```

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