Some Applications of Sturm's Comparison Theorem

BY

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DECLARATION

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Abstract

We consider several classes of orthogonal polynomials as well as the Bessel function where we study the convexity of zeros of these polynomials, satisfying either differential or real difference equations, by applying Sturm's comparison and convexity theorems as well as analogues of these theorems. In addition several results are obtained concerning the distances between consecutive zeros of some of these classes of polynomials. Further research possibilities concerning q-polynomials and polynomials satisfying complex difference equations are discussed.

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Chapter 1

Introduction

Since the first publication of the Sturm comparison and convexity theorems by Sturm in 1836 [27], many results following either directly or indirectly from these theorems have been obtained. All of these results have numerous applications, including finding certain properties of the zeros of second-order differential equations, as well as difference equations, some of which will be discussed in detail in this dissertation.

First we consider Sturm's comparison theorem, which provides information about the behaviour of solutions of two second-order differential equations in normal form

$$y'' + F(x)y = 0. \tag{1.0.1}$$

Theorem 1.1 (*Sturm's comparison theorem, cf. [27] and [29]*)
On the interval (a, b), *let* y_1 *and* y_2 *be non-trivial solutions of the differential equations*

$$y'' + g_1(x)y = 0$$

$$y'' + g_2(x)y = 0$$

respectively, where g_1 *and* g_2 *are continuous real-valued functions on* (a, b), *such that* $g_1 \leq g_2$.
Then between any two consecutive zeros of y_1 *in* (a, b), *there is at least one zero of* y_2.

Proof: Let x_k and x_{k+1} be the consecutive zeros of y_1 in (a, b). Since $y_1(x)$ is non-trivial we have that $y_1'(x_k)y_1'(x_{k+1}) \neq 0$. Without loss of generality we can assume that $y_1(x) > 0$ on (x_k, x_{k+1}) from which it follows that $y_1'(x_k) \geq 0$ and $y_1'(x_{k+1}) \leq 0$. Assume now that y_2 has no zero in (x_k, x_{k+1}). If $y_2(x) > 0$ on (x_k, x_{k+1}), then we consider the equations

$$y_1'' + g_1(x)y_1 = 0 \quad , \quad y_2'' + g_2(x)y_2 = 0$$

and multiply the first equation by y_2 and the second one by y_1. Subtracting these results give

$$y_1''(x)y_2(x) - y_2''(x)y_1(x) = [g_2(x) - g_1(x)]y_1(x)y_2(x).$$

Integrating this on $[x_k, x_{k+1}]$ we obtain

$$\begin{aligned}
& \int_{x_k}^{x_{k+1}} [g_2(x) - g_1(x)]y_1(x)y_2(x)dx \\
= & \int_{x_k}^{x_{k+1}} [y_1''(x)y_2(x) - y_2''(x)y_1(x)]dx \\
= & [y_1'(x)y_2(x)]_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} y_1'(x)y_2'(x)dx - [y_2'(x)y_1(x)]_{x_k}^{x_{k+1}} \\
& + \int_{x_k}^{x_{k+1}} y_2'(x)y_1'(x)dx \\
= & y_1'(x_{k+1})y_2(x_{k+1}) - y_1'(x_k)y_2(x_k).
\end{aligned}$$

The left-hand side of this equation is positive, since by assumption

$$g_1(x) \leq g_2(x), \quad y_1(x) > 0 \quad y_2(x) > 0$$

on (x_k, x_{k+1}). The right-hand side is non-positive, since

$$y_1'(x_k) \geq 0, \quad y_1'(x_{k+1}) \leq 0, \quad y_2(x_k) \geq 0, \quad y_2(x_{k+1}) \geq 0.$$

This follows from the assumption that $y_2(x) > 0$ on (x_k, x_{k+1}) and $y_2(x)$ is continuous, since it is differentiable, on (a, b). This leads to a contradiction. If $y_2(x) < 0$, we arrive at the same contradiction by an analogue argument. Hence, the function y_2 should have at least one zero in (x_k, x_{k+1}).

■

A result that immediately follows from the comparison theorem, which provides bounds on the distances between consecutive zeros, is summarised in the following theorem.

Theorem 1.2 (*cf. [2, Theorem 1]*)
Let $y''(x) + F(x)y(x) = 0$ be a second-order differential equation in normal

form, where F is continuous in (a,b). Let $y(x)$ be a non-trivial solution on (a,b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(x)$ in (a,b).

1. *If there exists an $M > 0$ such that $F(x) < M$ in (a,b) then*

$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{M}}.$$

2. *If there exists an $m > 0$ such that $F(x) > m$ in (a,b) then*

$$\Delta x_k \equiv x_{k+1} - x_k < \frac{\pi}{\sqrt{m}}.$$

Proof: Let $x_k < x_{k+1}$ be consecutive zeros of $y(x)$ which is a non-trivial twice differentiable solution of $y''(x) + F(x)y(x) = 0$ in (a,b), with $F(x)$ being continuous in (a,b). Without loss of generality we can suppose that $y(x)$ is positive in (x_k, x_{k+1}). Then $y'(x_k) > 0$ and $y'(x_{k+1}) < 0$ and therefore the function

$$h(x) = -\frac{y'(x)}{y(x)}$$

satisfies $\lim_{x \to x_k^+} h(x) = -\infty$ and $\lim_{x \to x_{k+1}^-} h(x) = +\infty$. Furthermore $h(x)$ is differentiable in (x_k, x_{k+1}) and

$$\begin{aligned} h'(x) &= \frac{-y''(x)y(x) + [y'(x)]^2}{y^2(x)} \\ &= \frac{-y''(x)}{y(x)} + \left[\frac{y'(x)}{y(x)}\right]^2 \\ &= F(x) + h^2(x). \end{aligned}$$

Assuming that $F(x) > m > 0$ in (a,b), it follows that $h' > m + h^2$ in (x_k, x_{k+1}) and then $g(x) \equiv h'(x)/(m + h^2(x)) - 1 > 0$. Therefore

$$\lim_{\varepsilon \to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} g(x)dx > 0$$

so that

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} \left[\frac{h'(x)}{(m + h^2(x))} - 1\right] dx \\ = &\lim_{\varepsilon \to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} \left[\frac{h'(x)}{(\sqrt{m})^2 + h^2(x)} - 1\right] dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon \to 0^+} \left[\frac{1}{\sqrt{m}} tan^{-1} \left(\frac{h(x)}{\sqrt{m}} \right) - x \right]_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} \\
&= \frac{1}{\sqrt{m}} \left[\frac{\pi}{2} - \frac{(-\pi)}{2} \right] - (x_{k+1} - x_k) \\
&= \frac{\pi}{\sqrt{m}} - (x_{k+1} - x_k) > 0.
\end{aligned}
$$

This proves the second part of the theorem. The proof of the first part follows in a similar way be assuming that $F(x) < M$ in (a, b) which implies that $h' < M + h^2$ in (x_k, x_{k+1}) and so $g(x) \equiv h'(x)/(M + h^2(x)) - 1 < 0$. Then

$$\lim_{\varepsilon \to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} g(x)dx < 0$$

which implies that

$$
\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} \left[\frac{h'(x)}{(M + h^2(x))} - 1 \right] dx \\
&= \frac{\pi}{\sqrt{M}} - (x_{k+1} - x_k) < 0.
\end{aligned}
$$

This proves the first part.

■

Another consequence of the comparison theorem, is Sturm's convexity theorem.

Theorem 1.3 (*Sturm's convexity theorem, cf. [2]*)
Let $y''(x) + F(x)y(x) = 0$ be a second-order differential equation in normal form, where F is continuous in (a, b). Let $y(x)$ be a non-trivial solution in (a, b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(x)$ in (a, b). Then

1. *if $F(x)$ is strictly increasing in (a, b), then $x_{k+2} - x_{k+1} < x_{k+1} - x_k$,*
2. *if $F(x)$ is strictly decreasing in (a, b), then $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.*

The following is given in [12],

Proof: (cf. [12])

Let $F(x)$ be strictly decreasing, then we need to show that $x_{k+2} - x_{k+1} > x_{k+1} - x_k$. This follows from the general result that if we rotate the graph of

y through 180° between the zeros x_k and x_{k+1} then the resulting arch will be contained in the arch between the zeros x_{k+1} and x_{k+2}. To prove this we need to show that

$$y(x) > -y(2x_{k+1} - x)$$

where

$$x_{k+1} < x < x_{k+1} + d$$

and $d = x_{k+1} - x_k$. Without loss of generality we may assume that $y(x) < 0$ for $x_k < x < x_{k+1}$. Now we can use the fact that

$$y_2(x) = -y(2x_{k+1} - x)$$

satisfies the differential equation

$$y_2''(x) + F(2x_{k+1} - x)y_2(x) = 0.$$

The result then follows from Sturm's comparison theorem, i.e. $y(x) > -y(2x_{k+1} - x)$.

■

A simpler proof for the case when $F(x) > 0$ is given in [2] and we include it for the convenience of the reader.

An alternative proof for Theorem 1.3 which holds for $F(x) > 0$ is as follows:

Proof: First, we prove part 1. If $F(x)$ is strictly increasing and we let $x_k < x_{k+1} < x_{k+2}$ be consecutive zeros of y, then in the interval (x_k, x_{k+1}) we have that $F(x) < F(x_{k+1})$ and in the interval (x_{k+1}, x_{k+2}), we have $F(x) > F(x_{k+1})$.
Then by Theorem 1.2,

$$x_{k+1} - x_k > \frac{\pi}{\sqrt{F(x_{k+1})}}$$

and

$$x_{k+2} - x_{k+1} < \frac{\pi}{\sqrt{F(x_{k+1})}}$$

which implies that

$$\begin{aligned}
\Delta^2 x_k &= \Delta(\Delta x_k) \\
&= \Delta x_{k+1} - \Delta x_k \\
&= x_{k+2} - 2x_{k+1} + x_k < 0.
\end{aligned}$$

Hence,

$$x_{k+2} - x_{k+1} < x_{k+1} - x_k.$$

To prove part 2, we let $F(x)$ be strictly decreasing. Then in the interval (x_k, x_{k+1}) we have that $F(x) > F(x_{k+1})$, and in the interval (x_{k+1}, x_{k+2}), we have $F(x) < F(x_{k+1})$. By Theorem 1.2

$$x_{k+1} - x_k < \frac{\pi}{\sqrt{F(x_{k+1})}}$$

and

$$x_{k+2} - x_{k+1} > \frac{\pi}{\sqrt{F(x_{k+1})}}$$

which implies that

$$\Delta^2 x_k = x_{k+2} - 2x_{k+1} + x_k > 0.$$

Hence,

$$x_{k+2} - x_{k+1} > x_{k+1} - x_k.$$

■

In summary we see that by comparing $y'' + F(x)y = 0$ with $y'' + m^2 y = 0$ and $y'' + M^2 y = 0$, Sturm's convexity theorem shows that if f is continuous on (a, b) and $m^2 < F(x) < M^2$, where m and M are positive numbers, then the zeros $x_1, x_2, \ldots$ of y on (a, b) satisfy

$$\frac{\pi}{M} < x_{i+1} - x_i < \frac{\pi}{m}$$

where $i = 1, 2, \ldots$ (cf. [12]). In other words, it provides information about the spacing of the zeros.

Furthermore, it was also mentioned in [12], that Sturm's convexity theorem provides information about the shape of the successive arches of the graph of a nontrivial solution of a second-order differential equation.

Another result, known as Sonin's theorem, also arises from Sturm's theorem and provides information on the monotonicity of the relative maxima of $|y(x)|$.

Theorem 1.4 (*Sonin's theorem, cf. [28]*)
Let $y = y(x)$ satisfy the differential equation

$$y'' + F(x)y = 0,$$

where $F(x)$ is a positive function having a continuous derivative of a constant sign in $x_0 < x < X_0$. Then the successive relative maxima of $|y|$, as x increases from x_0 to X_0, form an increasing (decreasing) sequence when $F(x)$ decreases (increases).

Proof:
Let

$$f(x) = [y(x)]^2 + [F(x)]^{-1}[y'(x)]^2 = [y(x)]^2 + r(x)[y'(x)]^2$$

then we have that

$$f(x) = [y(x)]^2$$

when $y'(x) = 0$. Also

$$\begin{aligned} f'(x) &= 2y'(x)[y(x) + r(x)y''(x) + \frac{1}{2}r'(x)y'(x)] \\ &= 2y'(x)[y(x) + F^{-1}(x)y''(x) + \frac{1}{2}r'(x)y'(x)] \\ &= r'(x)(y'(x))^2 \end{aligned}$$

since $y'' + F(x)y = 0$ implies that $y(x) + F^{-1}(x)y''(x) = 0$.
Also,

$$sgn[f'(x)] = -sgn[F'(x)]$$

since

$$r(x) = \frac{1}{F(x)}$$

which implies that

$$r'(x) = \frac{-F'(x)}{F^2(x)}.$$

This implies that when $F(x)$ is increasing (decreasing) then $f(x) = [y(x)]^2$ will be decreasing (increasing). Hence the statement follows.

■

Summarising, this means that if $F(x)$ is positive, continuous and increasing on an interval I, then the relative maxima of $|y|$ form a decreasing sequence. On the other hand, if $F(x)$ is decreasing, then the relative maxima of $|y|$ form an increasing sequence.

Sonin's theorem has been generalized by Redheffer (cf. [23]) for second-order differential equations of the form

$$P(x)y'' + Q(x)y' + y = 0.$$

Theorem 1.5 (*cf. [23, Theorem 5]*)
Let y be a nontrivial solution of

$$Py'' + Qy' + y = 0$$

on an interval (a, b) on which P is differentiable. Then the successive relative maxima of $|y|$ on (a, b) form an increasing, constant, or decreasing sequence when $P' > 2Q$, $P' = 2Q$, or $P' < 2Q$, respectively.

Proof: If $f = y^2 + P(y')^2$ the differential equation gives

$$\begin{aligned} f' &= 2yy' + P'(y')^2 + 2Py'y'' \\ &= 2yy' + P'(y')^2 + 2y'(-Qy' - y) \\ &= P'(y')^2 - 2(y')^2 Q \\ &= (y')^2(P' - 2Q). \end{aligned}$$

Hence f is increasing when $P' > 2Q$, decreasing when $P' < 2Q$ and constant when $P' = 2Q$. At a maximum x_k we have $y'(x_k) = 0$, therefore $f(x_k) = y(x_k)^2$, and the result follows.

■

Another related result on the solutions of a second-order differential equation in normal form is Sturm's separation theorem. The proof of this theorem follows from Theorem 1.1 (Sturm's comparison theorem) and in the literature the two theorems are often proved together. We will make use of the proof by Sista in [26]. An alternative proof can be found, for example, in [19].

Theorem 1.6 (*Sturm's separation theorem, cf. [26]*)
Suppose that y_1 and y_2 are a linearly independent pair of solutions of

$$y'' + F(x)y = 0.$$

If x_1 and x_2 are two consecutive zeros of y_1, then y_2 has exactly one zero in (x_1, x_2).

Proof: Assume that y_2 does not have a zero in (x_1, x_2). Then y_2 does not have a zero in $[x_1, x_2]$. This is due to the fact that y_1 and y_2 are linearly independent. Therefore we can define a function ψ on $[x_1, x_2]$ by

$$\psi = \frac{y_1(x)}{y_2(x)}.$$

which has the following properties:

1. ψ is continuous on $[x_1, x_2]$.
2. ψ' exists on (x_1, x_2).
3. $\psi(x_1) = \psi(x_2) = 0$.

By Rolle's theorem, there exists a c such that $x_1 < c < x_2$ and $\psi'(c) = 0$. This means that

$$y_1'(c)y_2(c) - y_1(c)y_2'(c) = 0,$$

i.e. the Wronskian is zero, but this is not possible as y_1 and y_2 are linearly independent. Therefore, y_2 has at least one zero on (x_1, x_2).

We claim that y_2 does not have more than one zero in (x_1, x_2). If this was not true, then there would exist at least two zeros of y_2 in (x_1, x_2). Let x_3 and x_4 be two consecutive zeros of y_2 in (x_1, x_2) such that $x_1 < x_3 < x_4 < x_2$. Then, by previous arguments, y_1 will have a zero in (x_3, x_4). This is a contradiction to the fact that x_1 and x_2 are two consecutive zeros of y_1. Hence, there is exactly one zero of y_2 in (x_1, x_2).

■

Thus, if a linear second-order differential equation has two linearly independent solutions y_1 and y_2, then the zeros of y_1 are separated by the zeros of y_2.

All of these theorems not only provide information about the spacing of the zeros, but also enable us to place a bound on the distance between consecutive zeros under certain conditions, and in some cases we know how the sequences of relative maxima of the absolute solutions will behave. In the next chapter we will investigate the possibilities of using these theorems to obtain information about the convexity of the zeros of solutions to second-order linear differential equations where the differential equation is not necessarily in normal form.

Chapter 2

Convexity of zeros

2.1 Introduction

In this chapter we will discuss and prove results that will provide us with information on the convexity of zeros of solutions of second-order differential equations. Convexity in relation to the zeros of an equation means that if the zeros are convex (concave), then the distance between successive zeros must increase (decrease).

The convexity theorem of Sturm (Theorem 1.3) can be summarised, as was done in [9], as follows.

Theorem 2.1 (*[9, Theorem 2.1]*)
Let $y''(x) + F(x)y(x) = 0$ be a second-order differential equation in the normal form, where F is continuous in (a,b). Let $y(x)$ be a nontrivial solution in (a,b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of y(x) in (a, b). Then

1. *if $F(x)$ is strictly increasing in (a,b), $x_{k+2} - x_{k+1} < x_{k+1} - x_k$,*
2. *if $F(x)$ is strictly decreasing in (a,b), $x_{k+2} - x_{k+1} > x_{k+1} - x_k$,*
3. *if there exists $M > 0$ such that $F(x) < M$ in (a,b) then*
$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{M}}$$
4. *if there exists $m > 0$ such that $F(x) > m$ in (a,b) then*
$$\Delta x_k < \frac{\pi}{\sqrt{m}}.$$

The zeros of y are concave on (a, b) for the first case and convex on (a, b) for the second case.

The convexity theorem is only applicable to functions which are solutions of second-order differential equations in normal form. To satisfy this condition, the differential equation can be transformed into normal form. This can be done by changing the dependent variable in the following way. Let

$$z'' + g(x)z' + f(x)z = 0$$

be a second-order differential equation and let

$$y = z \exp\Big(\frac{1}{2}\int^x g(s)ds\Big). \tag{2.1.1}$$

For this integral to exist, we assume that g is a continuous function to ensure that g is an integrable function. The corresponding equation for y in the normal form (1.0.1) is

$$y'' + F(x)y = 0$$

where

$$F(x) = f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x). \tag{2.1.2}$$

This follows from

$$\begin{aligned}
y' &= z'e^{\frac{1}{2}\int^x g(s)ds} + \frac{1}{2}zg(x)e^{\frac{1}{2}\int^x g(s)ds} \\
y'' &= z''e^{\frac{1}{2}\int^x g(s)ds} + g(x)z'e^{\frac{1}{2}\int^x g(s)ds} + \frac{1}{4}zg^2(x)e^{\frac{1}{2}\int^x g(s)ds} \\
&\quad + \frac{1}{2}zg'(x)e^{\frac{1}{2}\int^x g(s)ds} \\
&= -f(x)ze^{\frac{1}{2}\int^x g(s)ds} + \frac{1}{4}zg^2(x)e^{\frac{1}{2}\int^x g(s)ds} + \frac{1}{2}zg'(x)e^{\frac{1}{2}\int^x g(s)ds} \\
&= y(x)\Big[-f(x) + \frac{1}{4}g^2(x) + \frac{1}{2}g'(x)\Big] \\
&= -F(x)y(x).
\end{aligned}$$

Under this transformation the independent variable does not change and hence the zeros of y and z remain the same. This transformation (2.1.1) can be used to prove properties concerning the convexity of the zeros of some classes of hypergeometric orthogonal polynomials.

2.2 Hypergeometric polynomials

Firstly we define what is meant by hypergeometric polynomials.

Definition 2.2
The general ${}_pF_q$ hypergeometric function is defined by

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}; z\right) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k z^k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!}, \qquad |z| < 1, $$
$$ a_i, b_i \in \mathbb{R}, b_i \neq 0, -1, -2, \ldots \text{ for } i = 1, 2, \ldots, q $$

where

$$ \begin{aligned} (\alpha)_k &= \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k \geq 1, \quad k \in \mathbb{N} \\ (\alpha)_0 &= 1 \quad \text{when } \alpha \neq 0, \end{aligned} $$

a product known as Pochhammer's symbol or the shifted factorial function.

This series will terminate if one of the numerator parameters is equal to a negative integer, say $a_1 = -n, n \in \mathbb{N}$, and then the series is a polynomial of degree n.

2.3 Classical orthogonal polynomials

We define the concept of orthogonal polynomials.

Definition 2.3 (cf. [22])
If $\{P_n(x)\}_{n=0}^{\infty}$ form a simple set of real polynomials (polynomials that have only real coefficients) and $w(x) > 0$ on $a < x < b$, then the set $P_n(x)$ are orthogonal with respect to the weight function $w(x)$ over the interval $a < x < b$ if

$$ \int_a^b w(x) P_n(x) P_m(x) dx = 0, \quad m \neq n. $$

Examples of classical orthogonal polynomials include

Bessel

Bessel polynomials $y_n(a;x)$ can be defined by their hypergeometric representation (cf. [11, p.244, eqn.(9.13.1)])

$$y_n(a;x) = {}_2F_0\left(\begin{array}{c} -n, n+a+1 \\ - \end{array}; \frac{-x}{2}\right), \quad n = 0, 1, \ldots, N. \tag{2.3.3}$$

and are orthogonal on $(0, \infty)$ for $a < -2N - 1$ with respect to the weight function

$$w(x) = x^a e^{\frac{-2}{x}}.$$

Hermite

$H_\lambda(x)$ can be defined by their hypergeometric representation (cf. [11, p.250, eqn.(9.15.1)])

$$H_\lambda(x) = (2x)^\lambda \; {}_2F_0\left(\begin{array}{c} -\lambda/2, (-\lambda+1)/2 \\ - \end{array}; -\frac{1}{x^2}\right). \tag{2.3.4}$$

When $\lambda = n$, the Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function $w(x) = e^{-x^2}$.

Laguerre

The Laguerre polynomials (cf. [11, p.241, eqn.(9.12.1)])

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \; {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array}; x\right) \tag{2.3.5}$$

which are orthogonal on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}x^\alpha$ for $\alpha > -1$.

Jacobi

Jacobi polynomials (cf. [11, p.216, eqn.(9.8.1)])

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \; {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}; \frac{1-x}{2}\right) \tag{2.3.6}$$

are orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ when α, $\beta > -1$.

Ultraspherical

Ultraspherical polynomials may be defined by their hypergeometric representation (cf. [11, p.222, eqn.(9.8.19)]), where we let $\alpha = \beta = \lambda - \frac{1}{2}$ in the definition of the Jacobi polynomials.

$$C_n^{(\lambda)} = P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\begin{matrix} -n, n+2\alpha+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right) \tag{2.3.7}$$
$$= \frac{(2\lambda)_n}{n!}\ {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda+\frac{1}{2} \end{matrix}; \frac{1-x}{2}\right)$$

or

Hahn

Hahn polynomials are defined, as in (cf. [11, p.204, eqn.(9.5.1)]), by

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right) \tag{2.3.8}$$

with $n = 0, 1, 2, \ldots, N$.

They are orthogonal for $\alpha > -1$ and $\beta > -1$ or $\alpha < -N$ and $\beta < -N$ with respect to a discrete weight function, $w_1(x) = \begin{pmatrix} \alpha+x \\ x \end{pmatrix}\begin{pmatrix} \beta+N-x \\ N-x \end{pmatrix}$ with masses at $x = 0, 1, \ldots, N$. They have n simple zeros on $(0, N)$ with any two consecutive zeros being more than one unit apart (cf. [14]).

Meixner

Meixner polynomials $M_n(x;b,c)$ are orthogonal for $b > 0$ and $0 < c < 1$, with respect to the weight function $w_2(x) = \frac{(b)_x}{x!}c^x$ with $x = 0, 1, \ldots,$ and can be defined by (cf. [11, p.234, eqn.(9.10.1)])

$$M_n(x;b,c) = {}_2F_1\left(\begin{matrix} -n, -x \\ b \end{matrix}; 1-\frac{1}{c}\right). \tag{2.3.9}$$

Chebychev

Chebychev polynomials of the first kind, $T_n(x)$, are a special case of the Jacobi polynomials and can be found by setting $\alpha = \beta = -\frac{1}{2}$. Similarly, if we let $\alpha = \beta = \frac{1}{2}$ we obtain the Chebychev polynomials of the second kind, $U_n(x)$.

Chebychev polynomials of the first kind may be defined by their hypergeometric representation (cf. [11, p.225, eqn.(9.8.35)])

$$T_n(x) = {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix}; \frac{1-x}{2}\right) \tag{2.3.10}$$

and are orthogonal on $(-1,1)$ with respect to the weight function $w(x) = (1-x^2)^{-\frac{1}{2}}$.
Chebychev polynomials of the second kind can be defined by their hypergeometric representation (cf. [11, p.225, eqn.(9.8.36)])

$$U_n(x) = (n+1)\ {}_2F_1\left(\begin{array}{c} -n, n+2 \\ \frac{3}{2} \end{array}; \frac{1-x}{2}\right) \tag{2.3.11}$$

and are orthogonal on $(-1,1)$ with respect to the weight function $w(x) = (1-x^2)^{\frac{1}{2}}$.

Pseudo Jacobi

Pseudo Jacobi polynomials can be defined by (cf. [11, p.231, eqn.(9.9.1)])

$$P_n(x; v, N) = (x+i)^n\ {}_2F_1\left(\begin{array}{c} -n, N+1-n-iv \\ 2N+2-2n \end{array}; \frac{2}{1-ix}\right), \tag{2.3.12}$$

are orthogonal on $\mathbb{R}$ with respect to the weight function

$$w(x) = (1+x^2)^{-N-1} e^{2vtan^{-1}x}.$$

Dual Hahn

Dual Hahn polynomials can be defined by (cf. [11, p.208, eqn.(9.6.1)])

$$R_n(\lambda(x), \gamma, \delta, N) = \ {}_3F_2\left(\begin{array}{c} -n, -x, x+\gamma+\delta+1 \\ \gamma+1, -N \end{array}; 1\right) \tag{2.3.13}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1).$$

They are orthogonal for $\gamma > -1$ and $\delta > -1$ or $\gamma < -N$ and $\delta < -N$ with respect to the discrete weight function

$$w(x) = \frac{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_x N!}{(-1)^x (x+\gamma+\delta+1)_{N+1}(\delta+1)_x x!}$$

with masses at $x = 0, 1, \ldots, N$.

Charlier

Charlier polynomials can be defined by their hypergeometric representation (cf. [11, p.247, eqn.(9.14.1)])

$$C_n(x; a) = \ {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array}; -\frac{1}{a}\right) \tag{2.3.14}$$

and are orthogonal with respect to the weight function $w(x) = \frac{a^x}{x!}$ provided that $a > 0$.

Krawtchouk

Krawtchouk polynomials are defined by their hypergeometric representation (cf. [11, p.237, eqn.(9.11.1)])

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p}\right) \tag{2.3.15}$$

and are orthogonal for $0 < p < 1$ with respect to the discrete weight function $w(x) = \begin{pmatrix} N \\ x \end{pmatrix} p^x(1-p)^{N-x}$.

Further classes of orthogonal polynomials include the Wilson, Racah, Continuous Dual Hahn, Continuous Hahn and Meixner-Pollaczek polynomials whose definitions are not needed for the purpose of this dissertation.

All these classes of polynomials can be categorised according to whether they are continuous (satisfying a differential equation) or discrete (satisfying a difference equation).

2.3.1 Continuous classical orthogonal polynomials

Continuous orthogonal polynomials are polynomials which satisfy a second-order differential equation. This class consists of Hermite, Laguerre, Bessel, Jacobi and Pseudo Jacobi polynomials as well as the ultraspherical and Chebychev polynomials which are special cases of the Jacobi polynomials.

2.3.2 Discrete classical orthogonal polynomials

This class of polynomials can be divided into two groups where the polynomials either satisfy a real or a complex second-order difference equation. The class of discrete orthogonal polynomials satisfying a real second-order difference equation consists of Charlier, Meixner, Krawtchouk, Hahn and Dual Hahn polynomials and the class of discrete orthogonal polynomials satisfying a complex second-order difference equation consists of Wilson, Racah, Meixner-Pollaczek, Continuous Dual Hahn and Continuous Hahn polynomials.

In this dissertation we will be investigating the convexity of the zeros of all of these classes of polynomials.

Although we will be applying the theorems to orthogonal polynomials, we note that Sturm's theorems do not require the functions satisfying the differential (or difference) equations to be orthogonal or even polynomials. The only requirement is that the functions must be oscillating to ensure that the function will have zeros. This is discussed by J. Segura in [25] and [24] for Laguerre polynomials and Jacobi polynomials respectively.

2.4 Applications

In the literature Theorem 2.1 has been used in various cases. Hille [8] used the transformation (2.1.1) to prove the convexity of the zeros of Hermite polynomials while Jordaan and Toókos [9] used the same transformation to prove convexity properties of the zeros of Laguerre, Jacobi and ultraspherical polynomials. Further detail will be given in Chapter 3.

Szegő [28] considered a different change of variable to obtain information on the convexity of the transformed zeros of the ultraspherical polynomials and this change of variable was applied to hypergeometric functions by Deaño, Gil and Segura in [2].

In [20] Muldoon provides an overview of how the convexity theorem can be used to obtain information about convexity properties of the zeros of gamma, q-gamma and Bessel functions with respect to either a parameter or the order of the zeros and convexity of the Hermite functions is also briefly discussed. In Chapter 3 we will provide a more detailed exposition of these results for the Bessel and Hermite functions, supported by proofs.

2.5 Discrete versions of Sturm's theorems

So far we have only discussed the applications of Sturm's theorem for the continuous case of differential equations, but the comparison and convexity theorems have analogues for difference equations, i.e. for the discrete case. In [4] Gishe and Toókos discuss and apply the convexity and comparison theorems for difference equations to the Hahn and Meixner polynomials.

Although the classical comparison and convexity theorems can be directly applied to difference equations of the type $\Delta^2 y(x-1) + F(x)y(x) = 0$, this class of equations is too narrow. We know that any second-order linear differential equation can be transformed into normal form, while for difference equations this is too complicated and after such a transformation it is not possible to determine the monotonicity properties of the function $F(t)$ and therefore we need a more general approach. This will be discussed in more detail in Chapter 4.

In [4] second-order self-adjoint equations of the form

$$\Delta[p(x-1)\Delta y(x-1)] + q(x)y(x) = 0 \tag{2.5.16}$$

with $p(x) > 0$ are considered.

Lemma 2.4 (cf. [10, Lemma 6.1])
Let $y(t)$ be a nontrivial solution of (2.5.16) with $y(t_0) = 0$. Then $y(t_0 - 1)y(t_0 + 1) < 0$.

We note that the discrete versions of Sturm's separation and comparison theorems hold for self-adjoint equations. Before stating these theorems, we first need to define the concepts of a generalized zero and disconjugacy.

Definition 2.5 (cf. [10])
A solution $y(x)$ of (2.5.16) has a generalised zero at x_0 if either $y(x_0) = 0$ or $y(x_0 - 1)y(x_0) < 0$.

Definition 2.6 (cf. [7])
Equation (2.5.16) is called disconjugate on $[a, b]$ if no nontrivial solution has two or more generalised zeros on $[a, b]$. Otherwise it is called conjugate.

Discrete versions of Sturm's separation and comparison theorems are given below.

Theorem 2.7 (cf. [10])
Two linearly independent solutions of (2.5.16) cannot have a common zero. If a nontrivial solution of (2.5.16) has a zero at x_1 and a generalised zero at $x_2 > x_1$, then any second linearly independent solution has a generalised zero in $(x_1, x_2]$. If a nontrivial solution of (2.5.16) has a generalised zero at

x_1 and a generalised zero at $x_2 > x_1$, then any second linearly independent solution has a generalised zero in $[x_1, x_2]$.

Let
$$L_i y(x) = \Delta[p_i(x-1)\Delta y(x-1)] + q_i(x)y(x) = 0, \quad i = 1, 2$$
where $p_i(x) > 0$ in $[a, b+1]$ and $q_i(x)$ is defined on $[a+1, b+1]$.

Theorem 2.8 (cf. [10, Theorem 8.12])
Assume that $q_1(x) \geq q_2(x)$ on $[a+1, b+1]$ and $p_2(x) \geq p_1(x) > 0$ on $[a, b+1]$. If $L_1 y(x) = 0$ is disconjugate on $[a, b+2]$, then $L_2 y(x) = 0$ is disconjugate on $[a, b+2]$.

A new version of the Sturm comparison theorem and the consequence of that, which is the convexity theorem, are discussed in [4]. These theorems can be used to obtain results on the convexity of the zeros of solutions of self-adjoint difference equations. These results were also applied to the Hahn and Meixner polynomials, which we will discuss, together with the appropriate theorems, in Chapter 4.

2.6 Brief overview

We have observed that if a class of orthogonal polynomials satisfies a differential equation in normal form, then under certain conditions we will be able to determine where the zeros of these polynomials are convex (concave) and one can then often place bounds on the distances between consecutive zeros. We have also seen that these results might be extended to difference equations. In the next chapter we will apply these convexity results to Bessel functions and Hermite, Laguerre, Jacobi and ultraspherical polynomials as was done in [9] and [20].

In Chapter 4 we investigate results on the convexity of zeros of solutions of polynomials which satisfy difference equations with applications to Hahn and Meixner polynomials as was done in [4]. In Chapter 5 we consider Chebychev, Bessel, Pseudo Jacobi and Dual Hahn polynomials and obtain convexity results by making use of the known results for polynomials satisfying differential or difference equations. In Chapter 6 Charlier and Krawtchouk polynomials are investigated and we find that the existing theorems do not yield results on the convexity of these polynomials. We also briefly discuss polynomials which satisfy complex difference equations and q-orthogonal polynomials.

Chapter 3

Convexity of the zeros of some special functions

In this chapter we review the results from [9] and [20] on the convexity of the zeros, as well as bounds on the distances between consecutive zeros, of Bessel functions, Hermite, Laguerre, Jacobi and ultraspherical continuous orthogonal polynomials, together with detailed proofs. We will make use of Theorem 2.1, an application of Sturm's comparison theorem that can be used for second-order differential equations in normal form. This theorem is applicable to the above mentioned classes of polynomials since they all satisfy a second-order differential equation which is in normal form, or can be transformed into normal form.

We start by considering the Bessel functions for which various results on convexity, as well as on the spacing of its zeros, have been discussed in [12], [13], [20] and [25].

3.1 Bessel functions

Firstly, we consider convexity results for the zeros of Bessel functions.

Bessel functions $y(x) = J_\nu(x)$ of the first kind can be defined on $(0, \infty)$ by [28]

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\nu}}{n!\Gamma(\nu + n + 1)}$$

or by their hypergeometric representation (cf. [22, p.108, eqn.(1)])

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)} \; {}_0F_1\left(\begin{array}{c} - \\ \nu+1 \end{array} ; \frac{-x^2}{4} \right).$$

They satisfy the differential equation (cf. [22, p.109, eqn.(4)])

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \tag{3.1.1}$$

and have all their zeros real when $\nu \geq -1$ (cf. [30]). A second solution of the differential equation (3.1.1), the Bessel function of the second kind, is given by

$$Y_\nu(x) = \frac{J_\nu(x)\cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi}$$

where we take an appropriate limit when ν is an integer. The general solution is then given by

$$C_\nu(x) = \cos \alpha J_\nu(x) - \sin \alpha Y_\nu(x)$$

where $C_\nu(x)$ are referred to as cylinder functions (cf. [30]).

We will denote the respective k-th positive zeros of $J_\nu(x)$ and $C_\nu(x)$ by $j_{\nu k}$ and $c_{\nu k}$. It was shown by Sturm [27] that if $|\nu|$ is greater than or less than $1/2$, the positive zeros of the function $C_\nu(x)$ form a concave or convex sequence respectively.

The differential equation (3.1.1) for $y(x) = C_\nu(x)$ can be transformed to normal form

$$y'' + F(x) y = 0$$

as described as in Chapter 2 using equation (2.1.2) where

$$\begin{aligned} F(x) &= f(x) - \frac{1}{4} g^2(x) - \frac{1}{2} g'(x) \\ &= \frac{x^2 - \nu^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2} \\ &= \frac{4x^2 - 4\nu^2 + 1}{4x^2} \\ &= 1 + \frac{1/4 - \nu^2}{x^2} \end{aligned}$$

which results in the transformed differential equation

$$y'' + \left[1 + \frac{1/4 - \nu^2}{x^2} \right] y = 0$$

satisfied by the functions $y(x,\nu) = x^{\frac{1}{2}} C_\nu(x)$. It follows that the zeros $c_{\nu k}$ of $C_{\nu k}$ are also the zeros of $y(x,\nu)$.

Theorem 3.1 (cf. [20])
The zeros of $C_\nu(z)$ *are*

1. *convex on* $(0,\infty)$ *if* $|\nu| < \frac{1}{2}$
2. *concave on* $(0,\infty)$ *if* $|\nu| > \frac{1}{2}$.

In addition,

$$\Delta c_{\nu k} < \pi, \qquad \textit{for } |\nu| < \frac{1}{2}$$

and

$$\Delta c_{\nu k} > \pi, \qquad \textit{for } |\nu| > \frac{1}{2}.$$

Proof: We have that

$$F'(x) = \frac{4\nu^2 - 1}{2x^3}$$

Let $j(x) = 4\nu^2 - 1$, then $j(x) > 0$ when $|\nu| > \frac{1}{2}$ and $j(x) < 0$ when $|\nu| < \frac{1}{2}$. This implies that $F(x)$ will be increasing on $(0,\infty)$ when $|\nu| > \frac{1}{2}$ and decreasing on $(0,\infty)$ when $|\nu| < \frac{1}{2}$. Hence, the zeros of $C_\nu(z)$ are concave for $|\nu| > \frac{1}{2}$ and convex for $|\nu| < \frac{1}{2}$. Also

$$\begin{aligned} |\nu| &> \frac{1}{2} \\ \nu^2 &> \frac{1}{4} \\ \frac{1}{4} - \nu^2 &< 0 \\ 1 + \frac{1/4 - \nu^2}{x^2} &< 1 \\ F(x) &< 1. \end{aligned}$$

Similarly, if $|\nu| < 1/2$, $F(x) > 1$.
By Theorem 2.1 (3) and (4) we can now come to the conclusion that

$$\Delta c_{\nu k} < \pi, \qquad \text{for } |\nu| < \frac{1}{2}$$

and

$$\Delta c_{\nu k} > \pi, \qquad \text{for } |\nu| > \frac{1}{2}.$$

■

Lorch and Szegő [17] considered higher monotonicity properties of the Bessel function and proved that for $|\nu| > \frac{1}{2}$, we have

$$(-1)^n \Delta^{n+1} c_{\nu k} > 0, \quad n = 0, 1, \ldots, k = 1, 2, \ldots \tag{3.1.2}$$

where

$$\Delta^{n+1} c_{\nu k} = \Delta(\Delta(....(\Delta(c_{\nu k}))))$$

with Δ applied $n+1$ times to $c_{\nu k}$.
From this extended and more general results followed (cf. [15] and [16]). It was also proved by Gori, Laforgia and Muldoon in [5] that (3.1.2) remains valid when we replace $c_{\nu k}$ by $j_{\nu k}$ and the difference operation is replaced by a derivative operator. It is not as easy to derive higher monotonicity results for $0 \leq \nu \leq \frac{1}{2}$. In [15] it was conjectured that (3.1.2) should be replaced by

$$(-1)^n \Delta^{n+2} c_{\nu k} > 0, \quad n = 0, 1, \ldots, k = 1, 2, \ldots. \tag{3.1.3}$$

In [21] it was proved by Muldoon that (3.1.3) does hold for $\frac{1}{3} \leq \nu \leq \frac{1}{2}$.

An indirect application of Sturm's convexity theorem was used by Porter and Bôcher in [1] to show that each zero $j_{\nu k}$ of the Bessel function increases as ν increases, with $0 < \nu < \infty$. Makai [18] also used Sturm's convexity theorem to show that $j_{\nu k}/\nu$ decreases as ν increases.

The following general result concerning the convexity of the zeros $j_{\nu k}$ of the Bessel function $J_{\nu k}$ was proved by Elbert and Laforgia (cf. [3]).

Theorem 3.2
Let $j_{\nu k}$ be defined as above and let

$$k_0 = \inf\left\{k > 0 : j'_{\nu k} = \frac{d}{dv} j_{\nu k} > 1, \ \textit{for all } \nu \geq 0\right\}.$$

Then, $j^2_{\nu k}$ is convex for $\nu \geq 0$ and for every $k \geq k_0$.

3.2 Hermite polynomials

Hermite polynomials $y(x) = H_\lambda(x)$, as defined in (2.3.4), satisfy the differential equation (cf. [11, p.250, eqn.(9.15.5)])

$$y'' - 2xy' + 2\lambda y = 0.$$

From their orthogonality it follows that the zeros of $H_\lambda(x)$ are real, distinct and located symmetrically with respect to the origin.
Now

$$\begin{aligned} F(x) &= f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x) \\ &= 2\lambda - \frac{1}{4}(-2x)^2 - \frac{1}{2}(-2) \\ &= -x^2 + 2\lambda + 1 \end{aligned}$$

where $F(x)$ is defined in (2.1.2), the transformed differential equation in normal form (1.0.1) is

$$y'' + (-x^2 + 2\lambda + 1)y = 0$$

and the zeros of $F(x)$ are

$$x_{1,2} = \pm\sqrt{2\lambda + 1}$$

where x_1 denotes the negative zero and x_2 is the positive zero.

Theorem 3.3 (cf. [8])
The zeros of $H_\lambda(x)$ are concave on $(-\infty, 0)$ and convex on $(0, \infty)$. Moreover, we have the estimate

$$\Delta x_k > \frac{\pi}{\sqrt{2\lambda + 1}} \qquad \textit{if } \lambda > -\frac{1}{2}.$$

Proof:
Since $F'(x) = -2x$, $F(x)$ will have a local extremum at $x = 0$. $F(x)$ will be increasing for $x < 0$ and decreasing for $x > 0$. By Theorem 2.1 (1) and (2), the zeros of y will be concave on $(-\infty, 0)$ and convex on $(0, \infty)$.The leading coefficient of $F(x)$ is negative and $F(0) = 2\lambda + 1$, therefore

$$F(x) \leq 2\lambda + 1 \quad \text{for all } x \in \mathbb{R}.$$

Since $2\lambda + 1 > 0$ if $\lambda > -\frac{1}{2}$ it follows by Theorem 2.1(3) that

$$\Delta x_k > \frac{\pi}{\sqrt{2\lambda + 1}} \qquad \text{for } \lambda > -\frac{1}{2}.$$

■

3.3 Laguerre polynomials

Laguerre polynomials $y(x) = L_n^{(\alpha)}(x)$, as defined in (2.3.5), satisfy the differential equation (cf. [11, p.241, eqn.(9.12.5)])

$$xy'' + (\alpha + 1 - x)y' + ny = 0.$$

This differential equation can be transformed, by (2.1.1), to

$$y'' + F(x)y = 0$$

where

$$\begin{aligned} F(x) &= f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x) \qquad (3.3.4) \\ &= \frac{n}{x} - \frac{1}{4}(\frac{\alpha + 1 - x}{x})^2 - \frac{1}{2}(\frac{-\alpha - 1}{x^2}) \\ &= \frac{-x^2 + 2\alpha x + 2x + 4nx - \alpha^2 + 1}{4x^2} \end{aligned}$$

with $F(x)$ defined in (2.1.2). We also have that

$$F'(x) = \frac{(-2n - \alpha - 1)x^2 - (1 - \alpha^2)x}{2x^4}.$$

So when $F'(x) = 0$,

$$x_0 := \frac{\alpha^2 - 1}{\alpha + 2n + 1}. \qquad (3.3.5)$$

Hence $F'(x)$ changes sign at x_0.

Theorem 3.4 (cf. [9, Theorem 3.1])
The zeros of L_n^α on $(0, \infty)$ are

1. *all convex if $n > 0$ and $-1 < \alpha \leq 3$*
2. *all convex if $\alpha > 3$ and $0 < n < \frac{\alpha+1}{\alpha-3}$*
3. *concave for $x < x_0$ and convex for $x > x_0$ when $\alpha > 3$, $n > \frac{\alpha+1}{\alpha-3}$ and x_0 is defined by (3.3.5).*

Moreover, for the distance between consecutive zeros we have the general estimate

$$\Delta x_k > \frac{\pi\sqrt{2}}{\sqrt{2\alpha n + \alpha + 2n^2 + 2n + 1}}, \quad k = 1, \ldots, n-1$$

and also if $x_k > x_0$, then

$$\Delta x_k > \frac{\pi}{\sqrt{F(x_k)}}, \qquad k = 1, \ldots, n-1$$

and

$$\Delta x_k < \frac{\pi}{\sqrt{F(x_{k+1})}}, \qquad k = 1, \ldots, n-2$$

where F is defined by (3.3.4).

Proof:

If $|\alpha| < 1$, then

$$\begin{aligned} \alpha^2 &< 1 \\ \frac{\alpha^2 - 1}{\alpha + 2n + 1} &< 0 \\ x_0 &< 0 \end{aligned}$$

and

$$\begin{aligned} F'(x) &< 0 \\ (-2n - \alpha - 1)x^2 - (1 - \alpha^2)x &< 0. \end{aligned}$$

This quadratic equation will have a negative leading coefficient for all $n = 0, 1, \ldots$ if $-\alpha - 1 < 0$ and this is true for $|\alpha| < 1$. Therefore $F'(x) < 0$ for $x > 0$ or $x < x_0$ when $|\alpha| < 1$. Hence, $F(x)$ will be decreasing on $(-\infty, x_0) \bigcup (0, \infty)$ and increasing on $(x_0, 0)$.

For $\alpha \geq 1$, $x_0 \geq 0$, we have that the leading coefficient will be negative, hence $F(x)$ is increasing on $(0, x_0)$ and decreasing on $(-\infty, 0) \bigcup (x_0, \infty)$.

Let the smallest zero of L_n^α be denoted by x_1, then $x_1 > \frac{\alpha+1}{n}$ (cf. [6]). This implies that when $x_0 < \frac{\alpha+1}{n}$, $F(x)$ will be decreasing on the interval (x_1, ∞), since we have already shown that $F(x)$ is decreasing on (x_0, ∞). This will be true when

$$\begin{aligned} x_0 &< \frac{\alpha + 1}{n} \\ \frac{\alpha^2 - 1}{\alpha + 2n + 1} &< \frac{\alpha + 1}{n} \\ \frac{\alpha - 1}{\alpha + 2n + 1} &< \frac{1}{n}, \qquad \alpha \geq 1. \end{aligned}$$

Solving for α we find that $-1 < \alpha < \frac{3n+1}{n-1}$ and for all $n > 0$, $\frac{3n+1}{n-1} \geq 3$, hence $-1 < \alpha \leq 3$ and the zeros of L_n^α are all convex, by Theorem 2.1 (2). There is another solution for this inequality, if we solve for n in $\alpha < \frac{3n+1}{n-1}$, we find that $n < \frac{\alpha+1}{\alpha-3}$ if $\alpha > 3$, since $n > 0$.
From Theorem 2.1(3) we can estimate the distance Δx_k, where we take $M = F(x_0)$ since the maximum of F is at x_0 and $F(x_0) > 0$. We get

$$\begin{aligned}\Delta x_k &> \frac{\pi}{\sqrt{M}}\\ &= \frac{\pi}{\sqrt{F(x_0)}}\\ &= \frac{\pi\sqrt{2}}{\sqrt{2\alpha n + \alpha + 2n^2 + 2n + 1}}\end{aligned}$$

to obtain the first bound. Following from the facts that $F(x)$ is decreasing on (x_0, ∞) and $F'(x)$ changes sign at x_0, when $x_k > x_0$, F is monotone decreasing on (x_k, x_{k+1}). This means that on (x_k, x_{k+1}) we have

$$F(x_k) > F(x) > F(x_{k+1}).$$

In fact, F is monotone decreasing on $(0, \infty)$ and also

$$\lim_{t\to\infty} F(x) = -\frac{1}{4}$$

so there is exactly one point x_1 on (x_0, ∞) where F crosses the x-axis. From the form of the differential equation, this now implies that if $F(x) < 0$ and $y(x) > 0$,

$$y'' = -F(x)y(x) > 0$$

hence, the graph will be concave up and similarly, if $y(x) < 0$, the graph will be concave down. Hence, there can be at most one zero of the Laguerre polynomials to the right of x_1. This means that $F(x_{n-1})$ is positive, but $F(x_n)$ may be negative and therefore the index in the last bound on the zeros only runs up to $n-2$.

■

3.4 Jacobi polynomials

Jacobi polynomials $y(x) = P_n^{(\alpha,\beta)}(x)$, as defined in (2.3.6), satisfy the differential equation (cf. [11, p.218, eqn.(9.8.6)])

$$(1-t^2)y''(x) + (\beta - \alpha - (\alpha+\beta+2)x)y'(x) + n(n+\alpha+\beta+1)y(x) = 0.$$

This equation can be transformed to the normal form, $y''+F(x)y=0$, where

$$F(x) = \frac{-zx^2 - 2(w-y)x - 2w - 2y + z}{4(x^2-1)^2},$$

where
$$w = \alpha^2 - 1$$
$$y = \beta^2 - 1$$
$$z = (\alpha+\beta+2n)(\alpha+\beta+2n+2)$$

with $F(x)$ defined as in (2.1.2). Also

$$F'(x) = \frac{zx^3 + 3(w-y)x^2 + (4w+4y-z)x + (w-y)}{2(x^2-1)^3} := \frac{j(x)}{2(x^2-1)^3}$$

and the discriminant of $j'(x)$ is

$$D := 12(3w^2 + 3y^2 + z^2 - 6wy - 4wz - 4yz).$$

It is necessary to restrict the parameters to the following conditions (cf. [2]), since the solutions need to be oscillating in order to apply Sturm's convexity theorem,

$$n > 0, \quad n+\alpha+\beta > 0, \quad n+\alpha > 0, \quad n+\beta > 0.$$

In the following theorems we will assume that the parameters satisfy these conditions.

Theorem 3.5 (cf. [9, Theorem 4.1])
If $|\alpha| > 1$, $|\beta| < 1$ and $D < 0$, all the zeros of $P_n^{(\alpha,\beta)}$ on the interval $(-1,1)$ are convex.

Proof:
$F(x)$ is a rational function and it has vertical asymptotes at $x = \pm 1$, since

$$4(x^2-1)^2 = 0$$

implies that

$$x = \pm 1.$$

If $|\alpha| > 1$ and $|\beta| < 1$, then

$$w = \alpha^2 - 1 > 0$$

$$y = \beta^2 - 1 < 0$$

and

$$z = (\alpha + \beta + 2n)(\alpha + \beta + 2n + 2) > 0$$

so

$$\begin{aligned} j(-1) &= zx^3 + 3(w-y)x^2 + (4w+4y-z)x + (w-y) \\ &= -z + 3(w-y) - (4w+4y-z)x + (w-y) \\ &= -8y > 0 \end{aligned}$$

and also

$$\begin{aligned} j(1) &= zx^3 + 3(w-y)x^2 + (4w+4y-z)x + (w-y) \\ &= z + 3(w-y) + (4w+4y-z) + (w-y) \\ &= 8w > 0. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{x\to -1^+} F'(x) &= \lim_{x\to -1^+} \frac{j(x)}{2(x^2-1)^3} \\ &= \lim_{x\to -1^+} \frac{j(-1)}{2(x^2-1)^3} < 0 \end{aligned}$$

and similarly,

$$\lim_{x\to 1^-} F'(x) < 0.$$

This implies that

$$\lim_{x\to -1^+} F(x) = \infty$$

and

$$\lim_{x\to 1^-} F'(x) = -\infty.$$

$D < 0$ implies that $j'(x) \neq 0$ for $x \in (-1,1)$ and hence $j(x)$ will have no extreme values on this interval. It follows that $F(x)$ is monotone decreasing on $(-1,1)$ and Theorem 2.1 yields the result.

■

For fixed values of α and β the discriminant D is positive for large values of n and thus the conditions of Theorem 3.5 will not be satisfied. We investigate this possibility in the next theorem, where we let the degree of the polynomials be sufficiently large with α and β fixed.

Theorem 3.6 (cf. [9, Theorem 4.2])
Let α and β be fixed and let $n \to \infty$, then the convexity of the zeros of $P_n^{(\alpha,\beta)}$ on $(-1,1)$ changes in the following way (from left to right):

1. *if* $|\alpha| \leq 1$ *and* $|\beta| \leq 1$ *then convex-concave.*

2. *if* $|\alpha| \leq 1$ *and* $|\beta| > 1$ *then concave-convex-concave.*

3. *if* $|\alpha| > 1$ *and* $|\beta| \leq 1$ *then convex-concave-convex.*

4. *if* $|\alpha| > 1$ *and* $|\beta| > 1$ *then concave-convex-concave-convex.*

Proof:
If α and β are fixed and $n \to \infty$, then the extreme locations of $j(x)$ tend to $\pm 1/\sqrt{3}$. We can calculate this by first finding the $j'(x)$ as $n \to \infty$,

$$j'(x) = x^2(3z) + 6x(w - y) + (4w + 4y - z)$$

noting that $n \to \infty$ implies that $z \to \infty$, so to find the roots of $j'(x)$ we have

$$\begin{aligned} x^2(3z) + 6x(w - y) + (4w + 4y - z) &= 0 \\ 3x^2 + \frac{6x(w - y)}{z} + (4\frac{w}{z} + 4\frac{y}{z} - 1) &= 0 \end{aligned}$$

and if we now let $z \to \infty$ then

$$3x^2 - 1 = 0.$$

Hence the determining the extreme values of $j(x)$ are

$$x = \pm\frac{1}{\sqrt{3}}.$$

Since $z > 0$, the leading coefficient of $j(x)$ will be positive, which implies that the local extremum near $-1/\sqrt{3}$ will be the maximum and the local extremum near $1/\sqrt{3}$ will be the minimum.

$$j\Big(\frac{1}{\sqrt{3}}\Big) = z\Big(\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}\Big) + y\Big(-2 + \frac{4}{\sqrt{3}}\Big) + w\Big(2 + \frac{4}{\sqrt{3}}\Big).$$

$$j\Big(-\frac{1}{\sqrt{3}}\Big) = z\Big(-\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\Big) + y\Big(-2 - \frac{4}{\sqrt{3}}\Big) + w\Big(2 + \frac{4}{\sqrt{3}}\Big).$$

The maximum value tends to ∞ if we let $n \to \infty$, i.e. $z \to \infty$, and similarly, the minimum value tends to $-\infty$.

$$\begin{aligned} j''(x) &= 0 \\ x(6z) + 6(w - y) &= 0 \\ x &= \frac{y - w}{z}. \end{aligned}$$

This gives us the inflection point, x_0, of $j(x)$ and since $z \to \infty$,

$$x_0 = \frac{y - w}{z} \to 0$$

which implies that there is at least one change of concavity in $(-1, 1)$ (from convex to concave) and whether there are more, depends on the sign of $j(-1)$ and $j(1)$. We have that

$$j(-1) = -8y = 8(1 - \beta^2)$$

and

$$j(1) = 8w = 8(\alpha^2 - 1).$$

For case 1, we get that $j(-1) > 0$ and $j(1) < 0$ which implies that

$$\lim_{x \to 1^-} F'(x) = \infty \text{ and } \lim_{x \to -1^+} F'(x) = -\infty.$$

Hence, we have convex-concave. For case 2, $j(-1) < 0$ and $j(1) < 0$, which means that $\lim_{x \to 1^-} F'(x) = \infty$ and $\lim_{x \to -1^+} F'(x) = \infty$, so we have concave-convex-concave. Similarly the results for cases 3 and 4 follow.

■

A special case of the Jacobi polynomials is the ultraspherical, also known as Gegenbauer, polynomials.

3.5 Ultraspherical polynomials

Ultraspherical polynomials $y(x) = C_n^{(\lambda)}$, as defined in (2.3.7), satisfy a differential equation (cf. [11, p.223, eqn.(9.8.23)])

$$(1 - x^2)y''(x) - (2\lambda + 1)xy'(x) + n(n + 2\lambda)y(x) = 0. \tag{3.5.6}$$

Theorem 3.7 (cf. [9, Theorem 5.1])
If $|\lambda| \leq 1$, the zeros of $C_n^{(\lambda)}$ on $(-1, 0)$ are convex and those on $(0, 1)$ are concave. In addition

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}} = \frac{\pi}{\sqrt{2\lambda n + \lambda + n^2 + n + 1}}$$

and for the positive zeros we have

$$\frac{\pi}{\sqrt{F(x_{k+1})}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_k)}}.$$

Proof: First we need to write the differential equation (3.5.6) in the normal form. We find that

$$F(x) = \frac{-(\lambda+n)(\lambda+n+1)x^2 + (1+n+n^2+\lambda+2\lambda n)}{(x^2-1)^2}.$$

The numerator of $F'(x)$ is

$$j(x) = 4[(\lambda+n)(\lambda+n+1)x^3 - (2+n+n^2+\lambda+2\lambda n-\lambda^2)x]$$

and the discriminant of $j'(x)$ is

$$D = 192(\lambda+n)(\lambda+n+1)(2+n+n^2+\lambda+2\lambda n-\lambda^2).$$

The leading coefficient of $j(x)$ is positive when $\lambda > -1$. The point of inflection of $j(x)$ is $x_0 = 0$, since

$$\begin{aligned} j''(x) &= 0 \\ 24(\lambda+n)(\lambda+n+1)x &= 0 \\ x &= 0 \end{aligned}$$

and hence the convexity of the zeros changes exactly at the middle of the interval $(-1,1)$. The local extrema of $j(x)$ can be estimated as follows

$$\begin{aligned} 0 &= j'(x) \\ 0 &= 12[(\lambda+n)(\lambda+n+1)x^2 - (2+n+n^2+\lambda+2\lambda n-\lambda^2)] \\ x_{1,2} &= \pm\sqrt{\frac{(n+\lambda)(n+\lambda+1)-2(\lambda^2-1)}{3(n+\lambda)(n+\lambda+1)}} \end{aligned}$$

and the two remaining zeros of $j(x)$ are

$$\begin{aligned} 0 &= j(x) \\ 0 &= 4[(\lambda+n)(\lambda+n+1)x^3 - (2+n+n^2+\lambda+2\lambda n-\lambda^2)x] \\ 0 &= 4[(\lambda+n)(\lambda+n+1)x^2 - (2+n+n^2+\lambda+2\lambda n-\lambda^2)] \\ X_{1,2} &= \pm\sqrt{\frac{(n+\lambda)(n+\lambda+1)-2(\lambda^2-1)}{(n+\lambda)(n+\lambda+1)}} = x_{1,2}\sqrt{3} \qquad (3.5.7) \end{aligned}$$

where X_1 denotes the negative zero and X_2 the positive zero.
For $|\lambda| \leq 1$ we have $X_1 < -1$ and $X_2 > 1$, and since the leading coefficient of $j(x)$ is positive when $\lambda > -1$, $j(x)$ will be positive on $(-1,0)$ and negative on

$(0,1)$. But $F'(x)$ will be negative on $(-1,0)$ and positive on $(0,1)$, therefore, $F(x)$ is decreasing on $(-1,0)$ and increasing on $(0,1)$.
The convexity of the zeros now follows from Theorem 2.1(1) and (2). In addition,

$$F(0) = 1 + n + n^2 + \lambda + 2\lambda n > 0$$

is a minimum value and so we have an upper bound on the distance between any two consecutive zeros from Theorem 2.1(4),

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}} = \frac{\pi}{\sqrt{2\lambda n + \lambda + n^2 + n + 1}}.$$

Finally, since $F(x)$ is increasing on $(0,1)$, for $x \in (x_i, x_{i+1})$ we have

$$0 < F(x_i) < F(x) < F(x_{i+1})$$

where x_i and x_{i+1} are any two consecutive positive zeros and the last inequality follows from Theorem 2.1(3).

■

Theorem 3.8 (cf. [9, Theorem 5.2])
Let $|\lambda| > 1$ *and* $(n+\lambda)(n+\lambda+1) \leq 2(\lambda^2 - 1)$ *then the zeros of* $C_n^{(\lambda)}$ *on* $(-1,0)$ *are concave and those on* $(0,1)$ *are convex. Furthermore*

$$\Delta x_k > \frac{\pi}{\sqrt{F(0)}}.$$

Proof:

If $|\lambda| > 1$ and $D < 0$, then $j'(x)$ has no real roots, hence $j'(x) \neq 0$ for all real $x \in (-1,1)$, therefore $j(x)$ will have no local extremum and is monotone increasing on $(-1,1)$ since if $D < 0$ and we let $(n+\lambda)(n+\lambda+1) \leq 2(\lambda^2 - 1)$ then we must have $(\lambda+n)(\lambda+n+1) > 0$ which is the leading coefficient of $j(x)$. $F'(x)$ is decreasing on $(-1,1)$, but $j(0) = 0$, or equivalently $F'(0) = 0$, which implies that $F'(x) > 0$ on $(-1,0)$ and $F'(x) < 0$ on $(0,1)$.
Now, $F(x)$ is increasing on $(-1,0)$ and decreasing on $(0,1)$. From Theorem 2.1, the zeros of $P_n^{(\lambda,\lambda)}$ are concave on $(-1,0)$ and convex on $(0,1)$. Furthermore, $F(0)$ is a local maximum, therefore $F(x) < F(0)$ for $x \in (-1,1)$ and $F(0) > 0$. By Theorem 2.1(3),

$$\Delta x_k > \frac{\pi}{\sqrt{F(0)}}.$$

■

Theorem 3.9 (cf. [9, Theorem 5.3])
Let $|\lambda| > 1$ *and* $(n+\lambda)(n+\lambda+1) > 2(\lambda^2-1)$ *then the zeros of* $C_n^{(\lambda)}$ *are concave on* $(-1, X_1)$ *and* $(0, X_2)$ *and convex on* $(X_1, 0)$ *and* $(X_2, 1)$*, where* $X_{1,2}$ *are as in (3.5.7). We also have that*

$$\Delta x_k > \frac{\pi}{\sqrt{F(X_2)}} = \frac{\pi}{\sqrt{F(X_1)}},$$

moreover, if $(x_k, x_{k+1}) \subset (X_1, X_2)$ *then*

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}},$$

and if $(x_k, x_{k+1}) \subset (0, X_2)$*, then*

$$\frac{\pi}{\sqrt{F(x_{k+1})}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_k)}}.$$

Proof:

When $|\lambda| > 1$ and $D > 0$, then all the zeros of $j(x)$ are real, hence $j(x)$ has three zeros on $(-1, 1)$, namely $0, X_1$ and X_2 with

$$F(0) = (1 + n + n^2 + \lambda + 2\lambda n) = (\lambda + n)(\lambda + n + 1) - (\lambda^2 - 1)$$

and

$$\begin{aligned} F(X_1) &= \frac{-[(n+\lambda)(n+\lambda+1) - 2(\lambda^2-1)] + (1+n+n^2+\lambda+2\lambda n)}{\left(\frac{(n+\lambda)(n+\lambda+1)-2(\lambda^2-1)}{(n+\lambda)(n+\lambda+1)} - 1\right)^2} \\ &= \frac{F(0) + (\lambda+n)(\lambda+n+1) - 2(\lambda^2-1)}{\left(\frac{-2(\lambda^2-1)}{(n+\lambda)(n+\lambda+1)}\right)^2} \\ &> F(0) \end{aligned}$$

when $(n+\lambda)(n+\lambda+1) > 2(\lambda^2-1)$. Also, $F(X_1) = F(X_2)$, hence $F(x)$ has local maxima at $X_{1,2}$ and a local minimum at $x = 0$. $F(x)$ is decreasing on $(X_1, 0) \bigcup (X_2, 1)$ and increasing on $(-1, X_1) \bigcup (0, X_2)$. By Theorem 2.1, the zeros of $P_n^{\lambda,\lambda}$ are concave on $(-1, X_1)$ and $(0, X_2)$ and convex on $(X_1, 0)$ and $(X_2, 1)$.
We also have that $\lim_{x\to -1} F(x) = -\infty = \lim_{x\to 1} F(x)$, so $F(x) < F(X_2) = F(X_1)$ on $(-1, 1)$ and by Theorem 2.1(3)

$$\Delta x_k > \frac{\pi}{\sqrt{F(X_2)}} = \frac{\pi}{\sqrt{F(X_1)}}.$$

Also, if $(x_k, x_{k+1}) \subset (X_1, X_2)$ then $F(x) > F(0)$ where $x \in (x_k, x_{k+1})$ and by Theorem 2.1(4)

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}}.$$

If $(x_k, x_{k+1}) \subset (0, X_2)$, then $F(x_k) < F(x) < F(x_{k+1})$ where $x \in (x_k, x_{k+1})$ since $F(x)$ is increasing on $(0, X_2)$. Hence, by Theorem 2.1(3)(4),

$$\frac{\pi}{\sqrt{F(x_{k+1})}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_k)}}.$$

■

In this chapter we have discussed and proved results for various classes of orthogonal polynomials as well as Bessel functions, which all satisfy a second-order differential equation, where we applied theorems which followed from Sturm's comparison theorem. We have also seen that not only can we prove properties on convexity of zeros but we can also, in some cases, place bounds on the distances between consecutive zeros.

In the next chapter we will discuss results which can be applied to polynomials which satisfy difference equations and will use the same approach as was done with differential equations, this time applying analogues of Sturm's theorems.

Chapter 4

Convexity theorems and results for difference equations

In this chapter we explore the possibilities of obtaining results on the convexity of zeros of solutions of polynomials that satisfy a difference equation. We will do this in a similar way as we did for solutions of differential equations. In Section 2.4 we discussed analogues of Sturm's theorems that also hold for difference equations. We now consider further extensions by Gishe and Toókos [4].

The first theorem we state here may be considered a new version of the Sturm comparison theorem for difference equations (Theorem 2.8) which will provide better information about the behaviour of the zeros of solutions of such second-order difference equations.

Theorem 4.1 (cf. [4, Theorem 1])
For the following pair of second-order difference equations,

$$\Delta[p_1(x-1)\Delta y(x-1)] + q_1(x)y(x) = 0 \tag{4.0.1}$$

$$\Delta[p_2(x-1)\Delta z(x-1)] + q_2(x)z(x) = 0 \tag{4.0.2}$$

assume that $p_2(x) \geq p_1(x) > 0$ *and* $q_1(x) \geq q_2(x)$ *for* $x \in [x_0, x_0+n]$, $y(x_0) = z(x_0) = 0$, $y(x_0+1) > 0$ *and* $z(x_0+1) > 0$. *Then if* $y(x_0+2) > 0$, $y(x_0+3) > 0$, *…*, $y(x_0+n) > 0$ $(n \geq 2)$, *then* $z(x_0+2) > 0$, $z(x_0+3) > 0$,

$\ldots$, $z(x_0 + n) > 0$.

This means that "$z(x)$ cannot change sign before $y(x)$ does".

Proof:

Assume that $z(x)$ does change sign before $y(x)$ does, i.e. assume that $y(x_0 + 2) > 0$, $y(x_0 + 3) > 0$, $\ldots$, $y(x_0 + n) > 0$ and $z(x_0 + 2) > 0$, $z(x_0 + 3) > 0$, $\ldots$, $z(x_0 + k - 1) > 0$, $z(x_0 + k) < 0$, where $1 < k < n$. Then $z(x)$ has two generalised zeros on $[x_0, x_0 + k]$, since

$$z(x_0 + k - 1)z(x_0 + k) < 0$$

and $y(x_0) = 0$ which means that x_0 as well as $x_0 + k$ are generalised zeros. Therefore, equation (4.0.2) is conjugate on $[x_0, x_0 + k]$, by Definition 2.6. According to Theorem 2.8, equation (4.0.1) is also conjugate on $[x_0, x_0 + k]$, i.e. it has a nontrivial solution $w(x)$ with at least two generalised zeros on $[x_0, x_0 + k]$.

Since $k < n$, the functions $w(x)$ and $y(x)$ must be linearly independent. By Theorem 2.7, $w(x_0) \neq 0$ since two linearly independent solutions cannot have a common zero. Hence, $w(x)$ has two generalised zeros on $[x_0 + 1, x_0 + k]$. But then $y(x)$ also has to have a generalized zero on $[x_0 + 1, x_0 + k]$, which contradicts the fact that $y(x_0^*) \neq 0$ and $y(x_0^*)y(x_0^* - 1) > 0$ for all $x_0^* \in (x_0 + 1, x_0 + k)$.

Hence, we must have $z(x_0 + k) > 0$ for $1 < k < n$.

■

The following theorem may be considered as the Sturm convexity theorem for second-order difference equations.

Theorem 4.2 (cf. [4, Theorem 2])

Assume that in the equation

$$\Delta[p(x - 1)\Delta y(x - 1)] + q(x)y(x) = 0$$

$p(x)$ is monotone decreasing, $q(x)$ is monotone increasing on $[x_0 - n, x_0 + n]$, $y(x_0) = 0$ and $y(x_0 + 1) > 0$. Then if $y(x_0 + 2) > 0$, $y(x_0 + 3) > 0$, $\ldots$, $y(x_0 + n) > 0$, then $y(x_0 - 1) < 0$, $y(x_0 - 2) < 0$, $\ldots$, $y(x_0 - n) < 0$.

Proof:
Let $p_2(x) = p(2x_0 - x - 1)$ and $q_2(x) = q(2x_0 - x)$.

$$\begin{aligned}
& \Delta[p_2(x-1)\Delta z(x-1)] + q_2(x)z(x) \\
= \; & \Delta[p(2x_0 - x - 2)\Delta z(x-1)] + q(2x_0 - x)z(x) \\
= \; & z(x+1)p(2x_0 - x - 1) - z(x)[p(2x_0 - x - 1) + p(2x_0 - x) - q(2x_0 - x)] \\
& z(x-1)p(2x_0 - x)
\end{aligned}$$

Letting $z(x) = -y(2x_0 - x)$ equation (4.0.2) is satisfied and we have

$$\begin{aligned}
& -y(2x_0 - x - 1)p(2x_0 - x - 1) \\
& +y(2x_0 - x)[p(2x_0 - x - 1) + p(2x_0 - x) - q(2x_0 - x)] \\
& -y(2x_0 - x + 1)p(2x_0 - x) \\
= \; & -[y(2x_0 - x - 1)p(2x_0 - x - 1) \\
& -y(2x_0 - x)[p(2x_0 - x - 1) + p(2x_0 - x) - q(2x_0 - x)] \\
& +y(2x_0 - x + 1)p(2x_0 - x)].
\end{aligned}$$

From the monotonicity assumption, it now follows that $p_2(x) \geq p(x)$ and $q(x) \geq q_2(x)$ on $[x_0, x_0 + n]$.
By Theorem 4.1,

$$z(x_0 + 2) > 0, z(x_0 + 3) > 0, \ldots, z(x_0 + n) > 0$$

but $z(x) = -y(2x_0 - x)$, so

$$y(x_0 - 1) < 0, y(x_0 - 2) < 0, \ldots, y(x_0 - n) < 0.$$

■

Since we are investigating discrete polynomials we have to define a new type of convexity to ensure that we take all the zeros into consideration. Therefore we now define the concept of quasi-convexity.

Definition 4.3 (cf. [4, Definition 2.1])
Let $y(x)$ be a continuous function on an interval (a, b) and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(x)$ in (a, b). Then

1. *if* $\Delta^2 x_k < 1$ *for all* k*, then the zeros of* $y(x)$ *are called quasi-concave on* (a, b)*,*

2. *if* $\Delta^2 x_k > -1$ *for all* k*, then the zeros of* $y(x)$ *are called quasi-convex on* (a, b)*.*

We can show that a sufficient condition for quasi-convexity is monotonicity of the functions $p(x)$ and $q(x)$.

Corollary 4.4 (cf. [4, Corollary 2.2])
Let $y(x)$ *be a continuous function on an interval* (a, b) *and let* $x_1 < \ldots < x_k < x_{k+1} < \ldots$ *denote the consecutive zeros of* $y(x)$ *in* (a, b)*. Assume that* $\Delta x_k > 1$ *for all* k *and that* $y(x)$ *satisfies equation (2.5.16) on* (a, b)*.*

- *If* $p(x)$ *is monotone decreasing and* $q(x)$ *is monotone increasing on* (a, b)*, then the zeros of* $y(x)$ *are quasi-concave on* (a, b)*.*
- *If* $p(x)$ *is monotone increasing and* $q(x)$ *is monotone decreasing on* (a, b)*, then the zeros of* $y(x)$ *are quasi-convex on* (a, b)*.*

Proof:
Let x_{k+1} be a zero of $y(t)$ in (a, b) and assume that $y(x_{k+1} + 1) > 0$, then $y(x)$ must be positive between x_{k+1} and $x_{k+1} + 1$ as well, since $\Delta x_k > 1$ by assumption, which would imply that any two consecutive zeros are more than 1 unit apart, i.e. there cannot be a zero between x_{k+1} and $x_{k+1} + 1$.

Choose n such that

$$x_{k+1} + n < x_{k+2} \leq x_{k+1} + n + 1 \tag{4.0.3}$$

which implies that the difference between x_{k+1} and the next zero, x_{k+2}, is more than n. Then

$$y(x_{k+1} + 1) > 0, y(x_{k+1} + 2) > 0 \ldots y(x_{k+1} + n) > 0$$

and by Theorem 4.2

$$y(x_{k+1} - 1) < 0, y(x_{k+1} - 2) < 0 \ldots y(x_{k+1} - n) < 0.$$

This, together with the assumption that $\Delta x_k > 1$ and by Lemma 2.4 yields $x_{k+1} + n < x_{k+2}$ which is true for values of k, so we also have that

$$x_k < x_{k+1} - n. \tag{4.0.4}$$

On the other hand, by the choice of n in (4.0.3),

$$x_{k+2} - x_{k+1} \leq n + 1. \tag{4.0.5}$$

By combining equations (4.0.4) and (4.0.5) we obtain

$$\Delta^2 x_k < 1.$$

■

Next we consider the Hahn and Meixner polynomials, which both satisfy difference equations, and use the above mentioned theorems to obtain results on the convexity of their zeros as was done in [4].

4.1 Hahn polynomials

Hahn polynomials $y(x) = Q_n(x; \alpha, \beta, N)$, as defined in (2.3.8), satisfy the difference equation (cf. [11, p.205, eqn.(9.5.5)])

$$\begin{aligned} & n(n+\alpha+\beta+1)y(x) \\ = \; & B(x)y(x+1) - [B(x)+D(x)]y(x) + D(x)y(x-1), \end{aligned} \tag{4.1.6}$$

with

$$B(x) = (x+\alpha+1)(x-N)$$

and

$$D(x) = x(x-\beta-N-1).$$

As discussed in Section 2.5, the zeros of Hahn polynomials lie in the interval $(0, N)$ and hence we consider values of x such that $0 < x < N$.

If $\alpha > -1$ then

$$\alpha + 1 + x \quad > \quad 0 \quad \text{if } x > 0.$$

Also, if $x < N$ then

$$-\alpha - 1 < x < N$$

and we have that

$$B(x) = (x+\alpha+1)(x-N) \quad < \quad 0.$$

Similarly, if $\beta > -1$ then

$$\begin{aligned} \beta &> -N-1 \quad \text{for all } N \in \mathbb{Z}^+ \\ 0 &< x < \beta + N + 1 \quad \text{since } x < N \end{aligned}$$

which implies that

$$D(x) = x(x - \beta - N - 1) \quad < \quad 0.$$

Theorem 4.5 (cf. [4, Proposition 2.3])
Let $\alpha + \beta > 0$. The zeros of the Hahn polynomials $Q_n(x; \alpha, \beta, N)$ are quasi-convex or quasi-concave on certain intervals depending on the parameter values as follows

1. *If $-1 < \alpha < 0 < \beta$ then the zeros are quasi-convex on $(0, x_1)$.*
2. *Let $0 < \alpha < \beta$. If $N < \frac{\alpha}{\beta}$ then the zeros are quasi-concave on (x_1, N). If $\frac{\alpha}{\beta} < N < \frac{\alpha(\alpha+\beta+2)}{(\beta-\alpha)}$ then the zeros are quasi-concave on (x_1, x_2). If $N > \frac{\alpha(\alpha+\beta+2)}{(\beta-\alpha)}$ then the zeros are quasi-convex on (x_2, x_1).*
3. *If $-1 < \beta < 0 < \alpha$ then the zeros are quasi-concave on (x_1, N).*
4. *Let $0 < \beta < \alpha$. If $N < \frac{\alpha}{\beta}$ then the zeros are quasi-concave on (x_1, N). If $N > \frac{\alpha}{\beta}$ then the zeros are quasi-concave on (x_1, x_2).*

where x_1 and x_2 are defined as in (4.1.7).

Proof:
First, we notice that since any two consecutive zeros must be more than one unit apart, $\Delta x > 1$.
In order to apply the convexity theorem we bring the difference equation to self-adjoint form by multiplying both sides of (4.1.6) by

$$\prod_{s=0}^{x-1} \frac{B(s)}{D(s+1)},$$

which yields

$$p(x)y(x+1) - [p(x) + p(x-1)]y(x) + p(x-1)y(x-1) + q(x)y(x) = 0.$$

This is of the form (2.5.16) with

$$p(x) = \frac{-(\alpha+1)_{x+1}(-N)_{x+1}}{x!(-\beta-N)_x}$$

and

$$q(x) = n(n+\alpha+\beta+1)\frac{(\alpha+1)_x(-N)_x}{x!(-\beta-N)_x}.$$

Assume that $\alpha+\beta>0$ and let

$$x_1 = \frac{(\alpha+1)N}{\alpha+\beta+2} \text{ and } x_2 = \frac{\alpha(N+1)}{\alpha+\beta}. \tag{4.1.7}$$

Then $p(x)$ is monotone increasing when $x < x_1$ since

$$\begin{aligned}
x_1 = \frac{(\alpha+1)N}{\alpha+\beta+2} &> x \\
x(-\alpha-\beta-2)+N(\alpha+1) &> 0 \\
x(-\alpha-\beta-2)+N(\alpha+1)+Nx-Nx+x^2-x^2 &> 0 \\
x(-\beta-N+x-1)-(\alpha+1+x)(-N+x) &> 0.
\end{aligned}$$

Now, if we consider $p(x)-p(x-1)$ we find that

$$\begin{aligned}
& p(x)-p(x-1) \\
= & \frac{-(\alpha+1)_{x+1}(-N)_{x+1}}{x!(-\beta-N)_x} - \frac{-(\alpha+1)_x(-N)_x}{(x-1)!(-\beta-N)_{(x-1)}} \\
= & \frac{x(-\beta-N+x-1)(\alpha+1)_x(-N)_x-(\alpha+1)_{x+1}(-N)_{x+1}}{x!(-\beta-N)_x} \\
= & \frac{(\alpha+1)_x(-N)_x[x(-\beta-N+x-1)-(\alpha+1+x)(-N+x)]}{x!(-\beta-N)_x}.
\end{aligned}$$

Note that $x!>0$ and $(\alpha+1)_x>0$. Also for

$$\frac{(-N)_x}{(-\beta-N)_x} = \frac{(-N)(-N+1)\ldots(-N+x-1)}{(-\beta-N)(-\beta-N+1)\ldots(-\beta-N+x-1)}$$

all the factors are negative, but since the numerator and denominator have the same number of factors, the resultant fraction must be positive. Hence, $p(x)-p(x-1)>0$ which implies that $p(x)$ is monotone increasing when $x<x_1$.

Similarly, $q(x)$ is monotone decreasing for $x>x_2$, since

$$\begin{aligned}
x_2 = \frac{\alpha(N+1)}{\alpha+\beta} &< x \\
x(\alpha+\beta)-\alpha(N+1) &> 0 \\
x(\alpha+\beta)-\alpha(N+1)+xN-xN+x-x+x^2-x^2 &> 0 \\
(\alpha+x)(-N+x-1)-x(-\beta-N+x-1) &> 0.
\end{aligned}$$

Considering $q(x) - q(x-1)$ we find

$$\begin{aligned}
& q(x) - q(x-1) \\
= \; & n(n+\alpha+\beta+1)\frac{(\alpha+1)_x(-N)_x}{x!(-\beta-N)_x} \\
& -n(n+\alpha+\beta+1)\frac{(\alpha+1)_{x-1}(-N)_{x-1}}{(x-1)!(-\beta-N)_{x-1}} \\
= \; & \frac{n(n+\alpha+\beta+1)(\alpha+1)_{x-1}(-N)_{x-1}[(\alpha+x)(-N+x-1)]}{x!(-\beta-N)_x} \\
& -\frac{n(n+\alpha+\beta+1)(\alpha+1)_{x-1}(-N)_{x-1}[x(-\beta-N+x-1)]}{x!(-\beta-N)_x}.
\end{aligned}$$

Note that $n(n+\alpha+\beta+1)$ and $x!$ are positive. Furthermore,

$$\begin{aligned}
& \frac{[(\alpha+1)_{x-1}][(-N)_{x-1}]}{(-\beta-N)_x} \\
= \; & \frac{[(\alpha+1)\dots(\alpha+x-1)][(-N)\dots(-N+x-3)(-N+x-2)]}{(-\beta-N)\dots(-\beta-N+x-1)}
\end{aligned}$$

the numerator and denominator will have an unequal amount of positive and negative factors for all values of $0 < x < N$. Hence, $q(x) - q(x-1) < 0$ which implies that $q(x)$ is monotone decreasing for $x > x_2$ and monotone increasing for $x < x_2$.

This means that for $x \in (x_1, x_2)$, p and q have opposing monotonicity and Theorem 4.2 is applicable. We also note that $0 < x_1 < N$ because of our assumptions, however this isn't true for x_2, which may be negative or greater than N, but this depends on the values of α and β. Whether $x_1 < x_2$ or $x_2 < x_1$ will depend on the sign of $N(\beta-\alpha) - \alpha(\alpha+\beta+2)$, since

$$\begin{aligned}
x_1 - x_2 &= \frac{(\alpha+1)N}{\alpha+\beta+2} - \frac{\alpha(N+1)}{\alpha+\beta} \\
&= \frac{(\alpha+1)N(\alpha+\beta) - \alpha(N+1)(\alpha+\beta+2)}{(\alpha+\beta+2)(\alpha+\beta)} = 0
\end{aligned}$$

if and only if

$$\begin{aligned}
(\alpha+1)N(\alpha+\beta) - \alpha(N+1)(\alpha+\beta+2) &= 0 \\
N(\beta-\alpha) - \alpha(\alpha+\beta+2) &= 0.
\end{aligned}$$

■

4.2 Meixner polynomials

Considering the Meixner polynomials $M_n(x;b,c)$, as defined in (2.3.9), one can obtain convexity results using the same method that was used for the Hahn polynomials, however it is easier to consider the limit relation between the Meixner and Hahn polynomials to obtain these results.

Theorem 4.6 (cf. [4, Proposition 2.4])
The zeros of the Meixner polynomials $M_n(x;b,c)$ are quasi-convex on

1. $\left(0, \frac{bc}{1-c}\right)$ *if* $0 < b < 1$,
2. $\left(\frac{(b-1)c}{1-c}, \frac{bc}{1-c}\right)$ *if* $b \geq 1$.

Proof:
Let $\alpha = b-1$ and $\beta = N(1-c)/c$ in the definition of the Hahn polynomials (2.3.8). Then we have

$$Q_n\left(x; b-1, \frac{N(1-c)}{c}, N\right) = {}_3F_2\left(\begin{matrix} -n, n+b+N(1-c)/c, -x \\ b, -N \end{matrix}; 1\right)$$
$$= \sum_{k=0}^{\infty} \frac{(-n)_k(n+b+\frac{N(1-c)}{c})_k(-x)_k(1)^k}{(b)_k(-N)_k k!}.$$

Letting $N \to \infty$ yields, by the definition of hypergeometric polynomials in Section 2.2,

$$\begin{aligned} &\lim_{N\to\infty} Q_n\left(x; b-1, \frac{N(1-c)}{c}, N\right) \\ = &\lim_{N\to\infty} \sum_{k=0}^{\infty} \frac{(-n)_k(n+b+\frac{N(1-c)}{c})_k(-x)_k(1)^k}{(b)_k(-N)_k k!} \\ = &\sum_{k=0}^{\infty} \frac{(-n)_k(-x)_k(\frac{c-1}{c})^k}{(b)_k k!} \quad \text{by L'Hospital's rule} \\ = &{}_2F_1\left(\begin{matrix} -n, -x \\ b \end{matrix}; 1-\frac{1}{c}\right) \\ = &M_n(x;b,c). \end{aligned}$$

With these substitutions we have

$$x_1 = \frac{bcN}{bc+N(1-c)+c}$$

and

$$\lim_{N\to\infty} x_1 = \frac{bc}{1-c}.$$

Also,

$$x_2 = \frac{c(b-1)(N+1)}{bc-c+N(1-c)}$$

and taking the limit

$$\lim_{N\to\infty} x_2 = \frac{c(b-1)}{1-c}.$$

From Theorem 4.5(1), making the corresponding substitutions, we obtain the first result and considering the last case of Theorem 4.5(2) we obtain the second result.

■

We have seen that it is possible to determine the intervals of quasi-convexity of polynomials which satisfy difference equations under certain conditions. In this case we considered the Hahn and Meixner polynomials by applying theorems which resulted from analogues of Sturm's comparison and convexity theorems. These theorems can be further applied to other polynomials that satisfy second-order difference equations, some of which will be discussed in Chapter 5 and 6. Furthermore, similar theorems can be formulated for q-difference equations, which can then be applied to polynomials that satisfy a q-difference equation, such as the q-Laguerre polynomials (cf. [4]), to obtain results on the q-quasi-convexity of the zeros.

In the next chapter we will use the results in Chapter 3 and 4 to investigate the convexity of the zeros of the Chebychev, Bessel, Pseudo Jacobi and Dual Hahn polynomials.

Chapter 5

The Chebychev, Bessel, Pseudo Jacobi and Dual Hahn polynomials

In this Chapter we will consider Chebychev polynomials of the first and the second kind, Bessel polynomials, Pseudo Jacobi polynomials as well as Dual Hahn polynomials and make use of the same methods used in Chapters 3 and 4 to obtain some results on the convexity of the zeros of these polynomials.

Since Chebychev polynomials of the first kind, $T_n(x)$, are a special case of Jacobi polynomials and can be found by setting $\alpha = \beta = -\frac{1}{2}$ while $\alpha = \beta = \frac{1}{2}$ yields the Chebychev polynomials of the second kind, $U_n(x)$, the results in the next two sections are special cases of those in Section 3.5.

5.1 Chebychev polynomials of the first kind

These polynomials $y(x) = T_n(x)$, which are defined in (2.3.10), satisfy the differential equation (cf. [11, p.226, eqn.(9.8.43)])

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0.$$

In order to write the differential equation in the normal form (1.0.1), we first need to find the function $F(x)$. We use the definition in (3.3.4) to find

$$F(x) = f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x)$$

$$\begin{aligned} &= \frac{n^2}{1-x^2} - \frac{1}{4}\frac{x^2}{(1-x^2)^2} + \frac{1}{2}\frac{x^2+1}{(1-x^2)^2} \\ &= \frac{x^2(1-4n^2)+2(2n^2+1)}{4(1-x^2)^2}. \end{aligned}$$

Therefore, the differential equation in normal form is

$$y''(x) + \frac{x^2(1-4n^2)+2(2n^2+1)}{4(1-x^2)^2}y(x) = 0.$$

Also,

$$\begin{aligned} F'(x) &= \frac{8x(1-4n^2)(1-x^2)^2 - 8(1-x^2)(-2x)[x^2(1-4n^2)+2(2n^2+1)]}{16(1-x^2)^4} \\ &= \frac{x^3(1-4n^2)+x(5+4n^2)}{2(1-x^2)^3}. \end{aligned}$$

Let $j(x) = x^3(1-4n^2)+x(5+4n^2)$ then in order to find the zeros of $j(x)$ we set $j(x) = 0$

$$\begin{aligned} j(x) &= 0 \\ x^3(1-4n^2)+x(5+4n^2) &= 0 \\ x = \pm\sqrt{\frac{4n^2+5}{4n^2-1}} &\quad \text{or} \quad x = 0. \end{aligned} \tag{5.1.1}$$

Let X_1 be the negative zero, X_2 the positive zero and $X_3 = 0$. Note that

$$\begin{aligned} X_2 &= \sqrt{1+\frac{6}{4n^2-1}} \\ &\leq 1+\sqrt{\frac{6}{4n^2-1}} \end{aligned}$$

and similarly

$$X_1 \geq -1-\sqrt{\frac{6}{4n^2-1}}.$$

This implies that $X_1 < -1$ and $X_2 > 1$..
We can also find the local extrema of $j(x)$, by setting $j'(x) = 0$.

$$\begin{aligned} j'(x) &= 0 \\ 3x^2(1-4n^2)+(4n^2+5) &= 0 \\ x &= \pm\sqrt{\frac{4n^2+5}{12n^2-3}}. \end{aligned}$$

Let x_1 and x_2 be the negative and positive extrema respectively. These extreme values are proportional to the zeros of $j(x)$ with a factor of $\sqrt{3}$, i.e. $X_{1,2} = \sqrt{3}x_{1,2}$.

Theorem 5.1

The zeros of $T_n(x)$ *are convex on* $(0,1)$*, and concave on* $(-1,0)$*. In addition*

$$\Delta x_k < \frac{\pi}{\sqrt{F(X_1)}} = \frac{\pi}{\sqrt{F(X_2)}}$$

where X_1 *and* X_2 *are defined as in (5.1.1).*

Proof:

For $x < x_1$ and $x > x_2$, $j(x)$ is increasing and for $x_1 < x < x_2$ it is decreasing. This means that x_1 is a local maximum of $j(x)$ and x_2 is a local minimum. Also, considering the interval of orthogonality of $j(x)$, we find that $j(x)$ is positive on $(-1,0)$ and negative on $(0,1)$.

Now, by the definition of $F'(x)$, $F'(x)$ is positive on $(-1,0)$ and negative on $(0,1)$ for $x \in (-1,1)$. This means that $F(x)$ is increasing on $(-1,0)$ and decreasing on $(0,1)$.

By Theorem 2.1, the zeros of $T_n(x)$ are convex on $(0,1)$, and concave on $(-1,0)$.

We can also find a bound for the difference between consecutive zeros of $T_n(x)$. X_1, X_2 and X_3 are possible local extreme values of $F(x)$. Since

$$\lim_{x \to -1^+} F(x) = \infty \quad \text{and} \quad \lim_{x \to 1^-} F(x) = \infty.$$

X_1 and X_2 must be local minimum values of $F(x)$ for $x \in (-1,1)$, i.e. $F(x) > F(X_1) = F(X_2)$.

$$\begin{aligned} F(X_2) &= \frac{[X_2]^2(1-4n^2)+2(2n^2+1)}{4(1-[X_2]^2)^2} \\ &= \frac{(4n^2+5)+2(2n^2+1)}{[4(1-[X_2]^2)^2]}. \end{aligned}$$

The following holds true for all values of n

$$[4(1-[X_2]^2)^2] > 0$$

and

$$(4n^2+5)+2(2n^2+1) > 0.$$

Hence, $F(X_2) > 0$ for all values of n. Also note that in the expression for $F(x)$ we only have x^2 and X_1, X_2 are symmetric about the x-axis, therefore $F(X_1) = F(X_2)$.

It follows by Theorem 2.1(4) that

$$\Delta x_k < \frac{\pi}{\sqrt{F(X_1)}}.$$

■

5.2 Chebychev polynomials of the second kind

Chebychev polynomials of the second kind $y(x) = U_n(x)$, as defined in (2.3.11), satisfy a differential equation (cf. [11, p.225, eqn.(9.8.44)])

$$(1-x^2)y''(x) - 3xy'(x) + n(n+2)y(x) = 0.$$

In order to transform this differential equation to the normal form (1.0.1) we first need to calculate $F(x)$.

$$\begin{aligned} F(x) &= f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x) \\ &= \frac{n(n+2)}{1-x^2} - \frac{1}{4}\frac{9x^2}{(1-x^2)^2} + \frac{1}{2}\frac{3x^3+3}{(1-x^2)^2} \\ &= \frac{-x^2(4n(n+2)+3) + (4n(n+2)+6)}{4(1-x^2)^2}. \end{aligned}$$

We can now express the differential equation in the following way

$$y''(x) + \frac{-x^2(4n(n+2)+3) + (4n(n+2)+6)}{4(1-x^2)^2}y(x) = 0.$$

We find that

$$\begin{aligned} & F'(x) \\ = & \frac{-8x(4n(n+2)+3)(1-x^2)^2}{16(1-x^2)^4} \\ & -\frac{16x(1-x^2)[x^2(4n(n+2)+3) - (4n(n+2)+6)]}{16(1-x^2)^4} \\ = & \frac{-x^3(4n(n+2)+3) + x[(4n(n+2)+9]}{2(1-x^2)^3}. \end{aligned}$$

Now, in order to find the local extrema of $F(x)$ we find the values of x which satisfy $F'(x) = 0$.

$$\begin{aligned} F'(x) &= 0 \\ j(x) := -x^3(4n(n+2)+3) + x[4n(n+2)+9] &= 0 \\ x = \pm\sqrt{\frac{(4n(n+2)+9)}{(4n(n+2)+3)}} \quad &\text{or} \quad x = 0. \end{aligned}$$

These are the zeros of $j(x)$ where we let X_1 be the negative zero, X_2 the positive zero and $X_3 = 0$. Note that only X_3 lies in the interval of orthogonality $(-1, 1)$, since

$$\begin{aligned} X_1^2 = X_2^2 &= \frac{4n(n+2)+9}{4n(n+2)+3} \\ &> 1 \quad \text{for all values of } n. \end{aligned}$$

which implies that $X_1 < -1$ and $X_2 > 1$.

$$\begin{aligned} j'(x) &= 0 \\ -3x^2(4n(n+2)+3) + (4n(n+2)+9) &= 0 \\ x &= \pm\sqrt{\frac{4n(n+2)+9}{3(4n(n+2)+3)}}. \end{aligned}$$

Let x_1 be the negative value and x_2 be the positive value, then x_1 and x_2 are the local extrema of $j(x)$.

Theorem 5.2
The zeros of $U_n(x)$ are concave on $(0, 1)$, and convex on $(-1, 0)$. In addition,

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}} = \frac{\pi}{\sqrt{n(n+2)+6}}.$$

Proof:

The leading coefficient of $j'(x)$ is negative, which means that $j'(x) < 0$ on $(-1, x_1) \bigcup (x_2, 1)$ and positive on (x_1, x_2). Therefore, $j(x)$ is decreasing on $(-1, x_1) \bigcup (x_2, 1)$ and increasing on (x_1, x_2). Furthermore, x_1 is a local minimum value of $j(x)$ and x_2 is a local maximum value.

Considering the zero $X_3 = 0$ of $j(x)$, we can now also see that $j(x)$ is positive on $(0, 1)$ and negative on $(-1, 0)$, which implies that $F(x)$ is increasing on $(0, 1)$ and decreasing on $(-1, 0)$.
Hence, by Theorem 2.1, the zeros of $U_n(x)$ are concave on $(0, 1)$ and convex on $(-1, 0)$.

A possible extreme value of $F(x)$ is $x = 0$. If we consider the limits

$$\lim_{x \to -1^+} F(x) = \infty \quad \text{and} \quad \lim_{x \to 1^-} F(x) = \infty$$

and note the intervals of increase and decrease of $F(x)$ it follows that $X_3 = 0$ is a minimum value of $F(x)$ which further implies that $F(x) < F(0)$. Now, to be able to find bounds for the distances between consecutive zeros, i.e. applying Theorem 2.1(3) and (4), we need to establish whether $F(0) > 0$.

$$F(0) = n(n+2) + 6 > 0$$

for all n. Hence, by Theorem 2.1(3),

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}} = \frac{\pi}{\sqrt{n(n+2)+6)}}.$$

■

5.3 Bessel polynomials

Bessel polynomials $y(x) = y_n(a;x)$, as defined in (2.3.3), satisfy the second-order differential equation ([11, p.245, eqn.(9.13.5)])

$$x^2 y''(x) + [(a+2)x + 2]y'(x) - n(n+a+1)y(x) = 0. \qquad (5.3.2)$$

These polynomials are related to the Bessel functions (see Section 3.1) in the following way (cf. [22])

$$\begin{aligned}
y_n\left(\frac{1}{ir}\right) &= \left(\frac{1}{2}\pi r\right)^{\frac{1}{2}} e^{ir}\left[i^{-n-1}J_{n+\frac{1}{2}}(r) + i^n J_{-n-\frac{1}{2}(r)}\right], \\
J_{n+\frac{1}{2}}(r) &= (2\pi r)^{-\frac{1}{2}}\left[i^{-n-1}e^{ir}y_n\left(\frac{-1}{ir}\right) + i^{n+1}e^{-ir}y_n\left(\frac{1}{ir}\right)\right], \\
J_{-n-\frac{1}{2}}(r) &= (2\pi r)^{-\frac{1}{2}}\left[i^n e^{ir}y_n\left(\frac{-1}{ir}\right) + i^{-n}e^{-ir}y_n\left(\frac{1}{ir}\right)\right].
\end{aligned}$$

Theorem 5.3
Let $a < -2N - 1$, then the zeros of Bessel polynomials $y_n(a;x)$ are concave on the interval $(0, x_1)\bigcup(x_2, \infty)$ and convex on the interval (x_1, x_2), where x_2 is defined as in (5.3.3).

Proof:

The differential equation (5.3.2) can be transformed into the normal form (1.0.1) by first calculating $F(x)$ as it is defined in (2.1.2).

$$\begin{aligned} F(x) &= f(x) - \frac{1}{4}g^2(x) - \frac{1}{2}g'(x) \\ &= \frac{-n(n+a+1)}{x^2} - \frac{(a+2)^2x^2 + 4x(a+2) + 4}{4x^4} \\ &\quad -\frac{(a+2)x^2 - 2x((a+2)x+2)}{2x^4} \\ &= \frac{x^2(-4n^2 - 4na - 4n - a^2 - 2a) + x(-4a) - 4}{4x^4}. \end{aligned}$$

we then have

$$\begin{aligned} F'(x) &= \frac{x^2(-4n^2 - 4na - 4n - a^2 - 2a) - 2ax}{2x^5} \\ &\quad -\frac{2[x^2(-4n^2 - 4na - 4n - a^2 - 2a) - 4ax - 4]}{2x^5} \\ &= \frac{j(x)}{2x^5}. \end{aligned}$$

We now determine the zeros and the intervals of increase and decrease of $j(x)$.

$$\begin{aligned} 0 &= j(x) \\ 0 &= x^2(-4n^2 - 4na - 4n - a^2 - 2a) + 6ax \\ &\quad -2x^2(-4n^2 - 4na - 4n - a^2 - 2a) + 8 \\ 0 &= x^2(4n^2 + 4na + 4n + a^2 + 2a) + 6ax + 8 \\ x &= \frac{-3a \pm \sqrt{9a^2 - 8(4n^2 + 4na + 4n + a^2 + 2a)}}{(4n^2 + 4na + 4n + a^2 + 2a)} \end{aligned} \tag{5.3.3}$$

Since x is real we must have that $9a^2 - 8(4n^2 + 4na + 4n + a^2 + 2a) > 0$, but this is true for all values of n since $a < -2N - 1$. Both zeros of $j(x)$ are positive, so let $x_1 < x_2$ be the two zeros. Also,

$$\begin{aligned} j'(x) &= 0 \\ 0 &= 2x(4n^2 + 4na + 4n + a^2 + 2a) + 6a \\ x &= \frac{-3a}{(4n^2 + 4na + 4n + a^2 + 2a)} \end{aligned}$$

Let X_1 be the turning point of $j(x)$. Note that the leading coefficient of $j(x)$ is positive, therefore X_1 will be a minimum value and $j(x)$ will be positive on the interval $(0, x_1) \bigcup (x_2, \infty)$ and negative on (x_1, x_2), since the zeros of

$y_n(a;x)$ lie in $(0,\infty)$. This implies that $F(x)$ is increasing on $(0,x_1)\bigcup(x_2,\infty)$ and decreasing on (x_1,x_2). Hence, by Theorem 2.1, the zeros of $y_n(a;x)$ are convex on (x_1,x_2) and concave on $(0,x_1)\bigcup(x_2,\infty)$.

Possible extreme values of $F(x)$ are at $x = x_{1,2}$ and by the intervals of increase and decrease we find that x_1 is a maximum value and x_2 is a minimum value of $F(x)$. This implies that $F(x_2) < F(x) < F(x_1)$ for all $x \in (0,\infty)$.

$$F(x_2) \quad = \quad \frac{x_2^2(-4n^2-4na-4n-a^2-2a)+x_2(-4a)-4}{4x_2^4}$$

$F(x_2)$, and similarly also $F(x_1)$, will always be negative for all the possible values of a and n, therefore Theorem 2.1(3) and (4) are not applicable and we are not able to place any bounds on the distances between consecutive zeros of $y_n(a;x)$.

■

5.4 Pseudo Jacobi polynomials

Pseudo Jacobi polynomials $y(x) = P_n(x;v,N)$, as defined in (2.3.12), satisfy the second-order differential equation ([11, p.252, eqn.(9.9.5)])

$$(1+x^2)y''(x)+2(v-Nx)y'(x)-n(n-2N-1)y(x)=0$$

which can be transformed into the normal form

$$y''+F(x)y=0.$$

First we need to calculate $F(x)$.

$$\begin{aligned} F(x) &= f(x)-\frac{1}{4}g^2(x)-\frac{1}{2}g'(x) \\ &= \frac{n(1+2N-n)}{1+x^2}+\frac{-(v-Nx)^2+N(1+x^2)+2x(v-Nx)}{(1+x^2)^2} \\ &= \frac{x^2(-n^2+2nN+n-N^2-N)+x(2vN+2v)}{(1+x^2)^2} \\ &\quad +\frac{(-n^2+2nN+n-v^2+N)}{(1+x^2)^2} \end{aligned}$$

then

$$\begin{aligned} F'(x) &= \frac{[2x(-n^2+2nN+n-N^2-N)+(2vN+2v)](1+x^2)}{(1+x^2)^3} \\ &\quad -\frac{4x[x^2(-n^2+2nN+n-N^2-N)+x(2vN+2v)]}{(1+x^2)^3} \\ &\quad +\frac{4x[(N-n^2+2nN+n-v^2)]}{(1+x^2)^3} \\ &= \frac{j(x)}{(1+x^2)^3}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} j'(x) &= -6x^2(n-n^2+2nN-N^2-N)-6x(2vN+2v) \\ &\quad +2(-N^2-2nN-3N+n^2-n+2v^2). \end{aligned}$$

To determine the intervals of convexity we first need to find the values of x where $j'(x)=0$, which would also be the turning points of $F'(x)$.

$$\begin{aligned} j'(x) &= 0 \\ 0 &= -6x^2(-n^2+2nN+n-N^2-N)-6x(2vN+2v) \\ &\quad +2(-N^2-2nN-3N+n^2-n+2v^2) \end{aligned}$$

$$\begin{aligned} x &= \frac{-3(2vN+2v)}{6(-n^2+2nN+n-N^2-N)} \\ &\pm\frac{\sqrt{9(2vN+2v)^2+12(n-n^2+2nN-N^2-N)(n^2-N^2-2nN-3N-n+2v^2)}}{6(-n^2+2nN+n-N^2-N)}. \end{aligned}$$

Let x_1 be the negative value and x_2 the positive value. These values are real as long as

$$9(2vN+2v)^2-12(-n^2+2nN+n-N^2-N)(-N^2-2nN-3N+n^2-n+2v^2)>0$$

which will only hold true for certain values of n and v. To calculate the zeros of $j(x)$ we need to calculate the following

$$\begin{aligned} 0 &= j(x) \\ 0 &= -2x^3(-n^2+2nN+n-N^2-N)-3x^2(2vN+2v) \\ &\quad +2x(-N^2-2nN-3N+n^2-n+2v^2)+(2vN+2v). \end{aligned}$$

This cubic equation is not easily solvable, but making use of a suitable software program it does yield three zeros $X_{1,2,3}$ as solutions. If we assign specific values to the variables v and N we should be able to determine in which of the intervals $[-\infty, X_1]$, $[X_1, X_2]$, $[X_2, X_3]$ and $[X_3, \infty]$ $j(x)$ is positive or negative.

Whenever $j(x)$ is positive, $F'(x)$ will also be positive and $F(x)$ will be increasing, similarly, when $j(x)$ is negative $F(x)$ will be decreasing. According to Theorem 2.1(1) and (2), this implies that the zeros of $P_n(x; v, N)$ are concave when $F(x)$ is increasing and convex when $F(x)$ is decreasing.

Considering the remaining results of Theorem 2.1, we might also be able to place bounds on the distances of consecutive zeros of $P_n(x; v, N)$ if we are able to determine whether $F(X_1)$, $F(X_2)$ and $F(X_3)$ are positive extreme values of $F(x)$.

5.5 Dual Hahn polynomials

Dual Hahn polynomials $y(x) = R_n(\lambda(x), \gamma, \delta, N)$, as defined in (2.3.13), satisfy the following difference equation (cf. [11, p.209, eqn.(9.6.5)])

$$-ny(x) = B(x)y(x+1) - [B(x) + D(x)]y(x) + D(x)y(x-1) \qquad (5.5.4)$$

where

$$B(x) = \frac{(x+\gamma+1)(x+\gamma+\delta+1)(N-x)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}$$

and

$$D(x) = \frac{x(x+\gamma+\delta+N+1)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}.$$

Since we are dealing with a difference equation, we will make use of the methods in Chapter 4 to examine the convexity of the zeros of $R_n(\lambda(x), \gamma, \delta, N)$.

Theorem 5.4
Let $\gamma + \delta > 1$ and $\gamma > -1, \delta > -1$. The zeros of the Dual Hahn polynomials $R_n(\lambda(x), \gamma, \delta, N)$ are quasi-convex or quasi-concave on certain intervals depending on the parameter values as follows

1. *If $x_2 > \bar{x}_2$, $\gamma \leq \delta$ and $N > 2$ then the zeros are quasi-concave on $(\bar{x}_2, x_2)$.*

2. *If* $x_2 < \bar{x}_2$*,* $\gamma \leq \delta$ *and* $N > 2$ *then the zeros are quasi-concave on* $(x_2, \bar{x}_2)$*.*

3. *If* $x_2 > x_2^{\circ}$*,* $\gamma < \frac{N}{8}$ *and* $\gamma \geq \delta$ *then the zeros are quasi-convex on* (x_2°, x_2)*.*

4. *If* $x_2 < x_2^{*}$*,* $\gamma < \frac{N}{8}$ *and* $\gamma \geq \delta$ *then the zeros are quasi-convex on* (x_2^{*}, x_2)*.*

where x_2*,* $\bar{x}_2$*,* x_2^* *and* x_2° *are defined as in (5.5.5), (5.5.7), (5.5.8) and (5.5.10) respectively.*

Proof:
We multiply both sides of the difference equation (5.5.4) by

$$
\begin{aligned}
& \prod_{s=0}^{x-1} \frac{B(s)}{D(s+1)} \\
= \; & \prod_{s=0}^{x-1} \frac{(s+\gamma+1)(s+\gamma+\delta+1)(N-s)}{(2s+\gamma+\delta+1)(2s+\gamma+\delta+2)} \\
& / \left[\frac{(s+1)(s+\gamma+\delta+N+2)(s+\delta+1)}{(2s+\gamma+\delta+2)(2s+\gamma+\delta+3)} \right] \\
= \; & \frac{(\gamma+1)_x(\gamma+\delta+1)_x(-1)^x(-N)_x}{2^x(\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2})_x 2^x(\frac{1}{2}\gamma+\frac{1}{2}\delta+1)_x} / \left[\frac{x!(\gamma+\delta+N+2)_x(\delta+1)_x}{2^x(\frac{1}{2}\gamma+\frac{1}{2}\delta+1)_x 2^x(\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{3}{2})_x} \right] \\
= \; & \frac{(\gamma+\delta+1)_x(-1)^x(-N)_x(\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{3}{2})_x(\gamma+1)_x}{x!(\gamma+\delta+N+2)_x(\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2})_x(\delta+1)_x} \\
= \; & \frac{(\gamma+\delta+1)_x(-1)^x(-N)_x(\gamma+\delta+2x+1)(\gamma+1)_x}{x!(\gamma+\delta+N+2)_x(\gamma+\delta+1)(\delta+1)_x}.
\end{aligned}
$$

This gives the difference equation (5.5.4) in self-adjoint form (2.5.16), with

$$
\begin{aligned}
p(x) &= B(x) \left[\frac{(\gamma+\delta+1)_x(-1)^x(-N)_x(\gamma+\delta+2x+1)(\gamma+1)_x}{x!(\gamma+\delta+N+2)_x(\gamma+\delta+1)(\delta+1)_x} \right] \\
&= \frac{(\gamma+\delta+2)_x(-1)^x(-N)_x(N-x)(x+\gamma+1)(\gamma+1)_x}{x!(\gamma+\delta+N+2)_x(\delta+1)_x(2x+\gamma+\delta+2)}
\end{aligned}
$$

and

$$
q(x) = \frac{n(\gamma+\delta+1)_x(-1)^x(-N)_x(\gamma+\delta+2x+1)(\gamma+1)_x}{x!(\gamma+\delta+N+2)_x(\gamma+\delta+1)(\delta+1)_x}.
$$

To apply Corollary 4.4, we first have to determine where $p(x)$ and $q(x)$ are monotone increasing and decreasing. Assume that $\gamma + \delta > 0$.

$$
p(x) - p(x-1) = \frac{B(0)\ldots B(x)}{D(1)\ldots D(x)} - \frac{B(0)\ldots B(x-1)}{D(1)\ldots D(x-1)}
$$

$$= \frac{B(0)\dots B(x-1)[B(x)-D(x)]}{D(1)\dots D(x)}.$$

Note that $B(x) > 0$ and $D(x) > 0$ by our assumptions. This implies that $p(x) - p(x-1)$ is positive if and only if $B(x) - D(x)$ is positive.

$$\begin{aligned} & B(x) - D(x) \\ = & \frac{(x+\gamma+1)(x+\gamma+\delta+1)(N-x)(2x+\gamma+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \\ - & \frac{x(x+\gamma+\delta+N+1)(x+\delta)(2x+\gamma+\delta+2)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)} \end{aligned}$$

which is positive (negative) if and only if the numerator is positive (negative) since the denominator is always positive. Hence, $B(x)-D(x)$ will be positive if and only if

$$\begin{aligned} & (x+\gamma+1)(x+\gamma+\delta+1)(N-x)(2x+\gamma+\delta) \\ & -x(x+\gamma+\delta+N+1)(x+\delta)(2x+\gamma+\delta+2) \end{aligned}$$

is positive, which implies that

$$\begin{aligned} & [2x^2 + x(\delta+3\gamma+2) + (\gamma\delta+\gamma^2+\gamma+\delta)][-x^2 + x(N-\gamma-\delta-1)] \\ & +[2x^2 + x(\delta+3\gamma+2) + (\gamma\delta+\gamma^2+\gamma+\delta)][(\gamma N+\delta N+N)] \\ & -[2x^2 + x(\gamma+3\delta+2) + (\gamma\delta+\delta^2+2\delta)][x^2 + x(\gamma+\delta+N+1)] \end{aligned}$$

must be positive and if we further assume that $\gamma \geq \delta$ then

$$2x^2 + 2x(\gamma+\delta+1) - (\gamma N + \delta N + N) < 0$$

which is true whenever

$$x_1 < x < x_2$$

where

$$x_{1,2} = \frac{-(\gamma+\delta+1) \pm \sqrt{(\gamma+\delta+1)(\gamma+\delta+1+2N)}}{2}. \tag{5.5.5}$$

Hence, $p(x)$ is monotone increasing when $0 < x < x_2$ and $\gamma \geq \delta$, since $x = 0, 1, \dots, N$ and $x_1 < 0$. Similarly, $p(x)$ is monotone decreasing when $x > x_2 > 0$ and $\gamma \leq \delta$.

Now, we have to determine when $q(x)$ is increasing and decreasing.

$$\begin{aligned} q(x) - q(x-1) &= n\frac{B(0)\dots B(x-1)}{D(1)\dots D(x)} - n\frac{B(0)\dots B(x-2)}{D(1)\dots D(x-1)} \\ &= \frac{n[B(0)\dots B(x-2)][B(x-1)-D(x)]}{D(1)\dots D(x)} \end{aligned}$$

which is positive if and only if $B(x-1) - D(x)$ is positive.

$$\begin{aligned} & B(x-1) - D(x) \\ = & \frac{(x+\gamma)(x+\gamma+\delta)(N-x+1)(2x+\gamma+\delta+1)}{(2x+\gamma+\delta-1)(2x+\gamma+\delta)(2x+\gamma+\delta+1)} \\ & - \frac{x(x+\gamma+\delta+N+1)(x+\delta)(2x+\gamma+\delta-1)}{(2x+\gamma+\delta-1)(2x+\gamma+\delta)(2x+\gamma+\delta+1)} \end{aligned}$$

and noting that the denominator is positive for $\gamma + \delta > 1$, the fraction is positive if and only if

$$\begin{aligned} & (x+\gamma)(x+\gamma+\delta)(N-x+1)(2x+\gamma+\delta+1) \\ & -x(x+\gamma+\delta+N+1)(x+\delta)(2x+\gamma+\delta-1) > 0 \\ & (x+\gamma)(x+\gamma+\delta)(N-x+1) - x(x+\delta)(x+\gamma+\delta+N+1) > 0 \\ & (x+\gamma)(N-x+1) - x(x+\delta)(2N) > 0 \quad \text{for } N \geq 2 \end{aligned} \tag{5.5.6}$$

since $(2x+\gamma+\delta+1) > (2x+\gamma+\delta-1) > 0$. A further calculation shows that (5.5.6) is true if and only if $0 < x < \bar{x}_2$ where

$$\begin{aligned} & \bar{x}_{1,2} \\ = & \frac{(1+N-\gamma+2\delta N) \pm \sqrt{(1+N-\gamma+2\delta N)^2 + 4\gamma(1+2N)(1+N)}}{2(1+2N)} \end{aligned} \tag{5.5.7}$$

with $\bar{x}_1 < \bar{x}_2$ and $\bar{x}_1 < 0$. Hence $q(x)$ is monotone increasing when $0 < x < \bar{x}_2$. Similarly, $q(x)$ is monotone decreasing when $x > x_2^*$ where

$$x_{1,2}^* = \frac{(1+N-\gamma-\delta) \pm \sqrt{(\gamma+\delta-N-1)^2 + 8\gamma(N+1)}}{4} \tag{5.5.8}$$

and we also have the condition that

$$(x+\gamma+\delta)(N-x+1) < 0 \quad \text{and} \quad x > 0 \text{ (i.e. } x \geq 1) \tag{5.5.9}$$

which is true if and only if

$$x < x_1^\circ \text{ and } x > x_2^\circ$$

where

$$\begin{aligned} x_{1,2}^\circ = & -\frac{1}{2}[(\gamma+\delta-N-1) \\ & \pm\sqrt{(N+1-\gamma-\delta)^2 + 4(N\gamma+N\delta+\gamma+\delta)}] \end{aligned} \tag{5.5.10}$$

and $x_1^\circ < x_2^\circ$, which further implies that $q(x)$ is monotone decreasing when $x > x_2^*$ and $x > x_2^\circ$. Note that $x_2^* > 1$ and $x_2^\circ > 1$, so that (5.5.9) holds. Furthermore, we also have to assume that $\gamma < \frac{N}{8}$ to ensure that $\bar{x}_2$ and x_2^* are real numbers.

We now have to determine when $x_2^* > x_2^\circ$ and when $x_2^* < x_2^\circ$. We assume that $\gamma \geq \delta$, then we find that $x_2^* < x_2^\circ$ for $\gamma < 0$ and $x_2^* > x_2^\circ$ for $\gamma > 0$. Hence, $q(x)$ is monotone decreasing for $x > x_2^\circ$ when $\gamma < 0$ and monotone decreasing for $x > x_2^*$ when $\gamma > 0$.

Now, if we can find intervals where either $q(x)$ is decreasing and $p(x)$ is increasing, or where $q(x)$ is increasing and $p(x)$ is decreasing, then we will be able to determine intervals where the zeros of the Dual Hahn polynomials are quasi-convex or quasi-concave. To find these intervals we have to determine when $x_2 < x_2^*$, $x_2^* < x_2$, $x_2 < \bar{x}_2$, $x_2 > \bar{x}_2$, $x_2 < x_2^\circ$ and $x_2 > x_2^\circ$. We consider these six cases and the possible results that follow.

Case 1: $x_2 > \bar{x}_2$:
The zeros of $R_n(\lambda(x))$ are quasi-concave on $[\bar{x}_2, x_2]$, for $\gamma \leq \delta$ and $N > 2$.

Case 2: $x_2 < \bar{x}_2$:
The zeros of $R_n(\lambda(x))$ are quasi-concave on $[x_2, \bar{x}_2]$, for $\gamma \leq \delta$ and $N > 2$.

Case 3: $x_2 > x_2^\circ$:
The zeros of $R_n(\lambda(x))$ are quasi-convex on $[x_2^\circ, x_2]$, for $\gamma \geq \delta$ and $g < 0$.

Case 4: $x_2 < x_2^\circ$:
This yields no results as there is no interval in which $q(x)$ is decreasing and $p(x)$ is increasing.

Case 5: $x_2 > x_2^*$:.
This yields no results as there is no interval in which $q(x)$ is decreasing and $p(x)$ is increasing.

Case 6: $x_2 < x_2^*$:
The zeros of $R_n(\lambda(x))$ are quasi-convex on (x_2^*, x_2), for $\gamma \geq \delta$ and $\gamma < \frac{N}{8}$.

■

In this chapter we have applied the theorems from Chapter 2 and 4 to obtain some results on convexity of the zeros of the Chebychev, Bessel, Pseudo Jacobi and Dual Hahn polynomials. We have also seen that in some cases it is more complicated to find exact values for the intervals of convexity and that we might only be able to place bounds on the distances between consecutive zeros for certain parameter values. In the next chapter we will consider the Charlier and Krawtchouk polynomials, both satisfying difference equations, as well as further possible extensions of Sturm's comparison theorem.

Chapter 6

Other classes of orthogonal polynomials and further extensions

In the previous chapters we have discussed and proved various results on both convexity and distances between consecutive zeros of several classes of orthogonal polynomials as well as the Bessel function. Considering all the classes of hypergeometric polynomials, as listed in Section 2.3, there are still a few classes remaining which we haven't considered. These are the Charlier, Krawtchouk, Wilson, Racah, Continuous Hahn, Continuous Dual Hahn and Meixner-Pollaczek polynomials. We will now discuss these classes.

The Charlier and Krawtchouk polynomials satisfy real difference equations, whereas the Wilson, Racah, Continuous Dual Hahn, Continuous Hahn and Meixner-Pollaczek polynomials satisfy complex difference equations.

Applying the methods in Chapter 4 to the Charlier polynomials, we find that the existing theorems don't produce any results.

6.1 Charlier polynomials

Charlier polynomials $y(x) = C_n(x; a)$, as defined in (2.3.14), satisfy the following difference equation ([11, p.248, eqn.(9.14.5)])

$$-ny(x) = ay(x+1) - (x+a)y(x) + xy(x-1). \qquad (6.1.1)$$

First we write the difference equation (6.1.1) in self-adjoint form. If we multiply both sides of the equation with

$$\begin{aligned} & \prod_{s=0}^{x-1} \frac{B(s)}{D(s+1)} \\ = & \frac{a^x}{x!} \end{aligned}$$

then

$$\begin{aligned} \frac{a^{x+1}}{x!}y(x+1) - \frac{(x+a)a^x}{x!}y(x) + \frac{a^x}{(x-1)!}y(x-1) + \frac{a^x n}{x!}y(x) &= 0 \\ \frac{a^{x+1}}{x!}y(x+1) - \left[-\frac{a^x n}{x!} + \frac{a^x}{(x-1)!} + \frac{a^{x+1}}{x!} \right] y(x) + \frac{a^x}{(x-1)!}y(x-1) &= 0 \\ \frac{a^{x+1}}{x!}y(x+1) - \left[\frac{a^{x+1}}{x!} + \frac{a^x}{(x-1)!} \right] y(x) + \frac{a^x}{(x-1)!}y(x-1) + \frac{a^x n}{x!}y(x) &= 0. \end{aligned}$$

Now we have

$$p(x) = \frac{a^{x+1}}{x!} \quad \text{and} \quad q(x) = \frac{a^x n}{x!}.$$

Hence we can express equation (6.1.1) in self-adjoint form

$$\Delta \left[\frac{a^x}{(x-1)!} \Delta y(x-1) \right] + \frac{a^x n}{x!} y(x) = 0.$$

We can further see that $p(x)$ is monotone increasing for $a > x$ and monotone decreasing for $a < x$, since

$$\begin{aligned} a &> x \\ a - x &> 0 \\ \frac{a^x(a-x)}{x!} &> 0 \\ \frac{a^{x+1} - xa^x}{x!} &> 0 \\ p(x) - p(x-1) &> 0. \end{aligned}$$

Similarly, $q(x)$ is monotone increasing for $a > x$ and monotone decreasing for $a < x$, since

$$\begin{aligned} a &> x \\ a - x &> 0 \\ 1 - a^{-1}x &> 0 \quad \text{since } a > 0 \end{aligned}$$

$$\begin{aligned}
\frac{a^x n(1-a^{-1}x)}{x!} &> 0 \\
\frac{a^x n}{x!} - \frac{a^{x-1}n}{(x-1)!} &> 0 \\
q(x) - q(x-1) &> 0.
\end{aligned}$$

This implies that for all values of x, $p(x)$ and $q(x)$ have the same monotonicity, and therefore we cannot apply any of the theorems that we have for solutions of differential equations. Another possibility is to make use of the limit relation between the Meixner and Charlier polynomials, as we did in finding the results for the Meixner polynomials, but this method also yields no results.

6.2 Krawtchouk polynomials

Krawtchouk polynomials $y(x) = K_n(x; p, N)$, as defined in (2.3.15), satisfy the following difference equation (cf. [11, p.238, eqn.(9.11.5)])

$$-ny(x) = p(N-x)y(x+1) - [p(N-x) + x(1-p)]y(x) + x(1-p)y(x-1).$$

Let $B(x) = p(N-x)$ and $D(x) = x(1-p)$. In order to express this equation in the normal form (2.0.2) for difference equations, we multiply both sides by

$$\begin{aligned}
&\prod_{s=0}^{x-1} \frac{B(s)}{D(s+1)} \\
=& \prod_{s=0}^{x-1} \frac{p(N-s)}{(s+1)(1-p)} \\
=& \frac{p^x(-1)^x(-N)_x}{x!(1-p)^x}
\end{aligned}$$

then

$$\begin{aligned}
0 =& \frac{np^x(-1)^x(-N)_x}{x!(1-p)^x} y(x) + \frac{p^{x+1}(-1)^{x+1}(-N)_{x+1}}{x!(1-p)^x} y(x+1) \\
&- \left[\frac{p^{x+1}(-1)^{x+1}(-N)_{x+1}}{x!(1-p)^x} + \frac{p^x(-1)^x(-N)_x}{(x-1)!(1-p)^{x-1}} \right] y(x) + \frac{p^x(-1)^x(-N)_x}{(x-1)!(1-p)^{x-1}}.
\end{aligned}$$

Now the difference equation is in self-adjoint form (2.0.2) with

$$p(x) = \frac{p^{x+1}(-1)^{x+1}(-N)_{x+1}}{x!(1-p)^x}$$

and

$$q(x) = \frac{np^x(-1)^x(-N)_x}{x!(1-p)^x}.$$

We find that $p(x)$ is monotone increasing for $x < pN$ by considering

$$\begin{aligned} & x^2(1-p) + x(Np + p - N - 1) + p \\ = \quad & p - x(1-p)(N - x + 1). \end{aligned}$$

which will be positive for $x < pN$.
This implies that

$$\frac{p^x(-1)^{x+1}(-N)_{x+1}[p - x(1-p)(N-x+1)]}{x!(1-p)^x} > 0$$

since $x!$, $p^x(-1)^{x+1}(-N)_{x+1}$ and $(1-p)^x$ are all positive because we know that $0 < p < 1$. Hence

$$p(x) - p(x-1) \quad = \quad \frac{p^{x+1}(-1)^{x+1}(-N)_{x+1}}{x!(1-p)^x} - \frac{p^x(-1)^x(-N)_x}{(x-1)!(1-p)^{x-1}} > 0$$

for $x < pN$ which implies that $p(x)$ is monotone increasing for $x < pN$. Similarly, $q(x)$ is also monotone increasing for $x < p(N+1)$. Consider the equation

$$\begin{aligned} & x^2(1-p) + x(Np + 2p - N - 2) + p \\ = \quad & p - x(1-p)(N - x + 2) \end{aligned}$$

which is positive for $x < p(N+1)$. This further implies that

$$\frac{np^{x-1}(-1)^x(-N)_x[p - x(1-p)(N-x+2)]}{x!(1-p)^x} > 0$$

since n, $x!$, p^{x-1}, $(-1)^x(-N)_x$ and $(1-p)^x$ are all positive. Hence

$$q(x) - q(x-1) = \frac{np^x(-1)^x(-N)_x}{x!(1-p)^x} - \frac{np^{x-1}(-1)^{x-1}(-N)_{x-1}}{(x-1)!(1-p)^{x-1}} > 0$$

which implies that $q(x)$ is monotone in increasing for $x < p(N+1)$.
Now since $0 < p < 1$ $p(x)$ and $q(x)$ always have the same monotonicity and therefore the theorems that we have in Chapter 4 are not applicable.

6.3 Further extensions

It was mentioned in Chapter 4 that we can also obtain results on the q-convexity of zeros of the solutions of q-difference equations. To(ó)kos and Gishe [4] considered and found results for the q-Laguerre polynomials, but it may be possible to also obtain results for other q-orthogonal polynomials.

Furthermore, we cannot yet obtain any results on the convexity of the zeros of Wilson, Racah, Continuous Dual Hahn, Continuous Hahn and Meixner-Pollaczek polynomials which satisfy complex difference equations since the theorems that are currently available only apply to real difference equations. Thus, whether Sturm's theorems can be applied to complex difference equations in a similar manner as was done in Chapter 4, is a question for further research.

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