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Zeros of Jacobi, Meixner and Krawtchouk polynomials

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Zeros of Jacobi, Meixner and Krawtchouk Polynomials

by

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Submitted in partial fulfillment of the requirements
for the degree

Philosophiae Doctor

in the Department of Mathematics and Applied Mathematics
in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

September 2012

DECLARATION

I, the undersigned, declare that the thesis which I hereby submit for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not previously been submitted by me for any degree at this or any other tertiary institution.

Signature:

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Date: 3 September 2012

Abstract

A sequence of polynomials $\{P_n\}_{n=0}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, where P_n is of exact degree n, is orthogonal on the (finite or infinite) interval $[a, b]$ with respect to the weight function $\rho(x)$ if $\int_a^b x^k P_n(x)\rho(x)dx = 0$ for $k = 0, 1, \ldots, n-1$ and it is discrete orthogonal if $\sum_{i=0}^{M} (x_i)^k P_n(x_i)\rho_i = 0$, $k = 0, 1, \ldots, n-1$, where ρ_i are the values of the weight at the distinct points $x_i, i = 0, 1, 2, \ldots, M$, $M \in \mathbb{N} \cup \{\infty\}$. We study the zeros of ${}_2F_1$ hypergeometric polynomials, in particular the continuous orthogonal Jacobi polynomials and the discrete orthogonal Meixner and Krawtchouk polynomials. Knowledge of the location and behaviour of the zeros of these polynomials is relevant in various fields. Amongst many other applications, Jacobi polynomials are useful in the medical field where they are used in ECG data compression, Meixner polynomials are used for analysing discrete stochastic processes and Krawtchouk polynomials play a role in coding theory.

In the first place we investigate the interlacing of zeros of different sequences of each of these systems of polynomials, applying the results to obtain new bounds for the extreme zeros of the polynomials concerned. Interlacing of zeros of polynomials that belong to different sequences within the same family of orthogonal polynomials, was first studied in 1967 by Levit [48], who proved several separation results for the zeros of Hahn polynomials from different sequences. In 1989, Askey [8] proved that the zeros of Jacobi polynomials $P_n^{\alpha,\beta}$ and $P_n^{\alpha+1,\beta}$ interlace and he conjectured that the zeros of $P_n^{\alpha,\beta}$ and $P_n^{\alpha+2,\beta}$ interlace. The proof of Askey's conjecture is contained in a more general result, proved in [29], and the result we obtain in the Jacobi case can be considered as a further extension of this. Secondly, we study the zero location of Meixner and Krawtchouk polynomials for non-classical parameter values.

A result by Stieltjes [68, p. 46] proves that, within any orthogonal sequence $\{P_n\}_{n=0}^{N}$, the zeros of P_n and P_m, $n > m$, interlace in a well-defined way, a property called Stieltjes interlacing. Beardon [12] generalises the result by Stieltjes, showing that, if $m < n-1$ and P_m and P_n are co-prime, there exists a real polynomial S_{n-m-1} of degree $n-m-1$ whose real simple zeros provide a set of points that completes the interlacing picture. An important feature of the polynomials S_{n-m-1} is that they are completely determined by the coefficients in the three term recurrence relation satisfied by the orthogonal sequence $\{P_n\}_{n=0}^{N}$. We extend this result of Beardon to polynomials that belong to different orthogonal sequences, obtained by integer shifts of the appropriate parameters, as was done in [25] and [27] for the Gegenbauer and Laguerre polynomials. We consider different

sequences of Jacobi, Meixner and Krawtchouk polynomials, and specifically polynomials of the form $P_{n+1}(v_1, v_2; x)$ and $P_{n-1}(v_1 + s, v_2 + t; x)$, for different integer values of s and t, as well as $P_{n+1}(v_1, v_2; x)$ and $P_{n-k}(v_1+k, v_2+k; x),\ k = 1, 2, \ldots, n-1$. In the Meixner and Krawtchouk cases, we only consider integer shifts of one of their parameters. In each case, we identify the polynomial whose zeros complete the interlacing. Furthermore, we apply an immediate consequence due to Driver and Jordaan [28], on using the extra interlacing points as "inner" bounds for the extreme zeros of orthogonal polynomials, to obtain sharp lower (upper) bounds for the largest (smallest) zeros of each of the Jacobi, Meixner and Krawtchouk polynomials.

We make a comprehensive study of the zeros of Meixner and Krawtchouk polynomials for parameter values where (some of) the zeros are real. From the orthogonality relation satisfied by the Meixner polynomials $M_n(x; \beta, c),\ 0 < c < 1,\ \beta > 0$, we know that they have n real zeros on $(0, \infty.)$ We use a Sturmian sequence argument to prove that, for $n < 1-\beta$, the polynomials $M_n(x; \beta, c),\ \beta < 0,\ c < 0$, have n real zeros on $(0, -\beta)$. Furthermore, we prove results for the zero location of the quasi-orthogonal polynomials $M_n(x; \beta, c),\ -k < \beta < -k + 1,\ k = 1, \ldots, n-1$ and $0 < c < 1$ or $c > 1$, as well as the (non-orthogonal) polynomials $K_n(x; p, N)$ for $0 < p < 1$ and $n > N$. Finally, we show that the polynomials $M_n(x; \beta, c),\ \beta \in \mathbb{R}$, are real-rooted when $c \to 0$ and the zeros of the Krawtchouk polynomials $K_n(x; p, N),\ n = 1, 2, \ldots, N,\ 0 < p < 1$, tend to $x = 0, 1, \ldots, n-1$ when $p \to 0$.

Acknowledgements

I would like to express my thanks to the following: Prof Jean M-S Lubuma, Head of the Department of Mathematics and Applied Mathematics at the University of Pretoria, for reducing my teaching responsibilities while I was studying, as well as for financial support for attending a conference on Computational Methods and Function Theory in Turkey in 2010 and a research visit to the Helmholtz Zentrum München in January 2011; the University of Pretoria, for reducing my tuition fees; the National Research Foundation for financial support under grant number 2054423; the Institute for Biomathematics and Biometry at the Helmholtz Zentrum München for their support during a research visit; my family and my parents, Dr Jozef and Mrs Debora Henning, for financial and emotional support, as well as my father for his interest and intellectual input in the specific field of study; my three children, Henning, Mandé and Anli, for their understanding and co-operation.

Finally, I would like to express my thanks to Prof Kerstin Jordaan, my supervisor, for her support, inspiration, encouragement and guidance.

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Chapter 1

Introduction

1.1 Orthogonal polynomials

To define families of orthogonal polynomials, we use the scalar product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\ d\phi(x)$$

with positive measure ϕ supported on the real interval $[a, b]$, where a and/or b can be infinite.

A sequence of real polynomials $\{P_n(x)\}_{n=0}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, where $P_n(x)$ is of exact degree n, is orthogonal with respect to the measure ϕ if

$$\langle P_n, P_m\rangle = d_n^2\ \delta_{mn},\ d_n \neq 0,\ m, n = 0, 1, \ldots N, \tag{1.1}$$

$d_n^2 = \int_a^b P_n^2(x)\ d\phi(x)$ and δ_{mn} is Kronecker's symbol,

$$\delta_{mn} = \begin{cases} 0 & \text{if} \quad m \neq n, \\ 1 & \text{if} \quad m = n. \end{cases}$$

If the measure is absolutely continuous and the distribution $d\phi(x) = \rho(x)dx$, then (1.1) becomes

$$\int_a^b P_n(x)P_m(x)\ \rho(x)dx = d_n^2\ \delta_{mn},\ m, n = 0, 1, \ldots N,$$

or, equivalently (cf. [57, p. 148, Theorem 54] and [68, p. 28]),

$$\int_a^b x^k P_n(x)\rho(x)dx = 0,\ \text{for} \quad k = 0, 1, \ldots, n-1,$$

and the sequence $\{P_n(x)\}_{n=0}^N$ is said to be orthogonal on the interval (a, b) with respect to the weight or density function $\rho(x)$.

If the weight function $\rho(x)$ is discrete and $\rho_i > 0$ are the values of the weight at the distinct points x_i, $i = 0, 1, 2, \ldots, M$, $M \in \mathbb{N} \cup \{\infty\}$, then (1.1) takes the form of a sum [11, p. 182, eqn. 1.4]

$$\sum_{i=0}^{M} P_n(x_i)P_m(x_i)\ \rho_i \ = \ d_n^2\ \delta_{mn},\ \ m, n = 0, 1, \ldots N,$$

or, equivalently,

$$\sum_{i=0}^{M}(x_i)^k P_n(x_i)\rho_i = 0,\ \text{ for } \ k = 0, 1, \ldots, n-1,$$

and the sequence $\{P_n(x)\}_{n=0}^N$ is *discrete orthogonal*.

Throughout our discussion we will assume that the term *orthogonality* refers to orthogonality with respect to a measure that is supported on the real line, which implies that the polynomials are *real polynomials*, i.e., all coefficients are real.

1.2 Properties of orthogonal polynomials

Assume that $\{P_n(x)\}_{n=0}^N$, $N \in \mathbb{N}\cup\{\infty\}$, where $P_n(x)$ is of exact degree n, is a sequence of orthogonal polynomials. We list the properties of these polynomials that play an important role in this thesis.

(i) **Zeros of orthogonal polynomials**

The zeros of the orthogonal polynomials $P_n(x)$, associated with the positive measure ϕ on the interval $[a, b]$, are real and distinct and are located in (a, b) (cf. [68, p. 44, Theorem 3.3.1]).

The zeros of the polynomial P_n will be denoted by $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$.

(ii) **Three term recurrence relation**.

Every sequence of real orthogonal polynomials satisfies a three term recurrence relation

$$xP_n(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x)$$

for $n \geq 0$, the numbers A_n, B_n and C_n are real constants for $n = 0, 1, 2 \ldots$, such that $A_{n-1}C_n > 0$, $n = 1, 2, \ldots$ and $P_{-1}(x) = 0$ (cf. [6, p. 244-245, Theorem 5.2.2 and Remark 5.2.1]).

The converse is also true and is known as Favard's theorem ([16, p. 21, Theorem 4.4] and [57, p. 153]). If a sequence of polynomials satisfies a three term recurrence relation, the polynomials in that sequence are orthogonal with respect to a certain weight function, on the real line. The proof of this theorem is about existence of orthogonality and information about the weight function and interval of orthogonality is not explicitly given.

(iii) **Classic interlacing of zeros**.

A direct consequence of the three term recurrence relation is the Christoffel-Darboux formula [6, p. 246, Theorem 5.2.4], from which we obtain the inequality ([6, p. 247, Corollary 5.2.6] and [68, p. 45, eqn. 3.3.6])

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0, \; x \in \mathbb{R}.$$

As a first consequence of this inequality, the polynomials P_n and P_{n+1} cannot have common zeros. Furthermore, we have the following separation theorem.

Theorem 1.2.1 *(cf. [68, p. 46, Theorem 3.3.2]) Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ denote the zeros of P_n. The zeros of P_n and P_{n+1} separate each other in the following way:*

$$x_{n+1,1} < x_{n,1} < x_{n+1,2} < \cdots < x_{n,n} < x_{n+1,n+1}.$$

This separation of zeros will be called *classic interlacing of zeros.*

(iv) **Stieltjes interlacing of zeros**.

Another well-known result on interlacing of zeros of orthogonal polynomials is due to Stieltjes.

Theorem 1.2.2 *(cf. [68, p. 46, Theorem 3.3.3]) Between two zeros of $P_m(x)$ there is at least one zero of $P_n(x), m < n$.*

We will call this property *Stieltjes interlacing of zeros.*

Clearly, if $m < n-1$, there are not enough zeros of P_m to interlace fully with the n zeros of P_n. In a recent publication, Beardon [12] extends the result of Stieltjes on the interlacing of zeros of polynomials P_m and P_n, $m < n-1$ in an orthogonal sequence, by showing that, if P_m and P_n are co-prime, i.e., they do not have common zeros, there exists a real polynomial

S_{n-m-1} of degree $n-m-1$ whose real simple zeros provide a set of points that completes the interlacing picture. An important feature of the polynomials S_{n-m-1} is that they are completely determined by the coefficients in the three term recurrence relation satisfied by the orthogonal sequence $\{P_n\}_{n=0}^{\infty}$. The polynomials S_{n-m-1} are the dual polynomials introduced by de Boor and Saff in [19] or, equivalently, the associated polynomials analysed by Vinet and Zhedanov in [72].

We will refer to the zeros of the polynomial S_{n-m-1} as *extra interlacing points.*

The interlacing property of zeros of polynomials is important in, e.g., numerical quadrature applications, where the existence of positive interpolatory quadrature formulae, using zeros as nodes, is guaranteed by interlacing (cf. [54]). We now shift our focus to orthogonal polynomials that can be expressed as hypergeometric functions.

1.3 The hypergeometric function

The generalised hypergeometric series ${}_pF_q$ with p numerator and q denominator parameters, is defined by

$$ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k z^k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!}, $$

where $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are real or complex numbers, $b_1, \ldots, b_q \neq 0, -1, -2, \ldots$.

The symbol $(\ )_k$ is the shifted factorial, or Pochhammer symbol [36, p. 8, eqn. 1.3.6], defined by

$$ \begin{aligned} (\alpha)_k &= \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k \geq 1, \quad k \in \mathbb{N} \qquad (1.2) \\ (\alpha)_0 &= 1, \quad \alpha \neq 0. \end{aligned} $$

If $p \leq q$, the series ${}_pF_q$ converges for all finite z and if $p = q+1$, it converges if $|z| < 1$. In case $a_j = 0, -1, -2\ldots$, the series terminates and we have a polynomial of degree n in x.

The Gauss, or ${}_2F_1$, hypergeometric function was introduced by Gauss [32] in 1812 and is defined by

$$ {}_2F_1(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

a, b and c are complex parameters, $c \neq 0, -1, -2, \ldots$.

The Gauss hypergeometric series converges if $|z| < 1$, since

$$\lim_{n\to\infty}\left|\frac{t_{n+1}}{t_n}\right| = |z|,$$

where t_n is the nth term of the hypergeometric series. By analytic continuation, convergence for $|z| > 1$ can be obtained (cf. [73, p. 288]).

1.4 Classical orthogonal polynomials

Orthogonal polynomials that have hypergeometric representations will be referred to as *classical orthogonal polynomials*. These polynomials, together with some limit relations between them, form the *Askey scheme of hypergeometric orthogonal polynomials*, proposed by Richard Askey and compiled by Koekoek and Swarttouw in 1998; see Fig 1.1 for the most recent version.

1.4.1 Very classical orthogonal polynomials

The very classical orthogonal polynomials are named after Hermite, Laguerre and Jacobi and can be defined as the polynomial solutions of a second order differential equation of the type

$$\sigma(x){P_n}''(x) + \tau(x){P_n}'(x) + \lambda_n P_n(x) = 0, \tag{1.3}$$

where $\sigma(x)$ is a polynomial of degree at most two, $\tau(x)$ is a polynomial of degree at most one and λ_n depends only on n. The three infinite systems of Hermite, Laguerre and Jacobi polynomials, as well as the three finite systems of Jacobi, Bessel and Pseudo-Jacobi polynomials are the only polynomial solutions of (1.3) (cf. [44, p. xii]), that are orthogonal with respect to a measure that is supported on the real line (or part of the real line). We call these *finite systems*, because only a finite number of these polynomials are orthogonal (cf. [44, p. 93]).

1.4.2 Discrete classical orthogonal polynomials

The discrete classical orthogonal polynomials can be defined as the polynomial solutions of a second order difference equation with polynomial coefficients

$$\sigma(x)\triangle\bigtriangledown P_n(x) + \tau(x)\triangle P_n(x) + \lambda_n P_n(x) = 0$$

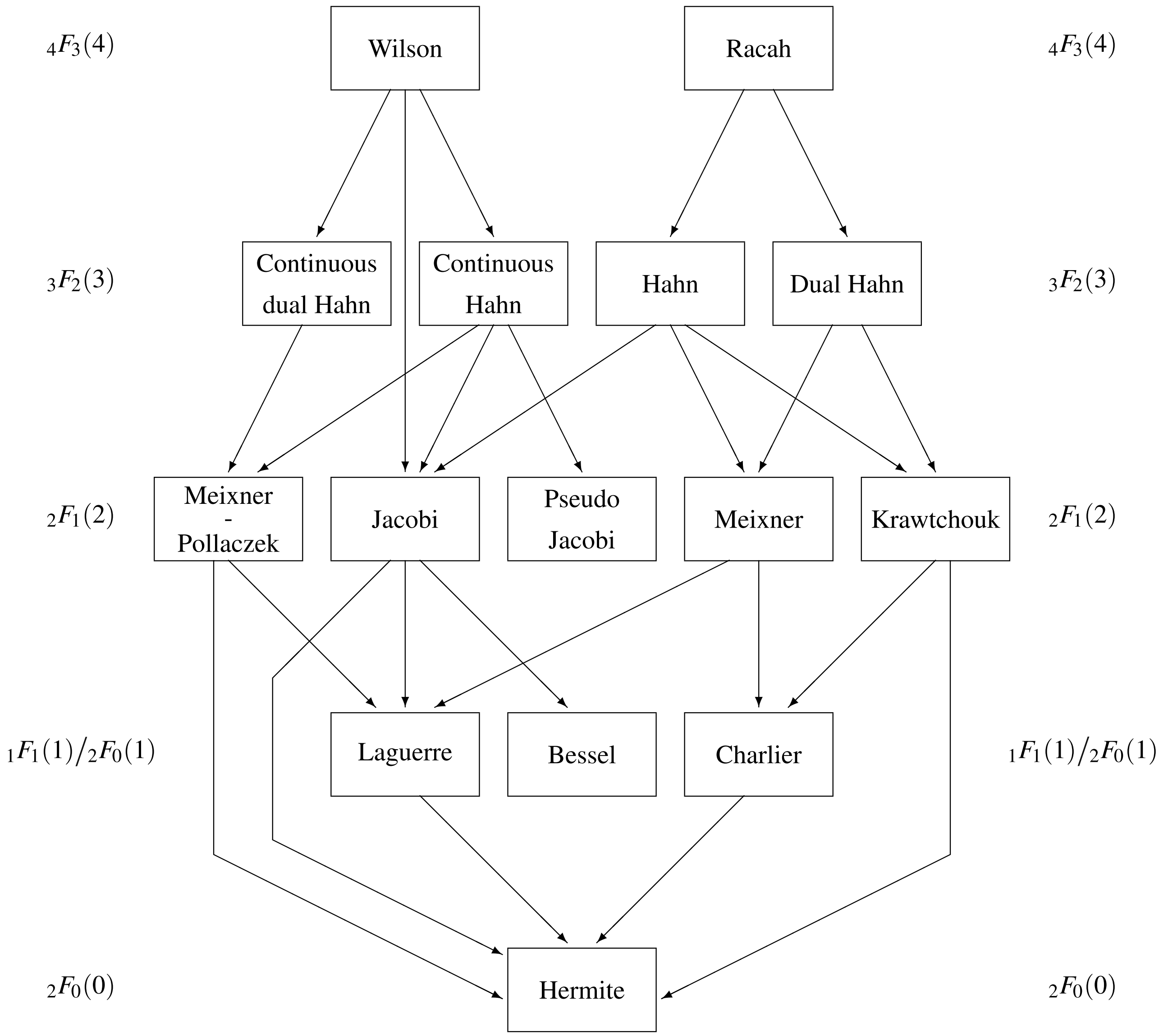

Figure 1.1: Askey Scheme of Hypergeometric Orthogonal Polynomials [44, p. 183]

where $\triangle f(x) = f(x+1) - f(x)$ and $\bigtriangledown f(x) = f(x) - f(x-1)$ denote the forward and backward difference operators respectively, $\sigma(x)$ is a polynomial of degree at most two, $\tau(x)$ is a polynomial of degree at most one and λ_n is a constant.

The polynomial solutions of this difference equation lead to the two infinite systems of Charlier and Meixner polynomials, and the two finite systems of Krawtchouk and Hahn polynomials (cf. [44, p. xii]).

1.5 The ${}_2F_1$ hypergeometric orthogonal polynomials

The ${}_2F_1$ hypergeometric orthogonal polynomials are the polynomials on the ${}_2F_1$ plane of the Askey scheme. We discuss the most important properties of the ${}_2F_1$ class of orthogonal polynomials and refer the reader to [44, Chapter 9], that deals with all families of hypergeometric orthogonal polynomials in the Askey Scheme.

1.5.1 Jacobi polynomials

Jacobi polynomials are named after the German mathematician Carl Jacobi (1804-1851). Jacobi is considered as one of the greatest mathematicians of all times, who made fundamental contributions in various fields, but specifically in number theory. He was the first to apply elliptic functions to number theory.

Jacobi polynomials can be used to approximate functions for which the Laplace transform is known [2]. They also play a role in the medical field, where they are used in ECG data compression [69], as well as in quantum physics, where they are applied in solving the Schrödinger equation (cf. [13]). The Jacobi polynomial of degree n may be defined by [44, p. 216, eqn. 9.8.1]

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{1.4}$$

The differential equation satisfied by Jacobi polynomials $\left(f = P_n^{\alpha,\beta}\right)$ is

$$(1-x^2)f''(x) + \Big(\beta - \alpha - (\alpha+\beta+2)x\Big)f'(x) + n(n+\alpha+\beta+1)f(x) = 0. \tag{1.5}$$

The Jacobi polynomials $\left\{P_n^{\alpha,\beta}\right\}_{n=0}^{\infty}$ satisfy the three term recurrence relation [44, p. 217, eqn. 9.8.4]

$$\begin{aligned}&\left(x-\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)P_n^{\alpha,\beta}(x) \qquad (1.6)\\ &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{\alpha,\beta}(x)+\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{\alpha,\beta}(x)\end{aligned}$$

and, for α, $\beta > -1$, they are orthogonal with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1, 1]$.

Symmetry

When polynomials $p_n(x)$ are orthonormal on $[a, b]$, an interval symmetric with respect to the origin, and the distribution $w(x)dx$ has an even weight function, i.e., $w(-x) = w(x)$, then the polynomial $p_n(x)$ is even or odd, as n is even or odd and thus $p_n(-x) = (-1)^n p_n(x)$ [68, p. 29, eqn. 2.3.3]. The weight function of the Jacobi polynomials satisfies the equation

$$w(x, \alpha, \beta) = w(-x, \beta, \alpha)$$

and therefore Jacobi polynomials with the standard normalisation satisfy the symmetry property [68, p. 59, eqn. 4.1.3]

$$P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x). \qquad (1.7)$$

The derivative of $P_n^{\alpha,\beta}(x)$

The very classical orthogonal polynomials have derivatives which again form orthogonal sequences (cf. [68, p. 63, eqn. 4.21.7] and [70, p. 81]) and

$$\frac{d}{dx}P_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{\alpha+1,\beta+1}(x).$$

By induction it follows that

$$D^k P_n^{\alpha,\beta}(x) = \frac{1}{2^k}(n+\alpha+\beta+1)_k P_{n-k}^{\alpha+k,\beta+k}(x), \qquad (1.8)$$

where D^k denotes the k-th derivative.

1.5.2 Meixner polynomials

Meixner polynomials are named after the German theoretical physicist, Josef Meixner (1908-1994), who taught at the Institute of Theoretical Physics in Aachen, Germany, from the 1950's until his death. These polynomials are associated with, e.g., stochastic processes [61] and in [34] they are used to analyse discrete stochastic processes in the context of spectral analysis in the Laplace domain.

Meixner polynomials may be defined in terms of the ${}_2F_1$ hypergeometric function (cf. [36, p. 174, 175] and [44, p. 234, eqn. 9.10.1])

$$\begin{aligned} M_n(x;\beta,c) &= (\beta)_n \; {}_2F_1\left(-n,-x;\beta;1-\frac{1}{c}\right) \\ &= (\beta)_n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(1-\frac{1}{c})^k}{(\beta)_k k!}, \end{aligned} \tag{1.9}$$

for $\beta, c \in \mathbb{R},\ \beta \neq -1,-2,\ldots,-n+1,\ c \neq 0$. We note that the position of the variable x differs from the position of x in the hypergeometric representation of the Jacobi polynomials.

Since $(\beta+k)_{n-k} = \dfrac{(\beta)_n}{(\beta)_k}$, (1.9) can be rewritten as

$$M_n(x;\beta,c) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(\beta+k)_{n-k}(1-\frac{1}{c})^k}{k!}, \tag{1.10}$$

a polynomial of degree n in x. In this way, we can define the Meixner polynomials for any $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$.

The sequence $\{M_n(x;\beta,c)\}_{n=0}^{\infty}$ satisfies the difference equation (cf. [44, p. 234, eqn. 9.10.5])

$$\big(n(c-1)+x+(x+\beta)c\big)M_n(x;\beta,c) = c(x+\beta)M_n(x+1;\beta,c) + xM_n(x-1;\beta,c), \tag{1.11}$$

and, for $0 < c < 1$ and $\beta > 0$, these polynomials satisfy the discrete orthogonality relation (cf. [44, p. 234, eqn. 9.10.2])

$$\sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} M_m(x;\beta,c)M_n(x;\beta,c) = \frac{(\beta)_n n!}{c^n(1-c)^\beta}\,\delta_{mn}, \tag{1.12}$$

hence the zeros are real, distinct and in $(0,\infty)$ for these values of the parameters β and c. We note that the weight function $\rho(x) = \frac{c^x(\beta)_x}{x!}$ is a step function that is constant on the open intervals

$(x, x+1)$, $x = 0, 1, 2, \ldots$ and therefore the zeros of $M_n(x;\beta,c)$ are separated by the numbers $0, 1, 2, \ldots$ (cf. [64, p. 1539] and [68, p. 50, Theorem 3.41.2]).

When we apply the Pfaff-Kummer transformation (7.1) to (1.9), we obtain the identity (cf. [16, p. 177, eqn. 3.6])

$$\begin{aligned} M_n(x;\beta,c) &= (\beta)_n \left(\frac{1}{c}\right)^n {}_2F_1\left(-n, x+\beta; \beta; 1-c\right) && (1.13) \\ &= \left(\frac{1}{c}\right)^n M_n\left(-x-\beta; \beta, \frac{1}{c}\right) && (1.14) \end{aligned}$$

for $\beta \neq -1, -2, \ldots, -n+1$, $c \neq 0$, a general symmetry property of the Meixner polynomials, since by continuity it holds for $\beta \in \mathbb{R}$.

The orthogonality relation when $c > 1$ and $\beta > 0$ [16, p. 177, eqn. 3.7],

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{c^x x!} M_m(-x-\beta;\beta,c) M_n(-x-\beta;\beta,c) = \left(\frac{c}{c-1}\right)^{\beta} c^{-n} (\beta)_n n!\, \delta_{mn},$$

can be derived from (1.12) and (1.14) and we conclude that, for $\beta > 0$ and $c > 1$, the zeros of $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ are real and distinct and in $(-\infty, -\beta)$.

For $n \geq 0$, Meixner polynomials satisfy the three term recurrence relation [44, p. 234, eqn. 9.10.3]

$$\left(x - \frac{n + (\beta+n)c}{1-c}\right) M_n(x;\beta,c) = \frac{c}{c-1} M_{n+1}(x;\beta,c) + \frac{n(\beta+n-1)}{c-1} M_{n-1}(x;\beta,c), \qquad (1.15)$$

where $M_0(x;\beta,c) = 1$ and $M_{-1}(x;\beta,c) = 0$.

The generating functions for the Meixner polynomials can be found in [44, p. 235].

The derivative of $M_n(x;\beta,c)$

The derivative of the polynomial $M_n(x;\beta,c)$ is defined in terms of the forward shift operator Δ, where $\Delta f(x) = f(x+1) - f(x)$. A direct calculation yields (cf. [44, p. 235, eqn. 9.10.7])

$$\Delta M_n(x;\beta,c) = n\left(1 - \frac{1}{c}\right) M_{n-1}(x;\beta+1,c)$$

and it follows by induction that

$$\Delta^k M_n(x;\beta,c) = \frac{n!(1-\frac{1}{c})^k}{(n-k)!} M_{n-k}(x;\beta+k,c), \; k = 0, 1, \ldots, n,$$

where $\Delta^k M_n(x;\beta,c) = \Delta(\Delta^{k-1} M_n(x;\beta,c))$.

The standard orthogonality of a finite number of Meixner polynomials $M_n(x;\beta,c)$ when $c<0$ and β is equal to a negative integer, say $\beta=-N$, $N\in\mathbb{N}$, is that of the Krawtchouk polynomials.

1.5.3 Krawtchouk polynomials

Mikhail Krawtchouk (1892-1942) was a Ukrainian mathematician and the author of around 180 articles. He introduced the Krawtchouk polynomials in 1929. Krawtchouk polynomials are a special case of Meixner polynomials and are applied in many areas of mathematics. The role they play in coding theory is briefly discussed in [36, p. 184]. They are also useful in graph theory [20, Chapter 11].

Krawtchouk polynomials are defined by ([36, p. 182] and [44, p. 237, eqn. 9.11.1])

$$K_n(x;p,N) = (-N)_n\ {}_2F_1\left(-n,-x;-N;\frac{1}{p}\right),\ n=0,1,\ldots,N, N\in\mathbb{N}, \tag{1.16}$$

that can be rewritten as

$$K_n(x;p,N) \quad = \quad \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(-N+k)_{n-k}}{k!p^k}, \tag{1.17}$$

which can be used to define Krawtchouk polynomials for any $n\in\mathbb{N}$.

The finite system of Krawtchouk polynomials satisfies the difference equation (cf. [44, p. 238, eqn. 9.11.5])

$$\big(p(N-x)+x(1-p)-n\big)K_n(x;p,N) = p(N-x)K_n(x+1;p,N)+x(1-p)K_n(x-1;p,N)$$

and, for $m<n\le N;\ m,n,N\in\mathbb{N}$ and $0<p<1$, they satisfy the orthogonality relation

$$\sum_{x=0}^{N} w(x;p,N)K_m(x;p,N)K_n(x;p,N)=0$$

on $[0,N]$, with respect to the finite binomial distribution $w(x;p,N)=\binom{N}{x}(p)^x(1-p)^{N-x}$, that is positive at the mass points $x=0,1,\ldots,N$ of the discrete measure for $0<p<1$. This implies that, for $0<p<1$ and $n\le N$, $n,N\in\mathbb{N}$, the zeros of $K_n(x;p,N)$ are real, distinct and in the interval

$(0, N)$. They are also separated by the mass points of the measure of orthogonality (cf. [24, p. 121] and [68, p. 50, Theorem 3.41.2]) and in the particular case where $n = N$, the zeros of $K_N(x; p, N)$, denoted by $x_{N,i}$, $i = 1, 2, \ldots, N$, interlace with the mass points as follows

$$0 < x_{N,1} < 1 < x_{N,2} < 2 < \cdots < x_{N,N} < N. \tag{1.18}$$

Furthermore, if $n > N$, the points $x = 0, 1, 2, \ldots, N$ are zeros of the (non-orthogonal) polynomials $K_n(x; p, N)$ [68, p. 36].

The three term recurrence relation for Krawtchouk polynomials is (cf. [44, p. 237, eqn. 9.11.3])

$$\begin{aligned} xK_n(x;p,N) &= A_nK_{n+1}(x;p,N) + B_nK_n(x;p,N) + C_nK_{n-1}(x;p,N) \\ K_0 &= 1, \quad K_{-1} = 0 \text{ and} \\ A_n &= p, \; B_n = p(N-n) + n(1-p) \text{ and } C_n = n(1-p)(N-n+1). \end{aligned} \tag{1.19}$$

We refer the reader to [44, p. 239], where the generating functions for the Krawtchouk polynomials are given.

From the forward shift operator (cf. [44, p. 238, eqn. 9.11.6])

$$\Delta K_n(x;p,N) = \frac{n}{p}K_{n-1}(x;p,N-1),$$

it follows by induction that, for each $n = 1, 2, \ldots, N$,

$$\Delta^k K_n(x;p,N) = \frac{n!}{(n-k)!p^k}K_{n-k}(x;p,N-k), \; k = 1, 2, \ldots, n.$$

Meixner polynomials are related to Krawtchouk polynomials in the following way:

$$M_n\left(x; -N, \frac{p}{p-1}\right) = K_n(x;p,N). \tag{1.20}$$

1.5.4 Meixner-Pollaczek polynomials

Meixner-Pollaczek polynomials [44, p. 213] were discovered by Josef Meixner [51] in 1934 and rediscovered by Pollaczek [55] in 1949. The hypergeometric representation of these polynomials is

$$p_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} \; {}_2F_1(-n, \lambda + ix; 2\lambda; 1 - e^{-2i\phi})$$

and, for $\lambda > 0,\ 0 < \phi < \pi$, they are orthogonal on the real line with respect to the weight function $w(x, \phi) =\mid \Gamma(\lambda - ix) \mid^2 e^{2\phi - \pi x}$.

The Meixner-Pollaczek polynomials satisfy a second order difference equation with complex coefficients [44, p. 132] and are connected to Lévy processes [62] in stochastics.

1.5.5 Pseudo Jacobi polynomials

Pseudo Jacobi polynomials [44, p. 231] refer to a finite system of continuous classical orthogonal polynomials and they were studied by Sir Edward John Routh [60] in 1884 and later rediscovered by Romanovski [59] in 1929. These polynomials satisfy the second order differential equation (1.3) and the polynomial $\sigma(x) = 1 + x^2$ has two complex roots. The hypergeometric representation of the Pseudo Jacobi polynomials is (cf. [44, p. 231, eqn. 9.9.1]).

$$p_n(x; v, N) = (x + i)^n \ {}_2F_1 \left(-n, N + 1 - n - iv; 2N + 2 - 2n; \frac{2}{1 - ix} \right), n = 0, 1, 2 \ldots, N$$

and they are orthogonal on the real line with respect to the weight function $(1 + x^2)^{-N-1} e^{2v \arctan x}$.

Romanovski polynomials, as discussed in [3, p. 148], are closely related to Pseudo-Jacobi polynomials and are applied to random matrix theory [58], as well as solutions of the Schrödinger equation, and we refer the reader to [3] where these applications are discussed in detail.

1.6 Brief overview

In this thesis we focus on zeros of Jacobi, Meixner and Krawtchouk polynomials.

We start by reviewing some recently published results on interlacing of zeros of different sequences of orthogonal polynomials of the same or adjacent degree. We show how Beardon's result on Stieltjes interlacing of zeros of polynomials p_m and p_n in an orthogonal sequence, for $m < n - 1$, was recently extended to zeros of different sequences of Laguerre and Gegenbauer polynomials and discuss results on upper and lower bounds for extreme zeros of a polynomial in an orthogonal sequence. The original contribution of this thesis is presented in chapters 3, 4 and 5.

The overarching theme of Chapters 3 and 4 is the investigation of Stieltjes interlacing between the zeros of polynomials of non-consecutive degree of different orthogonal sequences. We choose the different orthogonal sequences for which Stieltjes interlacing may hold, to lie within the same family of orthogonal polynomials, by considering different values of the appropriate parameter(s). The degrees of the polynomials we consider, say p_{n+1} and g_{n-1}, differ by two units and we make use of mixed three term recurrence relations of the form

$$(x - b_n)p_n(x) = a_n p_{n+1}(x) + c_n g_{n-1}(x), \tag{1.21}$$

satisfied by the polynomials under consideration, to prove these results. In each case we identify an extra interlacing point, b_n, uniquely determined by the coefficient of $p_n(x)$ in the appropriate mixed three term recurrence relation. We also consider the possibility that the polynomials under consideration can have common zeros. The extra interlacing points obtained, form upper (lower) bounds for the smallest (largest) zero of the polynomial p_{n+1}. We compare the different extra interlacing points, in order to determine the best *inner* bounds for the extreme zeros of each of the Jacobi, Meixner and Krawtchouk polynomials.

In Chapter 3 we consider zeros of different sequences of Jacobi polynomials, obtained by either shifting one parameter at a time or both simultaneously. The more general case of Stieltjes interlacing of zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}, k = 1, 2, \ldots, n-1$ is also discussed. We explain how some of these Stieltjes interlacing results can be interpreted electrostatically. The results in this chapter were published in 2011 [26].

In Chapter 4 we shift our attention to the discrete orthogonal infinite system of Meixner polynomials and the finite system of Krawtchouk polynomials. In the Meixner case, the different sequences of polynomials that we consider, are obtained by shifting the parameter β of the polynomial $M_n(x, \beta, c)$. The parameter c is restricted to the interval $(0, 1)$ and integer shifts of c will make no sense. We also consider the more general Stieltjes interlacing between zeros of $M_{n+1}(x, \beta, c)$ and $M_{n-k}(x, \beta + k, c), k = 1, 2, \ldots, n-1$. In the Krawtchouk case, we obtain the different sequences by shifting the parameter N of the polynomial $K_n(x; p, N),\ n = 0, 1, \ldots, N$. Shifting N implies a change in the interval of orthogonality and restrictions on the parameter p are necessary in some cases, to obtain interlacing. A paper on the results in this chapter is in preparation.

In Chapter 5 we make a comprehensive study of the zero location of Meixner and Krawtchouk polynomials, in particular for parameter values where (some of) the zeros are real. We use the three term recurrence relation satisfied by Meixner polynomials, as well as a Sturmian sequence argument, to prove that, for $n < 1 - \beta$, the Meixner polynomials $M_n(x;\beta,c)$, $c < 0$, have n real zeros on the interval $(0, -\beta)$. We prove results for the zero location of the quasi-orthogonal polynomials $M_n(x;\beta,c)$, $-k < \beta < -k+1$, $k = 1, \ldots, n-1$ and $0 < c < 1$ or $c > 1$, as well as the (non-orthogonal) polynomials $K_n(x;p,N)$, for $0 < p < 1$ and $n = N+1, N+2, \ldots$. We also show that the polynomials $M_n(x;\beta,c)$, $\beta \in \mathbb{R}$, are real-rooted when $c \to 0$. A paper on these results has been accepted for publication [41].

Chapter 2

Background

2.1 Introduction

In this chapter we provide an overview of some recently published results on interlacing, as well as Stieltjes interlacing, of zeros of orthogonal polynomials from sequences corresponding to different parameters, together with some background information necessary for our research.

2.2 Interlacing of zeros of orthogonal polynomials of the same or adjacent degree

Results on the interlacing of zeros of different sequences of Hahn polynomials of the same and adjacent degree, were published in 1967 by Levit [48, p. 199-202].

2.2.1 Jacobi polynomials

Various researchers, e.g., Askey, Driver, Jordaan and Mbuyi, studied zeros of Jacobi polynomials of the same or adjacent degree.

(i) In 1989, Askey [8, p. 28, 29] proved that the zeros of $P_n^{\alpha,\beta}(x)$ and $P_n^{\alpha+1,\beta}(x)$ interlace and he conjectured that the zeros of $P_n^{\alpha,\beta}(x)$ and $P_n^{\alpha+2,\beta}(x)$ interlace.

(ii) In 2008, Driver *et al* [29] proved that the zeros of $P_n^{\alpha,\beta}$ interlace with the zeros of polynomials from some different Jacobi sequences of the same and adjacent degree, namely

$P_n^{\alpha-t,\beta},\ P_n^{\alpha,\beta+t},\ P_n^{\alpha-t,\beta+k},\ P_n^{\alpha+t,\beta-k}$ for $0<t,k\leq 2$ and

$P_{n-1}^{\alpha+t,\beta+k},\ P_{n-1}^{\alpha+t,\beta},\ P_{n-1}^{\alpha,\beta+k}$ for $0\leq t,k\leq 2$,

that confirmed as well as extended Askey's conjecture. Numerical examples were given to illustrate that, in general, interlacing of zeros does not hold if t or k is greater then 2.

In the same year, Segura [63] proved that the zeros of the Jacobi *functions* $P_\nu^{\alpha,\beta}$ interlace with the zeros of the functions

- $P_{\nu\pm\epsilon}^{\alpha,\beta\pm k}$;
- $P_{\nu\pm\epsilon}^{\alpha+1,\beta\pm k}$;
- $P_{\nu\pm\epsilon}^{\alpha+2,\beta\pm k}$,

for $0<\epsilon<1,\ 0<k\leq 2$.

2.2.2 Meixner and Krawtchouk polynomials

In 1990, Chihara and Stanton [15] proved that, for $0<p<1$, the zeros of the Krawtchouk polynomials $K_n(x;p,N)$ and $K_n(x;p,N+1)$, as well as the zeros of $K_{n-1}(x;p,N)$ and $K_n(x;p,N+1)$, separate each other. Some special attention was paid to the case $p=\frac{1}{2}$.

A few years later, in 2009, contiguous relations satisfied by hypergeometric polynomials, were used in [42] to prove interlacing results between the zeros of different sequences of Meixner, Krawtchouk, Meixner-Pollaczek and Hahn polynomials of the same or adjacent degree. In the Meixner case, it was proved that, for $\beta>0$ and $0<c<1$, the zeros of the polynomials $M_n(x;\beta,c)$, $M_n(x;\beta+t,c)$ and $M_{n-1}(x;\beta+t,c)$ interlace when $0<t\leq 2$. In the Krawtchouk case, interlacing properties between the zeros of the Krawtchouk polynomials $K_n(x;p,N)$, $K_n(x;p,N+k)$ and $K_{n-1}(x;p,N+k),\ k\in\{-1,1\}$, were examined.

2.3 Stieltjes interlacing of zeros of Laguerre and Gegenbauer polynomials from different sequences

Stieltjes interlacing of zeros of different sequences of *one-parameter* orthogonal families, namely, Gegenbauer polynomials C_n^λ and Laguerre polynomials L_n^α, was recently studied.

In 2012, Driver [25] showed that, for $\lambda > -\frac{1}{2}$, the zeros $C_{n-1}^{\lambda+t}, 0 \leq t \leq 2$, together with the point $x = 0$, interlace in the Stieltjes sense with the zeros of C_{n+1}^{λ} when n is odd. When n is even, these polynomials have a common zero at $x = 0$ and the $(\frac{n}{2} - 1)$ positive (respectively negative) zeros of $C_{n-1}^{\lambda+t}$ interlace with the $(\frac{n}{2})$ positive (respectively negative) zeros of C_{n+1}^{λ}. Gegenbauer polynomials, whose degrees differ by 3, namely C_{n+1}^{λ} and $C_{n-2}^{\lambda+k}, k \in \{1, 2, 3\}$, were also considered. In this case, the polynomials under consideration have either no common zeros, or two symmetric common zeros. When these polynomials are co-prime, their zeros interlace in the Stieltjes sense and the two symmetric extra interlacing points are identified. A more general result, that Stieltjes interlacing holds between the zeros of the kth derivative of C_n^{λ} and the zeros of $C_{n+1}^{\lambda}, k \in \{1, 2, \ldots, n-1\}$, was also proved.

In [27], Driver and Jordaan showed that, for $\alpha > -1$, Stieltjes interlacing holds between zeros of Laguerre polynomials L_{n+1}^{α} and $L_{n-1}^{\alpha+t}$, when $t \in \{1, 2, 3, 4\}$, and, more generally, between the zeros of L_{n+1}^{α} and $L_{n-k}^{\alpha+k+t}, t \in \{0, 1, 2\}, k \in \{1, 2, \ldots, n-1\}$. In each of these cases, a polynomial whose zeros complete the interlacing process, was identified.

2.4 Bounds for extreme zeros of classical orthogonal polynomials

Bounds for the extreme zeros of the very classical polynomials can be obtained from the differential equations they satisfy. The following result, due to Laguerre, plays an important role in this regard. We show the proof for real polynomials.

Lemma 2.4.1 *(cf. [68, p. 117]) Let $f(x)$ be a polynomial of degree n, with zeros $x_1 < x_2 \cdots < x_n$ in the interval (a, b) and*

$$X_i = x_i - 2(n-1)\frac{f'(x_i)}{f''(x_i)}, \; i = 1, n. \tag{2.1}$$

Then

(i) $a < x_1 < x_2 < X_1 < x_n < b$

(ii) $a < x_1 < X_n < x_{n-1} < x_n < b$.

Proof. We will prove (ii), the proof of (i) is similar. Let

$$f(x) = (x - x_n)g(x)$$

and

$$g(x) = (x - x_1)(x - x_2)\dots(x - x_{n-1}),$$

then $f'(x) = g(x) + (x - x_n)g'(x)$, $f''(x) = 2g'(x) + (x - x_n)g''(x)$ and consequently

$$g(x_n) = f'(x_n) \quad \text{and} \quad g'(x_n) = \frac{1}{2}f''(x_n).$$

It follows that

$$\frac{g'(x_n)}{g(x_n)} = \frac{f''(x_n)}{2f'(x_n)} = \frac{1}{x_n - x_1} + \frac{1}{x_n - x_2} + \dots + \frac{1}{x_n - x_{n-1}}$$

and (2.1) becomes

$$\frac{1}{x_n - X_n} = \frac{1}{n-1}\left(\frac{1}{x_n - x_1} + \frac{1}{x_n - x_2} + \dots + \frac{1}{x_n - x_{n-1}}\right).$$

We know that, for any $a \in \mathbb{R}$, if $a = \frac{1}{n}(a_1 + a_2 + \dots + a_n)$ and $a_1 < a_2 < \dots < a_n$, then $a_1 < a < a_n$ and therefore

$$\frac{1}{x_n - x_1} < \frac{1}{x_n - X_n} < \frac{1}{x_n - x_{n-1}};$$

since all these denominators are positive, we can deduce that $x_1 < X_n < x_{n-1}$. The stated result follows.

■

We apply this result to Jacobi polynomials. Let $x_{n,1} < x_{n,2} \cdots < x_{n,n}$ be the zeros of $f(x) = P_n^{\alpha,\beta}(x)$, $\alpha, \beta > -1$. From the differential equation satisfied by the Jacobi polynomials (1.5), we deduce that

$$(1 - x_{n,j}^2)f''(x_{n,j}) + \Big(\beta - \alpha - (\alpha + \beta + 2)x_{n,j}\Big)f'(x_{n,j}) = 0, \; j = 1, 2, \dots, n,$$

and from (2.1), we get the values

$$X_j = x_{n,j} - \frac{2(n-1)(1 - x_{n,j}^2)}{\alpha - \beta + (\alpha + \beta + 2)x_{n,j}}, \; j = 1, n,$$

and consequently, from Lemma 2.4.1(i), we have

$$-1 < x_{n,1} < x_{n,2} < x_{n,1} - \frac{2(n-1)(1 - x_{n,1}^2)}{\alpha - \beta + (\alpha + \beta + 2)x_{n,1}} < x_{n,n} < 1,$$

which leads to

$$-1 < x_{n,1} < \frac{\beta - \alpha - 2n + 2}{\alpha + \beta + 2n} = B_{n,H}(\alpha, \beta). \tag{2.2}$$

Furthermore, should the lowest zero $x_{n,1}$ be known, one obtains an upper bound for $x_{n,2}$. In the same way, Lemma 2.4.1(ii) leads to (cf. [68, p. 119, eqn. 6.2.11])

$$B_{n,L}(\alpha, \beta) = \frac{\beta - \alpha + 2n - 2}{\alpha + \beta + 2n} < x_{n,n} < 1. \tag{2.3}$$

Laguerre's theorem thus provides us with an upper bound for the smallest zero and a lower bound for the largest zero of the Jacobi polynomial $P_n^{\alpha,\beta}$. For all $\alpha, \beta > -1$ and $n = 1, 2, \dots,$ we have

$$-1 < x_{n,1} < \frac{\beta - \alpha - 2(n-1)}{\alpha + \beta + 2n} < \frac{\beta - \alpha + 2(n-1)}{\alpha + \beta + 2n} < x_{n,n} < 1$$

and, should the extreme zero $x_{n,1}$ ($x_{n,n}$) be known, an upper (lower) bound for the zero $x_{n,2}$ ($x_{n,n-1}$) can also be found.

We note the following:

(i) $B_{n,L}(\alpha, \beta) = -B_{n,H}(\beta, \alpha)$.

(ii) Szegő provides an alternative formula for the lower bound for the largest zero when $\alpha \leq \beta$, namely $B^*_{n,L}(\alpha) = \dfrac{n-1}{n+\alpha}$ (cf. [68, p. 119, eqn. 6.2.12]).

Laguerre's theorem can only be applied to find *inner* bounds for zeros of polynomials that satisfy a second order differential equation, i.e., the polynomials mentioned in Section 1.4.1. Every family of orthogonal polynomials, however, satisfies a three term recurrence relation and in [38], the recurrence coefficients are used to find upper bounds for the largest zero and lower bounds for the smallest zero of inter alia Meixner and Jacobi polynomials.

More recently, sharp bounds for not only the extreme zeros, but all the zeros of the very classical polynomials, were obtained in [21] by a technique based on inequalities for real-root polynomials. These inequalities were proved by using the consecutive derivatives of the orthogonal polynomials provided by the differential equations they satisfy. A result due to Obrechkoff [53] was recently used in [7] to derive inequalities that determine the location of the zeros of Jacobi polynomials, and explicit bounds were obtained for all the zeros of Jacobi polynomials in terms of the extreme zeros of either Laguerre, or other families of Jacobi polynomials.

The results in Chapter 3 of this thesis, published in [26], together with the results in [25] and [27], were applied in [28] to obtain inner bounds for the extreme zeros of Jacobi, Laguerre and

Gegenbauer polynomials. The results in [28] in fact apply to any polynomial p_n, that is part of an orthogonal sequence satisfying a mixed three term recurrence relation of the form

$$f(x)g_{n-k}(x) = H_{k-1}(x)p_{n+1}(x) - G_k(x)p_n(x)$$

on an interval (c, d), where $f(x) \neq 0$ on (c, d) and H_k and G_k are polynomials of degree k.

2.5 Monotonicity of zeros

Andrey Markov (1856-1922), a Russian mathematician, showed in 1886 [49] how the zeros of a polynomial depend on the parameters of the weight function. A useful consequence of his monotonicity theorem is proved by Ismail:

Theorem 2.5.1 *[36, p. 205, Theorem 7.1.2] The zeros of a Jacobi polynomial $P_n^{\alpha,\beta}(x)$ or a Hahn polynomial $Q_n(x; \alpha, \beta, N)$ increase with β and decrease with α. The zeros of a Meixner polynomial $M_n(x; \beta, c)$ increase with β while the zeros of a Laguerre polynomial $L_n^{\alpha}(x)$ increase with α. In all these cases, increasing (decreasing) means strictly increasing (decreasing) and the parameters are such that the polynomials are orthogonal.*

In [22], the speed at which the value of the function $1 - x_{n,i},\ i = 1, 2, \ldots, n$, decreases as β increases, where $x_{n,i},\ i = 1, 2, \ldots, n$, are the zeros of $P_n^{\alpha,\beta}(x)$, is investigated. The monotonicity of the extreme zeros of the associated Jacobi polynomials was applied in [30], to prove interlacing of zeros of these polynomials with shifted parameters.

2.6 An electrostatic interpretation of the zeros of Jacobi polynomials

We discuss a one-dimensional energy model of Stieltjes (cf. [65], [66]) to show how the zeros of Jacobi polynomials can be interpreted electrostatically. This model is described by Szegö [68, p. 140-142], who also provides similar interpretations for the zeros of Hermite and Laguerre polynomials. We refer the reader to [37], where Ismail proved that the zeros of general orthogonal polynomials determine the equilibrium positions of n movable charges in an external electrostatic field determined by the weight function.

Consider two positive line charges, q_1 and q_2, positioned along the real line, with a distance of d units between them. If k_1 and k_2 are constants, the one-dimensional force between the two line charges is $F = k_1 \frac{q_1 q_2}{d}$ (cf. [31, p. 5-3, Unit 5-5]) and therefore each line charge generates a logarithmic energy or potential field $E = -k_2 q_1 q_2 \ln d$. We will use this formula to determine the energy in the following one-dimensional model.

Let n positive unit line charges, positioned at $x_1, x_2, \ldots, x_n$, be free to move between -1 and 1, i.e.,

$$-1 < x_1 < x_2 < \cdots < x_n < 1.$$

At -1 and 1 we position positive line charges q and p respectively. For convenience, we let $n = 3$. The energy of this system, which is the mutual energy of all these charges, is given by the expression

$$\begin{aligned} E(x) &= E(x_1, x_2, x_3) \\ &= -q \sum_{i=1}^{3} \ln(1 + x_i) - p \sum_{i=1}^{3} \ln(1 - x_i) - \sum_{i<j} \ln(x_j - x_i) - pq \ln 2. \end{aligned} \tag{2.4}$$

The positions of the unit charges where the energy of the system will reach a minimum, are fixed and given by

$$\frac{\partial E}{\partial x_i} = 0, i = 1, 2, 3.$$

Differentiating (2.4) with respect to x_1, yields

$$\frac{\partial E}{\partial x_1} = \frac{p}{x_1 - 1} + \frac{q}{x_1 + 1} + \frac{1}{x_1 - x_2} + \frac{1}{x_1 - x_3} = 0. \tag{2.5}$$

Let $f(x)$ be a polynomial that vanishes at the points x_1, x_2 and x_3:

$$f(x) = (x - x_1)(x - x_2)(x - x_3);$$

now equation (2.5) can be written as

$$\frac{f''(x_1)}{2f'(x_1)} + \frac{p}{x_1 - 1} + \frac{q}{x_1 + 1} = 0$$

and

$$(1 - x_1^2) f''(x_1) + \Big(2q - 2p - (2p + 2q)x_1\Big) f'(x_1) = 0.$$

We thus have the differential equation

$$(1-x^2)f''(x)+\Big(2q-2p-(2p+2q)x\Big)f'(x)=Kf(x)$$

which holds for all x and $f(x)$ is in this case a n dimensional polynomial that vanishes at $x_i, i = 1, 2, \ldots, n$.

Let $\alpha = 2p-1$ and $\beta = 2q-1$. By comparing coefficients of x^n, we find that $K = -n(n+\alpha+\beta+1)$ and obtain the differential equation satisfied by the Jacobi polynomials (1.5).

We conclude that the zeros of Jacobi polynomials $P_n^{\alpha,\beta}$ have a simple electrostatic application; they coincide with the equilibrium positions of n movable unit line charges, that are free to move between two charges $\frac{\beta+1}{2}$ and $\frac{\alpha+1}{2}$, fixed at -1 and 1, respectively.

Furthermore, increasing the parameter α corresponds to increasing the positive line charge at 1 and hence the electrostatic interpretation of the zeros beautifully illustrates the fact, due to Markov's monotonicity theorem, that the zeros of Jacobi polynomials are decreasing functions of α and increasing functions of β (see Theorem 2.5.1).

Chapter 3

Stieltjes interlacing of zeros of Jacobi polynomials from different sequences

3.1 Introduction

In this chapter we investigate the extent to which Stieltjes interlacing holds between the zeros of two Jacobi polynomials if each polynomial belongs to a sequence generated by a different value of one/both of the parameters α and β. We also identify, in each case, a polynomial that plays the role of the de Boor-Saff polynomial (cf. [12] and [19]) in the sense that its zeros provide a (non-unique) set of points that complete the interlacing process. These polynomials are completely determined by the coefficients in a mixed three term recurrence relation.

We know that within a sequence of orthogonal polynomials, two polynomials of adjacent degree cannot have common zeros. This is not the case when the degrees differ by two or more units and in our theorems we also consider the possibility that the polynomials under consideration can have common zeros.

In Section 3.2 we investigate the zeros of Jacobi polynomials where the degrees of the polynomials differ by two units. We consider the zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-1}^{\alpha+j,\beta+k}$ for some integer values of j and k. In each different case we determine a point that completes the interlacing process. For convenience of the reader, we list our results in this section.

In Section 3.3 we prove the results of Section 3.2. We use mixed three term recurrence relations, obtained from a useful *Maple* computer package, (cf. [71]), as well as the computer program *Mathematica*, to prove our results.

In Section 3.4 we prove the more general result that the zeros of $P_{n+1}^{\alpha,\beta}$ interlace with the zeros of the kth derivative of $P_n^{\alpha,\beta}$, $k = 1, 2, 3 \ldots, n-1$.

The results of sections 3.2 and 3.4 were applied by Driver and Jordaan [28] to obtain inner bounds for the extreme zeros of Jacobi polynomials. They showed that the extra interlacing points obtained from the mixed three term recurrence relations, form these inner bounds. In Section 3.5 we discuss these results and compare the different extra interlacing points with one another, as well as with the bounds for extreme zeros of Jacobi polynomials obtained by Szegö, as shown in Section 2.4.

In Section 3.6 we interpret our interlacing results from a one-dimensional electrostatic perspective.

3.2 Stieltjes interlacing of zeros of Jacobi polynomials from different sequences, whose degrees differ by two.

We recall that $P_n^{\alpha,\beta}$, $\alpha, \beta > -1$, is the Jacobi polynomial of degree n, orthogonal with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$ on the interval $[-1, 1]$.

Our first four results consider cases when Stieltjes interlacing occurs between the zeros of Jacobi polynomials from different sequences, whose degrees differ by two. In each theorem we firstly consider the case when the polynomials under consideration are co-prime, i.e., they do not have common zeros and secondly, we consider the possibility that the polynomials do have common zeros, in which case it directly follows from the mixed three term recurrence relation (1.21)

$$(x - b_n)p_n(x) = a_n p_{n+1}(x) + c_n g_{n-1}(x),$$

that there can be only one common zero that is equal to the point b_n, since p_n and p_{n+1} are co-prime.

In our first two results we change only one of the parameters, keeping the other one fixed, and consider the zeros of $P_{n-1}^{\alpha+t,\beta}$ (respectively $P_{n-1}^{\alpha,\beta+t}$) and $P_{n+1}^{\alpha,\beta}$ for $t \in \{0, 1, 2, 3, 4\}$.

Theorem 3.2.1

(i) If $P_{n-1}^{\alpha+t,\beta}$ *and* $P_{n+1}^{\alpha,\beta}$ *are co-prime, then*

(a) the zeros of $P_{n-1}^{\alpha+t,\beta}$*, together with the point* $B_n(t,0) = \frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$ *for fixed* $t \in \{0,1,2\}$*;*

(b) the zeros of $P_{n-1}^{\alpha+3,\beta}$*, together with the point* $B_n(3,0) = \frac{n(n+\alpha+\beta+2)+(\alpha+2)(n-\alpha+\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*

(c) the zeros of $P_{n-1}^{\alpha+4,\beta}$*, together with the point* $B_n(4,0) = \frac{2n(n+\alpha+\beta+3)+(\alpha+3)(\beta-\alpha)}{2n(n+\alpha+\beta+3)+(\alpha+3)(\alpha+\beta+2)}$ *interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(ii) If $P_{n-1}^{\alpha+t,\beta}$ *and* $P_{n+1}^{\alpha,\beta}$ *are not co-prime, they have one common zero located at the respective points identified in (i) (a) to (c) and the* $n-1$ *zeros of* $P_{n-1}^{\alpha+t,\beta}$ *interlace with the remaining* n *(non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

Since Jacobi polynomials satisfy the symmetry property (1.7), we immediately obtain the following Corollary of Theorem 3.2.1.

Corollary 3.2.2

(i) If $P_{n-1}^{\alpha,\beta+t}$ *and* $P_{n+1}^{\alpha,\beta}$ *are co-prime, then*

(a) The zeros of $P_{n-1}^{\alpha,\beta+t}$*, together with the point* $B_n(0,t) = \frac{\beta^2-\alpha^2-t(\alpha-\beta+2n(n+\alpha+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$ *for fixed* $t \in \{1,2\}$*;*

(b) The zeros of $P_{n-1}^{\alpha,\beta+3}$*, together with the point* $B_n(0,3) = -\frac{n(n+\alpha+\beta+2)+(\beta+2)(n-\beta+\alpha)}{(n+\beta+2)(n+\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*

(c) The zeros of $P_{n-1}^{\alpha,\beta+4}$*, together with the point* $B_n(0,4) = -\frac{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha-\beta)}{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(ii) If $P_{n-1}^{\alpha,\beta+t}$ *and* $P_{n+1}^{\alpha,\beta}$ *are not co-prime, they have one common zero located at the respective points identified in (i) (a) to (c) and the* $n-1$ *zeros of* $P_{n-1}^{\alpha,\beta+t}$ *interlace with the remaining* n *(non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

Remarks.

(1) The case $t = 0$ in Theorem 3.2.1(i) was proved by Segura in [63, p. 391, Theorem 1]. In his theorem, Segura provides a general formula for the extra interlacing point:

$$A_n = \frac{\int_a^b x p_n^2(x) w(x) dx}{\int_a^b p_n^2(x) w(x) dx} ,$$

where $p_n(x)$ is either a Hermite, Laguerre or Jacobi polynomial, orthogonal on $[a, b]$, with respect to the weight function $w(x)$. Segura also states that if p_{n+1} and p_{n-1} have a common zero, this zero is located at the extra interlacing point A_n and in the Jacobi case,

$$A_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta + 1)^2 - 1} = B_n(0, 0).$$

For completeness and convenience of the reader, we included this result in Theorem 3.2.1 together with an alternative proof.

(2) Segura [63, p. 391] also proved that when $t = 0$, Theorem 3.2.1(i) does not only hold for the classical parameter ranges, i.e., $\alpha, \beta > -1$, but for any values of the parameters such that $P_{n+1}^{\alpha,\beta}$ and $P_{n-1}^{\alpha,\beta}$ have zeros in the classical interval of orthogonality $(-1, 1)$.

(3) The extra interlacing point $B_n(2, 0) = \dfrac{\beta - \alpha + 2n}{2n + \alpha + \beta + 2}$ is equal to $B_{n+1,L}$, the lower bound for the largest zero of the polynomial $P_{n+1}^{\alpha,\beta}(x)$, as given by Szegő (cf. (2.3)), and $B_n(0, 2) = \dfrac{\beta - \alpha - 2n}{2n + \alpha + \beta + 2} = B_{n+1,H}$, the upper bound for the smallest zero of the polynomial $P_{n+1}^{\alpha,\beta}(x)$ as in (2.2).

Numerical experiments suggest that results analogous to those proved in Theorem 3.2.1 and its Corollary also hold as t varies continuously between 0 and 4.

Conjecture
For $t \in (0, 2)$, if $P_{n-1}^{\alpha+t,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime, the zeros of $P_{n-1}^{\alpha+t,\beta}$, together with the point

$$\frac{\beta^2 - \alpha^2 + t\,(\beta - \alpha + 2n(n + \beta + 1))}{(2n + \alpha + \beta + t)(2n + \alpha + \beta + 2)},$$

interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.

Our next two results prove that Stieltjes interlacing of the zeros of Jacobi polynomials from different sequences also holds when both the parameters α and β change within certain constraints.

Theorem 3.2.3

(i) For each fixed $j, k \in \{1, 2\}$*, if* $P_{n-1}^{\alpha+j,\beta+k}$ *and* $P_{n+1}^{\alpha,\beta}$

(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+j,\beta+k}$*, together with the point* $\frac{\beta-\alpha-n(k-j)}{\alpha+\beta+2+n(4-j-k)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*

(b) are not co-prime, they have one common zero located at the point identified in (i) (a) and the $n-1$ *zeros of* $P_{n-1}^{\alpha+j,\beta+k}$ *interlace with the* n *remaining (non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(ii) If $P_{n-1}^{\alpha+3,\beta+1}$ *and* $P_{n+1}^{\alpha,\beta}$

(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+3,\beta+1}$*, together with the point* $\frac{n^2+n(\alpha+\beta+3)-(\alpha+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\alpha+2)(\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*

(b) are not co-prime, then they have one common zero located at the point identified in (ii)(a) and the $n-1$ *zeros of* $P_{n-1}^{\alpha+3,\beta+1}$ *interlace with the* n *remaining (non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

(iii) If $P_{n-1}^{\alpha+1,\beta+3}$ *and* $P_{n+1}^{\alpha,\beta}$

(a) are co-prime, then the zeros of $P_{n-1}^{\alpha+1,\beta+3}$*, together with the point* $\frac{-n^2-n(\alpha+\beta+3)-(\beta+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\beta+2)(\alpha+\beta+2)}$*, interlace with the zeros of* $P_{n+1}^{\alpha,\beta}$*;*

(b) are not co-prime, then they have one common zero located at the point identified in (iii)(a) and the $n-1$ *zeros of* $P_{n-1}^{\alpha+1,\beta+3}$ *interlace with the* n *remaining (non-common) zeros of* $P_{n+1}^{\alpha,\beta}$*.*

Theorem 3.2.4

(i) If the respective pairs of polynomials are co-prime, then

(a) the zeros of $P_{n-1}^{\alpha-1,\beta+1}$*, together with the point* $\frac{\alpha+\beta}{2n+\alpha+\beta}$*,*

(b) the zeros of $P_{n-1}^{\alpha-1,\beta+2}$*, together with the point* $\frac{-n+\beta+1}{n+\beta+1}$*,*

(c) the zeros of $P_{n-1}^{\alpha+1,\beta-1}$*, together with the point* $\frac{-\alpha-\beta}{2n+\alpha+\beta}$*,*

(d) the zeros of $P_{n-1}^{\alpha+2,\beta-1}$*, together with the point* $\frac{n-\alpha-1}{n+\alpha+1}$*,*

interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.

(ii) *If the respective pairs of polynomials in (i) (a) to (d) are not co-prime, then they have one common zero located at the points identified in (i) (a) to (d) and the* $n-1$ *zeros of the respective polynomial of degree* $n-1$ *in each case, interlace with the* n *(non-common) zeros of* $P_{n+1}^{\alpha,\beta}$.

Remark. Restrictions on the ranges of t and k are required in our theorems since, in general, Stieltjes interlacing is not retained between the zeros of Jacobi polynomials from different sequences, whose degrees differ by two. Using *Mathematica*, we see that

- When $n = 5$, $\alpha = 0.1$ and $\beta = 0.1$, the zeros of $P_6^{\alpha,\beta}$ and $P_4^{\alpha+5,\beta}$ or $P_4^{\alpha,\beta-1}$ do not interlace, illustrating that Stieltjes interlacing does not hold in general for $t > 4$, $k = 0$ or $t = 0$, $k < 0$.
- When $t = k = -1$ and α, β and n are chosen as in the example above, the zeros of $P_4^{\alpha-1,\beta-1}$ and $P_6^{\alpha,\beta}$ do not interlace.
- The zeros of $P_8^{\alpha,\beta}$ and those of $P_6^{\alpha+4,\beta+1}$ or $P_6^{\alpha+3,\beta+2}$ do not interlace when $\alpha = -0.9$ and $\beta = 329.3$.

3.3 Proofs of results given in Section 3.2

In our proofs we make use of the connection between Jacobi and ${}_2F_1$ hypergeometric polynomials, as well as contiguous function relations satisfied by ${}_2F_1$ polynomials. These contiguous relations are given in Appendix A.

The following Lemma simplifies the proofs of Theorem 3.2.1 and Theorem 3.2.3.

Lemma 3.3.1 *Let* $\{p_n\}_{n=0}^{\infty}$ *be a sequence of polynomials orthogonal on the (finite or infinite) interval* (c,d). *Let* g_{n-1} *be any polynomial of degree* $n-1$ *that for each* $n \in \mathbb{N}$ *satisfies*

$$g_{n-1}(x) = a_n(x)p_{n+1}(x) - (x - A_n)b_n(x)p_n(x) \tag{3.1}$$

for some constant A_n *and some functions* $a_n(x)$ *and* $b_n(x)$ *defined on* (c,d), *such that* $b_n(x)$ *does not change sign in* (c,d). *Then, for each* $n \in \mathbb{N}$, *it follows that* *(i) if g_{n-1} and p_{n+1} are co-prime, the zeros of g_{n-1} are all real and simple and, together with the point A_n, they interlace with the zeros of p_{n+1};*

(ii) if g_{n-1} and p_{n+1} are not co-prime, they have one common zero located at $x = A_n$ and the $n-1$ zeros of g_{n-1} interlace with the n (non-common) zeros of p_{n+1}.

Proof. Let $x_{n+1,1} < x_{n+1,2} < \cdots < x_{n+1,n+1}$ denote the zeros of p_{n+1}.

(i) Since p_n and p_{n+1} are always co-prime while by assumption $b_n(x) \neq 0$ for $x \in (c,d)$ and p_{n+1} and g_{n-1} are co-prime, we deduce from (3.1) that $A_n \neq x_{n+1,k}$ for any $k \in \{1,2,\ldots,n+1\}$. Evaluating (3.1) at $x_{n+1,k}$ and $x_{n+1,k+1}$, we obtain

$$g_{n-1}(x_{n+1,k}) = -(x_{n+1,k} - A_n)b_n(x_{n+1,k})p_n(x_{n+1,k})$$

and

$$g_{n-1}(x_{n+1,k+1}) = -(x_{n+1,k+1} - A_n)b_n(x_{n+1,k+1})p_n(x_{n+1,k+1}).$$

We combine these two equations, to obtain

$$\frac{g_{n-1}(x_{n+1,k})g_{n-1}(x_{n+1,k+1})}{p_n(x_{n+1,k})p_n(x_{n+1,k+1})} = (x_{n+1,k} - A_n)(x_{n+1,k+1} - A_n)b_n(x_{n+1,k})b_n(x_{n+1,k+1}), \quad (3.2)$$

for each $k \in \{1,2,\ldots,n\}$. Since $x_{n+1,k}$ and $x_{n+1,k+1} \in (c,d)$ while b_n does not change sign in (c,d), we know that $b_n(x_{n+1,k})b_n(x_{n+1,k+1}) > 0$. Hence the right-hand side of (3.2) is positive if and only if $A_n \notin (x_{n+1,k}, x_{n+1,k+1})$. Since $p_n(x_{n+1,k})p_n(x_{n+1,k+1}) < 0$ for each $k \in \{1,2,\ldots,n\}$ because the zeros of p_n and p_{n+1} are interlacing, we deduce that, provided $A_n \notin (x_{n+1,k}, x_{n+1,k+1})$, g_{n-1} has a different sign at consecutive zeros of p_{n+1} and therefore has an odd number of zeros (counting multiplicity) in each interval $(x_{n+1,k}, x_{n+1,k+1})$, $k \in \{1,2,\ldots,n\}$, apart from one interval that may contain the point A_n. Since there are exactly n intervals $(x_{n+1,k}, x_{n+1,k+1})$, $k \in \{1,2,...,n\}$, it follows that the $n-1$ zeros of g_{n-1} are real, simple and, together with the point A_n, interlace with the $n+1$ zeros of p_{n+1}.

(ii) If p_{n+1} and g_{n-1} have common zeros, (3.1) implies that there can only be one common zero at $x = A_n$, since p_n and p_{n+1} are co-prime. For $x \neq A_n$ we can rewrite (3.1) as

$$\frac{g_{n-1}(x)}{x - A_n} = \frac{a_n(x)p_{n+1}(x)}{x - A_n} - b_n(x)p_n(x),$$

or

$$G_{n-2}(x) = a_n(x)P_n(x) - b_n(x)p_n(x), \tag{3.3}$$

where $(x - A_n)G_{n-2}(x) = g_{n-1}(x)$ and $(x - A_n)P_n(x) = p_{n+1}(x)$. Note that the zeros of P_n are exactly the n (non-common) zeros of p_{n+1}, i.e., the zeros of p_{n+1} excluding the single zero at A_n, that is also a zero of g_{n-1}. At most one interval of the form $(x_{n+1,k}, x_{n+1,k+1})$, $k \in \{1, \ldots, n-1\}$, can contain the point A_n. Evaluating (3.3) at $x_{n+1,k}$ and $x_{n+1,k+1}$, for each $k \in \{1, \ldots, n-1\}$ such that $A_n \notin (x_{n+1,k}, x_{n+1,k+1})$, we obtain

$$G_{n-2}(x_{n+1,k})G_{n-2}(x_{n+1,k+1}) = b_n(x_{n+1,k})b_n(x_{n+1,k+1})p_n(x_{n,k})p_n(x_{n+1,k+1}) < 0$$

and it follows that G_{n-2} has an odd number of zeros in each interval $(x_{n+1,k}, x_{n+1,k+1})$, $k \in \{1, 2, \ldots, n-1\}$, that does not contain A_n. Deg $(G_{n-2}) = n-2$ and there are $n-2$ of these intervals and therefore each interval $(x_{n+1,k}, x_{n+1,k+1})$, $k \in \{1, 2, \ldots, n-1\}$, that does not contain A_n, has exactly one zero of G_{n-2}. We deduce that $A_n = x_{n+1,j}$ where $j \in \{2, \ldots, n\}$ and the zeros of G_{n-2}, together with the point A_n, interlace with the n zeros of P_n. The stated result is then an immediate consequence of the definitions of G_{n-2} and P_n.

■

Remark 3.3.2 A theorem due to Gibson (cf. [33, p. 130]) proves that if $\{p_n\}_{n=0}^{\infty}$ is any orthogonal sequence, the polynomials p_{n+1} and p_m, $m = 1, 2, \ldots, n-1$ can have at most $\min\{m, n-m\}$ common zeros, consequently p_{n+1} and p_{n-1} can have at most $\min\{n-1, 1\} = 1$ common zero. Lemma 3.3.1 (ii) extends Gibson's result to any polynomials of degree $n-1$ and $n+1$ that satisfy a mixed three term recurrence relation of the form (3.1).

Proof of Theorem 3.2.1.

(i) (a) If $t = 0$, the result follows from (1.6) and Lemma 3.3.1(i). For $t = 1$, we use (7.9) with $b = n + \alpha + \beta + 1$ and $c = \alpha + 1$, together with (1.4) to obtain

$$\begin{aligned} &\left(x - \frac{\beta^2 - \alpha^2 + \beta - \alpha + 2n(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x) \\ &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) + \frac{(1-x)(n+\beta)}{2n+\alpha+\beta+1} P_{n-1}^{\alpha+1,\beta}(x) \end{aligned}$$

and the result follows from Lemma 3.3.1(i).

For $t = 2$, the stated result follows from

$$\begin{aligned}&\left(x - \frac{2n-\alpha+\beta}{2n+\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{2(\alpha+1)(n+1)}{(n+\alpha+1)(2n+\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) + \frac{(1-x)^2(n+\beta)}{2(n+\alpha+1)} P_{n-1}^{\alpha+2,\beta}(x),\end{aligned}$$

obtained from (7.10) and (1.4), together with Lemma 3.3.1(i).

(b) Replacing b by $n+\alpha+\beta+1$, c by $\alpha+1$ and z by $\frac{1-x}{2}$ in (7.16) and using (1.4), we obtain

$$\begin{aligned}&\left(x - \frac{n^2+(2\alpha+\beta+4)-(\alpha+2)(\alpha-\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{(n+1)A(x)P_{n+1}^{\alpha,\beta}(x)}{(n+\alpha+1)(n+\alpha+2)(n+\alpha+\beta+2)} + \frac{(1-x)^3(2n+\alpha+\beta+2)(n+\beta)}{4(n+\alpha+1)(n+\alpha+2)} P_{n-1}^{\alpha+3,\beta}(x),\end{aligned}$$

where $A(x) = n(n+\beta)(x-1) + 2(\alpha+1)(\alpha+2)$. Lemma 3.3.1(i) then yields the result.

(c) From (7.19) and (1.4) we have

$$\begin{aligned}&\left(x - \frac{2n^2-(\alpha+3)(\alpha-\beta)+2n(\alpha+\beta+3)}{C_n}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{-(n+1)B(x)}{2(n+\alpha+1)(\alpha+2)C_n} P_{n+1}^{\alpha,\beta}(x) + \frac{(1-x)^4 D_n}{8(n+\alpha+1)(\alpha+2)C_n} P_{n-1}^{\alpha+4,\beta}(x),\end{aligned}$$

where

$$\begin{aligned}C_n &= 2n(n+\alpha+\beta+3) + (\alpha+3)(\alpha+\beta+2) \text{ and}\\ D_n &= (2n+\alpha+\beta+2)(n+\beta)(n+\alpha+\beta+2)(n+\alpha+\beta+3)\end{aligned}$$

and $B(x)$ is a polynomial of degree 2 in x which depends on n, α and β,

$$\begin{aligned}B(x) &= 2n^3(x-1)^2 - 4(6+11\alpha+6\alpha^2+\alpha^3)\\ &\quad -n\beta(x-1)(10+5\alpha+\beta-x(2+\alpha+\beta)) - n^2(x-1)(10+5\alpha+3\beta-x(2+\alpha+3\beta)).\end{aligned}$$

The result follows from Lemma 3.3.1(i).

(ii) This follows immediately from Lemma 3.3.1(ii) and the proofs of Theorem 3.2.1(i)(a) to (c).

■

Proof of Theorem 3.2.3.

(i) (a) The case when $j = k = 1$ will be proved in Theorem 3.4.1. For $j = k = 2$, (7.17) and (1.4) yield

$$\begin{aligned}&\left(x - \frac{\beta-\alpha}{\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{2(n+1)C(x)}{(n+\alpha+1)(n+\beta+1)(\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) + E_n(1-x^2)^2 P_{n-1}^{\alpha+2,\beta+2}(x),\end{aligned}$$

where

$$\begin{aligned} E_n &= \frac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)(2n+\alpha+\beta+2)}{8(n+\alpha+1)(n+\beta+1)(\alpha+\beta+2)} \quad \text{and} \\ C(x) &= (1+\alpha)(1+n+\beta) + n(1-x)(1+n+\beta) - \frac{1}{4}n(1-x)^2(2+2n+\alpha+\beta). \end{aligned}$$

The result follows from Lemma 3.3.1(i). For $j = 1, k = 2$, the mixed three term recurrence relation

$$\begin{aligned}&\left(x + \frac{n+\alpha-\beta}{n+\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{(n(x+1)+2\beta+2)(n+1)}{(n+\alpha+\beta+2)(n+\beta+1)} P_{n+1}^{\alpha,\beta}(x) + \frac{(1+x)^2(1-x)(2n+\alpha+\beta+2)}{4(n+\beta+1)} P_{n-1}^{\alpha+1,\beta+2}(x)\end{aligned}$$

is obtained from (1.4) together with (7.11). Lemma 3.3.1(i) then yields the stated result. For $j = 2, k = 1$, the result follows from the symmetry property (1.7).

(b) This follows immediately from Lemma 3.3.1(ii) and the proof of (a).

(ii) (a) From (1.4) and (7.18), we obtain the mixed three term recurrence relation

$$\begin{aligned}&\left(x - \frac{n^2 - (\alpha+2)(\alpha-\beta) + n(\alpha+\beta+3)}{n^2 + n(\alpha+\beta+3) + (\alpha+2)(\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x)\\ &= \frac{D(x)(n+1)}{2(n+\alpha+1)(n^2+(\alpha+2)(\alpha+\beta+2)+n(\alpha+\beta+3))} P_{n+1}^{\alpha,\beta}(x)\\ &+ \frac{(1-x)^3(1+x)(n+\alpha+\beta+2)(n+\alpha+\beta+3)(2n+\alpha+\beta+2)}{8(n^2+(\alpha+2)(\alpha+\beta+2)+n(\alpha+\beta+3))(n+\alpha+1)} P_{n-1}^{\alpha+3,\beta+1}(x),\end{aligned}$$

where

$$D(x) = 4(\alpha+1)(\alpha+2) + (3\alpha-\beta+4)n - 2nx(n+2\alpha+3) + nx^2(2n+\alpha+\beta+2)$$

and Lemma 3.3.1(i) then yields the stated result.

(b) This follows immediately from Lemma 3.3.1(ii) and the proof of (a).

(iii) Both results follow directly from the symmetry property (1.7).

■

We omit the proof of Theorem 3.2.4 which follows exactly the same reasoning as the proofs of Theorems 3.2.1 and 3.2.3.

3.4 Stieltjes interlacing of zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ where $k = 1, 2, \ldots, n-1$.

We now state a general result for Stieltjes interlacing between the zeros of $P_{n+1}^{\alpha,\beta}$ and the $n-k$ zeros of $P_{n-k}^{\alpha+k,\beta+k}$.

Theorem 3.4.1 *Let $P_n^{\alpha,\beta}$, $\alpha, \beta > -1$, $n \in \mathbb{N}$, denote the Jacobi polynomial of degree n.*

(i) For each $k \in \{1, 2, \ldots, n-1\}$, there exist polynomials G_k and H_k of degree k such that

$$(1-x^2)^k Q_{n,k} P_{n-k}^{\alpha+k,\beta+k}(x) = (n+1) H_{k-1}(x) P_{n+1}^{\alpha,\beta}(x) + G_k(x) P_n^{\alpha,\beta}(x) \tag{3.4}$$

where $Q_{n,k} = \dfrac{(n+\alpha+\beta+2)_{k-1}(2n+\alpha+\beta+2)}{2^{2k}}$ and $(\ )_k$ denotes the Pochhammer symbol (1.2).

(ii) Let $k \in \{1, 2, \ldots, n-1\}$ be fixed. If $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ are co-prime, then the zeros of the kth derivative of $P_n^{\alpha,\beta}$, together with the k real zeros of G_k, interlace with the zeros of $P_{n+1}^{\alpha,\beta}$.

(iii) Let $k \in \{1, 2, \ldots, n-1\}$ be fixed. If $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ have r common zeros, then the $(n-2r)$ non-common zeros of the product $G_k P_{n-k}^{\alpha+k,\beta+k}$, together with the r common zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$, interlace with the $(n+1-r)$ non-common zeros of $P_{n+1}^{\alpha,\beta}$.

Proof.

(i) We use the mixed three term recurrence relations

$$\left(1-x^2\right) P_{n-1}^{\alpha+1,\beta+1}(x) = 2\left(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\right) P_n^{\alpha,\beta}(x) - \frac{4(n+1)}{2n+\alpha+\beta+2} P_{n+1}^{\alpha,\beta}(x) \tag{3.5}$$

and

$$\begin{aligned}&\left(1-x^2\right) P_n^{\alpha+1,\beta+1}(x) \qquad (3.6)\\ =\ & \frac{2}{n+\alpha+\beta+2}\left(\frac{2(n+\beta+1)(n+\alpha+1)}{2n+\alpha+\beta+2} P_n^{\alpha,\beta}(x)-(n+1)\left(x-\frac{\alpha-\beta}{2n+\alpha+\beta+2}\right) P_{n+1}^{\alpha,\beta}(x)\right)\end{aligned}$$

which can be obtained from (1.4), (7.12) and (7.13). We prove our result by induction on k.

For $k=1$, equation (3.4) is the same as (3.5) with $H_0(x)=-1$, $G_1(x)=\frac{1}{2}\left((2n+\alpha+\beta+2)x+\alpha-\beta\right)$ and $Q_{n,1}=\frac{1}{4}(2n+\alpha+\beta+2)$. Therefore (3.4) holds for $k=1$.

Next, we assume that the result holds for $m=1,2,\ldots,k$, i.e., we assume that

$$(1-x^2)^m Q_{n,m} P_{n-m}^{\alpha+m,\beta+m}(x)=(n+1)H_{m-1}(x)P_{n+1}^{\alpha,\beta}(x)+G_m(x)P_n^{\alpha,\beta}(x) \qquad (3.7)$$

with G_m and H_m polynomials of degree m and

$$Q_{n,m}=\frac{(n+\alpha+\beta+2)_{m-1}(2n+\alpha+\beta+2)}{2^{2m}} \quad \text{for } m=1,2,\ldots,k.$$

For $m=k+1$, the left-hand side of (3.4) is equal to

$$\begin{aligned}&(1-x^2)^{k+1}Q_{n,k+1}P_{n-k-1}^{\alpha+k+1,\beta+k+1}(x)\\ =\ & \frac{(1-x^2)^{k+1}}{2^{2(k+1)}}(\alpha+\beta+n+2)_k(\alpha+\beta+2n+2)P_{n-k-1}^{\alpha+k+1,\beta+k+1}(x)\\ =\ & \frac{1-x^2}{4}(\alpha_1+\beta_1+n_1+1)\frac{(1-x^2)^k}{2^{2k}}(\alpha_1+\beta_1+n_1+2)_{k-1}(\alpha_1+\beta_1+2n_1+2)P_{n_1-k}^{\alpha_1+k,\beta_1+k}(x),\\ & \qquad \text{where } \alpha_1=\alpha+1,\ \beta_1=\beta+1,\ n_1=n-1\\ =\ & \frac{1-x^2}{4}(\alpha_1+\beta_1+n_1+1)\Big((n_1+1)H_{k-1}(x)P_{n_1+1}^{\alpha_1,\beta_1}(x)+G_k(x)P_{n_1}^{\alpha_1,\beta_1}(x)\Big)\\ =\ & \frac{1-x^2}{4}(\alpha+\beta+n+2)\Big(nH_{k-1}(x)P_n^{\alpha+1,\beta+1}(x)+G_k(x)P_{n-1}^{\alpha+1,\beta+1}(x)\Big)\end{aligned}$$

by using the induction hypothesis.

Applying (3.5) and (3.6), a straightforward calculation shows that this equals

$$G_{k+1}(x)P_n^{\alpha,\beta}(x)+(n+1)H_k(x)P_{n+1}^{\alpha,\beta}(x)$$

with

$$H_k(x) = \frac{-n}{2}\left(x - \frac{\alpha-\beta}{2n+\alpha+\beta+2}\right)H_{k-1}(x) - \frac{n+\alpha+\beta+2}{2n+\alpha+\beta+2}G_k(x)$$

and

$$G_{k+1}(x) = \frac{n(n+\alpha+1)(n+\beta+1)}{2n+\alpha+\beta+2}H_{k-1}(x) + \frac{n+\alpha+\beta+2}{2}\left(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\right)G_k(x),$$

which is the righthand-side of (3.4) for $m = k+1$. It follows that (3.7) holds for $m = k+1$ and the result follows by induction on k.

(ii) We recall (1.8) that

$$D^k P_n^{\alpha,\beta}(x) = \frac{1}{2^k}(n+\alpha+\beta+1)_k P_{n-k}^{\alpha+k,\beta+k}(x).$$

From (3.4), provided $P_{n+1}^{\alpha,\beta}(x) \neq 0$, we have

$$\frac{(1-x^2)^k Q_{n,k} P_{n-k}^{\alpha+k,\beta+k}(x)}{P_{n+1}^{\alpha,\beta}(x)} = (n+1)H_{k-1}(x) + \frac{G_k(x)P_n^{\alpha,\beta}(x)}{P_{n+1}^{\alpha,\beta}(x)}. \tag{3.8}$$

Now, if $x_{n+1,1} < x_{n+1,2} < \cdots < x_{n+1,n+1}$ are the zeros of $P_{n+1}^{\alpha,\beta}$, we have

$$\frac{P_n^{\alpha,\beta}(x)}{P_{n+1}^{\alpha,\beta}(x)} = \sum_{j=1}^{n+1} \frac{A_j}{x - x_{n+1,j}}$$

where $A_j > 0$ for each $j \in \{1, \ldots, n+1\}$ (cf. [68, p. 47, Theorem 3.3.5]). Therefore (3.8) can be written as

$$\frac{(1-x^2)^k Q_{n,k} P_{n-k}^{\alpha+k,\beta+k}(x)}{P_{n+1}^{\alpha,\beta}(x)} = (n+1)H_{k-1}(x) + \sum_{j=1}^{n+1}\frac{G_k(x)A_j}{x - x_{n+1,j}}, \quad x \neq x_{n+1,j}. \tag{3.9}$$

Since $P_{n+1}^{\alpha,\beta}$ and $P_n^{\alpha,\beta}$ are always co-prime while $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ are co-prime by assumption, it follows from (3.4) that $G_k(x_{n+1,j}) \neq 0$ for any $j \in \{1, 2, \ldots, n+1\}$. Suppose that G_k does not change sign in the interval $I_j = (x_{n+1,j}, x_{n+1,j+1})$ where $j \in \{1, 2, \ldots, n\}$. Since $A_j > 0$ and the polynomial H_{k-1} is bounded on I_j while the right hand side of (3.9) takes arbitrarily large positive and negative values, it follows that $P_{n-k}^{\alpha+k,\beta+k}$ must have an odd number of zeros in each interval in which G_k does not change sign. G_k is of degree k and there is a total amount of n intervals $(x_{n+1,j}, x_{n+1,j+1})$, $j \in \{1, \ldots, n\}$, therefore there are at least $n-k$ intervals $(x_{n+1,j}, x_{n+1,j+1})$, $j \in \{1, \ldots, n\}$ in which G_k does not change sign and so each of these intervals must contain exactly one of the $n-k$ real, simple zeros of $P_{n-k}^{\alpha+k,\beta+k}$. We deduce that the k zeros of G_k are real and simple and, together with the zeros of $P_{n-k}^{\alpha+k,\beta+k}$, interlace with the $n+1$ zeros of $P_{n+1}^{\alpha,\beta}$.

(iii) Assume that $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ have r common zeros. From (3.4) it follows that if $P_{n-k}^{\alpha+k,\beta+k}$ and $P_{n+1}^{\alpha,\beta}$ have any common zeros, these must also be zeros of G_k since $P_n^{\alpha,\beta}$ and $P_{n+1}^{\alpha,\beta}$ are co-prime. It follows that $r \leq \min\{k, n-k\}$ and there are at least $(n-2r)$ open intervals $I_j = (x_{n+1,j}, x_{n+1,j+1})$ with endpoints at successive zeros $x_{n+1,j}$ and $x_{n+1,j+1}$ of $P_{n+1}^{\alpha,\beta}$ where neither $x_{n+1,j}$ or $x_{n+1,j+1}$ is a zero of $P_{n-k}^{\alpha+k,\beta+k}$ or $G_k(x)$. If G_k does not change sign in an interval I_j, it follows from (3.9), since $A_j > 0$ and H_{k-1} is bounded while the right hand side takes arbitrarily large positive and negative values for $x \in I_j$, that $P_{n-k}^{\alpha+k,\beta+k}$ must have an odd number of zeros in that interval, i.e., if G_k does not have a zero in I_j, then $P_{n-k}^{\alpha+k,\beta+k}$ must have an odd number of zeros in that interval. Since this applies to at least $(n-2r)$ intervals I_j and $P_{n-k}^{\alpha+k,\beta+k}$ has exactly $(n-k-r)$ simple zeros that are not zeros of $P_{n+1}^{\alpha,\beta}$ while G_k has at most $(k-r)$ zeros that are not zeros of $P_{n+1}^{\alpha,\beta}$, it follows that there must be exactly $(n-2r)$ intervals $I_j = (x_{n+1,j}, x_{n+1,j+1})$ with endpoints at successive zeros $x_{n+1,j}$ and $x_{n+1,j+1}$ of $P_{n+1}^{\alpha,\beta}$ where neither $x_{n+1,j}$ or $x_{n+1,j+1}$ is a zero of $P_{n-k}^{\alpha+k,\beta+k}$. This implies that the common zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k}$ cannot be two consecutive zeros of $P_{n+1}^{\alpha,\beta}$ and the stated result now follows using the same argument as in (ii).

■

3.5 Some recent applications

In a recent paper by Driver and Jordaan [28], mixed three term recurrence relations, typically satisfied by classical orthogonal polynomials from different sequences obtained from integer shifts of the parameters, are used to derive strict upper (lower) bounds for the smallest (largest) zeros of some classical orthogonal polynomials. Their results apply to any polynomial p_n that is part of an orthogonal sequence satisfying a mixed three term recurrence relation of the form (cf. [28, p. 1202, Corollary 2.2])

$$f(x)g_{n-k}(x) = H_{k-1}(x)p_{n+1}(x) - G_k(x)p_n(x)$$

on an interval (c,d), where $f(x) \neq 0$ on (c,d) and H_k and G_k are polynomials of degree k. They deduce that, since interlacing holds for such polynomials $g_{n-k}(x)$ and $p_{n+1}(x)$, irrespective of whether the polynomials have common zeros or not, the extra interlacing point(s), which in this case will be the zeros of $G_k(x)$, will form bound(s) for the extreme zeros of the polynomial p_{n+1}. The authors apply their results to the Jacobi, Gegenbauer and Laguerre polynomials and compare them with corresponding bounds obtained by other methods.

Applying their result to (1.6) and the mixed three term recurrence relations (7.9), (7.10), (7.16) and (7.19), yields in each case a point $B_n(k,0)$, $k = 0,1,\ldots,4$, that will be a strict lower bound for the largest zero as well as a strict upper bound for the smallest zero of $P_{n+1}^{\alpha,\beta}$, i.e.,

$$x_{n+1,1} < B_n(k,0) < x_{n+1,n+1}, \qquad \text{for each } \ k = 0,1,\ldots,4.$$

From the definitions of the points $B_n(k,0), k = 0,1,\ldots,4$, we can prove that

$$B_n < B_n(1,0) < B_n(2,0) < B_n(3,0) < B_n(4,0)$$

for all values of $\alpha, \beta > -1$.

Considering also the extra interlacing points obtained in Corollary 3.2.2, denoted by $B_n(0,4) < B_n(0,3) < B_n(0,2) < B_n(0,1)$, we find the following bounds for the extreme zeros $x_{n+1,1}$ and $x_{n+1,n+1}$ of the Jacobi polynomial $P_{n+1}^{\alpha,\beta}$

$$\begin{aligned} -1 < x_{n+1,1} < B_n(0,4) < B_n(0,3) < B_n(0,2) = B_{n+1,H}(\alpha,\beta) < B_n(0,1) < B_n \\ < B_n(1,0) < B_n(2,0) = B_{n+1,L}(\alpha,\beta) < B_n(3,0) < B_n(4,0) < x_{n+1,n+1} < 1, \end{aligned}$$

where B_n is the value obtained from the three term recurrence relation (1.6), $B_{n+1,L}(\alpha,\beta)$ and $B_{n+1,H}(\alpha,\beta)$ are the bounds obtained using classical methods as discussed in Section 2.4.

The formulas obtained in Theorem 3.2.1 and Corollary 3.2.2, provide us with bounds for the extreme zeros of $P_{n+1}^{\alpha,\beta}(x)$, polynomials of degree $n+1$. For simplification and the sake of future reference, we shift the parameter n and show explicitly the best upper bound for the smallest zero of $P_n^{\alpha,\beta}(x)$ resulting from this method, i.e.,

$$B_{n-1}(0,4) = -\frac{2(n-1)(n+\alpha+\beta+2) + (\beta+3)(\alpha-\beta)}{2(n-1)(n+\alpha+\beta+2) + (\beta+3)(\alpha+\beta+2)},$$

as well as the best lower bound for $x_{n,n}$, i.e.,

$$B_{n-1}(4,0) = \frac{2(n-1)(n+\alpha+\beta+2)+(\alpha+3)(\beta-\alpha)}{2(n-1)(n+\alpha+\beta+2)+(\alpha+3)(\alpha+\beta+2)}.$$

In Table 3.1 we provide numerical examples, in order to show that these new bounds (printed in bold), are more precise bounds, than the bounds given by Szegő, denoted by $B_{n,L}(\alpha,\beta)$ and $B_{n,H}(\alpha,\beta)$.

Table 3.1: Comparison of bounds for the extreme zeros of $P_6^{\alpha,\beta}(x)$ for different values of α and β.

$\alpha,\ \beta$	$x_{6,1}$	$B_5(0,4)$	$B_{6,L}(\alpha,\beta)$	$B_{6,H}(\alpha,\beta)$	$B_5(4,0)$	$x_{6,6}$
$\alpha=0.3,\beta=2.5$	-0.744	**-0.713**	-0.527	0.825	**0.931**	0.934
$\alpha=7.5,\beta=6$	-0.702	**-0.645**	-0.451	0.333	**0.527**	0.606
$\alpha=7.5,\beta=-0.06$	-0.971	**-0.970**	-0.903	0.126	**0.296**	0.392

3.6 Electrostatic interpretation

We refer to Section 2.6, where we discussed an electrostatic model to show how the zeros of Jacobi polynomials can be interpreted electrostatically. In this section we will look at a similar, one-dimensional electrostatic interpretation of the results in Section 3.2.

The result in Theorem 3.2.1(a) can be interpreted as follows. The zeros of $P_{n+1}^{\alpha,\beta}$ are in the interval $(-1,1)$ and we will consider them to be the equilibrium positions of $n+1$ positive unit charges, denoted by $q_1,q_1,\ldots,q_{n+1}$, spaced between the points -1 and 1, where the positive charges $\frac{\beta+1}{2}$ and $\frac{\alpha+1}{2}$ are positioned respectively. We consider these $n+1$ positions to be fixed.

We now replace the charge at 1 with a positive charge $\frac{\alpha+1}{2}+\frac{t}{2}$, $t\in\{0,1,2,\ldots\}$. When $n-1$ positive charges are allowed to move freely between -1 and 1, the equilibrium positions of these $n-1$ charges, $p_1,p_2,\ldots,p_{n-1}$ (which we assume do not coincide with $q_j, j=1,2,\ldots,n+1$), will interlace with the fixed positions $q_1,q_2,\ldots,q_{n+1}$, as long as $t\in\{0,1,2,3,4\}$. For each of these values of t, there will be one interval $(q_i,q_{i+1}), i\in\{1,2,\ldots,n\}$ that does not contain one of the positions $p_i, i=1,2,\ldots,n-1$. We find that, as soon as the value of t becomes greater than 4, i.e., as soon as the charge at 1 is greater than $\frac{\alpha+1}{2}+2$, the positions of the $n-1$ charges do not necessarily interlace with the fixed positions $q_1, q_2, \ldots, q_{n+1}$ any more. The interlacing property is retained until the repelling force of the charge at 1, is large enough to repel position p_i beyond the fixed position q_i for some $i = 1, 2, \ldots, n-1$.

From the results of Theorems 3.2.1, 3.2.3 and Corollary 3.2.2, we conjecture that the zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-1}^{\alpha+t,\beta+s}$, $s, t \in \{0, 1, 2, 3, 4\}$ interlace, as long as the absolute value of the difference between the sum of the parameters and the degree is less than or equal to 2. Let $r_k, k = 1, 2, \ldots, n-1$ be the equilibrium positions of $n-1$ unit charges, spaced between -1 and 1, where the charges $\frac{(\beta+s)+1}{2}$ and $\frac{(\alpha+t)+1}{2}$ are positioned. Electrostatically this conjecture would imply that there will be at most one position $r_k, k = 1, 2, \ldots, n-1$ between any two of the fixed positions $q_1, q_2, \ldots, q_{n+1}$, for all values of s and t in $\{0, 1, 2, 3, 4\}$, such that

$$|\alpha + t + \beta + s + (n-1) - \alpha - \beta - (n+1)| \leq 2, \quad \text{i.e.,} \quad 0 \leq s + t \leq 4.$$

3.7 Conclusion

Jacobi, Gegenbauer and Laguerre polynomials are infinite systems of continuous classical orthogonal polynomials. Unlike the Gegenbauer and Laguerre polynomials, Jacobi polynomials are two-parameter polynomials and in this chapter we showed that Stieltjes interlacing extends to the zeros of Jacobi polynomials from different sequences. We considered integer shifts of one of the parameters α or β at a time, as well as shifts of both parameters simultaneously and we identified the values of j and k for which the zeros of $P_{n+1}^{\alpha,\beta}$ and $P_{n-1}^{\alpha+j,\beta+k}$ interlace in the Stieltjes sense. In fact, Stieltjes interlacing is only retained for the specific values of j and k as mentioned in our theorems and examples were given to show that, in general, interlacing breaks down when these values are exceeded. We also proved that the zeros of $P_{n+1}^{\alpha,\beta}$ interlace with the zeros of the kth derivative of $P_n^{\alpha,\beta}$, for $k = 1, 2, \ldots, n-1$. In each case, the extra interlacing point, and in the latter case, the associated polynomial of degree k, whose zeros complete the interlacing process, was identified.

We showed how the extra interlacing points obtained from mixed three term recurrence relations, especially in the case where we shift only one parameter at a time, relate to each other and yield lower (upper) bounds for the largest (smallest) extreme zeros of the Jacobi polynomials and we compared these bounds to the bounds obtained by Szegő [68], using classical methods.

We identified the extra interlacing point

$$B_{n-1}(0,4) = -\frac{2(n-1)(n+\alpha+\beta+2)+(\beta+3)(\alpha-\beta)}{2(n-1)(n+\alpha+\beta+2)+(\beta+3)(\alpha+\beta+2)}$$

to be the most precise upper bound for the smallest zero and

$$B_{n-1}(4,0) = \frac{2(n-1)(n+\alpha+\beta+2)+(\alpha+3)(\beta-\alpha)}{2(n-1)(n+\alpha+\beta+2)+(\alpha+3)(\alpha+\beta+2)}$$

to be the best lower bound for the largest zero of the Jacobi polynomial $P_n^{\alpha,\beta}$ that can be obtained by using mixed three term recurrence relations to find inner bounds for the extreme zeros of Jacobi polynomials.

In the next chapter we will consider Stieltjes interlacing between the zeros of different sequences of discrete classical orthogonal polynomials.

Chapter 4

Stieltjes interlacing of zeros of Meixner and Krawtchouk polynomials

4.1 Introduction

In this chapter we discuss Stieltjes interlacing between zeros of different sequences of the infinite system of Meixner polynomials as well as between zeros of different sequences of the finite system of Krawtchouk polynomials.

As discussed in Section 1.5, Meixner polynomials $M_n(x;\beta,c)$ are orthogonal on $[0,\infty)$ with respect to a positive discrete measure for $\beta > 0, 0 < c < 1$ and $n \in \mathbb{N}$, while Krawtchouk polynomials $K_n(x,p,N),\ n = 1,2,\ldots,N,\ N \in \mathbb{N}$ are orthogonal with respect to a discrete measure on $[0,N]$ when $p \in (0,1)$. These polynomials are two-parameter polynomials, similar to the Jacobi polynomials, however, the parameters c and p of the Meixner and Krawtchouk polynomials respectively, are both restricted to the interval $(0,1)$ and we therefore only consider integer shifts of the parameter β in the Meixner case, and N in the Krawtchouk case.

In Section 4.2 we investigate the extent to which Stieltjes interlacing holds between the zeros of two Meixner polynomials, if each polynomial belongs to a sequence generated by a different value of the parameter β. We consider the zeros of $M_{n+1}(x;\beta,c)$ and $M_{n-1}(x;\beta+k,c)$ for different integer values k, as well as the zeros of $M_{n+1}(x;\beta,c)$ and $M_{n-k}(x;\beta+k,c)$, for $k = 1,2,\ldots,n-1$. In our theorems we also consider the possibility that the polynomials under consideration can have common zeros. In each case, we use a mixed three term recurrence relation to identify a polynomial, whose zeros complete the interlacing process. The proofs of our results are given in Section 4.3.

The extra interlacing points obtained in Sections 4.2, which provide inner bounds for the extreme zeros of the Meixner polynomials, are discussed in Section 4.4. We compare these points with one another, in order to identify the best upper (lower) bound for the smallest (largest) zero of each of the polynomials $M_n(x;\beta,c)$. Numerical examples are given in order to illustrate the accuracy of these bounds.

In Section 4.5 we provide the reader with Stieltjes interlacing results between the zeros of two Krawtchouk polynomials, where each polynomial is generated by a different value of the parameter N. Shifting the parameter N of the Krawtchouk polynomial $K_n(x;p,N)$, implies a change in the interval of orthogonality and we find that Stieltjes interlacing does not follow naturally as in the case of Jacobi and Meixner polynomials; additional restrictions on the parameter p are necessary in some instances to ensure the required interlacing. We identify the extra interlacing points that complete the interlacing process. The results of this section are proved in Section 4.6.

In Section 4.7 we compare the extra interlacing points obtained in Section 4.5 with one another, in order to determine which points will yield the best inner bounds for the extreme zeros of the polynomials $K_n(x;p,N)$. The bounds obtained are very sharp and numerical examples are provided to illustrate the quality of these bounds.

4.2 Stieltjes interlacing of zeros of Meixner polynomials from different sequences.

In our first theorem we examine Stieltjes interlacing between the zeros of Meixner polynomials from different sequences, whose degrees differ by two, with due attention to the possibility that the polynomials under consideration can have common zeros.

Theorem 4.2.1 *Let $M_n(x;\beta,c),\ \beta>0,\ 0<c<1,\ n\in\mathbb{N}$ denote a Meixner polynomial of degree n.*

(i) If $M_{n-1}(x;\beta+k,c), k\in\{0,1,\ldots,4\}$ and $M_{n+1}(x;\beta,c)$ are co-prime, then

(a) *for fixed* $k \in \{0, 1, 2, 3\}$, *the zeros of* $M_{n-1}(x; \beta + k, c)$, *together with the point*

$$B_n(k) = \frac{n + \beta c + (1 - k)nc}{1 - c} + \frac{k(1 - k)(2 - k)n(n + 1)c^2}{6(1 - c)(n + \beta + 1)}, \quad \textit{and}$$

(b) *the zeros of* $M_{n-1}(x; \beta + 4, c)$, *together with the point*

$$B_n(4) = \frac{n^3(c-1)^3 - (\beta)_3 c + n^2(c-1)^2(\beta c + 3c - 2\beta - 3) + n(c-1)((\beta+2)(c^2+\beta+1) - 3\beta c - 4c}{(1-c)\big(n^2(c^2-1) + n(c^2 - 2\beta - 3) - (\beta+1)(\beta+2)\big)},$$

interlace with the zeros of $M_{n+1}(x; \beta, c)$;

(ii) *If, for a fixed* $k \in \{0, 1, \ldots, 4\}$, $M_{n-1}(x; \beta + k, c)$ *and* $M_{n+1}(x; \beta, c)$ *are not co-prime, then*

(a) *the two polynomials under consideration have one common zero located at the respective points identified in (i) (a) and (b);*

(b) *the* $n - 1$ *zeros of* $M_{n-1}(x; \beta + k, c)$ *interlace with the remaining* n *(non-common) zeros of* $M_{n+1}(x; \beta, c)$.

Remarks.

1. The case $k = 0$ in Theorem 4.2.1(i)(a), extends the classic result of Stieltjes, as given in Section 1.2, that between any two zeros of $M_{n-1}(x; \beta, c)$ there is at least one zero of $M_{n+1}(x; \beta, c)$, by providing a formula for an extra interlacing point.

2. The zeros of Meixner polynomials from different sequences, whose degrees differ by two, no longer interlace when the values of k, as stated in Theorem 4.2.1, are exceeded. Using *Mathematica*, we find that

   - when $\beta = 2$, the zeros of $M_6(x; \beta, 0.9)$ and $M_4(x; \beta - 1, 0.9)$ do not interlace.
   - the zeros of $M_{n+1}(x; \beta, c)$ and $M_{n-1}(x; \beta + 5, c)$ do not interlace when $n = 5, c = 0.9$ and $\beta = 2$ because the largest zero of $M_4(x; \beta + 5, 0.9)$ exceeds the largest zero of $M_6(x; \beta, 0.9)$.

3. The interlacing results obtained in Theorem 4.2.1(i), are proved by using mixed three term recurrence relations that hold true for all real values of β and c. Meixner polynomials are orthogonal, and therefore real-rooted, on $(-\infty, -\beta)$ for parameter values $\beta > 0$ and $c > 1$

and Stieltjes interlacing results, similar to those in Theorem 4.2.1(i), can be obtained in an analogous way for these values of the parameters β and c. Numerical calculations indicate that for $\beta > 0$ and $c > 1$, Stieltjes interlacing of the zeros of $M_{n+1}(x;\beta,c)$ and $M_{n-1}(x;\beta+k,c)$ holds for $k \in \{1,2\}$. For $\beta = 4$ and $c = 3.7$, the zeros of $M_5(x;\beta,c)$ and $M_3(x;\beta+3,c)$ do not interlace in the Stieltjes sense.

4. The result in Theorem 4.2.1(ii)(a) extends the result of Gibson [33] on the maximum number of common zeros of two polynomials in an orthogonal sequence, to Meixner polynomials of degree $n-1$ and $n+1$ from different orthogonal sequences. We refer the reader to Remark 3.3.2.

Theorem 4.2.1 provides the integer values of k for which the zeros of $M_{n-1}(x;\beta+k,c)$ and $M_{n+1}(x;\beta,c)$ interlace in the Stieltjes sense, as well as a formula for an extra interlacing point. Numerical experiments involving animations of the zeros, lead us to conjecture that for each $n \geq 2$, $\beta > 0$ and $0 < c < 1$, there exists an $a \in \mathbb{R}$, such that, as t varies continuously from 0 to a, the zeros of $M_{n-1}(x;\beta+t,c)$ and $M_{n+1}(x;\beta,c)$ interlace in the Stieltjes sense. Furthermore, for at least one of these (non-integer) values of $t \in (0,a)$, the polynomials $M_{n-1}(x;\beta+t,c)$ and $M_{n+1}(x;\beta,c)$ have a zero in common. In this case, the n remaining (non-common) zeros of $M_{n+1}(x;\beta,c)$ interlace with the $n-1$ zeros of $M_{n-1}(x;\beta+t,c)$.

We now state a more general result for Stieltjes interlacing between the zeros of $M_{n+1}(x;\beta,c)$ and the $n-k$ zeros of $\Delta^k M_n(x;\beta,c)$ or, equivalently, $M_{n-k}(x;\beta+k,c)$, $k \in \{1,2,\ldots,n-1\}$.

Theorem 4.2.2 *Let $M_n(x;\beta,c)$, $n \in \mathbb{N}$, denote the Meixner polynomial of degree n.*

(i) For each $k \in \{1,2,\ldots,n-1\}$, there exist polynomials G_k and H_k of degree k such that

$$g_k(x)M_{n-k}(x;\beta+k,c) = H_{k-1}(x)M_{n+1}(x;\beta,c) + G_k(x)M_n(x;\beta,c), \tag{4.1}$$

where $g_k(x) = (1-\frac{1}{c})^k(-n)_k(x+\beta)_k$.

(ii) Let $k \in \{1,2,\ldots,n-1\}$ be fixed, $\beta > 0$ and $0 < c < 1$. If $M_{n+1}(x;\beta,c)$ and $M_{n-k}(x;\beta+k,c)$ are co-prime, then the zeros of $M_{n-k}(x;\beta+k,c)$, together with the k real zeros of G_k, interlace with the zeros of $M_{n+1}(x;\beta,c)$.

(iii) *Let* $k \in \{1, 2, \ldots, n-1\}$ *be fixed,* $\beta > 0$ *and* $0 < c < 1$. *If* $M_{n+1}(x; \beta, c)$ *and* $M_{n-k}(x; \beta + k, c)$ *have* r *common zeros, then*

- *(a)* $r \leq \min\{n-k, k\}$;
- *(b)* *the common zeros are the zeros of the polynomial* $G_k(x)$;
- *(c)* *the common zeros of* $M_{n+1}(x; \beta, c)$ *and* $M_{n-k}(x; \beta + k, c)$ *cannot be consecutive zeros of* $M_{n+1}(x; \beta, c)$ *and*
- *(d)* *the* $(n-2r)$ *non-common zeros of the product* $G_k(x) M_{n-k}(x; \beta + k, c)$, *together with the* r *common zeros of* $M_{n+1}(x; \beta, c)$ *and* $M_{n-k}(x; \beta, c)$, *interlace with the* $(n+1-r)$ *non-common zeros of* $M_{n+1}(x; \beta, c)$.

We note that the mixed three term recurrence relation (4.1) holds true for all values of the parameters β and c and, by replacing β with $-N$ and c with $\frac{p}{p-1}$, we obtain a similar relation for Krawtchouk polynomials, namely

$$\frac{(-n)_k(-N+x)_k}{p^k} K_{n-k}(x; p, N-k) = H_{k-1}(x) K_{n+1}(x; p, N) + G_k(x) K_n(x; p, N),$$

with $H_0(x) = -1$, $G_1(x) = x - n + np - Np$ and $k \in \{1, 2, \ldots, n-1\}$.

4.3 Proofs of results given in Section 4.2

In our proofs we make use of the connection between Meixner polynomials and the ${}_2F_1$ hypergeometric function, as well as contiguous function relations satisfied by ${}_2F_1$ polynomials.

Using (1.9), the identities (7.2), (7.8), (7.15), (7.21) and (7.22) can be written as the following mixed three term recurrence relations for Meixner polynomials, used in our proofs:

$$\left(1 - \frac{1}{c}\right)(x + \beta) M_n(x; \beta + 1, c) = M_{n+1}(x; \beta, c) - \frac{\beta + n}{c} M_n(x; \beta, c) \tag{4.2}$$

$$n\left(1 - \frac{1}{c}\right)(x + \beta) M_{n-1}(x; \beta + 1, c) = M_{n+1}(x; \beta, c) - \left(\beta + \frac{n}{c} + \left(1 - \frac{1}{c}\right) x\right) M_n(x; \beta, c) \tag{4.3}$$

$$\begin{aligned}\left(x - \frac{\beta c + n - nc}{1 - c}\right) M_n(x; \beta, c) &= \frac{c(n - nc + \beta)}{(c-1)(\beta + n)} M_{n+1}(x; \beta, c) \\ &+ \frac{n(c-1)(x+\beta)_2}{\beta + n} M_{n-1}(x; \beta + 2, c)\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\left(x - \frac{n^2(c-1)^2 + n(c-1)(c-\beta-1) + (\beta)_2 c}{(1-c)(\beta+n+1)}\right) M_n(x;\beta,c) \qquad (4.5)\\
&\qquad = \frac{cD_1(x)}{(c-1)(\beta+n)_2} M_{n+1}(x;\beta,c) - \frac{n(c-1)^2(x+\beta)_3}{(\beta+n)_2} M_{n-1}(x;\beta+3,c),\\
D_1(x) &= n^2(c-1)^2 + (\beta)_2 + n(c-1)((x+\beta+1)c - 2\beta - 1)
\end{aligned}$$

$$\begin{aligned}
\frac{(x-B_n(4))D_3}{(n+\beta+1)(n+\beta+2)} M_n(x;\beta,c) &= \frac{cD_2(x)}{(c-1)(\beta+n)_3} M_{n+1}(x;\beta,c) \qquad (4.6)\\
&- \frac{n(c-1)^3(x+\beta)_4}{(\beta+n)_3} M_{n-1}(x;\beta+4,c),
\end{aligned}$$

where

$$\begin{aligned}
B_n(4) &= \frac{n^3(c-1)^3 - (\beta)_3 c + n^2(c-1)^2(\beta c + 3c - 2\beta - 3) + n(c-1)((\beta+2)(c^2+\beta+1) - 3\beta c - 4c}{(1-c)D_3}\\
D_2(x) &= n^3(c-1)^3 - (\beta)_3 + n^2(c-1)^2((2x+2\beta+3)c - 3(\beta+1))\\
&+ n(c-1)(3\beta^2 + 6\beta + 2 + ((\beta+1)_2 + x(x+2\beta+3))c^2 - (3\beta^2 + 8\beta + 4 + x(x+4\beta+5))c)\\
D_3 &= n^2(c^2-1) + n(c^2 - 2\beta - 3) - (\beta+1)(\beta+2)
\end{aligned}$$

Proof of Theorem 4.2.1.

(i) (a) For $t = 0$, the result follows from the three term recurrence relation for Meixner polynomials (1.15) and Lemma 3.3.1(i).
For $t = 1, t = 2$ and $t = 3$, the stated results follow from the mixed three term recurrence relations (4.3), (4.4) and (4.5) respectively, as well as Lemma 3.3.1(i).

(b) We apply Lemma 3.3.1(i) to the mixed three term recurrence relation (4.6) to obtain the stated result.

(ii) This follows immediately from Lemma 3.3.1(ii) and the proofs of Theorem 4.2.1(i)(a) to (c).

■

Proof of Theorem 4.2.2.

(i) We use the mixed three term recurrence relations (4.2) and (4.3) and prove our result by induction on k.

If $k = 1$, (4.1) is the same as (4.3) with $H_0(x) = -1$ and $G_1(x) = \beta + \frac{n}{c} + (1 - \frac{1}{c})x$ and therefore (4.1) holds for $k = 1$.

We assume that (4.1) holds for $m = 1, 2, \dots, k$, i.e., we assume that

$$g_m(x)M_{n-m}(x;\beta + m, c) = H_{m-1}(x)M_{n+1}(x;\beta, c) + G_m(x)M_n(x;\beta, c) \tag{4.7}$$

with G_m and H_m polynomials of degree m and $g_m(x) = (1 - \frac{1}{c})^m(-n)_m(x + \beta)_m$ for $m = 1, 2, \dots, k$.

For $m = k + 1$, the left hand-side of (4.1) is equal to

$$\begin{aligned}
&g_{k+1}(x)M_{n-k-1}(x;\beta + k + 1, c)\\
&= \left(1 - \frac{1}{c}\right)^{k+1}(-n)_{k+1}(x + \beta)_{k+1}M_{n-k-1}(x;\beta + k + 1, c)\\
&= \left(1 - \frac{1}{c}\right)^{k+1}(-n_1 - 1)_{k+1}(x + \beta_1 - 1)_{k+1}M_{n_1-k}(x;\beta_1 + k, c) \text{ where } \beta_1 = \beta + 1,\ n_1 = n - 1\\
&= \left(1 - \frac{1}{c}\right)^{k}(-n_1)_k(x + \beta_1)_k\left(1 - \frac{1}{c}\right)(-n_1 - 1)(x + \beta_1 - 1)M_{n_1-k}(x;\beta_1 + k, c)\\
&= \left(1 - \frac{1}{c}\right)(-n_1 - 1)(x + \beta_1 - 1)\Big(H_{k-1}(x)M_{n_1+1}(x;\beta_1, c) + G_k(x)M_{n_1}(x;\beta_1, c)\Big)\\
&= \left(1 - \frac{1}{c}\right)(-n)(x + \beta)\Big(H_{k-1}(x)M_n(x;\beta + 1, c) + G_k(x)M_{n-1}(x;\beta + 1, c)\Big),
\end{aligned}$$

by using the induction hypothesis.

A straightforward calculation using (4.2) and (4.3), shows that this equals

$$H_k(x)M_{n+1}(x;\beta, c) + G_{k+1}(x)M_n(x;\beta, c)$$

with

$$H_k(x) \quad = \quad -nH_{k-1}(x) - G_k(x)$$

and

$$G_{k+1}(x) \quad = \quad G_k(x)\left(\beta + \frac{n}{c} + \left(n - \frac{1}{c}\right)x\right) - H_{k-1}(x)\frac{(-n)(\beta+n)}{c}$$

which is the righthand-side of (4.1) for $m = k+1$. It follows that (4.7) holds for $m = k+1$ and the result follows by induction on k.

(ii) From (4.1), provided $M_{n+1}(x;\beta,c) \neq 0$, we have

$$\frac{g_k(x)M_{n-k}(x;\beta+k,c)}{M_{n+1}(x;\beta,c)} = H_{k-1}(x) + \frac{G_k(x)M_n(x;\beta,c)}{M_{n+1}(x;\beta,c)}.$$

The stated result follows by the same argument used in the proof of Theorem 3.4.1 (ii).

(iii) The stated result is proved by using the same argument as in the proof of Theorem 3.4.1 (iii).

■

4.4 New inner bounds for the extreme zeros of Meixner polynomials

Let $x_{n+1,1} < x_{n+1,2} < \cdots < x_{n+1,n+1}$ be the zeros of $M_{n+1}(x;\beta,c)$. It follows from [28, p. 1202, Corollary 2.2] that each point $B_n(k), k = 0, 1, \ldots, 4$, obtained in Theorem 4.2.1(i), will be a strict lower bound for $x_{n+1,n+1}$ and a strict upper bound for $x_{n+1,1}$, i.e.,

$$0 < x_{n+1,1} < B_n(k) < x_{n+1,n+1}, \text{ for each } k = 0, 1, \ldots, 4.$$

The definitions of the points $B_n(k), k = 0, 1, \ldots, 4$, immediately imply that

$$0 < x_{n+1,1} < B_n(4) < B_n(3) < \cdots < B_n(0) < x_{n+1,n+1},$$

for all values of $\beta > 0$ and $0 < c < 1$, and consequently the point $B_n(0)$, obtained from the three term recurrence relation for Meixner polynomials (1.15), is a sharp lower bound for $x_{n+1,n+1}$ and $B_n(4)$ is a good upper bound for $x_{n+1,1}$.

The formulas obtained in Theorem 4.2.1 provide us with bounds for the extreme zeros of $M_{n+1}(x;\beta,c)$, polynomials of degree $n+1$. For the sake of future reference, we shift the parameter n and show the bounds for the extreme zeros of $M_n(x;\beta,c)$.

A good lower bound for the largest zero $x_{n,n}$ of $M_n(x;\beta,c)$, is the point

$$B_{n-1} = \frac{(n-1)(1+c)+\beta c}{1-c}$$

and a precise upper bound for the smallest zero of $M_n(x;\beta,c)$, is

$$B_{n-1}(4) = \frac{(\beta)_2\big((\beta+3)c-1\big)+n(c-1)\big((c-1)^2-\beta^2+\beta(1+c(c-3))\big)-\beta(c-2)(n(c-1))^2-(n(c-1))^3}{(1-c)\big(\beta(2n+\beta+1)+n(n+1-(n-1)c^2)\big)}, \tag{4.8}$$

i.e.,

$$0 < x_{n,1} < B_{n-1}(4) < B_{n-1}(0) < x_{n,n},$$

for all values of $\beta > 0$ and $0 < c < 1$. These bounds are the sharpest inner bounds that can be obtained using our method involving mixed three term recurrence relations, since Stieltjes interlacing is only retained for the integer values given in Theorem 4.2.1(i).

We provide numerical examples in Table 4.1, in order to indicate the sharpness of these bounds, and note that the best lower bounds for $x_{n,n}$ are obtained when $c \to 0$. The best upper bounds for $x_{n,1}$ are found for c close to 1.

Table 4.1: Comparison of bounds for the extreme zeros of $M_n(x;\beta,c)$, $n=8$, for different values of β and c.

Values of β and c	$x_{8,1}$	Upper bound for $x_{8,1}$	Lower bound for $x_{8,8}$	$x_{8,8}$
$\beta = 0.09,\ c = 0.02$	2.9×10^{-15}	6.727	**7.288**	7.913
β=0.09, c=0.99	1.118	**1.130**	1401.91	2114.7
β=0.09, c=0.5	0.0004	2.195	21.09	31.082
$\beta = 2.0,\ c = 0.99$	39.741	**43.894**	1591	2445.289
$\beta = 6.0,\ c = 0.01$	9.6×10^{-13}	6.921	**7.207**	7.825
$\beta = 20,\ c = 0.5$	5.234	16.474	41.00	65.935
$\beta = 20,\ c = 0.99$	892.097	1212.12	3373.00	5141.82

4.5 Stieltjes interlacing of zeros of Krawtchouk polynomials from different sequences

Krawtchouk polynomials are orthogonal with respect to a positive discrete measure on the interval $[0, N]$ and each polynomial $K_n(x; p, N)$, $n = 1, 2, \ldots, N$, $0 < p < 1$, has n real zeros in $(0, N)$. Our first result follows directly from the three term recurrence relation (1.19) for Krawtchouk polynomials.

Theorem 4.5.1 *Let $K_n(x; p, N)$, $0 < p < 1$, $n = 1, 2, 3, \ldots, N-1$, $N \in \mathbb{N}$, denote the Krawtchouk polynomial of degree n.*

(i) If $K_{n-1}(x; p, N)$ and $K_{n+1}(x; p, N)$ are co-prime, then the zeros of $K_{n-1}(x; p, N)$, together with the point $C_n = n + Np - 2np$, interlace with the zeros of $K_{n+1}(x; p, N)$;

(ii) If $K_{n-1}(x; p, N)$ and $K_{n+1}(x; p, N)$ are not co-prime, then

(a) they have one common zero at $x = C_n$;

(b) the $n-1$ zeros of $K_{n-1}(x; p, N)$ interlace with the n non-common zeros of $K_{n+1}(x; p, N)$.

Remark. Consider the case $n = N$.

(1) The results in Theorem 4.5.1 still hold because of the interlacing of the zeros of $K_n(x; p, N)$, $n = 1, 2, \ldots, N$, with the mass points of the discrete measure $x = 0, 1, 2 \ldots, N$, that are equal to the zeros of $K_{N+1}(x; p, N)$ [68, p. 36], as discussed in Section 1.5.3.

(2) If $K_{N-1}(x; p, N)$ and $K_{N+1}(x; p, N)$ have a zero in common, then this common zero is

$$C_N = \begin{cases} 1 & \text{if} \quad p = 1 - \frac{1}{N}; \\ 2 & \text{if} \quad p = 1 - \frac{2}{N}; \\ \vdots & \\ N-1 & \text{if} \quad p = \frac{1}{N}. \end{cases}$$

For Krawtchouk polynomials $K_n(x; p, N), 0 < p < 1$, $n = 1, 2, 3, \ldots, N$, shifting the parameter N means that we in fact change the interval of orthogonality and hence the interval in which the zeros may lie. We consider Stieltjes interlacing between the zeros of $K_{n+1}(x;p,N)$, $0<p<1$, $n=1,2,\ldots,N-1$, that lie in $(0,N)$ and the $n-1$ zeros of $K_{n-1}(x;p,N-k)$, $k\in\{-1,1,2\}$, that lie in $(0,N-k)$. We use mixed three term recurrence relations satisfied by the appropriate polynomials and restrictions on p are necessary in some cases to satisfy the conditions of Lemma 3.3.1.

We will denote the zeros of the polynomial $K_n(x;p,N)$ by $x_{n,1}<x_{n,2}<\cdots<x_{n,n}$ and when the parameter N shifts by k units, we will indicate the zeros of the resulting polynomial $K_n(x;p,N+k)$ by $x_{n,1}^{N+k}$, $x_{n,2}^{N+k},\ldots$.

In the next theorem we consider the zeros of $K_{n+1}(x;p,N)$ and $K_{n-1}(x;p,N-k)$, $k\in\{1,2\}$, $n=1,2,\ldots,N-1$, and we obtain new bounds for the two largest zeros of $K_{n+1}(x;p,N)$.

Theorem 4.5.2 *Let $K_n(x;p,N)$, $0<p<1$, $N\in\mathbb{N}$, denote the Krawtchouk polynomial of degree n, $C_n(k)=n+Np+(k-2)np$ and $p_n(k)=\dfrac{N-n-k}{N+(k-2)n}$, $k\in\{1,2\}$.*

(i) Let $n=1,2,\ldots,N-1$. If $K_{n-1}(x;p,N-1)$ and $K_{n+1}(x;p,N)$ are co-prime, then the zeros of $K_{n-1}(x;p,N-1)$, together with the point $C_n(1)=n+Np-np$, interlace with the zeros of $K_{n+1}(x;p,N)$.

Furthermore,

(a) if $0<p_n(1)<p<1$, then $x_{n-1,n-1}^{N-1}<x_{n+1,n}^{N}<N-1<C_n(1)<x_{n+1,n+1}^{N}<N$;

(b) if $p=p_n(1)=1-\frac{1}{N-n}$, then $x_{n-1,n-1}^{N-1}<x_{n+1,n}^{N}<C_n(1)=N-1<x_{n+1,n+1}^{N}<N$.

(ii) Let $K_{n-1}(x;p,N-2)$ and $K_{n+1}(x;p,N)$ be co-prime, $n=1,2,\ldots,N-2$. The zeros of $K_{n-1}(x;p,N-2)$, together with the point $C_n(2)=n+Np$, interlace with the zeros of $K_{n+1}(x;p,N)$ for $p<1-\frac{n+1}{N}$, if and only if the zeros of $K_{n+1}(x;p,N)$ lie in $(0,N-1)$.

Furthermore,

(a) if $0<p_n(2)<p<1-\frac{n+1}{N}$, then $x_{n-1,n-1}^{N-2}<x_{n+1,n}^{N}<N-2<C_n(2)<x_{n+1,n+1}^{N}<N-1$;

(b) if $p=p_n(2)=1-\frac{n+2}{N}$, then $x_{n-1,n-1}^{N-2}<x_{n+1,n}^{N}<C_n(2)=N-2<x_{n+1,n+1}^{N}<N-1$.

(iii) For $k\in\{1,2\}$, if $K_{n+1}(x;p,N)$ and $K_{n-1}(x;p,N-k)$ have a zero in common, then

(a) $0<p<p_n(k)<1$;

(b) the point $C_n(k)$ is the common zero of $K_{n-1}(x;p,N-k)$ and $K_{n+1}(x;p,N)$;

(c) the $n-1$ *zeros of* $K_{n-1}(x;p,N-k)$ *interlace with the* n *(non-common) zeros of* $K_{n+1}(x;p,N)$.

Remarks.

(1) Numerical investigations indicate that the extra-interlacing point $C_n(2)=n+Np$ lies between $x^N_{n+1,n}$ and $x^N_{n+1,n+1}$ for all $n=1,2,\ldots,N-2$ and $p<1-\frac{n+1}{N}$, and not only for the values of p as indicated in Theorem 4.5.2 (ii) (a) and (b).

(2) Consider the case where $n=N$.

(i) When $n=N$ in the mixed three term recurrence relation (4.9), we have

$$(x-N)K_N(x;p,N)=pK_{N+1}(x;p,N)+N(N-x)K_{N-1}(x;p,N-1),$$

and applying the well-known result stated in Lemma 5.4.1, we find that

$$K_N(x;p,N)+NK_{N-1}(x;p,N-1)=\frac{1}{p^N}x(x-1)\ldots(x-N+1).$$

Consequently the polynomials $K_{N-1}(x;p,N-1)$ and $K_{N+1}(x;p,N)$, $N\in\mathbb{N}$, are co-prime for all $p\in(0,1)$, since the zeros of $K_N(x;p,N)$ interlace with the mass points $x=0,1,\ldots,N$.

(ii) It follows directly from Lemma 5.4.1 that $K_{N+1}(x;p,N)$ and $K_{N-1}(x;p,N-2)$ have $N-1$ common zeros, $x=0,1,\ldots,N-2$, for all $p\in(0,1)$.

Theorem 4.5.3 *Let* $K_n(x;p,N)$, $0<p<1$, $n=1,2,\ldots,N-1$, $N\in\mathbb{N}$, *denote the Krawtchouk polynomial of degree* n *and let*

$$\begin{aligned}
q &= \frac{3n-N+\sqrt{5n^2-4n^3-2nN+4n^2N+N^2}}{2n(n+1)},\\
X_1 &= \frac{1}{2}\left(S_n-\sqrt{S_n^2+4((N+1)(3np-Np-n)-np^2(n+1)}\right) \quad \textit{and}\\
X_2 &= \frac{1}{2}\left(S_n+\sqrt{S_n^2+4((N+1)(3np-Np-n)-np^2(n+1)}\right), \quad \textit{where}\\
S_n &= 1+n+N-3np+Np.
\end{aligned}$$

(i) $X_2\in[N,N+1)$ *if* $p\le q$.

(ii) Let $K_{n-1}(x;p,N+1)$ and $K_{n+1}(x;p,N)$ be co-prime. For each fixed $n=1,2,\ldots,N-1$ and $p \leq q$, the zeros of $K_{n-1}(x;p,N+1)$ are in $(0,N)$ and, together with the point X_1, they interlace with the zeros of $K_{n+1}(x;p,N)$.

(iii) Let $n=1,2,\ldots,N$ and p be fixed, $p \leq q$. If $K_{n-1}(x;p,N+1)$ and $K_{n+1}(x;p,N)$ are not co-prime,

(a) they have one common zero at $x = X_1$;

(b) the $n-1$ zeros of $K_{n-1}(x;p,N+1)$ interlace with the n non-common zeros of $K_{n+1}(x;p,N)$.

Remark. For the sake of completeness, we mention that if $n = N$, then $q = \frac{2}{N+1}$ and for the zeros of $K_{N-1}(x;p,N+1)$ to be in $(0,N)$, we need $p \leq q$. For these values of p, the zeros of $K_{N+1}(x;p,N)$ are $x = 0,1,\ldots,N$ and they interlace with the zeros of $K_{N-1}(x;p,N+1)$ in the Stieltjes sense. If $K_{N-1}(x;p,N+1)$ and $K_{N+1}(x;p,N)$ do have a zero in common, then this common zero can only be one of the integers $0,1,\ldots,N$. When $p \to 0$, we will prove in Theorem 5.5.1 that the zeros of $K_{N-1}(x;p,N+1)$ tend to $x = 0,1,\ldots,N-2$.

4.6 Proofs of results given in Section 4.5

Proof of Theorem 4.5.1.

(i) We assume that $K_{n-1}(x;p,N)$ and $K_{n+1}(x;p,N)$ do not have any zeros in common. The three term recurrence relation satisfied by Krawtchouk polynomials (1.19), together with Lemma 3.3.1(i), yields the stated result.

(ii) If $K_{n-1}(x;p,N)$ and $K_{n+1}(x;p,N)$ are not co-prime, then both results (a) and (b) follow from Lemma 3.3.1(ii).

$\blacksquare$

From (1.16) and the contiguous relations (7.14) and (7.15) we obtain the mixed recurrence relations

$$K_{n-1}(x;p,N-1) = \frac{p}{n(x-N)}K_{n+1}(x;p,N) - \frac{x-(n+Np-np)}{n(x-N)}K_n(x;p,N) \quad \text{and} \tag{4.9}$$

$$K_{n-1}(x;p,N-2) = \frac{p(n+Np-N)}{n(x-N)_2}K_{n+1}(x;p,N) + (x-(n+Np))b_n(x)K_n(x;p,N), \tag{4.10}$$

$$\text{where} \quad b_n(x) = \frac{(N-n)(1-p)}{n(x-N)_2}.$$

Proof of Theorem 4.5.2.

(i) Let $K_{n-1}(x;p,N-1)$ and $K_{n+1}(x;p,N)$ be co-prime. We consider the mixed three term recurrence relation (4.9). The function $\frac{1}{n(x-N)}$ in this relation does not change sign on $(0,N)$ and consequently we can apply Lemma 3.3.1 (i) to (4.9), and deduce that the zeros of $K_{n-1}(x;p,N-1)$, together with the point $C_n(1) = n+Np-np$, interlace with the zeros of $K_{n+1}(x;p,N)$ on $(0,N)$.

Furthermore,

(a) if $p > p_n(1) = 1-\frac{1}{N-n}$, we have $N-1 < C_n(1) < x^N_{n+1,n+1}$ and since there is at most one zero of $K_{n+1}(x;p,N)$, $n = 1,2,\ldots,N-1$ in between any two consecutive mass points $0,1,2\ldots,N$, we have $x^N_{n+1,n} < N-1$.

(b) if $p = 1-\frac{1}{N-n}$, $C_n(1) = N-1$ and hence $x^N_{n+1,n} < N-1 < x^N_{n+1,n+1} < N$.

(ii) Let $K_{n-1}(x;p,N-2)$ and $K_{n+1}(x;p,N)$ be co-prime and consider the mixed three term recurrence relation (4.10). Firstly, we assume that the zeros of $K_{n+1}(x;p,N)$ lie in $(0,N-1)$. The function

$$b_n(x) = \frac{(N-n)(1-p)}{n(x-N)(x-N+1)}$$

in (4.10) is defined and does not change sign on the interval $(0,N-1)$. The same proof as that of Lemma 3.3.1(i) can be used for the interval $(0,N-1)$ and it follows that the zeros of $K_{n-1}(x;p,N-2)$, together with the point $C_n(2)$, interlace with the zeros of $K_{n+1}(x;p,N)$ on $(0,N-1)$. Furthermore, the point $C_n(2) = n+Np$ must lie in $(0,N-1)$, which means $p < 1-\frac{n+1}{N}$.

Next, assume that the zeros of $K_{n-1}(x;p,N-2)$, together with the point $C_n(2)$, interlace with the zeros of $K_{n+1}(x;p,N)$ and $p < 1-\frac{n+1}{N}$, i.e., $C_n(2) = n+Np < N-1$.

Suppose $x^N_{n+1,n+1} > N-1$. Evaluating (4.10) at $x^N_{n+1,n}$ and $x^N_{n+1,n+1}$, we obtain

$$\frac{K_n\left(x^N_{n+1,n};p,N\right)K_n\left(x^N_{n+1,n+1};p,N\right)d^2}{\left(x^N_{n+1,n}-N\right)\left(x^N_{n+1,n+1}-N\right)\left(x^N_{n+1,n}-N+1\right)\left(x^N_{n+1,n+1}-N+1\right)} = \frac{K_{n-1}\left(x^N_{n+1,n};p,N-2\right)K_{n-1}\left(x^N_{n+1,n+1};p,N-2\right)}{\left(x^N_{n+1,n}-C_n(2)\right)\left(x^N_{n+1,n+1}-C_n(2)\right)}, \quad (4.11)$$

where $d = \frac{(N-n)(1-p)}{n}$. The zeros of $K_{n+1}(x;p,N)$ lie in $(0,N)$ and there is only one zero of $K_{n+1}(x;p,N)$ between any two mass points, therefore $x^N_{n+1,n} < N-1$ and consequently the denominator on the left-hand side of (4.11) is negative. The numerator is also negative, since the zeros of $K_{n+1}(x;p,N)$ and $K_n(x;p,N)$ interlace, which implies that the left-hand side of (4.11) is positive.

By assumption, either a zero of $K_{n-1}(x;p,N-2)$ or the point $C_n(2)$ lies in the interval $\left(x^N_{n+1,n}, x^N_{n+1,n+1}\right)$. If $C_n(2) \in \left(x^N_{n+1,n}, x^N_{n+1,n+1}\right)$, the denominator on the right-hand side is negative. For the right-hand side of (4.11) to be positive, we need $K_{n-1}\left(x^N_{n+1,n};p,N-2\right)K_{n-1}\left(x^N_{n+1,n+1};p,N-2\right) < 0$, which means there is also a zero of $K_{n-1}(x;p,N-2)$ in $\left(x^N_{n+1,n}, x^N_{n+1,n+1}\right)$ and we have a contradiction.

Alternatively, if $K_{n-1}(x;p,N-2)$ has a zero in the interval $\left(x^N_{n+1,n}, x^N_{n+1,n+1}\right)$, the numerator on the right-hand side is negative and for the right-hand side of (4.11) to be positive, we need the denominator to be negative, i.e., $C_n(2) \in \left(x^N_{n+1,n}, x^N_{n+1,n+1}\right)$ and again, this contradicts our assumption. We conclude that $x^N_{n+1,n+1} < N-1$.

Furthermore,

(a) if $0 < 1-\frac{n+2}{N} < p < 1-\frac{n+1}{N}$, then $N-2 < C_n(2) < N-1$ and, because of the Stieltjes interlacing, $N-2 < C_n(2) < x^N_{n+1,n+1} < N-1$. Since there is at most one zero of $K_{n+1}(x;p,N)$ in between any two consecutive mass points, we have $x^N_{n+1,n} < N-2$.

(b) if $p = 1-\frac{n+2}{N}$, then $C_n(2) = N-2$ and $x^N_{n+1,n} < N-2 < x^N_{n+1,n+1} < N-1$.

(iii) For each $k \in \{1,2\}$, we have the following: If $K_{n-1}(x;p,N-k)$ and $K_{n+1}(x;p,N)$ have zeros in common, it follows from the mixed three term recurrence relations (4.9) and (4.10) (for $k=1$ and $k=2$ respectively), together with Lemma 3.3.1 (ii), that they can only have one common zero, that is equal the point $C_n(k)$ and since this common zero must lie in $(0, N-k)$, we have $C_n(k) < N - k$ and consequently $p < p_n(k)$.

■

Proof of Theorem 4.5.3. We use the contiguous relation (7.20) and the ${}_2F_1$ representation of the Krawtchouk polynomials (1.16) to obtain the mixed three term recurrence relation

$$n(N-n+1)_2(1-p)^2K_{n-1}(x;p,N+1) = p(x-N-1+np)K_{n+1}(x;p,N)-P_2(x)K_n(x;p,N) \quad (4.12)$$

$$\begin{aligned} \text{where } P_2(x) &= x^2-(1+n+N-3np+Np)x-(N+1)(3np-Np-n)+np^2(n+1) \\ &= (x-X_1)(x-X_2). \end{aligned}$$

It is easy to show that $X_1 \in (0, N)$ and $X_2 \in (0, N+1)$ for $0 < p < 1$. In order to apply Lemma 3.3.1 (i) to (4.12), we need to determine the parameter values for which $(x - X_2)$ does not change sign on $(0, N)$, i.e., we need to find the conditions on p so that $N \leq X_2 < N + 1$.

(i) A straight-forward calculation shows that $X_2 \geq N$ is equivalent to

$$(n^2+n)p^2+(N-3n)p+n-N \leq 0.$$

By solving this quadratic inequality and taking in consideration the assumption that $p > 0$, we find

$$0 < p \leq \frac{3n-N+\sqrt{5n^2-4n^3-2nN+4n^2N+N^2}}{2n(n+1)} = q.$$

(ii) Let $K_{n-1}(x;p,N+1)$ and $K_{n+1}(x;p,N)$ be co-prime, n and p fixed, and $p \leq q$. We apply Lemma 3.3.1 (i) to (4.12) and conclude that for $p \leq q$ the $n-1$ zeros of $K_{n-1}(x;p,N+1)$, together with the point X_1, interlace with the zeros of $K_{n+1}(x;p,N)$. A direct consequence of this interlacing is that the zeros of $K_{n-1}(x;p,N+1)$ lie in $(0, N)$ for the specified values of p, since $x^{N+1}_{n-1,n-1} < x^{N}_{n+1,n+1} < N$.

(iii) Let n and $p \leq q$ be fixed. When we assume that $K_{n-1}(x;p,N+1)$ and $K_{n+1}(x;p,N)$ have a common zero, this zero must lie in $(0, N)$ and the stated results follows from Lemma 3.3.1 (ii).

■

4.7 New bounds for the extreme zeros of Krawtchouk polynomials

From Theorems 4.5.1 and 4.5.2 (i) and (iii), we deduce that each one of the points $C_n(k) = n + Np + (k-2)np,\ k = 0, 1,$ will be an upper (lower) bound for the smallest (largest) zero of $K_{n+1}(x; p, N)$. Since $C_n < C_n(1)$ for all values of $n = 1, 2, \ldots, N-1$ and $p \in (0, 1)$, we can conclude that

$$0 < x_{n+1,1} < C_n < C_n(1) < x_{n+1,n+1} < N.$$

From Theorem 4.5.2 (ii) and (iii), as well as the definitions of $C_n(k),\ k = 0, 1, 2,$ it follows that, for $p < 1 - \frac{n+1}{N}$,

$$0 < x_{n+1,1} < C_n < C_n(1) < C_n(2) < x_{n+1,n+1} < N - 1.$$

The points $C_n(k),\ k = 0, 1, 2,$ are bounds for the extreme zeros of $K_{n+1}(x; p, N)$, polynomials of degree $n+1$. For easier reference, we state the bounds for the extreme zeros of $K_n(x; p, N)$:

$$C_{n-1} = Np + (n-1)(1-2p) \tag{4.13}$$

$$C_{n-1}(1) = Np + (n-1)(1-p) \tag{4.14}$$

$$C_{n-1}(2) = Np + n - 1 \tag{4.15}$$

and consequently, for $0 < p < 1$,

$$0 < x_{n,1} < C_{n-1} < C_{n-1}(1) < x_{n,n} < N.$$

The upper bound N for $x_{n,n}$, follows naturally from the orthogonality of the Krawtchouk polynomials and we draw the attention of the reader to the fact that, for $p < 1 - \frac{n}{N}$, we have obtained a sharper upper bound, $N - 1$, for $x_{n,n}$, since

$$0 < x_{n,1} < C_{n-1} < C_{n-1}(1) < C_{n-1}(2) < x_{n,n} < N - 1.$$

The lower bounds for the largest zeros are surprisingly good and we provide some numerical examples in Table 4.2 to illustrate this.

Table 4.2: Comparison of bounds for $x_{5,5}$, the largest zero of $K_5(x;p,N)$, for different values of p and N.

Values of $p,\ N$	Lower bound obtained from (4.14)	Lower bound obtained from (4.15)	Value of $x_{5,5}$	Upper bound for $x_{5,5}$
$p=0.1, N=10$	4.6	5	5.78	9
$p=0.45, N=10$	6.7	8.5	8.738	9
$p=0.9, N=10$	9.4	-	9.994	10
$p=0.1, N=7$	4.3	4.7	4.991	6
$p=0.9, N=7$	6.7	-	6.999	7

We observe that, for the special case when $p=1$, we have

$$\begin{aligned} K_n(x;1,N) &= (-N)_n\ {}_2F_1(-n,-x;-N;1) \\ &= (x-N)(x-N+1).....(x-N+n-1), \end{aligned}$$

which vanishes when $x=N, N-1, \ldots, N-n+1$, and this is consistent with the sharp bounds for the extreme zeros we obtain when $p \to 1$, as illustrated in Table 4.3.

Table 4.3: Bounds for the zeros of $K_n(x;0.99,N)$ for different values of n and N.

Values of $n,\ N$	Value of $x_{n,1}$	Upper bound for $x_{n,1}$ obtained from (4.13)	Lower bound for $x_{n,n}$ obtained from (4.14)	Value of $x_{n,n}$
$n=5,\quad N=8$	3.816	4	7.96	7.99999997
$n=5,\quad N=14$	9.573	9.94	13.9	13.9999991
$n=8,\quad N=12$	4.659	5.02	11.95	11.9999999998

4.8 Conclusion

In this chapter we proved Stieltjes interlacing between zeros of different sequences of discrete orthogonal polynomials and derived bounds for the extreme zeros of the Meixner and Krawtchouk polynomials.

We proved that the zeros of the Meixner polynomials $M_{n+1}(x;\beta,c),\ \beta>0, 0<c<1,\ n\in\mathbb{N}$, interlace in the Stieltjes sense with the zeros of the polynomials $M_{n-1}(x;\beta+k,c),\ k=0,1,\dots,4$, and $M_{n-k}(x;\beta+k,c)$, $k=1,2,\dots,n-1$, and the extra interlacing points we obtained, yield the following sharp bounds for the extreme zeros of $M_n(x;\beta,c)$:

$$0<x_{n,1}<B_{n-1}(4)<\frac{((2+\beta-2n)c+n-1)n(n-1)c^2}{1-c}<\frac{(n-1)(1+c)+\beta c}{1-c}<x_{n,n},$$

where $B_{n-1}(4)$ is given by (4.8).

In the Krawtchouk case we proved that the zeros of $K_{n+1}(x;p,N),\ n=0,1,\dots,N-1$, interlace in the Stieltjes sense with the zeros of $K_{n-1}(x;p,N)$, as well as $K_{n-1}(x;p,N-1)$, and we provided the conditions necessary for the zeros of the polynomials $K_{n+1}(x;p,N),\ n=0,1,\dots,N-1$, to interlace in the Stieltjes sense with the zeros of $K_{n-1}(x;p,N-2)$ and $K_{n-1}(x;p,N+1)$. In each case, we identified the extra interlacing points that complete the interlacing and which yield new inner bounds for the extreme zeros of the polynomial $K_{n+1}(x;p,N)$.

For $K_n(x;p,N),\ n=1,2,\dots,N$ and $p\in(0,1)$, we obtained the bounds

$$0<x_{n,1}<Np+(n-1)(1-2p)<Np+(n-1)(1-p)<x_{n,n}<N.$$

Furthermore, when $p<1-\frac{n}{N}$,

$$0<x_{n,1}<Np+(n-1)(1-2p)<Np+n-1<x_{n,n}<N-1.$$

In the next chapter we keep our focus on the zeros of Meixner and Krawtchouk polynomials and we deviate from the interlacing theme, by studying the zero location of the zeros of these polynomials for parameter values where the standard orthogonality does not hold.

Chapter 5

On the zeros of Meixner and Krawtchouk polynomials

5.1 Introduction

The standard orthogonality of Jacobi, Meixner and Krawtchouk polynomials was introduced in Section 1.5. In this chapter, we investigate the zeros of Meixner and Krawtchouk polynomials for non-classical parameters. Results on the orthogonality of Jacobi polynomials for parameter values different from those for which the standard orthogonality holds, were discussed in e.g. [1] and [47].

Extensions of discrete orthogonal polynomials beyond the orthogonality are considered in [4] and [17]. A non-standard (or Δ-Sobolev) orthogonality for Krawtchouk polynomials is obtained in [4], where it is shown that these polynomials are orthogonal with respect to a discrete measure involving difference operators. In [17], the authors use the three term recurrence relation satisfied by the Hahn polynomials to prove that these polynomials can be characterized by a Δ-Sobolev orthogonality, they also discuss the Δ-Sobolev orthogonality of the Krawtchouk polynomials, as well as a non-Hermitian orthogonality with respect to a complex weight function for Meixner polynomials $M_n(x;\beta,c)$ when $\beta, c \in \mathbb{C}$, $c \notin [0,\infty)$ and $\beta \notin \mathbb{Z}^-$ [17, Proposition 9]. Furthermore, in [52], a Δ-Sobolev inner product is used to obtain a generating function for the Δ-Sobolev orthogonal Meixner polynomials.

The asymptotic zero distribution of Meixner polynomials has been studied by various authors (cf.

[5], [40] and [45]). In [23] and [24], the limiting distribution of the zeros of Krawtchouk polynomials $K_n(x;p,N)$, for $p=\frac{1}{2}$ and $p\neq\frac{1}{2}$, respectively, is derived from the strong asymptotics as $n\to\infty$ and in [39], a saddle point method is applied to an integral representation of the Krawtchouk polynomial $K_n(x;p,N), p=\frac{1}{2}$, to derive the strong asymptotics of these polynomials as $N\to\infty$, $n\to\infty$, $\frac{N}{n}$ fixed and $x>0$. The more general case, $p\neq\frac{1}{2}$, is also discussed.

As discussed in Section 1.5, the polynomials $M_n(x;\beta,c)$ have the standard orthogonality of Meixner and Krawtchouk polynomials for the parameter ranges

$0<c<1,\ \beta>0,\ n=0,1,2,\ldots;$

$c>1,\ \beta>0,\ n=0,1,2,\ldots$ and

$c<0,\ \beta=-N,\ N\in\mathbb{N},\ n=0,1,2\ldots N.$

We examine the zeros of the polynomials $M_n(x;\beta,c)$ for different parameter values, i.e., for the values

(i) $c<0,\ \beta<0,\ n<1-\beta,\ \ n\in\mathbb{N};$

(ii) $0<c<1,\ \beta<0,\ n\in\mathbb{N};$

(iii) $c>1,\ \beta<0,\ n\in\mathbb{N};$

(iv) $c<0,\ \beta<0,\ n\in\mathbb{N};$

(v) $c<0,\ \beta=-N,\ \ n=N+1,N+2,\ldots;$

(vi) $c<0,\ \beta>0,\ n\in\mathbb{N}$ and

(vii) $c\to0,\ \beta\in\mathbb{R},\ n\in\mathbb{N},$

and the results obtained have been accepted for publication in [41].

We begin with case (i) in Section 5.2, where we extend the conclusion following from the discrete orthogonality of Krawtchouk polynomials $K_n(x;p,N)$ for integer values of the parameter N, $n<N+1,\ 0<p<1$, to prove that, for $c=\frac{p}{p-1}$, the zeros of the polynomials $M_n(x;\beta,c)$ are real, distinct and lie in the interval $(0,-\beta)$ for all real values of the parameter β, $n<1-\beta$ and $c<0$.

In Section 5.3 we consider cases (ii) and (iii), proving that the Meixner polynomials $M_n(x;\beta,c)$ are quasi-orthogonal of order k for $-k<\beta<-k+1$, $k=1,\dots,n-1$ and $0<c<1$ or $c>1$, as well as the related case (iv).

Cases (v) and (vi) are discussed in Section 5.4. Results obtained in [9], [10], [35] and [68] for the zeros of Krawtchouk polynomials $K_n(x;p,N)$, $0<p<1$, of degree $n=N+1$, are extended to polynomials of degree $n=N+2$ and $n=N+3$. We make use of the product decomposition (cf. [4, p. 17, Proposition 5.2] and [17, p. 448])

$$\begin{aligned} K_n(x;p,N) &= K_{N+1}(x;p,N)K_{n-N-1}(x-N-1;p,-N-2) && (5.1)\\ &= K_{N+1}(x;p,N)M_{n-N-1}\left(x-N-1;N+2,\frac{p}{p-1}\right) && (5.2)\end{aligned}$$

for $p\neq 0,1,\ n>N\in\mathbb{N}$, which also shows that for case (iv) it suffices to study polynomials $M_n(x;N,c)$ for $c<0$, $N=1,2,\dots$ (case (vi) for integer values of β).

In the last section we prove that the polynomials $\{M_n(x;\beta,c)\}_{n=0}^{\infty}$ are real-rooted for all $\beta\in\mathbb{R}$ when $c\to 0$ (case (vii)).

We observe that for the special case when $c\to\infty$, the polynomials $M_n(x;\beta,c)$ tend to

$$(\beta)_n\ {}_2F_1(-n,-x;\beta;1)=(x+\beta)(x+\beta+1).....(x+\beta+n-1)$$

which vanishes when $x=-\beta,-\beta-1,\dots,-\beta-n+1$, whereas when $c\to 1$ the polynomial $M_n(x;\beta,c)$ tends to

$$(\beta)_n\ {}_2F_1(-n,-x;\beta;0)=(\beta)_n$$

and has n zeros at infinity if it is considered as a polynomial of degree n in x (cf. [68, p. 145, eqn. 6.72.3]).

5.2 The zeros of $M_n(x;\beta,c)$, $c<0$ and $n<1-\beta$

The three term recurrence relation (1.15) for Meixner polynomials

$$xM_n(x;\beta,c)=A_nM_{n+1}(x;\beta,c)+B_nM_n(x;\beta,c)+C_nM_{n-1}(x;\beta,c), \qquad (5.3)$$

where $A_n=\frac{c}{c-1}, B_n=\frac{n+(\beta+n)c}{1-c}$ and $C_n=\frac{n(\beta+n-1)}{c-1}$, holds true for all $\beta,c\in\mathbb{R},\ c\neq 0,1$ and $n\in\mathbb{N}$, because it follows from the contiguous relation for hypergeometric functions (7.6). For the particular case when $\beta,c<0$, we will have $A_{n-1}C_n>0$ when $n<1-\beta$.

As discussed in Section 1.2 (ii), it follows from Favard's theorem that there is at least one positive measure α so that, for $\beta, c < 0$,

$$\int_{-\infty}^{\infty} M_n(x;\beta,c)M_m(x;\beta,c)d\alpha(x) = 0, \ m \neq n, \ m,n = 0,1,\ldots,-\lfloor\beta\rfloor,$$

and hence $\{M_n(x;\beta,c)\}_{n=0}^{-\lfloor\beta\rfloor}$ has n real, distinct zeros when $\beta, c < 0$, where $\lfloor a\rfloor$ denotes the greatest integer smaller than or equal to a. However, the set containing the real zeros does not follow immediately.

We prove that the zeros of $M_n(x;\beta,c)$ for $n < 1-\beta, \ \beta, c < 0$ are in $(0,-\beta)$, by using a generalised Sturmian sequence argument applied to solutions of difference equations (cf. [56]), as was done in [48] for Hahn polynomials. We begin by proving that if r denotes a zero of $M_n(x;\beta,c)$ in $(0,-\beta)$, then $r-1$ and $r+1$ cannot be zeros of $M_n(x;\beta,c)$ and, in addition, there will be an odd number of zeros of $M_n(x;\beta,c)$ in the interval $(r-1,r+1)$.

Lemma 5.2.1 *Let $\beta \in \mathbb{R}$, $n \in \mathbb{N}$, $n < 1-\beta$ and $c < 0$. If r is a zero of $M_n(x;\beta,c)$ and $r \in (0,-\beta)$, then $M_n(r-1;\beta,c)M_n(r+1;\beta,c) < 0$.*

Proof. Let $\beta \in \mathbb{R}$, $n < 1-\beta$ and $c < 0$. Consider the difference equation (1.11)

$$A(x)M_n(x+1;\beta,c) + C(x)M_n(x-1;\beta,c) = B(x)M_n(x;\beta,c) \tag{5.4}$$

where $A(x) = c(x+\beta)$ and $C(x) = x$. Note that $A(x) > 0$ and $C(x) > 0$ when $x \in (0,-\beta)$ and $c < 0$.

Suppose r is a zero of $M_n(x;\beta,c)$ in the interval $(0,-\beta)$, then

$$A(r)M_n(r+1;\beta,c) + C(r)M_n(r-1;\beta,c) = 0. \tag{5.5}$$

Assume that

$$M_n(r+1;\beta,c) = 0. \tag{5.6}$$

Letting $x = r+1$ in (5.4) we obtain $A(r+1)M_n(r+2;\beta,c) = 0$ and if $r+1 \in (0,-\beta)$, it follows that $A(r+1) > 0$ and $M_n(r+2;\beta,c) = 0$. By repeating this argument we can prove that

$$M_n(r+i;\beta,c) = 0 \text{ for all } i \text{ such that } \ 0 < r+i-1 < -\beta. \tag{5.7}$$

Under our assumption (5.6), it also follows from equation (5.5) that
$C(r)M_n(r-1;\beta,c) = 0$ if $r \in (0,-\beta)$ and since $C(r) > 0$ for these values of r, we have that $M_n(r-1;\beta,c) = 0$. In the same way as before we can prove that

$$M_n(r-j;\beta,c) = 0 \text{ for all } j \text{ such that } \ 0 < r-j+1 < -\beta. \tag{5.8}$$

In short, it follows from results (5.7) and (5.8) that $M_n(x;\beta,c)$ has as zeros all numbers $r+i$, $i \in \mathbb{Z}$ with $-1 < r+i < 1-\beta$. This means that $M_n(x;\beta,c)$ has a total of at most $\lfloor 1-\beta-(-1)\rfloor = 1 + \lfloor 1-\beta \rfloor \geq n+1 > n$ zeros unless both β and r are integers. In this case $M_n(x;\beta,c)$ has $1-\beta-(-1)-1 = 1-\beta > n$ zeros. In both cases, the number of zeros is greater than the degree of the polynomial and we have a contradiction. This means $M_n(r+1;\beta,c) \neq 0$.

The proof that $M_n(r-1;\beta,c) \neq 0$ is analogous. Now (5.5) implies that

$$M_n(r+1;\beta,c) = -\frac{C(r)}{A(r)}M_n(r-1;\beta,c)$$

and clearly $M_n(r+1;\beta,c)$ and $M_n(r-1;\beta,c)$ differ in sign. ∎

Theorem 5.2.2 *Let $\beta \in \mathbb{R}$, $n \in \mathbb{N}$, $n < 1-\beta$ and $c < 0$, the zeros of $M_n(x;\beta,c)$ lie in the open interval* $(0,-\beta)$.

Proof. Let $n < 1-\beta$, $c < 0$ and let n and N be integers, such that $N = \lceil -\beta \rceil$ where $\lceil a \rceil$ denotes the least integer larger than or equal to a. In the sequence

$$M_n(0;\beta,c), M_n(1;\beta,c), ..., M_n(N;\beta,c), \tag{5.9}$$

each term can be considered as a polynomial function of the parameter β with $c < 0$ fixed. When a numerical value is assigned to β, we denote the number of variations in sign in the resulting sequence by $V(\beta)$. We want to determine $V(\beta)$ for $N-1 < -\beta \leq N$.

When $-\beta = N$, it follows from (1.18) and (1.20) that the sequence of polynomials in (5.9) will have n sign changes since $K_n(x;\frac{c}{c-1},N)$ is orthogonal for $c < 0$. This means that $V(-N) = n$.

If $-\beta$ is assigned any value in the interval $(N-1,N]$, then Lemma 5.2.1 implies that in the resulting sequence

$$M_n(0;\beta,c), M_n(1;\beta,c), ..., M_n(N;\beta,c),$$

no two consecutive terms are zero and also that if $M_n(i;\beta,c)=0$ for $i=1,2,\dots N-1$, then the two adjacent terms: $M_n(i-1;\beta,c)$ and $M_n(i+1;\beta,c)$ differ in sign. Moreover, it follows directly from (1.9) that

$$M_n(0;\beta,c)=(\beta)_n \tag{5.10}$$

and the first term can never be zero for $-\beta$ in the interval $(N-1,N]$. The last term does not change sign on $(N-1,N]$ since by (1.13),

$$M_n(N;\beta,c) \quad = \quad (\beta)_n\left(\frac{1}{c}\right)^n\sum_{i=0}^{n}\frac{(-n)_i(\beta+N)_i(1-c)^i}{(\beta)_i i!}>0$$

for all $-\beta\in(N-1,N]$.

These conditions are sufficient to ensure that the sequence (5.9) forms a generalised Sturmian sequence and therefore $V(\beta)$ remains constant as $-\beta$ increases through the interval $(N-1,N]$. Hence $V(\beta)=n$ for all $-\beta\in(N-1,N]$.

Thus for $n<1-\beta$, $M_n(x;\beta,c)$ changes sign n times for x in $(0,N)$ and, since the degree is n, we conclude that $M_n(x;\beta,c)$ has n distinct roots in $(0,N)$.

If r is a root of $M_n(x;\beta,c)$, then $0<r<N$ and it follows from relation (1.14) that $-\beta-r$ will be a zero of $M_n(x;\beta,\frac{1}{c})$ with $0<-\beta-r<N$, i.e., $r<-\beta$. We conclude that the zeros of $M_n(x;\beta,c)$ are in the open interval $(0,-\beta)$. ∎

5.3 Quasi-orthogonality of $M_n(x;\beta,c)$

A polynomial P_n of exact degree $n\geq r$, is quasi-orthogonal of order r on $[a,b]$ with respect to a weight function $w(x)>0$, if (cf. [14, p. 159, Definition 1])

$$\int_a^b x^jP_n(x)w(x)dx\begin{cases}=0, & \text{for } j=0,1,\dots,n-r-1\\ \neq 0, & \text{for } j=n-r.\end{cases}$$

We say that a polynomial P_n of exact degree $n\geq r$, $n=0,1,\dots,N$, where N may be infinite, is discrete quasi-orthogonal of order r with ρ_i being the values of the weight at the points $x_i, i=0,1,\dots,M$, $M\in\mathbb{N}\cup\{\infty\}$, if

$$\sum_{i=0}^{M}(x_i)^jP_n(x_i)\rho_i\begin{cases}=0, & \text{for } j=0,1,\dots,n-r-1\\ \neq 0, & \text{for } j=n-r.\end{cases}$$

The Meixner polynomials $M_n(x;\beta,c)$ are orthogonal on $(0,\infty)$, for $0<c<1$, $\beta>0$, and as β decreases below 0, the zeros of $M_n(x;\beta,c)$ depart from the interval of orthogonality $(0,\infty)$. We prove the quasi-orthogonality of these polynomials in the following theorem.

Theorem 5.3.1 *The polynomials* $M_n(x;\beta-k,c)$ *with* $0<c<1$, $0<\beta<1$ *and* $k=1,2,\ldots,n-1$, *are quasi-orthogonal of order* k *with respect to the weight function* $\dfrac{c^x(\beta)_x}{x!}$ *on* $(0,\infty)$.

Proof. The recurrence relation

$$M_n(x;\beta,c)=nM_{n-1}(x;\beta,c)+M_n(x;\beta-1,c), \tag{5.11}$$

obtained from (7.2), shows that $M_n(x;\beta-k,c)$ can be expressed as a linear combination of $M_n(x;\beta,c)$, $M_{n-1}(x;\beta,c),\ldots,M_{n-k}(x;\beta,c)$ and since $\beta>0$, it follows from (1.12) that

$$\sum_{x=0}^{\infty}x^jM_n(x;\beta-k,c)\frac{c^x(\beta)_x}{x!}=0 \text{ for } j=0,1,\ldots,n-k-1.$$

■

Remark. By a change of variable, the result in Theorem 5.3.1 can be written as that the polynomials $M_n(x;\beta,c)$ are quasi-orthogonal of order k on $(0,\infty)$, for $0<c<1$ and $-k<\beta<-k+1$, $k=1,2,\ldots,n-1$, with respect to the weight function $\dfrac{c^x(\beta+k)_x}{x!}$.

The zeros of quasi-orthogonal polynomials are not necessarily all in the interval of orthogonality, but we can say the following from [14, Theorem 2].

Corollary 5.3.2 *The Meixner polynomials* $M_n(x;\beta,c)$, *with* $0<c<1, -k<\beta<-k+1$, *have at least* $n-k$ *zeros in* $(0,\infty)$ *when* $k=1,2,\ldots,n-1$.

The zeros of $M_n(x;\beta,c)$ for $c=0.75$, $n=10$ and $\beta=-4.8$ and -5.8, when the polynomial is quasi-orthogonal of order 5 and 6 respectively, are illustrated in Figure 5.1.

In order to specify the location of the remaining single zero of $M_n(x;\beta-1,c)$, $0<c<1, 0<\beta<1$, where we have quasi-orthogonality of order 1, we consider the monic polynomials

$$\tilde{M}_n(x;\beta,c)=\left(\frac{c}{c-1}\right)^nM_n(x;\beta,c).$$

Theorem 5.3.3 *If* $0<c<1$ *and* $0<\beta<1$, *then the smallest zero of* $M_n(x;\beta-1,c)$ *(or equivalently* $\tilde{M}_n(x;\beta-1,c)$*) is negative.*

Figure 5.1: The zeros of $M_{10}(x;-4.8,0.75)$ and $M_{10}(x;-5.8,0.75)$.

Proof. The recurrence relation (5.11) can be written as

$$\tilde{M}_n(x;\beta-1,c) = \tilde{M}_n(x;\beta,c) - n\Big(\frac{c}{c-1}\Big)\tilde{M}_{n-1}(x;\beta,c) \tag{5.12}$$

and according to [43, Theorem 4] we have to show that
$n\Big(\dfrac{c}{c-1}\Big) < \dfrac{\tilde{M}_n(0;\beta,c)}{\tilde{M}_{n-1}(0;\beta,c)} < 0$, which follows immediately from (5.10). ■

Joulak's results (cf. [43, Theorems 8, 9]) also give some information about the location of the zeros when we have quasi-orthogonality of order 2.

Theorem 5.3.4 *If $0 < c < 1$, $0 < \beta < 1$ and $n > \frac{\beta-2}{c-1}$, then all the zeros of $M_n(x;\beta-2,c)$ are nonnegative and simple.*

Proof. Iterating (5.12) we obtain

$$\tilde{M}_n(x;\beta-2,c) = \tilde{M}_n(x;\beta,c) - 2n\Big(\frac{c}{c-1}\Big)\tilde{M}_{n-1}(x;\beta,c) + b_n\tilde{M}_{n-2}(x;\beta,c)$$

where $b_n = n(n-1)\Big(\dfrac{c}{c-1}\Big)^2$. Replacing n by $n-1$ in (5.3) yields

$$\tilde{M}_n(x;\beta,c) = (x-B_{n-1})\tilde{M}_{n-1}(x;\beta,c) - \Big(\frac{c}{c-1}\Big)C_{n-1}\tilde{M}_{n-2}(x;\beta,c).$$

From [43, Theorem 8] all the zeros of $\tilde{M}_n(x;\beta-2,c)$ are real and simple if $b_n < \Big(\dfrac{c}{c-1}\Big)C_{n-1}$ which gives the condition $n > \dfrac{\beta-2}{c-1}$. Furthermore, the smallest zero (and hence all of the zeros) of $M_n(x;\beta-2,c)$ is nonnegative if and only if (cf. [43, Theorem 9])

$$\frac{\tilde{M}_n(0;\beta,c)}{\tilde{M}_{n-2}(0;\beta,c)} - 2n\Big(\frac{c}{c-1}\Big)\frac{\tilde{M}_{n-1}(0;\beta,c)}{\tilde{M}_{n-2}(0;\beta,c)} + n(n-1)\Big(\frac{c}{c-1}\Big)^2 \geq 0.$$

It follows from (5.10) that the left-hand side simplifies to $(\frac{c}{c-1})^2(1-\beta)(2-\beta)$, which is positive by the assumptions. This completes the proof. ■

Analogous results can be obtained for the polynomials $M_n(x;\beta,c)$, $\beta<0$ and $c>1$.

For $c<0$ and $n<1-\beta$, the polynomials $M_n(x;\beta,c)$ are orthogonal with respect to a positive weight function $w(x)$ on the interval $(0,-\beta)$ and

$$\int_0^{-\beta} M_n(x;\beta,c)M_m(x;\beta,c)w(x)dx=0,\ \beta\in\mathbb{R},\ m\neq n,\ m,n=1,2,\dots,-\lfloor\beta\rfloor.$$

Letting $b=-x$, $c=\beta$ and $z=1-\frac{1}{c}$ in the contiguous relation (7.3), we obtain

$$(x+\beta)M_n(x;\beta+1,c)=\frac{n(n+\beta-1)}{1-c}M_{n-1}(x;\beta,c)+\left(x+\frac{\beta(1-c)-nc}{1-c}\right)M_n(x;\beta,c)$$

and consequently (cf. [14, p. 160, eqn. 5]) the polynomial $(x+\beta)M_n(x;\beta+1,c)$ is quasi-orthogonal of degree $n+1$ and order 2 on $(0,-\beta)$ for $-n+1\le\beta<-n+2$.

Equivalently, by shifting the parameter β, $(x+\beta-1)M_n(x;\beta,c)$ is quasi-orthogonal of order 2 on $(0,-\beta+1)$ for $-n\le\beta-1<-n+1$.

It is easy to show by induction that

$$(x+\beta-1)(x+\beta-2)\dots(x+\beta-k)M_n(x;\beta,c)=\sum_{j=0}^{k}q_{k-j}(x)M_{n-j}(x;\beta-k,c), \tag{5.13}$$

where $q_{k-j}(x)$ are polynomials of degree $k-j$ and by applying [14, p. 160, eqn. 5] again, the polynomial $(x+\beta-1)(x+\beta-2)\dots(x+\beta-k)M_n(x;\beta,c)$ is quasi-orthogonal of degree $n+k$ and order $2k$ on $(0,-\beta+k)$ for $-n\le\beta-k<-n+1$, or alternatively, $n+\beta-1<k\le n+\beta$ and we conclude that the polynomial

$$(x+\beta-1)(x+\beta-2)\dots(x+\beta-\lfloor n+\beta\rfloor)M_n(x;\beta,c)$$

is quasi-orthogonal of degree $n+\lfloor n+\beta\rfloor$ and order $2\lfloor n+\beta\rfloor$ on $(0,-\beta+\lfloor n+\beta\rfloor)$.

In the next section, we turn our attention to the case $-\beta=N,\ n=N+k,\ n,k,N\in\mathbb{N}$ and $c<0$.

5.4 The zeros of $K_n(x;p,N)$, $0<p<1$ and $n=N+1,N+2,\ldots$

In this section we examine the zeros of the polynomials $K_n(x;p,N)$, $0<p<1$, for $n=N+k, k\in\mathbb{N}$. Note that for these parameter values, the coefficient C_n in the three term recurrence relation (1.19) becomes nonpositive, thus the polynomials are non-orthogonal on the real line. In the case $n=N+1$ we have $C_n=0$ and a degenerate version of Favard's theorem ensures a non-standard (Δ-Sobolev) orthogonality (cf. [18, p. 1250, Theorem 2.2]).

By substituting $-\beta$ with N and n with $N+k$ in (5.13), we find that the polynomial $(x-N-1)(x-N-2)\ldots(x-N-k)K_{N+k}(x;p,N)$ of degree $N+2k$ is quasi-orthogonal of order $2k$ on $(0,N+k)$. However, this does not lead to new information about the zeros of $K_{N+k}(x;p,N)$, since by (1.17), the three term recurrence relation (1.19) and the product formula (5.1), the following results are guaranteed.

Lemma 5.4.1 *(cf. [68, p. 36] and [4, p. 17, Proposition 5.1(v)]) For $0<p<1$ and N a positive integer, the polynomial $K_{N+1}(x;p,N)=(\frac{1}{p})^{N+1}(x)(x-1)\ldots(x-N)$ and has $N+1$ real zeros $x=0,1,\ldots,N$.*

Remark. The result in Lemma 5.4.1 is related to Sylvester type determinants, as stated by JJ Sylvester in 1854 (cf. [67, p. 305]). Both Askey [9] and Holtz [35] show different ways how to evaluate these determinants and obtain this specific result in [9, eqn. 3.25] and [35, p. 4, eqn. 5], where the connection of Krawtchouk polynomials with tridiagonal matrices, whose entries come from the recurrence coefficients of these discrete orthogonal polynomials, is made explicit.

Corollary 5.4.2 *For $0<p<1$ and N a positive integer, the polynomial $K_{N+2}(x;p,N)$ has $N+2$ real zeros $x=0,1,\ldots,N,N+1-p(N+2)$.*

Proof. Letting $n=N+1$ in (1.19), we obtain

$$pK_{N+2}(x;p,N) \quad = \quad \big(x-N-1+p(N+2)\big)K_{N+1}(x;p,N),$$

which, together with Lemma 5.4.1, yields the stated result. ∎

Corollary 5.4.3 *For $0 < p < 1$ and N a positive integer, the polynomial $K_{N+3}(x;p,N)$ has at least $N+1$ real zeros $x = 0,1,\dots,N$. Furthermore, the remaining two zeros will be real and distinct when*

$$0 < p < \frac{1}{2}\left(1 - \sqrt{\frac{N+2}{N+3}}\right) \quad \textit{or} \quad \frac{1}{2}\left(1 + \sqrt{\frac{N+2}{N+3}}\right) < p < 1.$$

Proof. It follows from (5.2) that $K_{N+3}(x;p,N) = K_{N+1}(x;p,N)\ p_2(x)$, where

$$\begin{aligned} p_2(x) &= M_2\Big(x - N - 1; N+2, \frac{p}{p-1}\Big) \\ &= \frac{x^2 + (2Np + 6p - 2N - 3)x + (N+2)(N+3)p^2 - 2p(N+1)(N+3) + (N+1)(N+2)}{p^2}. \end{aligned}$$

The zeros of the quadratic $p_2(x)$ are real and distinct when the discriminant

$$\Delta = \frac{1}{p^4}\Big(1 - 4p(N+3) + 4p^2(N+3)\Big)$$

is positive and this yields the stated result. ■

We refer the reader to the remark in [4, p. 18], where it is stated that the polynomial $K_{N+k}(x;p,N)$, $k \in \mathbb{N}$, has $N+1$ real zeros $x = 0,1,\dots,N$, as well as $k-1$ real zeros of odd multiplicity in the interval $[0,\infty)$. This is indeed true for $k = 1,2$, but from Corollary 5.4.3, we see that when $k = 3$, the remaining $k-1$ zeros can be non-real.

It is difficult to determine the exact location of the zeros in the general case $K_n(x;p,N)$, $n = N+k$, $k \in \mathbb{N}$. There will always be the $N+1$ real zeros $0,1,\dots,N$ and for N odd, another real zero is guaranteed, but from (5.1), we have

$$K_{N+k}(x;p,N) = K_{N+1}(x;p,N)K_{k-1}(x-N-1;p,-N-2)$$

and we can consider the polynomials $K_n(x;p,-N)$, for $0 < p < 1$ and $N \in \mathbb{N}$, instead.

In general, it suffices to investigate the zeros of $M_n(x;\beta,c)$ for $\beta > 0$ when $-1 \le c < 0$ (or $c \le -1$), because it follows from the symmetry relation (1.14) that if x is a zero of $M_n(x;\beta,c)$ then $-\beta - x$ is a zero of $M_n(x;\beta,\frac{1}{c})$. Taking into consideration the complex conjugate pairs, geometrically it means that the zeros of $M_n(x;\beta,\frac{1}{c})$ are the mirror image of the zeros of $M_n(x;\beta,c)$ with respect to the axis $\operatorname{Re} x = -\beta/2$ when $\beta > 0$ and $c < 0$.

Figure 5.2: The zeros of $M_{10}(x; 8.2, -4)$, $M_{10}(x; 8.2, -0.25)$, $M_{10}(x; 8.2, -15.667)$ and $M_{10}(x; 8.2, -0.064)$ clockwise.

Figure 5.2 shows the zeros of $M_n(x; \beta, c)$ when $n = 10$, $\beta = 8.2$ for different values of $c < 0$, clearly illustrating the symmetry with respect to $\operatorname{Re} x = -4.1$.
The numerical examples show that the zeros of polynomials $M_n(x; \beta, c)$, $\beta > 0$, $c < 0$, seem to lie on rays starting from the x axis. For the special case $c = -1$ that corresponds to Meixner-Pollaczek polynomials (cf. [24, (9.7.1)]) with $\lambda = \frac{\beta}{2}$, $\phi = \frac{\pi}{2}$ all the zeros of polynomials $M_n(x; \beta, c)$, $\beta > 0$, lie on the line $\operatorname{Re} x = -\beta/2$. This special case, when $c = -1$, is illustrated in Figure 5.3.

Figure 5.3: The zeros of $M_{10}(x; 8.2, -1)$

The asymptotic distribution of the zeros of $M_n(x; \beta, c)$, $\beta > 0$, $c < 0$, as $n \to \infty$, (after the necessary rescaling) could possibly be proved using a standard saddle point technique or the complex orthogonality (cf. [17, p. 450, Proposition 9]) and potential theoretical methods or a Riemann-Hilbert approach, as was done in the case of Jacobi polynomials for non-standard parameters (cf. [46], [50]).

5.5 The zeros of $M_n(x;\beta,c)$, $c \to 0$ and $\beta \in \mathbb{R}$

Lastly, we consider $M_n(x;\beta,c)$ when $\beta \in \mathbb{R}$ and prove that when $c \to 0$, all the zeros of $M_n(x;\beta,c)$, $n = 1, 2, \ldots$, approach non-negative integer values. Note that this theorem holds for any $\beta \in \mathbb{R}$, which implies that when $-\beta = N$, $N \in \mathbb{N}$ and $c < 0$, the zeros of the Krawtchouk polynomials $K_n(x; \frac{c}{c-1}, N)$ approach the mass points $x = 0, 1, \ldots, n-1$ of the discrete measure as $c \to 0$.

Theorem 5.5.1 *For $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$, the n zeros of the polynomial $M_n(x;\beta,c)$ approach the points $x = 0,\ 1, \ldots,\ n-1$ when $c \to 0$.*

Proof. From (1.10),

$$\begin{aligned} &M_n(x;\beta,c) \\ &= (\beta)_n + (\beta+1)_{n-1}(-n)(-x)\left(1-\frac{1}{c}\right) + \cdots + \\ &\quad \frac{(\beta+n-1)(-n)_{n-1}(-x)_{n-1}(1-\frac{1}{c})^{n-1}}{(n-1)!} + (x)(x-1)\ldots(x-n+1)\left(1-\frac{1}{c}\right)^n. \end{aligned}$$

For any $n \in \mathbb{N}$, the function

$$\begin{aligned} c^n M_n(x;\beta,c) &= c^n(\beta)_n + \cdots + \frac{c(\beta+n-1)(-n)_{n-1}(-x)_{n-1}(c-1)^{n-1}}{(n-1)!} \\ &+ (x)(x-1)\ldots(x-n+1)(c-1)^n, \end{aligned}$$

regarded as an nth degree polynomial in x with real parameters β and c, has the same zeros as $M_n(x;\beta,c)$. Since

$$\lim_{c\to 0} c^n M_n(x;\beta,c) = (x)(x-1)\ldots(x-n+1)(-1)^n,$$

the zeros of $c^n M_n(x;\beta,c)$ and hence the zeros of $M_n(x;\beta,c)$ tend to the zeros of $x(x-1)(x-2)\ldots(x-n+1)$, which is to say $x = 0, 1, 2, \ldots, n-1$. ∎

This theorem implies that for sufficiently small c, all the zeros of $M_n(x;\beta,c)$ are real. An analogous result can be proved for Charlier polynomials.

5.6 Conclusion

The Meixner polynomials $M_n(x;\beta,c),\ 0<c<1$ and $\beta>0$, are orthogonal with respect to the discrete measure $\frac{c^x(\beta)_x}{x!}$ on $(0,\infty)$ and by applying the Pfaff-Kummer transformation, an orthogonality relation can be obtained for these polynomials for parameter values $c>1$ and $\beta>0$, with respect to the measure $\frac{(\beta)_x}{c^x x!}$ on $(-\infty,-\beta)$. In this chapter we proved that the Meixner polynomials are quasi-orthogonal of order k for $-k<\beta<-k+1,\ k=1,2,\ldots,n-1$ and $0<c<1$, as well as $c>1$.

We used a Sturmian sequence argument to prove that, for $n<1-\beta$, the polynomials $M_n(x;\beta,c)$, $c<0$, are orthogonal with respect to an (unknown) weight function on the interval $(0,-\beta)$. Furthermore, we proved that, for $c<0$ and $1-\beta\leq n<0$, the polynomial

$$(x+\beta-1)(x+\beta-2)\ldots(x+\beta-\lfloor n+\beta\rfloor)M_n(x;\beta,c)$$

is quasi-orthogonal of degree $n+\lfloor n+\beta\rfloor$ and order $2\lfloor n+\beta\rfloor$.

We determined the location of the zeros of the polynomial $K_n(x;p,N)$, $0<p<1$, for $n=N+2$ and $n=N+3$ and finally, we showed that the zeros of the polynomials $M_n(x;\beta,c)$ tend to $0,1,\ldots,n-1$, as $c\to 0$ for all real values of β and this implies that the zeros of the Krawtchouk polynomials $K_n(x;p,N)$, $0<p<1$, tend to $0,1,\ldots,n-1$ as $p\to 0$.

Chapter 6

Conclusion, contribution to knowledge and future research

6.1 Conclusion

In this work we investigated the extent to which Stieltjes interlacing holds between the zeros of two Jacobi, Meixner and Krawtchouk polynomials if each polynomial belongs to a sequence generated by a different value of the parameters α and/or β in the Jacobi case, β in the Meixner case and N in the Krawtchouk case. These results differ from results obtained for Gegenbauer [25] and Laguerre [27] polynomials, since

- these polynomials are two-parameter polynomials, although it only makes sense to shift one parameter in the Meixner and Krawtchouk cases;
- Meixner and Krawtchouk polynomials are discrete orthogonal polynomials and
- the system of Krawtchouk polynomials is a finite system of orthogonal polynomials.

In each of the above cases, we identified a polynomial that plays the role of the de Boor-Saff polynomial [12, 19], in the sense that its zeros provide a (non-unique) set of points, that complete the interlacing process. The extra interlacing points obtained can be applied as inner bounds for the extreme zeros of the appropriate polynomials and we identified the specific points that are the best bounds for the extreme zeros of each of the Jacobi, Meixner and Krawtchouk polynomials.

Furthermore, we made a compehensive study of the zero location of Meixner and Krawtchouk polynomials, in particular for parameter values where (some of) the zeros are real.

6.2 Contribution to knowledge

Our contributions to knowledge are:

(1) We proved that Stieltjes interlacing holds between the zeros of the

(i) Jacobi polynomials $P_{n+1}^{\alpha,\beta},\ \alpha,\beta>0$, and

- $P_{n-1}^{\alpha+t,\beta},\ t\in\{0,1,2,3,4\}$ and $P_{n-1}^{\alpha,\beta+k},\ k\in\{0,1,2,3,4\}$;
- $P_{n-1}^{\alpha+t,\beta+k},\ t,k\in\{1,2\}$, $P_{n-1}^{\alpha+1,\beta+3}$ and $P_{n-1}^{\alpha+3,\beta+1}$;
- $P_{n-1}^{\alpha-1,\beta+t}$ and $P_{n-1}^{\alpha+t,\beta-1},\ t\in\{1,2\}$;

(ii) Meixner polynomials $M_{n+1}(x;\beta,c),\ \beta>0,\ 0<c<1$, and

- $M_{n-1}(x;\beta+t,c),\quad t\in\{0,1,2,3,4\}$;

(iii) Krawtchouk polynomials $K_{n+1}(x;p,N),\ 0<p<1,\ n=1,2,\dots,N-1,\ N\in\mathbb{N}$, and

- $K_{n-1}(x;p,N-k),\ k\in\{0,1\}$;
- $K_{n-1}(x;p,N-2)$, for $p<1-\frac{n+1}{N}$;
- $K_{n-1}(x;p,N+1)$, for $p<\frac{3n-N+\sqrt{5n^2-4n^3-2nN+4n^2N+N^2}}{2n(n+1)}$;

when the polynomials under consideration are co-prime.

(2) In each of the above cases, we identified an extra interlacing point, that completes the interlacing process.

(3) We identified the extra interlacing points that can be applied as sharp inner bounds for the extreme zeros of each of the Meixner and Krawtchouk polynomials.

(i) For the extreme zeros of Meixner polynomials $M_n(x;\beta,c),\ \beta>0,\ 0<c<1$, we have

$$0<x_{n,1}<B_{n-1}(4)<\frac{((2+\beta-2n)c+n-1)n(n-1)c^2}{1-c}<\frac{(n-1)(1+c)+\beta c}{1-c}<x_{n,n},$$

where

$$B_{n-1}(4)=\frac{(\beta)_2\big((\beta+3)c-1\big)+n(c-1)\big((c-1)^2-\beta^2+\beta(1+c(c-3))\big)-\beta(c-2)(n(c-1))^2-(n(c-1))^3}{(1-c)\big(\beta(2n+\beta+1)+n(n+1-(n-1)c^2)\big)}$$

(ii) For the extreme zeros of the Krawtchouk polynomials $K_n(x;p,N),\ n=1,2,\ldots,N$ and $p\in(0,1)$, we obtained the bounds

$$0<x_{n,1}<Np+(n-1)(1-2p)<Np+(n-1)(1-p)<x_{n,n}<N.$$

Furthermore, when $p<1-\frac{n}{N}$,

$$0<x_{n,1}<Np+(n-1)(1-2p)<Np+n-1<x_{n,n}<N-1.$$

(4) We considered the possibility that the pairs of polynomials, mentioned in (1), can have a common zero and we identified the common zero in each specific case.

(5) We extended Gibson's result [33], that determines the maximum amount of common zeros of two polynomials from the same orthogonal sequence, to the zeros of Jacobi, Meixner and Krawtchouk polynomials of degree $n-1$ and $n+1$ from different orthogonal sequences.

(6) We proved that Stieltjes interlacing holds between the zeros of the

- Jacobi polynomials $P_{n+1}^{\alpha,\beta}$ and $P_{n-k}^{\alpha+k,\beta+k},\ \alpha,\beta>0$;
- Meixner polynomials $M_{n+1}(x;\beta,c)$ and $M_{n-k}(x;\beta+k,c),\ \beta>0, 0<c<1$,

for $k\in\{1,2,\ldots,n-1\}$ when

(i) the polynomials under consideration are co-prime, in which case there will be k extra interlacing points;

(ii) the above-mentioned pairs of polynomials have common zero(s).

(7) We proved that Meixner polynomials $M_n(x;\beta,c)$

(i) are orthogonal for $n<1-\beta,\ \beta,c<0$, with respect to an (unknown) weight function on the interval $(0,-\beta)$;

(ii) are quasi-orthogonal of order k for $-k<\beta<-k+1,\ k=1,2,\ldots,n-1$ and $0<c<1$, as well as $c>1$;

(iii) $(x+\beta-1)(x+\beta-2)\ldots(x+\beta-\lfloor n+\beta\rfloor)M_n(x;\beta,c)$ are quasi-orthogonal of degree $n+\lfloor n+\beta\rfloor$ and order $2\lfloor n+\beta\rfloor$ for $c<0$.

(8) We proved that the zeros of $M_n(x;\beta,c)$ tend to the first n mass points of the discrete measure $\frac{c^x(\beta)_x}{x!}$, namely $x=0,1,\ldots,n-1$, for all $\beta\in\mathbb{R}$, when $c\to 0$.

(9) For the Krawtchouk polynomials $K_n(x;p,N),\ 0<p<1$, we proved that

(i) when $n=N+2$, the zeros are $x=0,1,\ldots,N,N+1-p(N+2)$;

(ii) when $n=N+3$, the zeros are $x=0,1,\ldots,N$, together with 2 more zeros and we provided the conditions on p necessary for these 2 zeros to be real.

(iii) when $p\to 0$, the zeros tend to $x=0,1,\ldots,n-1$.

6.3 Open problems

The following open problems could be studied in future research:

- The asymptotic distribution of the zeros of $M_n(x;\beta,c)$, $\beta>0$, $c<0$, as $n\to\infty$, could possibly be proved (after the necessary rescaling) using the complex orthogonality (cf. [17, p. 450, Proposition 9]) and potential theoretical methods or a Riemann-Hilbert approach, as was done in the case of Jacobi polynomials for non-standard parameters (cf. [46], [50]).

- Stieltjes interlacing between zeros of different sequences of the remaining classes of polynomials on the ${}_2F_1$ plane of the Askey scheme of hypergeometric orthogonal polynomials, i.e., Meixner-Pollaczek and Romanovski polynomials, can be investigated to obtain new bounds for the extreme zeros of these polynomials.

Chapter 7

Appendix

For the sake of convenience, we provide the reader with some equations that are frequently used in this thesis.

7.1 The Pfaff-Kummer transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{7.1} $$

[44, p. 10, eqn. 1.7.2].

7.2 Contiguous relations for ${}_2F_1$ hypergeometric polynomials

Gauss (cf. [57, p. 50]) defined as *contiguous to* ${}_2F_1(a,b;c;z)$ each of the six functions obtained by shifting one of the parameters by one unit, e.g., ${}_2F_1(a+1,b;c;z)$ and ${}_2F_1(a,b;c-1;z)$, and he proved that there is a relation, mostly linear in x, between the function ${}_2F_1(a,b;c;z)$ and any two of its contiguous functions. We list the contiguous function relations that are used in some of our proofs:

$$ (n+c-1)\ {}_2F_1(-n,b;c;z) = n\ {}_2F_1(-n+1,b;c;z) + (c-1)\ {}_2F_1(-n,b;c-1;z) \tag{7.2} $$

$$ \begin{aligned} (n-(b-c)z)\ {}_2F_1(-n,b;c;z) &= n(1-z)\ {}_2F_1(-n+1,b;c;z) \\ &+ \frac{(c+n)(c-b)}{c} z\ {}_2F_1(-n,b;c+1;z) \end{aligned} \tag{7.3} $$

$$ (1-z)\ {}_2F_1(-n,b;c;z) = {}_2F_1(-n-1,b;c;z) - \left(\frac{c-b}{c}\right) z\ {}_2F_1(-n,b;c+1;z) \tag{7.4} $$

$$(1-z)\ {}_2F_1(-n,b;c;z) = {}_2F_1(-n,b-1;c;z) - \left(\frac{c+n}{c}\right) z\ {}_2F_1(-n,b;c+1;z) \tag{7.5}$$

$$(2n+c-(n+b)z)\ {}_2F_1(-n,b;c;z) = n(1-z)\ {}_2F_1(-n+1,b;c;z) + (n+c)\ {}_2F_1(-n-1,b;c;z) \tag{7.6}$$

$$(2b-c-(n+b)z)\ {}_2F_1(-n,b;c;z) = b(1-z)\ {}_2F_1(-n,b+1;c;z) - (c-b)\ {}_2F_1(-n,b-1;c;z) \tag{7.7}$$

follow from (2), (3), (4), (5), (6) and (11) in [57, p. 71] respectively and

$$\left(1-\frac{n+b}{n+c}\ z\right)\ {}_2F_1(-n,b;c;z) = {}_2F_1(-n-1,b;c;z) - \frac{(c-b)n}{(n+c)c}\ z\ {}_2F_1(-n+1,b;c+1;z), \tag{7.8}$$

can be derived by combining (7.2) and (7.4).

7.2.1 More contiguous relations

A useful algorithm for computing contiguous relations for ${}_2F_1$ Gauss hypergeometric series, written by R. Vidunas in 2002, is available as a computer package (cf. [71]). In the following lemma, we provide the identities that are used in the proofs of our theorems; they follow from the contiguous relations for ${}_2F_1$ hypergeometric polynomials and can be easily verified by comparing equal powers of the corresponding coefficients.

Lemma 7.2.1 *Let $F_n = {}_2F_1(-n,b;c;z)$ and denote ${}_2F_1(-n-1,b+1;c;z)$ by $F_{n+1}(b+1)$, ${}_2F_1(-n+1,b+1;c-3;z)$ by $F_{n-1}(b+1,c-3)$ and so on. Then*

$$\left(\frac{b(c+n)}{(b+n)(b+n+1)}-z\right)F_n = \frac{b(c+n)}{(b+n)(b+n+1)}F_{n+1}(b+1) + \frac{n(b-c)z}{c(b+n)}F_{n-1}(c+1) \tag{7.9}$$

$$\left(\frac{c}{b+n+1}-z\right)F_n = \frac{c}{(b+n+1)}F_{n+1}(b+1) + \frac{(b-c)nz^2}{c(c+1)}F_{n-1}(b+1,c+2) \tag{7.10}$$

$$\begin{aligned}\left(\frac{c+n}{b+1}-z\right)\left(\frac{1+b-c}{b+n-1}\right)F_n &= \frac{(1+b-c-nz)(c+n)}{(b+1)(b+n-1)}F_{n+1}(b+1) \\ &+ \frac{nz(1-z)^2}{c}F_{n-1}(b+2,c+1)\end{aligned} \tag{7.11}$$

$$\left(\frac{c+n}{b+n+1}-z\right)F_n = \frac{c+n}{b+n+1}F_{n+1}(b+1) - \frac{z(z-1)n}{c}F_{n-1}(b+1,c+1) \tag{7.12}$$

$$\begin{aligned}\frac{(b-c+1)}{(b+1)(b+n+1)}F_n &= \frac{b-c+1-z(b+n+1)}{(b+1)(b+n+1)}F_{n+1}(b+1) \\ &- \left(\frac{z^2-z}{c}\right)F_n(b+2,c+1)\end{aligned} \tag{7.13}$$

$$
\begin{aligned}
(bz+nz-n-c)F_n(c+1) &= -cF_{n+1}+n(z-1)F_{n-1}(c+1) && (7.14)\\
(z-1)\left(z-\frac{c}{b+n}\right)F_n &= \frac{c-z(c+n)}{b+n}F_{n+1}+\frac{n(c-b)(c-b+1)}{(b+n)c(c+1)}z^2F_{n-1}(c+2) && (7.15)\\
\left(\frac{c(c+1)}{(b+1)(c+n+1)}-z\right)F_n &= \frac{c+c^2-bnz+cnz}{(b+1)(c+n+1)}F_{n+1}(b+1) && (7.16)\\
&+ \frac{n(b-c)(b+n+1)z^3}{c(c+1)(c+2)}F_{n-1}(b+2,c+3)\\
(c-z(b-n+1))\,F_n &= \left(c+2nz-nz^2\left(\frac{b+n+1}{b+1-c}\right)\right)F_{n+1}(b+1) && (7.17)\\
&+ \frac{n(b+1)(b+2)((z-1)z)^2(b+n+1)}{(b+1-c)c(c+1)}F_{n-1}(b+3,c+2)
\end{aligned}
$$

$$
\begin{aligned}
\left(z-\frac{c(c+1)}{(1+c+n)(1+b)-cn}\right)F_n &= -\frac{c+c^2-bnz+2cnz+nz^2+bnz^2+n^2z^2}{1+c-cn+n+b+bc+bn}F_{n+1}(b+1) && (7.18)\\
&+ \frac{(b+1)(b+2)(b+n+1)(c+n+1)n(z-1)z^3}{c(c+1)(c+2)(1+c-cn+n+b+bc+bn)}F_{n-1}(b+3,c+3)
\end{aligned}
$$

$$
\begin{aligned}
\left(1-\frac{(b+1)(c+2n+2)-cn)}{c(c+2)}z\right)F_n &= \left(1-\frac{2(b-c)n}{c(c+2)}z-\frac{n(b-c)(b+n+1)}{c(c+1)(c+2)}z^2\right)F_{n+1}(b+1)\\
&+ \frac{a}{c^2(c+1)^2(c+2)^2(c+3)}F_{n-1}(b+3,c+4) && (7.19)\\
\textit{where } a &= (b+1)(b+2)(b-c)(c+n+1)(c+n+2)(b+n+1)z^4n
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{-n(n+1)}{1+b-c}+(c+3n)z-(b+n)z^2\right)F_n &= (c+n)\left(z-\frac{n}{1+b-c}\right)F_{n+1} && (7.20)\\
&+ \frac{n(c-1)}{1+b-c}(z-1)^2F_{n-1}(c-1)
\end{aligned}
$$

$$
\begin{aligned}
&(z-1)\left((n+1)(b(z-1)+nz)z+c(z-1)(z(b+n)-c-1)\right)F_n && (7.21)\\
&= -(c^2(z-1)^2+nz(z(n+1)-b)+c(1-(n+2)z+(2n+1)z^2))F_{n+1}\\
&+ \frac{n(c-b)_3}{(c)_3}(c+n+1)z^3F_{n-1}(c+3)
\end{aligned}
$$

$$
\begin{aligned}
(1-z)D_1\ F_n &= D_2\ F_{n+1} + \frac{n(c-b)_4(n+c+1)(n+c+2)}{(c)_4} z^4 F_{n-1}(c+4), \qquad (7.22)\\
D_1 &= c^3(z-1)^2 - c^2(z-1)^2(bz+nz-3)\\
&- c((n+4+(2n+3)b)z - (2+3n+n^2+(4n+6)b)z^2 + (b+n)(2n+3)z^3 - 2)\\
&- (n+1)z(n(n+2)z^2 + b(2-2(n+2)z+(n+2)z^2))\\
D_2 &= -c^3(z-1)^3 - c^2((n+9)z - 3(n+3)z^2 + 3(n+1)z^3 - 3)\\
&+ c(2-(6+n+2bn)z + (6+4(b+1)n+n^2)z^2 - (2+6n+3n^2)z^3)\\
&- nz(b^2z + (2+3n+n^2)z^2 + b(2-5z-2nz))
\end{aligned}
$$

Proof. We will verify equations (7.9) and (7.15). The other identities can be verified in the same way.

For each $j = 1, 2, \ldots, n$, the coefficient of z^j on the left-hand side of (7.9) is

$$
\begin{aligned}
&\frac{b(c+n)(-n)_j(b)_j}{(b+n)(b+n+1)(c)_j(j)!} - \frac{(-n)_{j-1}(b)_{j-1}}{(c)_{j-1}(j-1)!}\\
&= \frac{(-n)_{j-1}(b)_{j-1}}{(b+n)(b+n+1)(c)_j j!}\left(b(c+n)(-n+j-1)(b+j-1) - j(c+j-1)(b+n)(b+n+1)\right)
\end{aligned}
$$

while the coefficient of z^j on the right-hand side of (7.9) is given by

$$
\begin{aligned}
&\frac{b(c+n)(-n-1)_j(b+1)_j}{(b+n)(b+n+1)(c)_j j!} + \frac{n(b-c)}{c(b+n)}\frac{(-n+1)_{j-1}(b)_{j-1}}{(c+1)_{j-1}(j-1)!}\\
&= \frac{(-n)_{j-1}(b)_{j-1}}{(b+n)(b+n+1)(c)_j j!}\left((c+n)(-n-1)(b+j-1)(b+j) - j(b-c)(-n+j-1)(b+n+1)\right).
\end{aligned}
$$

A straightforward calculation shows that these coefficients are equal and the result follows.

For each $j = 1, 2, \ldots, n$, the coefficient of z^j on the left-hand side of (7.15) is

$$
\begin{aligned}
&\frac{(-n)_{j-2}(b)_{j-2}}{(c)_{j-2}(j-2)!} - \frac{(c+b+n)(-n)_{j-1}(b)_{j-1}}{(b+n)(c)_{j-1}(j-1)!} + \frac{c(-n)_j(b)_j}{(b+n)(c)_j j!}\\
&= \frac{(-n)_{j-2}(b)_{j-2}}{(c)_{j-2}(j-2)!}\left(1 - \frac{(-n+j-2)(b+j-2)}{(b+n)(c+j-2)(j-1)}\left((c+b+n) + \frac{c(-n+j-1)(b+j-1)}{(c+j-1)j}\right)\right).
\end{aligned}
$$

The coefficient of z^j on the right-hand side of (7.15) is given by

$$\begin{aligned}
&\frac{c(-n-1)_j(b)_j}{(b+n)(c)_j j!} - \frac{(c+n)(-n-1)_{j-1}(b)_{j-1}}{(b+n)(c)_{j-1}(j-1)!} + \frac{(c-b)(c-b+1)n(-n+1)_{j-2}(b)_{j-2}}{c(c+1)(b+n)(c+2)_{j-2}(j-2)!} \\
&= \quad \frac{(-n)_{j-2}(b)_{j-2}}{(b+n)(c)_{j-2}(j-2)!(c+j-2)} \times \\
&\quad \left(\frac{c(n+1)(b+j-1)(b+j-2)(n-j+2)}{(c+j-1)j(j-1)} + \frac{(c+n)(n+1)(b+j-2)}{j-1} + \frac{(c-b)_2(n-j+2)}{c+j-1}\right).
\end{aligned}$$

Again, a straightforward calculation shows that these coefficients are equal and (7.15) follows. ■

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