FLOW OF SECOND–GRADE FLUIDS IN REGIONS WITH PERMEABLE BOUNDARIES.

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Summary

The equation of motion for the flows of incompressible Newtonian fluids (Navier Stokes equations) under no-slip boundary conditions have been studied deeply from many perspectives. The questions of existence and uniqueness of both classical and weak solutions have received more than a fair share of attention. In this study the same problem for non-Newtonian fluids of second grade has been studied from the point of view of weak solutions and classical solutions for non-homogeneous boundary data, i.e., dynamical boundary conditions in regions with permeable boundaries. We consider the situation where a container is immersed in a larger fluid body and the boundary admits fluid particles moving across it in the direction of the normal.

In this study we give alternative approaches through formulations of 'dynamics at the boundary', the idea being that the normal component of velocity at the boundary is viewed as an unknown function which satisfies a differential equation intricately coupled to the flow in the region 'enclosed' by the boundary.

We describe two mathematical models denoted by *Problem* $\mathcal{P}_1$ and *Problem* $\mathcal{P}_2$. These models lead to dynamics at a permeable boundary, and a kinematical boundary condition for normal flow through the boundary. These conditions take into account the curvature of the boundary which enforces certain stresses. We then show with the help of the energy method that for fluids of second grade, the dynamics at the boundary and the boundary condition lead to conditional stability of the rest state for *Problem* $\mathcal{P}_1$ and *Problem* $\mathcal{P}_2$. We also prove uniqueness of classical solutions for the two models. The existence of a weak solution for this system of evolution equations is proved only for *Problem* $\mathcal{P}_2$ with the help of the Faedo-Galerkin method with a special basis. In this case the special basis is formed by eigenfunctions.

The existence proof of at least one classical solution, local in time is established by means of a version of the Fixed-point Theorem of Bohnenblust and Karlin, and the Ascoli-Arzelà Theorem.

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Titel:	Vloei van tweede-graadse vloeistowwe in gebiede met deurlaatbare randte
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Opsomming

Die bewegingsvergelyking vir nie-saamdrukbare vloei vir Newtoniese vloeistowwe onder geen-glyding randvoorwaardes is al vele kere bestudeer. Die vrae oor bestaan en eenduidigheid van beide klassieke en swak oplossings het al baie aandag gekry. In hierdie werk word dieselfde probleem vir nie-Newtoniese vloeistowwe van graad twee met nie-homogene randdata, d.i. dinamiese randvoorwaardes in gebiede met deurlaatbare randte, aangespreek. Ons beskou die geval waar 'n liggaam in 'n groter liggaam vol vloeistof gedompel is, waarvan die rand vloeistof deurlaat in die rigting van die normaal.

Ons gee 'n alternatiewe benadering deur die formulering van 'n dinamiese randvoorwaarde waarvan die normaalkomponent van die snelheid by die rand as die onbekende beskou word wat hierdie differensiaalvergelyking bevredig. Ons beskryf wiskundige modelle *Probleem* $\mathcal{P}_1$ en *Probleem* $\mathcal{P}_2$. Hierdie modelle lei tot 'n dinamiese randvoorwaarde by die deurlaatbare rand, en kinematiese randvoorwaardes vir normaalvloei. Hierdie voorwaardes neem die kurwe van die rand in ag wat sekere kragte veroorsaak. Dan wys ons met die hulp van die energie-metode dat vir vloeistowwe van graad twee hierdie randvoorwaardes lei tot voorwaardelike stabiliteit van die rustoestand vir *Probleem* $\mathcal{P}_1$ en *Probleem* $\mathcal{P}_2$. Ons bewys eenduidigheid van klassieke oplossings vir die twee modelle.

Die bestaan van 'n swak oplossing van die stelsels van ewolusievergelykings vir *Probleem* $\mathcal{P}_2$ word bewys met die hulp van die Faedo-Galerkin metode met spesiale basis. In hierdie geval bestaan die spesiale basis uit eiefunksies. Die bestaansbewys van 'n klassieke oplossing, lokaal in tyd, word verkry deur gebruik te maak van 'n weergawe van die Dekpuntstelling van Bohnenblust en Karlin en die Ascoli-Arzelà Stelling.

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To my mother and to the loving memory of my father, who believed in me.

To my dearest husband Sam, for his constant love and encouragement.

To Karli and Lisa who will rejoice with me:

*Ps.*33:2,3.

> Praise the LORD with the harp, make melody to Him with an instrument of ten strings.
>
> Sing to Him a new song; play skillfully with a shout of joy.
>
> For the word of the LORD is right, and all His work *is done* in truth.

To my Father

Contents

Chapter 1

Introduction

1.1 Historical Outline.

The exact solution for the problem of the viscous fluid at rest was correctly given by the Greek mathematician Archimedes (287–212 B.C.).

In 1500, the equation of conservation of mass for incompressible one-dimensional viscous flow was correctly deduced by Leonardo da Vinci, the Italian painter, sculptor, musician, philosopher, anatomist, botanist, geologist, architect, engineer, and scientist. Leonardo's notes also contain accurate sketches and descriptions of wave motion, hydraulic jumps, free jets, reduction of drag by streamlining, and the velocity distribution in a vortex.

The next notable achievement was by Evangelista Torricelli (1608–1647) who published his theory that the velocity of efflux of a liquid from a hole in a tank is equal to the velocity which a liquid particle would attain in free fall from its surface.

The above achievements do not relate directly to viscous motion. It happens that these results are also true for a viscous or *real* fluid. Edme Marriotte (1620–1684) was probably the first to make a direct study of fluid friction. He invented a balance system to measure the drag of a model held stationary in a moving stream, the first wind tunnel. Mariotte's text, "Traité du mouvement des eaux", was published in 1686, a year before the excellent "Principia Mathematica" of Sir Isaac Newton.

In 1687 Newton published in his "Principia" the simple statement which delineates the viscous behavior of nearly all common fluids: "The resistance which arises from the lack of lubricity in the parts of a fluid — other things being equal — is proportional to the velocity by which the parts of the fluid are being separated from each other." Such fluids, water and air being prominent examples, are now called *newtonian* in his honour. With the law of linear viscosity thus proposed, Newton contributed the first viscous-flow analysis by deriving the correct velocity distribution about a rotating cylinder. Because of Newton's more famous discovery, the differential calculus, the world got sidetracked in another direction for some time.

Daniel Bernoulli was the first to bring the attention back to fluid mechanics when he demonstrated in 1738 the proportionality between pressure gradient and acceleration in inviscid flow. Subsequently, Leonhard Euler, who is said to be the master of calculus, derived in 1755 the famous frictionless equation which now bears Bernoulli's name. This magnificent derivation is essentially unchanged in ideal-fluid theory, or *hydrodynamics*, as Bernoulli termed it.

Jean d'Alembert published in 1752 his famous paradox, showing that a body immersed in a frictionless flow would have zero drag. Following d'Alembert, Lagrange (1736–1813), Laplace (1749–1827) and Gerstner (1756–1832) contributed priceless work to the new hydrodynamics.

The next significant analytical advance was the addition of frictional–resistance terms to Euler's inviscid equations. This was done by Navier in 1827, Cauchy in 1828, Poisson in 1829, St Venant in 1843 and Stokes in 1845. Stokes was the first to use the coefficient of viscosity μ, whereas the other four wrote their equations in terms of an unknown molecular function. Today these equations are called the *Navier-Stokes equations.* These equations, are nonlinear, complex and difficult to solve.

At the turn of the 19*th* century the biggest breakthrough was by Ludwig Prandtl in 1904. He demonstrated the existence of a thin *boundary layer* in fluid flow with high Reynolds numbers. Another breakthrough was the introduction of dimensional analysis by Osborne Reynolds (1842–1912), Lord Rayleigh (1842–1919) and Ludwig Prandlt (1875–1953).

The historic details in this section were abstracted from the excellent history of hydraulics by Rouse and Ince (1957)[R_4].

The stress tensor for the linear viscous Newtonian model is

$$\mathbf{T} = -p\mathbf{I} + \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$$

with p the pressure, μ the coefficient of viscosity and $\mathbf{v}$ the velocity of the fluid. This model describes the flow of fluids like water and other similar fluids. H Lamb [L_{10}], O. A. Ladyzhenskaya [L_1] wrote mathematical theories of viscous incompressible flow.

Nonlinear, or *non-Newtonian* fluids on the other hand are fluids like molten metals, multigrade oils, printing inks, paints, suspensions, polymer solutions, molten plastics, blood, protein solutions, ice[M_1], etc. These fluids cannot be described by the above model. The study of these interesting substances has proved to be very important with the growth of the polymer and plastics industry over the last four decades. Consequently an interest arose to study the flow of these nonlinear fluids and in the case of this model, second grade fluids, through permeable boundaries. The boundary conditions alone in such circumstances form an interesting study on its own. Works by R. Berker [B_2], Rajagopal and Gupta [R_1], are a few to mention in this regard.

In the study to follow a new approach is considered. The flow through permeable boundaries is modelled as a velocity normal through the boundary. A dynamic boundary condition forms a major part in the analysis of this problem.

1.2 Flow through Permeable Boundaries.

The equation of motion for incompressible flows in Newtonian fluids (Navier - Stokes equations) under no-slip boundary conditions have been studied deeply from many perspectives. Since the pioneering papers of Leray [L_6, L_7, L_8] and Hopf [H_2] questions of existence, stability [G_3, G_4] and uniqueness of both classical and weak solutions have received more than a fair share of attention.

Recently the same problem for non-Newtonian fluids of second grade has been studied from the point of view of weak solutions [$A_2, C_1, C_2, C_3, C_4, C_5$] and classical solutions for homogeneous Dirichlet boundary data [G_1], and non homogeneous boundary data [C_6, G_5, G_6].

Unlike Newtonian fluids, fluids of second grade (and other non-Newtonian species) have the property of developing "normal stresses differences" at boundaries. It was shown, for example, by R. Berker [B_2] that if an incompressible flow of a fluid of grade 2 satisfies the homogeneous Dirichlet boundary condition the stress at the boundary is given by

$$\mathbf{t} = (-p + \alpha|\boldsymbol{\omega}|^2)\mathbf{n} + [\mu\boldsymbol{\omega} + 2\alpha\partial_t\boldsymbol{\omega}] \wedge \mathbf{n}$$

where $\mathbf{n}$ is the unit exterior normal to the boundary and $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ the vorticity. The wedge denotes vector product. Thus there is a normal component of stress at the boundary in addition to the pressure.

We consider the situation where the container is immersed in a larger fluid body of the same kind and the boundary admits fluid particles moving across it in the direction of the normal.

The following figure illustrates the situation where the curvature of the boundary Γ of Ω' is non negative.

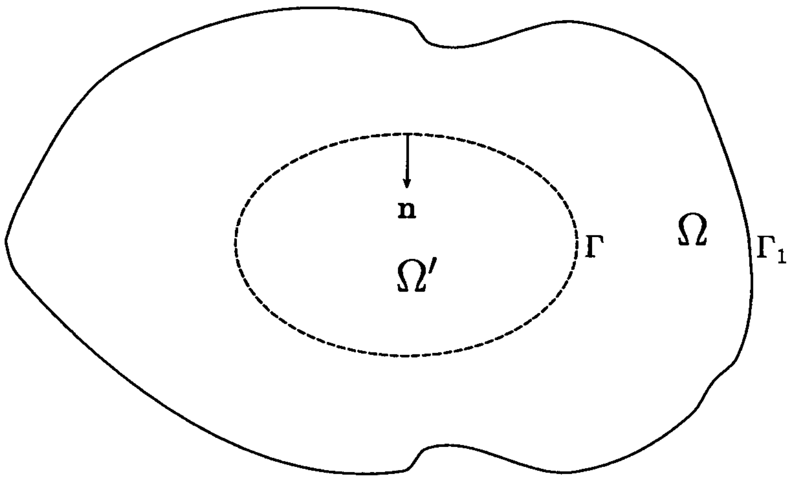

FIGURE 1. Profile for normal flow through the permeable wall Γ.

The question becomes, *what governs the flow across the boundary?* Possible attempts at circumventing this question may be to 'prescribe' the normal component of the velocity field at the boundary or to prescribe mass or momentum flux. Prescription of shear stress has also been suggested. $[B_4]$,

In this study we shall give an alternative approach through the formulation of 'dynamics at the boundary', the idea being that the normal component of velocity at the boundary is viewed as an unknown function which satisfies a differential equation intricately coupled to the flow in the region 'enclosed' by the boundary.

We describe two mathematical models denoted by *Problem* $\mathcal{P}_1$ and *Problem* $\mathcal{P}_2$. The models $\mathcal{P}_1$ and $\mathcal{P}_2$ has dynamics at a permeable boundary, and kinematical boundary condition for normal flow through the boundary. These conditions take into account the curvature of the boundary which enforces certain stresses. We then show with the help of the energy method that for fluids of second grade the dynamics at the boundary and the boundary condition leads to conditional stability of the rest state for *Problem* $\mathcal{P}_1$ and $\mathcal{P}_2$. We prove uniqueness of classical solutions for *Problem* $\mathcal{P}_1$ and $\mathcal{P}_2$. The existence of the solution of the system of evolution equations for $\mathcal{P}_2$ is proved with the help of the Faedo-Galerkin method with a special basis, as in the proof of $[T_1]$ for the Euler equations.

We shall deal here with isochoric motion (without change of volume) for which the condition is

$$\nabla \cdot \mathbf{v} = 0.$$

The Mathematical Model *Problem* $\mathcal{P}_1$**.**
We begin our study by introducing the mathematical model as an initial boundary value problem: The system of evolution equations, i.e. the equation of motion, $\nabla \cdot \mathbf{v} = 0$ because of incompressibility, and the three boundary conditions, because of the fact that we work with derivatives of the velocity to the third degree, form an important part of this study. Each equation in this model is discussed in detail in the sections that follow. If $\mathbf{T}$ denotes the stress tensor of the fluid, the equations of conservation of linear momentum and mass are expressed by

$$D_t[\rho\mathbf{v}(x,t)] = \nabla \cdot \mathbf{T}(p, \mathbf{v}) \quad \text{in} \quad \Omega \times (0, \infty) \tag{1.1}$$

and

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega \times (0, \infty),$$

where $D_t = \partial_t + \mathbf{v} \cdot \nabla$. The flow through the permeable boundary is assumed to be only in the direction of the unit exterior normal $\mathbf{n}$, which means that at the boundary $\mathbf{v}$ is of the form $\mathbf{v} = -\eta\mathbf{n}$. It is important to note that the function η defined on the boundary is also an unknown and is determined by the dynamical boundary condition:

$$\sigma\partial_t\eta + k\eta^2 = \mathbf{n} \cdot (\delta\mathbf{T})\mathbf{n} \quad \text{on} \quad \Gamma \times (0, T), \tag{1.2}$$

where $\delta\mathbf{T}$ denotes the difference between stress tensors on the two sides of the boundary. σ and k are defined in the next section.

The Mathematical Model $\mathcal{P}_2$**.**
Although it was possible to prove stability and uniqueness for the model of *Problem* $\mathcal{P}_1$ without any difficulty, we could not find a way to prove existence for a solution of *Problem* $\mathcal{P}_1$. We describe an alternative model which displays all the properties of *Problem* $\mathcal{P}_1$ with respect to stability and uniqueness, and for which existence can be proved. In the altenative model the dynamics at the boundary is formulated by assuming a 'shear flow' of the form

$$\mathbf{v}^*(y,t) = -\eta(s_1, s_2, t)\mathbf{n}(y)$$

with s_1, s_2 the surface parameters (like arc length). It is assumed that the 'body force' acting on the shearing fluid at the boundary is proportional to the difference between the pressures $\gamma_o p$ and $\ell(t)$. With γ_o we denote the value of certain commodeties on the boundary. Under these assumptions the equation governing the evolution of η is

$$\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta + \delta^{-1}\ell(t),$$

where $\gamma_o \mathbf{v} = -\eta\mathbf{n}$, and p the resulting pressure through the boundary with thickness δ. Δ_s is the Laplace–Beltrami operator ($\Delta_s = \nabla_s \cdot \nabla_s$) and ∇_s denotes the surface gradient. The parameter δ has the physical dimension of length, and may be thought of as the 'thickness' of the 'shear layer' (see [T_4], Sect 123, p. 506). The kinematical boundary condition is still imposed. For further detail see Chapter 5.

The form of the tensor $\mathbf{T}$ for second grade fluids will be discussed in Section 2.2. The meaning of the terms and the choice of the tensor $\delta\mathbf{T}$ in equation (1.2) will be made clear in Section 2.3 and 2.4.

In equation (1.1) time derivatives of both the velocity $\mathbf{v}$ and the symmetric part of the velocity gradient occur. As a result the system (1.1) – (1.2) has special initial conditions which will make the problem well-posed.

By well-posedness we mean existence and uniqueness of solutions as well as stability of the rest state. The question of stability and uniqueness for *Problem* $\mathcal{P}_1$ is investigated in Chapter 4 and for *Problem* $\mathcal{P}_2$ in Chapter 5. Existence for *Problem* $\mathcal{P}_2$ is proved in Chapter 6.

Stability and uniqueness are treated by means of energy methods which are specially adapted to accommodate the dynamical boundary conditions. In the proof of existence the Galdi-Grobbelaar-Sauer method [G_1] of reducing the equation (1.1) to a transport equation and a Stokes-type equation is modified so that the dynamical boundary condition is combined with the transport equation.

Existence of at least one solution for *Problem* $\mathcal{P}_2$ is proved by means of elliptic theory.

Treatment of the modified transport equation with the aid of the Galerkin method is done by choosing a special basis of eigenfunctions (see Appendix II) which deals with approximations in the domain and on the boundary at the same time.

In the system (1.1) – (1.2) the gradient of the pressure and the boundary value of the pressure occur, with the result that the traditional Helmholtz decomposition of a vector field is no longer appropriate . A modified decomposition theorem is used (Appendix III).

In this study a kinematical boundary condition, relating the vorticity $\boldsymbol{\omega}$ of the fluid to the surface gradient $\nabla_s \eta$ of the unknown function η, is imposed.

$$\boldsymbol{\omega} \wedge \mathbf{n} = 2\nabla_s \eta.$$

The meaning of this condition is that there are no tangential components of deformation at the interface boundary.

Chapter 2

Problem Formulation

In this chapter we give a mathematical formulation of the problem leading to the equations to be studied in Chapters 3 to 6.

This entails the following:
1. The choice of a stress tensor $\mathbf{T}$.
2. Dynamics at the boundary.
3. Choice of the stress tensors at the boundary.
4. The initial conditions.

2.1 Basic Notations.

We work in Euclidean 3-space. The following notation will be used throughout:

$$
\begin{aligned}
|\mathbf{x}| &:= \sqrt{\textstyle\sum_1^3 x_i^2} \quad \text{denotes the Euclidean norm.}\\
\partial_i &:= \partial/\partial x_i; i = 1,2,3.\\
\partial_t &:= \partial/\partial t.\\
[\nabla p]_i &:= \partial_i p \text{ if } p \text{ is a scalar field.}\\
[\nabla \mathbf{v}]_{ij} &:= \partial_j v_i; \quad i,j = 1,2,3, \quad \text{if } \mathbf{v} \text{ is a vector field.}\\
[\nabla \mathbf{v}]_{ij}^T &:= \partial_i v_j; \quad i,j = 1,2,3, \quad \text{if } \mathbf{v} \text{ is a vector field.}\\
\nabla \cdot \mathbf{v} &:= \textstyle\sum_{i=1}^3 \partial_i v_i \text{ if } \mathbf{v} \text{ is a vector field.}\\
\mathbf{v} \cdot \nabla &:= \textstyle\sum_{i=1}^3 v_i \partial_i \text{ if } \mathbf{v} \text{ is a vector field.}\\
[\nabla \cdot \mathbf{T}]_j &:= \textstyle\sum_{i=1}^3 \partial_i T_{ij}; \quad j = 1,2,3, \text{ if } \mathbf{T} \text{ is a matrix (tensor)}\\
&\quad\ \text{with Euclidean components } T_{ij}.\\
[\mathbf{v} \otimes \mathbf{v}]_{ij} &:= v_i v_j; \quad i,j = 1,2,3, \quad \text{if } \mathbf{v} \text{ is a vector.}\\
D_t &:= \partial_t + \mathbf{v} \cdot \nabla; D_t \text{ is the material time derivative.}\\
\mathbf{v} \wedge \mathbf{u} &:= \text{denotes the usual vector product of the vectors } \mathbf{v} \text{ and } \mathbf{u}\\
\nabla \wedge \mathbf{v} &:= \text{curl } \mathbf{v}.
\end{aligned}
$$

If $\mathbf{A}$ and $\mathbf{B}$ are second order tensors we shall use the notation

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij}$$

$$|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}.$$

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a smooth (at least $\mathbf{C}^2$) boundary Γ. Let $\mathbf{n} = \mathbf{n}(x)$ denote the unit exterior normal to Γ at x. We shall be concerned with smooth vector fields $\mathbf{v} = \mathbf{v}(x)$ defined on Ω such that on Γ it has the form

$$\gamma_o \mathbf{v}(x) = -\eta(x)\mathbf{n}(x),$$

where γ_o is the trace operator denoting boundary values and η is a smooth scalar field defined on Γ.

Associated with $\nabla \mathbf{v}$ we define the symmetric and skew-symmetric tensors $\mathbf{A}$ and $\mathbf{W}$ as

$$\mathbf{A} = \mathbf{A}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^T$$

and

$$\mathbf{W} = \mathbf{W}(\mathbf{v}) = \nabla \mathbf{v} - (\nabla \mathbf{v})^T,$$

where $(\nabla \mathbf{v})^T$ denotes the transpose of the gradient of $\mathbf{v}$. The rate of deformation tensor is related to $\mathbf{A}$ by $\mathbf{D}(\mathbf{v}) = \frac{1}{2}\mathbf{A}(\mathbf{v})$.
We remark that if $\mathbf{v}$ is solenoidal $(\nabla \cdot \mathbf{v} = 0)$ then

$$\text{trace } \mathbf{A}(\mathbf{v}) = 2\nabla \cdot \mathbf{v} = 0$$

and for any vector $\mathbf{a}$,

$$\mathbf{W}(\mathbf{v})\mathbf{a} = \boldsymbol{\omega} \wedge \mathbf{a}$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ denotes the vorticity associated with $\mathbf{v}$.

2.2 The Constitutive Equation for Fluids of Second Grade.

Fluids of *differential type* $[S_4, S_5, S_6]$, of which Rivlin-Ericksen fluids are a subclass, is a popular nonlinear model. Fluids of *complexity* n forms an important subclass of the fluids of differential type. For incompressible fluids of complexity n the Cauchy stress tensor is of the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{F}(\mathbf{A}_1, ..., \mathbf{A}_n).$$

The pressure p is not a thermodynamic variable and the term $-p\mathbf{I}$ reflects Pascal's law which is inherent to all fluids. $\mathbf{A}_1, ..., \mathbf{A}_n$ are the first n Rivlin-Ericksen tensors $[R_3]$ defined recursively by

$$\begin{aligned} \mathbf{A}_1 &= \nabla \mathbf{v} + (\nabla \mathbf{v})^T = \mathbf{A} \\ \mathbf{A}_n &= D_t \mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T \mathbf{A}_{n-1}, \quad n \geq 2. \end{aligned}$$

Fluids of *grade* n are examples of fluids of complexity n. The stress tensors for fluids of grades 1 and 2 respectively, are assumed to be of the form:

$$\begin{aligned} \mathbf{T}^{[1]} &= -p\mathbf{I} + \mu \mathbf{A}_1, \\ \mathbf{T}^{[2]} &= \mathbf{T}^{[1]} + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \end{aligned}$$

where μ and α_i are material coefficients (possibly temperature dependent).

For incompressible fluids of second grade, the stress-deformation relation then becomes

$$\mathbf{T} = \mathbf{T}^{[2]} = -p\mathbf{I} + \mu\mathbf{A} + \alpha_1 D_t\mathbf{A} + \alpha_1(\mathbf{A}\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{A}) + \alpha_2\mathbf{A}^2 \qquad (2.1)$$

where p and $\mathbf{v}$ are the pressure and the velocity fields. Here μ is the coefficient of viscosity and α_1 and α_2 are material coefficients or 'normal stress moduli'. In this case $\mathbf{A} = \mathbf{A}_1$.

To use the relations (2.1) for the modelling of a fluid, the fluid has to be compatible with thermodynamics in the sense that all flows of the fluid satisfy the Clausius - Duhem inequality, and the assumption that the specific Helmholtz free energy is a minimum when the fluid is in equilibrium. Under these assumptions, α_1 and α_2 [D_2] must satisfy

$$\alpha_1 + \alpha_2 = 0. \qquad (2.2)$$

Considerations on the stability of the rest state require that μ and α_1 be nonnegative. In what follows we assume they are positive constants, i.e.

$$\mu > 0, \quad \alpha_1 > 0.$$

See [D_3].

Under the assumption (2.2), which we shall follow throughout, the form of the stress tensor $\mathbf{T}$ given in (2.1) reduces to a more compact expression. To obtain this we note that

$$\nabla\mathbf{v} = \frac{1}{2}(\mathbf{A} + \mathbf{W})$$

and

$$(\nabla\mathbf{v})^T = \frac{1}{2}(\mathbf{A} - \mathbf{W}),$$

so that

$$\begin{aligned}\alpha_1(\mathbf{A}\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{A}) &= \frac{\alpha_1}{2}[\mathbf{A}(\mathbf{A} + \mathbf{W}) + (\mathbf{A} - \mathbf{W})\mathbf{A}] \\ &= \alpha_1\mathbf{A}^2 + \frac{\alpha_1}{2}(\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}). \qquad (2.3)\end{aligned}$$

Therefore by (2.1) and (2.3)

$$\begin{aligned}\mathbf{T} &= -p\mathbf{I} + \mu\mathbf{A} + \alpha_1 D_t\mathbf{A} + \frac{\alpha_1}{2}(\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}) + (\alpha_1 + \alpha_2)\mathbf{A}^2 \\ &= -p\mathbf{I} + \mu\mathbf{A} + \alpha D_t\mathbf{A} + \frac{\alpha}{2}(\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}), \qquad (2.4)\end{aligned}$$

where we have set $\alpha_1 = \alpha$.

2.3 Permeability.

The flow of incompressible fluids of second grade through permeable boundaries and the flow of these fluids past porous walls have been studied under various additional conditions.

A similarity solution is obtained for the two-dimensional creeping flow of a second-order fluid with non-parallel porous walls by Bourgin and Tichy [B_4]. An additional velocity boundary condition was needed. The other conditions they used were due to the usual no-slip conditions. This additional velocity boundary condition was to prescribe the rate of shear at the wall. The problem was then solved numerically by a standard routine.

K. R. Rajagopal and P. N. Kaloni wrote remarks on boundary conditions for flows of fluids of the differential type in 1989 [R_2]. [R_5] discusses a lot of related issues.

K. R. Rajagopal and A. S. Gupta [R_1], studied the flow of an incompressible fluid of second grade past an infinite porous plate subject to either suction or blowing at the plate. They studied fluids modelled by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A} + \alpha_1[D_t\mathbf{A} + \mathbf{A}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}] + \alpha_2\mathbf{A}^2. \tag{2.5}$$

No assumptions on the material moduli α_1 and α_2 were made. For the boundary value problem they considered, it was found that the velocity distributions do not depend on the normal stress modulus α_2 but the pressure does. They found that it is possible to exhibit an exact solution which is asymptotic in nature for both 'suction' and 'blowing' at the plate if the material modulus $\alpha_1 > 0$.

If $\alpha_1 < 0$, they found that such solutions cannot exist in the case of blowing, a result in keeping with the classical incompressible fluid. Fosdick and Rajagopal [F_1] have shown that the model (2.5) whose material modulus $\alpha_1 < 0$ exhibits anomalous behaviour not to be expected of any fluid of rheological interest, (see also [F_2]). Proudman studied an example of steady laminar flow at a large Reynolds number [P_1].

Beavers and Joseph [B_1], studied the flow of a Newtonian fluid over a porous surface in 1967. They determined that if the governing differential system is not to be underdetermined, it was necessary to specify some condition on the tangential component of the velocity of the free fluid at the porous interface. It is usual in these analysis to approximate the fluid motion near the true boundary by an adherence condition for the tangential component of velocity of the free fluid at some boundary.

Because of a certain ambiguity which is implied by the notion of a 'true' boundary for a permeable material, it was found useful to define a nominal boundary. They fixed a nominal boundary by first defining a smooth geometric surface and then assuming that the outermost perimeters of all the surface pores of the permeable material are in this surface. Thus if the surface pores were filled with solid material to the level of their respective perimeters a smooth impermeable boundary of the assumed shape would result. This definition is precise when the geometry is simple (planes, spheres, cylinders, etc.) but may not be fully adequate in more complex situations. Their experiment was designed to examine the tangential flow in the boundary region of a permeable interface. The results of this experiment indicate that the effects of viscous shear appear to penetrate into the permeable material in a boundary layer region, producing a velocity distribution similar to that depicted in the following figure. The tangential component of the velocity of the free fluid at the porous boundary can be considerably greater than the mean filter velocity within the body of the porous material.

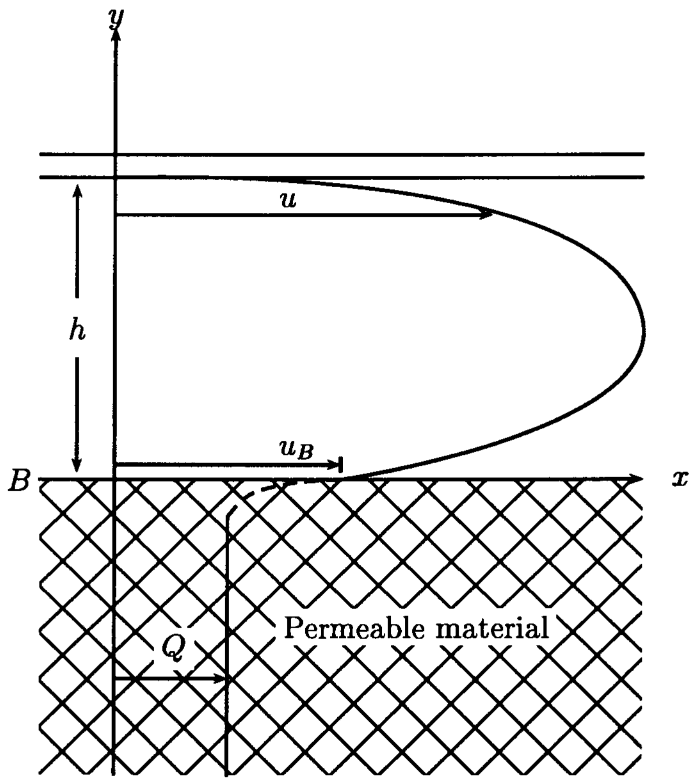

FIGURE 2. Velocity profile for the rectilinear flow in a horizontal channel formed by a permeable lower wall($y = 0$) and an impermeable upper wall ($y = h$).

In FIGURE 2 the plane $y = 0$ defines a nominal surface for the permeable material. The flow through the body of the permeable material, which is homogeneous and isotropic, is assumed to be governed by Darcy's Law. Read more on the status on Darcy's Law in $[R_6]$. In the absence of body forces Darcy's Law may be written as

$$Q = -\frac{k}{\mu}\frac{dP}{dx},$$

where k is the 'permeability' of the material and Q is a volume flow rate per unit cross-sectional area. As such, Q represents the filter velocity rather than the true velocity of the fluid in the pores. The measured pressure gradient is denoted by dP/dx.

2.3.1 Modelling of Permeability for *Problem* $\mathcal{P}_1$.

We study the motion of a fluid of second-grade around and through a fixed porous container filled with the same fluid. The interior of the porous container is a open bounded set $\Omega' \subset \mathbf{R}^3$ and the porous boundary, Γ, is smooth. The surrounding fluid domain, Ω is bounded and its outer boundary is denoted by Γ_1. The exterior normal to Ω on Γ is denoted by $\mathbf{n}$. (see Figure 1, p4).

Permeability of the walls of the container is described by assuming that at the boundary Γ the flow $\mathbf{v}$ has the direction of the normal:

$$\gamma_o \mathbf{v}(x,t) = -\eta(x,t)\mathbf{n}(x). \tag{2.6}$$

The velocity component η is treated as an unknown and an evolution equation for it has to be found.

We model the surface Γ as having an *effective area measure da* which has a density function $\zeta(x)$ with respect to the area measure ds.
Thus

$$da = \zeta(x)ds.$$

The effective area through which fluid can permeate is not more than the surface area and therefore

$$0 \leq \zeta(x) \leq 1 \text{ for any } x \in \Gamma.$$

If $\zeta(x) \equiv 0$, the wall is impermeable and if $\zeta(x) \equiv 1$, there is no wall.

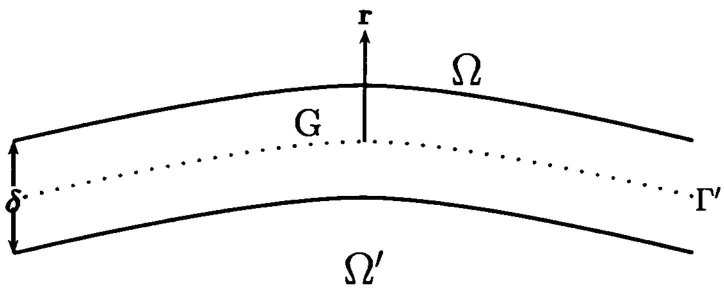

FIGURE 3. Heuristics of the permeable boundary.

In order to obtain expressions for mass and momentum in a boundary patch Γ', we heuristically represent the patch by a volume G built from copies of Γ' (Figure 3). This is in line with the Beavers-Joseph thinking which was discussed before. For this volume we set up a coordinate system consisting of a 'radial part' r which has the direction of the normal vector $\mathbf{n}$ and a 'surface part' made up by vectors tangential to Γ'. For the mass of G we obtain

$$\int_G \rho dx = \int_{\Gamma'} \int_o^\delta \rho dr da = \int_{\Gamma'} \int_o^\delta \rho dr \zeta ds = \int_{\Gamma'} \rho \zeta \delta ds$$

where δ is some measure of thickness. With the aid of these concepts we introduce the *surface density* of the fluid at $x \in \Gamma$ as

$$\sigma(x) = \delta(x)\zeta(x)\rho$$

where ρ is the volume density of the fluid.

To obtain the equation of motion for fluid in the boundary we assume that the rate of change of linear momentum in the boundary is explained by stress forces at both sides of the boundary. The particular form of these tensors at the boundary will be dealt with later.

Let $\mathbf{T}$ and $\mathbf{T}'$ denote the stress tensors at the sides of the boundary facing Ω and Ω' respectively, and let $\mathbf{P}$ and $\mathbf{P}'$ denote the transfer of momentum tensors on the two sides. On an arbitrary boundary patch $\Gamma' \subset \Gamma$ the law of conservation of linear momentum is stated in the following way:

$$\partial_t \int_{\Gamma'} \sigma(x)\gamma_o \mathbf{v} ds = \int_{\Gamma'} [\mathbf{P}\mathbf{n} - \mathbf{P}'\mathbf{n}] da + \int_{\Gamma'} [-\mathbf{T}'(-\mathbf{n}) - \mathbf{T}\mathbf{n}] ds$$

and it follows that

$$\sigma(x)\partial_t \gamma_o \mathbf{v} - \zeta[\mathbf{P} - \mathbf{P}']\mathbf{n} = [\mathbf{T}' - \mathbf{T}]\,\mathbf{n}. \tag{2.7}$$

From (2.7) we have

$$\sigma(x)\partial_t\eta(x,t) + \zeta\mathbf{n}\cdot[\mathbf{P}-\mathbf{P}']\mathbf{n} = \mathbf{n}\cdot\mathbf{T}\mathbf{n} - \mathbf{n}\cdot\mathbf{T}'\mathbf{n}.$$

In the domain Ω the momentum flux tensor is given by $\mathbf{P} = \rho\mathbf{v}\otimes\mathbf{v}$. In accordance with this we shall take $\mathbf{P} = \zeta\rho\eta^2\mathbf{n}\otimes\mathbf{n}$ at the boundary. The tensor $\mathbf{P}'$ will be taken as zero.

We take $\mathbf{T}' = \ell\mathbf{I}$ to obtain from (2.7)

$$\sigma(x)\partial_t\eta + \zeta\rho\eta^2 = \mathbf{n}\cdot\mathbf{T}\mathbf{n} - \ell(t). \tag{2.8}$$

From the incompressibility of the flow in Ω it follows that

$$-\int_\Gamma \eta ds = 0. \tag{2.9}$$

2.4 Explicit form of the Dynamical Boundary Condition for *Problem* $\mathcal{P}_1$.

In Appendix I it is shown that for a smooth two-dimensional manifold Γ contained in a domain $\Omega \subset \mathbf{R}^3$ the following is true for a vector field $\mathbf{v}$ which is of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ:

$$\gamma_o [\mathbf{A}(\mathbf{v})] = -2\eta\mathbf{M} + \mathbf{N}, \tag{2.10}$$

where

$$\mathbf{M} = K\mathbf{n} \otimes \mathbf{n} - (\kappa_1 \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 - \kappa_2 \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2), \tag{2.11}$$

and

$$\mathbf{N} = \mathbf{n} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \mathbf{n} - 2\theta\mathbf{n} \otimes \mathbf{n} \tag{2.12}$$

In these expressions the symbols have the following meaning:

κ_1, κ_2 are curvatures associated with orthogonal normal curves in Γ
$K = \kappa_1 + \kappa_2$, the mean curvature
$\theta = \nabla \cdot \mathbf{v}$
$\boldsymbol{\psi} = \boldsymbol{\omega} \wedge \mathbf{n} - 2\nabla_s \eta$
$\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$
∇_s is the surface gradient
$\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are orthogonal unit tangents to Γ.

If $\mathbf{v}$ is solenoidal, which is the case under consideration, $\theta = 0$. A straightforward application of Stokes' Theorem shows that $\boldsymbol{\omega}$ is tangential at the boundary, which implies that $\boldsymbol{\psi}$ is tangential at the boundary. Indeed, let Γ' be any patch of the surface Γ, then

$$\int_{\Gamma'} (\nabla \wedge \mathbf{f}) \cdot \mathbf{n}\, ds = \int_{\partial\Gamma'} \mathbf{f} \cdot d\boldsymbol{\tau}$$

where $d\boldsymbol{\tau}$ is a vector tangential to the boundary. Now if $\mathbf{f} = \gamma_o\mathbf{v} = -\eta\mathbf{n}$ then $\int_{\partial\Gamma'} \mathbf{f} \cdot d\boldsymbol{\tau} = 0$, and that implies that $\int_{\Gamma'} (\nabla \wedge \mathbf{v}) \cdot \mathbf{n}\, ds = 0$ for all $\Gamma' \subset \Gamma$, which in turn implies that $(\nabla \wedge \mathbf{v}) \cdot \mathbf{n} = 0$.

In the problem under consideration we shall assume that the "rate of deformation" tensor $\mathbf{A}$ has precisely the form (2.10) on the boundary Γ with $\mathbf{n}$ the unit exterior normal. (The traditional rate of deformation is defined as $\mathbf{D} = \frac{1}{2}\mathbf{A}$).

In a local coordinate system defined by $\mathbf{n}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, $\mathbf{M}$ and $\mathbf{N}$ have the following representations:

$$\mathbf{M} = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & K \end{pmatrix}, \tag{2.13}$$

$$\mathbf{N} = \begin{pmatrix} 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_1 \\ 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_2 \\ \boldsymbol{\psi}\cdot\boldsymbol{\tau}_1 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_2 & 0 \end{pmatrix}. \tag{2.14}$$

We shall consider a kinematical boundary condition, which has a physical meaning in that there are no tangential components of deformation at the interface boundary. This concerns the form of the tensor $\mathbf{N}$.

Towards this, we observe from (2.11), that

$$\gamma_0[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2K\eta\mathbf{n} + \boldsymbol{\psi}. \tag{2.15}$$

It follows from (2.15) that there are no tangential components of deformation at Γ if and only if $\boldsymbol{\psi} = 0$, i.e.

$$\boldsymbol{\omega} \wedge \mathbf{n} = 2\nabla_s\eta. \tag{2.16}$$

This is the kinematical boundary condition.

The various terms in $\mathbf{n}\cdot\mathbf{Tn}$ with $\mathbf{T}$ given by (2.4), on a surface Γ may be expressed as follows (see Appendix I, Lemma 7.2):

$$\begin{aligned} \mathbf{n}\cdot\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} &= -2K\eta && (2.17)\\ \mathbf{n}\cdot\gamma_o\partial_t[\mathbf{A}(\mathbf{v})\mathbf{n}] &= \partial_t[\mathbf{n}\cdot\mathbf{A}(\mathbf{v})\mathbf{n}] = -2K\eta_t && (2.18)\\ \gamma_o[\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}(\mathbf{v})]\mathbf{n}] &= +4K_G\eta^2 + \eta\Delta_s\eta && (2.19)\\ \mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\,\mathbf{n} &= 0 && (2.20) \end{aligned}$$

Guided by these expressions and (2.4), we assume that at Γ,

$$\mathbf{n}\cdot\mathbf{Tn} = -\left[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta\right]. \tag{2.21}$$

For the stress tensor $\mathbf{T}'$ in the fluid exterior to Ω we assume that $\mathbf{n}\cdot\mathbf{T}'\mathbf{n} = \ell(t)$. This amounts to the situation where the fluid in Ω' is at rest. As a result we have

$$\begin{aligned} \mathbf{n}\cdot(\delta\mathbf{T})\mathbf{n} &= \mathbf{n}\cdot[\mathbf{T}-\mathbf{T}']\,\mathbf{n} \\ &= -[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta] - \ell(t). \end{aligned} \tag{2.22}$$

From (2.7), (2.8) and (2.22) we obtain (see Appendix III):

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta) \tag{2.23}$$

with

$$s(\eta) = \sigma^{-1/2}[(-k + 4\alpha K_G)\eta^2 - 2\mu\eta K + \alpha\eta\Delta_s\eta - \ell(t)],$$

and $k = \zeta\rho$.

We shall assume throughout that the surface density is bounded and bounded away from zero: i.e. there exist constants s and S such that

$$0 < s \leq \sigma(x) \leq S \quad \text{for all} \quad x \in \Gamma. \tag{2.24}$$

Also, we assume $\sigma \in C^\infty(\Gamma)$.

Chapter 3

Preliminaries for Model $\mathcal{P}_1$.

In this section norms, bilinear forms, function spaces and additional assumptions are defined and explained. We derive some inequalities and identities which will be used in the following chapters.

Apart from the smoothness of Γ we make two additional assumptions regarding the shape of Ω', namely that the curvatures κ_1, κ_2 and K are constrained in the following way:

1. There exist constants g and G such that

$$0 < g \leq K(x) \leq G \text{ for all } x \in \Gamma. \tag{3.1}$$

2. There exists a constant H such that

$$0 \leq \kappa_1^2 + \kappa_2^2 \leq H^2 \text{ on } \Gamma. \tag{3.2}$$

Note that these assumptions allow cases where κ_1 and κ_2 can be of opposite sign.

3.1 Definitions.

All spaces of vector fields are denoted by boldface letters.

1. Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary Γ (of class C^∞), $\Omega_T = \Omega \times (0,T)$ and $\Gamma_T = \Gamma \times (0,T)$.

2. $H^{m,q}(\Omega)$, for m a nonnegative integer and $1 < q < \infty$, is the usual Sobolev space (of real valued functions) embedded in $L^q(\Omega)$ with norm $\|\cdot\|_{m,q}$. $H^m(\Omega)$, m a non-negative integer, denotes the Sobolev space $H^{m,2}(\Omega)$ of order m.

   By our agreement above $\mathbf{H}^m(\Omega)$ is the Sobolev space of three vector fields and the components elements of $H^m(\Omega)$. In particular, the norm and scalar product in $\mathbf{H}^1(\Omega)$ are defined by

$$\|\mathbf{u}\|_1^2 = \|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2$$

   and

$$(\mathbf{u},\mathbf{v})_1 = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}dx.$$

3. With the above notation $\mathbf{H}^o(\Omega)$ denotes the Hilbert space $\mathbf{L}^2(\Omega)$ of vector functions $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))$, with $x \in \Omega$, for which $|\mathbf{u}|^2$ is integrable on Ω. The norm and scalar product for $\mathbf{u},\mathbf{v} \in \mathbf{L}^2(\Omega)$ are defined as

$$\|\mathbf{u}\|^2 = \int_\Omega |\mathbf{u}|^2 dx$$

   and

$$(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx.$$

4. There exists a linear continuous operator $\gamma_o \in \mathcal{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\partial\Omega))$, called the *trace operator*, such that $\gamma_o\mathbf{u} =$ the 'restriction' of $\mathbf{u}$ to $\partial\Omega$ for every function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ which is continuous in $\overline{\Omega}$. The space $\mathbf{H}_o^1(\Omega)$ is the kernel of γ_o. The image space $\gamma_o(\mathbf{H}^1(\Omega))$ is a dense subspace of $\mathbf{L}^2(\Gamma)$ denoted by $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. The trace operator is bounded, indeed, there exists a constant $C_1 > 0$ such that

$$\|\gamma_o\mathbf{u}\|_\Gamma \le C_1\|\mathbf{u}\|_1 \quad \text{for all} \quad \mathbf{u} \in \mathbf{H}^1(\Omega). \tag{3.3}$$

   [L_3, Theorem 9.4, p.41]. We shall refer to this result (3.3) as the *Trace theorem*. Since $\partial\Omega = \Gamma \cup \Gamma_1$ and $\Gamma \cap \Gamma_1 = \emptyset$, we may write

$$\mathbf{L}^2(\partial\Omega) = \mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma_1).$$

   We shall assume throughout that $\gamma_o\mathbf{u} = 0$ on Γ_1, and shall usually refer to γ_o as if it maps to $\mathbf{L}^2(\Gamma)$ only.

5. We shall use the following notation in connection with $\mathbf{A}$:

$$(\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v})) = \int_\Omega \mathbf{A}(\mathbf{u}) : \mathbf{A}(\mathbf{v}) dx$$

$$\|\mathbf{A}\|^2 = \int_\Omega |\mathbf{A}|^2 dx.$$

6. We define the domain $\mathcal{D}$ by:

$$\mathcal{D} = \{\mathbf{v} \in \mathbf{H}^2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \gamma_o \mathbf{v} = \langle -\eta \mathbf{n}, 0\rangle \in \mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma_1),$$
$$\text{and } \mathbf{A}(\mathbf{v}) = -2\eta \mathbf{M} \text{ on } \Gamma\}. \tag{3.4}$$

   $\mathcal{D}$ is a closed subspace of $\mathbf{H}^2(\Omega)$. Note that because of (2.10) the tensor $\mathrm{N} = 0$ if $\mathbf{v} \in \mathcal{D}$. Thus elements of $\mathcal{D}$ satisfy the kinematical boundary conditions (2.16).

7. $\mathbf{H}^1_\varsigma(\Omega)$ denotes the closure of $\mathcal{D}$ in $\mathbf{H}^1(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_1$.

8. The norm of $\gamma_o \mathbf{v} \in \mathbf{L}^2(\Gamma)$ on the boundary Γ is chosen as

$$\|\gamma_o \mathbf{v}\|_\Gamma^2 = \|\eta\|_\Gamma^2 = \int_\Gamma \sigma(x) |\gamma_o \mathbf{v}|^2 ds.$$

   The associated scalar product is

$$(\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_\Gamma = \int_\Gamma \sigma(x) \gamma_o \mathbf{u} \cdot \gamma_o \mathbf{v} ds.$$

   According to assumption (2.24) this is equivalent to the usual $\mathbf{L}^2$ metric. It is assumed that the function $\sigma \in C^\infty(\Gamma)$.

9. We shall deal extensively with the *energy* associated with fluids of second grade defined for the purpose of this study by

$$E_v = \frac{\alpha}{2} \|\mathbf{A}(\mathbf{v})\|^2 + \rho \|\mathbf{v}\|^2 + \|\gamma_o \mathbf{v}\|_\Gamma^2. \tag{3.5}$$

   $E_v^{1/2}$ is evidently a norm on $\mathbf{H}^1_\varsigma(\Omega)$ (see also Lemma 3.6). We shall refer to the quantity $E_v^{1/2}$ as the energy norm of $\mathbf{v}$.

10. The constant C which appears in inequalities denotes a generic positive constant. This means that C may take different values even in the same calculation. Sometimes it is necessary to indicate the quantities on which a constant depends in brackets or by a subscript.

3.2 Important Identities

Identity 3.2.1 *For any symmetric tensor* $\mathbf{A}$ *and any arbitrary tensor* $\mathbf{B}$*, we have*

$$\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}_s,$$

with $\mathbf{B}_s = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$.

Proof.

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^T : \mathbf{B}^T = \mathbf{A} : \mathbf{B}^T,$$

thus

$$\mathbf{A} : \mathbf{B} = \frac{1}{2}\mathbf{A}(\mathbf{B} + \mathbf{B}^T) = \mathbf{A} : \mathbf{B}_s.$$

†

Expressions for inner products of the form $(D_t\mathbf{F}, \mathbf{F})$ where $\mathbf{F}$ is either a vector or a second order tensor are necessary. $D_t = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative. In order to obtain simple expressions for the scalar product we notice that if $\circ$ denotes either the usual scalar product or the "colon" product, then

$$[\partial_t\mathbf{F} + (\mathbf{v} \cdot \nabla)\mathbf{F}] \circ \mathbf{F} = \frac{1}{2}\partial_t|\mathbf{F}|^2 + \frac{1}{2}\nabla \cdot (|\mathbf{F}|^2\mathbf{v}).$$

provided $\nabla \cdot \mathbf{v} = 0$. Hence the following Identity:

Identity 3.2.2 *For any smooth vector or tensor quantity* $\mathbf{F}(x,t)$ *and any* $\mathbf{v} \in \mathcal{D}$ *we have*

$$(D_t\mathbf{F}, \mathbf{F}) = \frac{1}{2}\partial_t||\mathbf{F}||^2 - \frac{1}{2}\int_\Gamma |\mathbf{F}|^2\eta ds. \tag{3.6}$$

Proof.
By the divergence theorem

$$\begin{aligned}
(\partial_t\mathbf{F} + (\mathbf{v} \cdot \nabla)\mathbf{F}, \mathbf{F}) &= \frac{1}{2}\partial_t\int_\Omega |\mathbf{F}|^2 dx + \frac{1}{2}\int_\Omega \nabla \cdot (|\mathbf{F}|^2\mathbf{v})dx \\
&= \frac{1}{2}\partial_t||\mathbf{F}||^2 + \frac{1}{2}\int_\Gamma |\mathbf{F}|^2\mathbf{v} \cdot \mathbf{n}ds \\
&= \frac{1}{2}\partial_t||\mathbf{F}||^2 - \frac{1}{2}\int_\Gamma |\mathbf{F}|^2\eta ds.
\end{aligned}$$

†

Later in this study we shall employ the energy method. It will become necessary to use the various boundary conditions in order to prove stability. The following is important to obtain the required results:

Identity 3.2.3 *If $f \in H^1(\Omega)$ is a scalar field, and $\mathbf{v} \in \mathcal{D}$, then*

$$\int_\Omega (\mathbf{v}\cdot\nabla) f dx = -\int_\Gamma \eta f ds.$$

Proof.
Integrate by parts and use the fact that $\mathbf{v}$ is solenoidal:

$$\begin{aligned}\int_\Omega (\mathbf{v}\cdot\nabla) f dx &= \int_\Gamma f\mathbf{v}\cdot\mathbf{n}ds - \int_\Omega f\nabla\cdot\mathbf{v}dx \\ &= -\int_\Gamma \eta f ds.\end{aligned}$$

†

We note that in particular for $\mathbf{v} \in \mathcal{D}$, the imbedding of $\mathbf{H}^2(\Omega)$ in the space of bounded continuous functions makes the choice $f = |\mathbf{v}|^2$ possible, and it follows from Identity 3.2.3 that

$$\int_\Omega (\mathbf{v}\cdot\nabla)|\mathbf{v}|^2 ds = -\int_\Gamma |\eta|^3 ds. \tag{3.7}$$

For $\mathbf{v} \in \mathcal{D}$ we may also choose $f = |\mathbf{A}(\mathbf{v})|^2$, and it follows that

$$\begin{aligned}\int_\Omega (\mathbf{v}\cdot\nabla)|\mathbf{A}(\mathbf{v})|^2 dx &= -\int_\Gamma |\mathbf{A}(\mathbf{v})|^2\eta \, ds \\ &= -\int_\Gamma 4\eta^3|\mathrm{M}|^2 \, ds \end{aligned} \tag{3.8}$$

sinceN $= 0$ on $\mathcal{D}$.

The following will be of immediate importance.

Identity 3.2.4 *For any $\mathbf{v} \in \mathcal{D}$,*

$$||\mathbf{A}(\mathbf{v})||^2 = 2||\nabla\mathbf{v}||^2 + 2\int_\Gamma K(x)\eta^2 ds. \tag{3.9}$$

Proof.
From the definition of $\mathbf{A}$ it is evident that $|\mathbf{A}(\mathbf{v})|^2 = 2|\nabla\mathbf{v}|^2 + 2\nabla\mathbf{v} : \nabla^T\mathbf{v}$. Now $\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}] = \nabla\mathbf{v} : \nabla^T\mathbf{v} + (\mathbf{v}\cdot\nabla)(\nabla\cdot\mathbf{v})$, and, since $\nabla\cdot\mathbf{v} = 0$, $\nabla\mathbf{v} : \nabla^T\mathbf{v} = \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]$, whereupon integration over Ω and Identity 3.2.1 yields

$$||\mathbf{A}(\mathbf{v})||^2 = 2||\nabla\mathbf{v}||^2 + 2\int_\Omega \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]dx$$

$$\begin{aligned}
&= 2||\nabla \mathbf{v}||^2 + 2\int_\Gamma \mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]ds \\
&= 2||\nabla \mathbf{v}||^2 - 2\int_\Gamma \eta\mathbf{n}\cdot[\nabla\mathbf{v}]\mathbf{n}ds \\
&= 2||\nabla \mathbf{v}||^2 - 2\int_\Gamma \eta\mathbf{n}\otimes\mathbf{n} : [\nabla\mathbf{v}]ds \\
&= 2||\nabla \mathbf{v}||^2 - \int_\Gamma \eta\mathbf{n}\cdot\mathbf{A}(\mathbf{v})\mathbf{n}ds \\
&= 2||\nabla \mathbf{v}||^2 + 2\int_\Gamma \eta^2\mathbf{n}\cdot\mathbf{M}\mathbf{n}ds \\
&= 2||\nabla \mathbf{v}||^2 + 2\int_\Gamma K(x)\eta^2 ds.
\end{aligned}$$

†

Thus, if the curvature K is positive everywhere on Γ, it becomes apparent that $\mathbf{A}(\mathbf{v}) = \mathbf{0}$ if and only $\mathbf{v} = 0$.

Identity 3.2.5 *For any bilinear form b on a Hilbert space H we have for any $\mathbf{v}, \mathbf{w} \in H$ and with $\mathbf{u} = \mathbf{v} - \mathbf{w}$ that $b(\mathbf{v}, \mathbf{v}) - b(\mathbf{w}, \mathbf{w}) = b(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u})$.*

Identity 3.2.6 *Let $\mathbf{f}$ and $\mathbf{g}$ be tensor fields of the same order and let $\circ$ denote the 'scalar product' in such tensor fields. For $\mathbf{v} \in \mathcal{D}$ it is true that*

$$\int_\Omega [\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}]dx = -\int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}ds.$$

Proof.

$$\int_\Omega \mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g}dx = \int_\Gamma (\mathbf{v}\cdot\mathbf{n})\mathbf{f}\circ\mathbf{g}ds - \int_\Omega \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}dx.$$

Thus

$$\int_\Omega [\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}]dx = \int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}ds.$$

†

3.3 Inequalities

Lemma 3.1 *Under the assumptions* (3.1) *and* (2.24)*, for $\mathbf{v} \in \mathcal{D}$, $\mathbf{v} = 0$ if and only if $\mathbf{A}(\mathbf{v}) = 0$.*

Proof.
From (2.24), (3.9) and (3.1) we have

$$\frac{g}{S}||\eta||_\Gamma^2 + ||\nabla \mathbf{v}||^2 \leq \frac{1}{2}||\mathbf{A}(\mathbf{v})||^2 \leq ||\nabla \mathbf{v}||^2 + \frac{G}{s}||\eta||_\Gamma^2, \tag{3.10}$$

and the result follows. †

The following two lemmas are important in establishing a Poincaré inequality.

Lemma 3.2 *The bilinear forms* $a(\mathbf{u}, \mathbf{v}) = (\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v}))$ *and* $b(\mathbf{u}, \mathbf{v}) = \rho(\mathbf{u}, \mathbf{v}) + (\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_\Gamma$ *are bounded in the space* $\mathbf{H}^1_\varsigma(\Omega)$.

Proof.
For $\mathbf{u}$ and $\mathbf{v} \in \mathbf{H}^1_\varsigma(\Omega)$ and by (2.24), (3.1) and the Schwarz inequality

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &= |(\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v}))| \\ &= |2(\nabla \mathbf{u}, \nabla \mathbf{v}) + 2\int_\Gamma K(x)\eta_u \eta_v ds| \\ &\leq 2||\mathbf{u}||_1||\mathbf{v}||_1 + \frac{2G}{s^2}||\gamma_o \mathbf{u}||_\Gamma ||\gamma_o \mathbf{v}||_\Gamma. \end{aligned}$$

Hence, by the Trace theorem

$$|a(\mathbf{u}, \mathbf{v})| \leq C||\mathbf{u}||_1||\mathbf{v}||_1.$$

Furthermore

$$\begin{aligned} |b(\mathbf{u}, \mathbf{v})| &= |\rho(\mathbf{u}, \mathbf{v}) + (\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_\Gamma| \\ &\leq \rho||\mathbf{u}||_1||\mathbf{v}||_1 + ||\gamma_o \mathbf{u}||_\Gamma ||\gamma_o \mathbf{v}||_\Gamma \\ &\leq C||\mathbf{u}||_1||\mathbf{v}||_1, \end{aligned}$$

by the Trace theorem. †

Lemma 3.3 *The bilinear form* $|a(\mathbf{u}, \mathbf{v})| = (\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v}))$ *is coercive in the sense that there exist constants* $c_1 > 0$ *and* $c_o \geq 0$ *such that*

$$|a(\mathbf{u}, \mathbf{u})| \geq c_1||\mathbf{u}||_1^2 - c_o b(\mathbf{u}, \mathbf{u}).$$

Proof.
From (3.10) we have

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) &= (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u})) \\ &\geq 2\|\nabla\mathbf{u}\|^2 + \frac{2g}{S}\|\eta\|_\Gamma^2 \\ &= 2\|\mathbf{u}\|_1^2 - 2\|\mathbf{u}\|^2 + \frac{2g}{S}\|\eta\|_\Gamma^2 \\ &\geq 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}(\rho\|\mathbf{u}\|_J^2 - \frac{2}{\rho}\|\eta\|_\Gamma^2 \\ &= 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}b(\mathbf{u},\mathbf{u}). \end{aligned}$$

†

We may now obtain a generalized Poincaré inequality:

Lemma 3.4 *There exists a smallest possible constant $\beta > 0$ such that for every $\mathbf{v} \in \mathbf{H}^1_\varsigma(\Omega)$,*

$$\frac{\beta}{2}\|\mathbf{A}(\mathbf{v})\|^2 \geq \rho\|\mathbf{v}\|^2 + \|\gamma_o\mathbf{v}\|_\Gamma^2. \tag{3.11}$$

Proof.
From the smoothness of Γ (which is always assumed), it follows that the embedding $J : \mathbf{u} \in \mathbf{H}^1(\Omega) \to \langle\mathbf{u}, \gamma_o\mathbf{u}\rangle \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Gamma)$ is compact $[A_1]$. From the boundedness and coerciveness proved above follow that there exists a smallest eigenvalue λ and associated eigenfunction $\mathbf{u} \in \mathbf{H}^1_\varsigma(\Omega)$ for which $b(\mathbf{u},\mathbf{u}) = 1$:

$$\lambda = \inf\{a(\mathbf{v},\mathbf{v}) : \mathbf{v} \in \mathbf{H}^1_\varsigma(\Omega); b(\mathbf{v},\mathbf{v}) = 1\} = a(\mathbf{u},\mathbf{u}) \tag{3.12}$$

$[S_1]$. $\lambda > 0$, for if it is zero, it follows that $\mathbf{u} = 0$ which cannot be. It follows from (3.12) that for any $\mathbf{v} \in \mathbf{H}^1_\varsigma(\Omega)$ the inequality

$$a(\mathbf{v},\mathbf{v}) \geq \lambda[\rho\|\mathbf{v}\|^2 + \|\gamma_o\mathbf{v}\|_\Gamma^2]$$

holds and that λ is the largest such constant. Finally we set $\beta = 2/\lambda$. †

Remark 3.1 :
It is now easy to see that $\|\mathbf{A}(\cdot)\|$ is a norm on $\mathbf{H}^1_\varsigma(\Omega)$. In fact,

Lemma 3.5 *For all $\mathbf{v} \in \mathbf{H}^1_\varsigma(\Omega)$ we have*

$$\frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|^2 \leq E_v \leq \frac{\alpha+\beta}{2}\|\mathbf{A}(\mathbf{v})\|^2. \tag{3.13}$$

Proof.
Add $\frac{\alpha}{2}||\mathbf{A}(\mathbf{v})||^2$ to both sides of the inequality (3.11):

$$E_v = \frac{\alpha}{2}||\mathbf{A}(\mathbf{v})||^2 + \rho||\mathbf{v}||^2 + ||\eta||_\Gamma^2 \le \frac{\alpha+\beta}{2}||\mathbf{A}(\mathbf{v})||^2.$$

From the definition of the energy norm it is clear that

$$\frac{\alpha}{2}||\mathbf{A}(\mathbf{v})||^2 \le E_v,$$

and the result follows. †

From Lemma 3.4 it is clear that these are the best estimates of this form.

Lemma 3.6 *The norms* $||\mathbf{A}(\mathbf{v})||$ *and* $E_v^{1/2}$ *are equivalent to the norm in the Sobolev space* $\mathbf{H}^1(\Omega)$.

Proof.
From (3.10) and (3.11)

$$||\mathbf{A}(\mathbf{v})||^2 \ge 2||\nabla\mathbf{v}||^2,$$

and

$$||\mathbf{A}(\mathbf{v})||^2 \ge \frac{2\rho}{\beta}||\mathbf{v}||^2.$$

The addition of the two above inequalities yields

$$\begin{aligned} ||\mathbf{A}(\mathbf{v})||^2 &\ge ||\nabla\mathbf{v}||^2 + \frac{\rho}{\beta}||\mathbf{v}||^2 \\ &\ge ||\nabla\mathbf{v}||^2 + \frac{\rho}{\beta}||\mathbf{v}||^2. \end{aligned}$$

Let $k = \min\,(1, \rho/\beta)$ then

$$||\mathbf{A}(\mathbf{v})||^2 \ge k||\mathbf{v}||_1^2.$$

(3.10) yields

$$\begin{aligned} ||\mathbf{A}(\mathbf{v})||^2 &\le 2||\nabla\mathbf{v}||^2 + \frac{2G}{s}||\eta||_\Gamma^2 \\ &\le 2||\nabla\mathbf{v}||^2 + 2||\mathbf{v}||^2 + \frac{2G}{s}||\eta||_\Gamma^2, \end{aligned}$$

and from the Trace theorem

$$||\mathbf{A}(\mathbf{v})||^2 \le C||\mathbf{v}||_1^2$$

From (3.13) it is evident that the energy norm is equivalent to the norm $||\mathbf{A}(\mathbf{v})||$. †

Remark 3.2 *From the above Lemma we may claim from the embedding* $\mathbf{H}^1(\Omega) \to \mathbf{L}^3(\Gamma)$,[$A_1$], *that there exists a constant* $\tau > 0$ *such that*

$$\int_\Gamma |\gamma_o \mathbf{v}|_\Gamma^3 \, ds \leq \tau \|\mathbf{A}(\mathbf{v})\|^3 \quad \text{for every} \quad \mathbf{v} \in \mathbf{H}^1_\sigma(\Omega). \tag{3.14}$$

Chapter 4

Stability and Uniqueness for *Problem* $\mathcal{P}_1$.

4.1 Energy Identities.

The aim of this chapter is to develop tools for the study of stability and uniqueness of the solutions of *Problem* $\mathcal{P}_1$ under suitable initial conditions. For this purpose $\mathbf{v}$ is a solution of *Problem* $\mathcal{P}_1$ if $\mathbf{v} \in \mathbf{H}^3(\Omega) \cap \mathcal{D}$ which satisfies the system of evolution equations

$$\left.\begin{array}{rcl} D_t[\rho \mathbf{v}(x,t)] & = & \nabla \cdot \mathbf{T}(p, \mathbf{v}) \ \text{ in } \ \Omega \times (0,\infty) \\ \sigma \partial_t \eta + k\eta^2 & = & \mathbf{n} \cdot \mathbf{T}\mathbf{n} - \ell(t) \ \text{ on } \ \Gamma \times (0,\infty). \end{array}\right\} \tag{4.1}$$

In this section we shall derive an energy identity for solutions defined in this way. In these identities initial values are of no importance.

As stated before elements of $\mathcal{D}$ satisfy the kinematical boundary condition (2.16).

The energy method (see [M_2]) applied to this system is based on taking the $\mathrm{L}^2(\Omega)$ - scalar product with $\mathbf{v}$ on both sides of the first equation, and using the other equation and the properties of $\mathcal{D}$ in further calculations. This produces

$$(D_t(\rho \mathbf{v}), \mathbf{v}) = \rho(\partial_t \mathbf{v}, \mathbf{v}) + \rho((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v}) = (\nabla \cdot \mathbf{T}, \mathbf{v}). \tag{4.2}$$

From (3.7) with $((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v}) = \frac{1}{2}\int_\Omega (\mathbf{v} \cdot \nabla)|\mathbf{v}|^2 dx$ we have

$$(D_t(\rho \mathbf{v}), \mathbf{v}) \ = \ \frac{1}{2}\rho[\partial_t \|\mathbf{v}\|^2 - \int_\Gamma \eta^3 \, ds]$$

$$\begin{aligned}
&= (\nabla \cdot \mathbf{T}, \mathbf{v}) \\
&= \int_\Omega \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) \, dx \\
&= \int_\Gamma \gamma_o \mathbf{v} \cdot \mathbf{Tn} \, ds - (\mathbf{T}, \nabla \mathbf{v}) \\
&= -\int_\Gamma \eta \mathbf{n} \cdot \mathbf{Tn} \, ds - \frac{1}{2} \int_\Omega \mathbf{T} : \mathbf{A} \, dx \\
&= -\int_\Gamma \eta[\sigma \partial_t \eta + k\eta^2 + \ell(t)] \, ds - \frac{1}{2} \int_\Omega \mathbf{T} : \mathbf{A} \, dx \\
&= -\frac{1}{2} \partial_t \int_\Gamma \sigma \eta^2 \, ds - \int_\Gamma k\eta^3 \, ds - \frac{1}{2} \int_\Omega \mathbf{T} : \mathbf{A} \, dx.
\end{aligned}$$

It is important to bear in mind that $\int_\Gamma \eta \ell(t) ds = \ell(t) \int_\Gamma \eta ds = 0$.

Consider

$$\mathbf{T} : \mathbf{A} = [-p\mathbf{I} + \mu \mathbf{A} + \alpha D_t \mathbf{A} + \frac{1}{2} \alpha (\mathbf{AW} - \mathbf{WA})] : \mathbf{A}.$$

The trace of $\mathbf{A}$ is the divergence of $\mathbf{v}$ and therefore is zero. Hence $-p\mathbf{I} : \mathbf{A} = 0$. Since $\mathbf{A}$ is symmetric and $\mathbf{W}$ is skew-symmetric, we have

$$[\mathbf{AW} - \mathbf{WA}] : \mathbf{A} = -\mathbf{A}^2 : \mathbf{W} + \mathbf{A}^2 : \mathbf{W} = 0.$$

By Identity 3.2.2, the scalar product of the material derivative of $\mathbf{A}$ with $\mathbf{A}$ is:

$$\begin{aligned}
(D_t \mathbf{A}, \mathbf{A}) &= (\partial_t \mathbf{A}, \mathbf{A}) + ((\mathbf{v} \cdot \nabla) \mathbf{A}, \mathbf{A}) \\
&= \frac{1}{2} \partial_t ||\mathbf{A}||^2 - \frac{1}{2} \int_\Gamma |\mathbf{A}|^2 \eta \, ds \\
&= \frac{1}{2} \partial_t ||\mathbf{A}||^2 - 2 \int_\Gamma |\mathbf{M}|^2 \eta^3 ds,
\end{aligned}$$

from (3.8).

The general energy identity is therefore

$$\partial_t [\rho ||\mathbf{v}||^2 + \frac{\alpha}{2} ||\mathbf{A}||^2 + ||\eta||_\Gamma^2]$$

$$= -\mu ||\mathbf{A}||^2 + 2\alpha \int_\Gamma |\mathbf{M}|^2 \eta^3 \, ds + \rho \int_\Gamma \eta^3 \, ds - \int_\Gamma k\eta^3 \, ds. \tag{4.3}$$

Hence, the following calculations lead us to an energy identity for *Problem* $\mathcal{P}_1$:

$$(\mathbf{T}, \mathbf{A}) = \mu ||\mathbf{A}||^2 + \frac{\alpha}{2} \partial_t ||\mathbf{A}||^2 - 2\alpha \int_\Gamma |\mathbf{M}|^2 \eta^3 \, ds,$$

and

$$-\frac{1}{2}(\mathbf{T},\mathbf{A}) = -\frac{1}{2}\mu||\mathbf{A}||^2 - \frac{\alpha}{4}\partial_t||\mathbf{A}||^2 + \alpha\int_\Gamma |\mathbf{M}|^2\eta^3\, ds.$$

We therefore obtain the energy identity for *Problem* $\mathcal{P}_1$:

$$\begin{aligned}\partial_t[\rho||\mathbf{v}||^2 \quad &+ \quad \frac{\alpha}{2}||\mathbf{A}||^2 + ||\eta||^2] \\ &= \quad -\mu||\mathbf{A}||^2 + 2\alpha\int_\Gamma |\mathbf{M}|^2\eta^3\, ds + \rho\int_\Gamma \eta^3\, ds - \int_\Gamma k\eta^3\, ds.\end{aligned}$$

This identity can also be written in the form

$$\begin{aligned}\partial_t E_v(t) = \quad &- \quad \mu||\mathbf{A}||^2 + 2\alpha\int_\Gamma(\kappa_1^2+\kappa_2^2+K^2)\eta^3 ds \\ &+ \quad \rho\int_\Gamma \eta^3\, ds - \int_\Gamma k\eta^3\, ds. \end{aligned} \tag{4.4}$$

4.2 Energy and Stability for *Problem* $\mathcal{P}_1$.

In this section we study the stability of fluids of second grade in the situation described in Chapter 2. It is assumed that $\mathbf{v} \in \mathbf{H}^3(\Omega) \cap \mathcal{D}$.

The method of proof of stability is related to the energy method of Galdi and Padula [G_2]. See also Le Roux and Sauer [L_2] for a related treatment of implicit evolution equations which includes problems with dynamical boundary conditions. Let $\mathbf{v} \in \mathbf{H}^3(\Omega)\cap\mathcal{D}$, then the energy identity (4.4), with (3.1) and (3.2) in mind is used to obtain the following inequality:

$$\partial_t E_v(t) \leq -\mu||\mathbf{A}(\mathbf{v})||^2 + (2\alpha(G^2+H^2)+\rho)\int_\Gamma |\eta|^3\, ds. \tag{4.5}$$

The boundary integrals in (4.5) may be estimated in accordance with (3.13) and (3.14) to obtain

$$\int_\Gamma |\eta|^3 ds \leq \tau||\mathbf{A}||^3 \leq \tau\sqrt{\frac{2}{\alpha}}||\mathbf{A}||^2 E_v^{\frac{1}{2}}(t). \tag{4.6}$$

If we let $\varepsilon = \tau\sqrt{2/\alpha}[\rho + 2\alpha(G^2+H^2)]$, we obtain from (4.5) and (4.6) the following inequality:

$$\partial_t E_v(t) \leq -||\mathbf{A}||^2\left[\mu - \varepsilon E_v^{\frac{1}{2}}(t)\right]. \tag{4.7}$$

Theorem 4.1 (Stability) *If $E_v(0) < (\mu/\varepsilon)^2$, then the energy $E_v(t)$ decreases exponentially to zero as $t \to \infty$.*

Proof
From the hypothesis and (4.7) it follows that

$$E_v(t) \leq E_v(0) \quad \text{for} \quad \text{all} \quad t \geq 0$$

and it follows from (3.13) that

$$\partial_t E_v(t) \leq -[2/(\alpha+\beta)][\mu - \varepsilon E_v^{1/2}(0)]E_v(t) = -\Theta E_v(t),$$

from which it is clear that $E_v(t)$ decays to zero like exp(-Θt), with $\Theta > 0$.

†

Remark 4.1 *Decay of the energy to zero implies that velocity in the domain and at the boundary as well as the rate of deformation decay to zero, the rest and unstrained state.*

Stability therefore is obtained if the energy is initially small enough, which is a way of saying that the initial state is only slightly perturbed from the rest state. This implies that with an initially small velocity in the domain and at the boundary as well as a small initial rate of deformation, stability follows.

Remark 4.2 *For Problem $\mathcal{P}_1$ the expression for the energy appears to be quite natural and appropriate. Stability follows almost effortlessly. This suggests that the kinematical condition $\boldsymbol{\omega} \wedge \mathbf{n} = 2\nabla_s \eta$ is a natural condition.*

4.3 Uniqueness of Classical Solutions for *Problem $\mathcal{P}_1$*.

The purpose of this chapter is to establish uniqueness of classical solutions to *Problem $\mathcal{P}_1$* under the initial conditions

$$\left.\begin{array}{rcl} \mathbf{v}|_{t=o} & = & \mathbf{v}(0) \\ \eta_v|_{t=o} & = & \eta_o \equiv \eta(\mathbf{v}(0)) \\ \mathbf{A}|_{t=o} & = & \mathbf{A}_o \end{array}\right\}. \tag{4.8}$$

By a classical solution we mean that the derivatives of $\mathbf{v}$ exist in the traditional sense for $t > 0$ and $x \in \Omega$.

Proposition 4.1 *Assume that* $d\mathbf{A}(\mathbf{v}(\cdot,t))/dt$ *exists in* $\mathbf{L}^2(\Omega)$ *in the sense that*

$$\lim_{h\to 0}\left\|d\mathbf{A}(\mathbf{v}(\cdot,t))/dt - [\mathbf{A}(\mathbf{v}(\cdot,t+h)) - \mathbf{A}(\mathbf{v}(\cdot,t))]/h\right\| = 0,$$

then $d\mathbf{A}(\mathbf{v}(\cdot,t)/dt = \partial_t\mathbf{A}(\mathbf{v}(\cdot,t))$ *a.e. in* Ω.

We shall consider solutions which are bounded in the following sense: $\mathbf{v} \in \mathcal{D}\cap\mathbf{C}^3(\Omega)$ for which there exists a positive constant m such that

$$\max_{|\alpha|\le 3}\ \sup_{x\in\Omega}|\partial^\alpha\mathbf{v}(x,t)| < m \quad \text{for all} \quad t > 0. \tag{4.9}$$

Proposition 4.2 *If* $\mathbf{v}$ *satisfies condition* (4.9), *then* $\mathbf{v} \in \mathbf{H}^3(\Omega)$ *and there exist positive constants* k^* *and* m^*, *both dependent on* Ω, *such that*

$$\sup_{x\in\Gamma}|\eta_v| = \sup_{x\in\Gamma}|\gamma_o\mathbf{v}| \le k^*m^*|\Omega|^{1/2},$$

where $|\Omega|$ *denotes the volume of* Ω.

Proof.
From the Sobolev Imbedding theorem [A_1 p.97 Case C] and the compactness of the 2–dimensional manifold Γ, we have $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{C}(\Gamma)$. Thus by the Trace theorem

$$\sup_{x\in\Gamma}|\gamma_o\mathbf{v}| \le C||\gamma_o\mathbf{v}||_{1,\Gamma} \le k^*||\mathbf{v}||_{2,\Omega}.$$

We know that $||\mathbf{v}||_2^2 = \sum_{|\alpha|\le 2}||\partial^\alpha\mathbf{v}||^2$ and

$$||\partial^\alpha\mathbf{v}||^2 = \int_\Omega|\partial^\alpha\mathbf{v}|^2\,dx \le (m^*)^2\int_\Omega 1\,dx = (m^*)^2|\Omega|.$$

†

Theorem 4.2 *The initial-boundary value problem Problem* $\mathcal{P}_1$, *under* (4.8) *has at most one classical solution in* $\mathbf{H}^3(\Omega)\cap\mathcal{D}$ *which satisfies condition* (4.9).

Proof.
Let $\langle p,\mathbf{v},\ell(t)\rangle$ and $\langle q,\mathbf{w},h(t)\rangle$ be pressure-velocity trio's satisfying (4.1), (4.8) and (4.9) and let $\mathbf{u} = \mathbf{v}-\mathbf{w}$. The initial values for $\mathbf{u}$ in (4.8) are zero because $\mathbf{u}$ and $\nabla\cdot\mathbf{u}$ are zero in Ω.
Since $\langle p,\mathbf{v}\rangle$ and $\langle q,\mathbf{w}\rangle$ both satisfy (4.1), we have

$$\rho\partial_t\mathbf{v} = -\rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\cdot\mathbf{T}\langle p,\mathbf{v}\rangle \tag{4.10}$$

and

$$\rho\partial_t \mathbf{w} = -\rho(\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\cdot\mathbf{T}\langle q,\mathbf{w}\rangle, \tag{4.11}$$

where

$$T\langle p,\mathbf{v}\rangle = -pI + \mu\mathbf{A}(\mathbf{v}) + \alpha D_t\mathbf{A}(\mathbf{v}) + \frac{\alpha}{2}[\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})]$$

and

$$T\langle q,\mathbf{w}\rangle = -qI + \mu\mathbf{A}(\mathbf{w}) + \alpha D_t\mathbf{A}(\mathbf{w}) + \frac{\alpha}{2}[\mathbf{A}(\mathbf{w})\mathbf{W}(\mathbf{w}) - \mathbf{W}(\mathbf{w})\mathbf{A}(\mathbf{w})].$$

Subtraction of (4.11) from (4.10) yields

$$\rho\partial_t \mathbf{u} = -\rho[(\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{w}\cdot\nabla)\mathbf{w}] + \nabla\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle - \mathbf{T}\langle q,\mathbf{w}\rangle]. \tag{4.12}$$

Subtraction of the dynamic boundary conditions $\sigma\partial_t\eta_v - k\eta_v^2 = \mathbf{n}\cdot\mathbf{T}\langle p,\mathbf{v}\rangle\mathbf{n} - \ell(t)$ and $\sigma\partial_t\eta_w - k\eta_w^2 = \mathbf{n}\cdot\mathbf{T}(q,\mathbf{w})\mathbf{n} - h(t)$ yields

$$\sigma\partial_t\eta_u - k(\eta_v^2 - \eta_w^2) = \mathbf{n}\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle - \mathbf{T}\langle q,\mathbf{w}\rangle]\mathbf{n} - [\ell(t) - h(t)], \tag{4.13}$$

where $\ell(t)$ is the exterior pressure associated with $\mathbf{v}$ and $h(t)$ the exterior pressure associated with $\mathbf{w}$ on Γ. Furthermore, we obtain in the same way

$$\gamma_o\mathbf{u}(x,t) = -\eta_u(x,t)\mathbf{n}(x), \tag{4.14}$$

and

$$\gamma_o\mathbf{A}(\mathbf{u}) = -2\eta_u\mathbf{M}. \tag{4.15}$$

Furthermore,

$$(\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{w}\cdot\nabla)\mathbf{w} = (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{u}.$$

Clearly $\mathbf{u}\in\mathcal{D}$ satisfies the following equations:

$$\left.\begin{array}{rcl} \rho[\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{u}] &=& \nabla\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle - \mathbf{T}\langle q,\mathbf{w}\rangle] \text{ in } \Omega\times(0,\infty) \\ \sigma\partial_t\eta_u + k(\eta_v^2 - \eta_w^2) &=& \mathbf{n}\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle - \mathbf{T}\langle q,\mathbf{w}\rangle]\mathbf{n} \\ && -[\ell(t) - h(t)] \text{ on } \Gamma\times(0,\infty). \end{array}\right\} \tag{4.16}$$

We shall prove that $\mathbf{v}=\mathbf{w}$ by applying an energy method similar to the one used before to the first equation in (4.16). By Identity 3.2.6,

$$\left.\begin{array}{rcl} \frac{\rho}{2}\partial_t\|\mathbf{u}\|^2 & + & \rho((\mathbf{u}\cdot\nabla)\mathbf{v}+(\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{u}) \\ & = & (\nabla\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle-\mathbf{T}\langle q,\mathbf{w}\rangle],\mathbf{u}) \\ & = & \int_\Gamma \gamma_o\mathbf{u}\cdot[\mathbf{T}\langle p,\mathbf{v}\rangle-\mathbf{T}\langle q,\mathbf{w}\rangle]\mathbf{n}\; ds \\ & & -\frac{1}{2}\int_\Omega[\mathbf{T}\langle p,\mathbf{v}\rangle-\mathbf{T}\langle q,\mathbf{w}\rangle]:\mathbf{A}(\mathbf{u})\; dx. \end{array}\right\} \quad (4.17)$$

In a manner similar to the deduction of the energy identity for *Problem* $\mathcal{P}_1$, use of the boundary conditions (2.8) yields

$$\left.\begin{array}{rcl} \partial_t[\rho\|\mathbf{u}\|^2 & + & \|\eta_u\|_\Gamma^2+\frac{\alpha}{2}\|\mathbf{A}(\mathbf{u})\|^2] \\ & = & -\mu\|\mathbf{A}(\mathbf{u})\|^2-\rho[((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{u})+((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{u})] \\ & & +\alpha[((\mathbf{u}\cdot\nabla)\mathbf{A}(\mathbf{v}),\mathbf{A}(\mathbf{u}))+((\mathbf{w}\cdot\nabla)\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u}))] \\ & & +\frac{\alpha}{2}(\mathbf{A}(\mathbf{u})\mathbf{W}(\mathbf{v})+\mathbf{A}(\mathbf{w})\mathbf{W}(\mathbf{u}),\mathbf{A}(\mathbf{u})) \\ & & -\frac{\alpha}{2}(\mathbf{W}(\mathbf{u})\mathbf{A}(\mathbf{v})+\mathbf{W}(\mathbf{w})\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u})) \\ & & -\int_\Gamma k\eta_u[\eta_v^2-\eta_w^2]ds. \end{array}\right\} \quad (4.18)$$

Every term on the right hand side of (4.18) can be estimated in terms of E_u. We proceed to derive these estimates: The boundary integral in (4.18) is estimated as follows:

$$\left|\int_\Gamma k\eta_u[\eta_v^2-\eta_w^2]ds\right| = \left|\int_\Gamma k\eta_u^2[\eta_v+\eta_w]\; ds\right| \leq \frac{2km}{s}E_u.$$

By Identity 3.2.6 two of the four convective terms reduce to boundary integrals: If $\mathbf{f}=\mathbf{g}=\mathbf{A}(\mathbf{u})$ and $\mathbf{v}=\mathbf{w}$, we obtain

$$\begin{array}{rcl} ((\mathbf{w}\cdot\nabla)\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u})) & = & -(1/2)\int_\Gamma \eta_w|\mathbf{A}(\mathbf{u})|^2 ds \\ & = & -2\int_\Gamma \eta_w\eta_u^2|\mathbf{M}|^2\; ds, \end{array}$$

from the boundary value for $\mathbf{A}(\mathbf{u})$, (4.15).

The estimates (2.24), (3.1), (3.2) and (4.9) now lead to

$$\begin{array}{rcl} \alpha|((\mathbf{w}\cdot\nabla)\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u}))| & \leq & 2\alpha\int_\Gamma \eta_w\eta_u^2(\kappa_1^2+\kappa_2^2+K^2)\; ds \\ & \leq & \frac{2\alpha C_m}{s}(G^2+H^2)\|\eta_u\|_\Gamma^2 \\ & \leq & \frac{2\alpha C_m}{s}(G^2+H^2)E_u. \end{array} \quad (4.19)$$

Similarly with $\mathbf{f}=\mathbf{g}=\mathbf{u}$ and $\mathbf{v}=\mathbf{w}$,

$$\rho((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{u}) = -\frac{\rho}{2}\int_\Gamma \eta_w|\mathbf{u}|^2 ds. = -\frac{\rho}{2}\int_\Gamma \eta_w\eta_u^2 ds,$$

and the estimates (2.24) and (4.9) give

$$\rho|((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{u})| \le \frac{\rho C_m}{2s}\|\eta_u\|_\Gamma^2 \le \frac{\rho C_m}{2s}E_u. \tag{4.20}$$

The other two terms are estimated as follows:

$$\alpha|((\mathbf{u}\cdot\nabla)\mathbf{A}(\mathbf{v}),\mathbf{A}(\mathbf{u}))| \le \alpha C_m\|\mathbf{u}\|\|\mathbf{A}(\mathbf{u})\| \le \alpha\sqrt{\frac{\beta}{2\rho}}C_m\|\mathbf{A}(\mathbf{u})\|^2$$

$$\le \alpha\sqrt{\frac{\beta}{2\rho}}C_m E_u, \tag{4.21}$$

and $\rho((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{u}) = \rho\int_\Omega \mathbf{u}\cdot(\nabla\mathbf{v})\mathbf{u}\,dx = (\rho/2)\int_\Omega \mathbf{u}\cdot\mathbf{A}(\mathbf{v})\mathbf{u}\,dx$. Thus

$$\rho|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{u})| \le \frac{\rho}{2}C_m\|\mathbf{u}\|^2 \le \frac{1}{2}C_m E_u. \tag{4.22}$$

By making use of the Schwarz inequality and the estimate (3.13), the other terms in (4.18) are directly estimated in terms of the constant C_m, a generic constant which depends on m, defined in (4.9), for example:

$$\mathbf{A}(\mathbf{u})\mathbf{W}(\mathbf{v}):\mathbf{A}(\mathbf{u}) = -\mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{u}):\mathbf{A}(\mathbf{u})$$

and

$$|\mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{u}):\mathbf{A}(\mathbf{u})| \le |\mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{u})||\mathbf{A}(\mathbf{u})| \le |\mathbf{W}(\mathbf{v})||\mathbf{A}(\mathbf{u})|^2.$$

Thus

$$\frac{\alpha}{2}|(\mathbf{A}(\mathbf{u})\mathbf{W}(\mathbf{v}),\mathbf{A}(\mathbf{u}))| \le \frac{\alpha}{2}\sup_{x\in\Omega}|\mathbf{W}(\mathbf{v})|\|\mathbf{A}(\mathbf{u})\|^2 \le \frac{\alpha m}{2}\|\mathbf{A}(\mathbf{u})\|^2.$$

$$\le C_m E_u \tag{4.23}$$

The remaining terms in (4.18) are estimated in a similar way:

$$\frac{\alpha}{2}|(\mathbf{A}(\mathbf{w})\mathbf{W}(\mathbf{u}),\mathbf{A}(\mathbf{u}))| \le \frac{\alpha}{2}\sup_{x\in\Omega}|\mathbf{A}(\mathbf{w})|\|\mathbf{W}(\mathbf{u})\|.\|\mathbf{A}(\mathbf{u})\| \le \frac{\alpha C_m}{2}\|\mathbf{A}(\mathbf{u})\|^2$$

$$\le C_m E_u. \tag{4.24}$$

An

$$\begin{aligned}\frac{\alpha}{2}|(\mathbf{W}(\mathbf{u})\mathbf{A}(\mathbf{v})+\mathbf{W}(\mathbf{w})\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u}))| &\le \frac{\alpha}{2}[C_m\|\mathbf{A}(\mathbf{u})\|^2 + C_m\|\mathbf{A}(\mathbf{u})\|^2]\\ &\le (C_m)E_u.\end{aligned} \tag{4.25}$$

We are thus led to the differential inequality in terms of E_u:

$$\partial_t E_u(t) \leq -CE_u(t),$$

where $C = \left[\frac{2\mu}{\alpha+\beta} - C_m[\frac{2\alpha}{s}(G^2 + H^2) + \frac{\varrho}{s} + \alpha\sqrt{\frac{\beta}{2\rho}} + 4\frac{1}{2} + \frac{2k}{s}]\right]$,
which has the well known solution

$$0 \leq E_u(t) \leq E_u(0)e^{-Ct}.$$

Clearly, $E_u(0) = 0$, and hence $E_u(t) = 0$ for all t. Hence $\mathbf{u} = 0$. †

Remark 4.3 *In the uniqueness proof above it was impossible to rely solely on the boundedness of the eigenvalues of the tensor* $\mathbf{A}$ *as was done in [F_1] because of the more complex nature of the boundary conditions. However, in that paper the eigenvalues of* $\mathbf{A}(\Delta\mathbf{v})$ *played a role which essentially means an assumption on the boundedness of spatial derivatives up to order* 3.

Remark 4.4 *The dynamical boundary condition* $\sigma(x)\eta_t + k\eta^2 = \mathbf{n}\cdot\mathbf{Tn} - \ell(t)$ *may, with the boundary condition* $\mathbf{n}\cdot\mathbf{A}(\mathbf{v})\mathbf{n} = -2\eta K$ *on* Γ*, and the expressions* (2.17) *and* (2.18) *in mind, be expressed in the form*

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta)$$

with

$$s(\eta) = \sigma^{-1/2}[(-k + 4\alpha K_G)\eta^2 - 2\mu\eta K + \alpha\eta\Delta_s\eta - \ell(t)].$$

Chapter 5

Alternative Model: *Problem* $\mathcal{P}_2$

Although it was possible to prove stability and uniqueness for the model of *Problem* $\mathcal{P}_1$ without any difficulty, we could not find a way to prove existence for a solution of *Problem* $\mathcal{P}_1$. In this Chapter we describe an alternative model which displays all the properties of *Problem* $\mathcal{P}_1$ with respect to stability and uniqueness, and for which existence can be proved.

In the altenative model the dynamics at the boundary is formulated by assuming a 'shear flow' of the form

$$\mathbf{v}^*(y,t) = -\eta(s_1, s_2, t)\mathbf{n}(y)$$

with s_1, s_2 the surface parameters (like arc length). It is assumed that the 'body force' acting on the shearing fluid at the boundary is proportional to the difference between the pressures $\gamma_o p$ and $\ell(t)$. Under these assumptions the equation governing the evolution of η is

$$\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta + \delta^{-1}\ell(t), \tag{5.1}$$

where $\gamma_o \mathbf{v} = -\eta\mathbf{n}$, and p the resulting pressure through the boundary with thickness δ. Δ_s is the Laplace–Beltrami operator ($\Delta_s = \nabla_s \cdot \nabla_s$) and ∇_s denotes the surface gradient. The parameter δ has the physical dimension of length, and may be thought of as the 'thickness' of the 'shear layer' (see [T_4], Sect 123, p. 506). The equation (5.1) is derived by calculating the stress tensor for a shear flow and noticing that terms of the form $\mathbf{v}^* \cdot \nabla_s$ vanish. The term $\delta^{-1}\ell(t)$ may be left out since, as before, it disappears when projections are taken. The kinematical boundary condition is still imposed.

The Model *Problem* $\mathcal{P}_2$
$\mathbf{v} \in \mathbf{H}^3(\Omega) \cap \mathcal{D}$ satisfies the system of evolution equations

$$\left.\begin{array}{rl} D_t[\rho\mathbf{v}(x,t)] = \nabla\cdot\mathbf{T}(\rho,\mathbf{v}) & \text{in} \quad \Omega\times(0,\infty) \\ \partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta & \text{at} \quad \Gamma\times(0,\infty) \\ \gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta M & \text{at} \quad \Gamma\times(0,\infty). \end{array}\right\} \tag{5.2}$$

For the purpose of stability and uniqueness for *Problem* $\mathcal{P}_2$ we define the following norm

$$\int_\Gamma \eta ds = ||\eta||_{0,\Gamma}.$$

Now derive an energy identity for the solutions of (5.2). Take the $\mathbf{L}^2(\Omega)$–scalar product with $\mathbf{v}$ on both sides of $(5.2)_1$. This produces

$$\begin{aligned} (D_t(\rho\mathbf{v}),\mathbf{v}) &= \frac{\rho}{2}\partial_t||\mathbf{v}||^2 - \frac{\rho}{2}\int_\Omega \eta^3\, ds \\ &= (\nabla\cdot\mathbf{T},\mathbf{v}) \\ &= \int_\Gamma \gamma_o\mathbf{v}\cdot\mathbf{Tn}\, ds - (\mathbf{T},\nabla\mathbf{v}) \\ &= -\int_\Gamma \eta\mathbf{n}\cdot\mathbf{Tn}\, ds - \frac{1}{2}\int_\Omega \mathbf{T}:\mathbf{A}\, dx. \end{aligned}$$

According to the formulation of *Problem* $\mathcal{P}_1$ on the boundary (see Appendix III) where $s(\eta) = \mathbf{n}\cdot\mathbf{Tn}$, we obtain

$$-\int_\Gamma \eta\mathbf{n}\cdot\mathbf{Tn}\, ds = -\int_\Gamma \eta(-\gamma_o p - 2\mu\eta K - 2\alpha K\eta_t + 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta - \ell(t))\, ds. \tag{5.3}$$

From $(5.2)_2$ we obtain:

$$\gamma_o p = -\delta\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta\mu\Delta_s\eta. \tag{5.4}$$

Substitute (5.4) into (5.3) to obtain

$$\begin{aligned} -\int_\Gamma \eta\mathbf{n}\cdot\mathbf{Tn}\, ds \;=\; & -\partial_t\int_\Gamma \frac{\delta\rho}{2}|\eta|^2\, ds - \delta\alpha\partial_t||\nabla_s\eta||^2_{0,\Gamma} - \delta\mu||\nabla_s\eta||^2_{0,\Gamma} \\ & +2\mu\delta\int_\Gamma K\eta^2\, ds + \partial_t\int_\Gamma \delta\alpha K|\eta|^2\, ds \\ & -4\alpha\delta\int_\Gamma K_G\eta^3\, ds - \alpha\delta\int_\Gamma \eta|\nabla_s\eta|^2\, ds. \end{aligned} \tag{5.5}$$

Also

$$-\frac{1}{2}(\mathbf{T},\mathbf{A}) = -\frac{1}{2}\mu||\mathbf{A}||^2 - \frac{\alpha}{4}\partial_t||\mathbf{A}||^2 + \alpha\int_\Gamma |\mathbf{M}|^2\eta^3\, ds.$$

Therefore the energy identity for *Problem* $\mathcal{P}_2$ is

$$\begin{aligned} &\partial_t[\frac{\rho}{2}||\mathbf{v}||^2 + \frac{\alpha}{2}||\mathbf{A}(\mathbf{v})||^2 + \int_\Gamma (\frac{\rho\delta}{2} - \alpha K)|\eta|^2\, ds + \delta\alpha||\nabla_s\eta||^2_{0,\Gamma}] \\ = & -\frac{1}{2}\mu||\mathbf{A}(\mathbf{v})||^2 + \alpha\int_\Gamma (|M|^2 - 4\alpha\delta K_G)|\eta|^3\, ds \\ & -\delta\mu||\nabla_s\eta||^2 + 2\mu\delta\int_\Gamma K|\eta|^2\, ds - \alpha\delta\int_\Gamma \eta|\nabla_s\eta|^2\, ds. \end{aligned} \quad (5.6)$$

Now we can define an energy norm for *Problem* $\mathcal{P}_2$ as

$$\tilde{E}_v(t) = \rho||\mathbf{v}||^2 + \frac{\alpha}{2}||\mathbf{A}(\mathbf{v})||^2 + \int_\Gamma (\delta\rho - 2\alpha K)|\eta|^2\, ds + 2\delta\alpha||\nabla_s\eta||^2_{0,\Gamma} \quad (5.7)$$

Note that here we have to make the assumption that $\delta\rho - 2\alpha K > 0$ which gives us a restriction on K. We define a parameter

$$p_2 = \frac{\alpha K}{\delta\rho}.$$

Now it is clear that stability can only be proved under the assumption that $p_2 \in (0, 1/2)$.

The Poincaré (see $[L_3)$ inequality states that there exists a smallest constant c such that $||\eta||^2_{0,\Gamma} \geq c||\nabla_s\eta||^2_{0,\Gamma}$. With the use of the Schwarz inequality and the above Poincaré inequality we obtain

$$\begin{aligned} \partial_t\tilde{E}_v(t) \leq & -\mu||\mathbf{A}(\mathbf{v})||^2 + 2\alpha(G^2 + H^2 + 4\delta G^2)||\eta||^3_{0,\Gamma} + 2\delta||\nabla_s\eta||^2_{0,\Gamma}(\mu + \alpha||\eta||_{0,\Gamma}) \\ & +4\delta\mu G||\eta||^2_{0,\Gamma} \\ \leq & -\tilde{E}_v[\mu - 2\alpha(G^2 + H^2 + 2\delta G^2 - \delta)\tilde{E}_v^{1/2} - 2\delta\mu - 4\delta\mu G] \end{aligned} \quad (5.8)$$

With $2\alpha(G^2 + H^2 + 4\delta G^2 - \delta) = \epsilon^*$ and $\mu(1 - 2\delta - 4\delta G) = \epsilon^{**}$ we have

$$\partial_t\tilde{E}_v \leq -\tilde{E}_v[\epsilon^{**}\mu - \epsilon^*\tilde{E}_v^{1/2}].$$

Theorem 5.1 **(Stability for** ***Problem*** $\mathcal{P}_2$**)**
If $p_2 \in (0, 1/2)$ *and* $\tilde{E}_v(0) < (\epsilon^{**}\mu/\epsilon^*)^2$, *then the energy* $\tilde{E}_v(t)$ *decreases exponentially to zero as* $t \to \infty$.

The uniqueness of the solution of *Problem* $\mathcal{P}_2$ is treated in the same way as the uniqueness of the solution of *Problem* $\mathcal{P}_1$.

Chapter 6

Existence

The question of the existence of solutions to the general initial-boundary-value problem for an incompressible second grade fluid in a bounded domain, with no slip, has only been addressed recently. The pioneer in this subject was A.P. Oskolkov $[O_1]$, who proved the global (in time) existence and uniqueness of a *general solution* to a simplified version of the problem by formulating it in terms of

$$\mathbf{u} \equiv \mathbf{v} - \alpha \Delta \mathbf{v}.$$

and applying the Faedo-Galerkin method. Cioranescu & Girault $[C_3]$ followed a similar approach, but using the quantity $\mathbf{curl}(\mathbf{v} - \alpha_1 \Delta \mathbf{v})$ and applying the Faedo-Galerkin method to the full problem to obtain a unique solution, global in time if Ω is in R^2 and local in time if Ω is in R^3, for flows with

$$\alpha_1 \geq 0 \ \text{ and } \ \alpha_1 + \alpha_2 = 0.$$

Certain one-dimensional flows of a so-called power-law fluid of grade two, with shear-dependent viscosity, was studied by ManS $[M_1]$, where similar existence results were established.

Global existence of the weak solutions was successfully proved by Cioranescu & Girault $[C_2]$,– by this approach. This variational solution will also be classical if the data is sufficiently smooth.

In $[G_1]$ the problem of existence of classical solutions was formulated as a problem of the existence of a fixed point of a certain mapping by considering the change of variable $\mathbf{v} \to \mathbf{u} = \mathbf{v} - \alpha_1 \Delta \mathbf{v}$. This was the first proof of existence and uniqueness of classical solutions. The restriction that $\alpha_1 > 0$ as well as α_1 sufficiently large was imposed for the global existence result to hold. No restriction on α_2 was imposed. Cioranescu & Quazar $[C_4]$ proved that the stationary problem is well-posed for arbitrary values of $\alpha_1 > 0$ and α_2. Here orthogonal projections on solenoidal fields were used to annihilate the pressure term in the equation of motion in the region Ω. They considered a fixed point problem in $\mathbf{w}$, based on the Helmholtz decomposition $\Delta\mathbf{v} = \mathbf{w} + \nabla\boldsymbol{\pi}$.

For traditional boundary conditions, the classical Helmholtz decomposition is often useful. The Helmholtz decomposition results from orthogonal projections from the space of square integrable vector fields onto the subspaces consisting of gradients and solenoidal fields [T_3, Thm. 1.5, p 16].

When considering dynamical boundary conditions where interaction between the boundary and the fluid is taken into account, we need a modified projection theorem for the same results.

In our case we consider a canister filled with incompressible fluid, immersed in fluid of the same kind. It is assumed that the wall of the canister admits normal flow through it. Modelling of the situation (Chapter 5), has led to the equations of *Problem* $\mathcal{P}_2$ which are written as follows:

$$\left.\begin{array}{l}
\rho^{1/2}\mathbf{v}_t + \rho^{1/2}(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{-1/2}\nabla\cdot\mathbf{T}^* \quad \text{in} \quad \Omega \\
\nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega \\
\gamma_o\mathbf{v} = -\eta_v\mathbf{n} \quad \text{on} \quad \Gamma; \quad \mathbf{v} = 0 \text{ on } \Gamma_1 \\
\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta_v M \quad \text{on} \quad \Gamma \\
\rho^{1/2}\partial_t(\rho\eta_v - \alpha\Delta_s\eta_v) + \rho^{-1/2}\frac{\gamma_o p}{\delta} = \rho^{-1/2}\mu\Delta_s\eta_v \quad \text{on} \quad \Gamma.
\end{array}\right\} \tag{6.1}$$

where ρ denotes the density of the fluid. $\mathbf{T}^*$ is the part of the stress tensor which depends only on velocity i.e.

$$\mathbf{T}^* = \mu\mathbf{A} + \alpha D_t\mathbf{A} + \frac{\alpha}{2}(\mathbf{AW} - \mathbf{WA}).$$

In what follows we assume that the constants μ and α are strictly positive.

The projection we construct [S_2] is designed to keep the pair $\langle\rho^{\frac{1}{2}}\mathbf{v}, \rho^{1/2}\eta\rangle$ intact and at the same time eliminate the pressure term $\langle\rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p\rangle$. It is the following: $\langle\mathbf{v}, \eta\rangle \in \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ has the orthogonal (therefore unique) decomposition

$$\begin{array}{rcl}
\langle\mathbf{v}, \eta\rangle & = & \langle\rho^{1/2}\mathbf{w}, -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w}\rangle + \langle\rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o q\rangle \\
\nabla\cdot\mathbf{w} & = & 0
\end{array}$$

with $\mathbf{w} \in \mathbf{H}^k(\Omega)$, $q \in H^{k+1}(\Omega)$ provided that $\mathbf{v} \in \mathbf{H}^k(\Omega)$ and $\eta \in H^{k-1/2}(\Gamma)$.

The orthogonal projection associated with the term $\langle\rho^{1/2}\mathbf{w}, -\rho^{1/2}\gamma_o\mathbf{w}\cdot\mathbf{n}\rangle$ will be denoted by P.

6.1 Some Spaces, Operators and Definitions.

All spaces of vector fields are denoted by boldface letters. We define the space $\mathbf{V}(\Omega)$ as

$$\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{C}^\infty(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \ \gamma_o \mathbf{v} = -\eta_v \mathbf{n} \text{ on } \Gamma,$$

$$\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta_v M \text{ on } \Gamma, \ \ \gamma_o \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Note that $\mathbf{V}(\Omega) \subset \mathcal{D}$.

From now on the notation $\gamma_o \mathbf{v}$ will be used to denote the restriction of $\mathbf{v}$ to Γ.
We define the following Hilbert spaces:

$$\mathbf{X}_m = Cl_{\mathbf{H}^m(\Omega)}(\mathbf{V}(\Omega)); \quad \mathbf{X}_m^o = \{\mathbf{v} \in \mathbf{X}_m : \gamma_o \mathbf{v} = \mathbf{0}\}, \quad m = 1, 2, 3 \cdots.$$

Note that $\mathbf{X}_2 = \mathcal{D}$, by definition (3.4). The inner products of the above spaces are the usual inner products defined for the Sobolev space $\mathbf{H}^m(\Omega)$.
The metric of a product space is defined in the usual way: For $\mathbf{X}$ and $\mathbf{Y}$ Hilbert spaces, the scalar product in $\mathbf{X} \times \mathbf{Y}$ is defined as

$$(\langle \mathbf{p}, \mathbf{q} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle)_{X \times Y} = (\mathbf{p}, \mathbf{r})_X + (\mathbf{q}, \mathbf{s})_Y.$$

The corresponding norm is then defined by

$$||\langle \mathbf{p}, \mathbf{q} \rangle||^2_{X \times Y} = ||\mathbf{p}||^2_X + ||\mathbf{q}||^2_Y.$$

We define the canonical operators $\mathbf{C}_o$ and $\mathbf{C}_m$ in the following way:
Let

$$\mathbf{Y}_o = \mathbf{L}^2(\Omega) \times L^2(\Gamma), \ \text{ and } \ \mathbf{Y}_m = \mathbf{X}_m \times H^{m-\frac{1}{2}}(\Gamma).$$

The canonical operator $\mathbf{C}_o : \mathbf{X}_1 \to \mathbf{Y}_o := \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ is defined by

$$\mathbf{C}_o \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle. \tag{6.2}$$

According to the definition (6.2) above we have

$$\left. \begin{aligned} (\mathbf{C}_o \mathbf{v}, \mathbf{C}_o \mathbf{w})_{Y_o} &:= \rho(\mathbf{v}, \mathbf{w})_{L^2(\Omega)} + (\rho^{1/2}\eta_v, \rho^{1/2}\eta_w)_{L^2(\Gamma)} \\ ||\mathbf{C}_o \mathbf{v}||^2_{Y_o} &:= \rho||\mathbf{v}||^2 + \rho||\eta_v||^2_{L^2(\Gamma)}. \end{aligned} \right\} \tag{6.3}$$

And the canonical operator $\mathbf{C}_m : \mathbf{X}_m \to \mathbf{Y}_m \ \ (m \geq 1)$ is defined by

$$\mathbf{C}_m \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle. \tag{6.4}$$

With (6.3) in mind, the norm of $\mathbf{C}_m\mathbf{v}$ in $\mathbf{Y}_m$ is given by

$$||\mathbf{C}_m\mathbf{v}||^2_{Y_m} = \rho||\mathbf{v}||^2_{X_m} + \rho||\eta_v||^2_{H^{m-1/2}(\Gamma)}.$$

By the Trace Theorem $\mathbf{C}_o$ and $\mathbf{C}_m$ are continuous linear mappings.

Let $I = [0,T]$. For Y a Banach space with norm $||\cdot||_Y$, and $1 \leq p < \infty$ let

$$L^p(I;Y) = \{v : t \to v(t) \in Y; t \in I,\ v \text{ measurable, and } \int_o^T ||v(t)||^p_Y\,dt < \infty\}.$$

and denote by $W^{m,p}(I;Y)$ the space of functions such that the distributional time derivatives of order up to and including m are in $L^p(I;Y)$. For $p = \infty$, we denote by $L^\infty(I;Y)$ the Banach space of measurable and essentially bounded functions defined on I with values in Y. The norms in $W^{k,\infty}(I;\mathbf{H}^m(\Omega))$ and in $W^{k,\infty}(I;H^{m-1/2}(\Gamma))$, $k \geq 0$, are denoted by $||\cdot||_{k,m,T}$ and $||\cdot||_{k,m-1/2,T,\Gamma}$, respectively. For $k = 0$ we write $||\cdot||_{m,T}$ and $||\cdot||_{m-1/2,T,\Gamma}$.

The linear operator $\mathbf{B}_m : \mathbf{X}_{m+2} \to \mathbf{Y}_m$ is defined by

$$\mathbf{B}_m\mathbf{v} := \langle \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}), \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v)\rangle.$$

Now for $\mathbf{v} \in \mathbf{X}_{m+2}$

$$\begin{aligned} \mathbf{u}_1 &= \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}) \\ u_2 &= \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) \\ \mathbf{u} &= \langle \mathbf{u}_1, u_2\rangle \in \mathbf{Y}_m, \end{aligned}$$

so that

$$\mathbf{B}_m\mathbf{v} = \mathbf{u}.$$

We note that since $\mathbf{X}_m$ is embedded in $\mathbf{X}_1$ and $\mathbf{Y}_m$ is embedded in $\mathbf{Y}_o$, $\mathbf{C}_m\mathbf{v} = \mathbf{C}_o\mathbf{v}$, $\mathbf{v} \in \mathbf{X}_m$ for $m \geq 1$.

In motivation of what is to follow we obtain a formal energy identity by taking the scalar product in $\mathbf{Y}_o$ of $\mathbf{u} = \mathbf{B}_m\mathbf{v}$ with $\mathbf{C}_m\boldsymbol{\phi}$. This identity holds for $\boldsymbol{\phi}$, $\mathbf{v} \in \mathbf{X}_m$, $m \geq 1$:

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_m\boldsymbol{\phi})_{Y_o} &= (\mathbf{B}_m\mathbf{v}, \mathbf{C}_o\boldsymbol{\phi})_{Y_o} \\ &= (\langle \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}), \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v)\rangle, \langle \rho^{1/2}\boldsymbol{\phi}, -\rho^{1/2}\gamma_o\boldsymbol{\phi}\cdot\mathbf{n}\rangle)_{Y_o} \\ &= \rho(\mathbf{v},\boldsymbol{\phi})_{L^2(\Omega)} - \alpha(\boldsymbol{\phi},\Delta\mathbf{v})_{L^2(\Omega)} + \rho(\eta_v,\eta_\phi)_{L^2(\Gamma)} - \delta\alpha(\Delta_s\eta_v,\eta_\phi)_{L^2(\Gamma)}.. \end{aligned}$$

But with $\gamma_o\boldsymbol{\phi} = -\eta_\phi\mathbf{n}$ and $\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} = -2K\eta_v\mathbf{n}$ we have

$$\begin{aligned} -(\boldsymbol{\phi}, \Delta\mathbf{v})_{L^2(\Omega)} &= -(\boldsymbol{\phi}, \nabla\cdot\mathbf{A}(\mathbf{v}))_{L^2(\Omega)} \\ &= \frac{1}{2}(\mathbf{A}(\boldsymbol{\phi}), \mathbf{A}(\mathbf{v}))_{L^2(\Omega)} - (\gamma_o\boldsymbol{\phi}, \gamma_o\mathbf{A}(\mathbf{v})\mathbf{n})_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\boldsymbol{\phi}), \mathbf{A}(\mathbf{v}))_{L^2(\Omega)} - (-\eta_\phi\mathbf{n}, -2K\eta_v\mathbf{n})_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\boldsymbol{\phi}), \mathbf{A}(\mathbf{v}))_{L^2(\Omega)} - 2(K\eta_v, \eta_\phi)_{L^2(\Gamma)}. \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_o\boldsymbol{\phi})_{Y_o} &= \rho(\mathbf{v}, \boldsymbol{\phi})_{L^2(\Omega)} + \delta\rho(\eta_v, \eta_\phi)_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\boldsymbol{\phi}))_{L^2(\Omega)} \\ &\quad -2\alpha(K\eta_v, \eta_\phi)_{L^2(\Gamma)} + \delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \end{aligned}$$

for any $\boldsymbol{\phi}$, $\mathbf{v} \in \mathbf{X}_m$; $m \geq 1$.

For $\boldsymbol{\phi} = \mathbf{v}$,

$$(\mathbf{u}, \mathbf{C}_o\mathbf{v})_{Y_o} = \rho\|\mathbf{v}\|^2_{L^2(\Omega)} + \|(\delta\rho - 2\alpha K)\eta_v\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|^2_{L^2(\Omega)} + \alpha\|\nabla_s\eta_v\|^2_{L^2(\Gamma)}.$$

We define, accordingly, the bilinear form b_1 by

$$\begin{aligned} b_1(\mathbf{v}, \boldsymbol{\phi}) &= \rho(\mathbf{v}, \boldsymbol{\phi})_{L^2(\Omega)} + ((\delta\rho - 2\alpha K)\gamma_o\mathbf{v}, \gamma_o\boldsymbol{\phi})_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\boldsymbol{\phi}))_{L^2(\Omega)} \\ &\quad +\delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \quad \text{for} \quad \mathbf{v}, \boldsymbol{\phi} \in \mathbf{X}_1. \end{aligned} \tag{6.5}$$

As in Chapter 5 we assume that the parameter $p_2 \in (0, 1/2)$.

Let the operator $\mathbf{N}_v$ be defined by $\mathbf{N}_v\mathbf{u} = \langle(\mathbf{v}\cdot\nabla)\mathbf{u}_1, 0\rangle$, and $\mathbf{u}_1 = \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v})$.

The pressure operator Π is defined by

$$\Pi p = \langle\rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p\rangle.$$

According to Theorem 7.1 in Appendix III

$$\langle\mathbf{v}, \eta_v\rangle = \mathbf{C}_o\mathbf{w} + \Pi q$$

if $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $q \in H^{1/2}(\Gamma)$, i.e. $P\mathbf{C}_o\mathbf{w} = \mathbf{C}_o\mathbf{w}$, $P\Pi q = 0$.

6.2 Preparatory Results.

The fact that $\mathbf{H}^m(\Omega)$ is a Banach function algebra under pointwise multiplication and addition $[A_1]$, provided $m \geq 2$, has, together with the Trace Theorem important consequences in the present study.

In the first place we note that there exists a constant $C_1 = C_1(\Omega)$ such that if $\mathbf{u}, \mathbf{v} \in \mathbf{H}^m(\Omega)$ then $\mathbf{u} \cdot \mathbf{v} \in H^m(\Omega)$, and

$$\|\mathbf{u} \cdot \mathbf{v}\|_m \leq C_1 \|\mathbf{u}\|_m \|\mathbf{v}\|_m.$$

By the Trace Theorem, we have for $\mathbf{u} \in \mathbf{H}^m(\Omega)$, that $\gamma_o \mathbf{u} \in H^{m-\frac{1}{2}}(\Gamma)$, and

$$\|\gamma_o \mathbf{u}\|_{H^{m-\frac{1}{2}}(\Gamma)} \leq C_2(\Omega, m) \|\mathbf{u}\|_m \quad \forall \ \mathbf{u} \in \mathbf{H}^m(\Omega),$$

provided $m \geq 1$.

6.3 The Auxiliary Problems.

Let us write the equations (6.1) in the form

$$\left.\begin{aligned} \partial_t \left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] + (\mathbf{v}\cdot\nabla)\left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] & \\ + \rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \ \text{ in } \Omega \times (0,T) & \\ \mathbf{v} = 0 \ \text{ on } \ \Gamma_1 \times (0,T) & \\ \gamma_0 \mathbf{v} = -\eta_v \mathbf{n} = -\eta \mathbf{n} \ \text{ on } \Gamma \times (0,T) & \\ \gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta M \quad \text{on} \quad \Gamma \times (0,T) & \\ \delta\partial_t(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_0 p = s^*(\eta), \quad \text{on } \Gamma & \end{aligned}\right\} \tag{6.6}$$

with

$$\left.\begin{aligned} \mathbf{S}(\mathbf{v}) = \rho^{-1/2}[\tfrac{\alpha}{2}\nabla\cdot[\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})] + & \\ \alpha\nabla\cdot(\nabla\mathbf{v}\mathbf{A}(\mathbf{v})) + \mu\Delta\mathbf{v}] & \\ s^*(\eta) = \rho^{-1/2}\delta\mu\Delta_s\eta_v & \end{aligned}\right\} \tag{6.7}$$

(Appendix III, Sect. 6.3.1).

Note that in general $\gamma_o[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2K\eta_v\mathbf{n} + \boldsymbol{\psi}$ from (2.15). Hence the condition $\gamma_o[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2K\eta_v\mathbf{n}$ is equivalent to the kinematical boundary condition $\boldsymbol{\psi} = 0$.

Under the substitution

$$\left.\begin{aligned} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} = \mathbf{u}_1 \ \text{ in } \ \Omega & \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) = u_2 \ \text{ on } \ \Gamma & \\ \mathbf{v} \in \mathbf{X}_{m+2}, \quad m \geq 1 & \end{aligned}\right\} \tag{6.8}$$

the first and last equations of (6.6) become

$$\left.\begin{aligned}\partial_t \mathbf{u}_1 + (\mathbf{v}\cdot\nabla)\mathbf{u}_1 + \rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \ \text{ in } \ \Omega\\ \partial_t u_2 + \rho^{-1/2}\gamma_o p = s^*(\eta) \ \text{ on } \ \Gamma.\end{aligned}\right\} \tag{6.9}$$

The existence proof which we develop in the ensuing sections is based roughly on the following algorithm:

1. With $\mathbf{v}$ given, at least one solution $\mathbf{u} = \langle \mathbf{u}_1, u_2\rangle$ of (6.9) under the initial condition $\mathbf{u}(x,0) =: \mathbf{u}_o(x)$ is found.
2. With each $\mathbf{u} = \langle \mathbf{u}_1, u_2\rangle$ in hand, a unique solution $\mathbf{v}^*$ of (6.8) is found.
3. It is proved that the composite mapping

$$\mathbf{\Phi} : \mathbf{v} \xrightarrow{g} \mathbf{u} \xrightarrow{f} \mathbf{v}^*$$

   has a fixed point which is a solution of (6.6).

The rough algorithm described above, will be developed in detail, after some refinement of the equations (6.8) and (6.9).

6.3.1 The Stokes Problem $(\mathcal{SP})$.

The problem (6.8) with $\mathbf{u}_1$ and u_2 given, leads to the Stokes Problem. Since $\mathbf{v}$ has to be divergence–free, we shall write the system (6.8) as a Stokes–like system. To this end, we use the Helmholtz–decomposition $[S_2]$ in the following form

Theorem 6.1 (Appendix III, Sect. 6.3.3)
Let $\mathbf{u}_1 \in \mathbf{H}^k(\Omega)$ and $u_2 \in H^{k-1/2}(\Gamma)$; $k \geq 1$. Then there exists $q \in H^{k+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^k(\Omega)$ such that

$$\left.\begin{aligned}&\mathbf{u}_1 = \rho^{1/2}\mathbf{w} + \rho^{-1/2}\nabla q \ \text{ in } \ \Omega\\ &u_2 = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} + \rho^{-1/2}\gamma_o q \ \text{ on } \ \Gamma\\ &\nabla\cdot\mathbf{w} = 0 \ \text{ in } \ \Omega.\end{aligned}\right\}$$

The operator $P : \langle \mathbf{u}_1, u_2\rangle \in \mathbf{L}^2(\Omega)\times L^2(\Gamma) \to \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w}\rangle \in \mathbf{L}^2(\Omega)\times L^2(\Gamma)$ is an orthogonal projection.

Remark 6.1 *Note that for* $\mathbf{v} \in \mathbf{X}_m$; $m \geq 1$, $P\mathbf{C}_m\mathbf{v} = \mathbf{C}_m\mathbf{v}$, (*see* (6.4)).

Indeed, we have the orthogonal decomposition

$$\begin{aligned} \mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle &= \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\gamma_o \mathbf{n}\cdot\mathbf{w}\rangle + \langle \rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o q\rangle \\ &= \mathbf{C}_o\mathbf{w} + \Pi q. \end{aligned}$$

From this result, we can rewrite (6.8) in the form

$$\left.\begin{aligned} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{1/2}\mathbf{w} \;\; &\text{in } \Omega\times(0,T) \\ \nabla\cdot\mathbf{v} = 0 \;\; &\text{in } \Omega\times(0,T) \\ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \;\; \text{on } \Gamma\times(0,T); \;\; \mathbf{v} = 0 \;\; &\text{on } \Gamma_1\times(0,T) \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_o p = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} \;\; &\text{on } \Gamma\times(0,T) \\ \gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} = -2\eta_v K\mathbf{n} \;\; &\text{on } \Gamma\times(0,T) \end{aligned}\right\} \tag{6.10}$$

(with $p = -q$), which is Stokes–like. We shall refer to (6.10) as the Stokes Problem $(\mathcal{SP})$.

Note that the kinematical boundary condition is incorporated in the $(\mathcal{SP})$.

According to the definition of the operators $\mathbf{C}_m$, $\mathbf{B}_m$ and Π, (6.10) may also be written in the form

$$\mathbf{B}_m\mathbf{v} + \Pi p = \mathbf{C}_m\mathbf{w}, \quad \mathbf{v}\in\mathbf{X}_{m+2}, \quad m \geq 1.$$

Led by the discussion above we show the existence and uniqueness for $(\mathcal{SP})$:

Theorem 6.2 *Let Γ be of class C^∞, and suppose that $p_2 < 1/2$. Let $m \geq 0$ and let there be given $\mathbf{w}\in\mathbf{W}^{k,\infty}(I;\mathbf{H}^m(\Omega))$ and $\gamma_o\mathbf{w}\cdot\mathbf{n}\in W^{k,\infty}(I;H^{m+1/2}(\Gamma))$. Then the problem (6.10) has a solution for which $\mathbf{v}$ is unique and ∇p is unique. $\mathbf{v}\in\mathbf{W}^{k,\infty}(I;\mathbf{X}_{m+2})$, $\eta_v\in W^{k,\infty}(I;H^{m+3/2}(\Gamma))$ and $p\in H^{m+1}(\Omega)$.*

Proof.
Consider the system (6.10) of equations

$$\left.\begin{aligned} \rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{1/2}\mathbf{w} \;\; &\text{in } \Omega\times(0,T) \\ \delta\rho^{-1/2}(\rho - \alpha\Delta_s\eta_v) + \rho^{-1/2}\gamma_o p = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} \;\; &\text{on } \Gamma\times(0,T) \\ \gamma_o\mathbf{v} = -\eta_v\mathbf{n} \;\; &\text{on } \Gamma\times(0,T) \\ \gamma_o[\mathbf{A}(\mathbf{v})\mathbf{n}] = -2\eta_v K\mathbf{n} \;\; &\text{on } \Gamma\times(0,T). \end{aligned}\right\} \tag{6.11}$$

We prove first that (6.11) has a unique weak solution: We use the projection P in Theorem 7.1 to eliminate p, (Appendix III, Sect. 6.3.3). If we formally take the $\mathbf{L}^2(\Omega)$-inner product of the first two equations of (6.11) with $\mathbf{C}_o\boldsymbol{\phi}$ we obtain the weak formulation of the problem

$$b_1(\mathbf{v},\boldsymbol{\phi}) = \rho(\mathbf{w},\boldsymbol{\phi}) + \rho(\mathbf{n}\cdot\gamma_o\mathbf{w}, \eta_\phi)_{L^2(\Gamma)}, \tag{6.12}$$

with b_1 defined in (6.5).

Since all the spaces $\mathbf{X}_m$ have a common dense subspace $\mathbf{V}(\Omega)$, we define a weak solution of (6.10) if $\mathbf{v} \in \mathbf{X}_1$ such that (6.12) holds for all $\boldsymbol{\phi} \in \mathbf{X}_1$. The bilinear form b_1 being equivalent to the metric in $\mathbf{H}^1(\Omega)$, is clearly positive definite on $\mathbf{X}_1$, and therefore a unique weak solution exists.

Regularity results for $(\mathcal{SP})$ need to be investigated. A classical result is that the solution $\mathbf{u} \in \mathbf{H}_o^1(\Omega)$ for the Dirichlet problem $-\Delta\mathbf{u} + \mathbf{u} = \mathbf{f}$ belongs to $\mathbf{H}^{m+2}(\Omega)$ whenever $\mathbf{f} \in \mathbf{H}^m(\Omega)$(with Ω sufficiently smooth). The question is whether similar results exist for $(\mathcal{SP})$.

A priori estimates for the solution $\mathbf{v}$ of (6.11) is is obtained in a proposition with the use of Theorem 10.5 of $[A_3]$ in the following way:

Proposition 6.1 *Let Ω be an open bounded set in $\mathbf{R}^3$, with boundary $\Gamma \cup \Gamma_1$ of class C^{m+3}, m a nonnegative integer. Suppose that $\mathbf{v} \in \mathbf{X}_2$ is a weak solution of the Stokes–like problem* (6.10). *If $\mathbf{w} \in \mathbf{H}^m(\Omega)$ and $\gamma_o\mathbf{w}\cdot\mathbf{n} \in H^{m-1/2}(\Gamma)$, then $\mathbf{v} \in \mathbf{H}^{m+2}(\Omega)$, and there exists a $p \in H^{m+1}(\Omega)$ and a constant $C_o(\alpha,\rho,m,\Omega)$ such that*

$$||\mathbf{v}||_{m+2} + ||p||_{m+1} \le C_o\{||\mathbf{w}||_m + ||\gamma_o\mathbf{w}\cdot\mathbf{n}||_{H^{m-1/2}(\Gamma)}\}. \tag{6.13}$$

Proof.
This proof results from the paper of Agmon-Douglis-Nirenberg $[A_3]$, giving *a priori* estimates of solutions of general elliptic systems. This proof is given in Appendix II. †

Proposition 6.2 *Let $\Gamma \cup \Gamma_1$ be of class C^∞. Let $k \ge 0$ and let $\mathbf{w} \in \mathbf{W}^{k,\infty}(I;\mathbf{H}^m(\Omega))$ and $\gamma_o\mathbf{w}\cdot\mathbf{n} \in W^{k,\infty}(I;H^{m-1/2}(\Gamma))$ be given. Then there exists a unique vector field $\mathbf{v} \in \mathbf{W}^{k,\infty}(I;\mathbf{X}_{m+2})$ satisfying* (6.11). *Moreover, there is a constant $C_1 = C_{1(\Omega,m,\rho,\alpha)}$ such that*

$$||\mathbf{v}||_{k,m+2,T} \le C_1\{||\mathbf{w}||_{k,m,T} + ||\gamma_o\mathbf{w}\cdot\mathbf{n}||_{W^{k,\infty}(I;H^{m-1/2}(\Gamma))}\}.$$

Proof.
By Proposition 6.1 it is only necessary to prove that $\mathbf{v} \in \mathbf{X}_2$. This was proved for the general $\mathbf{L}^p$ case by Miyakawa in $[M_4]$.

This conclude the proof of Theorem 6.2. †

Let the mapping $f : \mathbf{Y}_m \to \mathbf{X}_{m+2}$ be defined as

$$f : \mathbf{u} \to \mathbf{w} \to \mathbf{v}^* = f(\mathbf{u})$$

where $\mathbf{v}^* \in \mathbf{X}_{m+2}$ is the unique solution of (6.10) for a given $\mathbf{u} \in \mathbf{Y}_m$.

Theorem 6.3 *f is a bounded linear operator.*

Proof.
The result follows from (6.13) and the fact that $\mathbf{w}$ is obtained from $\mathbf{u}$ by the orthogonal projection P. †

6.3.2 The Transport Problem $(\mathcal{TP})$.

The formulation of the transport equation in the product space $\mathbf{Y}_m$ involves two transport-like equations, one in the region Ω and the other equation on the boundary Γ. The first equation represents the equation of motion in Ω and the second the dynamical boundary condition on the boundary Γ. In terms of the notations of Section 5.1, the system of equations (6.9), with initial conditions added, can be written as

$$\left.\begin{aligned} \partial_t \mathbf{u} + \mathbf{N}_v \mathbf{u} + \Pi p = \mathbf{F}, \\ \mathbf{u}(0) = (\mathbf{u}_{1,o}, u_{2,o}), \end{aligned}\right\} \tag{6.14}$$

where $\mathbf{u} \in \mathbf{Y}_m$ and $p \in H^{m+1}(\Omega)$, with $\mathbf{F} = \langle \mathbf{S}(\mathbf{v}), s^*(\eta)\rangle \in \mathbf{X}_m \times H^{m-1/2}(\Gamma)$. We recall that

$$\begin{aligned} \rho^{1/2}\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v} &= \mathbf{u}_1 \\ \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) &= u_2 \end{aligned}$$

From $\gamma_o \mathbf{v}\cdot\mathbf{n} = -\eta_v \in H^{m+3/2}(\Gamma)$ on the boundary Γ and Lemma 7.2 in Appendix I, we obtain

$$-\gamma_o \mathbf{u}_1 \cdot \mathbf{n} = \rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v \tag{6.15}$$

with $\gamma_o \mathbf{u}_1 \cdot \mathbf{n} \in H^{m-1/2}(\Gamma)$. We now formulate the Transport Problem $(\mathcal{TP})$ in the following way

$$\left.\begin{aligned} \partial_t \mathbf{u}_1 + (\mathbf{v}\cdot\nabla)\mathbf{u}_1 + \rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \quad \text{in} \quad \Omega. \\ \delta\tfrac{d}{dt}[\gamma_o \mathbf{u}_1 \cdot \mathbf{n}] + \rho^{-1/2}\gamma_o p = s^*(\eta_v) \quad \text{at} \quad \Gamma \\ \mathbf{u}_1|_o = \mathbf{u}_1(0). \end{aligned}\right\} \tag{6.16}$$

The formulation (6.14) includes both the equations in Ω and the equations on the part Γ of the boundary. These problems will be solved by means of the Galerkin method in a way similar to Temam's construction of solutions of the Euler equations [T_1].

The following estimates are of great importance:

Lemma 6.1 *Suppose* $m \geq 1$. *If* $\mathbf{v}$, $\mathbf{v}' \in \mathbf{X}_{m+2}$, *and* η_v, $\eta_{v'} = \eta' \in H^{m+3/2}(\Gamma)$, *then* $\mathbf{S}(\mathbf{v}) \in \mathbf{X}_m$, $s^*(\eta) \in \mathbf{H}^{m-\frac{1}{2}}(\Gamma)$ *and*

$$\left.\begin{array}{r} ||\mathbf{S}(\mathbf{v})||_{X_m} \leq C||\mathbf{v}||_{X_{m+2}}[||\mathbf{v}||_{X_{m+1}} + 1] \\ ||s^*(\eta)||_{m-1/2} \leq C||\mathbf{v}||_{X_{m+2}} \end{array}\right\} \tag{6.17}$$

and

$$\left.\begin{array}{r} ||\mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v}')||_{X_m} \leq C||\mathbf{v} - \mathbf{v}'||_{X_{m+2}}[||\mathbf{v}||_{X_{m+2}} + ||\mathbf{v}'||_{X_{m+2}} + 1] \\ ||s^*(\eta) - s^*(\eta')||_{m-1/2} \leq C||\mathbf{v} - \mathbf{v}'||_{X_{m+2}} \end{array}\right\} \tag{6.18}$$

with C *depending on* α, S, G, μ, δ, ρ *and* Ω.

Proof.
From (6.7) we see that $\mathbf{S}(\mathbf{v})$ is composed of sums of products of second order and first order derivatives of $\mathbf{v}$, the function $\mathbf{v}$ with third order derivatives of $\mathbf{v}$ plus a term consisting of a second order derivative of $\mathbf{v}$. Also

$$\partial^\alpha \mathbf{v} \in \mathbf{X}_{m+2-|\alpha|} \quad m \geq 1.$$

Thus from the Banach Algebra property of $\mathbf{H}^m(\Omega)$ for $m \geq 2$ we obtain

$$\begin{aligned} ||\mathbf{S}(\mathbf{v})||_{X_m} &\leq C[4||\mathbf{v}||_{X_{m+2}}||\mathbf{v}||_{X_{m+1}} + 2\alpha||\mathbf{v}||_{X_{m+2}}||\mathbf{v}||_{X_{m+1}} + \mu||\mathbf{v}||_{X_{m+2}}] \\ &\leq C(4, \alpha, \mu, \rho)||\mathbf{v}||_{X_{m+2}}[||\mathbf{v}||_{X_{m+1}} + 1]. \end{aligned}$$

The estimate for $s^*(\eta_v)$ is obvious:

$$||s^*(\eta)||_{m-1/2} \leq C(\delta, \rho, \mu)||\Delta_s \eta||_{m+1/2} \leq C(\delta, \rho, \mu)||\mathbf{v}||_{m+2},$$

with the aid of the Trace Theorem.

For $\mathbf{v}$ and $\mathbf{v}' \in \mathbf{X}_{m+2}$, we consider the following:

$$\begin{aligned} &\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v}) - \mathbf{A}(\mathbf{v}')\mathbf{W}(\mathbf{v}') + \mathbf{W}(\mathbf{v}')\mathbf{A}(\mathbf{v}') \\ &= [\mathbf{A}(\mathbf{v}) - \mathbf{A}(\mathbf{v}')]\mathbf{W}(\mathbf{v}) + \mathbf{A}(\mathbf{v}')[\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v}')] + \mathbf{W}(\mathbf{v}')[\mathbf{A}(\mathbf{v}') - \mathbf{A}(\mathbf{v})] \\ &\quad + [\mathbf{W}(\mathbf{v}') - \mathbf{W}(\mathbf{v})]\mathbf{A}(\mathbf{v}). \end{aligned}$$

With the definition of $\mathbf{A}(\mathbf{v})$ and $\mathbf{W}(\mathbf{v})$ in mind and application of the Schwarz inequality, it is easily deduced that

$$||\nabla \cdot [\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v}) - \mathbf{A}(\mathbf{v}')\mathbf{W}(\mathbf{v}') + \mathbf{W}(\mathbf{v}')\mathbf{A}(\mathbf{v}')]||_m$$

$$\leq C\|\mathbf{v}-\mathbf{v}'\|_{m+2}\left[\|\mathbf{v}\|_{m+2}+\|\mathbf{v}'\|_{m+2}\right],$$

where $C = C(\Omega)$. In a completely analogous way we show that

$$\|\alpha[\nabla\cdot(\Delta\mathbf{v}\mathbf{A}(\mathbf{v})-\Delta\mathbf{v}'\mathbf{A}(\mathbf{v}')]\|_m \leq C(\Omega,\alpha)\ \|\mathbf{v}-\mathbf{v}'\|_{m+2}\left[\|\mathbf{v}\|_{m+2}+\|\mathbf{v}'\|_{m+2}\right],$$

$$\|\mu(\Delta\mathbf{v}-\Delta\mathbf{v}')\|_m \leq C(\mu)\|\mathbf{v}-\mathbf{v}'\|_{m+2}.$$

By the Trace Theorem we derive

$$\begin{aligned}\|s^*(\eta)-s^*(\eta')\|_{m-1/2} &\leq C(\delta,\mu,\rho)\|\Delta_s\eta-\Delta_s\eta'\|_{m-1/2}\\ &\leq C(\delta,\mu,\rho)\|\mathbf{v}-\mathbf{v}'\|_{m+2}\end{aligned}$$

†

Corollary
If $\mathbf{v} \in (\mathbf{L}^\infty(I,\mathbf{X}_{m+2}))$ then $\mathbf{F} = \langle \mathbf{S}(\mathbf{v}), s^*(\eta)\rangle$ belongs to $(\mathbf{L}^\infty(I,\mathbf{Y}_m))$ where $\mathbf{Y}_m = \mathbf{X}_m \times H^{m-\frac{1}{2}}(\Gamma)$.

We shall now derive *a priori* estimates for solutions of the Transport Problem $(\mathcal{TP})$ (6.16).

To this end we give the following preparatory result based on the theory of linear elliptic equations:

Lemma 6.2 *Let Ω be of class C^{m+1}, $m \geq 1$. Then given*

$$\left.\begin{aligned}G &= \nabla\cdot[\mathbf{S}(\mathbf{v})-(\mathbf{v}\cdot\nabla)\mathbf{u}_1]\\ g &= \gamma_o[\mathbf{S}(\mathbf{v})-(\mathbf{v}\cdot\nabla)\mathbf{u}_1]\cdot\mathbf{n}-s^*(\eta_v).\end{aligned}\right\}$$

with

$$G \in H^{m-1}(\Omega),\quad g \in H^{m-1/2}(\Gamma)$$

the Neumann problem

$$\left.\begin{aligned}\rho^{-1/2}\Delta p &= G \ \text{ in } \ \Omega\\ \rho^{-1/2}\gamma_1 p-\rho^{-1/2}\gamma_o p &= g \ \text{ at } \ \Gamma\end{aligned}\right\} \qquad (6.19)$$

admits a unique solution $p \in H^m(\Omega)$ such that

$$\|\nabla p\|_{m-1} \leq C[\|G\|_{m-1}+\|g\|_{H^{m-1/2}(\Gamma)}]. \qquad (6.20)$$

Proof.
(See proof of estimate (6.20) in Appendix II).

To prove the uniqueness let $\tilde{p}$ satisfy the Neumann problem:

$$\left.\begin{aligned} \rho^{-1/2}\nabla\cdot\nabla\tilde{p} = 0 \\ \rho^{-1/2}\gamma_1\tilde{p} - \rho^{-1/2}\gamma_o\tilde{p} = 0. \end{aligned}\right\} \tag{6.21}$$

We take the scalar product over Ω with $\tilde{p}$ on both sides of $(6.21)_1$ and with $\gamma_o\tilde{p}$ over Γ on both sides of $(6.21)_2$, we add the two equations and he result follows.

If $p \in H^2(\Omega)$ and $\gamma_o p \in H^2(\Gamma)$ then Theorem 10.5 of $[A_3]$ can be applied.

To prove $p \in H^2(\Omega)$ we use results from a doctoral thesis, ($[J_1]$ Theorem 5, p.26).

†

Lemma 6.3 *Assume* $m \geq 1$, $\mathbf{v} \in \mathbf{L}^\infty(I, \mathbf{X}_{m+2})$, *with* $||\mathbf{v}||_{m+2} \leq D$, $\mathbf{S} \in \mathbf{L}^\infty(I, \mathbf{X}_m)$ *and* $||\mathbf{S}||_{X_m,T} \leq \gamma D$ *and that* $\mathbf{u}_1|_o = \mathbf{u}_1(0) \in \mathbf{X}_m$. *Then if there exist solutions* $\mathbf{u}_1 \in (\mathbf{L}^\infty(I, \mathbf{X}_m))$, $p \in H^{m+1}(\Omega)$ *to* (6.16), *the following a priori estimate is true*

$$||\mathbf{u}_1||_{X_m,T} + \left\|\frac{d\mathbf{u}_1}{dt}\right\|_{X_{m-1},T} \leq C_T,$$

with $C_T = C_T(\Omega, m, \gamma, D, D_1, D_2, T, ||\mathbf{u}_1(0)||_{X_m,T})$, *where* $\gamma = CD + C$*(see* (6.17)*).*

Proof.
We apply the derivative operator ∂^κ with respect to x, (κ is a multi-index) to both sides of $(6.16)_1$, take the $\mathbf{L}^2(\Omega)$ scalar product and sum over κ, with $0 \leq \kappa \leq m$. We thus obtain

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}_1||^2_{X_m} = -((\mathbf{v}\cdot\nabla)\mathbf{u}_1, \mathbf{u}_1)_{X_m} + (\mathbf{S}(\mathbf{v}), \mathbf{u}_1)_{X_m} - \rho^{-1/2}(\nabla p, \mathbf{u}_1)_{X_m}. \tag{6.22}$$

We now find estimates of all the terms on the right–hand side of (6.22).

By the Leibnitz rule we have

$$((\mathbf{v}\cdot\nabla)\mathbf{u}_1, \mathbf{u}_1)_{X_m} = \sum_{0\leq|\kappa|\leq m} ((\mathbf{v}\cdot\nabla)\partial^\kappa\mathbf{u}_1, \partial^\kappa\mathbf{u}_1)$$

$$+ \sum_{\substack{1\le|\beta|\le|\kappa| \\ 0\le|\kappa|\le m}} C_{(\kappa,\beta)}(\partial^{\beta}(\mathbf{v}\cdot\nabla)\partial^{\kappa-\beta}\mathbf{u}_1, \partial^{\kappa}\mathbf{u}_1). \tag{6.23}$$

With $C_{(\kappa,\beta)} = \binom{\kappa 1}{\beta_1}\binom{\kappa 2}{\beta_2}\binom{\kappa 3}{\beta_3}$.

Furthermore, by (6.23), an application of the Schwarz inequality and Sobolev's imbedding theorem (Lemma 3.6 of[L_5])

$$\sum_{\substack{1\le|\beta|\le|\kappa| \\ 0\le|\kappa|\le m}} C_{(\kappa,\beta)}|(\partial^{\beta}(\mathbf{v}\cdot\nabla)\partial^{\kappa-\beta}\mathbf{u}_1, \partial^{\kappa}\mathbf{u}_1)| \le C_{(\kappa,\beta)}||\mathbf{v}||_{X_{m+2}}||\mathbf{u}_1||^2_{X_m}. \tag{6.24}$$

The first term of the right–hand side of (6.23) can be reduced to a boundary integral and by the Trace Theorem we obtain

$$|((\mathbf{v}\cdot\nabla)\partial^{\kappa}\mathbf{u}_1, \partial^{\kappa}\mathbf{u}_1)| \le \frac{1}{2}\int_{\Gamma} |\partial^{\kappa}\mathbf{u}_1|^2|\mathbf{v}\cdot\mathbf{n}|\, ds; \quad 0\le\kappa\le m,$$

therefore

$$\sum_{0\le|\kappa|\le m} ((\mathbf{v}\cdot\nabla)\partial^{\kappa}\mathbf{u}_1, \partial^{\kappa}\mathbf{u}_1) \le C||\mathbf{u}_1||^2_{X_m}||\mathbf{v}||_{X_{m+2}}. \tag{6.25}$$

From (6.24) and (6.25) it follows that

$$\frac{d}{dt}||\mathbf{u}_1||_{X_m} \le C_{(\rho,\alpha,\kappa,\beta)}[||\mathbf{u}_1||_{X_m}||\mathbf{v}||_{X_{m+2}} + ||\mathbf{S}||_{X_m} + ||\nabla p||_m]. \tag{6.26}$$

It remains to estimate the pressure term ∇p. Take the divergence on both sides of (6.16), and recall that $\mathbf{v}$ and $\mathbf{u}_1$ are solenoidal, we derive that p obeys the Dirichlet problem (6.19) at time $t = 0$ in I.

In view of Lemma's 6.1 and 6.2 and of the Trace Theorem we then find

$$||G||_{X_{m-1}} \le C[||\mathbf{S}(\mathbf{v})||_{X_m} + ||\mathbf{v}||_{X_m}||\mathbf{u}_1||_{X_m}], \tag{6.27}$$

and

$$\begin{aligned} ||g||_{H^{m-1/2}(\Gamma)} &\le ||\gamma_o[\mathbf{S}(\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{u}_1]\cdot\mathbf{n}||_{H^{m-1/2}(\Gamma)} + ||s^*(\eta_v)||_{H^{m-1/2}(\Gamma)} \\ &\le C[D||\mathbf{u}_1||_{X_m} + D^2 + D]. \end{aligned} \tag{6.28}$$

From (6.27) and (6.28) we obtain

$$||\nabla p||_{X_m} \le C[\gamma D + D||\mathbf{u}_1||_{X_m} + (D^2 + D)]. \tag{6.29}$$

From (6.22), (6.25) and (6.29) we deduce

$$\frac{d}{dt}||\mathbf{u}_1||_{X_m} \le CD||\mathbf{u}_1||_{X_m} + C(D^2 + D + \gamma D). \tag{6.30}$$

Integrate this inequality over I and apply the version (d) of Gronwall's lemma in $[L_9]$, p.54, to find that

$$||\mathbf{u}_1||_{X_m,T} \leq D_1, \tag{6.31}$$

with $D_1 = D_1(\Omega, m, D, T, \gamma, ||\mathbf{u}_1(0)||_{X_m},)$, more specific $D_1 = e^{kT}[||\mathbf{u}_1(0)||_{X_m} + (C_o e^{-kT})/(-k)]$ with $k = CD$ and $C_o = C(D^2 + D + \gamma D)$.

Furthermore from (6.14) and (6.29)

$$\begin{aligned} ||\frac{d\mathbf{u}_1}{dt}||_{X_{m-1}} &\leq ||(\mathbf{v}\cdot\nabla)\mathbf{u}||_{X_{m-1}} + ||\nabla p||_{m-1} + ||\mathbf{S}||_{X_{m-1}} \\ &\leq C[||\mathbf{v}||_{X_{m+2}}||\mathbf{u}_1||_{X_m} + D^2 + D + ||\mathbf{S}||_{X_m}] \end{aligned} \tag{6.32}$$

thus

$$||\frac{d\mathbf{u}_1}{dt}||_{X_{m-1}} \leq C[D^2 + (D_1 + 1 + \gamma)D] = D_2 \tag{6.33}$$

Let $C_T = D_1 + D_2$. †

In the following sections we will show that $(\mathcal{TP})$ has at least one solution. We proceed to study the existence of solutions of the Transport Problem $(\mathcal{TP})$ with the aid of the Galerkin method.

T The projection we construct is designed to keep the pair $\langle \mathbf{u}_1, -\gamma_o \mathbf{u}_1 \cdot \mathbf{n}\rangle$ intact and at the same time will eliminate the pressure couple $\langle \rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p\rangle$. Here we refere to $[S_2]$ and Appendix III. Accordingly we choose $H_1 = H^1(\Omega)$ and $H_2 = \mathbf{L}^2(\Omega) \times L^2(\Gamma)$. In Appendix III we have the decomposition theorem:

Theorem 6.4 *Let* $\mathbf{v} \in \mathbf{H}^k(\Omega)$; $k \geq 1$ *and* $\eta \in H^{k-1/2}(\Gamma)$. *Then there exists a unique* $q \in H^{k+1}(\Omega)$ *and* $\mathbf{w} \in \mathbf{H}^k(\Omega)$ *such that*

$$\left.\begin{aligned} &\mathbf{v} = \rho^{1/2}\mathbf{w} + \rho^{-1/2}\nabla q \\ &\eta = -\sigma^{1/2}\gamma_o \mathbf{w}\cdot\mathbf{n} + \rho^{-1/2}\gamma_o q \\ &\nabla\cdot\mathbf{w} = 0. \end{aligned}\right\} \tag{6.34}$$

6.4 The Galerkin Method.

We are going to study the $(\mathcal{TP})$ in the following abstract form:

$$\left.\begin{array}{r}\frac{d}{dt}\mathbf{y} + P\mathbf{N}_v\mathbf{y} = \mathbf{G} \\ \mathbf{y}|_{t=o} = \mathbf{y}_o.\end{array}\right\} \tag{6.35}$$

In the proof of existence of solutions of $(\mathcal{TP})$ we shall use an expansion in terms of the eigenfunctions of a very particular 'operator'. The eigenfunction–eigenvalue problem is discussed in detail in Appendix III, Sect. 6.3.2. We give a brief description here.

For a fixed $m \geq 3$, we consider the space $\mathbf{Y}_m = \mathbf{X}_m \times H^{m-\frac{1}{2}}(\Gamma)$(Section 5.1). There exists an increasing sequence of positive real numbers $\{\lambda_j\}$ and a corresponding sequence $\{\boldsymbol{\psi}_j\} \subset \mathbf{X}_m$ such that

$$(\boldsymbol{\psi}_j, \boldsymbol{\phi})_m = \lambda_j(\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\phi})_{Y_o} \qquad \text{for all} \quad \boldsymbol{\phi} \in \mathbf{X}_m \tag{6.36}$$

and $\{\mathbf{C}_o\boldsymbol{\psi}_j\} \subset \mathbf{Y}_m$ is a complete orthonormal basis in $\mathbf{Y}_o$, (see Section 6.3.2). This means that

$$(\mathbf{C}_o\boldsymbol{\psi}_j\mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \tag{6.37}$$

and for any $\mathbf{v} \in \mathbf{X}_m$, we define the Fourier coefficients $f_k = (\mathbf{C}_o\mathbf{v}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}$. Then

$$\lim_{n\to\infty} ||\mathbf{C}_o\mathbf{v} - \sum_{k=1}^{n} f_k\mathbf{C}_o\boldsymbol{\psi}_k||_{Y_m} = 0.$$

Moreover the series $\mathbf{v} = \sum_{k=1}^{\infty} f_k\boldsymbol{\psi}_k$ converges in $\mathbf{X}_m$. Also, since the range of $\mathbf{C}_o$ is, by definition, dense in $\mathbf{Y}_o$, the series $\mathbf{u} = \sum_{k=1}^{\infty}(\mathbf{u}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\mathbf{C}_o\boldsymbol{\psi}_k$ converges in $\mathbf{Y}_o$ for any $\mathbf{u} \in \mathbf{Y}_o$, if $f_k = (\mathbf{u}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}$, $[S_1]$.

Consider $\mathbf{y}^{[n]}$ defined by

$$\mathbf{y}^{[n]} = \sum_{j=1}^{n} g_{jn}\mathbf{C}_o\boldsymbol{\psi}_j = \langle \mathbf{y}_1^{[n]}, y_2^{[n]} \rangle,$$

for some undetermined coefficients g_{jn}.

It will turn out that the decomposition of $\mathbf{y}^{[n]}$ into a 'volume' part $\mathbf{y}_1^{[n]} \in \mathbf{X}_m$ and a 'boundary' part $y_2^{[n]} \in H^{m-1/2}(\Gamma)$ is extremely useful. Let us remind ourselves that

$$\mathbf{C}_o\boldsymbol{\psi}_j = \langle \rho^{1/2}\boldsymbol{\psi}_j, \rho^{1/2}\eta_j \rangle$$

with $\eta_j = \eta_{\psi_j}$, and accordingly, we have

$$\mathbf{y}_1^{[n]} = \sum_{j=1}^{n} g_{jn}\rho^{1/2}\boldsymbol{\psi}_j,$$

and

$$y_2^{[n]} = \sum_{j=1}^{n} g_{jn}\rho^{1/2}\eta_j.$$

Then we have

$$\mathbf{y}^{[n]} = \sum_{j=1}^{n} g_{jn}\langle\rho^{1/2}\boldsymbol{\psi}_j, \rho^{1/2}\eta_j\rangle.$$

$\mathbf{C}_o$ is linear and therefore

$$\begin{aligned}\mathbf{y}^{[n]} = \mathbf{C}_o\mathbf{y}_1^{[n]} &= \mathbf{C}_o\sum_{j=1}^{n} g_{jn}\boldsymbol{\psi}_j = \sum_{j=1}^{n} g_{jn}\mathbf{C}_o\boldsymbol{\psi}_j \\ &= \sum_{j=1}^{n} g_{jn}\langle\rho^{1/2}\boldsymbol{\psi}_j, \rho^{1/2}\eta_j\rangle. \end{aligned} \tag{6.38}$$

Let us approximate the right hand side of (6.35), ($\mathbf{G} \in \mathbf{Y}_m$), by:

$$\mathbf{G}^{[n]} = \sum_{j=1}^{n} h_j\mathbf{C}_o\boldsymbol{\psi}_j,$$

with $h_j := (\mathbf{G}, \mathbf{C}_o\boldsymbol{\psi}_j)_{Y_o}$. Let $\mathbf{g}_1^{[n]} = \sum_{j=1}^{n} h_j\boldsymbol{\psi}_j$. Then $\mathbf{G}^{[n]} = \mathbf{C}_o\mathbf{g}_1^{[n]}$.
Let $\mathbf{Y}_{[n]} = span\{\mathbf{C}_o\boldsymbol{\psi}_1, \cdots \mathbf{C}_o\boldsymbol{\psi}_n\}$ in $\mathbf{Y}_o$,. At this point we define a finite dimensional space $\mathbf{X}_{[n]} \subset \mathbf{X}_m$ by $\mathbf{X}_{[n]} = span\{\boldsymbol{\psi}_1, \cdots \boldsymbol{\psi}_n\}$. We seek an approximate solution $\mathbf{y}$ of (6.35) in $\mathbf{Y}_{[n]}$ as the solution of

$$\left.\begin{aligned} \tfrac{d}{dt}\mathbf{y}^{[n]} + P\mathbf{N}_v(\mathbf{y}^{[n]}) = \mathbf{G}^{[n]} \\ \mathbf{y}^{[n]}(0) = \mathbf{y}_o^{[n]} = P^{[n]}\mathbf{y}_o = \textstyle\sum_{k=1}^{n}(\mathbf{y}(0), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\mathbf{C}_o\boldsymbol{\psi}_k. \end{aligned}\right\} \tag{6.39}$$

$P^{[n]}$ is the orthogonal projection in $\mathbf{Y}_o$ onto $\mathbf{Y}_{[n]}$, and $\mathbf{G}^{[n]} = P^{[n]}(\mathbf{G})$ is the projection of $\mathbf{G}$ in $\mathbf{Y}_o$. It is important to note that

$$\begin{aligned} P_X^{[n]}\mathbf{v} &= \textstyle\sum_{k=1}^{n}(\mathbf{C}_o\mathbf{v}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\rho^{1/2}\boldsymbol{\psi}_k \\ &= P_1^{[n]}\mathbf{y}, \end{aligned}$$

where

$$\begin{aligned} P^{[n]}\mathbf{y} &= \textstyle\sum_{k=1}^{n}(\mathbf{y}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\mathbf{C}_o\boldsymbol{\psi}_k \\ &= \textstyle\sum_{k=1}^{n}(\mathbf{y}, \mathbf{C}_o\boldsymbol{\psi}_k)\langle\rho^{1/2}\boldsymbol{\psi}_k, -\rho^{1/2}\gamma_o\boldsymbol{\psi}_k\rangle \\ &= \langle P_1^{[n]}, P_2^{[n]}\rangle\mathbf{y}. \end{aligned}$$

The solution of (6.39) must be of the form $\mathbf{y}^{[n]} = \sum_{k=1}^{n} g_{kn}\mathbf{C}_o\boldsymbol{\psi}_k$. Since $\mathbf{Y}_{[n]}$ is finite dimensional. Thus, (6.39) should reduce to a finite system of equations. Indeed, if we take the scalar product of (6.39) with $\mathbf{C}_o\boldsymbol{\psi}_k$, we obtain:

$$\left.\begin{array}{ll}(\frac{d}{dt}\mathbf{y}^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} + (P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = (\mathbf{G}^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} & 1 \leq k \leq n. \\ \mathbf{y}^{[n]}(0) = \mathbf{y}_o^{[n]} = P^{[n]}\mathbf{y}_o. & \end{array}\right\} \tag{6.40}$$

The equations (6.40) reduce to a finite system of ordinary differential equations for g_{jn} in the following way:

$$(\frac{d}{dt}\sum_{j=1}^{n} g_{jn}(t)\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} + (P\mathbf{N}_v(\sum_{j=1}^{n} g_{jn}(t)\mathbf{C}_o\boldsymbol{\psi}_j), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}$$

$$= (\sum_{j=1}^{n} h_j(t)\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}.$$

From the orthonormality (6.37) of the $\mathbf{C}_o\boldsymbol{\psi}_k$'s, we obtain

$$\left.\begin{array}{l}\frac{d}{dt}g_{kn}(t) + \sum_{j=1}^{n} g_{jn}(t)(\mathbf{N}_v(\mathbf{C}_o\boldsymbol{\psi}_j), P\mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = h_k(t) \\ g_{kn}(0) = g_{kn}^o = (\mathbf{y}(0), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}.\end{array}\right\}$$

This is a linear system of ordinary differential equations and the solution exists on any interval $[0, T]$, $[H_3,\ p79]$. This defines a unique solution of (6.39).

We now show that the sequence $\{\mathbf{y}^{[n]}\}$ is bounded in $L^\infty(I; \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Gamma))$.

Lemma 6.4 *If* $\mathbf{G} \in L^\infty(0, T; \mathbf{Y}_o)$ *and* $\|\mathbf{v}\|_{m+2} \leq D$, *then*

$$\sup_{0\leq t\leq T} \|\mathbf{y}^{[n]}(\cdot, t)\|^2_{L^\infty(0,T;Y_o)} \leq e^{CT}\left[\|\mathbf{y}_o\|^2_{Y_o} + C\|\mathbf{G}\|^2_{L^\infty(0,T;Y_o)}\right]$$

for $n = 1, 2, \cdots$ *and* C *depending on* D.

Proof.
Multiply (6.40) with $g_{kn}(t)$ and sum with respect to k:

$$(\frac{d}{dt}\mathbf{y}^{[n]}, \sum_{k=1}^{n} g_{nk}(t)\mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} + (P\mathbf{N}_v(\mathbf{y}^{[n]}), \sum_{k=1}^{n} g_{nk}(t)\mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}$$

$$= (\mathbf{G}^{[n]}, \sum_{k=1}^{n} g_{nk}(t)\mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}, \ 1 \leq k \leq n.$$

We have

$$\begin{aligned}(P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{y}^{[n]})_{Y_o} &= ((\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]}) \\ &= -\frac{1}{2}\int_\Gamma |\mathbf{y}_1^{[n]}|^2\eta_v \, ds.\end{aligned}$$

We use the Schwarz inequality and the Trace Theorem to obtain:

$$|(P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{y}^{[n]})_{Y_o}| \leq C||\mathbf{y}_1^{[n]}||_{Y_o}^2||\mathbf{v}||_{H^1(\Omega)}.$$

We use the Schwarz inequality to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\mathbf{y}^{[n]}||^2 &= (\mathbf{G}^{[n]}, \mathbf{y}^{[n]})_{Y_o} - (P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{y}^{[n]})_{Y_o} \\ &\leq ||\mathbf{G}^{[n]}||\,||\mathbf{y}^{[n]}||_{Y_o} + C||\mathbf{y}_1^{[n]}||_{Y_o}^2||\mathbf{v}||_{H^1(\Omega)} \\ &\leq \frac{1}{2}||\mathbf{G}^{[n]}||_{Y_o}^2 + C_D||\mathbf{y}^{[n]}||_{Y_o}^2.\end{aligned}$$

Now we obtain

$$\frac{d}{dt}||\mathbf{y}^{[n]}||_{Y_o}^2 - C_D||\mathbf{y}^{[n]}||_{Y_o}^2 \leq ||\mathbf{G}^{[n]}||_{Y_o}^2.$$

Thus by Gronwall's lemma and the assumption on $\mathbf{G}$ we have

$$\begin{aligned}||\mathbf{y}^{[n]}(\cdot,t)||_{Y_o}^2 &\leq e^{Ct}\Big[||\mathbf{y}^{[n]}(0)||_{Y_o}^2 + \int_o^t e^{-Cs}||\mathbf{G}^{[n]}(\cdot,s)||_{Y_o}^2 ds\Big] \\ &\leq e^{Ct}\Big[||\mathbf{y}_o||_{Y_o}^2 + ||\mathbf{G}||_{L^\infty(0,T;Y_o)}^2(1/C - 1/Ce^{-Ct})\Big] \\ &\leq e^{CT}\Big[||\mathbf{y}_o||_{Y_o}^2 + C||\mathbf{G}||_{L^\infty(0,T;Y_o)}^2\Big].\end{aligned}$$

for C depending on D, $n = 1, 2, \cdots$ and $t \in [0, T]$, taking into account that $||\mathbf{y}^{[n]}(0)||_{Y_o} \leq ||\mathbf{y}_o||_{Y_o}$.

†

The proof of existence also requires the boundedness of $\mathbf{y}^{[n]}$ in $\mathbf{Y}_m$. The next step is to obtain bounds in terms of $||\cdot||_m^2$: To this end we multiply (6.40) with the eigenvalue λ_k in order to use the identity $(\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\phi})_m = \lambda_j(\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\phi})_{Y_o}$ to obtain

$$\lambda_k(\frac{d}{dt}\mathbf{y}^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} + \lambda_k(P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = \lambda_k(\mathbf{G}^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}, \quad (6.41)$$

We consider each term of (6.41) on its own: After various calculations the first term of (6.41) can be written as

$$(\frac{d}{dt}\mathbf{y}_1^{[n]}, \sum_{k=1}^{n} g_{nk}\boldsymbol{\psi}_k)_m = \frac{1}{2}\frac{d}{dt}||\mathbf{y}_1^{[n]}||_m^2. \tag{6.42}$$

To show that (6.42) is true we consider

$$\lambda_k(\frac{d}{dt}\mathbf{y}^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = \lambda_k(\frac{d}{dt}\mathbf{C}_o\mathbf{y}_1^{[n]}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} = (\frac{d}{dt}\mathbf{y}_1^{[n]}, \boldsymbol{\psi}_k)_m,$$

from (6.36). If, in addition we multiply with g_{nk} and sum over k, the first term of (6.41) becomes:

$$(\frac{d}{dt}\mathbf{y}_1^{[n]}, \sum_{k=1}^{n} g_{nk}\boldsymbol{\psi}_k)_m = (\frac{d}{dt}\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m = \frac{1}{2}\frac{d}{dt}(\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m. \tag{6.43}$$

$$= \frac{1}{2}\frac{d}{dt}||\mathbf{y}_1^{[n]}||_m^2$$

The second term of (6.41) can be estimated as follows:

$$\begin{aligned} & || \sum_{k=1}^{n} \lambda_k(P\mathbf{N}_v(\mathbf{y}^{[n]}), g_{nk}\mathbf{C}_o\boldsymbol{\psi}_k)||_{Y_o} \\ = \; & |(\boldsymbol{\varpi}^{[n]}, \mathbf{y}_1^{[n]})_m| \\ \leq \; & |(\rho^{-1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m| + |(\rho^{-1}\nabla q^{[n]}, \mathbf{y}_1^{[n]})_m| \\ \leq \; & C||\mathbf{v}||_{m+2}||\mathbf{y}_1^{[n]}||_m^2 \end{aligned} \tag{6.44}$$

where we write $P\mathbf{N}_v(\mathbf{y}^{[n]})$ in the form $\mathbf{C}_o(\boldsymbol{\varpi}^{[n]})$.

To show that (6.44) is true
We do the same with the term $(P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}$,:

$$\begin{aligned} \lambda_k(P\mathbf{N}_v(\mathbf{y}^{[n]}), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} &= \lambda_k(\mathbf{C}_o(\boldsymbol{\varpi}^{[n]}), \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o} \\ &= (\boldsymbol{\varpi}^{[n]}, \boldsymbol{\psi}_k)_m \end{aligned}$$

Multiply with g_{nk} and sum over k, the second term of (6.40) becomes $(\boldsymbol{\varpi}^{[n]}, \mathbf{y}_1^{[n]})_m$.

The problem is to find $\boldsymbol{\varpi}^{[n]}$: We know that

$$\mathbf{N}_v(\mathbf{y}^{[n]}) = \langle(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, 0\rangle,$$

and to make calculations easier, we set

$$\mathbf{N}_v(\mathbf{y}^{[n]}) = \langle(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, 0\rangle = \langle\bar{\mathbf{v}}, 0\rangle. \tag{6.45}$$

Next we use the Helmholtz projection on the pair $\langle\bar{\mathbf{v}}, 0\rangle$ which is constructed in the following way(also see Appendix III, Sect. 6.3.3 and $[S_2]$). For each $t \in I$, $q^{[n]}(t)$ solves the problem

$$\left.\begin{aligned} \Delta q^{[n]} &= \rho^{1/2}\nabla\cdot\bar{\mathbf{v}} \quad \text{on} \quad \Omega \\ \rho^{-1/2}\gamma_1 q^{[n]} + \rho^{-1/2}\gamma_o q^{[n]} &= \bar{\mathbf{v}}\cdot\mathbf{n} = 0 \qquad \text{on} \quad \Gamma \end{aligned}\right\} \tag{6.46}$$

The projection $P^{\perp} : \mathbf{Y}_m \to \mathbf{Y}_m$ is defined by

$$P^{\perp}\langle\bar{\mathbf{v}}, 0\rangle = \langle\rho^{-1/2}\nabla q^{[n]}, \rho^{-1/2}\gamma_o q^{[n]}\rangle$$

where $q^{[n]}$ is the weak solution of (6.46). Let $\bar{\mathbf{v}} \in \mathbf{H}^m(\Omega)$, $q^{[n]} \in H^{m+1}(\Omega)$ and $\gamma_o q \in H^{m+1/2}(\Gamma)$. Now

$$\mathbf{C}_o\boldsymbol{\varpi}^{[n]} = P\langle\bar{\mathbf{v}}, 0\rangle = \langle\bar{\mathbf{v}} - \rho^{-1/2}\nabla q^{[n]}, \rho^{-1/2}\gamma_1 q^{[n]}\rangle$$

By definition we have $\mathbf{C}_o\boldsymbol{\varpi}^{[n]} = \langle\rho^{1/2}\boldsymbol{\varpi}^{[n]}, -\rho^{1/2}\gamma_o\boldsymbol{\varpi}^{[n]}\cdot\mathbf{n}\rangle$, thus

$$\begin{aligned} (\boldsymbol{\varpi}^{[n]}, \mathbf{y}_1^{[n]})_m &= (\rho^{-1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]} - \rho^{-1}\nabla q^{[n]}, \mathbf{y}_1^{[n]})_m \\ &= (\rho^{-1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m - (\rho^{-1}\nabla q^{[n]}, \mathbf{y}_1^{[n]})_m. \end{aligned}$$

In order to obtain an estimate for $(\boldsymbol{\varpi}^n, \mathbf{y}_1^n)_m$ we obtain estimates separately for the terms $(\rho^{-1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m$ and $-(\rho^{-1}\nabla q^{[n]}, \mathbf{y}_1^{[n]})_m$.

Let $\mathbf{v} \in \mathbf{X}_{m+2}$ and $\mathbf{y}_1^{[n]}$ be as defined previously. By the Leibnitz rule we have

$$\begin{aligned} ((\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m &= \sum_{0\le|\kappa|\le m} ((\mathbf{v}\cdot\nabla)\partial^\kappa\mathbf{y}_1^{[n]}, \partial^\kappa\mathbf{y}_1^{[n]}) \\ &+ \sum_{\substack{1\le|\beta|\le|\kappa| \\ 0\le|\kappa|\le m}} C_{(\kappa,\beta)}(\partial^\beta(\mathbf{v}\cdot\nabla)\partial^{\kappa-\beta}\mathbf{y}_1^{[n]}, \partial^\kappa\mathbf{y}_1^{[n]}). \end{aligned} \tag{6.47}$$

With $C_{(\kappa,\beta)} = \binom{\kappa_1}{\beta_1}\binom{\kappa_2}{\beta_2}\binom{\kappa_3}{\beta_3}$.

We obtain an estimate for the term

$$|((\mathbf{v}\cdot\nabla)\partial^\kappa\mathbf{y}_1^{[n]}, \partial^\kappa\mathbf{y}_1^{[n]})| \le \frac{1}{2}\int_\Gamma |\partial^\kappa\mathbf{y}_1^{[n]}|^2|\mathbf{v}\cdot\mathbf{n}|\, ds \quad 0 \le \kappa \le m,$$

therefore

$$\sum_{0\le|\kappa|\le m}((\mathbf{v}\cdot\nabla)\partial^{\kappa}\mathbf{y}_1^{[n]},\partial^{\kappa}\mathbf{y}_1^{[n]})\le||\mathbf{y}_1^{[n]}||^2_{X_m}||\mathbf{v}||_{X_{m+2}}. \tag{6.48}$$

Furthermore, by (6.47), an application of the Schwarz inequality and Sobolev's imbedding theorem (Lemma 3.6 of[L_5]) yields

$$\begin{aligned}|\rho^{-1/2}((\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]},\mathbf{y}_1^{[n]})_m| &\le C_{(\rho,\alpha)}\sum_{\substack{1\le|\beta|\le|\kappa|\\0\le|\kappa|\le m}}C_{(\kappa,\beta)}|(\partial^{\beta}(\mathbf{v}\cdot\nabla)\partial^{\kappa-\beta}\mathbf{y}_1^{[n]},\partial^{\kappa}\mathbf{y}_1^{[n]})|\\ &\le C_{\rho,\alpha}||\mathbf{v}||_{m+2}||\mathbf{y}_1^{[n]}||^2_m. \end{aligned} \tag{6.49}$$

We conclude that

$$|(\rho^{1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]},\mathbf{y}_1^{[n]})_m|\le C||\mathbf{v}||_{m+2}||\mathbf{y}_1^{[n]}||^2_m \tag{6.50}$$

where C is a generic constant.

Now for each $t\in I$, $q^n(t)$ satisfies the mixed boundary value problem

$$\begin{cases}\Delta q^{[n]}=\rho^{1/2}\nabla\cdot\bar{\mathbf{v}} & \text{on} \quad \Omega\\ \rho^{-1/2}\gamma_1 q^{[n]}+\rho^{-1/2}\gamma_o q^{[n]}=0 & \text{on} \quad \Gamma\end{cases}$$

with $\bar{\mathbf{v}}=(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}$. Consequently we have

$$\begin{aligned}\Delta q^{[n]} &= \rho^{1/2}\nabla\cdot(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}\\ &= \rho^{1/2}\nabla\mathbf{v}^T:\nabla\mathbf{y}_1^{[n]}.\end{aligned}$$

Since $\bar{\mathbf{v}}\in\mathbf{H}^m$, it is clear that $\Delta q^{[n]}\in\mathbf{H}^{m-1}(\Omega)$ and $\nabla q^{[n]}\in\mathbf{H}^m(\Omega)$. Then (see[$A_1$]) and the fact that $\mathbf{H}^{m-1}(\Omega)$ is an algebra for $m\ge 3$ is used to show that

$$||\nabla q^{[n]}||_m=||q^{[n]}||_{m+1}\le C||\rho^{1/2}\nabla\mathbf{v}^T:\nabla\mathbf{y}_1^{[n]}||_{m-1}\le C_{(\rho,\Omega,m)}||\mathbf{v}||_{m+2}||\mathbf{y}_1^{[n]}||_m. \tag{6.51}$$

Note that the regularity result for q is similar to the result (6.20).

Now we use (6.51) to obtain

$$\begin{aligned}|\rho^{-1}(\nabla q^{[n]},\mathbf{y}_1^{[n]})_m| &\le C_{(\rho,\alpha)}|(|\nabla q^{[n]}|,|\mathbf{y}_1^{[n]}|)_m|\\ &\le C||\nabla q^{[n]}||_m||\mathbf{y}_1^{[n]}||^2_m\\ &\le C||\mathbf{v}||_{m+2}||\mathbf{y}_1^{[n]}||^2_m.\end{aligned} \tag{6.52}$$

†

Following the same procedures as before we may prove the following result. The right hand side of (6.41) can be written in terms of the scalar product in $\mathbf{H}^m(\Omega)$:

$$\sum_{k=1}^{n} \lambda_k(\mathbf{C}_o\mathbf{g}_1^{[n]}, g_{kn}\mathbf{C}_o\boldsymbol{\psi}_k) = (\mathbf{g}_1^{[n]}, \mathbf{y}_1^{[n]})_m \tag{6.53}$$

where

$$\mathbf{g}_1^{[n]} = \sum_{j=1}^{n} h_j(t)\boldsymbol{\psi}_j.$$

Hence, by virtue of (6.43),(6.48), (6.50) and (6.53), we finally arrive at

Lemma 6.5 *Let* $m \geq 3$. *If* $\mathbf{v} \in \mathbf{X}_{m+2}$ *and* $\|\mathbf{v}\|_{X_{m+2}} \leq D$, *and* $\mathbf{y}^{[n]}|_o = \mathbf{y}^{[n]}(0) \in \mathbf{Y}_m$, *then*

$$\sup_{0\leq t\leq T} \|\mathbf{y}_1^{[n]}(\cdot,t)\|_m^2 \leq Ce^{cT}\left[\|\mathbf{y}^{[n]}(0)\|_m^2 + c\|\mathbf{S}\|^2_{L^\infty(I;X_m)}\right],$$

for $n = 1, 2\cdots$.

Proof.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{y}_1^{[n]}\|_m^2 &= -(\rho^{-1/2}(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}, \mathbf{y}_1^{[n]})_m + (\rho^{-1}\nabla q^{[n]}, \mathbf{y}_1^{[n]})_m + (\mathbf{g}_1^{[n]}, \mathbf{y}_1^{[n]})_m \\
&\leq C\|\mathbf{v}\|_{m+2}\|\mathbf{y}_1^{[n]}\|_m^2 + (\mathbf{g}_1^{[n]}, \mathbf{y}_1^{[n]})_m \\
&\leq C\|\mathbf{v}\|_{m+2}\|\mathbf{y}_1^{[n]}\|_m^2 + \|\mathbf{g}_1^{[n]}\|_m\|\mathbf{y}_1^{[n]}\|_m \\
&\leq C(\|\mathbf{v}\|_{m+2} + \frac{1}{2})\|\mathbf{y}_1^{[n]}\|_m^2 + \frac{1}{2}\|\mathbf{g}_1^{[n]}\|_m^2.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\|\mathbf{y}_1^{[n]}\|_m^2 \leq C(2\|\mathbf{v}\|_{m+2} + 1)\|\mathbf{y}_1^{[n]}\|_m^2 + \|\mathbf{g}_1^{[n]}\|_m^2.$$

Now we have for

$$\|\mathbf{y}_1^{[n]}(t)\|_m \leq a(t) \quad \text{for all} \quad t < T,$$

where a is the solution of the differential inequality

$$\frac{da^2(t)}{dt} \leq ca^2(t) + \|\mathbf{g}_1^{[n]}(t)\|_m^2,$$

$$a(0) \leq C||\mathbf{y}_1^{[n]}(0)||_m.$$

and $[0,T]$ is the interval of existence of a. By Gronwall's lemma a satisfies

$$\begin{aligned} a^2(t) &\leq e^{ct}[a^2(0) + \int_o^t e^{-cs}||\mathbf{g}_1^{[n]}||_m^2 ds] \\ &\leq e^{ct}[a^2(0) + ||\mathbf{S}||^2_{L^\infty(I;X_m)}(\frac{1}{c} - \frac{1}{c}e^{-ct})] \\ &\leq e^{cT}[a^2(0) + C||\mathbf{S}||^2_{L^\infty(I;X_m)}]. \end{aligned}$$

with c dependant on D.

Now there exists a constant C such that

$$\begin{aligned} \sup_{0\leq t\leq T} ||\mathbf{y}_1^{[n]}(\cdot,t)||_m^2 &\leq C||\mathbf{y}_1^{[n]}(\cdot,t)||_m^2 \\ &\leq Ce^{cT}\Big[||\mathbf{y}^{[n]}(0)||^2_{Y_m} + C||\mathbf{S}||^2_{L^\infty(I;X_m)}\Big]. \end{aligned} \tag{6.54}$$

Thus with a boundedness condition on $\sup_{t\in I}||\mathbf{S}||_m$, and the initial condition on $\mathbf{y}_1^{[n]}$, $\{\mathbf{y}_1^{[n]}\}$ is a bounded sequence in $\mathbf{L}^\infty(I, \mathbf{H}^m(\Omega))$. Therefore there exists a subsequence, which we shall also denote by $\{\mathbf{y}_1^{[n]}\}$ such that $\mathbf{y}_1^{[n]} \rightharpoonup \mathbf{y}_1$ in $\mathbf{H}^m(\Omega)$ and $y_2^{[n]} \rightharpoonup y_2$ in $\mathbf{H}^{m+1/2}(\Gamma)$. Thus as $n \to \infty$, $\{\mathbf{y}^{[n]}\}$ is bounded in $\mathbf{L}^\infty(I, \mathbf{H}^m(\Omega) \times \mathbf{H}^{m+1/2}(\Gamma))$. By the Trace Theorem we have $||y_2||_{H^{m-1/2}(\Gamma)} = ||\gamma_o \mathbf{y}_1||_{H^{m-1/2}(\Gamma)} \leq C||\mathbf{y}_1||_{X_m} \leq Cc$ and therefore

$$\sup_{0\leq t\leq T} ||\mathbf{y}^{[n]}(\cdot,t)||_m^2 \leq Ce^{cT}\Big[||\mathbf{y}^{[n]}(0)||^2_{Y_m} + c||\mathbf{S}||^2_{L^\infty(I;X_m)}\Big], \tag{6.55}$$

†

To conclude we obtain an estimate for $d\mathbf{y}^{[n]}/dt$:

Lemma 6.6 *Let* $\mathbf{y}^{[n]}$ *be the solution of* (6.39) *and* P *and* $P^{[n]}$ *as defined. Then for a* $n > 0$ *and a* $\mathbf{G}^{[n]}$ *, and in some time interval* $I = [0,T]$, $\{\frac{d\mathbf{y}^{[n]}(t)}{dt}\}$ *is bounded in* $\mathbf{L}^\infty(I;\mathbf{Y}_o)$.

Proof.
Since the $(\mathbf{C}_o\psi_k)'s$ are orthogonal in $\mathbf{Y}_m$ we see from (6.39) that

$$\frac{d\mathbf{y}^{[n]}(t)}{dt} = P^{[n]}P(\mathbf{G}^{[n]} - \mathbf{N}_v(\mathbf{y}^{[n]})).$$

Hence

$$\begin{aligned}\left\|\frac{d\mathbf{y}^{[n]}(t)}{dt}\right\|_{L^\infty(I;Y_o)} &\leq \|\mathbf{G}^{[n]} - \mathbf{N}_v(\mathbf{y}^{[n]})\|_{Y_o,T}\\ &\leq \|\mathbf{G}\|_{Y_o,T} + \|\mathbf{N}_v(\mathbf{y}^{[n]})\|_{Y_o,T}\\ &\leq \|\mathbf{G}\|_{Y_o,T} + C\|\mathbf{y}^{[n]}\|_{Y_o,T}.\end{aligned}$$

The above result implies that $\{\frac{d\mathbf{y}^{[n]}(t)}{dt}\}$ is bounded in $\mathbf{L}^\infty(I;\mathbf{Y}_o)$. †

6.5 Existence of a Solution of $(\mathcal{TP})$.

In this section we will denote *weak* convergence by $\rightharpoonup$, *weak**- convergence by $\rightharpoonup^*$ and convergence by $\longrightarrow$. We shall assume throughout that $m \geq 3$. The method used here is similar to the method in $[L_5, L_9]$.

Theorem 6.5 Existence of a Solution of $(\mathcal{TP})$.
If $\mathbf{v} \in \mathbf{L}^\infty(I, \mathbf{X}_{m+2})$ *and* $\mathbf{y}(0) \in \mathbf{Y}_m$, $m \geq 3$, *with* $\|\mathbf{v}\|_{X_{m+2}} \leq D$, *then there exists a solution* $\mathbf{y} \in \mathbf{L}^\infty(I, \mathbf{Y}_m)$ *of* $(\mathcal{TP})$ *and* $d\mathbf{y}/dt \in \mathbf{L}^\infty(I, \mathbf{Y}_{m-1})$. *Moreover*

$$\|\mathbf{y}\|_{Y_m,T} + \left\|\frac{d\mathbf{y}}{dt}\right\|_{Y_{m-1},T} \leq C_T,$$

with $\mathbf{y} = \mathbf{C}_o\mathbf{w}$ *a classical solution of* $(\mathcal{TP})$ *and* $\mathbf{w}$ *is obtained from* $\mathbf{u}$ *by the orthogonal projection* P. $C_T = C_T(D, \Omega, m, M, T, \alpha, \|\mathbf{y}(0)\|_{Y_m}, \|\mathbf{G}\|_{Y_m,T})$.

Proof.
We shall make use of the following results:

Lemma 6.7 $[B_3$, *p.*68$]$ *Let* $T > 0$ *and let* X *and* Y *be Hilbert spaces with dual spaces* X' *and* Y'. *Suppose that* Y *is continuously and densely imbedded in* X. *If*

$$\mathbf{u}_n \rightharpoonup^* \mathbf{u} \quad in \quad \mathbf{L}^\infty(I;\mathbf{X}')$$

and

$$\frac{d\mathbf{u}_n}{dt} \rightharpoonup^* \chi \quad in \quad \mathbf{L}^\infty(I;\mathbf{Y}'),$$

then

$$\chi = \frac{d\mathbf{u}}{dt} \quad in \quad \mathbf{L}^\infty(I;\mathbf{Y}').$$

Lemma 6.8 *[L4, p.57] or [T2, pp.274 – 278]. Let* X_o, X, X_1 *be three Hilbert spaces, such that* $X_o \subset X \subset X_1$ *and*

$$X_o \hookrightarrow\hookrightarrow X \hookrightarrow X_1,$$

which means that the injection of X *into* X_1 *is continuous and the injection of* X_o *into* X *is compact. Then, for any* $0 < T < \infty$, *the space*

$$W = W(I; X_o, X_1) \equiv \{\mathbf{v} \in \mathbf{L}^2(I; \mathbf{X}_o) : \frac{d\mathbf{v}}{dt} \in \mathbf{L}^2(I; \mathbf{X}_1)\}$$

is a Hilbert space with inner product

$$(\mathbf{v}, \mathbf{w})_W = (\mathbf{v}, \mathbf{w})_{\mathbf{L}^2(I;\mathbf{X}_o)} + (\frac{d\mathbf{v}}{dt}, \frac{d\mathbf{w}}{dt})_{\mathbf{L}^2(I;\mathbf{X}_1)}.$$

From Lemma 6.5 and Lemma 6.6 we have

$$\{\mathbf{y}^{[n]}\} \text{ is bounded in } \mathbf{L}^\infty(I; \mathbf{Y}_m), \tag{6.56}$$

$$\{\frac{d\mathbf{y}^{[n]}}{dt}\} \text{ is bounded in } \mathbf{L}^\infty(I; \mathbf{Y}_o). \tag{6.57}$$

From (6.56) and the fact that $\mathbf{L}^\infty(I; \mathbf{Y}_m)$ (where $\mathbf{Y}_m$ is identified with its dual $\mathbf{Y}'_m$ via the Riesz representation theorem) is the dual of $\mathbf{L}^1(I; \mathbf{Y}_m)$ which is separable, it follows that $\{\mathbf{y}^{[n]}\}$ has a weakly convergent subsequence $\{\mathbf{y}^q\}$ and that there is a function $\mathbf{y}^* \in \mathbf{L}^\infty(I; \mathbf{Y}_m)$ such that

$$\mathbf{y}^q \rightharpoonup^* \mathbf{y}^* \text{ in } \mathbf{L}^\infty(I; \mathbf{Y}_m). \tag{6.58}$$

This also implies that $\mathbf{y}^q \rightharpoonup^* \mathbf{y}^* \in \mathbf{L}^\infty(I; \mathbf{Y}_o)$. Now by the Riesz representation theorem, for any given $\boldsymbol{\phi} \in \mathbf{L}^1(I; \mathbf{Y}_o)$, we have $\mathbf{y} \mapsto (\mathbf{y}, \boldsymbol{\phi})_{Y_o}$ on $\mathbf{Y}_m$, therefore there exists a function $\boldsymbol{\xi} \in \mathbf{L}^1(I; \mathbf{Y}_m)$ (with $||\boldsymbol{\xi}(t)||_{Y_m} = ||\boldsymbol{\phi}(t)||_{Y_o}$) such that

$$\int_o^T (\mathbf{v}(t), \phi(t))_{Y_o}\, dt = \int_o^T (\mathbf{v}(t), \boldsymbol{\xi}(t))_{Y_m}\, dt \text{ for all } \mathbf{v} \in \mathbf{L}^\infty(I; \mathbf{Y}_m).$$

Hence, using (6.57) and Lemma 6.7 (with $X = Y = \mathbf{Y}_o$) in a similar argument as above, one can extract a subsequence $\{\mathbf{y}^r\}$ of $\{\mathbf{y}^q\}$ such that

$$\frac{d\mathbf{y}^r}{dt} \rightharpoonup^* \frac{d\mathbf{y}^*}{dt} \text{ in } \mathbf{L}^\infty(I; \mathbf{Y}_o). \tag{6.59}$$

Furthermore, as T is finite, (6.56) and (6.57) implies that

$$\{\mathbf{y}^r\} \text{ is bounded in } \mathbf{L}^2(I; \mathbf{Y}_m),$$

$$\{\frac{d\mathbf{y}^r}{dt}\} \text{ is bounded in } \mathbf{L}^2(I;\mathbf{Y}_o),$$

and therefore, by Lemma 6.8, there exists a subsequence $\{\mathbf{y}^s\}$ of $\{\mathbf{y}^r\}$ and a function $\mathbf{y} \in W(I;\mathbf{Y}_m,\mathbf{Y}_o)$ such that

$$\mathbf{y}^s \rightharpoonup \mathbf{y} \text{ in } W(I;\mathbf{Y}_m,\mathbf{Y}_o) \tag{6.60}$$

$$\mathbf{y}^s \longrightarrow \mathbf{y} \text{ in } \mathbf{L}^\infty(I;\mathbf{Y}_{m-1}). \tag{6.61}$$

From (6.58) we have that for each $\boldsymbol{\phi} \in \mathbf{L}^1(I;\mathbf{Y}_m) \supset \mathbf{L}^2(I;\mathbf{Y}_m)$,

$$\int_o^T (\mathbf{y}^q(t) - \mathbf{y}^*(t), \boldsymbol{\phi}(t))_{Y_m}\, dt \longrightarrow 0, \quad q \longrightarrow \infty,$$

i.e. $\mathbf{y}^q \rightharpoonup \mathbf{y}^*$ in $\mathbf{L}^2(I;\mathbf{Y}_m)$. On the other hand (6.60) implies that $\mathbf{y}^s \rightharpoonup \mathbf{y}$ in $\mathbf{L}^2(I;\mathbf{Y}_m)$. Hence $\mathbf{y}^* = \mathbf{y}$.
Now let $\varphi \in C^o[0,T]$ and $\bar{\mathbf{y}} \in \mathbf{Y}_o$. Then there is a sequence $\{\bar{\mathbf{y}}^n\}$ in $\mathbf{Y}^{[n]}$ such that $\bar{\mathbf{y}}^n \longrightarrow \bar{\mathbf{y}}$ in $\mathbf{Y}_o$. Thus, defining $\boldsymbol{\vartheta}_n \equiv \varphi(t)\bar{\mathbf{y}}^n$ and $\boldsymbol{\vartheta} \equiv \varphi(t)\bar{\mathbf{y}}$,

$$\boldsymbol{\vartheta}_n \longrightarrow \boldsymbol{\vartheta} \text{ in } \mathbf{L}^2(I;\mathbf{Y}_o). \tag{6.62}$$

From (6.40) one deduces

$$\int_o^T ((\mathbf{y}^{[n]})'(t) + P\mathbf{N}_v(\mathbf{y}^{[n]}(t)), \boldsymbol{\vartheta}_s(t))\, dt = \int_o^T (\mathbf{G}^{[n]}(t), \boldsymbol{\vartheta}_s(t))\, dt \quad \text{for all } s.$$

By virtue of (6.60) and (6.62),

$$\int_o^T ((\mathbf{y}^{[n]})'(t), \boldsymbol{\vartheta}_s(t))_{Y_o}\, dt \longrightarrow \int_o^T (\mathbf{y}'(t), \boldsymbol{\vartheta}(t))_{Y_o}\, dt, \quad s \longrightarrow \infty. \tag{6.63}$$

The next term to be consider is

$$\int_o^T (P\mathbf{N}_v(\mathbf{y}^{[n]}(t)), \boldsymbol{\vartheta}_s(t))\, dt = \int_o^T (P(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}(t), \boldsymbol{\vartheta}_{s,1}(t))\, dt$$

Since (6.61) implies $\mathbf{y}^{[n]} \longrightarrow \mathbf{y}$ in $\mathbf{L}^2(I;\mathbf{Y}_o)$, the estimate

$$\|P(\mathbf{v}\cdot\nabla)(\mathbf{y}^{[n]} - \mathbf{y})\|_{Y_o} \le C(\Omega)\|\mathbf{v}\|_{Y_2}\|\mathbf{y}^{[n]} - \mathbf{y}\|_{y_1}$$

shows that $(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]} \longrightarrow (\mathbf{v}\cdot\nabla)\mathbf{y}_1$ in $\mathbf{L}^2(I;\mathbf{L}^2(\Omega))$ and therefore

$$\int_o^T (P(\mathbf{v}\cdot\nabla)\mathbf{y}_1^{[n]}(t), \boldsymbol{\vartheta}_{s,1}(t))\, dt \longrightarrow \int_o^T (P(\mathbf{v}\cdot\nabla)\mathbf{y}_1(t), \boldsymbol{\vartheta}_1(t))\, dt, \quad n \longrightarrow \infty. \tag{6.64}$$

To obtain convergence for $\mathbf{G} = \langle \mathbf{S}(\mathbf{v}), s^*(\eta)\rangle$ with

$$S(\mathbf{v}) = \rho^{-1/2}[\frac{\alpha}{2}\nabla \cdot [\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})] + \alpha\nabla \cdot (\nabla\mathbf{v}\mathbf{A}(\mathbf{v}))$$

$$+\mu\Delta\mathbf{v}]$$

and

$$s^*(\eta) = \rho^{-1/2}\delta\mu\Delta_s\eta_v,$$

for *Problem* $(\mathcal{P}_2)$, we apply estimates (6.18). By Lemma 6.5, if $\mathbf{v} \in \mathbf{X}_{m+2}$, then $\{\mathbf{v}'^n\}$ converges weakly in $\mathbf{H}^{m+2}(\Omega)$ and therefore $||\mathbf{v}'^n - \mathbf{v}||_{m+2} \to 0$ as $n \to \infty$, and the results follow.

Lastly, it follows from (6.62) that

$$\int_o^T (\mathbf{G}^{[n]}, \boldsymbol{\vartheta}_s)_{Y_o} dt \longrightarrow \int_o^T (\mathbf{G}, \boldsymbol{\vartheta})_{Y_o} dt, \qquad n, s \quad \longrightarrow \infty. \tag{6.65}$$

Hence we obtained from (6.63), (6.64) and (6.65) that

$$\int_o^T (\frac{d}{dt}\mathbf{y}(t) + P\mathbf{N}_v(\mathbf{y}(t)), \bar{\mathbf{y}})_{Y_o}\varphi(t)\ dt = \int_o^T (\mathbf{G}(t), \bar{\mathbf{y}})_{Y_o}\varphi(t)\ dt$$

$$\text{for all } \bar{\mathbf{y}} \in \mathbf{Y}_o, \quad \text{and for all } \varphi \in C^o[0,T].$$

By the density of $C^o[0,T]$ in $L^2(0,T)$

$$\partial_t\mathbf{y} + P\mathbf{N}_v\mathbf{y} = \mathbf{G}, \quad \text{for a.e. } t \in (0,T), \quad .$$

†

Thus we have proved that $(\mathcal{TP})$ has at least one solution. In order to obtain the estimates we need, the following two propositions is important:

Proposition 6.3 *Let $\partial\Omega$ be of class C^∞. Assume $\mathbf{v} \in L^\infty(I;\mathbf{X}_{m+2})$ with $||\mathbf{v}||_{m+2,T} \le D$, $\quad \mathbf{G} \in L^\infty(I;\mathbf{Y}_m)$ and $\mathbf{y}(0) \in \mathbf{Y}_m$. Then*

$$||\mathbf{y}||_{Y_m,T} + \left\|\frac{d\mathbf{y}}{dt}\right\|_{Y_{m-1},T} \le C_T, \tag{6.66}$$

with $\mathbf{y}$ a classical solution of $(\mathcal{TP})$. $C_T = C_T(D,\Omega,m,M,T,\alpha,||\mathbf{y}(0)||_{Y_m},||\mathbf{G}||_{Y_m,T})$, and to be more precise $C_T = (K_T+\gamma+C)D+D^2$, where K_T depends on the initial data.

Proof
From the boundedness of $\mathbf{y}^{[n]}$ in $\mathbf{L}^\infty(I;\mathbf{Y}_m)$ (see (6.55)) we have $\mathbf{y}^{[n]} \rightharpoonup^* \mathbf{y}$ in $\mathbf{L}^\infty(I;\mathbf{Y}_m)$ and by the same argument and Lemma 6.7, $\frac{d\mathbf{y}^{[n]}}{dt} \rightharpoonup^* \frac{d\mathbf{y}}{dt}$.
Thus

$$C \geq \text{lim inf } ||\mathbf{y}^{[n]}||_{Y_m} \geq ||\mathbf{y}||_{Y_m}$$

and

$$C \geq \text{lim inf } \left\|\frac{d\mathbf{y}}{dt}^{[n]}\right\|_{Y_{m-1}} \geq \left\|\frac{d\mathbf{y}}{dt}\right\|_{Y_{m-1}}.$$

†

Proposition 6.4 *Assume that*

$$\mathbf{v} \in L^\infty(I, \mathbf{X}_{m+2}), \quad m \geq 3,$$

$$\mathbf{G} \in L^\infty(I, \mathbf{Y}_m) \ \ \textit{with} \ \ ||\mathbf{G}||_{Y_m,T} \leq \gamma D,$$

$$\mathbf{y}|_o = \mathbf{y}(0) \in \mathbf{Y}_m \ \ \textit{with} \ \ ||\mathbf{y}(0)||_m \leq \beta D$$

where $C, \gamma, D \geq 0$ and $\beta < 1$. Then, if

$$T = \frac{1}{c}\, ln\, \Big[\frac{K_T^2 D^2 - C_2 D}{CC_1[\beta^2 D^2 + C\gamma^2 D^2]}\Big]. \tag{6.67}$$

with c a suitable constant depending only on Ω, m and α and $t \in [0,T]$, the solution $\mathbf{y}$ *determined in Proposition* 6.3 *satisfies*

$$||\mathbf{y}||_{Y_m,T} \leq K_T D.$$

where K_T depends on the initial data.

This proposition is proved in Appendix III, Sect 6.3.1. Thus the proof of Theorem 6.5 is complete.

6.6 The Existence of a Classical Solution.

We shall prove existence by linking the solutions of $(\mathcal{TP})$ and solutions of $(\mathcal{SP})$. That means that in the transport problem

$$\left.\begin{array}{l}\partial_t \mathbf{y} + P\mathbf{N}_v\mathbf{y} = \mathbf{G} \\ \mathbf{y}(0) = \mathbf{y}_o\end{array}\right\} \tag{6.68}$$

$\mathbf{G} = P\langle \mathbf{S}(\mathbf{v}), s^*(\eta)\rangle$ with $\mathbf{S}$ and s^* defined in (6.7), $\mathbf{v}$ is thought of as a solution of the Stokes problem, and that in the Stokes problem

$$\left.\begin{array}{l}\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v} + \rho^{-1/2}\nabla p = \rho^{1/2}\mathbf{w} = \mathbf{y}_1 \;\text{ in } \Omega\times(0,T) \\ \delta(\rho^{1/2}\eta_v - \alpha\rho^{-1/2}\Delta_s\eta_v) + \rho^{-1/2}\gamma_o p = -\rho^{1/2}\mathbf{n}\cdot\gamma_o\mathbf{w} = y_2 \;\text{ on } \Gamma\times(0,T),\end{array}\right\}$$

$\mathbf{y} = \langle \mathbf{y}_1, y_2\rangle$ is a solution of the transport problem.

In this section we show that (6.68) with initial data $\mathbf{y}(x,0) = \mathbf{y}(0) = \langle \mathbf{y}_{1,o}, y_{2,o}\rangle \in \mathbf{Y}_m$, where

$$\begin{array}{rcl}\mathbf{y}_{1,o} &=& \lim_{t\to 0^+}\{\rho^{1/2}\mathbf{v}(t) - \alpha\rho^{-1/2}\Delta\mathbf{v}(t)\} + \rho^{-1/2}\nabla p|_{t=0} \\ y_{2,1} &=& \lim_{t\to 0^+}\{\delta(\rho^{1/2}\eta_v - \alpha\rho^{-1/2}\Delta_s\eta_v(t))\} + \rho^{-1/2}\gamma_o p|_{t=0},\end{array}$$

admits at least one classical solution in $I = [0,T]$ for all $t \geq 0$ in I and T sufficiently small depending on the size of the initial data.

Our method is to consider the composed mapping $\mathbf{\Phi} : \mathbf{v} \to \mathbf{y} \to \mathbf{v}^*$, and to prove that it has a fixed point. Although $\mathbf{\Phi}$ is well–defined we do not know if the $(\mathcal{TP})$ is uniquely solvable, therefore the mapping $\mathbf{\Phi}$ is *a priori* multi–valued, because of the mapping $\mathbf{v} \to \mathbf{y}$ which can be many–valued. So we have to resort to results concerning fixed points of such mappings. The one we shall use is the Bohnenblust–Karlin theorem (see $[B_5]$ and $[Z_1]$, Corollary 9.8, p. 542).

We have to define some concepts:

Definition: Fixed Point: A multi–valued map $\mathbf{\Phi}$ has a fixed point in a non–empty set G, if the set $\mathbf{\Phi}(\phi)$ is non–empty for all $\phi \in G$ and there exists $\phi \in G$ such that $\phi \in \mathbf{\Phi}(\phi)$.

Definition: Upper semi-continuity: If $\{\phi_n\} \subset G$ and $\phi_n \to \phi$ in G, if $\wp_n \in \mathbf{\Phi}(\phi_n)$ and $\wp_n \to \wp$ then $\wp \in \mathbf{\Phi}(\phi)$.

The following theorem gives sufficient conditions for the existence of a fixed point of a multi–valued map.

Theorem 6.6 Bohnenblust–Karlin Fixed-Point Theorem.
Suppose that

(i) G is a non–empty closed, convex subset of a Banach space X;

(ii) the map $\mathbf{\Phi} : G \subset X \to 2^G$ is upper semi-continuous.

(iii) the set

$$\bigcup_{\phi \in G} \mathbf{\Phi}(\phi)$$

is relatively compact [K_1];

(iv) the set $\mathbf{\Phi}(\phi)$ is non–empty, closed, and convex for all $\phi \in G$.

Then there exists $\bar{\phi} \in G$ such that $\bar{\phi} \in \mathbf{\Phi}(\bar{\phi})$.

Note that 2^G denotes the set of all subsets of G.

In the process of verifying the conditions of above theorem we shall need the Ascoli-Arzelà theorem in the following form:

Theorem 6.7 The Ascoli-Arzelà Theorem.
Let (K, d) be a compact metric space and $\mathcal{H}$ a bounded subset of $C(K, X)$, with X a Banach space. Assume that $\mathcal{H}$ is uniformly equicontinuous, that is, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \quad d(x_1, x_2) < \delta \quad \text{then} \quad |f(x_1) - f(x_2)| < \varepsilon \quad \text{for} \quad \text{all} \quad f \in \mathcal{H}.$$

Then $\mathcal{H}$ is relatively compact in $C(K, X)$.

See [B_5].

We define the multi–valued mapping $g : G \to Y_y$ by:

$$g(\mathbf{v}) = \{y : y \;\; \text{satisfies} \;\; (\mathcal{TP}) \;\; \text{for} \;\; \text{given} \;\; \mathbf{v} \in G\}$$

and fixed initial state $\mathbf{y}(0) \in \mathbf{Y}_m$

We prove that g is *upper semi-continuous:*

Lemma 6.9 *The multi–valued mapping $g : \mathbf{X}_{m+2} \to \mathbf{Y}_m$ for fixed $m \geq 3$, is upper semi-continuous. I*

Proof.
We let $\mathbf{y} \in g(\mathbf{v})$ and $\mathbf{y}_n \in g(\mathbf{v}_n)$ be solutions of $(\mathcal{TP})$. Subtract the equations written for $\mathbf{y}$ and $\mathbf{y}_n$, set $(\mathbf{y} - \mathbf{y}_n) = \tilde{\mathbf{y}}$, and $\mathbf{G}(\langle \mathbf{v}, \eta \rangle) - \mathbf{G}(\langle \mathbf{v}_n, \eta_n \rangle) = \widetilde{\mathbf{G}}$, with corresponding data $\langle \mathbf{v}, \eta \rangle$ and $\langle \mathbf{v}_n, \eta_n \rangle$. $(\mathcal{TP})$ now becomes

$$\frac{d}{dt}\tilde{\mathbf{y}} + P\mathbf{N}_{v_o}\tilde{\mathbf{y}} = \widetilde{\mathbf{G}}.$$

Take the inner product with $\tilde{\mathbf{y}}$ in $\mathbf{Y}_o$ to obtain

$$\frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{y}}||^2_{Y_o} = (\widetilde{\mathbf{G}}, \tilde{\mathbf{y}})_{Y_o} + (P\mathbf{N}_{(v-v_n)}\mathbf{y}_n, \tilde{\mathbf{y}})_{Y_o}.$$

The Schwarz inequality, and some additional calculations, yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{y}}||^2_{Y_o} &\leq C||\widetilde{\mathbf{G}}||_{Y_m}||\tilde{\mathbf{y}}||_{Y_o} + C||\mathbf{v}_o - \mathbf{v}_{o,n}||_m||\mathbf{y}_1||_1||\tilde{\mathbf{y}}||_{Y_o} \\ &\leq C||\widetilde{\mathbf{G}}||_{Y_m}||\tilde{\mathbf{y}}||_{Y_o} + C||\mathbf{v} - \mathbf{v}_n||_m||\mathbf{y}_1||_1||\tilde{\mathbf{y}}||_{Y_o} \\ &\leq C||\tilde{\mathbf{y}}||^2_{Y_o} + C[||\widetilde{\mathbf{G}}||^2_{Y_m} + ||\mathbf{v} - \mathbf{v}_n||^2_m]. \end{aligned}$$

After applying Gronwall's method, and the fact that $\tilde{\mathbf{y}}(0) = 0$ we deduce for all $t \in I$ the estimate

$$||\tilde{\mathbf{y}}||^2_{Y_o} \leq C\int_o^t e^{C(T-s)}\,[||\widetilde{\mathbf{G}}||^2_{Y_m} + ||\mathbf{v} - \mathbf{v}_n||^2_m]\,ds.$$

Apply the estimates (6.18) from Lemma 6.1 and the result follows. †

We define the following:

As Banach space X we choose

$$X = X(I, \Omega, m) = C(I; \mathbf{X}_{m+2})$$

with the norm

$$||\phi||_X = ||\phi||_{X_{m+2},T},$$

and for $D \geq 0$, we set

$$\begin{aligned} G &= G(T, \Omega, m, D) \\ &= \{\mathbf{v} \in X : \mathbf{v} \in L^\infty(I; \mathbf{X}_{m+2}) \text{ with } ||\mathbf{v}||_{X_{m+2},T} \leq D\}. \end{aligned}$$

The following figure illustrates the map $\mathbf{\Phi}$.

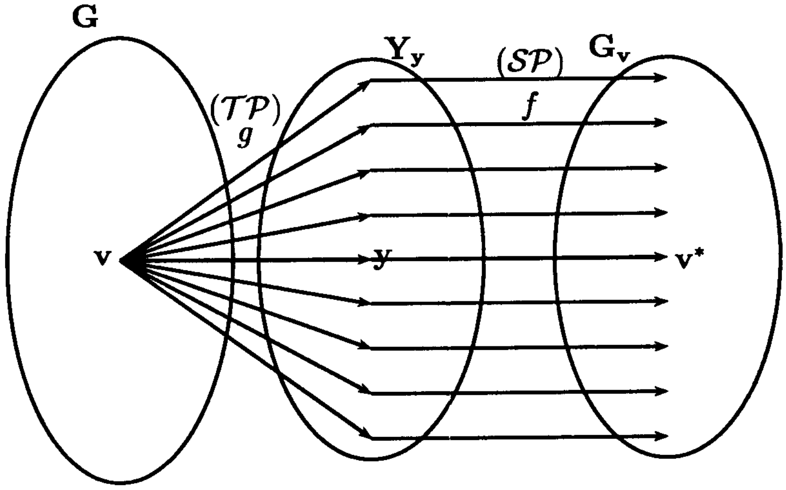

FIGURE 4. Profile as an illustration how the mapping $\mathbf{\Phi}$ is multi-valued.

Let $\langle \mathbf{v}, \eta \rangle$ denote the solution of $(\mathcal{SP})$ with $\mathbf{y} = \langle \mathbf{y}_1, y_2 \rangle$ on the right hand side, where $\mathbf{y}$ is the solution of $(\mathcal{TP})$. We denote the correspondence by $\langle \mathbf{v}, \eta \rangle = f(\langle \mathbf{y}_1, y_2 \rangle$. We note that for a given $\mathbf{y} \in Y_y$, $\langle \mathbf{v}^*, \eta^* \rangle \in G_v$ is the uniquely determined solution of $(\mathcal{SP})$ while

$$G_v = \{\mathbf{v}^* \in \mathbf{L}^\infty(I, \mathbf{X}_{m+2}) : \mathbf{v}^* \text{ solves } (\mathcal{SP}) \text{ corresponding to } \langle \mathbf{v}^*, \eta^* \rangle = f(\langle \mathbf{y}_1, y_2 \rangle)\}.$$

The following two estimates are essential in showing that $\mathbf{\Phi}$ satisfies all assumptions of Theorem 6.6:

From (6.66), (6.32) and from Proposition 6.2, we obtained the estimates

$$\left. \begin{array}{l} ||\mathbf{y}||_{Y_m,T} \leq K_T D, \quad \left\| \frac{d\mathbf{y}}{dt} \right\|_{Y_{m-1},T} \leq CD + \gamma D + D^2, \\ ||\mathbf{v}||_{\mathcal{X}_{m+2},T} \leq C\{||\mathbf{y}_1||_{\mathcal{X}_m,T} + ||y_2||_{H^{m-1/2}(\Gamma),T}\}. \end{array} \right\} \tag{6.69}$$

The theorem of Bohnenblust and Karlin will be applied in the following way:

$\mathbf{\Phi}$ is a composite set–valued mapping and can be denoted by $\mathbf{\Phi} = f \circ g$, where

$$\left. \begin{array}{l} g : \mathbf{v} \in G \to \langle \mathbf{v}, \eta \rangle \to \langle \mathbf{y}_1, y_2 \rangle \in Y_y \\ f : \langle \mathbf{y}_1, y_2 \rangle \in Y_y \to \langle \mathbf{v}^*, \eta^* \rangle \to \mathbf{v}^* \in G_v. \end{array} \right\}$$

We show that $\mathbf{\Phi}$ satisfies all assumptions of Theorem 6.6:

(i) From the definition of G condition (i) is verified.

(ii) From Lemma 6.9 we have that g is upper semi-continuous and from Theorem 6.3 we have the continuity of f, therefore $\mathbf{\Phi} = f \circ g$ is upper semi-continuous.

(iii) Consider the well-known compact imbedding operator $J : \mathbf{v} \in \mathbf{X}_{m+2} \to \mathbf{v} \in \mathbf{X}_{m+1}$. We define

$$B' = J\ B\ = J\{\mathbf{v} \in \mathbf{X}_{m+2} : ||\mathbf{v}||_{m+2} \leq D\} \subset \mathbf{X}_{m+1}$$

Lemma 6.10 *B' is a compact metric space.*

Proof.
We need to prove the following:

- B' is equentially compact.
- B' is closed.

For $\mathbf{v}'_n \in B'$ there exists a $\mathbf{v}_n \in B$ such that $\mathbf{v}'_n = J\mathbf{v}_n$. $\{\mathbf{v}_n\}$ is bounded, therefore there exist a subsequence $\mathbf{v}_{n'} \rightharpoonup \mathbf{v}$ in $\mathbf{X}_{m+2}$. J is compact, therefore $J\mathbf{v}_{n'} \to J\mathbf{v}$ in $\mathbf{X}_{m+1}$, which implies $\mathbf{v}'_{n'} \to \mathbf{v}'$ in B'. Hence B' is sequentially compact.

Suppose that $\mathbf{v}'_n \in B'$ and $\mathbf{v}'_n \to \mathbf{v}'$ in $\mathbf{X}_{m+1}$, and suppose $\mathbf{v}'_n = J\mathbf{v}_n$, and $\{\mathbf{v}_n\}$ is bounded in $\mathbf{X}_{m+2}$ which implies that there exist a subsequence such that $\mathbf{v}_n \rightharpoonup \mathbf{v}$ in $\mathbf{X}_{m+2}$, and $\mathbf{v} \in B$ because $D \geq \liminf ||\mathbf{v}_{n'}||_{m+2} \geq ||\mathbf{v}||_{m+2}$ and $J\mathbf{v}_{n'} \to J\mathbf{v}$ in $\mathbf{X}_{m+1}$, thus $\mathbf{v}'_{n'} \to J\mathbf{v} = \mathbf{v}'$, and this concludes that B' is closed. †

For $D \geq 0$ we set

$$G' = \{\mathbf{v} \in B' : \quad ||\mathbf{v}||_{m+1,T} \leq D\}$$

Lemma 6.11

$$\mathcal{G}' := \bigcup_{v \in G'} G'_v = \bigcup_{v \in G'} \mathbf{\Phi}(G')$$

is relatively compact in $C(B', X)$.

Proof.
We use the Ascoli-Arzelà Theorem to prove that G' is relatively compact: First we prove that G' is equicontinuous:

Assume $\mathbf{v}(t) \in L^\infty(I, G')$, where $I = [0, T]$ and G' is a bounded subset of $C(B', X)$. From the Mean Value Theorem there exists a $\xi \in (s,t) \subset I$ such that for $\mathbf{v} \in G'$

$$\mathbf{v}(t) - \mathbf{v}(s) = \int_s^t \frac{d\mathbf{v}(\tau)}{d\tau}\, d\tau = (t-s)\frac{d\mathbf{v}(\xi)}{dt},$$

thus

$$\|\mathbf{v}(t)-\mathbf{v}(s)\|_{X_{m+1}} = |t-s|\left\|\frac{d\mathbf{v}(\xi)}{dt}\right\|_{X_{m+1}} \leq |t-s|\left\|\frac{d\mathbf{y}(\xi)}{dt}\right\|_{Y_{m-1}} \leq |t-s|C_T.$$

From Proposition 6.2. 0 It remains to prove that $\mathcal{G}' := \bigcup_{v \in G'} G'_v$ is precompact. Suppose $\mathbf{\Phi}(G') \subset G'$ (which is proved in (iv)) then $\mathbf{\Phi}(G') \to \mathcal{G}' \subset G'$. G' is compact, Therefore $\mathcal{G}' := \bigcup_{v \in G'} G'_v$ is precompact in $C(B', X)$. †

(iv) Via the Galerkin method we proved that there exist at least one solution for the $(\mathcal{TP})$, and we proved uniqueness for the $(\mathcal{SP})$. This proves that $\mathbf{\Phi}(\phi)$ is non-empty.

Lemma 6.12 *The set $\mathbf{\Phi}(\phi)$ is closed and convex for all $\phi \in G$.*

Proof.
Let $\mathbf{v}_n^* \in G_v$ with $\mathbf{v}_n^* \to \mathbf{v}^*$ in $C[I, \mathbf{X}_{m+2}]$, $m \geq 3$.
There exists a $\mathbf{y}_n \in Y_y$ such that $\mathbf{v}_n^* = f(\mathbf{y}_n)$; where the function f represents the solution of the Stokes problem. There exist therefore a $\mathbf{y}$ such that $\mathbf{y}_n \to \mathbf{y}$ in $\mathbf{Y}_{m+2}$ with $\mathbf{v}^* = f(\mathbf{y})$.

Also there exists a $\mathbf{v} \in G$ such that $\mathbf{y}_n = g(\mathbf{v})$ and $lim_{n\to\infty}\mathbf{y}_n = lim_{n\to\infty} g(\mathbf{v}) = g(\mathbf{v})$. Therefore $\mathbf{y} \in Y_y$ because of the semi–continuity of g and it follows that $\mathbf{v}^* \in G_v$.

For the convexity of G_v take $\mathbf{v}^*$ and $\mathbf{w}^* \in G_v$ and let

$$\mathbf{y}^* = \alpha \mathbf{y}_{v^*} + (1-\alpha)\mathbf{y}_{w^*}$$

where $f(\mathbf{v}^*) = \mathbf{y}_{v^*}$ and $f(\mathbf{w}^*) = \mathbf{y}_{w^*}$ and $0 < \alpha < 1$.
Also there exist a $\mathbf{v} \in G_v$ such that

$$g(\mathbf{v}) = \mathbf{y}_{v^*} \quad \text{and} \quad g(\mathbf{v}) = \mathbf{y}_{w^*}.$$

Now $\alpha \mathbf{y}_{v^*} + (1-\alpha)\mathbf{y}_{w^*} = \alpha g(\mathbf{v}) + (1-\alpha) g(\mathbf{v}) = g(\mathbf{v})$, and it follows that $f(\mathbf{y}^*) = f(g(\mathbf{v})) \in G_v$. †

According to Proposition 6.2 we then have :

$$\begin{aligned} \|g(\langle \mathbf{v}, \eta\rangle)\|_{Y_m,T} &= \|\langle \mathbf{y}_1, y_2\rangle\|_{Y_m,T} \\ &\leq K_T D \end{aligned}$$

Also from $f(\mathbf{y}) = \langle \mathbf{v}^* \eta^* \rangle$, Proposition 6.2 and the Trace Theorem, we deduce

$$\begin{aligned} \|f(\mathbf{y})\|^2_{Y_m} = \|\langle \mathbf{v}^*, \eta^*\rangle\|^2_{Y_m} &\leq \sup_{t\in I} \|\mathbf{v}^*\|^2_m + \sup_{t\in I} \|\eta^*\|^2_{H^{m-1/2}(\Gamma)} \\ &\leq \sup_{t\in I} (C(\|\mathbf{y}_1\|^2_{m-2} + \|y_2\|^2_{H^{m-1/2}(\Gamma)})) \\ &\leq C\|\mathbf{y}\|^2_{Y_m,T} \end{aligned}$$

Now

$$\|\mathbf{\Phi}\langle \mathbf{v}, \eta\rangle\|_{Y_m} = \|\langle \mathbf{v}^*, \eta^*\rangle\|_{Y_m} \leq C K_T D.$$

To prove that $\mathbf{\Phi}(G) \subset G$ we need a K_T such that $CK_T \leq 1$. From the definition of K_T this is possible if T is sufficiently small.

Since $G_v \subset G$ for all $\mathbf{v} \in G$, we have that $\mathbf{\Phi}$ has values in 2^G and that there exists a $\bar{\phi} \in G$ such that $\bar{\phi} \in \mathbf{\Phi}(\bar{\phi})$.

We can now conclude that $\mathbf{\Phi}$ has a fixed point in $C(I, G')$ and $G' \subset G$. We have thus proved the following existence theorem for classical solutions.

Theorem 6.8 Local Existence of a Classical Solution.
Let Ω be a bounded domain in $\mathbf{R}^3$, $m \geq 3$, Γ be of class C^∞ then the initial value problem

$$\left.\begin{aligned} \partial_t \left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] + (\mathbf{v}\cdot\nabla)\left[\rho^{1/2}\mathbf{v} - \alpha\rho^{-1/2}\Delta\mathbf{v}\right] & \\ +\rho^{-1/2}\nabla p = \mathbf{S}(\mathbf{v}) \ \text{ in } \Omega \times (0,T) & \\ \mathbf{v} = 0 \ \text{ on } \ \Gamma_1 \times (0,T) & \\ \gamma_0 \mathbf{v} = -\eta_v \mathbf{n} = -\eta\mathbf{n} \ \text{ on } \Gamma \times (0,T) & \\ \gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta M \quad \text{on} \quad \Gamma \times (0,T) & \\ \partial_t \delta(\rho^{1/2}\eta_v - \alpha\rho^{-1/2}\Delta_s\eta_v) + \rho^{-1/2}\gamma_0 p = s^*(\eta), \quad \text{on } \Gamma & \end{aligned}\right\} \tag{6.70}$$

with

$$\left.\begin{aligned} \mathbf{S}(\mathbf{v}) = \rho^{-1/2}[\tfrac{\alpha}{2}\nabla\cdot[\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})] + & \\ \alpha\nabla\cdot(\nabla\mathbf{v}\mathbf{A}(\mathbf{v})) + \mu\Delta\mathbf{v}] & \\ s^*(\eta) = \delta\rho^{-1/2}\mu\Delta_s\eta_v, & \end{aligned}\right\} \tag{6.71}$$

and with initial data

$$\begin{aligned}\mathbf{y}_{1,o} &= \lim_{t\to 0^+}\{\rho^{1/2}\mathbf{v}(t) - \alpha\rho^{-1/2}\Delta\mathbf{v}(t)\} + \rho^{-1/2}\nabla p|_{t=0} \quad \text{in } \mathbf{X}_m \\ y_{2,o} &= \lim_{t\to 0^+}\{\rho^{1/2}\eta_v(t) - \alpha\rho^{-1/2}\Delta_s\eta_v(t)\} + \rho^{-1/2}\gamma_0 p|_{t=0} \quad \text{in } H^{m-1/2}(\Gamma),\end{aligned}$$

and under the assumption that $p_2 \in (0, 1/2)$ *has at least one classical solution local in time in* $\mathbf{X}_{m+2}$ *provided that* $\mathbf{y}_{1,0} \in \mathbf{X}_m$ *and* $y_{2,0} \in H^{m-1/2}(\Gamma)$.

Proof.

The multi–valued mapping $\mathbf{\Phi}$ has a fixed point in G. That means there exists $\mathbf{v} \in \mathbf{\Phi}(\mathbf{v})$, which in turn means that there is a solution $\mathbf{y}$ of $(\mathcal{TP})$ such that the solution of $(\mathcal{SP})$ for this $\mathbf{y}$ as data, is again $\mathbf{v}$.

To prove that the fixed point $\mathbf{v}$ is indeed a classical solution we consider the following:

If we start with $\mathbf{v} \in G$ then $\mathbf{u}_1$ is the solution of the $(\mathcal{TP})$

$$\frac{d}{dt}P\langle \mathbf{u}_1, -\gamma_o(\mathbf{u}_1)\cdot\mathbf{n}\rangle + P\mathbf{N}_v\mathbf{u}_1 = \mathbf{G}(\mathbf{v}) \tag{6.72}$$

where the projection $P : \langle \rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o p\rangle \in L^2(\Omega) \times L^2(\Gamma) \to \langle 0, 0\rangle$, and

$$\left.\begin{aligned}\mathbf{u}_1 &= \rho^{1/2}\alpha\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v} \\ -\gamma_o\mathbf{u}_1\cdot\mathbf{n} &= \rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v.\end{aligned}\right\} \tag{6.73}$$

Substitute (6.73) into (6.72) and use the fact that the projection P is linear to obtain

$$P\left[\frac{d}{dt}\begin{pmatrix}\rho^{1/2}\alpha\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v} \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v)\end{pmatrix} + \begin{pmatrix}(\mathbf{v}\cdot\nabla)(\rho^{1/2}\alpha\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v}) \\ 0\end{pmatrix}\right] = \begin{pmatrix}0\\0\end{pmatrix}.$$

The result is that there exist a $\mathbf{v}$ and a unique p such that

$$\frac{d}{dt}\begin{pmatrix}\rho^{1/2}\alpha\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v} \\ \delta(\rho^{1/2}\eta_v - \rho^{-1/2}\alpha\Delta_s\eta_v)\end{pmatrix} + \begin{pmatrix}(\mathbf{v}\cdot\nabla)(\rho^{1/2}\alpha\mathbf{v} - \rho^{-1/2}\alpha\Delta\mathbf{v}) \\ 0\end{pmatrix} = \begin{pmatrix}\rho^{-1/2}\nabla p \\ \rho^{-1/2}\gamma_o p\end{pmatrix}.$$

The proof is complete. †

Chapter 7

Appendices

7.1 Appendix I: Expressions at the Interface.

In order to obtain expressions for the stress tensors $\mathbf{T}$ and $\mathbf{T}'$ as well as the acceleration at the boundary through which only normal flow occurs, we obtain a formal expression for the symmetric tensor $\mathbf{A}$ on a surface which is immersed in fluid. We shall eventually use these expressions in postulating the form of $\mathbf{T}$ and $\mathbf{T}'$ and in formulating a boundary condition which expresses zero tangential acceleration at a wall.

We consider a smooth vector field $\mathbf{v}(\mathbf{x})$ defined on a domain $\Omega \subset \mathbf{R}^3$ and a smooth two - dimensional (at least class C^2) manifold $\Gamma \subset \Omega$ so that $\mathbf{v}$ and $\nabla\mathbf{v}$ are defined on Γ. Let $\mathbf{n}(\mathbf{x})$ be the unit normal to Γ at the point $\mathbf{x} \in \Gamma$.

At any point $\mathbf{x}$ on Γ we consider two orthogonal curves c_1 and c_2 in a neighbourhood of x parametrized by arc length s_1 and s_2 respectively. Let $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ be the unit tangents to the principal normal curves at a point on the surface. For local coordinates we use the orthogonal system formed by $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$ and $\mathbf{n}$. Under the convention that

$$\boldsymbol{\tau_1} \wedge \boldsymbol{\tau_2} = \mathbf{n}$$

we have

$$\mathbf{n} \wedge \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$$

and

$$\mathbf{n} \wedge \boldsymbol{\tau}_2 = -\boldsymbol{\tau}_1.$$

Let κ_1 and κ_2 represent the principal curvatures at a point on the surface and let $K = \kappa_1 + \kappa_2$ denote twice the mean curvature. The Frenet-Serret

$[S_3, W_1, W_2]$ formulae in this case is then, providing that there is no torsion are

$$\begin{array}{rclrcl} \frac{\partial \mathbf{n}}{\partial s_1} &=& -\kappa_1 \boldsymbol{\tau}_1 & \frac{\partial \mathbf{n}}{\partial s_2} &=& -\kappa_2 \boldsymbol{\tau}_2 \\ \frac{\partial \boldsymbol{\tau}_1}{\partial s_1} &=& \kappa_1 \mathbf{n} & \frac{\partial \boldsymbol{\tau}_1}{\partial s_2} &=& 0 \\ \frac{\partial \boldsymbol{\tau}_2}{\partial s_1} &=& 0 & \frac{\partial \boldsymbol{\tau}_2}{\partial s_2} &=& \kappa_2 \mathbf{n}. \end{array}$$

The surface gradient ∇_s of a scalar function f may be written as

$$\gamma_o(\nabla f) = \nabla_s(\gamma_o f) + \mathbf{n}\gamma_1 f, \tag{7.1}$$

where the trace operator γ_1 denotes the normal derivative. Also consider

$$\nabla_s f = \frac{\partial f}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial f}{\partial s_2}\boldsymbol{\tau}_2$$

and

$$\Delta_s f = \nabla_s \cdot (\nabla_s f) = \frac{\partial^2 f}{\partial^2 s_1^2} + \frac{\partial^2 f}{\partial^2 s_2^2}.$$

If $\mathbf{f}$ is a vector field defined on Γ, the surface gradient ∇_s is defined as the tensor

$$\nabla_s \mathbf{f} = \frac{\partial \mathbf{f}}{\partial s_1} \otimes \boldsymbol{\tau}_1 + \frac{\partial \mathbf{f}}{\partial s_2} \otimes \boldsymbol{\tau}_2. \tag{7.2}$$

Surface divergence and surface curl are defined as

$$\nabla_s \cdot \mathbf{f} = \boldsymbol{\tau}_1 \cdot \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau}_2 \cdot \frac{\partial \mathbf{f}}{\partial s_2}, \tag{7.3}$$

$$\nabla_s \wedge \mathbf{f} = \boldsymbol{\tau}_1 \wedge \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau}_2 \wedge \frac{\partial \mathbf{f}}{\partial s_2}. \tag{7.4}$$

The relationship between the surface operators and the volume operators for a function defined in Ω is given by

$$\gamma_o(\nabla \mathbf{f}) = \nabla_s \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}] \otimes \mathbf{n}. \tag{7.5}$$

$$\gamma_o(\nabla \cdot \mathbf{f}) = \nabla_s \cdot \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}] \cdot \mathbf{n}, \tag{7.6}$$

$$\gamma_o(\nabla \wedge \mathbf{f}) = \nabla_s \wedge \gamma_o \mathbf{f} + \mathbf{n} \wedge \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}]. \tag{7.7}$$

We use (7.1) — (7.3) to prove more important results to make the calculations easier:

Lemma 7.1 *Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be two orthogonal unit tangential vectors and let* $\mathbf{n}$ *be the exterior unit normal vector to Γ. Let α, β and γ be scalar functions, then*

$$\begin{aligned} &\text{(i)} \quad \nabla_s \cdot (\alpha\boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1} \\ &\text{(ii)} \quad \nabla_s \cdot (\beta\boldsymbol{\tau}_2) = \frac{\partial \beta}{\partial s_2} \\ &\text{(iii)} \quad \nabla_s \cdot (\gamma\mathbf{n}) = -\gamma K \end{aligned}$$

Proof

(i) $\nabla_s \cdot (\alpha\boldsymbol{\tau}_1) = \boldsymbol{\tau}_1 \cdot \frac{\partial}{\partial s_1}(\alpha\boldsymbol{\tau}_1) + \boldsymbol{\tau}_2 \cdot \frac{\partial}{\partial s_2}(\alpha\boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1}$.

(ii) Similar to (i).

(iii) $\nabla_s \cdot (\gamma\mathbf{n}) = \boldsymbol{\tau}_1 \cdot [\frac{\partial \gamma}{\partial s_1}\mathbf{n} - \kappa_1\gamma\boldsymbol{\tau}_1] + \boldsymbol{\tau}_2 \cdot [\frac{\partial \gamma}{\partial s_2}\mathbf{n} - \gamma\kappa_2\boldsymbol{\tau}_2]$

$$= -\gamma[\kappa_1 + \kappa_2] = -\gamma K.$$

†

We shall apply the expressions above to $\mathbf{v}$.
By the Frenet-Serret formulae (torsion is zero)

$$\begin{aligned} \frac{\partial}{\partial s_1}(\gamma_o\mathbf{v}) &= -\frac{\partial \eta}{\partial s_1}\mathbf{n} - \eta\frac{\partial \mathbf{n}}{\partial s_1} \\ &= -\frac{\partial \eta}{\partial s_1}\mathbf{n} + \kappa_1\boldsymbol{\tau}_1\eta, \end{aligned}$$

and, similarly,

$$\frac{\partial \mathbf{v}}{\partial s_2} = -\frac{\partial \eta}{\partial s_2}\mathbf{n} + \kappa_2\boldsymbol{\tau}_2\eta.$$

Hence

$$\begin{aligned} \nabla_s\gamma_o\mathbf{v} &= \eta[\kappa_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] - [\frac{\partial \eta}{\partial s_1}\mathbf{n} \otimes \boldsymbol{\tau}_1 + \frac{\partial \eta}{\partial s_2}\mathbf{n} \otimes \boldsymbol{\tau}_2] \\ &= \eta[\kappa_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] - \mathbf{n} \otimes [\frac{\partial \eta}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial \eta}{\partial s_2}\boldsymbol{\tau}_2] \\ &= \eta[\kappa_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] - \mathbf{n} \otimes \nabla_s\eta. \end{aligned}$$

The transpose is given by

$$(\nabla_s\gamma_o\mathbf{v})^T = \eta[\kappa_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] - \nabla_s\eta \otimes \mathbf{n}.$$

To find an expression for $\mathbf{A}(\mathbf{v})$ at Γ we need to look at $\nabla\mathbf{v}$:

$$\gamma_o(\nabla\mathbf{v}) = \nabla_s(\gamma_o\mathbf{v}) + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\otimes\mathbf{n}. \tag{7.8}$$

Although we know that the divergence of $\mathbf{v}$ will be zero, it is helpful to observe that

$$\theta = \gamma_o(\nabla\cdot\mathbf{v}) = \nabla_s\cdot\gamma_o\mathbf{v} + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n},$$

where

$$\begin{aligned}
\nabla_s\cdot\gamma_o\mathbf{v} &= -\boldsymbol{\tau}_1\cdot\frac{\partial}{\partial s_1}(\eta\mathbf{n}) - \boldsymbol{\tau}_2\cdot\frac{\partial}{\partial s_2}(\eta\mathbf{n}) \\
&= -\boldsymbol{\tau}_1\cdot[\frac{\partial\eta}{\partial s_1}\mathbf{n} - \eta\kappa_1\boldsymbol{\tau}_1] - \boldsymbol{\tau}_2\cdot(\frac{\partial\eta}{\partial s_2}\mathbf{n} - \eta\kappa_2\boldsymbol{\tau}_2) \\
&= \eta(\kappa_1+\kappa_2) \\
&= \eta K.
\end{aligned}$$

And

$$\theta = \eta K + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n}.$$

We proceed to find expressions for $\gamma_o(\mathbf{v}\cdot\nabla)\mathbf{v}$, $\gamma_o(\nabla\mathbf{v})$ hence $\gamma_o(\nabla\mathbf{v})^T$:

We know that

$$\boldsymbol{\omega}\wedge\mathbf{n} = \mathbf{W}(\mathbf{v})\mathbf{n} = (\mathbf{n}\cdot\nabla)\mathbf{v} - (\nabla\mathbf{v})^T\mathbf{n},$$

and

$$\begin{aligned}
\gamma_o(\nabla\mathbf{v})^T\mathbf{n} &= (\nabla_s\gamma_o\mathbf{v})^T\mathbf{n} + \gamma_o([\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{v}]\mathbf{n}) \\
&= -(\nabla_s\eta) + (\theta - K\eta)\mathbf{n}.
\end{aligned}$$

Therefore,

$$(\mathbf{n}\cdot\nabla)\gamma_o\mathbf{v} = (\boldsymbol{\omega}\wedge\mathbf{n}) - \nabla_s\eta + (\theta - K\eta)\mathbf{n}. \tag{7.9}$$

Multiply (7.9) with $-\eta$ to obtain

$$\gamma_o(\mathbf{v}\cdot\nabla)\mathbf{v} = -\eta(\mathbf{n}\cdot\nabla)\gamma_o\mathbf{v} = K\eta^2\mathbf{n} + \eta\,[\nabla_s\eta - \boldsymbol{\omega}\wedge\mathbf{n}]. \tag{7.10}$$

From (7.8) we now have

$$\begin{aligned}
\gamma_o(\nabla\mathbf{v}) &= \nabla_s\mathbf{v} + [\boldsymbol{\omega}\wedge\mathbf{n} - \nabla_s\eta + (\theta - K\eta)\mathbf{n}]\otimes\mathbf{n} \\
&= \eta[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2] - \mathbf{n}\otimes\nabla_s\eta - \nabla_s\eta\otimes\mathbf{n} + (\boldsymbol{\omega}\wedge\mathbf{n})\otimes\mathbf{n} \\
&\quad +(\theta - K\eta)\mathbf{n}\otimes\mathbf{n} \\
&= \eta[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1 + \kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2 - K\mathbf{n}\otimes\mathbf{n}] - [\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}] \\
&\quad +(\boldsymbol{\omega}\wedge\mathbf{n})\otimes\mathbf{n} + \theta\mathbf{n}\otimes\mathbf{n}.
\end{aligned}$$

The transpose is

$$\gamma_o(\nabla \mathbf{v})^T = \eta[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}-K\mathbf{n}\otimes\mathbf{n}]-[\mathbf{n}\otimes\nabla_s\eta+\nabla_s\eta\otimes\mathbf{n}]+\mathbf{n}\otimes(\boldsymbol{\omega}\wedge\mathbf{n})+\theta\mathbf{n}\otimes\mathbf{n}.$$

Thus we have

$$\begin{aligned}\gamma_o(\mathbf{A}(\mathbf{v})) &= \gamma_o(\nabla \mathbf{v}) + \gamma_o(\nabla \mathbf{v})^T\\ &= \nabla_s\gamma_o\mathbf{v} + (\nabla_s\gamma_o\mathbf{v})^T + \gamma_o([(\mathbf{n}\cdot\nabla)\mathbf{v}]\otimes\mathbf{n}) + \gamma_o(\mathbf{n}\otimes[(\mathbf{n}\cdot\nabla)\mathbf{v}])\\ &= 2\eta[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2} - K\mathbf{n}\otimes\mathbf{n}] - 2[\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}]\\ &\quad +\mathbf{n}\otimes(\boldsymbol{\omega}\wedge\mathbf{n}) + (\boldsymbol{\omega}\wedge\mathbf{n})\otimes\mathbf{n}. + 2\theta\mathbf{n}\otimes\mathbf{n}. \qquad (7.11)\end{aligned}$$

It is important to note that we get an expression for the acceleration at the boundary from (7.10):

$$[-\partial_t\eta + K\eta^2]\mathbf{n} + \eta[\nabla_s\eta - \boldsymbol{\omega}\wedge\mathbf{n}]. \qquad (7.12)$$

Let us define the symmetrical tensors M and N by

$$\mathbf{M} = [K\mathbf{n}\otimes\mathbf{n} - (\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} - \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2})],$$

$$\begin{aligned}\mathbf{N} &= \mathbf{n}\otimes(\boldsymbol{\omega}\wedge\mathbf{n}) + (\boldsymbol{\omega}\wedge\mathbf{n})\otimes\mathbf{n} + 2\theta\mathbf{n}\otimes\mathbf{n} - 2(\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n})\\ &= \mathbf{n}\otimes[\boldsymbol{\omega}\wedge\mathbf{n} - 2\nabla_s\eta] + [\boldsymbol{\omega}\wedge\mathbf{n} - 2\nabla_s\eta]\otimes\mathbf{n} + 2\theta\mathbf{n}\otimes\mathbf{n}\\ &= \mathbf{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{n} - 2\theta\mathbf{n}\otimes\mathbf{n}\end{aligned}$$

with

$$\boldsymbol{\psi} = \boldsymbol{\omega}\wedge\mathbf{n} - 2\nabla_s\eta, \qquad (7.13)$$

a tangential vector. Then, for a vector field of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ, we have

$$\mathbf{A}(\mathbf{v}) = -2\eta\mathbf{M} + \mathbf{N} \quad \text{on } \Gamma. \qquad (7.14)$$

In local coordinates we have the representations

$$\mathbf{M} = \begin{pmatrix} -\kappa_1 & 0 & 0\\ 0 & -\kappa_2 & 0\\ 0 & 0 & K \end{pmatrix}, \qquad (7.15)$$

$$\mathrm{N} = \begin{pmatrix} 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau_1} \\ 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau_2} \\ \boldsymbol{\psi}\cdot\boldsymbol{\tau_1} & \boldsymbol{\psi}\cdot\boldsymbol{\tau_2} & -2\theta \end{pmatrix}. \tag{7.16}$$

If $\nabla\cdot\mathbf{v}=0$ it follows that $\mathrm{tr}\mathbf{A}=0$ which is in line with incompressibility.

We would further like to obtain expressions for the terms $\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}$ and $\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}$ on the boundary Γ:

Lemma 7.2 *(Problem ($\mathcal{P}$).) Let* $\mathbf{n}$ *be the exterior normal to the boundary* Γ, $\mathbf{v}\in\mathcal{D}$ *and* $\mathbf{A}=-2\eta\mathrm{M}+\mathrm{N}$ *with* M *and* N *as defined in* Section 2.4. *We assume that* $\nabla\cdot\mathbf{v}=0$ *and* $\boldsymbol{\omega}\wedge\mathbf{n}=2\nabla_s\eta$, *which imply that* $\mathrm{N}=0$, *then*

$$\begin{array}{ll} (a) & \gamma_o(-\mathbf{n}\cdot\Delta\mathbf{v}) = \Delta_s\eta \\ (b) & \gamma_o\,[\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}] = -4\eta K_G - \Delta_s\eta \\ (c) & \gamma_o\,[\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}] = 0, \end{array}$$

where K_G *denotes the Gauss-curvature.*

Proof

(a) We have chosen $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$ and $\mathbf{n}$ so that $\boldsymbol{\tau_1}\wedge\boldsymbol{\tau_2}=\mathbf{n}$. From incompressibility and the fact that there is zero tangential velocity:

$$\begin{aligned} -\Delta\mathbf{v} &= \nabla\wedge\boldsymbol{\omega} \\ &= \boldsymbol{\tau_1}\wedge\partial_{s_1}[\eta_1\boldsymbol{\tau_2}-\eta_2\boldsymbol{\tau_1}] + \boldsymbol{\tau_2}\wedge\partial_{s_2}[\eta_1\boldsymbol{\tau_2}-\eta_2\boldsymbol{\tau_1}] + \text{a tangential term} \\ &= \boldsymbol{\tau_1}\wedge[\partial_{s_1}\eta_1\boldsymbol{\tau_2}-\eta_2\kappa_1\mathbf{n}] + \boldsymbol{\tau_2}\wedge[\eta_1\kappa_2\mathbf{n}-\partial_{s_1}\eta_2\boldsymbol{\tau_1}] + \ldots \\ &= (\partial_{s_1}\eta_1+\partial_{s_2}\eta_2)\mathbf{n} + \ldots \\ &= \Delta_s\eta\mathbf{n} \end{aligned}$$

(b) Consider the tensor built from 'row vectors'

$$\mathbf{A} = \begin{pmatrix} \mathbf{Ae}_1 \\ \mathbf{Ae}_2 \\ \mathbf{Ae}_3 \end{pmatrix},$$

with $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ a basis for $\mathbf{R}^3$. Therefore

$$\nabla \cdot \mathbf{A} = \begin{pmatrix} \nabla \cdot \mathbf{A}\mathbf{e}_1 \\ \nabla \cdot \mathbf{A}\mathbf{e}_2 \\ \nabla \cdot \mathbf{A}\mathbf{e}_3 \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_s \cdot \mathbf{A}\mathbf{e}_1 \\ \nabla_s \cdot \mathbf{A}\mathbf{e}_2 \\ \nabla_s \cdot \mathbf{A}\mathbf{e}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{A}\mathbf{e}_1] \\ \mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{A}\mathbf{e}_2] \\ \mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{A}\mathbf{e}_3] \end{pmatrix}$$

$$= \nabla_s \cdot \mathbf{A} + [(\mathbf{n} \cdot \nabla)\mathbf{A}]\mathbf{n}.$$

Hence,

$$[(\mathbf{n} \cdot \nabla)\mathbf{A}]\mathbf{n} = \nabla \cdot \mathbf{A} - \nabla_s \cdot \mathbf{A}.$$

The value of $\mathbf{A}$ on Γ:

$$\begin{aligned} \gamma_o[\mathbf{A}(\mathbf{v})] &= -2\eta M \\ &= -2\eta[\kappa_1 \boldsymbol{\tau_1} \otimes \boldsymbol{\tau_1} + \kappa_2 \boldsymbol{\tau_2} \otimes \boldsymbol{\tau_2} - K\mathbf{n} \otimes \mathbf{n}]. \end{aligned}$$

Furthermore,

$$\nabla_s \cdot [\mathbf{A}(\mathbf{v})] = -2(M\nabla_s \eta + \eta \nabla_s \cdot M)$$

and now

$$\begin{aligned} \mathbf{n} \cdot [\nabla_s \cdot [\gamma_o \mathbf{A}]] &= -2(\mathbf{n} \cdot M\nabla_s \eta + \eta \mathbf{n} \cdot (\nabla_s \cdot M)) \\ &= -2\eta \mathbf{n} \cdot (\nabla_s \cdot M). \end{aligned}$$

(Here we used the fact that $M\mathbf{n} = -K\mathbf{n}$). Determine $\mathbf{n} \cdot [\nabla_s \cdot M]$ term by term to obtain

$$\begin{aligned} \mathbf{n} \cdot [\nabla_s \cdot [\gamma_o \mathbf{A}]] &= -2\eta(\kappa_1^2 + \kappa_2^2 - K^2) \\ &= -2\eta(2\kappa_1\kappa_2) = -4\kappa_1\kappa_2\eta \\ &= -4K_G\eta. \end{aligned}$$

K_G denotes the Gauss-curvature and is bounded by assumptions (3.1) and (3.2).

Hence

$$\mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{A}]\mathbf{n} = \mathbf{n} \cdot \Delta \mathbf{v} - 4\eta K_G = -4\eta K_G - \Delta_s \eta.$$

The term we use in the proof (2.19) is therefore

$$-\eta \mathbf{n} \cdot [(\mathbf{n} \cdot \nabla)\mathbf{A}]\mathbf{n} = +4\eta^2 K_G + \eta \Delta_s \eta.$$

(c) $\mathbf{n} \cdot (\mathbf{AW} - \mathbf{WA})\mathbf{n} = \mathbf{An} \cdot \mathbf{Wn} + \mathbf{Wn} \cdot \mathbf{An} = 2\mathbf{An} \cdot \mathbf{Wn}$.

Here we make use of the additional boundary conditions (2.10) and (2.16), and the fact that $\mathbf{Wn} = \boldsymbol{\omega} \wedge \mathbf{n}$ to obtain that

$$\begin{aligned} \mathbf{An} \cdot \mathbf{Wn} &= \mathbf{An} \cdot (\boldsymbol{\omega} \wedge \mathbf{n}) \\ &= (\boldsymbol{\omega} \wedge \mathbf{n}) \cdot [-2\eta K \mathbf{n}] \\ &= 0 \end{aligned}$$

†

7.2 Appendix II: Agmon, Douglis & Nirenberg $[A_3]$.

Proof of Proposition 6.1:
We write equation (6.10), after multiplying both sides with $\rho^{1/2}$, in the form

$$\begin{array}{cccccl} (\Delta - \frac{\rho}{\alpha})v_1 & \cdot & \cdot & -\frac{1}{\alpha}p_{,1} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_1 \\ \cdot & (\Delta - \frac{\rho}{\alpha})v_2 & \cdot & -\frac{1}{\alpha}p_{,2} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_2 \\ \cdot & \cdot & (\Delta - \frac{\rho}{\alpha})v_3 & -\frac{1}{\alpha}p_{,3} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_3 \\ v_{1,1} & +v_{2,2} & +v_{3,3} & & = & 0 \end{array}$$

Let $v_4 = -\frac{p}{\alpha}$ and $\mathbf{f} = (-\frac{\rho^{1/2}}{\alpha}(y_1)_1, -\frac{\rho^{1/2}}{\alpha}(y_1)_2, -\frac{\rho^{1/2}}{\alpha}(y_1)_3, 0)$.
Then this becomes

$$\sum_{j=1}^{4} \ell_{ij}(\boldsymbol{\partial})v_j(\mathbf{x}) = f_i(\mathbf{x}) \quad \text{in} \quad \Omega, \quad i = 1, 2, 3, 4$$

where $\boldsymbol{\partial} = (\partial_1, \partial_2, \partial_3)$ and the matrix $[\ell_{ij}(\boldsymbol{\xi})], \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, is given by

$$\begin{aligned} \ell_{ij}(\boldsymbol{\xi}) &= |\boldsymbol{\xi}|^2\delta_{ij} - \frac{\rho}{\alpha}, \quad |\boldsymbol{\xi}|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad i, j = 1, 2, 3, \\ \ell_{4j}(\boldsymbol{\xi}) &= -\ell_{j4}(\boldsymbol{\xi}) = \xi_j, \quad j = 1, 2, 3, \\ \ell_{4,4}(\boldsymbol{\xi}) &= 0. \end{aligned}$$

In accordance with the proof of Proposition 2.2 [T_3, *p*.34], we define two systems of weights by $s_1 = s_2 = s_3 = 0$, $s_4 = -1$, and $t_1 = t_2 = t_3 = 2$, $t_4 = 1$. Then $s_i \leq 0$ and degree$(\ell_{ij}(\boldsymbol{\xi})) \leq s_i + t_j$, as required by [$A_4$, *p*.38].

The matrix $[\ell'_{ij}(\boldsymbol{\xi})]$, where $\ell'_{ij}(\boldsymbol{\xi})$ consists of the terms in $\ell_{ij}(\boldsymbol{\xi})$ that are of order $s_i + t_j$ in $\boldsymbol{\xi}$, is identical to the corresponding matrix in [T_3]:

$$[\ell'_{ij}(\boldsymbol{\xi})] = \begin{bmatrix} |\boldsymbol{\xi}|^2 & 0 & 0 & -\xi_1 \\ 0 & |\boldsymbol{\xi}|^2 & 0 & -\xi_2 \\ 0 & 0 & |\boldsymbol{\xi}|^2 & -\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{bmatrix}$$

It is easily shown that $\mathcal{L}(\boldsymbol{\xi}) \equiv \det[\ell'_{ij}(\boldsymbol{\xi})] = |\boldsymbol{\xi}|^6$, so that $\mathcal{L}(\boldsymbol{\xi}) \neq 0$ for nonzero real $\boldsymbol{\xi}$, i.e. (6.10) is *elliptic*. Moreover , the ***supplementary condition on*** $\mathcal{L}$ is satisfied: $\mathcal{L}(\boldsymbol{\xi})$ is of even degree 6, and for every pair of linearly independent real vectors $\boldsymbol{\xi}, \boldsymbol{\xi}'$; in particular for each point $\mathbf{x}$ on Γ, $\boldsymbol{\xi}$ is a tangent and $\boldsymbol{\xi}'$ is a normal at $\mathbf{x}$; the polynomial $\mathcal{L}(\boldsymbol{\xi} + \tau\boldsymbol{\xi}')$ in τ has exactly 3 roots with positive imaginary part, namely $\tau^+(\boldsymbol{\xi}, \boldsymbol{\xi}') = i|\boldsymbol{\xi}|/|\boldsymbol{\xi}'|$:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\xi} + \tau\boldsymbol{\xi}') &= [(\boldsymbol{\xi} + \tau\boldsymbol{\xi}') \cdot (\boldsymbol{\xi} + \tau\boldsymbol{\xi}')]^3 \\ &= (|(\boldsymbol{\xi}|^2 + |\boldsymbol{\xi}'|^2\tau^2))^3 \\ &= |\boldsymbol{\xi}|^6(\tau - i|\boldsymbol{\xi}|/|\boldsymbol{\xi}'|)^6(\tau + i|\boldsymbol{\xi}|/|(\boldsymbol{\xi}'|)^6. \end{aligned}$$

Now concerning the boundary conditions we set

$$\begin{aligned} \gamma_o v_{1,3} + \gamma_o v_{3,1} &= 0 \\ \gamma_o v_{2,3} + \gamma_o v_{3,2} &= 0 \\ (\frac{\rho}{\alpha} - \Delta_s)v_3 + \gamma_o p &= y_2. \end{aligned}$$

These boundary conditions can be expressed as

$$[\mathcal{B}_{hj}(\mathbf{x}, \boldsymbol{\xi})] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 \\ 0 & 0 & \frac{rho}{\alpha} - \xi_3^2 & 1 \end{bmatrix}$$

in other words

$$\begin{aligned} \mathcal{B}_{hj} &= \xi_3\delta_{hj} \quad \text{for} \quad h = 1,2,3 \quad \text{and} \quad j = 1,2,3 \\ \mathcal{B}_{h3} &= \xi_h, \quad \text{for} \quad h = 1,2, \\ \mathcal{B}_{h4} &= 0, \quad \text{for} \quad h = 1,2 \\ \mathcal{B}_{34} &= 1 \\ \mathcal{B}_{33} &= \xi_3^2. \end{aligned}$$

Take $r_1 = r_2 = -1, r_3 = 0$ and $t_1 = t_2 = t_3 = 2$, then degree$(\mathcal{B}_{hj}) \leq r_h + t_j$ and $[\mathcal{B}'_{hj}] = [\mathcal{B}_{hj}]$, where $\mathcal{B}'_{hj}(\mathbf{x}, \boldsymbol{\xi})$ consists of the terms in $\mathcal{B}_{hj}(\mathbf{x}, \boldsymbol{\xi})$ that are of order $r_h + t_j$ in $\boldsymbol{\xi}$.

$$[\mathcal{B}'_{hj}(\mathbf{x},\boldsymbol{\xi})] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 \\ 0 & 0 & \xi_3^2 & 0 \end{bmatrix}$$

Now it remains to check the *Complementing Boundary Condition*: For an arbitrary $\mathbf{x} \in \Gamma$ let $\mathbf{n}$ denote the outward unit normal vector at $\mathbf{x}$, let $\boldsymbol{\xi}$ be any nonzero real tangent vector to Γ at $\mathbf{x}$ and define $\mathcal{L}^{jk}(\cdot) \equiv \ell'_{jk}(\cdot)$, $j,k = 1,2,3,4$. Then
$[\mathcal{B}'_{hj}(\boldsymbol{\xi}+\tau\mathbf{n})\mathcal{L}^{jk}(\boldsymbol{\xi}+\tau\mathbf{n})] =$

$$\begin{bmatrix} \tau[|\boldsymbol{\xi}|^2+\tau^2] & 0 & \xi_1[|\boldsymbol{\xi}|^2+\tau^2] & -2\xi_1\tau \\ 0 & \tau[|\boldsymbol{\xi}|^2+\tau^2] & \xi_2[|\boldsymbol{\xi}|^2+\tau^2] & -2\xi_2\tau \\ 0 & 0 & \tau^2[|\boldsymbol{\xi}|^2+\tau^2] & -\tau^3 \end{bmatrix}.$$

The rows of the latter matrix are required to be linearly independent modulo $\mathcal{M}^+$. Let $\tau^+ = \tau^+(\boldsymbol{\xi},\mathbf{n}) = i|\boldsymbol{\xi}|$, and set $\mathcal{M}^+ = (\tau-\tau^+)^3$ and suppose that $\boldsymbol{C} = (C_1, C_2, C_3)$ is a constant vector with the property that, as polynomials in τ,

$$\sum_{h=1}^{3} C_h\Big(\sum_{j=1}^{4}\mathcal{B}'_{hj}\mathcal{L}^{jk}\Big) \equiv 0 \pmod{\mathcal{M}^+}, \quad k = 1,2,3,$$

i.e.

$$\begin{aligned} C_1\tau(|\boldsymbol{\xi}|^2+\tau^2) &= 0 \\ C_2\tau(|\boldsymbol{\xi}|^2+\tau^2) &= 0 \\ C_1\xi_1(|\boldsymbol{\xi}|^2+\tau^2) + C_2\xi_2(|\boldsymbol{\xi}|^2+\tau^2) + C_3\tau^2(|\boldsymbol{\xi}|^2+\tau^2) &= 0. \end{aligned}$$

Now it is easy to verify that $\boldsymbol{C} = \mathbf{0}$, and that the *Complimenting Condition* holds.

We then apply Theorem 10.5, page 78 of $[A_3]$ in order to get the final result.(For a similar application see $[L_9]$.)

Proof of estimate (6.20) of Lemma 6.2

We are going to apply ADN to the Neumann problem (6.19)(see $[A_4]$, p 40). We introduce new variable $p_1 = \partial_1 p$, $p_2 = \partial_2 p$ and $p_3 = \partial_3 p$. In terms of these variables we reduce $(6.19)_1$ to the first order system

$$\left.\begin{aligned} \partial_1 p - p_1 &= 0 \\ \partial_2 p - p_2 &= 0 \\ \partial_3 p - p_3 &= 0 \\ \rho^{-1/2}\partial_1 p + \rho^{-1/2}\partial_2 p + \rho^{-1/2}\partial_3 p &= G \end{aligned}\right\} \qquad (7.17)$$

From (7.17) we obtain the following matrix:

$$[\ell_{ij}(\boldsymbol{\xi})] = \begin{bmatrix} 0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & -1 & 0 & 0 \\ \xi_2 & 0 & -1 & 0 \\ \xi_3 & 0 & 0 & -1 \end{bmatrix}$$

We define two systems of weights by $s_o = 0$, $s_1 = s_2 = s_3 = -1$ and $t_o = 2$, $t_1 = t_2 = t_3 = 1$. The matrix $[\ell'_{ij}(\boldsymbol{\xi})]$ consisting of the terms in $[\ell_{ij}(\boldsymbol{\xi})]$ that are of order $s_i + t_j$ in $\boldsymbol{\xi}$, is identical to the corresponding matrix $[\ell_{ij}(\boldsymbol{\xi})]$. It is easily shown that $\mathcal{L}(\boldsymbol{\xi}) \equiv det[\ell'_{ij}(\boldsymbol{\xi})] = \xi_1^2 + \xi_2^2 + \xi_3^2$, so that $\mathcal{L}(\boldsymbol{\xi}) \neq 0$ and is of even degree. Therefore the *supplementary condition on* $\mathcal{L}$ is satisfied.

Now concerning the boundary conditions we have the following conditions on Γ

$$\left.\begin{aligned} \partial_1 p - p_1 = 0 \\ \partial_2 p - p_2 = 0 \\ \partial_3 p - p_3 = 0 \\ \rho^{-1/2} p_3 + \rho^{-1/2} p = g, \end{aligned}\right\} \tag{7.18}$$

which can be expressed as

$$[\mathcal{B}_{hj}(\mathbf{x}, \boldsymbol{\xi})] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ \xi_1 & -1 & 0 & 0 \\ \xi_2 & 0 & -1 & 0 \\ \xi_3 & 0 & 0 & -1 \end{bmatrix}$$

Take $r_o = 0$, $r_1 = r_2 = r_3 = -1$ and $t_o = 2$, $t_1 = 1, t_2 = 1, t_3 = 0$, then

$$[\mathcal{B}'_{hj}(\mathbf{x}, \boldsymbol{\xi})] = \begin{bmatrix} 0 & -\xi_1 & -\xi_2 & 1 \\ \xi_1 & -1 & 0 & 0 \\ \xi_2 & 0 & -1 & 0 \\ \xi_3 & 0 & 0 & 0 \end{bmatrix}$$

Now it remains to check the *Complementing Boundary Condition*:

$[\mathcal{B}'_{hj}(\boldsymbol{\xi} + \tau\mathbf{n})\mathcal{L}^{jk}(\boldsymbol{\xi} + \tau\mathbf{n})] =$

$$\begin{bmatrix} -\tau & 0 & 0 & -1 \\ -\xi_1 & \xi_1^2 + 1 & \xi_1\xi_2 & \xi_1\tau \\ -\xi_2 & \xi_1\xi_2 & \xi_2^2 + 1 & \tau\xi_2 \\ -\tau & \tau\xi_1 & \tau\xi_2 & \tau^2 \end{bmatrix}.$$

The rows of the latter matrix are required to be linearly independent modulo $\mathcal{M}^+$, which is easily verified, therefore the *Complimenting Condition* holds.

We then apply Theorem 10.5, page 78 of [A_4]by choosing $\ell = m-1$ to obtain

$$||\nabla p||_m \leq ||p||_{m+1} \leq C[||G||_{X_{m-1}} + ||g||_{H^{m-3/2}(\Gamma)}].$$

(For a similar application see [L_9].)

7.3 Appendix III.

7.3.1 The Auxiliary Problems

Lemma 7.3 :**Formulation in Ω.** :
Let $\mathbf{v} \in \mathbf{X}_{m+2}$, then the equation of motion

$$D_t(\rho \mathbf{v}) = \nabla \cdot \mathbf{T} \quad in \ \ \Omega$$

is equivalent to

$$\partial_t \rho^{-1/2}(\rho \mathbf{v} - \alpha \Delta \mathbf{v}) + (\mathbf{v} \cdot \nabla)\rho^{-1/2}(\rho \mathbf{v} - \alpha \Delta \mathbf{v}) + \rho^{-1/2}\nabla p = S(\mathbf{v})$$

in Ω. With

$$S(\mathbf{v}) = \rho^{-1/2}\Big[\frac{\alpha}{2}\nabla \cdot [\mathbf{A}(\mathbf{v})\mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v})\mathbf{A}(\mathbf{v})] + \alpha \nabla \cdot (\nabla \mathbf{v} \mathbf{A}(\mathbf{v})) + \mu \Delta \mathbf{v}\Big].$$

Proof
Seeing that $\mathbf{A}$ and $\mathbf{W}$ are both functions of $\mathbf{v}$, we will write $\mathbf{A}$ instead of $\mathbf{A}(\mathbf{v})$ and $\mathbf{W}$ instead of $\mathbf{W}(\mathbf{v})$.

$$\begin{aligned} \nabla \cdot \mathbf{T} &= \nabla \cdot [-p\mathbf{I} + \mu \mathbf{A} + \alpha D_t \mathbf{A} + \frac{\alpha}{2}[\mathbf{AW} - \mathbf{WA}]] \\ &= -\nabla p + \mu \nabla \cdot \mathbf{A} + \alpha \nabla \cdot (\partial_t \mathbf{A} + (\mathbf{v} \cdot \nabla)\mathbf{A}) + \frac{\alpha}{2}\nabla \cdot [\mathbf{AW} - \mathbf{WA}]. \end{aligned}$$

The term

$$\begin{aligned} \alpha \nabla \cdot (\partial_t \mathbf{A} + (\mathbf{v} \cdot \nabla)\mathbf{A}) &= \alpha \nabla \cdot \partial_t \mathbf{A} + \alpha \nabla \cdot (\mathbf{v} \cdot \nabla)\mathbf{A} \\ &= \alpha \partial_t \Delta \mathbf{v} + \alpha(\partial_i v_k \partial_k A_{i,j}) \\ &= \alpha \partial_t \Delta \mathbf{v} + \alpha(\partial_i \partial_k v_k A_{i,j}) \\ &= \alpha \partial_t \Delta \mathbf{v} + \alpha(\partial_k \partial_i (v_k A_{i,j})) \\ &= \alpha \partial_t \Delta \mathbf{v} + \alpha(\partial_k \partial_i v_k A_{i,j}) + \alpha(\partial_k v_k \partial_i A_{i,j}) \\ &= \alpha \partial_t \Delta \mathbf{v} + \alpha(\mathbf{v} \cdot \nabla)\Delta \mathbf{v} + \alpha \nabla \cdot (\nabla \mathbf{v} \mathbf{A}). \\ &= \alpha D_t \Delta \mathbf{v} + \alpha \nabla \cdot (\nabla \mathbf{v} \mathbf{A}). \end{aligned}$$

And thus we obtain

$$D_t(\rho \mathbf{v}) - \alpha(D_t \Delta \mathbf{v}) - \mu \Delta \mathbf{v} + \nabla p = \frac{\alpha}{2}\nabla \cdot [\mathbf{AW} - \mathbf{WA}] + \alpha \nabla \cdot (\nabla \mathbf{v} \mathbf{A}).$$

Hence

$$\rho \partial_t \mathbf{v} - \alpha \partial_t \Delta \mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \alpha(\mathbf{v} \cdot \nabla)\Delta \mathbf{v} + \nabla p = S(\mathbf{v}).$$

†

Lemma 7.4 :Formulation on Γ for *Problem $\mathcal{P}_1$.*
The equation of motion

$$\sigma(x)\eta_t + k\eta^2 = \mathbf{n}\cdot\mathbf{Tn} - \ell(t)$$

on the boundary Γ is equivalent to

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta),$$

on Γ, With

$$s(\eta) = \sigma^{-1/2}[(-k + 4\alpha K_G)\eta^2 - 2\mu\eta K + \alpha\eta\Delta_s\eta - \ell(t)],$$

for Problem $\mathcal{P}_1$.

Proof.
Take $\mathbf{n}$ to be the unit exterior normal to Γ to get $\mathbf{n}\cdot(-pI)\mathbf{n} = -\gamma_o p$. Also bear in mind that $\mathbf{v} = -\eta\mathbf{n}$ on the boundary, then

$$\begin{aligned}\sigma(x)\eta_t + k\eta^2 &= \mathbf{n}\cdot\mathbf{Tn} - \ell(t)\\ &= -\gamma_o p + \mu(\mathbf{n}\cdot\mathbf{An}) + \alpha\mathbf{n}\cdot\partial_t\mathbf{An} + \alpha\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}\\ &\quad+\frac{\alpha}{2}\mathbf{n}\cdot[\mathbf{AW} - \mathbf{WA}]\mathbf{n} - \ell(t).\end{aligned}$$

We use (2.17) - (2.20) to obtain

$$\begin{aligned}\sigma(x)\eta_t + k\eta^2 &= -\gamma_o p - 2\mu\eta K - 2\alpha K\eta_t - \alpha\eta\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}\\ &\quad-\ell(t).\\ &= -\gamma_o p - 2\mu\eta K - 2\alpha K\eta_t + 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta - \ell(t)\end{aligned}$$

for *Problem* ($\mathcal{P}$). Then

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta).$$

†

The Proof of Proposition 6.4: *Assume that*

$$\mathbf{v} \in L^\infty(I, \mathbf{X}_{m+2}), \quad m \geq 3,$$

$$\mathbf{G} \in L^\infty(I, \mathbf{Y}_m) \;\; with \;\; \|\mathbf{G}\|_{m,T} \leq \gamma D,$$

$$\mathbf{y}(0) \in \mathbf{Y}_m \;\; with \;\; \|\mathbf{y}(0)\|_m \leq \beta D$$

where $C, \gamma, D \geq 0$ *and* $\beta < 1$. *Then, if*

$$T = \frac{1}{c}\, ln\, \Big[\frac{K_T^2 D^2 - C_2 D}{[\beta^2 D^2 + C\gamma^2 D^2]}\Big].$$

with c *a suitable constant depending only on* Ω, m *and* α *and* $t \in [0, T]$, *the solution* $\mathbf{y} \in \mathbf{Y}_m$, *determined in Proposition* 6.3 *satisfies*

$$||\mathbf{y}||_{Y_m,T} \leq K_T D.$$

Proof.
By the estimates (2.24), (3.1) and (6.54), the Trace Theorem and the definition of $\mathbf{y}$

$$\begin{aligned} ||\mathbf{y}||^2_{Y_m} &= ||\mathbf{y}_1||^2_{X_m} + ||y_2||^2_{H^{m-1/2}(\Gamma)} \\ &\leq ||\mathbf{y}_1||^2_{X_m} + C_1(\rho, \alpha)D \\ &\leq e^{cT}\Big[||\mathbf{y}(0)||^2_m + C||\mathbf{S}||^2_{L^\infty(I,X_m)}\Big] + C_1 D \\ &\leq e^{cT}\Big[\beta^2 D^2 + C\gamma^2 D^2\Big] + C_1 D \end{aligned}$$

To obtain a value for T we set

$$e^{cT}\Big[\beta^2 D^2 + C\gamma^2 D^2\Big] + C_2 D \leq K_T^2 D^2.$$

Solve for T to obtain the required result. †

7.3.2 The Eigenvalue Problem.

Let H_o and H_s be two Hilbert spaces with inner products and norms denoted by $(\ ,\ )_o; (\ ,\ )_s; ||\ \ ||_o; ||\ \ ||_s$ respectively. Weak and strong convergence in H_o and H_s respectively will be denoted by $\underset{o}{\rightharpoonup}, \underset{s}{\rightharpoonup}, \underset{o}{\longrightarrow}, \underset{s}{\longrightarrow}$.

We need the canonical operator $\mathbf{C}_o$ (see (6.2) – (6.4)) to be compact and invertible and the bilinear form b_m needs to be bounded, symmetric and $\mathbf{C}_o$ - *coercive* as it is defined in [S_1 p.9]. Thus:
The form b_m will be called $\mathbf{C}_o$ - *coercive* if there exist constants $\mu_o \geq 0$ and $\mu_1 > 0$ such that

$$b_m(u, u) \geq \mu_1 ||u||^2_m - \mu_o ||\mathbf{C}u||^2_o \quad \text{for} \quad \text{all} \quad u \in H_m.$$

If b_m is only *coercive* the reader is referred to $[H_1]$.

For the purpose of this study we work with a special bilinear form which is the inner product in $\mathbf{H}^m(\Omega)$ and is positive definite and will satisfy all the conditions for the eigenvalue problem to exist.

Lemma 7.5 *The operator* $\mathbf{C}_o$ *is compact.*

Proof
Ω is a bounded region with smooth boundary Γ. If $\{\mathbf{v}_n\}$ is a sequence in $\mathbf{X}_1$ and $\mathbf{v}_n \underset{T}{\rightharpoonup} 0$ in $\mathbf{H}^1(\Omega)$ then $\mathbf{v}_n \underset{o}{\longrightarrow} 0$ in $\mathbf{L}^2(\Omega)$ and $\gamma_o\mathbf{v} \underset{o}{\longrightarrow} 0$ in $L^2(\Gamma)$. (The embedding $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ is compact $[L_3,\ p.101]$).
Therefore $\eta_v \to 0$ in $L^2(\Gamma)$ and

$$\mathbf{C}_o\mathbf{v}_n = \langle \rho^{1/2}\mathbf{v}_n, \rho^{1/2}\eta_v\rangle \to \langle 0, 0\rangle \quad \text{in} \quad \mathbf{Y}_o.$$

†

We now have that the inner product of $\mathbf{H}^m$ is symmetrical and $\mathbf{C}_o$-*coersive* on $\mathbf{H}^m(\Omega)$ and that the linear operator $\mathbf{C}_o$ is compact and invertible. Therefore we can use the results from $[S_1]$ where the eigenvalues and eigenvectors are constructed according to a recursive scheme. Now we have a real number λ_j which is called the eigenvalue of $(\cdot,\cdot)_m$ and $\boldsymbol{\psi}_j \in \mathbf{H}^m(\Omega)$ the corresponding eigenvector such that

$$(\boldsymbol{\psi}_j, \boldsymbol{\phi})_m = \lambda_j(\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\phi})_{Y_o} \qquad \text{for all} \quad \boldsymbol{\phi} \in \mathbf{X}_m$$

with $\mathbf{C}_o\boldsymbol{\psi}_j$ and $\mathbf{C}_o\boldsymbol{\phi} \in \mathbf{Y}_m$.

7.3.3 The Helmholtz Projection.

We consider a canister filled with incompressible fluid, immersed in fluid of the same kind. It is assumed that the wall of the canister admits normal flow through it. Modelling of the situation leads to the following equations:

$$\left.\begin{array}{l} \rho^{1/2}\mathbf{v}_t + \rho^{1/2}(\mathbf{v}\cdot\nabla)\mathbf{v} = -\rho^{-1/2}\nabla p + \rho^{1/2}\nabla\cdot\mathbf{T} \ \text{ in } \ \Omega \\ \gamma_o\mathbf{v} = -\eta\mathbf{n} \ \text{ on } \ \Gamma; \quad \mathbf{v} = 0 \text{ on } \Gamma' \\ \sigma^{1/2}\eta_t = -\sigma^{-1/2}\gamma_o p + \sigma^{-1/2}\mathbf{n}\cdot\mathbf{T}\mathbf{n} - \sigma^{-1/2}\ell(t) \ \text{ at } \ \Gamma. \end{array}\right\} \tag{7.19}$$

Here $\sigma(x)$ represents a surface density of fluid particles which is assumed to be bounded below and above by positive numbers. The dynamical boundary condition as seen in (7.19) models the stress-induced forces at the boundary.

The projection we construct is designed to keep the pair $\langle \rho^{1/2}\mathbf{v}, \sigma^{1/2}\eta\rangle$ intact and the same time will eliminate the pressure couple $\langle \rho^{-1/2}\nabla p, \sigma^{-1/2}\gamma_o p\rangle$. To this end we let $H_2 = \mathbf{L}^2(\Omega) \times L^2(\Gamma)$, and define the operator M by $Mq = \mathbf{N}_p\langle \nabla q, \gamma_o q\rangle$ where

$$\mathbf{N}_p = \begin{pmatrix} \rho^{-1/2} I_3 & 0 \\ 0 & \sigma^{-1/2} \end{pmatrix}.$$

For a given $\mathbf{y} = \langle \mathbf{v}, \eta\rangle \in H_2$ we define a linear functional $M^*\mathbf{y} \in H_1^*$ such that

$$\begin{aligned} M^*\mathbf{y}(\phi) &= (\mathbf{y}, M\phi)_2 \\ &= (\mathbf{v}, \rho^{-1/2}\nabla\phi) + (\eta, \gamma_o\phi)_\Gamma. \end{aligned}$$

I_3 is the identity matrix in $\mathbf{R}^3$. The space H_1 will be the Sobolev space $H^1(\Omega)$. We consider the bilinear form b defined on $H_1 \times H_1$ by $b(q,\phi) = (Mq, M\phi)_2$. Since M is bounded b is bounded in the norm of H_1 and since $||M_q||_2^2 = \rho^{-1}||\nabla q||^2 + ||\sigma^{-1/2}\gamma_o q||_\Gamma^2$, it is readily seen that $||Mq||_2^2 = 0$ if and only if $q = 0$, which implies that *ker* M is trivial and that b is positive definite. Hence for $\mathbf{y} = \langle \mathbf{v}, \eta\rangle \in H_2$ the following functional equation has a unique solution q:

$$b(q,\phi) = (Mq, M\phi)_2 = (\mathbf{v}, \rho^{-1/2}\nabla\phi) + (\eta, \gamma_o\phi)_\Gamma. \tag{7.20}$$

Subsequently the function q is a weak solution for the following boundary value problem:

$$\left.\begin{aligned} &\Delta q = \rho^{1/2}\nabla\cdot\mathbf{v} \quad \text{on} \quad \Omega \\ &\rho^{-1}\gamma_1 q + \sigma^{-1}(x)\gamma_o q = \rho^{-1/2}\mathbf{n}\cdot\gamma_o\mathbf{v} + \sigma^{-1/2}(x)\eta \quad \text{on} \quad \Gamma. \end{aligned}\right\}$$

We now define a linear operator $P^\perp : H_2 \to H_2$ by $P^\perp\mathbf{y} = Mq = \langle \rho^{1/2}\nabla q, \sigma^{1/2}\gamma_o q\rangle$, where q is the solution of (7.20). $P^\perp$ is bounded, and if $\mathbf{y}$ is of the form Mp then $P^\perp\mathbf{y} = \mathbf{y}$. $P^\perp$ is indempotent and $||P|| \leq 1$, and is therefore the orthogonal projection on a closed subspace of H_2 that consists of all images under M of solutions of (7.20).

Let $P := I - P^\perp$ be the projection orthogonal to $P^\perp$. We see immediately that the range of P consists of all those $\mathbf{y} \in H_2$ for which $M^*\mathbf{y} = 0$. We thus have the following form of the Helmholtz decomposition:

Theorem 7.1 *Let* $\mathbf{v} \in \mathbf{H}^k$; $k \geq 1$ *and* $\eta \in H^{k-1/2}(\Gamma)$. *Then there exists a unique* $q \in H^{k+1}(\Omega)$ *and* $\mathbf{w} \in \mathbf{H}^k(\Omega)$ *such that*

$$\begin{aligned} &\mathbf{v} = \rho^{1/2}\mathbf{w} + \rho^{-1/2}\nabla q \\ &\eta = -\sigma^{1/2}\mathbf{n} \cdot \gamma_o \mathbf{w} + \sigma^{-1/2}\gamma_o q \\ &\nabla \cdot \mathbf{w} = 0 \end{aligned}$$

We refer the reader to $[S_2]$ for precise detail.

7.3.4 The Projection $P^{[n]}$.

The eigenvalue problem (6.36):

$$(\boldsymbol{\psi}_j, \boldsymbol{\phi})_m = \lambda_j(\mathbf{C}_o\boldsymbol{\psi}_j, \mathbf{C}_o\boldsymbol{\phi})_{Y_o}$$

We know that the eigenfunctions are orthogonal:

$$(\mathbf{C}_o\boldsymbol{\psi}_k, \mathbf{C}_o\boldsymbol{\psi})_{y_o} = \delta_{k,l}$$

hence

$$(\boldsymbol{\psi}_k, \boldsymbol{\psi}_l)_m = \lambda_k\delta_{k,l}.$$

This makes $\{\boldsymbol{\psi}_k\}$ m–orthogonal. Also

$$\mathbf{X}^{[n]} = span\{\boldsymbol{\psi}_1 \ldots \boldsymbol{\psi}_n\} \subset \mathbf{X_m}$$

and

$$\mathbf{Y}^{[n]} = span\{\mathbf{C}_o\boldsymbol{\psi}_1 \ldots \mathbf{C}_o\boldsymbol{\psi}_n\} \subset \mathbf{Y}_o.$$

Let $P^{[n]}$ be the projection from $\mathbf{Y}_o \to \mathbf{Y}^{[n]}$ and

$$\begin{aligned} P^{[n]}\mathbf{y} &= \Sigma_{k=1}^n(\mathbf{y}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\mathbf{C}_o\boldsymbol{\psi}_k \\ &= \Sigma_{k=1}^n(\mathbf{y}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\langle\rho^{1/2}\boldsymbol{\psi}_k, -\sigma^{1/2}\gamma_o\boldsymbol{\psi}_k\rangle \\ &= \langle P_1^{[n]}, P_2^{[n]}\rangle\mathbf{y}. \end{aligned}$$

Let $\mathbf{v} \in \mathbf{X}_m$, then $\mathbf{v}^{[n]} := \Sigma_{k=1}^n a_k\rho^{1/2}\boldsymbol{\psi}_k$. This must be the best approximation for $\mathbf{v} \in \mathbf{X}^{[n]}$ and $||\rho^{1/2}\mathbf{v} - \mathbf{v}^{[n]}||^2_{X_m}$ must be a minimum. Therefore

$$a_k = \frac{1}{\lambda_k}(\mathbf{v}, \boldsymbol{\psi}_k)_m = (\mathbf{C}_o\mathbf{v}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}.$$

Now we can consider

$$P_X^{[n]}\mathbf{v} = \Sigma_{k=1}^n(\mathbf{C}_o\mathbf{v}, \mathbf{C}_o\boldsymbol{\psi}_k)_{Y_o}\rho^{1/2}\boldsymbol{\psi}_k = P_1^{[n]}\mathbf{y}$$

where $\mathbf{y} = \mathbf{C}_o\mathbf{v}$.

Chapter 8

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