Finite element analysis of vibration models with interface conditions

by

Lizette Zietsman

Submitted in partial fulfilment of the requirements for the degree

Philosophiae Doctor

in the Faculty of Natural and Agricultural Sciences
University of Pretoria
Pretoria

June 2000

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Title	Finite element analysis of vibration models with interface conditions
Name	Lizette Zietsman
Supervisor	Dr N F J van Rensburg
Co-supervisor	Dr A J van der Merwe
Department	Mathematics and Applied Mathematics
Degree	Philosophiae Doctor

Synopsis

We are concerned with vibration models with interface conditions. Due to interaction or damage, one is confronted by interface conditions or dynamical boundary conditions instead of classical boundary conditions.

For the analysis as well as implementation of the finite element method, the model problems must be written in variational form. We found the process more manageable if we start with the equations of motion and the constitutive equations. Consequently, we formulate each problem specifying the equations of motion and constitutive equations separately. For the sake of completeness and comparison with the literature, the models are also given in terms of the displacement, i.e. a partial differential equation with boundary and interface conditions. Due to the fact that our problems are not standard, it is necessary to discuss these aspects in some detail.

Model problems with interface conditions is a relatively new subject, and we could not find adequate derivations of the variational form in standard references. Hence we found it necessary to present rigorous derivations.

For the finite element analysis it is necessary to consider product spaces. The basis of the finite dimensional subspace for the Galerkin approximation consists of ordered pairs or triplets of functions instead of ordinary functions.

Our main concern is error analysis. As a result finite element interpolation had to be adapted for product spaces. A projection operator is defined and the approximation error derived from the interpolation error.

We consider three typical problems: Equilibrium problem, Eigenvalue problem and Vibration problem. In each case we show that the convergence theory can be adapted for product spaces.

We also present two case studies namely the model problems for a damaged beam and plate beam. The finite element method is used to find approximations for equilibrium problems, eigenvalue problems and vibration problems. The results show that the method is highly effective and that the errors correspond to the predictions of the theory.

Titel Finite element analysis of vibration models with interface conditions
Naam Lizette Zietsman
Promotor Dr N F J van Rensburg
Mede-promotor Dr A J van der Merwe
Departement Wiskunde en Toegepaste Wiskunde
Graad Philosophiae Doctor

Samevatting

Die ondersoek handel oor vibrasie modelle met tussenvlakvoorwaardes. As gevolg van interaksie of skade ontstaan tussenvlakvoorwaardes of dinamiese randvoorwaardes in plaas van die klassieke randvoorwaardes.

Vir die analise sowel as implimentering van die eindige element metode, moet die wiskundige modelle in variasievorm geskryf word. Ons het gevind dat die proses meer hanteerbaar is as ons met die bewegingsvergelykings en die samestellingsvergelykings begin. Gevolglik het ons vir elke probleem die bewegingsvergelykings en die samestellingsvergelykings afsonderlik gegee. Vir volledigheid, en ook om met die literatuur te vergelyk, word elk van die modelle ook in terme van die verplasing gegee, naamlik 'n parsiële differensiaalvergelyking met randvoorwaardes en tussenvlakvoorwaardes. Aangesien die probleme nie standaard is nie, is dit nodig om van hierdie aspekte in besonderhede te bespreek.

Wiskundige modelle met tussenvlakvoorwaardes is 'n betreklik nuwe onderwerp en ons kon nie voldoende afleidings van die variasievorm in standaard bronne kry nie. Om hierdie rede het ons dit nodig geag om wiskundig korrekte afleidings aan te bied.

Produkruimtes is nodig vir die eindige element analise. Die basis van die eindig dimensionale deelruimte vir die Galerkin benadering bestaan uit geordende pare of geordende drietalle van funksies, in plaas van die gewone funksies.

Ons hoofbelangstelling is die foutanalise. Gevolglik moes eindige element interpolasie aangepas word vir produkruimtes. 'n Projeksie operator is gedefinieer en die benaderingsfout word afgelei in terme van die interpolasiefout.

Ons beskou drie tipiese probleme: Ewewigsprobleem, Eiewaardeprobleem en Vibrasieprobleem. In elk van die gevalle toon ons aan dat die konvergensieteorie aangepas kan word vir produkruimtes.

Ons bied ook twee gevalle studies aan, naamlik die wiskundige modelle vir die beskadigde balk en die plaat-balk. Die eindige element metode word gebruik om benaderings vir die ewewigsprobleem, eiewaardeprobleem en vibrasieprobleem te kry. Hierdie resultate toon dat die metode hoogs effektief is en dat die foute ooreenstem met die voorspellings uit die teorie.

Contents

Chapter 1

Introduction

As the title indicates, this thesis is about the application of the finite element method to vibration problems. We have in mind the implementation of the method (computation of approximations) as well as error analysis.

To be more specific, we are concerned with the vibration of beams and plates. Partial differential equations that model the vibration of beams and plates are classical topics. However, new mathematical problems appear from time to time. One reason is that mathematical models are changed to provide a more accurate description of reality. Another is that new situations arise in industrial applications.

The stabilization and control of beams and plates lead to model problems with non-standard boundary conditions. We give but three examples [BK], [CDKP] and [LL] from a vast literature.

Due to interaction or damage one is confronted by interface conditions or dynamical boundary conditions instead of classical boundary conditions. Our main examples are the vibration of a damaged beam and a plate beam model. These problems are presented in Chapter 2 with the necessary references.

It is necessary to adapt the finite element method to accommodate these problems. The derivation of the variational form is one aspect treated at length in Chapter 3.

For the analysis of the model problems it is necessary to consider product spaces. The basis of the finite dimensional subspace for the Galerkin approximation consists of ordered pairs or triple of functions instead of ordinary functions. As a result finite element interpolation had to be adapted for product spaces.

Our main concern is error analysis. In Chapter 5 we consider the three typical problems: Equilibrium problem, Eigenvalue problem and Vibration problem. In each case we show that the convergence theory can be adapted to product spaces.

To implement the method, it is necessary to adapt the basis functions to avoid the imposition of invalid constraints. To match the theory we rather used the basis elements mentioned earlier. The construction of these basis functions is discussed in Chapter 4.

In Chapters 6 and 7 we have two case studies. Here we consider the damaged beam and plate beam problem and demonstrate the implementation of the finite element method to find approximations for equilibrium problems, eigenvalue problems and vibration problems.

Chapter 2

Model problems

2.1 Introduction

In this chapter we present the one and two dimensional vibration models that form the basis of this study. The common feature in the models is the presence of interface conditions—although these interface conditions may occur for different reasons.

For the analysis as well as implementation of the finite element method, the model problems must be written in variational form. We found the process more manageable if we start with the equations of motion and the constitutive equations. Consequently, we formulate each problem specifying the equations of motion and constitutive equations separately. For the sake of completeness and comparison with the literature, the models are also given in terms of the displacement, i.e. a partial differential equation with boundary and interface conditions. Due to the fact that our problems are not standard, it is necessary to discuss these aspects in some detail.

We also write the model problems in dimensionless form to simplify numerical experimentation. It also facilitates the interpretation of numerical results.

In Sections 2.2 and 2.5 the equations of motion and the constitutive equations for a beam and a plate are discussed. This paves the way for the presentation of the model problems in Sections 2.3, 2.4 and 2.6. The models for a damaged beam in Section 2.3 and a plate beam system in Section 2.6 are the main topics. In Section 2.4 other beam models involving interface conditions are presented.

2.2 Motion of a beam

In this section we focus on the small transverse vibration of a beam modelled as a one dimensional continuum, i.e. the reference configuration is an interval on the real line. The beam has length ℓ, density ρ, cross sectional area A and area moment of inertia I. The position of point x at time t is denoted by $u(x,t)$. The shear force is denoted by F and the bending moment by M. P denotes an external lateral load on the beam, k_1 and k_2 are damping constants and E is Young's modulus. For our approach the equations of motion are important.

The equations of motion are given by

$$\rho A \partial_t^2 u = \partial_x F - k_1 \partial_t u + P \tag{2.2.1}$$

and

$$\rho I \partial_t^2 \partial_x u = F + \partial_x M. \tag{2.2.2}$$

A constitutive equation for M is required to complete the model:

$$M = EI\partial_x^2 u + k_2 I \partial_t \partial_x^2 u. \tag{2.2.3}$$

Viscous damping is included in the equation of motion (2.2.1) by the term $k_1\partial_t u$, and the Kelvin-Voigt damping as the term $k_2 I \partial_t \partial_x^2 u$, in the constitutive equation (2.2.3). The term $\rho I \partial_t^2 \partial_x u$ in the second equation of motion represents the angular momentum density of the cross section relative to the centroid. In the literature it is usually referred to as the rotary inertia term. See, for example, [I], [Fu] or [Se] for background on the modelling procedure.

The mathematical model is given by equations (2.2.1), (2.2.2) and (2.2.3).

Dimensionless form

Choose dimensionless variables $\xi = x/\ell$ and $\tau = t/T$ with T a chosen time which will be specified later. It follows that if $f(x,t) = g(\xi,\tau)$, then

$$\partial_t f = \partial_\tau g \frac{d\tau}{dt} = \frac{1}{T}\partial_\tau g,$$

and, similarly,

$$\partial_x f = \frac{1}{\ell}\partial_\xi g.$$

Introduce the following dimensionless quantities:

$$u^*(\xi,\tau) = \frac{u(x,t)}{\ell}, \quad F^*(\xi,\tau) = \frac{F(x,t)}{EA}, \quad M^*(\xi,\tau) = \frac{\ell M(x,t)}{EI},$$

$$P^*(\xi,\tau) = \frac{\ell^3 P(x,t)}{EI}, \ \lambda = \frac{k_1}{ET}, \ \mu = \frac{k_2}{ET}.$$

In terms of these dimensionless quantities (2.2.1), (2.2.2) and (2.2.3) become

$$\partial_\tau^2 u^* = \frac{ET^2}{\rho\ell^2}\partial_\xi F^* - \frac{ET^2}{\rho A}\lambda\partial_\tau u^* + \frac{EIT^2}{\rho A\ell^4}P^*, \tag{2.2.4}$$

$$\frac{\ell^2\rho}{ET^2}\partial_\tau^2\partial_\xi u^* = \frac{A\ell^2}{I}F^* + \partial_\xi M^*, \tag{2.2.5}$$

$$M^* = \partial_\xi^2 u^* + \mu\partial_\tau\partial_\xi^2 u^*. \tag{2.2.6}$$

We choose

$$T = \ell^2\sqrt{\frac{\rho A}{EI}},$$

and introduce dimensionless constants

$$r = \frac{\rho\ell^2}{ET^2} \text{ and } k = \frac{\lambda\ell^2}{rA}.$$

If we return to the original notation, i.e. use x and t for the spatial and time variables and u, F, M and P for the dimensionless quantities, the dimensionless form of the model is given by

$$\partial_t^2 u = \frac{1}{r}\partial_x F - k\partial_t u + P, \tag{2.2.7}$$

$$r\partial_t^2\partial_x u = \frac{1}{r}F + \partial_x M, \tag{2.2.8}$$

$$M = \partial_x^2 u + \mu\partial_t\partial_x^2 u. \tag{2.2.9}$$

These equations yield the following partial differential equation describing small transverse vibration of a beam in terms of the dimensionless displacement u:

$$\partial_t^2 u - r\partial_t^2\partial_x^2 u = -\partial_x^4 u - \mu\partial_t\partial_x^4 u - k\partial_t u + P.$$

In Section 2.6 we have the situation where a plate is supported by beams. In this case (2.2.2) must be modified to include a couple L to allow for the bending moment density transmitted to the beam by the plate. To obtain the dimensionless form we set

$$L^*(\xi,\tau) = \frac{\ell^2 L(x,t)}{EI}.$$

The second equation of motion (2.2.8) changes to

$$r\partial_t^2\partial_x u = \frac{1}{r}F + \partial_x M + L, \tag{2.2.10}$$

if we write L for L^*.

2.3 Model for a damaged beam

In this section we consider a model for small transverse vibration of a cantilever beam damaged at a single point. The model was proposed by Viljoen *et al* [VV]. See also [JVRV] for the model that includes Kelvin-Voigt damping. In this model the interface condition is due to the mathematical description of the damage.

We start with the equations, in dimensionless form, that describe the dynamical behaviour of an undamaged cantilever beam. The reference configuration is the interval $I = [0, 1]$ and the displacement of x at time t is denoted by $u(x, t)$.

From Section 2.2 the equations of motion are

$$\partial_t^2 u = \frac{1}{r}\partial_x F - k\partial_t u + P \qquad (2.3.1)$$

and

$$r\partial_t^2 \partial_x u = \frac{1}{r}F + \partial_x M. \qquad (2.3.2)$$

The constitutive equation is

$$M = \partial_x^2 u + \mu\partial_t\partial_x^2 u. \qquad (2.3.3)$$

For a cantilever beam the standard boundary conditions at the endpoints are

$$u(0, t) = \partial_x u(0, t) = 0, \qquad (2.3.4)$$

$$F(1, t) = M(1, t) = 0. \qquad (2.3.5)$$

Suppose now that we have a damaged beam with the damage located at a single point $x = \alpha$. (This is of course impossible but it is a convenient model for approximating the effect of damage.)

At $x = \alpha$ the following interface conditions are prescribed:

$$u(\alpha^+, t) = u(\alpha^-, t), \qquad (2.3.6)$$

$$F(\alpha^+, t) = F(\alpha^-, t), \qquad (2.3.7)$$

$$M(\alpha^+, t) = M(\alpha^-, t). \qquad (2.3.8)$$

Right and left limits are denoted by the superscripts $+$ and $-$.

Condition (2.3.6) specifies the continuity of the beam and (2.3.7) and (2.3.8) follow from the action-reaction principle for the shear force F and the bending moment M at $x = \alpha$.

The effect of the damage is modelled by the jump condition at $x = \alpha$:

$$\begin{aligned} M(\alpha, t) &= \frac{1}{\delta}\left(\partial_x u(\alpha^+, t) - \partial_x u(\alpha^-, t)\right) \\ &\quad + \frac{\mu}{\delta}\left(\partial_t \partial_x u(\alpha^+, t) - \partial_t \partial_x u(\alpha^-, t)\right). \end{aligned} \tag{2.3.9}$$

Right and left derivatives are denoted by the superscripts $+$ and $-$.

The jump condition (2.3.9) allows for discontinuities in the derivatives $\partial_x u$ and $\partial_t \partial_x u$ at $x = \alpha$. Note that the magnitude of δ indicates the extent of the damage and that $\delta = 0$ corresponds to a beam with no damage. It is clearly impossible to use $\delta = 0$ in this problem. However, our numerical experimentation in [ZVV] showed that solutions for relative small values of δ correspond to solutions of an undamaged beam. If a point force is applied at the free end of the beam it will result in an increase of the gradient. This increase, as a factor of the gradient, is exactly δ. The second term represents internal "friction".

The mathematical model is given by (2.3.1) to (2.3.9). In Section 3.1 we will derive a variational formulation for the problem from these equations.

In terms of the dimensionless displacement u, an equivalent form of the model is given on the next page.

Problem 1

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) \\
&\quad +P(x,t), \text{ for } 0 < x < 1,\ x \neq \alpha,\ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
\partial_x^2 u(1,t) + \mu\partial_t\partial_x^2 u(1,t) &= 0, \\
r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t), \\
u(\alpha^+,t) &= u(\alpha^-,t), \\
\partial_x^2 u(\alpha^+,t) + \mu\partial_t\partial_x^2 u(\alpha^+,t) &= \partial_x^2 u(\alpha^-,t) + \mu\partial_t\partial_x^2 u(\alpha^-,t), \\
r\partial_t^2\partial_x u(\alpha^+,t) - \partial_x^3 u(\alpha^+,t) & \\
-\mu\partial_t\partial_x^3 u(\alpha^+,t) &= r\partial_t^2\partial_x u(\alpha^-,t) - \partial_x^3 u(\alpha^-,t) \\
&\quad -\mu\partial_t\partial_x^3 u(\alpha^-,t), \\
\partial_x^2 u(\alpha,t) + \mu\partial_t\partial_x^2 u(\alpha,t) &= \frac{1}{\delta}\left(\partial_x u(\alpha^+,t) - \partial_x u(\alpha^-,t)\right) \\
&\quad +\frac{\mu}{\delta}\left(\partial_t\partial_x u(\alpha^+,t) - \partial_t\partial_x u(\alpha^-,t)\right).
\end{aligned}$$

Instead of a cantilever beam other boundary conditions can be considered. For example, if the beam is clamped at both ends we have

$$u(1,t) = \partial_x u(1,t) = 0 \tag{2.3.10}$$

instead of (2.3.5).

2.4 Beam models with dynamical boundary conditions

When some object of interest interacts with another object at some part of the boundary, standard boundary conditions are not applicable. The simplest case (which can be found in books on partial differential equations) is probably a rod or spring, executing longitudinal vibrations with a mass attached to one end. See [BST], [GV], [BI] and [V1] for examples of models of this type.

2.4.1 Tip body

In this section we consider a cantilever beam with a body, of mass m_B and moment of inertia I_B, attached to the free end at $x = \ell$. In this case the shear force and bending moment are no longer zero at $x = \ell$, but the following so called dynamical boundary conditions are prescribed.

$$m_B\partial_t^2 u(\ell,t) = -F(\ell,t), \tag{2.4.1}$$

$$I_B\partial_t^2\partial_x u(\ell,t) = -M(\ell,t). \tag{2.4.2}$$

We assume that the angle θ through which the tip body rotates can be approximated by $\partial_x u(\ell,t)$.

These boundary conditions are also converted into dimensionless form by using the dimensionless quantities introduced in Section 2.2:

$$\frac{\ell m_B}{T^2}\partial_\tau^2 u^*(1,\tau) = -EAF^*(1,\tau),$$

$$\frac{I_B}{T^2}\partial_\tau^2\partial_\xi u^*(1,\tau) = -\frac{EI}{\ell}M^*(1,\tau).$$

Choosing dimensionless mass m and moment of inertia I_m as

$$m = \frac{m_B\ell}{rEAT^2} = \frac{m_B}{\rho A\ell}$$

and

$$I_m = \frac{I_B\ell}{IET^2} = \frac{rI_B}{\rho\ell I}$$

and returning to the original notation, yield the dimensionless boundary conditions

$$rm\partial_t^2 u(1,t) = -F(1,t), \tag{2.4.3}$$

$$I_m\partial_t^2\partial_x u(1,t) = -M(1,t). \tag{2.4.4}$$

The mathematical model is given by the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), the standard boundary conditions at $x = 0$,

$$u(0,t) = \partial_x u(0,t) = 0, \tag{2.4.5}$$

and the dynamical boundary conditions (2.4.3) and (2.4.4).

In terms of the dimensionless displacement u, the mathematical model follows as:

Tip body problem

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\qquad \text{for } 0 < x < 1,\ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
m\partial_t^2 u(1,t) &= -r\partial_t^2\partial_x u(1,t) + \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t), \\
I_m\partial_t^2\partial_x u(1,t) &= -\partial_x^2 u(1,t) - \mu\partial_t\partial_x^2 u(1,t).
\end{aligned}$$

2.4.2 Boundary control

For a cantilever beam it is possible to suppress vibration by boundary feedback controls. See [C]. In this case the shear force and bending moment are not zero at $x = \ell$ and the situation is modelled by boundary feedback control conditions:

$$\begin{aligned}
F(\ell,t) &= -k_0\partial_t u(\ell,t), \\
M(\ell,t) &= -k_1\partial_t\partial_x u(\ell,t).
\end{aligned}$$

Choosing the dimensionless quantities

$$\mu_0 = \frac{k_0}{EAT\ell r} \text{ and } \mu_1 = \frac{k_1\ell}{EIT}$$

and returning to the original notation yields the dimensionless boundary conditions:

$$\begin{aligned}
F(1,t) &= -\mu_0\partial_t u(1,t), \\
M(1,t) &= -\mu_1\partial_t\partial_x u(1,t).
\end{aligned}$$

The mathematical model is given by these boundary conditions, the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), and the standard boundary conditions at $x = 0$ (2.4.5).

In terms of the dimensionless displacement u the model follows as:

Boundary damping problem

$$
\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\quad \text{for } \ 0 < x < 1, \ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t) - \mu_0\partial_t u(1,t), \\
\partial_x^2 u(1,t) + \mu\partial_t\partial_x^2 u(1,t) &= -\mu_1\partial_t u(1,t).
\end{aligned}
$$

2.4.3 General model

For theoretical purposes and without regard for the physical meaning of the model, we will formulate a generalization which contains both the two previous models as special cases. This general model will be used to derive a variational formulation.

The general model is given by the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), the standard boundary conditions at $x = 0$ (2.4.5), and the following dynamical boundary conditions

$$
\begin{aligned}
F(1,t) &= -\mu_0\partial_t u(1,t) - rm\partial_t^2 u(1,t), && (2.4.6) \\
M(1,t) &= -\mu_1\partial_t\partial_x u(1,t) - I_m\partial_t^2\partial_x u(1,t). && (2.4.7)
\end{aligned}
$$

By setting either μ_0 and μ_1 or m and I_m equal to zero, these boundary conditions reduce to the boundary conditions of the appropriate problem.

In terms of the dimensionless displacement u the model follows as:

Problem 2

$$
\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\quad \text{for } \ 0 < x < 1, \ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
m\partial_t^2 u(1,t) + r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t) - \mu_0\partial_t u(1,t), \\
I_m\partial_t^2\partial_x u(1,t) &= -\partial_x^2 u(1,t) - \mu\partial_t\partial_x^2 u(1,t) - \mu_1\partial_t u(1,t).
\end{aligned}
$$

2.5 Motion of a thin plate

When a plate interacts with a beam, one is confronted by more complicated dynamical boundary conditions than those encountered in Section 2.4. To prepare, we discuss the equations of motion and constitutive equation for a plate. We used [Fu], [TW], [Rei], [V1] and [V2].

Consider the transverse motion of a thin plate with density ρ and thickness h. The reference configuration for the plate is a domain Ω in the plane. The transverse displacement of $\mathbf{x} = (x_1, x_2)$ at time t is $u(\mathbf{x}, t)$. This means that the position $\mathbf{r}$ of $\mathbf{x}$ at time t is $\mathbf{r} = (x_1, x_2, u(\mathbf{x}, t))$.

The equations of motion are

$$\rho h \partial_t^2 u = \operatorname{div} \mathbf{T} + q$$

and

$$-R\partial_t \mathbf{H} = \mathbf{T} - R \operatorname{div} M$$

with $\mathbf{T}$ the contact force and q an external lateral load. $\mathbf{H}$ is the angular momentum density relative to the centroid and M the moment or contact couple. For more detail concerning the meaning of $\mathbf{T}$ and M see [Rei], [Fu] or [TW]. R and M are square matrices:

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Note that $R^{-1} = R^T = -R$.

The constitutive equation is given by

$$M = D \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}.$$

(See [Fu, p 461] and [TW, p 81].) D is a measure of stiffness for the plate given by

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

where E is Young's modulus and ν Poisson's ratio.

For small vibrations where shear is ignored, the angular momentum density $\mathbf{H}$ is given by

$$\mathbf{H} = \rho I \partial_t(\partial_2 u, -\partial_1 u)$$

where $I = h^3/12$ is the length moment of inertia.

Dimensionless form

We introduce the dimensionless variables: $\xi_1 = x_1/a$, $\xi_2 = x_2/a$ and $\tau = t/\eta$ where η will be specified later, and a is some typical length dimension of the plate.

We introduce the following dimensionless quantities:

$$u^*(\boldsymbol{\xi}, \tau) = \frac{u(\mathbf{x}, t)}{a},\ \mathbf{T}^*(\boldsymbol{\xi}, \tau) = \frac{I}{hD}\mathbf{T}(\mathbf{x}, t),\ \mathbf{H}^*(\boldsymbol{\xi}, \tau) = \frac{\eta}{\rho I}\mathbf{H}(\mathbf{x}, t),$$

$$q^*(\boldsymbol{\xi}, \tau) = \frac{a^3}{D}q(\mathbf{x}, t),\ M^*(\boldsymbol{\xi}, \tau) = \frac{a}{D}M(\mathbf{x}, t).$$

In terms of these quantities the equations of motion are given by

$$\begin{aligned} \partial_\tau^2 u^* &= \frac{\eta^2 D}{\rho a^2 I}\operatorname{div}\mathbf{T}^* + \frac{\eta^2 D}{\rho h a^4}q^*, \\ -\frac{\rho a^2 I}{D\eta^2}R\partial_\tau \mathbf{H}^* &= \frac{ha^2}{I}\mathbf{T}^* - R\operatorname{div} M^* \end{aligned}$$

with

$$\mathbf{H}^* = \partial_\tau(\partial_2 u^*, -\partial_1 u^*),$$

and the constitutive equation is given by

$$M^* = \begin{bmatrix} (1-\nu)\partial_1\partial_2 u^* & \partial_2^2 u^* + \nu\partial_1^2 u^* \\ -\partial_1^2 u^* - \nu\partial_2^2 u^* & -(1-\nu)\partial_1\partial_2 u^* \end{bmatrix}.$$

If we choose

$$\eta^2 = \rho a^4 h/D \text{ and } r = I/a^2 h,$$

and return to the original notation, the equations of motion are given by

$$\partial_t^2 u = \frac{1}{r}\operatorname{div}\mathbf{T} + q, \tag{2.5.1}$$

$$-r\,R\partial_t \mathbf{H} = \frac{1}{r}\mathbf{T} - R\operatorname{div} M \tag{2.5.2}$$

with

$$\mathbf{H} = \partial_t(\partial_2 u, -\partial_1 u), \tag{2.5.3}$$

and the constitutive equation by

$$M = \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}. \tag{2.5.4}$$

Substituting (2.5.2) into (2.5.1) yields

$$r \operatorname{div}(R\partial_t \mathbf{H}) + \partial_t^2 u = \operatorname{div}(R \operatorname{div} M) + q.$$

Substituting (2.5.3) here we obtain the partial differential equation

$$\partial_t^2 u - r\,\partial_t^2(\partial_1^2 u + \partial_2^2 u) = -\left(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u\right) + q. \tag{2.5.5}$$

2.6 Plate beam model

We consider a thin plate as in Section 2.5 which interacts with beams at the boundary. See [V1], [V2] and [ZVGV1]. The boundary $\partial\Omega$ consists of two parts, Σ and Γ. The section Σ is rigidly supported and the section Γ elastically supported by a beam. The end points of the beam are also rigidly supported. The orientation of the boundary $\partial\Omega$ is important. The domain Ω is on the "left" of the tangent. To be precise, we require that

$$\mathbf{n} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{\tau} = R\boldsymbol{\tau}$$

where $\mathbf{n}$ is the unit exterior normal and $\boldsymbol{\tau}$ is the unit tangent.

For the mathematical model of the plate we use the dimensionless equations of Section 2.5:

$$\partial_t^2 u = \frac{1}{r} \operatorname{div} \mathbf{T} + q, \tag{2.6.1}$$

$$-rR\partial_t \mathbf{H} = \frac{1}{r}\mathbf{T} - R \operatorname{div} M, \tag{2.6.2}$$

$$\mathbf{H} = \partial_t(\partial_2 u, -\partial_1 u), \tag{2.6.3}$$

$$M = \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}. \tag{2.6.4}$$

From Section 2.2 the equations of motion for the supporting beam, in dimensionless form and without damping terms, are given by

$$\partial_t^2 u = \frac{1}{r_b}\partial_s F + P, \tag{2.6.5}$$

and

$$r_b \partial_t^2 \partial_s u = \frac{1}{r_b} F + \partial_s M_b + L. \tag{2.6.6}$$

In applications $\partial_s = \pm\partial_1$ or $\partial_s = \pm\partial_2$ depending on the orientation of the beam. A subscript b will be used, where necessary, to indicate quantities associated with the beam.

The length of the beam is a. In terms of this notation the dimensionless quantities chosen in Section 2.2 are:

$$F^* = \frac{1}{E_b A}F,\ P^* = \frac{a^3}{E_b I_b}P,\ L^* = \frac{a^2}{E_b I_b}L,\ M_b^* = \frac{a}{E_b I_b}M_b.$$

For the rotary inertia constant r_b we have

$$r_b = \frac{a^2 \rho_b}{E_b T^2}.$$

In addition, we choose the following dimensionless constants:

$$\alpha = \frac{E_b I_b}{aD} \text{ and } \beta = \frac{\rho_b A}{\rho a h}.$$

It is necessary to adapt (2.6.5) and (2.6.6) to allow for the difference in dimensionless time scale, between the plate model and the beam model. The time derivatives have to be multiplied by a factor T/η. As $T^2/\eta^2 = \beta/\alpha$, (2.6.5) and (2.6.6) change to

$$\beta \partial_t^2 u = \frac{\alpha}{r_b} \partial_s F + \alpha P, \tag{2.6.7}$$

$$\beta r_b \partial_t^2 \partial_s u = \frac{\alpha}{r_b} F + \alpha \partial_s M_b + \alpha L. \tag{2.6.8}$$

The constitutive equation for the beam is:

$$M_b = \partial_s^2 u. \tag{2.6.9}$$

Next we formulate the interface conditions. In this case the force density P is the contact force density that the plate exerts on the beam. It follows that

$$P = -\mathbf{T} \cdot \mathbf{n}. \tag{2.6.10}$$

The moment density L is the moment that the plate exerts on the beam. This implies, for the moment density $-M\mathbf{n}$ of the plate on Γ, that

$$L = -M\mathbf{n} \cdot \mathbf{n}. \tag{2.6.11}$$

Since the plate is merely supported by the beam it also follows that

$$M\mathbf{n} \cdot \boldsymbol{\tau} = 0. \tag{2.6.12}$$

The interface conditions change to the following (again using the original notation to refer to the dimensionless quantities):

$$\alpha P = -\frac{1}{r} \mathbf{T} \cdot \mathbf{n}, \tag{2.6.13}$$

$$M\mathbf{n} \cdot \boldsymbol{\tau} = 0, \tag{2.6.14}$$

$$\alpha L = -M\mathbf{n} \cdot \mathbf{n}. \tag{2.6.15}$$

To complete the model we have to add the boundary conditions for the rigidly supported section Σ:

$$u = 0, \tag{2.6.16}$$
$$M\mathbf{n}\cdot\tau = 0; \tag{2.6.17}$$

and for the rigidly supported end points of the beam:

$$u = 0, \tag{2.6.18}$$
$$M_b = 0. \tag{2.6.19}$$

The mathematical model is given by (2.6.1) to (2.6.4), (2.6.7) to (2.6.9) and (2.6.13) to (2.6.19). Alternatively, the mathematical model can be given in terms of the dimensionless displacement u. We illustrate the procedure for obtaining the boundary conditions for a special domain Ω.

Consider a rectangular plate rigidly supported at two opposing sides and supported by two identical beams at the remaining sides. To find the dimensionless form for the model we choose a as the length of the supporting beams. Then the reference configuration Ω is the rectangle with $0 < x_1 < 1$ and $0 < x_2 < d$. Σ_0 and Σ_1 are those parts of the boundary where $x_1 = 0$ and $x_1 = 1$ respectively, and correspond to the rigidly supported parts of the boundary. Γ_0 and Γ_1 are those parts of the boundary where $x_2 = 0$ and $x_2 = d$ respectively, and correspond to the sections of the boundary supported by beams.

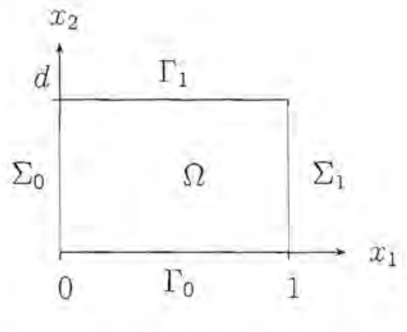

Figure 2.1: *Reference configuration of the plate.*

As in Section 2.5, the partial differential equation (2.5.5) is obtained.

On Σ_0 and Σ_1, (2.6.17) reduces to $M_{21} = 0$. The conditions at the end points of the beams (2.6.18) and (2.6.19) are included by extending the conditions on Σ_0 and Σ_1 to $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$.

On Γ_1, $\mathbf{n} = \mathbf{e}_2$ and $\boldsymbol{\tau} = -\mathbf{e}_1$. Using (2.6.2), we get

$$-\frac{1}{r}\mathbf{T} \cdot \mathbf{n} = -\frac{1}{r}T_2 = \partial_1 M_{11} + \partial_2 M_{12} - r\partial_t^2 \partial_2 u$$

and

$$M\mathbf{n} \cdot \boldsymbol{\tau} = M_{12} \text{ and } M\mathbf{n} \cdot \mathbf{n} = M_{22}.$$

Assuming that the beams are merely supporting the plate, we get

$$M_{12} = 0 \text{ and } L = -\frac{1}{\alpha}M_{22} \text{ on } \Gamma_1.$$

Similarly, on Γ_0, $\mathbf{n} = -\mathbf{e}_2$ and $\boldsymbol{\tau} = \mathbf{e}_1$. Hence

$$-\frac{1}{r}\mathbf{T} \cdot \mathbf{n} = \frac{1}{r}T_2 = -\partial_1 M_{11} - \partial_2 M_{12} + r\partial_t^2 \partial_2 u$$

and

$$M_{12} = 0 \text{ and } L = -\frac{1}{\alpha}M_{22}.$$

For the two beams, the first equation of motion (2.6.7) reduces to the following two equations. Note that $\partial_s = -\partial_1$ on Γ_1, and $\partial_s = \partial_1$ on Γ_0.

$$\begin{aligned} \beta\partial_t^2 u &= \frac{\alpha}{r_b}\partial_1 F - \partial_1 M_{11} - \partial_2 M_{12} + r\partial_t^2\partial_2 u \text{ on } \Gamma_0 \\ \beta\partial_t^2 u &= -\frac{\alpha}{r_b}\partial_1 F + \partial_1 M_{11} + \partial_2 M_{12} - r\partial_t^2\partial_2 u \text{ on } \Gamma_1. \end{aligned}$$

The second equation of motion (2.6.8) reduces to

$$\beta r_b \partial_t^2 \partial_1 u = \frac{\alpha}{r_b}F + \alpha\partial_1 M_b - M_{22} \text{ on } \Gamma_0$$
$$-\beta r_b \partial_t^2 \partial_1 u = \frac{\alpha}{r_b}F - \alpha\partial_1 M_b - M_{22} \text{ on } \Gamma_1.$$

Finally, (2.6.14) implies that

$$M_{12} = 0 \text{ on } \Gamma_0 \text{ and } \Gamma_1.$$

In terms of the dimensionless displacement u the model is given by:

Problem 4

$$
\begin{aligned}
\partial_t^2 u - r\partial_t^2(\partial_1^2 u + \partial_2^2 u) &= -\left(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u\right) + q \text{ in } \Omega, \\
u &= 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1, \\
\partial_1^2 u + \nu\partial_2^2 u &= 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1, \\
\partial_2^2 u + \nu\partial_1^2 u &= 0 \text{ on } \Gamma_0 \text{ and } \Gamma_1, \\
\beta\partial_t^2 u - \beta r_b\partial_t^2\partial_1^2 u - r\partial_t^2\partial_2 u &= -\partial_2^3 u - (2-\nu)\partial_1^2\partial_2 u - \alpha\partial_1^4 u \text{ on } \Gamma_0, \\
\beta\partial_t^2 u - \beta r_b\partial_t^2\partial_1^2 u + r\partial_t^2\partial_2 u &= \partial_2^3 u + (2-\nu)\partial_1^2\partial_2 u - \alpha\partial_1^4 u \text{ on } \Gamma_1.
\end{aligned}
$$

Chapter 3

Variational form and weak solutions

Our main concern is the analysis and implementation of the finite element method, i.e. Chapters 5, 6 and 7. For this we need the model problems in (weak) variational form. Model problems with interface conditions is a relatively new subject, and we could not find adequate derivations of the variational form in standard references. Hence we found it necessary to present rigorous derivations. Much of the material is from [V1], [V2], [VVZ], [ZVGV1] and [ZVGV2].

In each case we find a variational formulation as a first step. This is done by multiplying the equation of motion by an arbitrary smooth function, and integrating over the reference configuration. The variational form is sufficient for the implementation of the finite element method, but for existence theory and analysing the convergence of the finite element method, a weak formulation of the model problem is required. In each case we will define the necessary function spaces and present such a weak form. At this stage a unified approach will become possible as we show that the model problems all have similar weak forms.

In Section 3.4 we discuss existence results for weak solutions. The free response of a system is determined by the natural frequencies and natural modes. These are determined by the eigenvalues and eigenvectors of a bilinear form. This topic is treated in Section 3.5.

3.1 The damaged beam

3.1.1 Variational formulation

Multiplying the equation of motion (2.3.1) by an arbitrary function v and integrating gives

$$\int_0^1 \partial_t^2 u(\cdot,t)v = \frac{1}{r}\int_0^1 \partial_x F(\cdot,t)v - k\int_0^1 \partial_t u(\cdot,t)v + \int_0^1 P(\cdot,t)v. \quad (3.1.1)$$

We use the notation $u(\cdot,t)$ for the function

$$u(\cdot,t) : [0,1] \to \mathbb{R} \text{ with } u(\cdot,t)(x) = u(x,t).$$

As the jump condition allows for discontinuities in $\partial_x u$ and $\partial_t\partial_x u$ at $x = \alpha$, the integration must be performed separately on the subintervals $(0,\alpha)$ and $(\alpha,1)$. Due to the discontinuity of $\partial_x u(\cdot,t)$ at α, the function $\partial_x^2 u(\cdot,t)$ will not exist—not even in a generalized sense. (We exclude δ-functions.) For this reason it is necessary to consider product spaces with pairs of functions as elements. With each function u, we associate a pair $\bar{u} = \langle u_1, u_2\rangle$ with u_1 the restriction of u to the interval $[0,\alpha]$, and u_2 the restriction to $[\alpha,1]$. For simplicity of notation we will write u for $\bar{u}$.

For any open interval $I = (a,b)$ the function spaces $C^i(I)$, $C^i(\bar{I})$, $C_0^\infty(I)$ and $L^2(I)$ are defined in Appendix A.

Let $I = (0,1)$, $I_1 = (0,\alpha)$ and $I_2 = (\alpha,1)$. Define the following product spaces:

$$\begin{aligned} L^2 &:= L^2(I_1) \times L^2(I_2), \\ C^i &:= C^i(\bar{I}_1) \times C^i(\bar{I}_2),\ i = 0,1,\ldots, \\ C_0^\infty &:= C_0^\infty(I_1) \times C_0^\infty(I_2). \end{aligned}$$

In terms of this notation $\partial_x u(\alpha^-,t) = \partial_x u_1(\alpha,t)$ and $\partial_x u(\alpha^+,t) = \partial_x u_2(\alpha,t)$, etcetera.

Notation For any $u = \langle u_1,u_2\rangle \in L^2$ and any $v = \langle v_1,v_2\rangle \in L^2$,

$$(u,v)_0 := \int_0^\alpha u_1 v_1 + \int_\alpha^1 u_2 v_2.$$

We will also use the notation $u' := \langle u_1', u_2'\rangle$, etcetera.

In terms of the new notation, (3.1.1) can be written as

$$\left(\partial_t^2 u(\cdot,t),v\right)_0 = \left(\frac{1}{r}\partial_x F(\cdot,t),v\right)_0 - (k\partial_t u(\cdot,t),v)_0 + (P(\cdot,t),v)_0. \quad (3.1.2)$$

In the following results the term $\left(\frac{1}{r}\partial_x F(\cdot,t),v\right)_0$ is examined. For simplicity of notation we write F and u for $F(\cdot,t)$ and $u(\cdot,t)$ in the following results.

Lemma 3.1.1 *If the equation of motion* (2.3.2) *is satisfied, then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F,v\right)_0 &= -(M,v'')_0 - (r\partial_t^2\partial_x u,v')_0 + \left[\frac{1}{r}F_1 v_1\right]_0^\alpha + \left[\frac{1}{r}F_2 v_2\right]_\alpha^1 \\ &\quad + [M_1 v_1']_0^\alpha + [M_2 v_2']_\alpha^1 \text{ for all } v \in C^2.\end{aligned}$$

Proof The result is obtained by performing integration by parts twice. □

Define the space of test functions T as

$$T = \{v = \langle v_1, v_2\rangle \in C^2 : v_1(0) = v_1'(0) = 0,\ v_1(\alpha) = v_2(\alpha)\}.$$

Corollary 3.1.1 *Assume that the equation of motion* (2.3.2) *is satisfied. If, in addition, F and M satisfy the boundary conditions at $x = 1$,* (2.3.5)*, as well as the interface conditions at $x = \alpha$,* (2.3.6) *to* (2.3.8)*, then*

$$\left(\frac{1}{r}\partial_x F,v\right)_0 = -(M,v'')_0 - (r\partial_t^2\partial_x u,v')_0 - M(\alpha,t)\left(v_2'(\alpha) - v_1'(\alpha)\right)$$

for all $v \in T$

From the constitutive equation (2.3.3) and the jump condition (2.3.9), the term $\left(\frac{1}{r}\partial_x F,v\right)_0$ can be expressed in terms of u.

Corollary 3.1.2 *If u is a solution of Problem 1, then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F,v\right)_0 &= -(\partial_x^2 u,v'')_0 - (\mu\partial_t\partial_x^2 u,v'')_0 - (r\partial_t^2\partial_x u,v')_0 \\ &\quad -\frac{1}{\delta}\left(\partial_x u_2(\alpha,t) - \partial_x u_1(\alpha,t)\right)\left(v_2'(\alpha) - v_1'(\alpha)\right) \\ &\quad -\frac{\mu}{\delta}\left(\partial_x\partial_t u_2(\alpha,t) - \partial_x\partial_t u_1(\alpha,t)\right)\left(v_2'(\alpha) - v_1'(\alpha)\right)\end{aligned}$$

for all $v \in T$.

We define bilinear forms a, b and c by

$$\begin{aligned}
b(u,v) &:= (u'',v'')_0 + \frac{1}{\delta}\left(u_2'(\alpha)-u_1'(\alpha)\right)\left(v_2'(\alpha)-v_1'(\alpha)\right) \text{ for all } u,v\in C^2,\\
a(u,v) &:= \mu b(u,v) + (ku,v)_0 \text{ for all } u,v\in C^2,\\
c(u,v) &:= (u,v)_0 + (ru',v')_0 \text{ for all } u,v\in C^1.
\end{aligned}$$

The variational form of Problem 1 can be expressed in terms of these bilinear forms. In the following sections we will show that all the model problems can be reduced to the same abstract form if appropriate bilinear forms are introduced.

Problem 1b: Variational formulation

Find u such that, for all $t>0$, $u(\cdot,t)\in T$ and

$$c(\partial_t^2 u(\cdot,t),v) + a(\partial_t u(\cdot,t),v) + b(u(\cdot,t),v) = (P(\cdot,t),v)_0$$

for all $v\in T$.

Theorem 3.1.1 *If u is a solution of Problem 1, then u is a solution of Problem 1b.*

Proof The proof follows directly from substituting the result in Corollary 3.1.2 into (3.1.2). Note that if u is a solution of Problem 1, it follows from (2.3.4) and (2.3.6) that $u\in T$. □

Theorem 3.1.2 *If u is a solution of Problem 1b and $\partial_t u(\cdot,t)\in C^4$ and $\partial_t^2 u(\cdot,t)\in C^2$, then u is a solution of Problem 1.*

Proof For simplicity of notation we will write u for $u(\cdot,t)$ in this proof. Let $v\in T$ such that $v_1\in C_0^\infty(I_1)$ and $v_2=0$. Performing integration by parts, we find that

$$\int_0^\alpha (\partial_t^2 u_1 - r\partial_t^2\partial_x^2 u_1 + \partial_x^4 u_1 + \mu\partial_t\partial_x^4 u_1 + k\partial_t u_1 - P_1)v_1 = 0. \qquad (3.1.3)$$

Since $C_0^\infty(I_1)$ is dense in $L^2(I_1)$, it follows that u satisfies the partial differential equation on $I_1=(0,\alpha)$. The same is obviously true on $I_2=(\alpha,1)$.

A direct consequence is that

$$\begin{aligned}
&\int_0^\alpha (-r\partial_t^2\partial_x^2 u_1 + \partial_x^4 u_1 + \mu\partial_t\partial_x^4 u_1)v_1 + \int_\alpha^1 (-r\partial_t^2\partial_x^2 u_2 + \partial_x^4 u_2 + \mu\partial_t\partial_x^4 u_2)v_2\\
&= (r\partial_t^2\partial_x u, v')_0 + b(u,v) + \mu b(\partial_t u, v) \text{ for each } v\in T.
\end{aligned}$$

Performing integration by parts once on the terms $r\partial_t^2\partial_x^2 u_1 v_1$ and $r\partial_t^2\partial_x^2 u_2 v_2$ and twice on the terms $\partial_x^4 u_1 v_1$, $\partial_x^4 u_2 v_2$, $\mu\partial_t\partial_x^2 u_1 v_1$, and $\mu\partial_t\partial_x^2 u_2 v_2$ yield that

$$\begin{aligned} & -\left[r\partial_t^2\partial_x u_1 v_1\right]_0^\alpha - \left[r\partial_t^2\partial_x u_2 v_2\right]_\alpha^1 + \left[\partial_x^3(u_1 + \mu\partial_t u_1)v_1\right]_0^\alpha \\ & + \left[\partial_x^3(u_2 + \mu\partial_t u_2)v_2\right]_\alpha^1 - \left[\partial_x^2(u_1 + \mu\partial_t u_1)v_1'\right]_0^\alpha - \left[\partial_x^2(u_2 + \mu\partial_t u_2)v_2'\right]_\alpha^1 = 0 \end{aligned}$$

for each $v \in T$.

Recall that $v_1(0) = v_1'(0) = 0$. Choosing $v_1'(\alpha) = v_2'(\alpha)$ and $v_2(1) = v_2'(1) = 0$, we have

$$\begin{aligned} r\partial_t^2\partial_x u(\alpha^+, t) - \partial_x^3 u(\alpha^+, t) - \mu\partial_t\partial_x^3 u(\alpha^+, t) = & \; r\partial_t^2\partial_x u(\alpha^-, t) \\ & -\partial_x^3 u(\alpha^-, t) - \mu\partial_t\partial_x^3 u(\alpha^-, t). \end{aligned}$$

All the other conditions follow from suitable choices for the values of v_1', v_2 and v_2' at $x = \alpha$ and $x = 1$. □

3.1.2 Weak formulation

L^2 is a Hilbert space with the inner product $(\cdot,\cdot)_0$. The norms in $L^2(I_j)$ and L^2 will all be denoted by $\|\cdot\|_0$ with the relevant space being clear from the context.

Define the following product spaces:

$$H^i := H^i(I_1) \times H^i(I_2), \; i = 1, 2, \ldots$$

where $H^i(I_j)$ is the Sobolev space of order i on I_j. See Appendix B. The norms in $H^i(I_j)$ and H^i are all denoted by $\|\cdot\|_i$. For the product space H^i we use the usual product space inner product and norms, i.e. if $u = \langle u_1, u_2\rangle \in H^i$, then $\|u\|_i^2 = \|u_1\|_i^2 + \|u_2\|_i^2$.

For the weak formulation of the vibration problem we consider the closure of T in H^2. We denote this closure by V and note that V is a Hilbert space with the inner product of H^2.

The bilinear form c can be extended to H^1 and is an inner product on H^1. We define the space W as the closure of T with respect to the norm induced by c. We refer to this norm as the *inertia norm*. If rotary inertia is ignored (i.e. $r = 0$), it follows that $c(\cdot,\cdot) = (\cdot,\cdot)_0$ and then $W = L^2$.

For $v = \langle v_1, v_2\rangle \in H^2$, the mappings

$$v_1' \longrightarrow v_1'(\alpha) \text{ and } v_2' \longrightarrow v_2'(\alpha)$$

are well defined in the sense of trace. See Appendix B. We have the following result.

Lemma 3.1.2 *For $i = 1,\ 2$ and $K = \max\{\alpha^{-1},\ (1-\alpha)^{-1}\}$,*

$$|v_i'(\alpha)| \leq K\|v_i\|_2 \text{ for all } v \in V. \tag{3.1.4}$$

Proof From Lemma B.2.1 in Appendix B follows that

$$|v_i'(\alpha)| \leq K\|v_i'\|_1 \leq K\|v_i\|_2 \text{ for all } v \in V.$$

□

It follows that the domains of the bilinear forms a and b can be extended to H^2.

For the definition of $C^m((0,\tau), X)$ see Appendix B.

Let $f(t) = \langle P_1(t), P_2(t)\rangle$.

Problem 1c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_0$$

for all $v \in V$.

The initial conditions $u(0)$ and $u'(0)$ will be discussed later. Note that $u'(0)$ refers to the right derivative of u' at $t = 0$.

Notation With a function u we associate a function u^* such that

$$u^* : [0,\tau] \rightarrow L^2 \text{ with } u^*(t)(x) = u(t,x).$$

Theorem 3.1.3 *If u is a solution of Problem 1b, then u^* is a solution of Problem 1c.*

Proof Suppose u is a solution of Problem 1b. Then $(u_1^*)''(t) = \partial_t^2 u_1(\cdot, t)$ and $(u_2^*)''(t) = \partial_t^2 u_2(\cdot, t)$. (See Appendix B.) Obviously the same will be true in the product space. The result now follows from the fact that T is dense in V. □

Notation With $u \in C([0,\tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : [0,\tau] \times (0,1) \to \mathbb{R} \text{ with } \tilde{u}(x,t) = u(t)(x).$$

If the weak solution is smooth enough, it satisfies the variational problem.

Theorem 3.1.4 *If u is a solution of Problem 1c and $u \in C^2([0,\tau], C^2)$, then $\tilde{u}$ is a solution of Problem 1b.*

Proof If $u_1 \in C^2([0,\tau], C^2[0,\alpha])$, then $\partial_t \tilde{u}_1$ exists and $\partial_t^2 \tilde{u}_1(x,t) = u_1''(t)(x)$ for each point (x,t). It is now clear that $\tilde{u}_1 \in C^2([0,\tau) \times [0,\alpha])$. Similarly, $\tilde{u}_2 \in C^2([0,\tau) \times [\alpha, 1])$. □

3.1.3 The energy norm

The following lemma gives some inequalities of Poincaré type for the space V.

Lemma 3.1.3 *For any $u = \langle u_1, u_2 \rangle \in V$,*

$$\|u_1\|_0 \leq \|u_1'\|_0 \leq \|u_1''\|_0, \tag{3.1.5}$$

$$\|u\|_2^2 \leq 14\left(\|u''\|_0^2 + (u_2'(\alpha) - u_1'(\alpha))^2\right). \tag{3.1.6}$$

Proof We assume first that $u \in T$. For $x \in (0,\alpha)$ we choose $[a,b] = [0,x]$ in Lemma B.2.1 and note that $u_1(0) = u_1'(0) = 0$. It follows that

$$|u_1(x)| \leq \|u_1'\|_0 \text{ and } |u_1'(x)| \leq \|u_1''\|_0$$

and the inequalities (3.1.5) are direct consequences.

For $x \in (\alpha, 1)$, we choose $[a,b] = [\alpha, x]$ in Lemma B.2.1, and find that

$$|u_2(x) - u_2(\alpha)| \leq \|u_2'\|_0.$$

As $|u_2(\alpha)| = |u_1(\alpha)| \leq \|u_1'\|_0$ it follows that

$$|u_2(x)| \leq \|u_1'\|_0 + \|u_2'\|_0.$$

Hence

$$\|u_2\|_0^2 \leq (\|u_1'\|_0 + \|u_2'\|_0)^2.$$

Similarly, from Lemma B.2.1,

$$|u_2'(x) - u_2'(\alpha)| \leq \|u_2''\|_0$$

and as $|u_1'(\alpha)| \leq \|u_1''\|_0$ it follows (using the inverse triangle inequality) that

$$|u_2'(x)| \leq \|u_1''\|_0 + \|u_2''\|_0 + |u_2'(\alpha) - u_1'(\alpha)|.$$

Hence

$$\|u_2'\|_0^2 \leq (\|u_1''\|_0 + \|u_2''\|_0 + |u_2'(\alpha) - u_1'(\alpha)|)^2.$$

It is now easy to prove that (3.1.6) holds. The inequalities will also hold on V as it is the closure of T in H^2. □

Theorem 3.1.5 *The bilinear form b is bounded and positive definite on V.*

Proof For any $u, v \in V$,

$$\begin{aligned} |b(u,v)| \leq \; & \|u_1\|_2\|v_1\|_2 + \|u_2\|_2\|v_2\|_2 \\ & + \frac{1}{\delta}\left(|u_2'(\alpha)| + |u_1'(\alpha)|\right)\left(|v_2'(\alpha)| + |v_1'(\alpha)|\right). \end{aligned}$$

Using also Lemma 3.1.2, it is easy to see that b is bounded. Clearly, from (3.1.6), there exists a constant C such that

$$b(u,u) \geq C^2\|u\|_2^2 \text{ for all } u \in V.$$ □

Due to the fact that b is symmetric we have the following result.

Corollary 3.1.3 *The bilinear form b defines an inner product on V.*

Define the *energy norm* $\|\cdot\|_E$ in V by

$$\|u\|_E^2 = b(u,u) \text{ for any } u \in V.$$

Corollary 3.1.4 *The energy norm is equivalent to the H^2-norm on V.*

Lemma 3.1.4 *V is dense in W.*

Proof W is the closure of T with respect to the inertia norm and $T \subset V \subset W$. □

Lemma 3.1.5 *V is dense in L^2.*

Proof For $v = \langle v_1, v_2 \rangle \in C_0^\infty = C_0^\infty(I_1) \times C_0^\infty(I_2)$ it is clear that $v_1(\alpha) = v_2(\alpha) = 0$, and hence that $v \in T$. Thus $C_0^\infty \subset T \subset V \subset L^2$, and as C_0^∞ is dense in L^2, it follows that V is dense in L^2. □

The following result is required to prove that bounded subsets of V are precompact in W.

Lemma 3.1.6 *Let $X_1 \subset Y_1$ and $X_2 \subset Y_2$, be four Hilbert spaces. Let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. If bounded sequences in X_1 and X_2, respectively, have convergent subsequences in Y_1 and Y_2, then any bounded subset in X is precompact in Y.*

Proof Suppose the subset A of X is bounded and $\{u^n\} = \{\langle u_1^n, u_2^n \rangle\}$ is a sequence in A. Then $\{u_1^n\}$ and $\{u_2^n\}$ are bounded sequences in X_1 and X_2 respectively. This means that there exists a convergent subsequence $\{u_n^{n_k}\}$ of $\{u_1^n\}$ in Y_1. Consider the sequence $\{u^{n_k}\} = \{\langle u_1^{n^k}, u_2^{n^k} \rangle\}$. It is now obvious that this sequence possesses a convergent subsequence in Y which is a subsequence of $\{u^n\}$. We conclude that A is precompact in Y. □

Lemma 3.1.7 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof Assume that $\{u_1^n\}$ and $\{u_2^n\}$ are bounded sequences in $H^2(I_1)$ and $H^2(I_2)$, respectively. Using the Rellich imbedding theorem (See [Fr, p 31-32]), we can find convergent subsequences in $H^1(I_1)$ and $H^2(I_2)$, respectively. The result follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.2 Beam models with dynamical boundary conditions

3.2.1 Variational formulation

The equation of motion (2.2.7) is multiplied by an arbitrary function v and integrated to get

$$\int_0^1 \partial_t^2 u(\cdot,t)v = \frac{1}{r}\int_0^1 \partial_x F(\cdot,t)v - k\int_0^1 \partial_t u(\cdot,t)v + \int_0^1 P(\cdot,t)v. \quad (3.2.1)$$

Let $I = (0,1)$.

Notation For any $u \in L^2(I)$ and any $v \in L^2(I)$,

$$(u,v)_I := \int_0^1 uv.$$

In the following results the term $(\frac{1}{r}\partial_x F, v)_I$ is examined.

Lemma 3.2.1 *If the equation of motion* (2.2.8) *is satisfied, then*

$$\left(\frac{1}{r}\partial_x F, v\right)_I = -(M,v'')_I - (r\partial_t^2\partial_x u, v')_I + \left[\frac{1}{r}Fv\right]_0^1 + [Mv']_0^1$$

for all $v \in C^2(I)$.

Proof The result is obtained by performing integration by parts twice. □

Define the space of test functions $T(I)$ as

$$T(I) = \{v \in C^2(\bar{I}) : v(0) = v'(0) = 0\}.$$

The following result follows from the constitutive equation (2.2.9) and the boundary conditions (2.4.6) and (2.4.7).

Corollary 3.2.1 *If u is the solution of Problem* 2, *then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F, v\right)_I &= -(\partial_x^2 u, v'')_I - (\mu\partial_t\partial_x^2 u, v'')_I - (r\partial_t^2\partial_x u, v')_I \\ &\quad - m\partial_t^2 u(1,t)v(1) - I_m\partial_t^2\partial_x u(1,t)v'(1) \\ &\quad - \mu_0\partial_t u(1,t)v(1) - \mu_1\partial_t\partial_x u(1,t)v'(1) \quad \textit{for all } v \in T(I).\end{aligned}$$

We define bilinear forms a, b and c by

$$
\begin{aligned}
b(u,v) &:= (u'',v'')_I \text{ for all } u,v \in C^2(\bar{I}),\\
a(u,v) &:= \mu b(u,v) + (ku,v)_I + \mu_0 u(1)v(1) + \mu_1 u'(1)v'(1) \text{ for all } u,v \in C^2(\bar{I}),\\
c(u,v) &:= (u,v)_I + (ru',v')_I + mu(1)v(1) + I_m u'(1)v'(1) \text{ for all } u,v \in C^1(\bar{I}).
\end{aligned}
$$

The variational form of Problem 2 can be expressed in terms of these bilinear forms.

Problem 2b: Variational formulation

Find u such that, for all $t > 0$, $u(\cdot,t) \in T(I)$ and

$$c(\partial_t^2 u(\cdot,t),v) + a(\partial_t u(\cdot,t),v) + b(u(\cdot,t),v) = (P(\cdot,t),v)_I$$

for all $v \in T(I)$.

Theorem 3.2.1 *If u is a solution of Problem 2, then u is a solution of Problem 2b.*

Proof The proof follows directly from substituting the result of Corollary 3.2.1 into (3.2.1). Note that if u is a solution of Problem 2, it follows from (2.4.5) that $u \in T(I)$. □

Theorem 3.2.2 *If u is a solution of Problem 2b and $\partial_t u(\cdot,t) \in C^4(\bar{I})$ and $\partial_t^2 u(\cdot,t) \in C^2(\bar{I})$, then u is a solution of Problem 2.*

Proof The proof is virtually the same as that of Theorem 3.1.2. □

3.2.2 Weak formulation

The product spaces $\mathcal{L}^2$ and $\mathcal{H}^m$ are defined by

$$\mathcal{L}^2 := L^2(I) \times \mathbb{R} \times \mathbb{R} = \{v = \langle v_1, v_2, v_3 \rangle : v_1 \in L^2(I), v_2 \in \mathbb{R}, v_3 \in \mathbb{R}\}$$

and

$$\mathcal{H}^m := H^m(I) \times \mathbb{R} \times \mathbb{R} = \{v = \langle v_1, v_2, v_3 \rangle : v_1 \in H^m(I), v_2 \in \mathbb{R}, v_3 \in \mathbb{R}\}.$$

The inner product in L^2 is given by

$$(u,v)_0 = \left(\int_0^1 u_1 v_1\right) + u_2 v_2 + u_3 v_3$$

and in H^m by

$$(u,v)_m = \sum_{i=0}^{m} \left(\int_0^1 u_1^{(i)} v_1^{(i)}\right) + u_2 v_2 + u_3 v_3.$$

The notation $\|\cdot\|_m$ is used for the associated norm in H^m.

The definitions of the bilinear forms a and b are extended to

$$\begin{aligned} b(u,v) &= (u_1'', v_1'')_I \text{ for all } u, v \in H^2, \\ a(u,v) &= \mu b(u_1, v_1) + (k u_1, v_1)_I + \mu_0 u_2 v_2 + \mu_1 u_3 v_3 \text{ for all } u, v \in H^2. \end{aligned}$$

Define the space $\widetilde{T}_2(I)$ as the closure of $T(I)$ in $H^2(I)$. Recall the fact that the boundary values of $v_1^{(i)}$ are defined in the sense of trace, for example $v_1^{(i)}(1)$ is well defined if $v_1 \in H^{i+1}(I)$. (See Appendix B.)

Define the subspace V of H^2 by

$$V = \{v \in H^2 : v_1 \in \widetilde{T}_2(I), v_2 = v_1(1), v_3 = v_1'(1)\}.$$

Lemma 3.2.2 *V is a closed subspace of H^2.*

Proof If $\{v^n\}$ is a sequence in V with limit $v \in H^2$, it follows that

$$\|v_1^n - v_1\|_2 \to 0 \text{ as } n \to \infty.$$

The boundedness of the trace operator yields that

$$|(v_1^n)'(1) - v_1'(1)| \leq C\|v_1^n - v_1\|_2$$

and

$$|v_1^n(1) - v_1(1)| \leq C\|v_1^n - v_1\|_2.$$

Uniqueness of limits implies that $v \in V$. □

Define $\widetilde{T}_1(I)$ as the closure of $T(I)$ in $H^1(I)$. Define the subspace W of H^1 by

$$W = \{v \in H^1 : v_1 \in \widetilde{T}_1(I), v_2 = v_1(1)\}.$$

Lemma 3.2.3 *W is a closed subspace of H^1.*

Proof The proof is virtually identical to that of Lemma 3.2.2. □

The definition of c is extended to

$$c(u,v) = (u_1, v_1)_I + (ru_1', v_1')_I + mu_2v_2 + I_m u_3 v_3 \text{ for all } u, v \in W.$$

This bilinear form defines an inner product on W, even if $m = I_m = 0$. The *inertia norm* induced by c on W is equivalent to the H^1 norm. If rotary inertia is ignored (i.e. $r = 0$), we again have $W = L^2$.

Choose $f(t) = \langle P(\cdot, t), 0, 0\rangle$.

Problem 2c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_0$$

for all $v \in V$

Notation With the function u we associate a function u^* such that

$$u^* : [0,\tau] \to L^2 \text{ with } u^*(t)(x) = u(t,x).$$

Theorem 3.2.3 *If u is a solution of Problem 2b, then u^* is a solution of Problem 2c.*

Proof The proof is the same as that of Theorem 3.1.3. □

Notation With $u \in C([0,\tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : [0,\tau] \times (0,1) \to \mathbb{R} \text{ with } \tilde{u}(x,t) = u(t)(x).$$

Theorem 3.2.4 *If u is a solution of Problem 2c and $u_1 \in C^2([0,\tau], C^2(\bar{I}))$, then $\tilde{u}$ is a solution of Problem 2b.*

Proof The proof is the same (even simpler) than the proof of Theorem 3.1.4.

□

3.2.3 Energy norm

Lemma 3.2.4 *The bilinear form b is bounded and positive definite on V.*

Proof Clearly, b is bounded in H^2. From Lemma B.2.3, follows that for any $w \in V$,

$$\|w_1\|_0 \leq \|w_1'\|_0 \leq \|w_1''\|_0,$$

as $w_1(0) = w_1'(0) = 0$.

Also, from Lemma B.2.1, $|w_2| \leq \|w_1'\|_0$ and $|w_3| \leq \|w_1''\|_0$. Consequently there exists a constant c, such that

$$\|w\|_2^2 \leq cb(w, w) \text{ for all } w \in V.$$

□

We define the *energy norm* on V by

$$\|w\|_E^2 = b(w, w) \text{ for all } w \in V.$$

It remains to show that V is dense in L^2 and that V is dense in W. This can be done by adapting the proof of [Sa, Prop 8.1], to this situation.

Lemma 3.2.5 *V is dense in L^2.*

Proof Let $f \in C^\infty(0,1)$ and $g \in C^\infty(0,1)$ be such that

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \frac{1}{3}, \\ 1 & \text{for} \quad \frac{2}{3} < x \leq 1, \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \frac{1}{3}, \\ x & \text{for} \quad \frac{2}{3} < x \leq 1. \end{cases}$$

Put $f_n(x) = f(x^n)$ and $g_n(x) = g(x^n)/n$ for $0 \leq x \leq 1$.

Let $w = \langle w_1, w_2, w_3 \rangle \in L^2$. Then there exists a sequence of functions $\{p_n\}$ in $C_0^\infty(0,1)$ such that

$$\|p_n - w_1\|_I \to 0 \text{ as } n \to \infty.$$

Let $v_n = p_n + w_2 f_n + w_3 g_n$. Then $v_n \in T(I)$ and $y_n = \langle v_n, v_n(1), v_n'(1)\rangle \in V$ Now,

$$\|v_n - w_1\|_I \leq \|p_n - w_1\|_I + |w_2|\|f_n\|_I + |w_3|\|g_n\|_I \to 0 \text{ as } n \to \infty.$$

We also have

$$v_n(1) = w_2 + \frac{w_3}{n} \to w_2 \text{ as } n \to \infty$$

and

$$v_n'(1) = w_3 \text{ for all } n.$$

Hence $\|y_n - w\|_0 \to 0$ as $n \to \infty$. □

Lemma 3.2.6 *V is dense in W.*

Proof Consider any $w \in W$. From the definitions of $\widetilde{T}_1(I)$ and $\widetilde{T}_2(I)$ it is clear that there exists a sequence $\{p_n\} \subset \widetilde{T}_2(I)$ such that $\|w_1 - p_n\|_1^I \to 0$ as $n \to \infty$. Using the sequence of functions $\{f_n\}$ defined in the proof of Lemma 3.2.5, we let $v_n = p_n + w_2 f_n$. The rest of the proof in the same as the proof of Lemma 3.2.5. □

Lemma 3.2.7 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof Suppose $\{w^n\}$ is a bounded sequence in V. This implies that $\{w_1^n\}$ is a bounded sequence in $H^2(I)$ and that $\{w_2^n\}$ and $\{w_3^n\}$ are bounded sequences of real numbers. Using the Rellich imbedding theorem (See [Fr, p 31-32]) yields a convergent subsequence of $\{w^n\}$ in $H^1(I)$, and from the Weierstrass theorem we find convergent subsequences of $\{w_2^n\}$ and $\{w_3^n\}$. The result then follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.3 Plate beam model

3.3.1 Variational formulation

The variational form is obtained by multiplying the dimensionless form of the equation of motion (2.6.1) by an arbitrary scalar valued function v and integrating to get

$$\int_\Omega \partial_t^2 uv = \frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v + \int_\Omega qv. \tag{3.3.1}$$

We start by quoting a general Green formula on a domain Ω in the plane:

For any scalar valued function v and any vector valued function $\mathbf{F}$,

$$\int_\Omega (\operatorname{div}\mathbf{F})v = -\int_\Omega \mathbf{F}\cdot\operatorname{grad} v + \int_{\partial\Omega}(\mathbf{F}\cdot\mathbf{n})v\,ds. \tag{3.3.2}$$

We also need a similar result for a matrix valued function A. We define $\operatorname{div} A$ to be a vector with the ith component equal to $\operatorname{div} A^{\text{row } i}$. The trace of the matrix A is denoted by $\operatorname{tr}(A)$.

Lemma 3.3.1 *For any vector valued function* $\mathbf{w}$ *and any matrix valued function* A,

$$\int_\Omega \operatorname{div} A\cdot\mathbf{w} = -\int_\Omega \operatorname{tr}(AW) + \int_{\partial\Omega} A\mathbf{w}\cdot\mathbf{n}\,ds, \tag{3.3.3}$$

where $W = \begin{bmatrix} \partial_1 w_1 & \partial_2 w_1 \\ \partial_1 w_2 & \partial_2 w_2 \end{bmatrix}$.

Proof The proof follows directly if (3.3.2) is applied for each row. □

Lemma 3.3.2 *If the equation of motion* (2.6.2) *is satisfied and* $v \in C^2(\bar{\Omega})$, *then*

$$\begin{aligned} \frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v &= \int_\Omega \operatorname{tr}(RMV) - r\int_\Omega \left(\partial_t^2(\operatorname{grad} u)\right)\cdot(\operatorname{grad} v) \\ &\quad - \int_{\partial\Omega}(RM\mathbf{n})\cdot(\operatorname{grad} v)\,ds + \frac{1}{r}\int_{\partial\Omega}(\mathbf{T}\cdot\mathbf{n})v\,ds \end{aligned} \tag{3.3.4}$$

where $V = \begin{bmatrix} \partial_1^2 v & \partial_1\partial_2 v \\ \partial_1\partial_2 v & \partial_2^2 v \end{bmatrix}$.

Proof From (3.3.2), with $\mathbf{F} = \mathbf{T}$, follows that

$$\int_\Omega (\operatorname{div} \mathbf{T})v = -\int_\Omega \mathbf{T} \cdot \operatorname{grad} v + \int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{n})v\, ds.$$

From (2.6.2), $1/r\mathbf{T} = R \operatorname{div} M - rR\partial_t \mathbf{H}$, and hence

$$\begin{aligned} \frac{1}{r}\int_\Omega (\operatorname{div} \mathbf{T})v &= -\int_\Omega (R \operatorname{div} M) \cdot \operatorname{grad} v - r\int_\Omega R\partial_t \mathbf{H} \cdot \operatorname{grad} v \\ &\quad + \frac{1}{r}\int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{n})v\, ds. \end{aligned} \quad (3.3.5)$$

Using (3.3.3) with $\mathbf{w} = \operatorname{grad} v$ and $A = RM$ gives

$$\int_\Omega R \operatorname{div} M \cdot \operatorname{grad} v = -\int_\Omega \operatorname{tr}(RMV) + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds, \quad (3.3.6)$$

where $V = \begin{bmatrix} \partial_1^2 v & \partial_1\partial_2 v \\ \partial_1\partial_2 v & \partial_2^2 v \end{bmatrix}$.

Note that RM is symmetric and $\operatorname{div}(RM) = R \operatorname{div} M$.

From (2.6.3) follows that

$$\int_\Omega R\partial_t \mathbf{H} \cdot \operatorname{grad} v = \int_\Omega \partial_t^2 (\operatorname{grad} u) \cdot \operatorname{grad} v.$$

□

Choose the space of test functions $T(\Omega)$ as

$$T(\Omega) = \{v \in C^2(\bar{\Omega}) : v = 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1\}.$$

It is necessary to analyze the line integrals in (3.3.4). Let $v \in T(\Omega)$ throughout the discussion that follows.

The boundary $\partial\Omega$ consists of four parts. Consider first the part Σ_1. We have $v = 0$, hence

$$\int_{\Sigma_1} (\mathbf{T} \cdot \mathbf{n})v\, ds = 0. \quad (3.3.7)$$

Since $\mathbf{n} = \mathbf{e}_1$ and $\operatorname{grad} v = \partial_1 v \mathbf{e}_1$,

$$(RM\mathbf{n}) \cdot (\operatorname{grad} v) = -(\partial_1^2 u + \nu\partial_2^2 u)\partial_1 v = 0. \quad (3.3.8)$$

As a consequence the boundary terms vanish. The same will happen on Σ_0.

For the domain Ω the line integrals on Γ_0 and Γ_1 reduce to two one-dimensional integrals on $(0,1)$. Formally, $ds = dx_1$ on Γ_0 and $ds = -dx_1$ on Γ_1, because of the orientation of the line integral. This means that for any function v

$$\int_{\Gamma_0} v\,ds = \int_0^1 v(x_1,0)dx_1 \text{ and } \int_{\Gamma_1} v\,ds = \int_0^1 v(x_1,d)dx_1.$$

We will use subscripts 0 and 1 to differentiate between functions defined on Γ_0 and Γ_1.

Now, consider Γ_0. From (2.6.13),

$$\frac{1}{r}\int_{\Gamma_0}(\mathbf{T}\cdot\mathbf{n})v\,ds = -\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0). \tag{3.3.9}$$

Since $\mathbf{n} = -\mathbf{e}_2$ and $M\mathbf{e}_2\cdot\mathbf{e}_1 = 0$, we have from (2.6.15) that

$$-(RM\mathbf{n})\cdot(\text{grad}\,v) \;=\; (M\mathbf{e}_2\cdot\mathbf{e}_2)\partial_1 v = -\alpha L_0\partial_1 v. \tag{3.3.10}$$

If (2.6.8) is satisfied and $w \in C^2[0,1]$ with $w(0) = w(1) = 0$, then

$$\begin{aligned}\frac{\alpha}{r_b}\int_0^1 \partial_x F_0(\cdot,t)w \;&=\; \int_0^1 \left(\alpha\partial_1 M_{b0}(\cdot,t) + \alpha L_0(\cdot,t) - \beta r_b\partial_t^2\partial_1 u(\cdot,0,t)\right)w' \\ &\quad + \left[\frac{\alpha}{r_b}F_0(\cdot,t)w\right]_0^1 \\ &= -\int_0^1 \alpha M_{b0}(\cdot,t)w'' - \beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)w' \\ &\quad + \int_0^1 \alpha L_0(\cdot,t)w' + \left[\frac{\alpha}{r_b}F_0(\cdot,t)w\right]_0^1 + [\alpha M_{b0}(\cdot,t)w']_0^1.\end{aligned}$$

Clearly, the boundary terms vanish as $M_{b0}(0,t) = M_{b0}(1,t) = 0$ from (2.6.19).

Choosing $w(x_1) = v(x_1,0)$ and combining this result with (3.3.9) and (3.3.10) yield that

$$\begin{aligned}&\int_{\Gamma_0}\left((-RM\mathbf{n})\cdot\text{grad }v + \frac{1}{r}\mathbf{T}\cdot\mathbf{n}\right)ds \\ &= -\frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 M_{b0}(\cdot,t)\partial_1^2 v(\cdot,0) \\ &\quad -\beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \alpha\int_0^1 P_0(\cdot,t)v(\cdot,0). \end{aligned} \tag{3.3.11}$$

An analogous result is true for Γ_1.

Lemma 3.3.3 *If u is a solution of Problem 3 and $v \in T(\Omega)$, then*

$$\begin{aligned}
&\frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_1(\cdot,t)v(\cdot,d) \\
= &\int_\Omega \operatorname{tr}(RMV) - r\int_\Omega \partial_t^2(\operatorname{grad} u)\cdot \operatorname{grad} v \\
&-\beta r_b \int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \beta r_b \int_0^1 \partial_t^2\partial_1 u(\cdot,d,t)\partial_1 v(\cdot,d) \\
&-\alpha\int_0^1 \partial_1^2 u(\cdot,0,t)\partial_1^2 v(\cdot,0) - \alpha\int_0^1 \partial_1^2 u(\cdot,d,t)\partial_1^2 v(\cdot,d) \\
&-\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 P_1(\cdot,t)v(\cdot,d).
\end{aligned}$$

Proof Substitute (3.3.7), (3.3.8) and (3.3.11) for the boundary integrals in Lemma 3.3.2. □

Notation For any $u \in L^2(\Omega)$ and $v \in L^2(\Omega)$,

$$(u,v)_\Omega = \int_\Omega uv.$$

Define a bilinear form b on $C^2(\bar{\Omega})$ by

$$b(u,v) = b_\Omega(u,v) + \alpha b_0(u,v) + \alpha b_1(u,v)$$

with

$$\begin{aligned}
b_\Omega(u,v) = \int_\Omega \operatorname{tr}(RMV) &= (\partial_1^2 u, \partial_1^2 v)_\Omega + 2(1-\nu)(\partial_1\partial_2 u, \partial_1\partial_2 v)_\Omega \\
&\quad + (\partial_2^2 u, \partial_2^2 v)_\Omega + \nu(\partial_2^2 u, \partial_1^2 v)_\Omega + \nu(\partial_1^2 u, \partial_2^2 v)_\Omega
\end{aligned}$$

and

$$\begin{aligned}
b_0(u,v) &= \int_0^1 \partial_1^2 u(\cdot,0)\partial_1^2 v(\cdot,0), \\
b_1(u,v) &= \int_0^1 \partial_1^2 u(\cdot,d)\partial_1^2 v(\cdot,d).
\end{aligned}$$

Lemma 3.3.4 *If u is a solution of Problem 3, then*

$$
\begin{aligned}
& \frac{1}{r}\int_\Omega (\operatorname{div} \mathrm{T})v + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_1(\cdot,t)v(\cdot,d) \\
= \; & -b(u,v) - r\int_\Omega \partial_t^2(\operatorname{grad} u)\cdot \operatorname{grad} v \\
& -\beta r_b \int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \beta r_b \int_0^1 \partial_t^2 \partial_1 u(\cdot,d,t)\partial_1 v(\cdot,d) \\
& -\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 P_1(\cdot,t)v(\cdot,d)
\end{aligned}
$$

for all $v \in T(\Omega)$.

Proof A direct substitution yields that $\int_\Omega \operatorname{tr}(RMV) = b_\Omega(u,v)$. □

Define a bilinear form c on $C^1(\bar{\Omega})$ by

$$c(u,v) = c_\Omega(u,v) + \beta c_0(u,v) + \beta c_1(u,v)$$

with

$$c_\Omega(u,v) \;=\; (u,v)_\Omega + r(\partial_1 u, \partial_1 v)_\Omega + r(\partial_2 u, \partial_2 v)_\Omega.$$

and

$$
\begin{aligned}
c_0(u,v) &= \int_0^1 u(\cdot,0)v(\cdot,0) + r_b\int_0^1 \partial_1 u(\cdot,0)\partial_1 v(\cdot,0), \\
c_1(u,v) &= \int_0^1 u(\cdot,d)v(\cdot,d) + r_b\int_0^1 \partial_1 u(\cdot,d)\partial_1 v(\cdot,d).
\end{aligned}
$$

Problem 3b: Variational formulation

Find u such that for all $t > 0$, $u(\cdot,t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot,t), v) + b(u(\cdot,t), v) = (q(\cdot,t), v)_\Omega$$

for all $v \in T(\Omega)$.

Theorem 3.3.1 *If u is a solution of Problem 3, then u is a solution of Problem 3b.*

Proof If (2.6.7) is multiplied by an arbitrary function $v \in T(\Omega)$ and integrated over Γ_0, it follows that

$$\beta \int_0^1 \partial_t^2 u(\cdot, 0, t) v(\cdot, 0) = \frac{\alpha}{r_b} \int_0^1 \partial_1 F_0(\cdot, t) v(\cdot, 0) + \alpha \int_0^1 P_0(\cdot, t) v(\cdot, 0).$$

A similar result holds on Γ_1.

Combining these results with (3.3.1) and Lemma 3.3.4 completes the proof.

□

Theorem 3.3.2 *If u is a solution of Problem 3b and $\partial_t^2 u(\cdot, t) \in C^2(\bar{\Omega})$, then u is a solution of Problem 3.*

Proof The proof follows the same pattern as in the previous cases and is tedious rather than difficult. We will write u for $u(\cdot, t)$ in this proof.

First let $v \in C_0^\infty(\Omega)$. Using integration by parts and the fact that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we see that u satisfies the partial differential equation in Problem 3.

This in turn implies that, for each $v \in T(\Omega)$,

$$\begin{aligned} &\beta c_0(\partial_t u(\cdot, 0, t), v) + \beta c_1(\partial_t u(\cdot, d, t), v) \\ &+ \int_\Omega r(\partial_t^2 \nabla^2 u) v - \int_\Omega (\partial_1^4 u + 2\partial_1^2 \partial_2^2 u + \partial_2^4 u) v \\ &= -r(\partial_t^2 \partial_1 u, \partial_1 v)_\Omega - r(\partial_t^2 \partial_2 u, \partial_2 v)_\Omega - b(u, v). \end{aligned}$$

Using integration by parts again, we have

$$r \int_\Omega (\nabla^2 \partial_t^2 u) v = -r(\partial_t^2 \partial_1 u, \partial_1 v)_\Omega - r(\partial_t^2 \partial_2 u, \partial_2 v)_\Omega + \int_{\partial\Omega} r v (\operatorname{grad} \partial_t u) \cdot \mathbf{n} \, ds.$$

Hence,

$$\begin{aligned} \beta c_0(\partial_t u(\cdot, 0, t), v) + \beta c_1(\partial_t u(\cdot, d, t), v) - \int_\Omega (\partial_1^4 u + 2\partial_1^2 \partial_2^2 u + \partial_2^4 u) v \\ = -b(u, v) - \int_{\partial\Omega} r v (\operatorname{grad} \partial_t u) \cdot \mathbf{n} \, ds. \end{aligned}$$

For convenience, define a matrix M by (2.5.4) (regardless of any physical interpretation). We have

$$\begin{aligned} b_\Omega(u,v) &= \int_\Omega \operatorname{tr}(RMV) \\ &= -\int_\Omega R\operatorname{div} M \cdot \operatorname{grad} v + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds \\ &= \int_\Omega \operatorname{div}(R\operatorname{div} M) v\, ds - \int_{\partial\Omega} (R\operatorname{div} M \cdot \mathbf{n}) v\, ds \\ &\quad + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds. \end{aligned}$$

But $\operatorname{div}(R\operatorname{div} M) = -(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u)$, (see (2.5.4) and (2.5.5)).

We are left with

$$\begin{aligned} \beta c_0(\partial_t u(\cdot,0,t),v) + \beta c_1(\partial_t u(\cdot,d,t),v) &= -\alpha b_0(u(\cdot,0,t),v) - \alpha b_1(u(\cdot,d,t),v) \\ &\quad - \int_{\partial\Omega} rv(\operatorname{grad}\partial_t u) \cdot \mathbf{n}\, ds \\ &\quad + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds \\ &\quad - \int_{\partial\Omega} (R\operatorname{div} M \cdot \mathbf{n}) v\, ds \\ &\quad \text{for each } v \in T(\Omega). \end{aligned}$$

Recall that for $v \in T(\Omega)$, $v = 0$ on $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$. All the boundary conditions are obtained by suitable choices of the values of v and $\operatorname{grad} v$ on the boundary $\partial\Omega$. As an example we consider the dynamical boundary condition on Γ_0.

Choose $v \in T(\Omega)$ such that $v = \partial_2 v = 0$ on Γ_1. Then $\partial_1 v = 0$ on Γ_1. In addition, choose $\partial_1 v = 0$ on $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$ and $\partial_2 v = 0$ on Γ_0.

We are left with,

$$\begin{aligned} \beta\int_0^1 \partial_t^2 uv + \beta\int_0^1 r_b\partial_t^2\partial_1 u\partial_1 v &= -\alpha\int_0^1 \partial_1^2 u\partial_1^2 v + \int_0^1 r\partial_t^2\partial_2 uv \\ &\quad + \int_0^1 M_{22}\partial_1 v - \int_0^1 \partial_1 M_{11} v - \int_0^1 \partial_2 M_{12} v. \end{aligned}$$

Applying integration by parts twice to the term $\partial_1^2 u\partial_1^2 v$ and once to the terms $\partial_t^2\partial_1 u\partial_1 v$ and $M_{22}\partial_1 v$ gives the dynamical boundary condition on Γ_0. □

3.3.2 Weak formulation

Define the following product spaces:

$$\begin{aligned} L^2 &:= L^2(\Omega) \times L^2(I) \times L^2(I), \\ H^k &:= H^k(\Omega) \times H^k(I) \times H^k(I). \end{aligned}$$

We use the product space inner products $(\cdot,\cdot)_0$ on L^2 and $(\cdot,\cdot)_k$ on H^k defined by:

$$\begin{aligned} (u,v)_0 &= (u_1,v_1)_\Omega + (u_2,v_2)_I + (u_3,v_3)_I \\ &= \int_\Omega u_1 v_1 + \int_0^1 u_2 v_2 + \int_0^1 u_3 v_3, \\ (u,v)_k &= (u_1,v_1)_k^\Omega + (u_2,v_2)_k^I + (u_3,v_3)_k^I \end{aligned}$$

with $(\cdot,\cdot)_k^\Omega$ and $(\cdot,\cdot)_k^I$ the standard inner products on $H^k(\Omega)$ and $H^k(I)$ respectively.

Define $\widetilde{T}_2(\Omega)$ as the closure of $T(\Omega)$ in $H^2(\Omega)$.

The trace operators γ_0 and γ_1 are defined by

$$\gamma_0 : u \to u(\cdot, 0)$$

and

$$\gamma_1 : u \to u(\cdot, d)$$

for any $u \in H^1(\Omega)$. See Appendix B.

The bilinear form b can be extended to H^2

$$b(u,v) = b_\Omega(u_1,v_1) + \alpha b_0(u_2,v_2) + \alpha b_1(u_3,v_3) \text{ for all } u,v \in H^2.$$

The definition of c is extended to

$$c(u,v) = c_\Omega(u_1,v_1) + \beta c_0(u_2,v_2) + \beta c_1(u_2,v_3) \text{ for all } u,v \in H^1.$$

Define a subspace V of H^2 by

$$V = \{v \in H^2 : v_1 \in \widetilde{T}_2(\Omega), v_2 = \gamma_0 v_1, v_3 = \gamma_1 v_1\}.$$

Lemma 3.3.5 *V is a closed subspace of H^2.*

Proof If $\{v^n\}$ is a sequence in V with limit $v \in H^2$, it follows that

$$\|v_1^n - v_1\|_2^\Omega \to 0 \text{ as } n \to \infty.$$

Hence $v_1 \in \widetilde{T}_1(\Omega)$. Also,

$$\|\gamma_0 v_1^n - v_2\|_2^I \to 0 \text{ as } n \to \infty$$

and

$$\|\gamma_1 v_1^n - v_3\|_2^I \to 0 \text{ as } n \to \infty.$$

The boundedness of the trace operators yields that

$$\|\gamma_0 v_1^n - \gamma_0 v_1\|_0^I \to 0 \text{ as } n \to \infty$$

and

$$\|\gamma_1 v_1^n - \gamma_0 v_1\|_0^I \to 0 \text{ as } n \to \infty.$$

Uniqueness of limits implies that $v \in V$. □

Define $\widetilde{T}_1(\Omega)$ as the closure of $\widetilde{T}(\Omega)$ in $H^1(\Omega)$. Define the subspace W of H^1 by

$$W = \{v \in H^1 : v_1 \in \widetilde{T}_1(\omega), v_2 = \gamma_0 v_1, v_3 = \gamma_1 v_1\}.$$

The bilinear form c defines an inner product on W. The *inertia norm* induced by c on W is equivalent to the H^1 norm. If rotary inertia is ignored (i.e. $r = 0$), we again get $W = L^2$.

Lemma 3.3.6 *W is a closed subspace of H^1.*

Proof The proof is virtually identical to that of Lemma 3.3.5. □

Let $f(t) = \langle q(t), 0, 0\rangle$.

Problem 3c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (f(t), v)_0 \text{ for all } v \in V.$$

Notation

With the function $u \in C(\bar{\Omega})$ we associate functions u_1 and u^* such that

$$u_1 : [0, \tau] \to L^2(\Omega) \text{ with } u_1(t)(x) = u(x, t)$$

and

$$u^* : [0, \tau] \to L^2 \text{ with } u^*(t) = (u_1(t), (\gamma_0 u_1)(t), (\gamma_1 u_1)(t)).$$

Theorem 3.3.3 *If u is a solution of Problem 3b, then u^* is a solution of Problem 3c.*

Proof A solution of Problem 3b is in $C^2\left([0, \tau] \times \overline{\Omega}\right)$. In this case the operators γ_0 and γ_1 merely indicate the restriction of a function to Γ_0 or Γ_1. As before (see the proof of Theorem 3.1.3),

$$u_1''(t) = \partial_t^2 u(\cdot, t), \; (\gamma_0 u_1)''(t) = \gamma_0 \partial_t^2 u(\cdot, t) \text{ and } (\gamma_1 u_1)''(t) = \gamma_1 \partial_t^2 u(\cdot, t).$$

We conclude that $(u^*)''(t) = \langle \partial_t^2 u(\cdot, t), \gamma_0 \partial_t^2 u(\cdot, t), \gamma_1 \partial_t^2 u(\cdot, t) \rangle$. □

For $u_1 \in C^2([0, \tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : \bar{\Omega} \times [0, \tau] \to \mathbb{R} \text{ with } \tilde{u}(x, t) = \begin{cases} u_1(t)(x), & u \in \Omega \\ \gamma_0 u_1(t)(x), & x \in \Gamma_0 \\ \gamma_1 u_1(t)(x), & x \in \Gamma_1. \end{cases}$$

Theorem 3.3.4 *If u is a solution of Problem 3c and $u_1 \in C^2([0, \tau], C^2(\bar{\Omega}))$, then $\tilde{u}$ is a solution of Problem 3b.*

Proof See the proof of Theorem 3.1.4. □

3.3.3 Energy norm

Lemma 3.3.7 *The bilinear form b is positive definite on V.*

Proof We start with a Poincaré type inequality (Lemma B.2.3):

If $f \in C^1[0, a]$ and $f(0) = f(a) = 0$, then

$$\int_0^a f^2 \le a^4 \int_0^a (f'')^2.$$

Clearly,

$$\|u(\cdot,0)\|_I \leq [b_0(u(\cdot,0),v(.,0))]^{1/2}$$

and

$$\|u(\cdot,d)\|_I \leq [b_1(u(\cdot,d),v(\cdot,d))]^{1/2}.$$

Also

$$\int_0^1 [u(\cdot,y)]^2 \leq \int_0^1 [\partial_x^2 u(\cdot,y)]^2 \text{ for each } y.$$

Evaluating the double integral over Ω, we have

$$\|u\|_\Omega^2 \leq \|\partial_x^2 u\|_\Omega^2.$$

Hence $\|u\|_\Omega \leq a_1^2 b_\Omega(u,u)^{1/2}$. It is now straightforward to derive the desired estimate. □

Lemma 3.3.8 *V is dense in L^2.*

Proof Let $f \in C^\infty(0,d)$ be such that

$$f(y) = \begin{cases} 0 & \text{for} \quad 0 \leq y < \frac{d}{3}, \\ 1 & \text{for} \quad \frac{2d}{3} < y \leq d, \end{cases}$$

let $f_n(y) = f(y^n))$ and let $g_n(y) = f_n(d-y)$ for $0 \leq y \leq d$.

Let $w = \langle w_1, w_2, w_3 \rangle \in L^2$. Then there exists a sequence of functions $\{p_n\}$ in $C_0^\infty(\Omega)$ such that

$$\|p_n - w_1\|_\Omega \to 0 \text{ as } n \to \infty.$$

Let $v_n = p_n + w_2 f_n + w_3 g_n$. Then $v_n \in T(\Omega)$ and $y_n = \langle v_n, \gamma_0 v_n, \gamma_1 v_n \rangle \in V$. Now,

$$\|v_n - w_1\|_\Omega \leq \|p_n - w_1\|_\Omega + \|w_2 f_n\|_\Omega + \|w_3 g_n\|_\Omega \to 0 \text{ as } n \to \infty.$$

We also have $\gamma_0 v_n = w_2$ for each n and $\gamma_1 v_n = w_3$ for each n. Hence $\|y_n - w\|_0 \to 0$ as $n \to \infty$. □

Lemma 3.3.9 *V is dense in W.*

Proof Consider any $w \in W$. From the definitions of $\widetilde{T}_1(I)$ and $\widetilde{T}_2(I)$ it is clear that there exists a sequence $\{p_n\} \subset \widetilde{T}_2(I)$ such that $\|w_1 - p_n\|_1^I \to 0$ as $n \to \infty$. The rest of the proof is the same as the proof of Lemma 3.3.8. □

Lemma 3.3.10 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof $V \subset H^2 = H^2(\Omega) \times H^2(I) \times H^2(I)$ and $W \subset H^1 = H^1(\Omega) \times H^1(I) \times H^1(I)$. The Rellich imbedding theorem (See [Fr, [p 31-32]) yields that bounded sequences in $H^2(\Omega)$ and $H^2(I)$ have convergent subsequences in $H^1(\Omega)$ and $H^1(I)$ respectively. The result follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.4 Abstract differential equation for model problems

All our model problems have now been written in the same weak form. We have two Hilbert spaces V and W with inner products b and c respectively. It will be more convenient to denote $c(\cdot,\cdot)$ by $(\cdot,\cdot)$ and we will reserve the notation $\|\cdot\|$ for the associated norm which is called the *inertia norm*. Recall that the norm associated with the inner product b is called the *energy norm* and denoted by $\|\cdot\|_E$. The inner product in L^2 is denoted by $(\cdot,\cdot)_0$ and the associated norm by $\|\cdot\|_0$.

The properties of the spaces V, W and L^2 are of critical importance in the theory. For convenience we present a summary:

Space	Inner product	Norm
Energy space V	$b(\cdot,\cdot)$	Energy norm $\|\cdot\|_E$
Inertia space W	$c(\cdot,\cdot) = (\cdot,\cdot)$	Inertia norm $\|\cdot\|$
L^2	$(\cdot,\cdot)_0$	$\|\cdot\|_0$

The energy norm is equivalent to the norm of H^2 on V. The inertia norm is equivalent to the norm of H^1 on W if rotary inertia is included.

Estimates

There exist constants C_E and C_I such that

$$\begin{aligned} \|u\|_E &\geq C_E\|u\| \text{ for all } u \in V, \\ \|u\| &\geq C_I\|u\|_0 \text{ for all } u \in W. \end{aligned}$$

Topological properties

V is dense in W with respect to the inertia norm. These spaces are also dense in the underlying Hilbert space L^2.
If a subset of V is bounded with respect to the energy norm $\|\cdot\|_E$, it is precompact with respect to the inertia norm $\|\cdot\|$, and if it is bounded with respect to the inertia norm $\|\cdot\|$, it is precompact with respect to the L^2 norm $\|\cdot\|_0$.

We consider the following problem:

Problem A (Vibration problem)

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ *such that, for all* $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ *and*

$$\begin{aligned} (u''(t), v) + a(u'(t), v) + b(u(t), v) &= (f(t), v) \text{ for all } v \in V, \\ u(0) = \alpha,\ u'(0) &= \beta. \end{aligned}$$

Remarks

1. In general the damping term a is a non-negative bounded bilinear form on V. In our model problems, we have that

$$a(u,v) = \mu b(u,v) + k(u,v)_0$$

and k or μ or both can be zero.

2. For the model problems in Section 3.1 to Section 3.3 the forcing term is $(f(t), v)_0$. In the cases where $W \neq L^2$, it is proved in Lemma 3.4.1 that there exists a function $\tilde{f} : [0,\tau] \to W$ with

$$(\tilde{f}(t), v) = (f(t), v)_0 \text{ for all } v \in W.$$

We use the notation f for $\tilde{f}$.

The following results are special cases of the Lax-Milgram Lemma. See [Fr, p 41].

Lemma 3.4.1 *For each* $y \in L^2$ *there exists a unique* $w \in W$ *such that*

$$(w, v) = (y, v)_0 \text{ for all } v \in W.$$

Proof The Riesz Theorem yields that, for any F in the dual of W, there exists a unique $w \in W$ such that

$$(w, v) = F(v) \text{ for all } v \in W.$$

Now define F by $F(v) = (y, v)_0$ for all $v \in W$, then F is a continuous linear functional on W. □

Lemma 3.4.2 *For each $y \in W$ there exists a unique $u \in V$ such that*

$$b(u,v) = (y,v) \text{ for all } v \in V.$$

Proof The proof is exactly the same as that of Lemma 3.4.1. □

It is easy to show that Problem A can have at most one solution for given $u(0)$ and $u'(0)$. If w is a solution of the associated homogeneous problem (i.e. $f(t) = 0$ for $t > 0$, $w(0) = w'(0) = 0$), it follows that

$$(w''(t), v) + a(w'(t), v) + b(w(t), v) = 0 \text{ for all } v \in V.$$

Thus

$$(w''(t), w'(t)) + a(w'(t), w'(t)) + b(w(t), w'(t)) = 0 \text{ for all } t > 0.$$

This means that

$$\frac{d}{dt}\left(\|w'(t)\|^2 + \|w(t)\|_E^2\right) = -2a(w'(t), w'(t)) \le 0 \text{ for all } t > 0.$$

We conclude that $w(t) = 0$ for all $t > 0$, as $w(0) = 0$ and the uniqueness of solutions for Problem A follows.

Even though Problem A is a typical weak formulation of a vibration model, we were unable to find any directly applicable existence result. Available existence results which follow from standard semigroup theory are all formulated for abstract differential equations. This means that operators associated with the bilinear forms a and b have to be constructed. We present this construction as it will also be used for the analysis of the eigenvalue problem.

Define an operator

$$\Lambda : W \to V \text{ by } b(\Lambda f, v) = (f, v) \text{ for all } v \in V.$$

Lemma 3.4.3 *The operator Λ is a bounded linear operator with trivial null-space and range $R(\Lambda)$ dense in V.*

Proof For any $f \in W$,

$$\|\Lambda f\|_E^2 = b(\Lambda f, \Lambda f) = (f, \Lambda f) \le \|f\|\|\Lambda f\| \le C_E^{-1}\|f\|\|\Lambda f\|_E.$$

This implies that Λ is bounded.

Also, if $\Lambda f = 0$, it follows that $0 = b(\Lambda f, v) = (f, v)$ for all $v \in V$. As V is dense in W, this implies that $(f, f) = 0$ and that Λ has a trivial nullspace.

Suppose that the closure $\overline{R}$ of $R(\Lambda)$ with respect to the energy norm is not equal to V. Then there exists a $y \in V$, $y \neq 0$ such that $b(v, y) = 0$ for all $v \in \overline{R}$. As $y \in V$, $(y, y) = b(\Lambda y, y) = 0$ and hence $y = 0$, which is a contradiction. □

Define an operator

$$A : D(A) = R(\Lambda) \subset V \to W \text{ by } A = -\Lambda^{-1}.$$

The idea for the construction of the operator A is due to Lax and Milgram [LM].

Corollary 3.4.1 *The operator A is a closed densely defined symmetric operator with range $R(A) = W$ and*

$$b(u, v) = -(Au, v) \text{ for all } u \in D(A) \text{ and } v \in V.$$

For the damping term a, we can define an associated operator in a similar way. From the model problems in Section 3.1 and Section 3.2 it is clear that there are two special cases to consider.

In the case of Kelvin-Voigt damping, a is positive definite with respect to the energy norm, i.e. there exists a constant c such that

$$a(u, u) \geq c\, b(u, u) \text{ for all } u \in V.$$

Then an operator J can be constructed in exactly the same way as A with $R(J) = W$ and

$$a(u, v) = -(Ju, v) \text{ for all } u \in D(J) \text{ and } v \in V.$$

In this case J will be a closed densely defined symmetric linear operator.

In the case of viscous damping, a is bounded in the inertia norm. The bilinear form a can then be extended to W. From the Riesz Theorem (see Lemma 3.4.1) follows that for any $u \in W$ there exists a unique $w \in W$ with

$$a(u, v) = (w, v) \text{ for all } v \in W.$$

Choose $Ju = -w$. Then

$$a(u, v) = -(Ju, v) \text{ for all } u \in D(J) \text{ and } v \in W.$$

In terms of the operators A and J, the weak formulation of the problem can be represented as follows:

Initial value problem for second order abstract differential equation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ *such that, for all* $t > 0$, $u(t) \in D(A)$, $u'(t) \in D(J)$ *and*

$$\begin{aligned} u''(t) - Ju'(t) - Au(t) &= f(t), \\ u(0) = \alpha,\ u'(0) &= \beta. \end{aligned}$$

In a report [VVZ], existence results for Problem A are proved. Some restrictions had to be placed on the initial conditions. A general existence result is proved in [BI] using results of [P] or [Sh]. In the latter references one may also find existence results. See [Sh, Section VI.2, Theorems 2A, 2B and 2C] and also [K, Section III.1, Theorem 1.3]. However, the transition from the abstract existence result to a concrete example is far from trivial. See for instance the treatment of the wave equation in [P, Section 7.4]. We consider this topic to be beyond the scope of this thesis.

As a final remark on the dynamic problem we mention that general existence results do not include satisfactory regularity results. The regularity of the solution is of great importance in convergence theory. (See Section 5.3.) For the one-dimensional wave equation it is a fact that the regularity of the solution depends on the regularity of the initial conditions. This fact is clear from either D'Alemberts method or a Fourier series solution. (See [W].) For two-dimensional problems the shape of the physical domain is also a factor. In Section 5.3 we will allow for different possibilities as far as regularity is concerned.

The following equilibrium problem is associated with Problem A. The solvability of this problem follows from Lemmas 3.4.1 and 3.4.2.

Problem B (Equilibrium problem)

For $f \in L^2$, *find* $u \in V$ *such that,* $b(u,v) = (f,v)_0$ *for all* $v \in V$.

3.5 Eigenvalue problem

Consider the undamped homogeneous problem associated with Problem A:

$$(u''(t), v) + b(u(t), v) = 0 \textit{ for all } v \in V. \tag{3.5.1}$$

Applying separation of variables to (3.5.1) (i.e. assuming that $u(t) = \phi(t)w$ with ϕ a real valued function and $w \in V$), yields two problems, namely an eigenvalue problem and an ordinary differential equation.

Problem C (Eigenvalue problem)

Find a complex number λ and $w \in V$, $w \neq 0$, such that

$$b(w, v) = \lambda(w, v) \textit{ for all } v \in V. \tag{3.5.2}$$

The differential equation is

$$\phi'' + \lambda\phi = 0.$$

The function u is a solution of (3.5.1) if and only if w is an eigenvector of b, λ is a corresponding eigenvalue and ϕ is a solution of the differential equation.

The constant λ is called an eigenvalue of b and the subspace of solutions w is called the eigenspace E_λ of b corresponding to λ. The elements of E_λ are called eigenvectors. (Recall the fact that b is symmetric.)

$\sqrt{\lambda}$ is called a natural frequency and w a natural mode of vibration. (We will prove that all the eigenvalues are positive.)

Theorem 3.5.1 *Let Λ be the operator defined in Section* 3.4.

1. *λ is an eigenvalue of b if and only if λ^{-1} is an eigenvalue of Λ. The eigenspace of b corresponding to λ is the same as the eigenspace of Λ corresponding to λ^{-1}.*

2. *All the eigenvalues of b are positive.*

3. *Suppose λ and μ are eigenvalues of b with $\lambda \neq \mu$. If $w \in E_\lambda$ and $u \in E_\mu$, then*

$$b(u, w) = (u, w) = 0.$$

Proof

1. Note that b cannot have a zero eigenvalue. Zero is also not an eigenvalue of Λ as the nullspace of Λ is trivial. The definition of Λ implies that

$$b(w, v) = \lambda(w, v) \text{ for all } v \in V$$

if and only if

$$\Lambda w = \lambda^{-1} w.$$

2. The operator Λ is symmetric since b is symmetric. It is well known that the eigenvalues of a symmetric operator are real. Finally, $\lambda > 0$, since $b(w, w) > 0$ and $(w, w) > 0$.

3. From $\lambda(w, u) = b(w, u) = \mu(w, u)$, it follows that $(\lambda - \mu)(w, u) = 0$. As $\lambda \neq \mu$, this yields that $(w, u) = 0$ and as a consequence, $b(w, u) = 0$. □

Lemma 3.5.1 *The operator Λ from W to W is compact.*

Proof The operator Λ maps a bounded subset of W onto a bounded subset of V. But this set is precompact in W. □

Theorem 3.5.2

1. *The set of eigenvalues of b is countable.*

2. *If the sequence of eigenvalues are ordered as a non-decreasing sequence $\lambda_1, \lambda_2, \ldots,$ then $\lambda_n \to \infty$ if $n \to \infty$.*

3. *E_λ is finite dimensional for each λ.*

Proof Due to Theorem 3.5.1, it is sufficient to consider the operator Λ. Since Λ is compact, we have immediately the facts that the eigenvalues are at most countable and the finite dimensionality of the eigenspaces. (See any text on Functional Analysis, for example [Kr, Section 8.3].) For a symmetric compact operator on a Hilbert space we have more: There exists an orthonormal sequence of eigenvectors for which the corresponding sequence of eigenvalues converge to zero. See [Sh, Theorem 7c] or [Ze, Theorem 4A].

Since the Hilbert space V is not finite dimensional, it follows that there must be an infinite number of different eigenvalues. □

Remark The dimension of the eigenspace E_λ is called the multiplicity of the eigenvalue λ.

Definition 3.5.1 *Rayleigh quotient*

The Rayleigh quotient R is defined as

$$R(v) = \frac{b(v,v)}{(v,v)} = \frac{\|v\|_E^2}{\|v\|^2}.$$

The eigenvalues can be characterised in terms of the Rayleigh quotient.

Theorem 3.5.3 *The smallest eigenvalue λ_1, is given by*

$$\lambda_1 = \min\{R(v) : v \in V\}.$$

Proof See [SF, p 220]. □

Remarks

1. Theorem 3.5.3 may be used to order the eigenvalues of b. If u is an eigenvector corresponding to λ_1, we consider the orthogonal complement of u in V, which is again a Hilbert space.

2. For the eigenvalue problem (3.5.2), various bounds for the eigenvector w, associated with an eigenvalue λ, can be obtained. Clearly, in general,

$$\|w\|_E = \lambda^{1/2}\|w\|. \tag{3.5.3}$$

3. We may also consider eigenvalue problems for other bilinear forms, for example

$$(u,v) = c(u,v) = \lambda(u,v)_0.$$

Exactly the same results will be true—this time in the Hilbert space W.

Regularity For beam problems where rotary inertia is ignored (i.e. $W = L^2$), the eigenfunction satisfies the differential equation

$$w^{(4)} = \lambda w.$$

As a consequence we have

$$\|w^{(4)}\|_0 = \lambda\|w\|_0, \tag{3.5.4}$$

Also, from the differential equation,

$$w^{(6)} = \lambda w''.$$

Since the energy norm is equivalent to the H^2-norm, there exists a constant C_b such that

$$\|w^{(6)}\|_0 \leq C_b\lambda^{3/2}\|w\|_0. \tag{3.5.5}$$

For the case where rotary inertia is included,

$$w^{(4)} = \lambda(rw'' + w).$$

In this case it follows that

$$\|w^{(4)}\|_0 = r\lambda\|w''\|_0 + \lambda\|w\|_0 \leq C_b\lambda^{3/2}\|w\|, \tag{3.5.6}$$

if $\lambda > 1$.

Remark For the plate beam problem we do not know if the eigenvectors are in H^k for $k > 2$.

Chapter 4

Discretization

4.1 Galerkin approximation

In Chapter 3 we showed that all the model problems lead to three typical abstract problems. In this section we will formulate a Galerkin approximation for the general vibration problem, Problem A, as well as for the associated equilibrium problem, Problem B, and the eigenvalue problem, Problem C.

To formulate these Galerkin approximations, it is necessary to choose a finite dimensional subspace S^h of V. (At this stage the symbol h is used only to indicate that we are considering approximation in a finite dimensional space.)

Galerkin approximation for the vibration problem

Problem AG

Find $u_h \in C^2((0,\infty), S^h)$, such that for all $t > 0$,

$$\begin{aligned} (u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) &= (f(t), v)_0 \text{ for all } v \in S^h, \\ u_h(0) = \alpha_h,\ u_h'(0) &= \beta_h. \end{aligned}$$

The initial conditions α_h and β_h are approximations in S^h for α and β.

The Galerkin approximation yields a semi-discrete problem which can be written as a system of ordinary differential equations. Let $\{\phi_1, \phi_2, \dots, \phi_n\}$ be a basis for S^h. Then there exist functions $u_i(t)$ such that

$$u_h(t) = \sum_{i=1}^{n} u_i(t)\phi_i.$$

Let $\bar{u}$ be a function with values in $\mathbb{R}_n$ such that the vector $\bar{u}(t)$ has components $u_i(t)$.

Problem AD

Find $\bar{u} \in C^2([0,\infty), \mathbb{R}_n)$, *such that*

$$\begin{aligned} M\bar{u}''(t) + L\bar{u}'(t) + K\bar{u}(t) &= \bar{f}(t),\ t > 0, \\ \bar{u}(0) = \bar{\alpha},\ \bar{u}'(0) &= \bar{\beta}, \end{aligned}$$

with $\bar{\alpha}$ *and* $\bar{\beta}$ *the coefficients of* α_h *and* β_h.

The matrices K, L, M and M_0 are defined as follows

$$K_{ij} = b(\phi_i, \phi_j),\quad L_{ij} = a(\phi_i, \phi_j),\quad M_{ij} = (\phi_i, \phi_j) \text{ and } [M_0]_{ij} = (\phi_i, \phi_j)_0.$$

The vector $\bar{f}(t)$ has components $(f(t), \phi_i)_0$.

It is easy to see that Problem AG is equivalent to Problem AD. For instance, if $v = \sum_1^n v_i\phi_i$, then $(u_h''(t), v) = M\bar{u}''(t) \cdot \bar{v} = \bar{v}^T M\bar{u}''(t)$.

Remark Problem *AD* is an initial value problem for a system of differential equations. It will have a unique solution if $\bar{f}$ is continuous, but the differentiability properties of $\bar{u}$ (and hence u_h) will depend on the differentiability properties of $\bar{f}$.

Galerkin approximation for the equilibrium problem

Problem BG

Find $u_h \in S^h$, *such that* $b(u_h, v) = (f, v)_0$ *for all* $v \in S^h$.

Since u_h and v are linear combinations of $\{\phi_1,\ \phi_2,\ \ldots,\ \phi_n\}$, the Galerkin approximation reduces to the system of linear equations:

Problem BD

Find $\bar{u} \in \mathbb{R}_n$, *such that* $K\bar{u} = \overline{F}$, *where* $F_i = (f, \phi_i)_0$.

Galerkin approximation for the eigenvalue problem

Problem CG

Find $w_h \in S^h$, $w_h \neq 0$, *and a complex number* λ^h, *such that*

$$b(w_h, v) = \lambda^h(w_h, v) \textit{ for all } v \in S^h.$$

Since u_h and v are linear combinations of $\{\phi_1,\ \phi_2,\ \ldots,\ \phi_n\}$, the Galerkin approximation reduces to the generalised eigenvalue problem:

Problem CD

Find $\bar{w} \in \mathbb{R}_n$, $\bar{w} \neq 0$, *and a complex number* λ, *such that*

$$K\bar{w} = \lambda M\bar{w}.$$

The vector $\bar{w}$ has components w_i where $w_h = \sum_{i=1}^{n} w_i\phi_i$.

More will be said concerning the computation of these matrices in Section 4.4 and Chapters 5 and 6.

4.2 Finite dimensional subspaces. Beam problems

4.2.1 Hermite cubics and Hermite quintics

The well-known Hermite piecewise cubics (see for instance [SF] or [Re]) are used successfully as basis functions for the Galerkin approximation in beam problems. Although cubics are sufficiently accurate for beam problems, we also use Hermite piecewise quintics. The main reason is that cubics will not be compatible with reduced quintics in plate beam models. As a bonus we find that quintics are extremely efficient. (See Chapter 5.)

The interval $I = [a, b]$ is divided into subintervals by nodes x_i, $i = 0, 1, \ldots, n$, with

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Consequently we have elements $\Omega_i = [x_{i-1}, x_i]$ of length h_i.

We proceed to define Hermite piecewise quintic polynomials. For $k = 0,\ 1,\ 2$ and for each element Ω_i there exist six quintic polynomials $\psi^{(k)}_{i-1,i}$ and $\psi^{(k)}_{i,i}$ with the following properties.

For $j = i - 1$ or i:

$$\text{If } m \neq j\text{:}\quad \psi^{(k)}_{ji}(x_m) = D\psi^{(k)}_{ji}(x_m) = D^2\psi^{(k)}_{ji} = 0.$$

$$\text{If } m = j\text{:}\quad D^\ell\psi^{(k)}_{ji}(x_m) = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k. \end{cases}$$

Next these polynomials are "pieced" together. The basis function $\phi^{(k)}_i$ is defined by

$$\phi^{(k)}_i = \begin{cases} \psi^{(k)}_{i,i+1} & \text{on } \Omega_{i+1}, \\ \psi^{(k)}_{i,i} & \text{on } \Omega_i, \\ 0 & \text{elsewhere.} \end{cases}$$

For piecewise Hermite cubics the construction is virtually the same. The only difference is that $k = 0,\ 1$.

Instead of piecewise Hermite cubics and piecewise Hermite quintics we will refer to cubics and quintics. Note that the cubics are elements of $H^2(a,b)$ and the quintics elements of $H^3(a,b)$.

Definition 4.2.1 *Interpolation operator.*

Let $r = 1$ for cubics and $r = 2$ for quintics. Then we define

$$\Pi u = \sum_{k=0}^{r} \sum_{i=1}^{n} D^k u(x_i) \phi_i^{(k)}.$$

Remark

1. For cubics it is necessary that $u \in H^2(I)$, for then $u \in C^1(\bar{I})$. (See Appendix B.) For quintics it is necessary that $u \in H^3(I)$.
2. Note that if $v = \Pi u$, then $D^k v(x_i) = D^k u(x_i)$.
3. Cubic splines are often used as basis functions. This has advantages over cubics (see [Pr]), but for reasons already mentioned we choose quintics as an alternative to cubics.

4.2.2 The damaged beam

The variational form of the damaged beam is derived in Section 3.1. In this section we construct a finite dimensional subspace for the Galerkin approximation. The interval $I = [0,1]$ is divided in such a way that $x = \alpha$ (location of damage) coincides with an interior node x_p. This means that $I_1 = (0, x_p)$ and $I_2 = (x_p, 1)$.

We construct a finite dimensional subspace of the space V by specifying a basis. The basis elements must be pairs of functions. Suppose $\phi_i^{(k)}$ is a quintic or cubic. Let $\phi_{i1}^{(k)}$ denote the restriction of $\phi_i^{(k)}$ to $I_1 = [0, \alpha]$ and $\phi_{i2}^{(k)}$ the restriction of $\phi_i^{(k)}$ to $I_2 = [\alpha, 1]$. We then define the basis elements $\tilde{\phi}_i^{(k)}$ by $\tilde{\phi}_i^{(k)} = \langle \phi_{i1}^{(k)}, \phi_{i2}^{(k)} \rangle$. These elements are in H^2 since $\phi_{i1}^{(k)} \in H^2(I_1)$ and $\phi_{i2}^{(k)} \in H^2(I_2)$. $\phi_p^{(1)}$ is an exception and may not be used. Instead we have $\tilde{\phi}_{pL}^{(1)} = \langle \phi_{p1}^{(1)}, 0 \rangle$ and $\tilde{\phi}_{pR}^{(1)} = \langle 0, \phi_{p2}^{(1)} \rangle$.

Consequently, $\tilde{\phi}_i^{(k)} \in V$ if $i \neq 0$. Also $\tilde{\phi}_{pL}^{(k)}$ and $\tilde{\phi}_{pR}^{(k)} \in V$. (They all satisfy the forced boundary conditions. For $k = 0$ and $k = 1$, $\tilde{\phi}_0^{(k)}$ are not admissible on account of the forced boundary conditions at $x = 0$.)

Let $h = \max h_i$. The finite dimensional subspace S^h is chosen as the span of all the admissible basis elements. Although the space S^h is determined by the partition of the interval, and not h, the notation S^h is commonly used. Note that S^h is a subspace of V, since the basis of S^h consists of elements of V.

Next we define an interpolation operator Π on H^k. (For any $u \in H^k$, $u_1 \in H^k(I_1)$ and $u_2 \in H^k(I_2)$.) We use the usual interpolation operators for these spaces (defined in the previous subsection) and denote them by Π_1 and Π_2:

Definition 4.2.2 *Interpolation operator*

$$\Pi u := \langle \Pi_1 u_1, \Pi_2 u_2 \rangle \quad \textit{for each} \quad u \in H^k.$$

$k = 3$ *for quintics and* $k = 2$ *for cubics.*

Note that $\Pi u \in V$ for all $u \in V$ as

$$\left(\Pi_i u_i\right)(x_j) = u_i(x_j) \quad \text{and} \quad \left(\Pi_i u_i\right)'(x_j) = u_i'(x_j).$$

4.2.3 Beam with dynamical boundary conditions

The construction of a finite dimensional subspace is simpler in this case. Suppose $\phi_i^{(k)}$ is either a cubic or a quintic. The basis elements for S^h are

$$\tilde{\phi}_i^{(k)} = \langle \phi_i^{(k)}, \phi_i^{(k)}(1), (\phi_i^{(k)})'(1) \rangle.$$

Note that $\phi_0^{(0)}$ and $\phi_0^{(1)}$ are excluded. Clearly $\tilde{\phi}_i^{(k)} \in V$ for each i and each k.

Definition 4.2.3 *Interpolation operator*

Let Π_1 *denote the usual interpolation operator.*

$$\Pi u := \langle \Pi_1 u_1, (\Pi_1 u_1)(1), (\Pi_1 u_1)'(1) \rangle \textit{ for } u \in H^k.$$

$k = 3$ *for quintics and* $k = 2$ *for cubics.*

Note that $\left(\Pi_1 u_1\right)(x_i) = u_1(x_i)$ and $\left(\Pi_1 u_1\right)'(x_i) = u_1'(x_i)$. Again, it is easy to see that $\Pi u \in V$ if $u \in V$.

4.3 Finite dimensional subspaces. Plate problems

4.3.1 Reduced quintics

We use only reduced quintics developed by Cowper *et al.* (See for instance [SF], [CKLO1], [CKLO2] and [CKLO3].) These basis functions are highly accurate for the Galerkin approximation in plate problems.

The rectangle Ω is divided into triangles or elements Ω_i. With each node we associate six basis functions. To define these functions the following notation for derivatives is convenient:

$$\begin{array}{ll} \partial^{(0)}u = u, & \partial^{(3)}u = \partial_1^2 u, \\ \partial^{(1)}u = \partial_1 u, & \partial^{(4)}u = \partial_1\partial_2 u, \\ \partial^{(2)}u = \partial_2 u, & \partial^{(5)}u = \partial_2^2 u. \end{array}$$

For each node $\mathbf{x}_j$ and for each element Ω_i (with $\mathbf{x}_j$ as a vertex) there exist six reduced quintics $\psi_{ji}^{(k)}$ for $k = 0, 1, \ldots, 5$ with the following properties:

$$\text{For } j = m: \qquad \partial^{(\ell)}\psi_{ji}^{(k)}(\mathbf{x}_m) = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k. \end{cases}$$

$$\text{For } j \neq m: \qquad \partial^{(\ell)}\psi_{ji}^{(k)}(\mathbf{x}_m) = 0.$$

Next these polynomials are "pieced" together. We define the basis function $\phi_j^{(k)}$ by:

> The restriction of $\phi_j^{(k)}$ to element Ω_i is $\psi_{ji}^{(k)}$ if $\mathbf{x}_j$ is a vertex of Ω_i.
> If $\mathbf{x}_j$ is not a vertex of Ω_i then $\phi_j^{(k)} = 0$ on Ω_i.

The piecewise polynomial functions $\phi_j^{(k)}$ are continuous and have continuous partial derivatives. The second order partial derivatives are not continuous. However, these functions are elements of $H^2(\Omega)$.

Definition 4.3.1 *Interpolation operator* Π_Ω

$$\Pi_\Omega u := \sum_{j=1}^{n}\sum_{k=0}^{5} \partial^{(k)}u(x_j)\phi_j^{(k)}.$$

Consequently, if $v = \Pi_\Omega u$, then $\partial^{(k)} v(x_j) = \partial^{(k)} u(x_j)$.

4.3.2 Plate beam model

The elements of S^h must be ordered triples of the form $\langle u, \gamma_0 u, \gamma_1 u\rangle$. The basis elements for S^h are

$$\tilde{\phi}_i^{(k)} = \langle \phi_i^{(k)}, \gamma_0 \phi_i^{(k)}, \gamma_1 \phi_i^{(k)} \rangle.$$

Note that some basis functions must be excluded to satisfy the forced boundary conditions. Clearly $\tilde{\phi}_i^{(k)} \in V$ for each i and each k. The finite dimensional space S^h is the span of all the admissible basis functions.

Definition 4.3.2 *Interpolation operator*

$$\Pi u := \langle \Pi_\Omega u_1, \gamma_0 (\Pi_\Omega u_1), \gamma_1 (\Pi_\Omega u_1) \rangle \text{ for } u \in H^2.$$

Remark Let Π_Γ denote the interpolation operator defined for quintics in Section 4.2.1. Note that

$$\gamma_0 (\Pi_\Omega u) = \Pi_\Gamma (\gamma_0 u).$$

The same is true for γ_1.

4.4 Implementation

We now reconsider the three types of problems posed in Section 4.1. Having defined finite dimensional subspaces, the matrices K, L, M and M_0 are also defined.

The equilibrium problem, Problem BD, is trivial and needs no further discussion.

The eigenvalue problem, Problem CD, is a generalized eigenvalue problem. For the different model problems, this is solved, after the matrices have been computed, using standard Matlab subroutines. In the one-dimensional case, this is a straightforward procedure. (Chapter 6.) The two-dimensional case is more interesting because of the presence of repeated eigenvalues and the irregular pattern in which they occur. As the multiplicity of eigenvalues for the abstract eigenvalue problem (Problem B) and the Galerkin approximation (Problem BG) do not correspond, it can be difficult to interpret the numerical results. We will elaborate on this in Chapter 7.

The vibration problem, Problem AD, is an initial value problem for a second order system of ordinary differential equations

$$\begin{aligned} M\bar{u}''(t) + L\bar{u}'(t) + K\bar{u}(t) &= \bar{f}(t), \\ \bar{u}(0) = \bar{\alpha},\ \bar{u}'(0) &= \bar{\beta}, \end{aligned}$$

which is to be solved on an interval $[0,\tau]$. The initial conditions, $\bar{\alpha}$ and $\bar{\beta}$, depend on the choice of approximations for α and β. One possibility is to use $\alpha_h = \Pi\alpha$ and $\beta_h = \Pi\beta$. Another possibility is to use projections—to be defined in Section 4.6. This problem is solved using a finite difference method. The interval $[0,\tau]$ is partitioned into subintervals of length δt and $t_k = k\delta t$.

Let $\bar{u}_k$ denote the approximation for $\bar{u}(t_k)$. We use the following scheme:

$$\begin{aligned} (\delta t)^{-2}M(\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}) & \\ +(2\delta t)^{-1}L(\bar{u}_{k+1} - \bar{u}_{k-1}) & \\ +K(\rho_1\bar{u}_{k+1} + \rho_0\bar{u}_k + \rho_1\bar{u}_{k-1}) &= \rho_1\bar{f}_{k+1} + \rho_0\bar{f}_k + \rho_1\bar{f}_{k-1}. \end{aligned} \tag{4.4.1}$$

More detail about the weights, ρ_0 and ρ_1, will be given in Section 5.4 and Chapter 6. The scheme is started with initial conditions $\bar{u}_0 = \bar{\alpha}$ and $(2\delta t)^{-1}(\bar{u}_1 - \bar{u}_{-1}) = \bar{\beta}$.

It is clear that as far as implementation is concerned, the real challenge is the computation of the matrices K, L and M. This is complicated by the interface conditions which result in non-standard basis elements. We write our own code to assemble the matrices.

As an example we illustrate the computation for the simplest case, namely the beam with tip body. The dimensionless model is given in Section 2.4.

The interval $I = [0, 1]$ is divided into subintervals by nodes x_i, $i = 0, 1, \ldots, n$, with

$$0 = x_0 < x_1 < \cdots < x_n = 1.$$

Consequently, we have elements $\Omega_i = [x_{i-1}, x_i]$ of length h_i. Suppose we use Hermite piecewise cubics defined in Section 4.2 and the basis elements are ordered as follows:

$$\tilde{\phi}_i = \begin{cases} \tilde{\phi}_i^{(0)} & \text{for } i = 1,\ 2,\ \ldots,\ n, \\ \tilde{\phi}_{i-n}^{(1)} & \text{for } i = n+1,\ n+2,\ \ldots,\ 2n. \end{cases}$$

First consider the computations for an undamaged beam. We denote the standard cubics by ϕ_i. Also, the matrices M^A, M_1 and K are the usual matrices used for an undamaged beam:

$$M_{ij}^A = \int_0^1 \phi_i \phi_j, \quad (M_1)_{ij} = \int_0^1 \phi_1' \phi_j' \quad \text{and} \quad K_{ij}^A = \int_0^1 \phi_i'' \phi_j''.$$

Next we show how to adapt these matrices for the beam with tip body. (The bilinear forms are defined in Section 3.2 and the matrices in Section 4.1.)

The K-matrix does not change since

$$K_{ij} = b(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \phi_i'' \phi_j''.$$

Since $\mu_0 = \mu_1 = 0$,

$$L_{ij} = a(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \left(\mu \phi_i'' \phi_j'' + k \phi_i \phi_j\right)$$

which yields that $L = \mu K + k M_A$. Lastly,

$$\begin{aligned} M_{ij} &= c(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \left(\phi_i \phi_j + r \phi_i' \phi_j'\right) + m\phi_i(1)\phi_j(1) + I_m \phi_i'(1)\phi_j'(1) \\ &= M_{ij}^A + r\,[M_1]_{ij} + m\phi_i(1)\phi_j(1) + I_m \phi_i'(1)\phi_j'(1). \end{aligned}$$

Thus, $M = M_A + rM_1 + M^*$ where M^* is the $2n \times 2n$ zero matrix, except for two non-zero entries namely $M_{n,n}^*$ and $M_{2n,2n}^*$.

4.5 Interpolation error

4.5.1 Standard estimates

In this subsection we quote standard interpolation estimates, as found in, for instance, [SF], [OR] and [OC]. The standard Sobolev spaces $H^k(I)$ and $H^k(\Omega)$ are used and $\|\cdot\|$ denotes the $L^2(I)$ or $L^2(\Omega)$ norm.

We introduce two parameters for an interpolation operator Π:
$r(\Pi)$ is the highest degree of polynomials left invariant by the operator Π,
$s(\Pi)$ is the highest order derivative used in the definition of Π.

We will use $\widehat{C}$ to denote a generic constant which depends on the constants in Sobolev's lemma and the constants in the Bramble Hilbert lemma.

In the following result $|\cdot|_k$ denotes the seminorm of order k, i.e. $|u|_k = \|u^{(k)}\|$, on an interval I.

Lemma 4.5.1 *Suppose $s(\Pi)+1 \leq k \leq r(\Pi)+1$. Then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \ m = 0, 1, \ldots, k.$$

Notation If $u \in H^k(I)$, let k^* denote the minimum of k and $r(\Pi) + 1$.

Corollary 4.5.1 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$, with $k \geq s(\Pi) + 1$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

Proof $H^k(I) \subset H^{r(\Pi)+1}$ for $k \geq r(\Pi) + 1$. □

Remark For cubics, $k^* = 4$ if $k \geq 4$. For quintics, $k^* = 6$ if $k \geq 6$.

In the following result, for a two-dimensional domain Ω, $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

In the two-dimensional case, the constant of the estimate also depends on the shape of the elements. Care should be taken that the minimum angle of any triangle element does not become too small. See for instance [SF, p 138].

Lemma 4.5.2 *Suppose $s(\Pi)+2 \leq k \leq r(\Pi)+1$. Then there exists a constant $\widehat{C}$ such that, for $u \in H^k(\Omega)$,*

$$\|u - \Pi u\|_m \leq \widehat{C}h^{k-m}|u|_k, \; m = 0, 1, \ldots, k.$$

Remark For reduced quintics $r(\Pi) = 4$, hence $k^* = 5$ if $k \geq 5$.

Corollary 4.5.2 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)$, with $k \geq s(\Pi) + 2$,*

$$\|u - \Pi u\|_m \leq \widehat{C}h^{k^*-m}|u|_{k^*}, \; m = 0, 1, \ldots, k^*.$$

4.5.2 Damaged beam model

The seminorm of order k for the product space $H^k = H^k(I_1) \times H^k(I_2)$ is defined by $|u|_k^2 = |u_1|_k^2 + |u_2|_k^2$.

Lemma 4.5.3 *There exists a constant $\widehat{C}$ and an s^* such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C}h^{k^*-m}|u|_{k^*}, \; m = 0, 1, \ldots, k^*.$$

Proof The result is a direct consequence of the definition of seminorms, norms and the interpolation operator on the product space. For this interpolation operator, $s^* = 2$ and $k^* = \min\{k, 4\}$, if adapted cubic basis functions are used, and $s^* = 3$ and $k^* = \min\{k, 6\}$, if adapted quintic basis functions are used. □

4.5.3 Beam with dynamical boundary conditions

The seminorm of order k for the product space $H^k = H^k(I) \times \mathbb{R} \times \mathbb{R}$ is defined by $|u|_k^2 = |u_1|_k^2$.

Lemma 4.5.4 *There exists a constant $\widehat{C}$ and an s^* such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C}h^{k^*-m}|u|_{k^*}, \; m = 0, 1, \ldots, k^*.$$

Proof The result is a direct consequence of the definition of seminorms, norms and the interpolation operator on the product space. Also for this interpolation operator, $s^* = 2$ and $k^* = \min\{k, 4\}$ if adapted cubic basis functions are used, and $s^* = 3$ and $k^* = \min\{k, 6\}$ if adapted quintic basis functions are used. □

4.5.4 Plate beam model

The seminorm of order k on the product space $H^k = H^k(\Omega) \times H^k(I) \times H^k(I)$ is defined by $|u|_k^2 = |u_1|_k^2 + |u_2|_k^2 + |u_3|_k^2$.

Consider the interpolation operator Π defined in Section 4.3.2. In this case k^* is the minimum of k and 5.

Lemma 4.5.5 *There exists a constant $\widehat{C}$ and an s^* such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

Proof Note that for any $u \in H^m$, we have that

$$\|u - \Pi u\|_m^2 = \|u_1 - \Pi_\Omega u_1\|_m^2 + \|u_2 - \Pi_\Gamma u_2\|_m^2 + \|u_3 - \Pi_\Gamma u_3\|_m^2.$$

Use the results of Corollaries 4.5.1 and 4.5.2. From Corollary 4.5.2 it follows that $s^* = 4$ as $s(\Pi) = 2$ for reduced quintics. □

4.5.5 Abstract error estimates

At this stage a unified approach is possible. We have a Hilbert space V, a finite dimensional subspace S^h, an interpolation operator Π and require an estimate for the interpolation error $u - \Pi u$.

In Subsections 4.5.2 to 4.5.4 we showed that for all the model problems we can define parameters s^* and k^* for each interpolation operator, such that the following general interpolation estimate holds.

Lemma 4.5.6 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

As $V \subset H^2$ for all the model problems, and the energy norm is equivalent to the H^2 norm, the following interpolation estimate holds.

Corollary 4.5.3 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$, with $k \geq s^*$,*

$$\|u - \Pi u\|_E \leq \widehat{C} h^{k^*-2} |u|_{k^*}.$$

For all the model problems $s^* < 4$ and for $k = 4$ it follows that $k^* = 4$. This means that the following result applies to all the interpolation operators that we use.

Corollary 4.5.4 *There exists a constant $\widehat{C}$ such that, for all $u \in H^4 \cap V$,*

$$\|u - \Pi u\|_E \leq \widehat{C} h^2 |u|_4.$$

4.6 Approximation

We have a Hilbert space V, a finite dimensional subspace S^h, an interpolation operator Π and an estimate for the interpolation error $u - \Pi u$. We now introduce a projection of V onto the subspace S^h. This projection will feature in every convergence proof in Chapter 4. For more information on this projection, see any Functional Analysis text, for example [Kr, Section 3.3].

Definition 4.6.1 *Projection P*

P is a projection of V onto S^h with respect to the inner product b.

The definition implies that for any $x \in V$,

$$b(x - Px, v) = 0 \text{ for all } v \in S^h.$$

Due to the important role that P will play in the theory, we display the properties of this projection:

$$\|x - Px\|_E \leq \|x - v\|_E \text{ for all } v \in S^h,$$

$$\|Px - v\|_E \leq \|x - v\|_E \text{ for all } v \in S^h$$

and

$$\|Px\|_E \leq \|x\|_E.$$

Lemma 4.6.1 *There exists a constant $\widehat{C}$ such that, for any $u \in H^k \cap V$ with $k \geq s^*$,*

$$\|Pu - u\|_E \leq \widehat{C}|u|_{k^*}h^{k^*-2} \text{ and } \|\Pi u - Pu\|_E \leq \widehat{C}|u|_{k^*}h^{k^*-2}.$$

Proof

$$\|Pu - u\|_E \leq \|\Pi u - u\|_E$$

and

$$\|\Pi u - Pu\|_E \leq \|u - \Pi u\|_E.$$

Now use the results for the interpolation error in Corollary 4.5.3. □

The next result is convenient due to the fact that it applies to all the interpolation operators that we use.

Corollary 4.6.1 *There exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,*

$$\|Pu - u\|_E \leq \widehat{C}|u|_4 h^2 \quad and \quad \|\Pi u - Pu\|_E \leq \widehat{C}|u|_4 h^2.$$

Lemma 4.6.2 *For any $\varepsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that*

$$\|u - Pu\|_E < \varepsilon, \quad if \ h < \delta.$$

Proof For any $u \in V$ there exists a $w \in H^4 \cap V$ such that

$$\|u - w\|_E \leq \varepsilon.$$

Now,

$$\begin{aligned} \|Pu - u\|_E &\leq \|u - w\|_E + \|w - Pw\|_E + \|Pw - Pu\|_E \\ &\leq \varepsilon + \widehat{C}|w|_4 h^2 + \varepsilon \\ &< 3\varepsilon \text{ for } h \text{ sufficiently small.} \end{aligned}$$

□

In a final result we show that the Aubin-Nitsche trick, [N], can also be applied to find estimates in the inertia norm for the discretization error.

Lemma 4.6.3 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$, with $k \geq s^*$,*

$$\|u - Pu\| \leq \widehat{C} h^{k^*} |u|_{k^*}.$$

Proof Set $e_p = u - Pu$. As b defines an inner product on V it follows from the Riesz theorem that there exists a unique $y \in V$ such that

$$b(y, v) = (e_p, v) \text{ for all } v \in V. \tag{4.6.1}$$

Regularity results yield that $y \in H^4 \cap V$ and that there exists a c_b such that

$$\|y\|_4 \leq c_b \|e_p\|. \tag{4.6.2}$$

Since P is a projection

$$b(e_p, v) = 0 \text{ for all } v \in S^h. \tag{4.6.3}$$

Let $v = e_p$ in (4.6.1) and $v = Py$ in (4.6.3). This yields that

$$\|e_p\|^2 = b(y - Py, e_p) \leq \|y - Py\|_E \|e_p\|_E.$$

From Corollary 4.6.1,

$$\|e_p\|^2 \leq \widehat{C}|y|_4 h^2 \|e_p\|_E.$$

We conclude from (4.6.2) that

$$\|e_p\| \leq \widehat{C}h^2 \|e_p\|_E.$$

The result now follows from Lemma 4.6.1. □

For all our model problems the following result applies.

Corollary 4.6.2 *There exists a constant $\widehat{C}$ such that, for all $u \in H^4 \cap V$,*

$$\|u - Pu\| \leq \widehat{C}h^4 |u|_4.$$

Chapter 5

Convergence

In Section 3.4 we presented three general problems, Problems A, B and C. In Section 4.1 we formulated the problems for the Galerkin approximations. These are Problems AG, BG and CG.

5.1 Equilibrium problem

In this section we consider the convergence of the solution of Problem BG to the solution of Problem B.

Assume that $u^h \in S^h$ is the solution of

$$b(u^h, v) = (f, v) \text{ for all } v \in S^h \tag{5.1.1}$$

and $u \in V$ is the solution of

$$b(u, v) = (f, v) \text{ for all } v \in V. \tag{5.1.2}$$

In the proof of the theorem, we will use the projection P, defined in Section 4.6.

Theorem 5.1.1

1. $\|u^h - u\|_E \to 0$ *if* $h \to 0$.

2. *If* $u \in H^4 \cap V$, *then*

$$\|u^h - u\|_E \leq \widehat{C} h^{k^*-2} |u|_4,$$

and

$$\|u^h - u\|_0 \leq \widehat{C} h^{k^*} |u|_4.$$

Proof If (5.1.2) is subtracted from (5.1.1), it follows that

$$b(u - u^h, v) = 0 \text{ for all } v \in S^h.$$

This means that $u^h = Pu$. The first part of the theorem follows directly from Lemma 4.6.2. The estimates in the second part of the theorem follow from Lemma 4.6.1 and Lemma 4.6.3. □

5.2 Eigenvalue problem

Our approach is based on the ideas of [BDSW], [BF] and [SF] and we follow the presentation in [SF]. The theory in the book of Strang and Fix, [SF, Section 6.3], concerns eigenvalue problems for general symmetric elliptic operators. Most of the presentation is written in a style which encourage abstraction. In collaboration with others, [ZVGV2], we verified that the theory is valid for abstract eigenvalue problems such as Problem C. In this thesis we present this abstract version, and also offer a number of modest improvements.

The rate of convergence for eigenvalue problems also depends on the regularity of the eigenvectors. In the absence of such theory for interface problems, we pose the following assumption which we showed to be true in the one-dimensional case. (See Section 3.5.)

Regularity Assumption The eigenvectors of the eigenvalue problem, Problem C, are in $H^k \cap V$ for $k = 4$ or 6, and there exists a constant C_b—depending on the bilinear forms b and $(\cdot,\cdot)$—such that for each eigenvector y

$$|y|_k \leq C_b \lambda_i^\alpha \|y\|_0, \text{ where } \alpha = \frac{k}{4}.$$

5.2.1 The Rayleigh quotient and the Minmax principle

To analyse the convergence of eigenvalues and eigenvectors, some preparation is necessary. First we establish bounds for the approximate eigenvalues using the Rayleigh quotient and the minimax principle.

It is well-known that the eigenvalues are the stationary values of the Rayleigh quotient. However, the following result gives a more convenient characterization of the eigenvalues. See [SF, p 221].

Lemma 5.2.1 *Minmax principle*

Let $\mathcal{T}$ denote the class of subspaces of V having dimension j, then

$$\lambda_j = \min_{S \in \mathcal{T}} \max_{v \in S} R(v).$$

We may assume that the eigenvalues are ordered

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots.$$

For some integer m, consider the eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_m$ and corresponding eigenvectors $y_1,\ y_2,\ \ldots,\ y_m$. Equal eigenvalues are possible but we assume that $\lambda_j \neq \lambda_m$ for each $j > m$.

In the finite dimensional subspace S^h we have $\lambda_1^h,\ \lambda_2^h,\ \ldots,\ \lambda_m^h$ (also ordered) and corresponding eigenvectors $y_1^h,\ y_2^h,\ \ldots,\ y_m^h$. (Equal eigenvalues do not matter. In the case of multiplicity, y_j^h is not uniquely determined, but it does not influence any proof.)

5.2.2 Bounds for the approximate eigenvalues

The minimax principle yields lower bounds for the approximate eigenvalues.

Lemma 5.2.2 $\lambda_j^h \geq \lambda_j$ *for each* j.

Proof The minmax principle is also true for the space S^h. Any subspace of S^h is a subspace of V. □

Notation y_i will be used to denote the normalised eigenvectors, i.e. $\|y_i\| = 1$.

For $j = 1,\ 2,\ \ldots,\ m$, let E_j denote the subspace of V spanned by $\{y_1,\ y_2,\ \ldots,\ y_j\}$.

Consider the subspaces S_j where

$$S_j = PE_j \text{ for } j = 1,\ 2,\ \ldots,\ m.$$

$P = P_h$ is the projection defined in Section 4.6.

An upper bound for some approximate eigenvalue depends on the construction of S^h. Clearly the construction of S^h must be such that $\dim S^h = N > m$. However, it is still possible that $\dim S_m < m$.

Assumption The construction of S^h is such that $\dim S_m = m$.

We define a quantity μ_m^h to measure the "distortion" of the projection of the unit ball $B_m = \{y \in E_m : \|y\| = 1\}$: Set

$$\mu_m^h = \inf\{\|Py\|^2 : y \in B_m\}.$$

Proposition 5.2.1 $\mu_m^h > 0$ *if and only if* $\dim S_m = m$.

Proof The function $\|Py\|^2$ has a minimum on the compact set B_m. Hence $\mu_m^h > 0$ if and only if $Py \neq 0$ for each $y \in B_m$. But this is so if and only if the vectors $Py_1,\ Py_2,\ \ldots, Py_m$ form a linearly independent set. □

Proposition 5.2.2 $\lambda_m^h \leq \max\{R(Py) : y \in B_m\}$.

Proof

Since $\dim S_m = m$, it follows from the minimax principle that

$$\lambda_m^h \leq \max\{R(v) : v \in S_m\}.$$

Consider any $v \in S_m,\ v \neq 0$. There exists a vector $y \in E_m$ such that $Py = v$.

Note that $R(P(\alpha y)) = R(\alpha v) = R(v)$. Choose α such that $\alpha y \in B_m$. Consequently,

$$\max\{R(Pz) : z \in B_m\} = \max\{R(v) : v \in S_m\}.$$

□

The following result is crucial.

Lemma 5.2.3 $\lambda_m^h \leq \dfrac{\lambda_m}{\mu_m^h}$.

Proof If $y = \sum_{i=1}^m c_i y_i$, then $b(y,y) = \displaystyle\sum_{i=1}^m c_i^2 \lambda_i$, since $\{y_1,\ y_2,\ \ldots, y_m\}$ is an orthonormal set. Hence

$$b(y,y) \ \leq \ \lambda_m \sum_{i=1}^m c_i^2 = \lambda_m \|y\|^2 \text{ for each } y \in E_m.$$

Since P is a projection with respect to the inner product b,

$$b(Py, Py) \leq \lambda_m \quad \text{for each} \quad y \in B_m.$$

From the definition of μ_m^h, we have

$$R(Py) = \frac{b(Py, Py)}{\|Py\|^2} \leq \frac{\lambda_m}{\mu_m^h}.$$

Now use Proposition 5.2.2. □

Corollary 5.2.1 $\mu_m^h \leq 1$.

This is a direct consequence of Lemmas 5.2.2 and 5.2.3. It is convenient to formulate error estimates in terms of the quantity $1 - \mu_m^h$.

Notation $\sigma_m^h = 1 - \mu_m^h$.

Corollary 5.2.2 $0 \leq \sigma_m^h < 1$ *and* $\lambda_m^h - \lambda_m \leq \lambda_m^h \sigma_m^h$.

Since the eigenvalue error is bounded by σ_m^h, it is sufficient to estimate σ_m^h and prove that $\sigma_m^h \to 0$.

5.2.3 Estimates

Proposition 5.2.3 $\sigma_m^h = \max\{2(y, y - Py) - \|y - Py\|^2 : y \in B_m\}$.

Proof

$$\begin{aligned} \|y - Py\|^2 &= \|Py\|^2 + \|y\|^2 - 2(y, Py) \\ &= \|Py\|^2 + 2(y, y) - 2(y, Py) - 1 \quad (\text{since } \|y\|^2 = 1). \end{aligned}$$

As a consequence $1 - \|Py\|^2 = 2(y, y - Py) - \|y - Py\|^2$. The result follows from the definition of σ_m^h. □

Remark In [SF, Section 6.3] σ_m^h is defined by

$$\sigma_m^h = \max\{|2(y, y - Py) - \|y - Py\|^2| : y \in B_m\}.$$

The absolute value is not necessary since

$$\max\{2(y, y - Py) - \|y - Py\|^2 : y \in B_m\} \geq 0.$$

The assumption is then made that $\sigma_m^h < 1$, and they prove that $\dim S_m = m$.

We proved the fact that $\dim S_m = m$ is equivalent to $\sigma_m^h < 1$ and we believe that it is important to take note of this equivalence.

Notation For any $y \in E_m$, let $y^* = \sum_{i=1}^m c_i \lambda_i^{-1} y_i$ where $y = \sum_{i=1}^m c_i y_i$.

Proposition 5.2.4 *For any $y \in E_m$,*

$$(y, y - Py) = b(y^* - Py^*, y - Py).$$

Proof

$b(y_i - Py_i, y - Py) = b(y_i, y - Py)$ since $b(y - Py, Py_i) = 0$.

Hence,

$$\lambda_i(y_i, y - Py) = b(y_i - Py_i, y - Py).$$

Multiply by $c_i\lambda_i^{-1}$ and sum over i. We have

$$\begin{aligned} (y, y - Py) &= \sum_{i=1}^{m} c_i\lambda_i^{-1} b(y_i - P_i, y - Py) \\ &= b\left(\sum_{i=1}^{m} c_i\lambda_i^{-1} y_i - \sum_{i=1}^{m} c_i\lambda_i^{-1} Py_i, y - Py\right). \end{aligned}$$

□

The following result also differ from [SF].

Lemma 5.2.4 $\sigma_m^h \leq \max\{2\|y^* - Py^*\|_E \|y - Py\|_E : y \in B_m\}$.

Proof Consider the result of Proposition 5.2.3. We have demonstrated that the quantity

$$2(y, y - Py) - \|y - Py\|^2$$

must have a non negative maximum (Corollary 5.2.2). Consequently

$$\sigma_m^h \leq \max\{2(y, y - Py) : y \in B_m\}.$$

Use Proposition 5.2.4 and the Schwartz inequality for the inner product b. □

Proposition 5.2.5 *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for $h < \delta$,*

$$\begin{aligned} \|y^* - Py^*\|_E &< \varepsilon \text{ for each } y \in B_m, \\ \|y - Py\|_E &< \varepsilon \text{ for each } y \in B_m. \end{aligned}$$

Proof From Lemma 4.6.2 there exist positive numbers $\delta_1,\ \delta_2,\ \ldots,\delta_n$ such that for each i

$$\|y_i - Py_i\|_E \leq \varepsilon \text{ if } h < \delta_i.$$

Now, suppose $h < \min_i \delta_i$, then

$$\|y - Py\|_E \dot{\leq} \sum_{i=1}^{m} |c_i| \|y_i - Py_i\|_E \leq \varepsilon m, \text{ since } |c_i| < 1.$$

The same arguments are valid for $\|y^* - Py^*\|_E$. □

Lemma 5.2.5 *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sigma_m^h < \varepsilon \quad \text{if} \quad h < \delta.$$

Proof

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $h < \delta$, then

$$\|u - Pu\|_E < \varepsilon \quad \text{for each} \quad u \in B_m.$$

The result follows from Lemma 5.2.4 and Proposition 5.2.5. □

Proposition 5.2.6 *If Problem C satisfies the regularity assumption, then for any $y \in B_m$*

$$\|y^* - Py^*\|_E \leq \widehat{C} C_b \lambda_m^{\alpha-1} h^{k^*-2}$$

and

$$\|y - Py\|_E \leq \widehat{C} C_b \lambda_m^{\alpha} h^{k^*-2}.$$

Proof We may assume that $c_i \geq 0$ for each i.

First estimate:

$$\begin{aligned}
\|y^* - Py^*\|_E &\leq \sum_{i=1}^{m} c_i \lambda_i^{-1} \|y_i - Py_i\|_E \\
&\leq \widehat{C} \sum_{i=1}^{m} c_i \lambda_i^{-1} |y_i|_{k^*} h^{k^*-2} \\
&\leq C_b \widehat{C} \sum_{i=1}^{m} c_i \lambda_i^{\alpha-1} \|y_i\| h^{k^*-2} \\
&\leq \widehat{C} C_b \lambda_m^{\alpha-1} h^{k^*-2}.
\end{aligned}$$

Second estimate:

$$\begin{aligned} \|y - Py\|_E &\leq \sum_{i=1}^{m} c_i \|y_i - Py_i\|_E \\ &\leq \widehat{C} C_b \lambda_i^{\alpha} h^{k^*-2}, \end{aligned}$$

using the same arguments as for the first estimate. □

Lemma 5.2.6 *If Problem C satisfies the regularity assumption, then*

$$\sigma_m^h \leq \widehat{C} C_b \lambda_m^{2\alpha-1} h^{2(k^*-2)}.$$

Proof

Use Lemma 5.2.4 and Proposition 5.2.6. □

5.2.4 Convergence of eigenvalues

We may now use the results of the previous subsection to establish the convergence of λ_m^h to λ_m.

Lemma 5.2.7 *There exists a $\delta > 0$ such that for $h < \delta$,*

$$\lambda_m^h - \lambda_m \leq 2\lambda_m \sigma_m^h.$$

Proof

Choose δ such that $\sigma_m^h < \frac{1}{2}$. Consequently $\lambda_m^h < 2\lambda_m$. □

Theorem 5.2.1

1. $\lambda_m^h - \lambda_m \to 0$ *as* $h \to 0$.
2. *If Problem C satisfies the regularity assumption, then*
$$\lambda_m^h - \lambda_m \leq \widehat{C} C_b \lambda_m^{2\alpha} h^{2(k^*-2)}.$$

Proof

1. This is a direct consequence of Lemmas 5.2.5 and 5.2.7.
2. Use Lemmas 5.2.6 and 5.2.7. □

5.2.5 Convergence of eigenvectors

To estimate the error $\|y_m - y_m^h\|$, we need to estimate the difference $\|Py_m - y_m^h\|$. It is necessary to consider the possibility that λ_m has multiplicity more than one. Suppose that the multiplicity of λ_m is r and let $\Lambda = 1,\ 2,\ \ldots,\ m-r,\ m+1,\ \ldots,\ N$. From Theorem 5.2.1 it follows that there exist real numbers $\rho > 0$ and $\delta > 0$ such that if $h < \delta$, then

$$|\lambda_m - \lambda_j^h| > \rho \text{ for each } j \in \Lambda. \tag{5.2.1}$$

Assumption Assume that h is sufficiently small for (5.2.1) to hold.

Suppose $\{y_{m-r+1}^h,\ y_{m-r+2}^h,\ \ldots,\ y_m^h\}$ is an orthonormal set of eigenvectors corresponding to $\lambda_{m-r+1}^h,\ \lambda_{m-r+2}^h,\ \ldots,\ \lambda_m^h$. The strategy now is to estimate the distance between y_{m-r+i}^h and some (uniquely defined) vector in E_{λ_m}, the eigenspace corresponding to λ_m.

We define a projection P_m with domain $P(E_{\lambda_m})$:

$$P_m w = \sum_{j=m-r+1}^{m} (w, y_j^h) y_j^h \text{ for each } w \in P(E_{\lambda_m}).$$

This projection enables us to deal with the case of a repeated eigenvalue.

Here we differ from [SF]. Although most of the computations are the same, we believe that our construction of the projection P_m is a worthwhile contribution. We will show that $P_m P$ (and hence P_m) is invertible for h sufficiently small.

Proposition 5.2.7 *For each $j \in \Lambda$ and each $y \in E_{\lambda_m}$,*

$$(\lambda_j^h - \lambda_m)(Py, y_j^h) = \lambda_m (y - Py, y_j^h).$$

Proof

It is only necessary to prove that

$$\lambda_j^h (Py, y_j^h) = \lambda_m (y, y_j^h) \tag{5.2.2}$$

since the term $-\lambda_m (Py, y_j^h)$ appears on both sides of the equation.

Since y_j^h and y are eigenvectors, it follows that

$$\lambda_j^h (Py, y_j^h) = b(Py, y_j^h) \quad \text{and} \quad \lambda_m (y, y_j^h) = b(y, y_j^h).$$

But $b(Py - y, y_j^h) = 0$ for each j, thus (5.2.2) follows. ☐

Lemma 5.2.8

$$\|Py - P_m Py\| \leq \lambda_m \rho^{-1} \|y - Py\| \text{ for each } y \in E_{\lambda_m}.$$

Proof From the assumption we have the estimate

$$\frac{\lambda_m}{|\lambda_j^h - \lambda_m|} \leq \rho_m \quad \text{for each} \quad j \in \Lambda, \tag{5.2.3}$$

where $\rho_m = \lambda_m \rho^{-1}$.

The set $y_1^h,\ y_2^h,\ \ldots,\ y_N^h$ form an orthonormal basis for S^h, hence

$$Py = \sum_{j=1}^{N} (Py, y_j^h) y_j^h.$$

Consequently,

$$Py - P_m Py = \sum_{j \in \Lambda} (Py, y_j^h) y_j^h.$$

If $y \in E_{\lambda_m}$, then

$$\|Py - P_m Py\|^2 = \sum_{j \in \Lambda} (Py, y_j^h)^2.$$

We now use Proposition 5.2.7.

$$\begin{aligned} \|Py - P_m Py\|^2 &= \sum_{j \in \Lambda} \left(\frac{\lambda_m}{|\lambda_j^h - \lambda_m|} \right)^2 (y - Py, y_j^h)^2 \\ &\leq \rho_m^2 \sum_{j \in \Lambda} (y - Py, y_j^h)^2 \quad \text{(Inequality (5.2.3))} \\ &\leq \rho_m^2 \sum_{j=1}^{N} (y - Py, y_j^h)^2 \\ &= \rho_m^2 \|y - Py\|^2. \end{aligned}$$

□

Lemma 5.2.9

$$\|y - P_m Py\| \le (1 + \lambda_m \rho^{-1})\|y - Py\| \textit{ for each } y \in E_{\lambda_m}.$$

Proof

$$\begin{aligned}\|y - P_m Py\| &\le \|y - Py\| + \|Py - P_m Py\| \\ &\le (1 + \lambda_m \rho^{-1})\|y - Py\|.\end{aligned}$$

□

Corollary 5.2.3 *$P_m P$ is invertible for h sufficiently small.*

Proof Let $y \in B_m \cap E_{\lambda_m}$. Then

$$\|Py - P_m Py\| \le \lambda_m \rho^{-1}\|y - Py\| < \frac{1}{2},$$

for h sufficiently small. Since

$$\|Py\|^2 = \|P_m Py\|^2 + \|Py - P_m Py\|^2,$$

it follows that $\|P_m Py\| > \frac{1}{4}$. Consequently

$$\|P_m Py\| > \frac{1}{4}\|y\| \text{ for each } \quad y \in E_{\lambda_m}.$$

□

Corollary 5.2.4 *If h is sufficiently small, then for each j, $j = 1, 2, \ldots, r$ there exists a unique $x_j \in E_{\lambda_m}$ with $\|x_j\| = 1$ such that*

$$\|x_j - y^h_{m-r+j}\| \le 4(1 + \rho^{-1}\lambda_m)\|x_j - Px_j\|.$$

Proof There exists a unique $y \in E_{\lambda_m}$ such that $y = (P_m P)^{-1} y_{m-r+j}$. Hence,

$$\|y - y^h_{m-r+j}\| \le (1 + \rho^{-1}\lambda_m)\|y - Py\|.$$

Let β be a real number such that $|\beta| = \|y\|$ and let $x_j = \beta^{-1} y$. We can choose x_j such that $\beta > 0$. As a consequence $\|y\| = \beta$.

It follows that

$$\|x_j - y\| = |\beta - 1| = \big|\|y\| - \|y^h_{m-r+j}\|\big| \le \|y - y^h_{m-r+j}\|.$$

Hence

$$\begin{aligned}\|x_j - y^h_{m-r+j}\| &\le \|x_j - y\| + \|y - y^h_{m-r+j}\| \\ &\le 2(1+\rho^{-1}\lambda_m)\|y - Py\|.\end{aligned}$$

□

It is important to realise that one compute the approximation y^h_{m-r+j}. The result above guarantees the existence of an exact eigenvector, with norm one, close to the approximate one.

The following result from [SF] shows that an error estimate in the energy norm depends on error estimates in the norm $\|\cdot\|$ and eigenvalue errors. We modified it slightly to make it useful for the case of repeated eigenvalues.

Lemma 5.2.10

$$\|y_m - y^h_j\|^2_E = \lambda_m\|y_m - y^h_j\|^2 + \lambda^h_j - \lambda_m, \; j \notin \Lambda.$$

Proof

$$\begin{aligned} b(y_m - y^h_j, y_m - y^h_j) &= b(y_m, y_m) - 2b(y_m, y^h_j) + b(y^h_j, y^h_j) \\ &= \lambda_m\|y_m\|^2 - 2\lambda_m(y_m, y^h_j) + \lambda^h_j\|y^h_j\|^2 \\ &= \lambda_m - 2\lambda_m(y_m, y^h_j) + \lambda^h_j \\ &= \lambda_m[2 - 2(y_m, y^h_j)] + \lambda^h_j - \lambda_m \\ &= \lambda_m\left[\|y_m\|^2 - 2(y_m, y^h_j)\right] + \|y^h_j\|^2\big] + \lambda^h_j - \lambda_m \\ &= \lambda_m\|y_m - y^h_j\|^2 + \lambda^h_j - \lambda_m. \end{aligned}$$

□

Theorem 5.2.2

1. *Let $\varepsilon > 0$ be arbitrary. If h is sufficiently small, then for each j, $j = 1, 2, \ldots, r$ there exists a unique $x_j \in E_{\lambda_m}$ with $\|x_j\| = 1$ such that*

$$\|x_j - y^h_{m-r+j}\|_E \le \varepsilon.$$

2. *Suppose Problem C satisfies the regularity assumption. If h is sufficiently small, then for each j, $j = 1, 2, \ldots, r$ there exists a unique $x_j \in E_{\lambda_m}$ with $\|x_j\| = 1$ such that*

$$\|x_j - y^h_{m-r+j}\|_E \leq \widehat{C} C_b \lambda^{\alpha}_m h^{(k^*-2)}.$$

Proof

1. Use Corollary 5.2.4: There exists a unique $x_j \in E_{\lambda_m}$ such that

$$\|x_j - y^h_{m-r+j}\| \leq 2(1 + \rho^{-1}\lambda_m)\|x_j - Px_j\|.$$

But

$$\|x_j - y^h_{m-r+j}\|^2_E \;\; \leq \;\; \lambda_m \|x_j - y^h_{m-r+j}\|^2 + \lambda^h_{m-r+j} - \lambda_m$$

(Lemma 5.2.10). Hence

$$\|x_j - y^h_{m-r+j}\|^2_E \;\; \leq \;\; 4\lambda_m(1 + \rho^{-1}\lambda_m)^2\|x_j - Px_j\|^2 + \lambda^h_{m-r+j} - \lambda_m \tag{5.2.4}$$

Now use Proposition 5.2.5 and Theorem 5.2.1.

2. Consider the Inequality (5.2.4). We have the estimates

$$\|x_j - Px_j\| \leq \widehat{C} C_b \lambda^{\alpha}_m h^{k^*-2} \tag{5.2.5}$$

from Proposition 5.2.6 and

$$\lambda^h_{m-r+j} - \lambda_m \leq \widehat{C} C_b \lambda^{2\alpha}_m h^{2(k^*-2)} \tag{5.2.6}$$

from Theorem 5.2.1. The result follows from (5.2.5) and (5.2.6).

□

5.3 Vibration problem

Our concern is the difference between the solution u of Problem A and the solution u_h of Problem AG. It is possible to estimate this error in terms of the projection error (Section 4.6) and errors for the initial conditions. See [SF, Section 7.3]. This is called a projection method and was first used for parabolic problems. For second order hyperbolic problems, it appear that credit is due to [D], [De] and [SF]. Research in this direction was also done by [Ba].

After this it appear that abstract methods became popular. See for example [Sh, Section 6.4]. In Section 3 of an invited paper, [BIt], very general results are given. (Incidentally they use results in [Sh].)

A general approximation theory, using functional analysis, is obviously important. However, we found that the basic error inequality mentioned before ([SF, Section 7.3] and [D, Lemma 1]) is valid for an abstract problem as general as Problem A. As a final remark we mention the paper [FXX] where the authors also use what they term a "partial projection" method to obtain L^2-error estimates.

5.3.1 Discretization error

In this section we show that the convergence proof sketched by Strang and Fix [SF, Section 7.3] can be applied to Problem A in Section 3.4 and Problem AG in Section 4.1. In this proof the projection operator P defined in Section 4.6 is used to find an estimate on the discretization error

$$\|u(t) - u_h(t)\|_E \quad \text{for} \quad t \in [0, \infty).$$

We also use the symbol P to denote the "projection" Pu of the solution u of Problem A , i.e. $(Pu)(t) = Pu(t)$ for each $t \geq 0$.

Let $e(t) = Pu(t) - u_h(t)$ and $e_p(t) = u(t) - Pu(t)$. Then

$$\|u(t) - u_h(t)\|_E \leq \|e_p(t)\|_E + \|e(t)\|_E. \tag{5.3.1}$$

The following result is required for the main result of this section. Note that differentiability with respect to the energy norm is required to prove that the projection function Pu is differentiable. This regularity requirement is not stated by [SF].

Lemma 5.3.1 *If $u \in C^2([0,\infty), V)$, then $Pu \in C^2([0,\infty), V)$ with*

$$(Pu)'(t) = Pu'(t) \text{ and } (Pu)''(t) = Pu''(t).$$

Proof As the projection operator P is a bounded linear operator with norm less than one, it follows that

$$\|(\delta t)^{-1}(Pu(t+\delta t) - Pu(t)) - Pu'(t)\|_E \leq \|(\delta t)^{-1}(u(t+\delta t) - u(t)) - u'(t)\|_E.$$

This implies that $Pu \in C^1([0,\infty), V)$ and $(Pu)'(t) = Pu'(t)$.

In exactly the same way we prove that $(Pu)' \in C^1([0,\infty), V)$ and $(Pu)''(t) = Pu''(t)$.

□

Since we already have an estimate for the projection error $e_p(t)$, it is only necessary to estimate the other part of the error.

In the next proof the following "energy" expression will be convenient:

$$\begin{aligned} E(t) &= \frac{1}{2}(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) \\ &= \frac{1}{2}\|e'(t)\|^2 + \frac{1}{2}\|e(t)\|_E^2. \end{aligned} \tag{5.3.2}$$

Lemma 5.3.2 *Assume that $u \in C^2([0,\infty), V)$. Then, for any $t \geq 0$,*

$$\|e(t)\|_E \leq \|P\alpha - \alpha_h\|_E + \|P\beta - \beta_h\| + \int_0^t \|e_p''(s)\| + \frac{k}{C_l}\|e_p'(s)\|_0 \, ds.$$

Proof From Problem A and the Galerkin approximation (Problem AG) we deduce that for any $v \in S^h$,

$$(u''(t) - u_h''(t), v) + a(u'(t) - u_h'(t), v) + b(u(t) - u_h(t), v) = 0. \tag{5.3.3}$$

Since P is a projection, we have

$$b(u(t) - Pu(t), v) = b(u'(t) - Pu'(t), v) = 0 \text{ for all } v \in S^h.$$

Using the fact that $Pu''(t) = (Pu)''(t)$, (5.3.3) can be written as

$$\begin{aligned} (e''(t), v) + b(e(t), v) = -(e_p''(t), v) - k(e_p'(t), v)_0 - k(e'(t), v)_0 \\ -\mu b(e'(t), v) \text{ for all } v \in S^h. \end{aligned} \tag{5.3.4}$$

(Note that $a(u,v) = \mu b(u,v) + k(u,v)_0$ where μ or k or both can be zero.)

We will use the fact that

$$E'(t) = (e''(t), e'(t)) + b(e(t), e'(t)).$$

As $e(t) \in S^h$ it follows that $e'(t) \in S^h$. Choose $v = e'(t)$ in (5.3.4), then

$$\begin{aligned} E'(t) &= -(e_p''(t), e'(t)) - (ke_p'(t), e'(t))_0 - k(e'(t), e'(t))_0 - \mu b(e'(t), e'(t)) \\ &\leq \left(\|e_p''(t)\| + \frac{k}{C_I}\|e_p'(t)\|_0 \right) \|e'(t)\|. \end{aligned}$$

From (5.3.2), $\|e'(t)\| \leq \sqrt{2E(t)}$. Thus

$$E'(t) \leq \sqrt{2E(t)} \left(\|e_p''(t)\| + \frac{k}{C_I}\|e_p'(t)\|_0 \right)$$

and consequently

$$\frac{d}{dt}\sqrt{E(t)} \leq \frac{1}{\sqrt{2}} \left(\|e_p''(t)\| + \frac{k}{C_I}\|e_p'(t)\|_0 \right).$$

This yields that

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \frac{1}{\sqrt{2}} \int_0^t \left(\|e_p''(s)\| + \frac{k}{C_I}\|e_p'(s)\|_0 \right) ds. \qquad (5.3.5)$$

As

$$E(0) = \frac{1}{2}\|P\beta - \beta_h\|^2 + \frac{1}{2}\|P\alpha - \alpha_h\|_E^2$$

and $\|e(t)\|_E \leq \sqrt{2E(t)}$, again from (5.3.2), the result follows from (5.3.5). □

Theorem 5.3.1 *Assume that $u \in C^2([0,\infty), V)$. Then, for any $t \geq 0$,*

$$\begin{aligned} \|u(t) - u_h(t)\|_E &\leq \|e_p(t)\|_E + \|P\alpha - \alpha_h\|_E + \|P\beta - \beta_h\| \\ &\qquad + \int_0^t \left(\|e_p''(s)\| + \frac{k}{C_I}\|e_p'(s)\|_0 \right) ds. \end{aligned}$$

Proof Use Lemma 5.3.2 and Equation (5.3.1).

□

To prove the convergence results, it is now necessary to consider the terms on the right side of the inequality in this theorem.

5.3.2 Convergence

The main factor that determines the rate of convergence of the solution u_h of Problem AG to the solution u of Problem A as h tends to zero, is the regularity of the weak solution u. The regularity of u depends on the regularity of the initial values α and β, as we pointed out in Section 3.4. [Ra] gave an example to show that the regularity of the solution is necessary to obtain optimal order convergence.

The rate of convergence is also directly influenced by the choice of the initial values α_h and β_h for the solution u_h of Problem AG. We will consider two cases, i.e. $\alpha_h = \Pi\alpha$, $\beta_h = \Pi\beta$ and $\alpha_h = P\alpha$, $\beta_h = P\beta$. In the following result we show that the rate of convergence in the energy norm is of order h^2 if certain regularity conditions are satisfied. The estimates are expressed in terms of the constants C_I and C_E defined in Section 3.4 as well as $\widehat{C}$ defined in Section 4.5.

Theorem 5.3.2 *Let $\alpha_h = \Pi\alpha$ and $\beta_h = \Pi\beta$. Assume that $u \in C^2([0,\infty), V)$ and that $u(t)$, $u'(t)$ and $u''(t)$ are in $H^4 \cap V$ for $t \geq 0$. Then,*

$$\|u(t) - u_h(t)\|_E \leq \widehat{C}\left(|\alpha|_4 + C_E^{-1}|\beta|_4 + |u(t)|_4 + k(C_EC_I)^{-1}t \max_{s\in[0,t]} |u'(s)|_4 \right.$$
$$\left. + C_E^{-1}t \max_{s\in[0,t]} |u''(s)|_4\right) h^2 \text{ for } t \in [0,\infty).$$

Proof From Theorem 5.3.1,

$$\begin{aligned}\|u(t) - u_h(t)\|_E \leq \; & \|e_p(t)\|_E + \|P\alpha - \Pi\alpha\|_E + \|P\beta - \Pi\beta\| \\ & + \int_0^t \left(\|e_p''(s)\| + \frac{k}{C_I}\|e_p'(s)\|_0\right) ds.\end{aligned}$$

All that remains to be done is to apply the approximation results from Corollary 4.6.1 to each of the terms in this expression:

$$\|P\alpha - \Pi\alpha\|_E \leq \widehat{C}|\alpha|_4 h^2,$$

$$\|P\beta - \Pi\beta\| \leq C_E^{-1}\|P\beta - \Pi\beta\|_E \leq C_E^{-1}\widehat{C}|\beta|_4 h^2$$

and

$$\|e_p(t)\|_E = \|u(t) - Pu(t)\|_E \leq \widehat{C}|u(t)|_4 h^2.$$

From Lemma 5.3.1, $(Pu)' = Pu'$ and hence $e_p'(t) = u'(t) - Pu'(t)$. This yields that

$$\|e_p'(s)\|_0 \leq (C_E C_I)^{-1}\|e_p'(s)\|_E \leq (C_E C_I)^{-1}\widehat{C}|u'(s)|_4 h^2$$

and

$$\int_0^t \|e_p'(s)\|_0\, ds \leq (C_E C_I)^{-1}\widehat{C}t \max_{s\in[0,t]} |u'(s)|_4 h^2.$$

Similarly,

$$\int_0^t \|e_p''(s)\|\, ds \leq C_E^{-1}\widehat{C}t \max_{s\in[0,t]} |u''(s)|_4 h^2.$$

□

Under less strict regularity conditions we can still show that the solution u_h of Problem AG converges to the solution u of Problem A in the energy norm if h tends to zero.

Theorem 5.3.3 *Let $\alpha_h = \Pi\alpha$ and $\beta_h = \Pi\beta$. Assume that $\alpha \in V$, $\beta \in V$ and $u \in C^2([0,\infty), V)$, then*

$$\lim_{h\to 0} \|u(t) - u_h(t)\|_E = 0 \text{ for } t \in [0,\tau].$$

Proof From Theorem 5.3.1,

$$\begin{aligned}\|u(t) - u_h(t)\|_E \;\leq\; & \|e_p(t)\|_E + \|P\alpha - \Pi\alpha\|_E + \|P\beta - \Pi\beta\| \\ & + \int_0^t \left(\|e_p''(s)\| + \frac{k}{C_I}\|e_p'(s)\|_0\, ds \right).\end{aligned}$$

From the approximation results we know that for any $\varepsilon > 0$, each term is less than ε, provided that h is sufficiently small. □

5.3.3 Inertia norm estimate

In a final result we show that the Aubin-Nitsche trick can also be applied to this problem to find inertia norm estimates for the discretization error.

Theorem 5.3.4 *Let $\alpha_h = P\alpha$ and $\beta_h = P\beta$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are all in $V \cap H^4$ for all $t \geq 0$. Then,*

$$\|u(t) - u_h(t)\| \leq \widehat{C}\left(|u(t)|_4 + kt(C_I^2 C_E)^{-1}\max_{s\in[0,t]}|u'(s)|_4 + tC_E^{-1}\max_{s\in[0,t]}|u''(s)|_4\right)h^4$$

$$\text{for } t \in [0,\infty).$$

Proof From Theorem 5.3.2,

$$\begin{aligned}\|u(t) - u_h(t)\| &\leq \|e_p(t)\| + \|e(t)\| \\ &\leq \|e_p(t)\| + C_E^{-1}\|e(t)\|_E \\ &\leq \|e_p(t)\| + C_E^{-1}\int_0^t \left(\|e_p''(s)\| + \frac{k}{C_I}\|e_p'(s)\|_0\right) ds.\end{aligned}$$

For a fixed $t \geq 0$, we consider $e_p(t) = u(t) - Pu(t)$.

We conclude from Corollary 4.6.2 that

$$\|e_p(t)\| \leq \widehat{C}|u(t)|_4 h^4.$$

Similar arguments yield that

$$\|e_p'(t)\|_0 \leq C_I^{-1}\|e_p'(t)\| \leq C_I^{-1}\widehat{C}|u'(t)|_4 h^4 \text{ and } \|e_p''(t)\| \leq \widehat{C}|u''(t)|_4 h^4. \quad (5.3.6)$$

□

A useful result is also obtained if the Aubin-Nitsche trick is used only for the terms containing the integrals.

Theorem 5.3.5 *Let $\alpha_h = \Pi\alpha$ and $\beta_h = \Pi\beta$. Assume that α, β, $u(t)$, $u'(t)$ and $u''(t)$ are all in $V \cap H^4$ for all t. Then,*

$$\|u(t) - u_h(t)\|_E \leq \widehat{C}\left(|\alpha|_4 + C_E^{-1}|\beta|_4 + |u(t)|_4\right)h^2 +$$

$$\widehat{C}\left(ktC_I^{-2}\max_{s\in[0,t]}|u'(s)|_4 + t\max_{s\in[0,t]}|u''(s)|_4\right)h^4 \text{ for } t \in [0,\infty).$$

Proof The proof is exactly the same as the proof of Theorem 5.3.2. The estimates in (5.3.6) are used for the terms containing the integral. □

Remark We consider this result to be significant. It is advantageous to have an error estimate in the energy norm, while the terms containing t are "suppressed" by h^4.

5.4 Finite Differences

In this section we consider the system of ordinary differential equations, Problem AD in Section 4.1, and the finite difference method for approximating the solution. The objective is to prove that the solution of the discretized problem converges to the solution of the Galerkin approximation. This method has been extensively studied—even in the context of finite difference methods for second order hyperbolic partial differential equations. However, one must be careful when matching the estimates. Although all norms are equivalent in the finite dimensional space S^h, the "constants" may depend on the dimension of S^h. Presenting error estimates for semi-discrete and fully discrete systems in the same presentation is a line also followed by others. See for example [D], [Ba] and [FXX].

We consider Problem AG in Section 4.1 and the finite difference scheme proposed in Section 4.4. In the first subsection we estimate the local error and then proceed to establish stability results.

5.4.1 Local error

The first step is to derive finite difference formulas similar to the Newmark schemes [Zi]. Since we need error estimates in terms of the unknown function or its derivatives, it is necessary to derive the formulas.

We will use Taylor's theorem in the following form:

$$g(t) = g(t_0) + (t-t_0)g'(t_0) + \cdots + \frac{(t-t_0)^{n-1}}{(n-1)!}g^{(n-1)}(t_0) + R(t)$$

where $R(t) = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-\theta)^{n-1}g^{(n)}(\theta)\,d\theta$. It is also true for $t < t_0$.

See [Cl, p 179] or [Ap, p 279].

The following notation is introduced for convenience.

Notation $R_n^+(t) = \dfrac{1}{(n-1)!}\displaystyle\int_t^{t+\delta t}(t+\delta t-\theta)^{n-1}g^{(n)}(\theta)\,d\theta$ and

$$R_n^-(t) = \frac{1}{(n-1)!}\int_t^{t-\delta t}(t-\delta t-\theta)^{n-1}g^{(n)}(\theta)\,d\theta.$$

The first proposition contains well-known results and the proofs are trivial.

Proposition 5.4.1

1. *If the real valued function* g *is in* $C^3[t-\delta t, t+\delta t]$, *then*
$$g(t+\delta t) - g(t-\delta t) = 2\delta t g'(t) + R_3^+(t) - R_3^-(t). \tag{5.4.1}$$

2. *If the real valued function* g *is in* $C^4[t-\delta t, t+\delta t]$, *then*
$$g(t+\delta t) - 2g(t) + g(t-\delta t) = (\delta t)^2 g''(t) + R_4^+(t) + R_4^-(t). \tag{5.4.2}$$

Proof

1. Use Taylor's theorem to get:
$$g(t+\delta t) = g(t) + \delta t\, g'(t) + \frac{(\delta t)^2}{2}\, g''(t) + R_3^+(t)$$
and
$$g(t-\delta t) = g(t) - \delta t\, g'(t) + \frac{(\delta t)^2}{2}\, g''(t) + R_3^-(t).$$
Clearly
$$g(t+\delta t) - g(t-\delta t) \;=\; 2\delta t\, g'(t) + R_3^+(t) - R_3^-(t).$$

2. Approximate g by a polynomial of degree three and compute $g(t+\delta t) + g(t-\delta t)$.

□

We gave the proof of part one in detail because we use the result in the next proposition.

Proposition 5.4.2 *Let* ρ_0 *and* ρ_1 *be real numbers such that* $\rho_0 + 2\rho_1 = 1$.

1. *If the real valued function* g *is in* $C^4[t-\delta t, t+\delta t]$, *then*
$$g(t+\delta t) - g(t-\delta t)$$
$$= 2\delta t\left(\rho_1 g'(t+\delta t) + \rho_0 g'(t) + \rho_1 g'(t-\delta t)\right) + \widetilde{R}_4(t), \tag{5.4.3}$$
where
$$\widetilde{R}_4(t) = \omega_1\left\{R_3^+(t) - R_3^-(t)\right\} + \omega_2\Big\{R_4^+(t) + R_4^-(t) -$$
$$\frac{\delta t}{6}\int_t^{t+\delta t}(t+\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta - \frac{\delta t}{6}\int_t^{t-\delta t}(t-\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta\Big\}.$$

2. *Suppose the real valued function g is in $C^5[t-\delta t, t+\delta t]$, then*

$$g(t+\delta t) - 2g(t) + g(t-\delta t)$$
$$= (\delta t)^2 \left(\rho_1 g''(t+\delta t) + \rho_0 g''(t) + \rho_1 g''(t-\delta t)\right) + \widetilde{R}_5(t), \qquad (5.4.4)$$

where $\widetilde{R}_5(t) = \omega_1 \left\{ R_4^+(t) + R_4^-(t) \right\} + \omega_2 \Big\{ R_5^+(t) + R_5^-(t) -$

$$\frac{(\delta t)^2}{24} \int_t^{t+\delta t} (t+\delta t-\theta)^2 g^{(5)}(\theta)\, d\theta - \frac{(\delta t)^2}{24} \int_t^{t-\delta t} (t-\delta t-\theta)^2 g^{(5)}(\theta)\, d\theta \Big\}.$$

Proof

1. Use Taylor's theorem to get:

$$g(t+\delta t) = g(t) + \delta t\, g'(t) + \frac{(\delta t)^2}{2} g''(t) + \frac{(\delta t)^3}{6} g'''(t) + R_4^+(t)$$

and

$$g(t-\delta t) = g(t) - \delta t\, g'(t) + \frac{(\delta t)^2}{2} g''(t) - \frac{(\delta t)^3}{6} g'''(t) + R_4^-(t).$$

This yields

$$g(t+\delta t) - g(t-\delta t) \;=\; 2\delta t\, g'(t) + \frac{(\delta t)^3}{3} g'''(t) + R_4^+(t) - R_4^-(t). \qquad (5.4.5)$$

Applying Taylor's theorem once more on g' we obtain

$$g'(t+\delta t) = g'(t) + \delta t\, g''(t) + \frac{(\delta t)^2}{2} g'''(t) + \frac{1}{2}\int_t^{t+\delta t} (t+\delta t-\theta)^2 g^{(4)}(\theta)\, d\theta$$

and

$$g'(t-\delta t) = g'(t) - \delta t\, g''(t) + \frac{(\delta t)^2}{2} g'''(t) + \frac{1}{2}\int_t^{t-\delta t} (t-\delta t-\theta)^2 g^{(4)}(\theta)\, d\theta.$$

The two equations yield

$$g'(t+\delta t) + g'(t-\delta t) = 2g'(t) + (\delta t)^2 g'''(t) +$$
$$\frac{1}{2}\int_t^{t+\delta t} (t+\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta + \frac{1}{2}\int_t^{t-\delta t} (t-\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta.$$

From this we get an expression for $(\delta t)^2 g'''(t)$ which can substituted into (5.4.5). The result is

$$\begin{aligned} g(t+\delta t) - g(t-\delta t) &= \frac{(\delta t)}{3}\left[g'(t+\delta t) + 4g'(t) + g'(t-\delta t)\right] + R_4^+(t) \\ &\quad + R_4^-(t) - \frac{\delta t}{6}\int_t^{t+\delta t} (t+\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta \\ &\quad - \frac{\delta t}{6}\int_t^{t-\delta t} (t-\delta t-\theta)^2 g^{(4)}(\theta)\,d\theta. \qquad (5.4.6) \end{aligned}$$

Finally we combine (5.4.1) and (5.4.6) with weights ω_1 and ω_2 to get the desired result.

2. This proof is similar to the proof in (1). □

Remark The results above will also be used in the case where the function g is not defined for $t < 0$. In this case we may extend g by using the polynomial approximation on $[t-\delta t, 0)$. This will only influence the result in so far as there will be fewer remainder terms.

The second step is to apply the difference formulas to Problem AG and to estimate the errors.

Assumption We assume that $f \in C^3[0,\tau]$ so that the solution u_h of Problem AG is in $C^5[0,\tau]$.

Notation $\|u_h\|^E_{5,\max} = \sum_{k=0}^{5} \max_{t\in[0,\tau]} \|u_h^{(k)}(t)\|_E.$

Notation In the rest of this section C_b will denote a generic constant that depends on the bilinear forms, i.e. C_b is a combination of C_E and C_I.

Notation $\|f\|_{3,\max} = \sum_{k=0}^{3} \max_{t\in[0,\tau]} \|f^{(k)}(t)\|.$

Proposition 5.4.3 *Suppose* $u_i \in C^5[t-\delta t, t+\delta t]$ *for* $i = 1,\ 2,\ \ldots,\ n$ *and* $\{\phi_1,\ \phi_2,\ \ldots,\ \phi_n\}$ *is the basis for* S^h *and let* $u_h(t) = \sum_{i=1}^n u_i(t)\phi_i$. *Suppose also that* ρ_0 *and* ρ_1 *are real numbers such that* $\rho_0 + 2\rho_1 = 1$.

If $u_h(t+\delta t) - 2u_h(t) + u_h(t-\delta t)$

$$= (\delta t)^2(\rho_1 u_h''(t+\delta t) + \rho_0 u_h''(t) + \rho_1 u_h''(t-\delta t)) + e_1^h, \tag{5.4.7}$$

$$u_h(t+\delta t) - u_h(t-\delta t)$$

$$= 2\delta t\,(\rho_1 u_h'(t+\delta t) + \rho_0 u_h'(t) + \rho_1 u_h'(t-\delta t)) + e_2^h \tag{5.4.8}$$

and

$$u_h(t+\delta t) - u_h(t-\delta t) = 2\delta t\, u_h'(t) + e_3^h, \tag{5.4.9}$$

then

$$\|e_i^h\| \le K(\delta t)^3 \left\{ \max_{\theta\in[t,t+\delta t]} \|u_h^{(3)}(\theta)\| + \max_{\theta\in[t,t+\delta t]} \|u_h^{(4)}(\theta)\| + \max_{\theta\in[t,t+\delta t)} \|u_h^{(5)}(\theta)\| \right\}$$

and

$$\|e_i^h\|_E \le K(\delta t)^3 \left\{ \max_{\theta\in[t,t+\delta t]} \|u_h^{(3)}(\theta)\|_E + \max_{\theta\in[t,t+\delta t]} \|u_h^{(4)}(\theta)\|_E \right.$$

$$\left. + \max_{\theta\in[t,t+\delta t)} \|u_h^{(5)}(\theta)\|_E \right\}.$$

Proof

Consider (5.4.7) as an example: Use (5.4.4) in Proposition 5.4.2 for u_i and denote the remainder by $\widetilde{R}_{5i}(t)$. Now, each term in (5.4.7)can be written as a linear combination, for example,

$$u_h(t+\delta t) = \sum_{i=1}^n u_i(t+\delta t)\phi_i.$$

Consequently we have (5.4.7), if we set $e_1^h(t) = \sum_{i=1}^n \widetilde{R}_{5i}(t)\phi_i$.

It remains to estimate the error term $e_1^h(t)$, which is actually the sum of six error terms. Consider one of the terms: For any $v \in S^h$

$$
\begin{aligned}
& \left|\left(\omega_1 \sum_{i=1}^{n} \int_t^{t+\delta t} (t+\delta t-\theta)^3 u_i^{(4)}(\theta)\phi_i \, d\theta, v\right)\right| \\
= & \left|\int_t^{t+\delta t} \omega_1 (t+\delta t-\theta)^3 (u_h^{(4)}(\theta), v)\, d\theta\right| \\
\leq & \int_t^{t+\delta t} |\omega_1|(t+\delta t-\theta)^3 \|u_h^{(4)}(\theta)\|\|v\|\, d\theta \\
\leq & \frac{1}{4}(\delta t)^4 |\omega_1| \|v\| \max_{\theta\in[t,t+\delta t)} \|u_h^{(4)}(\theta)\|.
\end{aligned}
$$

Hence there exists a constant K, which depends only on the weights ω_1 and ω_2 such that $|(e_i^h(t), v)| \leq K(\delta t)^3 \|v\| \left\{ \max_{\theta\in[t,t+\delta t]} \|u_h^{(3)}(\theta)\| + \max_{\theta\in[t,t+\delta t)} \|u_h^{(4)}(\theta)\| \right.$

$$\left. + \max_{\theta\in[t,t+\delta t)} \|u_h^{(5)}(\theta)\| \right\}.$$

Note that the worst of the errors are of order $(\delta t)^3$. Since v is arbitrary, we have the desired result. The same procedure yields estimates in the energy norm.

□

Lemma 5.4.1 *Suppose u_h is the solution of Problem AG. Let ρ_0 and ρ_1 be real numbers such that $\rho_0 + 2\rho_1 = 1$. If $u^*(t,\delta t)$ is defined by*

$$
\begin{aligned}
& (u^*(t,\delta t) - 2u_h(t) + u_h(t-\delta t), v) + \frac{(\delta t)}{2} a(u^*(t,\delta t) - u_h(t-\delta t), v) \\
& + (\delta t)^2 b(\rho_1 u^*(t,\delta t) + \rho_0 u_h(t) + \rho_1 u_h(t-\delta t), v) \\
= & (\delta t)^2 (\rho_1 f(t+\delta t) + \rho_0 f(t) + \rho_1 f(t-\delta t), v)_0 \quad \text{for eaxh } v \in S^h,
\end{aligned}
\tag{5.4.10}
$$

then $\|u_h(t+\delta t) - u^*(t,\delta t)\| \leq C_b (\delta t)^3 \|u_h\|_{5,\max}^E$.

Proof Using Proposition 5.4.3 we have

$$
\begin{aligned}
& (u_h(t+\delta t) - 2u_h(t) + u_h(t-\delta t), v) + \frac{(\delta t)}{2} a(u_h(t+\delta t) - u_h(t-\delta t), v) \\
= & (\delta t)^2 (\rho_1 u_h''(t+\delta t) + \rho_0 u_h''(t) + \rho_1 u_h''(t-\delta t), v) + (e_1^h, v) \\
& + (\delta t)^2 a(\rho_1 u_h'(t+\delta t) + \rho_0 u_h'(t) + \rho_1 u_h'(t-\delta t), v) + \frac{(\delta t)}{2} a(e_2^h, v).
\end{aligned}
$$

Now use the fact that u_h is the solution of Problem AG to prove that u_h satisfies (5.4.10) with $u(t+\delta t)$ in stead of $u^*(t,\delta t)$ provided that the error terms (e_1^h, v) and $\frac{(\delta t)}{2}a(e_2^h, v)$ are included.

Consequently, $(u(t+\delta t) - u^*(t,\delta t), v) = (e_1^h, v) + \frac{(\delta t)}{2}\, a(e_2^h, v)$ for each $v \in S^h$. Replace v by $u^*(t,\delta t) - u_h(t+\delta t)$ to obtain the estimate. □

Reconsider the semi discrete system in Section 4.4.

$$M\bar{u}''(t) + L\bar{u}'(t) + K\bar{u}(t) = \bar{f}(t) \tag{5.4.11}$$

$$\bar{u}(0) = \bar{\alpha}, \quad \bar{u}'(0) = \bar{\beta}.$$

To estimate the local errors for a finite difference scheme, we consider a one-to-one correspondence between S^h and $\mathbb{R}_n$.

Definition 5.4.1 *For $u^h \in S^h$, the vector $\bar{u} = Qu^h$ has components u_i where* $u^h = \sum_{i=1}^{n} u_i\phi_i$.

If we use the norm $\|\bar{u}\|_M = (M\bar{u}\cdot\bar{u})^{\frac{1}{2}}$ for $\mathbb{R}_n$, then $\|Qu^h\|_M = \|u^h\|$.

In our next result use the fact that $\bar{u}$ is a solution of (5.4.11) if and only if u_h is a solution of Problem AG.

Corollary 5.4.1 *If $\bar{u}$ is a solution of the system of differential equations (5.4.11) and $\bar{u}^*(t,\delta t)$ is defined by*

$$\begin{aligned} & M\left[\bar{u}^*(t,\delta t) - 2\bar{u}(t) + \bar{u}(t-\delta t)\right] + \frac{(\delta t)}{2} L\left[\bar{u}^*(t,\delta t) - \bar{u}(t-\delta t)\right] \\ & +(\delta t)^2 K\left[\rho_1\bar{u}^*(t,\delta t) + \rho_0\bar{u}(t) + \rho_1\bar{u}(t-\delta t)\right] \\ = \;& (\delta t)^2[\rho_1\bar{f}(t+\delta t) + \rho_0\bar{f}(t) + \rho_1\bar{f}(t-\delta t)], \end{aligned} \tag{5.4.12}$$

then $\|\bar{u}^*(t,\delta t) - \bar{u}(t+\delta t)\|_M \le C_b(\delta t)^3\|u_h\|^E_{5,\max}$.

Proof Consider the terms in (5.4.10). If $\bar{v} = Qv^h$, then

$$(u_h(t+\delta t), v^h) = M\bar{u}(t+\delta t)\cdot v.$$

In this way we can associate each term in (5.4.12) with a corresponding term in (5.4.10). The result follows fom the fact that $\bar{u}^*(t,\delta t) = Qu^*(t,\delta t)$ and $\bar{u}(t+\delta t) = Qu_h(t+\delta t)$. □

2. If we assume that f is merely continuous and hence u_h twice continuously differentiable, we could still estimate the local errors but not obtain the same order. The results would be of the form: *Given $\varepsilon > 0$, there exists a real number $\triangle > 0$ such that the error will be less than $\varepsilon \delta t K$ for $\delta t < \triangle$.* (K a constant depending on u_h and f.)

5.4.2 Transformation

Due to symmetry considerations, it will be more convenient to consider a transformed system for stability analysis. Since M is symmetric and positive definite, there exists a symmetric positive definite matrix N such that $N^2 = M$. Set $\bar{v}(t) = N\bar{u}(t)$, then $\bar{v}$ is a solution of the problem

$$\bar{v}'' + N^{-1}LN^{-1}\bar{v}' + N^{-1}KN^{-1}\bar{v} = N^{-1}\bar{f}$$

or

$$\bar{v}'' + \widetilde{L}\bar{v}' + \widetilde{K}\bar{v} = \bar{g}. \tag{5.4.14}$$

where $\widetilde{L} = N^{-1}LN^{-1}$, $\widetilde{K} = N^{-1}KN^{-1}$ and $\bar{g} = N^{-1}\bar{f}$.

The advantage of the transformation is that the matrix $\widetilde{K}$ is symmetric, and hence has orthogonal eigenvectors.

Let $\bar{y} = N\bar{x}$, then

$$K\bar{x} = \lambda M\bar{x}$$

if and only if

$$\widetilde{K}\bar{y} = N^{-1}KN\bar{y} = \lambda\bar{y}.$$

The eigenvalues of $\widetilde{K}$ are the eigenvalues of the eigenvalue problem Problem CG (See Sections 4.1 and 5.2.)

We use the norm $\|\bar{x}\|_2 = (\bar{x}\cdot\bar{x})^{\frac{1}{2}}$, and in the remaining part of this section $\|\cdot\|$ will refer to $\|\cdot\|_2$ unless stated otherwise.

Corollary 5.4.3 *If $\bar{v}$ is a solution of the system of differential equations (5.4.14), and $\bar{v}^*(t,\delta t)$ is defined by*

$$\begin{aligned}
&[\bar{v}^*(t,\delta t) - 2\bar{v}(t) + \bar{v}(t-\delta t)] + \frac{(\delta t)}{2}\widetilde{L}\left[\bar{v}^*(t,\delta t) - \bar{v}(t-\delta t)\right] \\
&+(\delta t)^2\widetilde{K}\left[\rho_1\bar{v}^*(t,\delta t) + \rho_0\bar{v}(t) + \rho_1\bar{v}(t-\delta t)\right] \\
=\ &(\delta t)^2[\rho_1\bar{g}(t+\delta t) + \rho_0\bar{g}(t) + \rho_1\bar{g}(t-\delta t)],
\end{aligned}$$

then $\|\bar{v}^*(t,\delta t) - \bar{v}(t+\delta t)\| \le C_b(\delta t)^3 \left(\|u_h\|^E_{5,\max} + (\delta t)^3 \|f\|_{3,\max}\right)$.

Proof Direct from Corollary 5.4.1, since $\bar{v}^*(t,\delta t) = N\bar{u}^*(t,\delta t)$. □

Corollary 5.4.4 *If* $\bar{v}$ *is a solution of the system of differential equations* (5.4.14), *and* $\bar{v}^{**}(t,\delta t)$ *is defined by*

$$\begin{aligned} & 2\left[\bar{v}^{**}(t,\delta t) - \bar{v}(t)\right] + \frac{(\delta t)^2}{2}\widetilde{K}\left[\bar{v}^{**}(t,\delta t) + \bar{v}(t)\right] \\ = \; & \frac{(\delta t)^2}{2}\left[\bar{g}(t+\delta t) + \bar{g}(t)\right] + 2\delta t\, \bar{v}'(t) - (\delta t)^2\, \widetilde{L}\bar{v}'(t) + \frac{(\delta t)^3}{2}\widetilde{K}\bar{v}'(t) \\ & -\frac{(\delta t)^3}{2}\bar{g}'(t), \end{aligned}$$

then $\|\bar{v}^{**}(t,\delta t) - \bar{v}(t+\delta t)\| \le C_b(\delta t)^3 \left(\|u_h\|^E_{5,\max} + (\delta t)^3 \|f\|_{3,\max}\right)$.

Proof See Corollary 5.4.3. □

Remark The result remains true for $t = 0$.

5.4.3 Global error

We approximate the solution of (5.4.14) on the interval $[0,\tau]$. Let δt indicate the time step length, i.e. $\delta t = \tau/N$, and let $\bar{w}_k$ denote the approximation for $\bar{v}(t_k)$.

We use the difference scheme (which corresponds to (5.4.12))

$$\begin{aligned} & (\bar{w}_{k+1} - 2\bar{w}_k + \bar{w}_{k-1}) + \frac{(\delta t)}{2}\widetilde{L}(\bar{w}_{k+1} - \bar{w}_{k-1}) \\ & + (\delta t)^2\, \widetilde{K}(\rho_1\bar{w}_{k+1} + \rho_0\bar{w}_k + \rho_1\bar{w}_{k-1}) = (\delta t)^2(\rho_1\bar{g}_{k+1} + \rho_0\bar{g}_k + \rho_1\bar{g}_{k-1}). \end{aligned} \tag{5.4.15}$$

The initial conditions for the system of differential equations are

$$\bar{v}(0) = N\bar{\alpha} \text{ and } \bar{v}'(0) = N\bar{\beta},$$

and the initial conditions of the finite difference system are

$$w_0 = N\bar{\alpha} \text{ and } (2\delta t)^{-1}(\bar{w}_1 - \bar{w}_{-1}) = N\bar{\beta}.$$

To estimate local errors the following scheme will also be used:

$$\begin{aligned} &2(\bar{w}_{k+1} - 2\bar{w}_k + \bar{w}_{k-1}) + \delta t\,\widetilde{L}(\bar{w}_{k+1} - \bar{w}_{k-1}) + \frac{(\delta t)^2}{2}\widetilde{K}(\bar{w}_{k+1} + \bar{w}_k) \\ = \;& \frac{(\delta t)^2}{2}(\bar{g}_{k+1} + \bar{g}_k) + 2\delta t\,\bar{v}'(t_k) + (\delta t)^2 L\bar{v}'(t_k) + \frac{(\delta t)^3}{2}\widetilde{K}\bar{v}'(t_k) - \frac{(\delta t)^3}{2}g'(t_k). \end{aligned} \tag{5.4.16}$$

To estimate the global error $\bar{w}_N - \bar{v}(\tau)$, we introduce artificial numerical solutions $\bar{w}_k^{(i)}$. For each i, $\bar{w}_k^{(i)}$ satisfies (5.4.14) with $\bar{w}_i^{(i)} = \bar{v}(t_i)$ and $\bar{w}_{i-1}^{(i)} = \bar{w}_{i+1}^{(i)} - 2\delta t\bar{v}'(t_i)$. Note that $\bar{w}_k = \bar{w}_k^{(0)}$.

For the global error we have

$$\|\bar{v}(\tau) - \bar{w}_N\| \le \|\bar{v}(\tau) - \bar{w}_N^{(N-1)}\| + \|\bar{w}_N^{(N-1)} - \bar{w}_N^{(N-2)}\| + \ldots + \|\bar{w}_N^{(1)} - \bar{w}_N\|. \tag{5.4.17}$$

(Note that the global error for the original system can be derived from this error.)

It is clearly necessary to estimate $\|\bar{w}_N^{(i)} - \bar{w}_N^{(i-1)}\|$. The next two subsections will be devoted to the estimation of the differences between "neighbouring numerical solutions".

5.4.4 Consistency

In this subsection we consider the differences $\|\bar{w}_{i+1}^{(i)} - \bar{v}(t_{i+1})\|$ and $\|\bar{w}_{i+2}^{(i)} - \bar{w}_{i+2}^{(i+1)}\|$. For simplicity we denote $\bar{v}(t_i)$ by $\bar{v}_i$. The first lemma deals with the "starting" error.

Lemma 5.4.3

$$\|\bar{v}_{i+1} - \bar{w}_{i+1}^{(i)}\| \le C_b(\delta t)^3\left(|\!|\!| u_h |\!|\!|_{5,\max}^E + (\delta t)^3 |\!|\!| f |\!|\!|_{3,\max}\right).$$

Proof This is a direct consequence of Corollary 5.4.4. □

Next we have the error at the second step.

Lemma 5.4.4

$$\|\bar{w}_{i+2}^{(i)} - \bar{v}_{i+2}\| \leq C_b(\delta t)^3 \left(\|u_h\|_{5,\max}^E + (\delta t)^3 \|f\|_{3,\max}\right).$$

Proof Combine the results of Corollary 5.4.3 and Lemma 5.4.3. □

Lemma 5.4.3 provide an estimate for the difference $\bar{w}_i^{(i)} - \bar{w}_{i+1}^{(i)}$. The following result provide an estimate for the difference at the second step.

Corollary 5.4.5

$$\|w_{i+2}^{(i)} - w_{i+2}^{(i+1)}\| \leq C_b(\delta t)^3 \left(\|u_h\|_{5,\max}^E + \|f\|_{3,\max}\right).$$

Proof Use Lemmas 5.4.3 and 5.4.4.

$$\|w_{i+2}^{(i)} - w_{i+2}^{(i+1)}\| \leq \|\bar{w}_{i+2}^{(i)} - \bar{v}_{i+2}\| + \|\bar{v}_{i+1} - \bar{w}_{i+2}^{(i+1)}\|.$$

□

5.4.5 Stability

For the stability analysis we introduce the following matrices:

$$\begin{aligned}
A &= I + \frac{(\delta t)}{2}\widetilde{L} + \rho_1(\delta t)^2\widetilde{K}, \\
B &= -2I + \rho_0(\delta t)^2\widetilde{K}, \\
C &= I - \frac{(\delta t)}{2}\widetilde{L} + \rho_1(\delta t)^2\widetilde{K}.
\end{aligned}$$

The system (5.4.15) is now

$$A\bar{w}_{k+1} + B\bar{w}_k + C\bar{w}_{k-1} = (\delta t)^2(\rho_1\bar{g}_{k+1} + \rho_0\bar{g}_k + \rho_1\bar{g}_{k-1}) \qquad (5.4.18)$$

As mentioned at the end of Subsection 5.4.3, we need to estimate the difference $\bar{w}_N^{(i)} - \bar{w}_N^{(i+1)}$ for each i. Since both $\bar{w}_j^{(i)}$ and $\bar{w}_j^{(i+1)}$ satisfy the system (5.4.18) it follows that the error $\bar{e}_j = \bar{w}_j^{(i)} - \bar{w}_j^{(i+1)}$ must satisfy

$$A\bar{e}_{j+1} + B\bar{e}_j + C\bar{e}_{j-1} = \bar{0}, \qquad (5.4.19)$$

with the starting values, the local errors $\bar{e}_{i+1}$ and $\bar{e}_{i+2}$, already estimated.

For the case $L = \mu K$, we derive the eigenvalues of A, B and C. If $\widetilde{K}\bar{y} = \lambda\bar{y}$, then

$$\begin{aligned} A\bar{y} &= \bar{y} + \mu\frac{(\delta t)}{2}\widetilde{K}\bar{y} + \rho_1(\delta t)^2\widetilde{K}\bar{y} = \left(1 + \frac{\delta t}{2}(\mu\lambda) + \rho_1(\delta t)^2\lambda\right)\bar{y}, \\ B\bar{y} &= -2\bar{y} + \rho_0(\delta t)^2\widetilde{K}\bar{y} = \left(-2 + \rho_0(\delta t)^2\lambda\right)\bar{y}, \\ C\bar{y} &= \left(1 - \frac{\delta t}{2}(\mu\lambda) + \rho_1(\delta t)^2\lambda\right)\bar{y}. \end{aligned}$$

It is now possible to solve (5.4.19). Let $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$ denote the normalized eigenvectors of $\widetilde{K}$ and suppose $\bar{e}_{i+1} = \sum_{i=1}^{n}\eta_i\bar{y}_i$ and $\bar{e}_{i+2} = \sum_{i=1}^{n}\xi_i\bar{y}_i$.

Since the eigenvectors are orthogonal, it is sufficient to solve difference equations of the form

$$\alpha_i r_{k+1} + \beta_i r_k + \gamma_i r_{k-1} = 0 \text{ for } i = 1,\ 2,\ \ldots,\ n,$$

where α_i, β_i and γ_i denote the eigenvalues of the matrices A, B and C respectively.

The following result can be obtained by elementary calculations. Note that we do not use the subscripts for the coefficients α, β and γ. We take $r_1 = \xi$ and $r_0 = \eta$.

Solution of the difference equation

Case 1 $\beta^2 < 4\alpha\gamma$.

Note that in this case $\gamma > 0$. The solution is of the form

$$r_k = p^k(A\cos\omega k + B\sin\omega k),$$

where $p = \sqrt{\gamma/\alpha}$, $\cos\omega = -\beta/(2\sqrt{\alpha\gamma})$, $A = \eta$ and $B = (\xi - p\eta\cos\omega)/(\sin\omega)$.

Case 2 $\beta^2 = 4\alpha\gamma$.

The solution is of the form

$$r_k = \eta(1-k)r^k + \xi k r^{k-1}, \text{ with } r = -\frac{\beta}{2\alpha}.$$

Case 3 $\beta^2 > 4\alpha\gamma$.

The solution is of the form

$$r_k = Ar_1^k + Br_2^k,$$

with r_1 and r_2 the real roots of the equation $\alpha r^2 + \beta r + \gamma = 0$. The constants are $A = (\xi - \eta r_2)/(r_1 - r_2)$ and $B = (\xi - \eta r_1)/(r_2 - r_1)$.

We now prove the stability result. Bear in mind that $\lambda_k \to \infty$ as $k \to \infty$.

Lemma 5.4.5 *Stability*

If $\rho_0 \leq 2\rho_1$, then there exists a constant K —independent of the dimension of S^h—such that

$$\|w_N^{(i)} - w_N^{(i+1)}\| \leq K\left(\|v_{i+1} - w_{i+1}^{(i)}\| + \|w_{i+2}^{(i)} - w_{i+2}^{(i+1)}\|\right).$$

Proof For the eigenvalues

$$\begin{aligned}
\alpha &= 1 + \rho_1(\delta t)^2\lambda + \frac{\delta t}{2}\lambda\mu, \\
\beta &= -2 + \rho_0(\delta t)^2\lambda, \\
\gamma &= 1 + \rho_1(\delta t)^2\lambda - \frac{\delta t}{2}\lambda\mu
\end{aligned}$$

of A, B and C, we get

$$\begin{aligned}
\beta^2 - 4\alpha\gamma &= -4\rho_0(\delta t)^2\lambda + \rho_0^2(\delta t)^4\lambda^2 - 8\rho_1(\delta t)^2\lambda - 4\rho_1^2(\delta t)^4\lambda^2 + (\delta t)^2\mu^2\lambda^2 \\
&= \lambda(\delta t)^2\left[\mu^2\lambda - 4 + (\rho_0^2 - 4\rho_1^2)(\delta t)^2\lambda\right].
\end{aligned}$$

Consider the different cases:

Case 1 If $\beta^2 < 4\alpha\gamma$ then r_k is bounded if $\gamma \leq \alpha$.

This case, $\beta^2 - 4\alpha\gamma < 0$, is possible only for a finite number of small eigenvalues and only if $\rho_0 \geq 2\rho_1$. Since $\gamma/\alpha < 1$, the corresponding modes will not cause error growth.

Case 2 If $\beta^2 = 4\alpha\gamma$ then r_k is bounded if $|\beta/\alpha| < 2$.

If $\rho_0(\delta t)^2\lambda < 2$, we have $|\beta|/|\alpha| < \frac{2}{1}$.

If $\rho_0(\delta t)^2\lambda > 2$, we have $|\beta|/|\alpha| < \dfrac{\rho_0(\delta t)^2\lambda}{\rho_1(\delta t)^2\lambda} = \dfrac{\rho_0}{\rho_1} \leq 2$ if $\rho_0 \leq 2\rho_1$.

Case 3 If $\beta^2 > 4\alpha\gamma$ then r_k is bounded if both roots of $\alpha r^2 + \beta r + \gamma = 0$ are less than one in absolute value. Let

$$\triangle = \beta^2 - 4\alpha\gamma \;\; \leq \;\; \lambda^2(\delta t)^2\left[d^2 + (\rho_0^2 - 4\rho_1^2)(\delta t)^2\right]$$

and let $r_{\max}$ denote the absolute value of the root largest in absolute value.

$$\begin{aligned} r_{\max} &= \frac{\beta + \sqrt{\triangle}}{2\alpha} \\ &\leq \frac{\rho_0(\delta t)^2\lambda + \lambda\delta t\sqrt{\mu^2 + (\rho_0^2 - 4\rho_1^2)(\delta t)^2}}{2\rho_1(\delta t)^2\lambda + \delta t\,\lambda\mu} \\ &= \frac{\rho_0\delta t + \sqrt{\mu^2 + (\rho_0^2 - 4\rho_1^2)(\delta t)^2}}{2\rho_1\delta t + \mu} \\ &\leq \frac{\rho_0\delta t + \mu}{2\rho_1\delta t + \mu} \qquad (\text{ if } \rho_0 \leq 2\rho_1) \\ &\leq 1 \qquad (\text{ if } \rho_0 \leq 2\rho_1). \end{aligned}$$

□

Remark If damping is excluded, the difference system is considered to be unconditionally stable for $\rho_1 = \frac{1}{4}$ and $\rho_0 = \frac{1}{2}$, see [RM] or [Zi]. However, the bound may depend on the eigenvalues.

$$\begin{aligned} \cos\omega = \frac{-\beta}{2\sqrt{\alpha\gamma}} &= \frac{-\rho_0(\delta t)^2\lambda + 2}{2(1 + \rho_1(\delta t)^2\lambda)} \\ &= \frac{-\rho_0(\delta t)^2 + 2\lambda^{-1}}{2\rho_1(\delta t)^2 + 2\lambda^{-1}} \to \frac{-\rho_0}{2\rho_1} \text{ as } \lambda \to \infty. \end{aligned}$$

Consequently, $\sin\omega \to 0$ as $\lambda \to \infty$ and $\sin\omega$ is present in the numerator of a constant.

Remarks

1. Exactly the same results hold if we assume that rotary inertia can be ignored and we have only viscous damping. In this case $L = kM_0$ and $M = M_0$.

2. The eigenvalues of $K\bar{x} = \lambda M_0\bar{x}$ are much larger than the eigenvalues of $K\bar{x} = \lambda M\bar{x}$ (with rotary inertia).

3. Rotary inertia and Kelvin-Voigt damping both enhance stability.

5.4.6 Convergence

Lemma 5.4.6 *Global error*

$$\|\bar{e}_N\| \leq C_b N(\delta t)^3 \left(\|u_h\|_{5,\max}^E + \|f\|_{3,\max} \right)$$

(where N is the number of steps).

Proof

$$\begin{aligned} \|\bar{e}_N\| &\leq \|\bar{v}(\tau) - \bar{w}_N^{N-1}\| + \cdots + \|w_N^{(1)} - \bar{w}_N\| \\ &\leq KN \max_i \{\|\bar{v}_{i+1} - \bar{w}_{i+1}^{(i)}\| + \|\bar{v}_{i+1} - \bar{w}_{i+2}^{(i)}\|\}. \end{aligned}$$

Now use Lemmas 5.4.3 and 5.4.4.

□

To the sequence of finite difference vectors $\bar{v}_k$, correspond a sequence of approximations for u^h: $u_h^{(k)} = (QJ)^{-1}\bar{v}_k \in S^h$.

Theorem 5.4.1 *If u_h is the solution of Problem AG, then*

$$\|u_h(\tau) - u_N^h\| \leq C_b(\delta t)^2 \left(\|u_h\|_{5,\max}^E + \|f\|_{3,\max} \right),$$

where $u_N^h = Q^{-1}\bar{u}_N$.

Proof Since $\bar{e}_N = QN(u_n(\tau) - u_n^N)$, we have

$$\|u_h(\tau) - u_h^N\| = \|\bar{e}_N\|.$$

Now use Lemma 5.4.6.

□

Remark Error estimates for the fully discrete system is obtained by combining Theorem 5.4.1 with the results of Section 5.3. Note that the error estimates are with respect to the inertia norm.

Chapter 6

Application. Damaged beam

6.1 Introduction

We consider Problem 1 (from Section 2.3). This model for a damaged beam was proposed in [VV]. See also [JVRV].

The detection of damage in structures or materials is clearly of great importance. Ideally it should be possible to infer the location and extent of damage from indirect measurements or signals. To facilitate such deduction, a mathematical model of the object or structure is necessary. See [VV] for details and numerous other references.

Viljoen *et al.* [VV] use changes in the natural frequencies of the beam to locate and quantify the damage. The natural angular frequencies for the damaged beam are calculated from the characteristic equation obtained from the associated eigenvalue problem. As is well-known, only the first few natural angular frequencies and modes are usually calculated with this method, because of computational difficulties with the hyperbolic functions. Due to this limitation, the need arises for a numerical method to simulate the dynamical behaviour of the beam.

In a joint paper, [ZVV], we developed a finite element method (FEM) to approximate the solution of the model problem for arbitrary initial conditions. (Ironically we also found it possible to calculate eigenvalues and eigenfunctions more accurately with the FEM.) It was necessary to adapt standard procedures to deal with the discontinuity in the derivative that arises as a result of the elastic joint. We made the assumption that damping would not influence the solution significantly on a small time scale. We now investigate the validity of this assumption, and also deem it prudent to include the effect of rotary inertia.

In the paper [ZVV], only Hermite piecewise cubics were used as basis functions. In this investigation we also demonstrate the effectiveness of Hermite piecewise quintics.

From Section 3.1 we have the variational formulation. The Galerkin approximations for the eigenvalue and initial value problems are given in Section 4.1. (We do not consider the equilibrium problem.) In Section 4.2 we showed how the standard basis functions are adapted to deal with the discontinuity in the derivative.

In Section 6.2 we compute the natural angular frequencies and modes of vibration from the characteristic equation for comparison purposes. The computation of the matrices is discussed in Section 6.3.

In Sections 6.4 and 6.5 numerical results are presented that demonstrate not only the effect of damage on the motion of a beam but also the effect of damping and rotary inertia. We also investigate the use of Hermite piecewise quintics as basis functions instead of Hermite piecewise cubics.

6.2 Natural frequencies and modes of vibration

One way to calculate the natural angular frequencies and modes of vibration for the damaged beam is to apply the method of separation of variables directly to Problem 1 (from Section 2.3).

For the case $r = 0$ (without rotary inertia), we have the following eigenvalue problem:

$$\begin{aligned}
w^{(4)} - \lambda w &= 0, \;\; 0 < x < 1, \; x \neq \alpha, \\
w(0) &= w'(0) = w''(1) = w'''(1) = 0, \\
w(\alpha^+) &= w(\alpha^-), \\
w''(\alpha^+) &= w''(\alpha^-), \\
w'''(\alpha^+) &= w'''(\alpha^-), \\
w''(\alpha) &= \frac{1}{\delta}(w'(\alpha^+) - w'(\alpha^-)).
\end{aligned}$$

For this eigenvalue problem it is possible to find so called exact solutions. It is convenient to introduce the positive real number ν, with $\lambda = \nu^4$. Consequently $\nu^2 = \sqrt{\lambda}$ is a natural angular frequency. Analogous to the case of the undamaged beam, the corresponding mode is of the form

$$w(x) = \begin{cases} A\sin(\nu x) - A\sinh(\nu x) + B\cos(\nu x) - B\cosh(\nu x) & \text{for } 0 < x < \alpha, \\ (C + A)\sin(\nu x) + (D - A)\sinh(\nu x) & \\ \qquad +(E + B)\cos(\nu x) + (F - B)\cosh(\nu x) & \text{for } \alpha < x < 1. \end{cases}$$

Note that the boundary conditions at $x = 0$ have already been taken into account.

From the continuity conditions and the jump condition at $x = \alpha$, the constants C, D, E, and F can be expressed in terms of A and B. Finally, from the two boundary conditions at $x = 1$, the characteristic equation for ν can be constructed from

$$\det \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} = 0 \qquad (6.2.1)$$

where

$$\begin{aligned}
d_1 &= -(\sin\nu + \sinh\nu) + \frac{\delta\nu}{2}(\sin\nu\alpha + \sinh\nu\alpha) \times \\
&\quad (\sin\nu\cos\nu\alpha - \sinh\nu\cosh\nu\alpha - \cos\nu\sin\nu\alpha + \cosh\nu\sinh\nu\alpha), \\
d_2 &= -(\cos\nu + \cosh\nu) + \frac{\delta\nu}{2}(\cos\nu\alpha + \cosh\nu\alpha) \times \\
&\quad (\sin\nu\cos\nu\alpha - \sinh\nu\cosh\nu\alpha - \cos\nu\sin\nu\alpha + \cosh\nu\sinh\nu\alpha), \\
d_3 &= -(\cos\nu + \cosh\nu) + \frac{\delta\nu}{2}(\sin\nu\alpha + \sinh\nu\alpha) \times \\
&\quad (\cos\nu\cos\nu\alpha - \cosh\nu\cosh\nu\alpha + \sin\nu\sin\nu\alpha + \sinh\nu\sinh\nu\alpha), \\
d_4 &= (\sin\nu - \sinh\nu) + \frac{\delta\nu}{2}(\cos\nu\alpha + \cosh\nu\alpha) \times \\
&\quad (\cos\nu\cos\nu\alpha - \cosh\nu\cosh\nu\alpha + \sin\nu\sin\nu\alpha + \sinh\nu\sinh\nu\alpha).
\end{aligned}$$

Solving equation (6.2.1) numerically using the Newton-Raphson method, yields the natural angular frequencies for the damaged beam. For each natural angular frequency a corresponding mode can then be obtained. As is expected, only the first few natural angular frequencies and modes could be calculated, as it is difficult to handle the hyperbolic functions numerically for large values of ν.

Numerical results obtained using the finite element method—using cubics as well as quintics as basis functions— are given in Section6.4.

6.3 Computation of Matrices

The matrices K, L and M are defined in Section 4.1 in terms of the bilinear forms defined in Section 3.1. The computation of the matrices is complicated by the interface conditions which results in non-standard basis elements. In this section we give an indication of how we went about in computing these matrices. The first step is to reorder the basis elements constructed in Subsection 4.2.2.

Consider the matrix M_0:

$$[M_0]_{ij} = (\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^\alpha \phi_{i1}\phi_{j1} + \int_\alpha^1 \phi_{i2}\phi_{j2}.$$

Note that $\tilde{\phi}_i = \langle 0, \phi_i \rangle$ or $\langle \phi_i, 0 \rangle$ except when we are dealing with a basis element associated with the node $x_p = \alpha$, the location of the damage. In general then, the entries will be those of the standard mass matrix for an undamaged beam. Now suppose one of the basis elements are associated with x_p:

If $\tilde{\phi}_i = \tilde{\phi}_p^{(0)}$, then

$$[M_0]_{ij} = \int_0^\alpha \phi_{i1}\phi_{j1} + \int_\alpha^1 \phi_{i2}\phi_{j2} = \int_0^1 \phi_i\phi_j.$$

Again the result will be the same as in the standard case. The same for $\tilde{\phi}_p^{(2)}$.

On the other hand, suppose $\tilde{\phi}_i = \tilde{\phi}_{pL}^{(1)}$, then

$$[M_0]_{ij} = \int_0^\alpha \phi_{j1}\phi_{i1} + 0,$$

which is not the same as for an undamaged beam. Similarly for $\tilde{\phi}_i = \tilde{\phi}_{pR}^{(1)}$. Thus the standard matrix has to be modified for the damaged beam.

We have the same situation for the matrix M_r where we define

$$[M_r]_{ij} = (\tilde{\phi}'_i, \tilde{\phi}'_j) = \int_0^\alpha \phi'_{i1}\phi'_{j1} + \int_\alpha^1 \phi'_{i2}\phi'_{j2}.$$

There is an additional complication for the K-matrix:

$$K_{ij} = b(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^\alpha \phi''_{i1}\phi''_{j1} + \int_\alpha^1 \phi''_{i2}\phi''_{j2} + \frac{1}{\delta}\left(\phi'_{j2}(\alpha) - \phi'_{j1}(\alpha)\right)\left(\phi'_{i2}(\alpha) - \phi'_{i1}(\alpha)\right).$$

Only four entries in the standard K-matrix will change due to the additional term, $(u_2'(\alpha) - u_1'(\alpha))(v_2'(\alpha) - v_1'(\alpha))/\delta$, in the bilinear form b.

For greater clarity we will explain the procedure in another way. In the discussion that follows, we refer to $\tilde{\phi}_i^{(k)}$ as a Type k basis function.

In modifying the matrices for an undamaged beam to the matrices for a damaged beam, we have to keep in mind that the Type 1 basis function associated with $x_p = \alpha$, has changed. By replacing the row and column associated with the Type 1 basis function at x_p, in the matrix of the undamaged beam, by two rows and columns respectively, provision is made for the modified basis function. The values in the matrix in these two rows and columns have to be modified accordingly. For the K-matrix one must also keep the additional term in mind.

Having computed M_0, M_r and K we are done since $L = \mu K + kM_0$ and $M = M_0 + M_r$.

6.4 Numerical results. Eigenvalue problem

Cubics as basis functions, are usually sufficiently accurate in solving one-dimensional vibration problems with the finite element method. In a joint paper [ZVV] we discussed the use of cubics as basis functions for the damaged beam.

In this section of the thesis we also consider numerical convergence of the eigenvalues. The order of convergence that is suggested by the numerical results is also compared to the order obtained from the theory. Additionally, quintics are considered as basis functions. The main reason for this is that cubics are not compatible with reduced quintics in plate beam models. We also investigate the effect of rotary inertia.

6.4.1 Cubics

Natural angular frequencies and modes for the vibration problem are calculated by solving the eigenvalue problem with the FEM. We developed the code to construct the relevant matrices in Matlab and use standard Matlab subroutines to calculate the eigenvalues and eigenvectors of the generalised eigenvalue problem.

It is possible to compare only the first few FEM eigenvalues to the so called exact eigenvalues calculated from the characteristic equation. Thereafter the exact values can not be computed accurately and the FEM is used to calculate the eigenvalues.

In Table 6.1 we list values for the eigenvalues obtained from the characteristic equation (see Section 6.2) and values obtained by the FEM using cubics as basis functions with 20, 40, 80 and 160 subintervals respectively. This give approximations for respectively the first 40, 80, 160 and 320 eigenvalues.

i	λ_i	$\lambda_i^{(20)}$	$\lambda_i^{(40)}$	$\lambda_i^{(80)}$	$\lambda_i^{(160)}$
1	11.81469	11.81469	11.81469	11.81469	11.81469
2	406.01614	406.01757	406.01623	406.01615	406.01615
3	3806.05283	3806.17742	3806.06067	3806.05332	3806.05287
4	12544.12940	12545.47137	12544.21439	12544.13473	12544.12972
5	39943.82322	39957.35763	39944.68387	39943.87724	39943.82661

Table 6.1: *Eigenvalues from the characteristic equation as well as FEM eigenvalues using cubics as basis functions with $\delta = 0.1$ and $\alpha = 0.5$.*

Throughout this section n denote the number of subintervals. (All of equal length.)

To investigate the convergence of the FEM eigenvalues, we calculate the relative difference between FEM approximations, that is $(\lambda^{(2n)} - \lambda^{(n)})/\lambda^{(2n)}$. These differences are calculated and listed in Table 6.2 for $n = 20$, 40, 80 and 160 subintervals respectively.

	$\lvert\lambda_i^{(2n)} - \lambda_i^{(n)}\rvert/\lambda_i^{(2n)}$			
i	$n = 20$	$n = 40$	$n = 80$	$n = 160$
6	6.1×10^{-4}	3.9×10^{-5}	2.5×10^{-6}	1.6×10^{-7}
12	1.1×10^{-2}	7.6×10^{-4}	4.9×10^{-5}	3.1×10^{-6}
24	2.2×10^{-1}	1.2×10^{-2}	8.6×10^{-4}	5.5×10^{-5}
48	—	2.2×10^{-1}	1.3×10^{-2}	9.2×10^{-4}

Table 6.2: *Relative differences for FEM eigenvalues using cubics as basis functions.*

The tendency of the relative difference to decrease (by roughly a factor 10) each time that the number of subintervals is doubled, is empirical verification that there is convergence of the FEM eigenvalues. We found that the eigenvalues computed from the characteristic equation were less dependable.

It is necessary to determine a relationship between the number of FEM eigenvalues that is sufficiently accurate (criterion to be specified) and the number of subintervals used.

A relative difference strictly less than 10^{-3} is considered sufficiently accurate for our purpose. Using this as criterion, we find that approximately a seventh of the $2n$ eigenvalues calculated using n subintervals, yields a relative difference, $(\lambda^{(2n)} - \lambda^{(n)})/\lambda^{(2n)}$, strictly less than 10^{-3}, see Table 6.2.

The relative difference between the FEM eigenvalues with 160 and 320 subintervals is an indication of the relative error between the exact eigenvalue and the FEM eigenvalue using 320 subintervals.

Since we use $(\lambda^{(320)} - \lambda^{(160)})/\lambda^{(320)}$ as measure of the relative error, $(\lambda - \lambda^{(320)})/\lambda$, we conclude that the first 90 eigenvalues obtained using 320 subintervals yield a relative error that is sufficiently accurate.

An indication of the order of convergence of the FEM eigenvalues can be obtained from the ratio of two successive differences

$$|\lambda_i^{(2n)} - \lambda_i^{(n)}|/|\lambda_i^{(4n)} - \lambda_i^{(2n)}|.$$

Typical results are listed in Table 6.3.

	$\lvert\lambda_i^{(2n)} - \lambda_i^{(n)}\rvert/\lvert\lambda_i^{(4n)} - \lambda_i^{(2n)}\rvert$		
i	$n = 20$	$n = 40$	$n = 80$
1	10.25	0.03	0.14
3	15.88	16.41	6.57
6	15.53	15.88	15.72
12	14.23	15.57	15.90
24	18.29	14.47	15.62

Table 6.3: *Relationship between successive relative differences with cubics as basis functions.*

These relative differences decrease by roughly a factor 16 if the number of subintervals is doubled. From this it would appear that the convergence is of order h^4 which matches the theory, Section 5.2.

It is observed that those differences not yielding a factor 16 typically occur in the right top part as well as the left bottom part of Table 6.3. These deviations are illustrated by the first, third and 24th eigenvalues:

Firstly, the accuracy of an approximation can decrease if the number of subintervals is increased. This is due to an increase in the roundoff error and has significant effects in situations where the errors are already small. For example, FEM approximations for the first eigenvalue yield

$$(\lambda_1^{(40)} - \lambda_1^{(20)}) = -1.1 \times 10^{-13} \quad \text{while} \quad (\lambda_1^{(80)} - \lambda_1^{(40)}) = 3.8 \times 10^{-12}.$$

From the theory, Section 5.2, we know that the FEM approximations of an eigenvalue will decrease if the number of subintervals is increased. This can be used to detect cases where the effect of the roundoff error is greater than the advantageous effect of an increase in the number of subintervals used.

Rounding error also explain the decrease in the ratios for the third eigenvalue from roughly a factor 16 to 6.5. This situation differ from the first eigenvalue in that the decrease (improvement) in the relative difference was just partially cancelled by the increase in the roundoff error.

Secondly, as we have showed previously, there is a relationship between the number of FEM eigenvalues that can be calculated sufficiently accurately and the number of subintervals used. The 24th eigenvalue is such an example. The effect of the poor approximation of $\lambda_{24}^{(20)}$, is seen in Table 6.3 in that $18.29 > 14.47$. This is expected as only the first six eigenvalues obtained, using 20 subintervals, yield relative errors less than 10^{-3}.

6.4.2 Quintics

We now consider quintics as basis functions, and compare the results to the case where we used cubics.

In Table 6.4 we list values for the eigenvalues obtained from the characteristic equation and values obtained by the FEM using quintics as basis functions with 2, 4, 8 and 16 subintervals respectively. This gives approximations for respectively the first 6, 12, 24 and 48 eigenvalues.

i	λ_i	$\lambda_i^{(2)}$	$\lambda_i^{(4)}$	$\lambda_i^{(8)}$	$\lambda_i^{(16)}$
1	11.81469	11.81469	11.81469	11.81469	11.81469
2	406.01614	406.01954	406.01618	406.01614	406.01614
3	3806.05283	3822.51900	3806.09344	3806.05297	3806.05283
4	12544.12940	12844.88875	12544.53719	12544.13441	12544.12941
5	39943.82322	41569.18041	40042.72518	39944.02589	39943.82387

Table 6.4: *Eigenvalues from the characteristic equation as well as FEM eigenvalues using quintics as basis functions with $\delta = 0.1$ and $\alpha = 0.5$.*

If these values are compared to those in Table 6.1, it seems as if the same accuracy can be obtained, using quintics as basis functions, with less subintervals, than in the case where cubics were used as basis functions. For example, the fifth FEM eigenvalue using quintics as basis functions with

16 subintervals, already yields a better approximation than using cubics with 80 subintervals.

As in the case with cubics as basis functions, we investigate the convergence of the FEM eigenvalues by considering relative differences, $(\lambda^{(2n)}-\lambda^{(n)})/\lambda^{(2n)}$. These values are are listed in Table 6.5 for 2, 4, 8 and 16 subintervals respectively.

	$\|\lambda_i^{(2n)} - \lambda_i^{(n)}\|/\lambda_i^{(2n)}$			
i	$n=2$	$n=4$	$n=8$	$n=16$
1	2.1×10^{-8}	7.9×10^{-11}	1.2×10^{-11}	1.7×10^{-11}
2	8.3×10^{-6}	9.5×10^{-8}	3.4×10^{-10}	1.3×10^{-11}
4	2.4×10^{-2}	3.2×10^{-5}	4.0×10^{-7}	1.4×10^{-9}
8	—	3.9×10^{-2}	7.2×10^{-5}	8.2×10^{-7}

Table 6.5: *Relative differences for FEM eigenvalues using quintics as basis functions.*

The numerical results suggest convergence of the FEM eigenvalues since the relative error decreases (by roughly a factor 100) each time that the number of subintervals is doubled, see Table 6.5.

For approximately a third of the $3n$ eigenvalues computed, using n subintervals, the relative difference $(\lambda^{(2n)} - \lambda^{(n)})/\lambda^{(2n)}$ is strictly less than 10^{-3}.

As with the cubics, we now consider the ratio of two successive differences

$$|\lambda_i^{(2n)} - \lambda_i^{(n)}|/|\lambda_i^{(4n)} - \lambda_i^{(2n)}|$$

to get an idea of the order of convergence. Typical results are listed in Table 6.6.

As was the case in Table 6.3, the values in the top right of Table 6.6 exhibit effect of roundoff error and the values in the bottom left the result of eigenvalues not calculated sufficiently accurately. From this it would appear that the order of convergence is h^8 which matches the theory, Section 5.2.

To compare the accuracy of the FEM eigenvalues using quintics as basis functions to the case using cubics as basis functions, we choose the number of subintervals in each of the cases such that the sizes of the matrices in the two cases are equal. For example, using 30 subintervals for cubics yield 61×61 matrices and 20 subintervals for quintics 62×62 matrices. We then

	$\lvert\lambda_i^{(2n)} - \lambda_i^{(n)}\rvert / \lvert\lambda_i^{(4n)} - \lambda_i^{(2n)}\rvert$		
i	$n = 2$	$n = 4$	$n = 8$
1	265.60	6.75	0.69
2	87.21	278.22	25.45
3	405.90	278.67	319.70
4	745.69	80.59	291.89
5	15.43	488.57	311.65
6	16.24	330.71	307.27
7	304.98	166.95	286.96
8	302.37	540.94	87.70

Table 6.6: *Relationship between successive relative differences with quintics basis functions.*

compare the eigenvalues calculated in the two cases with the eigenvalues computed using cubics with 320 subintervals. (We use the first 90 FEM eigenvalues using cubics as basis functions with 320 subintervals as the FEM approximation to the first 90 exact eigenvalues.)

Our numerical experiments indicate that using quintics with n subintervals, yield at least double the number of eigenvalues to the prescribed accuracy (relative error strictly less than 10^{-3}) than when cubics are used with $3n/2$ subintervals. In Table 6.7 we give an example of results obtained.

In Table 6.7 we use the following notation:

- Let λ_i denote the ith FEM eigenvalue that we use as approximation for the exact eigenvalue. (In this case those FEM eigenvalues obtained using cubics as basis functions with 320 subintervals.)
- To distinguish between the FEM eigenvalues computed using quintics and cubics as basis functions, we denote the ith FEM eigenvalue using cubics with 30 subintervals by $\lambda_i^{(c)}$ and using quintics with 20 subintervals by $\lambda_i^{(q)}$.

Note that the FEM approximations for the first eigenvalue are identical in both cases.

From Table 6.7 we see that using cubics, the first 9 eigenvalues (approximately a seventh of the number of eigenvalues calculated, $61/7 \approx 8.7$) have

i	$(\lambda_i - \lambda_i^{(c)})/\lambda_i$	$(\lambda_i - \lambda_i^{(q)})/\lambda_i$
1	2.6×10^{-6}	2.6×10^{-6}
5	6.8×10^{-5}	9.6×10^{-9}
9	8.5×10^{-4}	4.3×10^{-7}
10	1.2×10^{-3}	8.6×10^{-7}
15	6.8×10^{-3}	4.6×10^{-5}
20	1.9×10^{-2}	1.4×10^{-4}

Table 6.7: *Comparing FEM eigenvalues using quintics with 20 subintervals to FEM eigenvalues using cubics with 30 subintervals.*

relative difference less than 10^{-3}. Using quintics, the first 20 eigenvalues, that is approximately a third of the number of eigenvalues calculated, have relative difference less than 10^{-3}.

In conclusion, for the same computational effort (same size of the matrices), quintics yield twice as many eigenvalues sufficiently accurate than when cubics are used i.e. to obtain the first k FEM eigenvalues with relative difference less than 10^{-3}, $7k/2$ subintervals must be used with cubics as basis functions and k subintervals with quintics.

6.4.3 The effect of rotary inertia

We now consider the effect of rotary inertia on the eigenvalues and use quintics as basis functions.

Note that this eigenvalue problem differs from the one excluding rotary inertia, Section 3.5. The parameter r is a measure of the effect of rotary inertia, Section 2.2.

We start by establishing convergence of the FEM eigenvalues for the case where rotary inertia is included, thereafter, we investigate the effect of rotary inertia on the eigenvalues.

As for the case without rotary inertia, the numerical results indicate convergence of the FEM eigenvalues. In Table 6.8 typical results for the relative differences, $(\lambda^{(2n)} - \lambda^{(n)})/\lambda^{(2n)}$, including rotary inertia, are listed for 2, 4, 8 and 16 subintervals respectively.

	$\lvert\lambda_i^{(2n)} - \lambda_i^{(n)}\rvert/\lambda_i^{(2n)}$			
i	$n = 2$	$n = 4$	$n = 8$	$n = 16$
1	2.1×10^{-8}	7.8×10^{-11}	2.9×10^{-11}	7.6×10^{-10}
2	7.8×10^{-6}	8.9×10^{-8}	3.2×10^{-10}	1.4×10^{-11}
4	1.6×10^{-3}	4.0×10^{-6}	2.0×10^{-8}	7.9×10^{-11}
8	—	1.6×10^{-2}	9.5×10^{-5}	3.3×10^{-7}

Table 6.8: *Relative differences for FEM eigenvalues including rotary inertia with* $1/r = 4800$.

The numerical results again suggests convergence of the FEM eigenvalues. The same pattern with respect to the order of convergence is observed as for the case without rotary inertia.

The presence of rotary inertia decreases the values of corresponding eigenvalues in comparison to the case without rotary inertia. Furthermore, the bigger the parameter r, the greater the change in the eigenvalues in comparison to the case without rotary inertia. In Table 6.9 we list eigenvalues for different values of r as well as for the case without rotary inertia ($r = 0$). We use 32 subintervals for these approximations.

In Table 6.9 λ_i denotes the ith FEM eigenvalue.

i	$\lambda_i$with $r = 0$	$1/r = 19200$	$1/r = 4800$	$1/r = 1200$
1	11.81469	11.81138	11.80145	11.76186
2	406.01614	403.97925	397.71100	370.06960
4	12544.12940	11057.05461	5401.71657	3576.19925
8	273293.79309	169832.12061	158744.64019	125970.79578

Table 6.9: *FEM eigenvalues for different effects of rotary inertia using* 32 *subintervals.*

These results are for the dimensionless case. Where rotary inertia is included, two dimensionless constants, T and r, must be calculated if the results is to be connected to a specific beam, Section 2.2.

Modes

In [ZVV] we showed that only up to the seventh so called exact mode can be computed before computational difficulties are encountered. Therefore we consider the convergence of the FEM modes using quintics as basis functions and include rotary inertia.

Let $\bar{w}_i^{(n)}$ denote the FEM approximation for the ith mode using n elements, normalised with respect to the infinity norm, $\|\cdot\|_\infty$.

The way in which we ordered our basis elements implies that the first $n+1$ components of $\bar{w}_i^{(n)}$ are associated with the function values at the $n+1$ nodes. The next $n+2$ values represent the values of the first order derivatives at the nodes. Two values are associated with the point where the damage occurs. Quintics as basis functions also yield approximations for the values of the second order derivatives, and the last $n+1$ values of $\bar{w}_i^{(n)}$ represent the values of the second order derivatives at the nodes.

In Table 6.10 the numerical convergence of the FEM modes are illustrated. We list the differences $\|\bar{w}_i^{(2n)} - \bar{w}_i^{(n)}\|_\infty$, $\|(\bar{w}_i^{(2n)})' - (\bar{w}_i^{(n)})'\|_\infty$ and $\|(\bar{w}_i^{(2n)})'' - (\bar{w}_i^{(n)})''\|_\infty$ for different values of n.

	$\|\bar{w}_i^{(2n)} - \bar{w}_i^{(n)}\|_\infty$ $\|(\bar{w}_i^{(2n)})' - (\bar{w}_i^{(n)})'\|_\infty$ $\|(\bar{w}_i^{(2n)})'' - (\bar{w}_i^{(n)})''\|_\infty$		
i	$n = 4$	$n = 8$	$n = 16$
1	6.83140×10^{-6}	4.56592×10^{-7}	2.578292×10^{-8}
	9.65478×10^{-6}	6.45299×10^{-7}	3.64876×10^{-8}
	2.48714×10^{-6}	6.33818×10^{-8}	8.43408×10^{-9}
2	2.49236×10^{-5}	2.09579×10^{-6}	1.47615×10^{-7}
	1.18092×10^{-4}	9.93009×10^{-6}	6.99423×10^{-7}
	1.96527×10^{-4}	6.46058×10^{-6}	2.06282×10^{-7}
4	1.46719×10^{-5}	6.47026×10^{-6}	5.44719×10^{-7}
	1.61154×10^{-4}	7.01558×10^{-5}	5.90054×10^{-6}
	8.16804×10^{-3}	2.53189×10^{-4}	8.28369×10^{-6}
8	3.70776×10^{-4}	6.16539×10^{-6}	1.75076×10^{-6}
	7.90080×10^{-3}	1.30886×10^{-4}	3.71925×10^{-5}
	6.26608×10^{-2}	1.00243×10^{-3}	2.90161×10^{-5}
14	3.20162×10^{-4}	9.48592×10^{-5}	9.65450×10^{-7}
	1.68312×10^{-2}	4.23303×10^{-3}	4.29824×10^{-5}
	1.17602	1.65933×10^{-1}	1.00174×10^{-2}
16	—	9.03027×10^{-5}	4.81429×10^{-6}
	—	4.78628×10^{-3}	2.60210×10^{-4}
	—	1.91004×10^{-1}	2.44992×10^{-2}

Table 6.10: *Convergence of FEM modes with* $\delta = 0.1$, $\alpha = 0.5$ *and* $1/r = 4800$.

The rate at which convergence of the function values and the first order derivatives occur, differ from the convergence rate of the second order derivatives, which is much slower. Those modes that are associated with the first eigenvalues, starting with the smallest, converges faster than the modes associated with later eigenvalues. (Convergence in the energy norm implies that the second order derivative converges in the mean.)

6.5 Numerical results. Initial value problem

Consider the initial value problem. From Section 4.1 we have the following system of differential equations

$$M\bar{u}''(t) = -L\bar{u}'(t) - K\bar{u}(t).$$

For the numerical experimentations, we choose the following initial conditions: $u_h'(0) = 0$ and $u_h(0)$ a quintic "solitary wave".

To approximate the solution of this problem we use the difference scheme in Section 5.4 with $\rho_0 = 2\rho_1 = 1/2$.

$$\left(\frac{M}{\delta t^2} + \frac{L}{2\delta t} + \frac{1}{4}K\right)\bar{u}_{k+1} + \left(-2\frac{M}{\delta t^2} + \frac{1}{2}K\right)\bar{u}_k + \left(\frac{M}{\delta t^2} - \frac{L}{2\delta t} + \frac{1}{4}K\right)\bar{u}_{k-1} = 0.$$

Since the initial velocity is zero, we have $\bar{u}_1 = \bar{u}_{-1}$.

The results obtained for the eigenvalue problem motivated us to use quintics as basis functions.

Convergence

The verify convergence, we choose a fixed spacial discretization and a fixed final time τ. Then, starting with 10 time intervals, we increased the number of intervals until the relative difference is strictly less than 10^{-3}. This approximation is then considered as sufficiently accurate for the system of differential equations.

Decreasing the time step size, we found the first order derivatives needed approximately double the number of time steps to yield the same relative difference in $\|\cdot\|_\infty$ than the function values do. It seems as if the the second order derivatives do not converge point wise, if they do, the convergence is very slow. This is not altogether surprising (see Section 5.4).

To establish the number of elements needed for our approximation, we choose a fixed final time, τ, and time step size, δt. Then the number of elements, starting with 10, is doubled until the relative difference satisfy our criterion.

Simulation of the motion of beam

We are primarily concerned with the detection of damage. In this section we give an indication of the effect of respectively damage, damping and rotary inertia on the motion of a beam.

Our experiments indicate that measurable differences between the undamaged and damaged beams occur in displacements as well as gradients. (Table 6.11.) Viscous damping has no significant effect on the motion. Looking at the modal analysis this was expected, since it only effects the first few modes. Adding Kelvin-Voigt damping, the differences between the damaged and undamaged cases decrease, but is still clearly detectable. (Table 6.12.) The presence of rotary inertia can have a more significant effect on the difference between the motion of the damaged and undamaged beams. (Tables 6.13 and 6.14).

To illustrate the above effects we compare the motion of an undamaged beam to that of a damaged beam where the initial velocity is zero and the initial position a 'solitary wave". For this simulation we choose $\alpha = 0.4$, $\delta = 0.1$ which is rather excessive, 80 elements, $\tau = 0.02$ and 400 time subintervals.

Almost immediately after the first wave front pass through the damaged point, measurable differences in displacements as well as gradients between the two cases occur. (See Figure 6.1.) In Table 6.11 we compare the displacement of the damaged and undamaged beams on $\tau = 0.02$ at $x = 0.3$ and $x = 0.7$.

x	Undamaged beam	Damaged beam	% difference
0.3	2.325×10^{-1}	1.505×10^{-1}	8.2
0.7	4.733×10^{-1}	5.508×10^{-1}	7.8

Table 6.11: *Effect of damage during motion where* $\delta = 0.1$, $\alpha = 0.4$, $\tau = 0.02$.

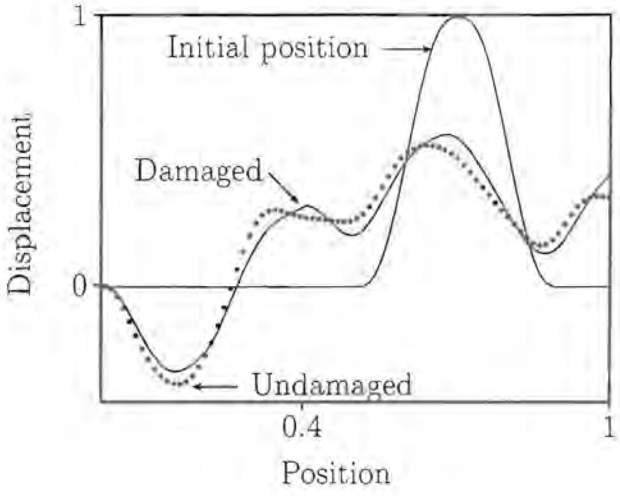

Figure 6.1: *Comparing the motion of an undamaged beam to that of a damaged beam where* $\delta = 0.1$, $\alpha = 0.4$ *and* $\tau = 0.02$.

We now add Kelvin-Voigt damping to the same situation as in the previous case. We use $\mu = 3.469 \times 10^{-5}$. This value for μ was obtained from [JVRV].

x	Undamaged beam	Damaged beam	% difference
0.3	2.099×10^{-1}	1.378×10^{-1}	7.2
0.7	4.716×10^{-1}	5.436×10^{-1}	7.2

Table 6.12: *Effect of Kelvin-Voigt damping on the damage during motion where* $\delta = 0.1$, $\alpha = 0.4$, $\tau = 0.02$ *and* $\mu = 3.469 \times 10^{-5}$.

The presence of rotary inertia can make the differences more difficult to detect. An example is given in Tables 6.13 and 6.14.

x	Undamaged beam	Damaged beam	% difference
0.2	-2.525×10^{-1}	-2.065×10^{-1}	4.6
0.6	4.762×10^{-1}	5.248×10^{-1}	4.9

Table 6.13: *Effect of Rotary inertia on the damage during motion where* $\delta = 0.1$, $\alpha = 0.4$, $\tau = 0.02$ *and* $1/r = 19200$.

x	Undamaged beam	Damaged beam	% difference
0.2	-2.236×10^{-1}	-2.092×10^{-1}	1.4
0.6	4.140×10^{-1}	4.287×10^{-1}	1.5

Table 6.14: *Effect of Rotary inertia on the damage during motion where* $\delta = 0.1$, $\alpha = 0.4$, $\tau = 0.02$ *and* $1/r = 4800$.

Chapter 7

Application. Plate beam model

7.1 Introduction

We consider Problem 3 (from Section 2.6). It is a mathematical model for a plate connected to two beams. Problems of this type are clearly of great practical importance. The plate can be rigidly connected to the beams or simply supported by the beams. The same model can be used for an I-shaped structural member (depending on the type of vibration). For simplicity we restrict our investigation to the case of a plate supported by beams. If the plate is rigidly connected to the beams, it may result in a problem with six unknown functions (excluding shear) due to dynamical effects. Even in our restricted case, one may easily encounter very large matrices.

In collaboration with others, [ZVGV1], we considered the equilibrium and eigenvalue problems of a rectangular plate supported by two beams at the boundary. In this thesis we extend the investigation and include the effect of rotary inertia.

The computation of the matrices is explained in Section 7.2. We use reduced quintics for the plate, which necessitates the use of quintics for the beams. We treat the equilibrium problem in Section 7.3 and the eigenvalue problem Section 7.4.

7.2 Computation of matrices

For the numerical experimentation we consider a square plate, Ω, rigidly supported at two opposing sides and supported by identical beams at the remaining sides. The plate has thickness h and the beams are of square profile with thickness d. Furthermore, we assume the plate and beams are of the same material. (These restrictions are evidently not necessary.)

The reference configuration Ω is the rectangle with $0 < x_1 < 1$ and $0 < x_2 < 1$. Those parts of the boundary where $x_1 = 0$ and $x_1 = 1$ are denoted by Σ_0 and Σ_1 respectively and correspond to the rigidly supported parts of the boundary. Those parts where $x_2 = 0$ and $x_2 = 1$ are denoted by Γ_0 and Γ_1 respectively and correspond to the sections of the boundary supported by beams.

7.2.1 Basis elements

For the plate we use only reduced quintics as basis functions. These functions are in $H^2(\Omega)$ or fully conforming, in finite element language. They are defined on a triangular mesh. The mesh for the rectangle $\overline{\Omega}$ is generated in the following way: The interval $[0,1]$ is divided into n_1 subintervals and the interval $[0,1]$ into n_2 subintervals. This partition of the intervals yields $n_1 \times n_2$ rectangles. The final triangular mesh is then obtained by dividing each of these rectangles into two triangles by connecting the lower left corner with the upper right corner. The rectangle Ω is divided into $2n_1 \times n_2$ triangles. Consequently we have $2n_1 \times n_2$ elements Ω_i.

Reduced quintics are defined in Section 4.3.1. The computation of the coefficients is not trivial and we describe it in Appendix C. The choice of reduced quintics "force" one to use quintics for the beams. Hermite piecewise quintics are defined in Section 4.2.1.

7.2.2 Standard Matrices

First we compute standard matrices for the two beams with quintics. The procedure is the same as with cubics.

$$\left[M_0^{\Gamma_0}\right]_{ij} = \int_0^1 (\gamma_0\phi_i)(\gamma_0\phi_j), \qquad \left[M_1^{\Gamma_0}\right]_{ij} = \int_0^1 (\gamma_0\phi_i)'(\gamma_0\phi_j)',$$

$$\left[M_0^{\Gamma_1}\right]_{ij} = \int_0^1 (\gamma_1\phi_i)(\gamma_1\phi_j), \qquad \left[M_1^{\Gamma_1}\right]_{ij} = \int_0^1 (\gamma_1\phi_i)'(\gamma_1\phi_j)',$$

as well as

$$K_{ij}^{\Gamma_0} = \int_0^1 (\gamma_0\phi_i)''(\gamma_0\phi_j)'' \quad \text{and} \quad K_{ij}^{\Gamma_1} = \int_0^1 (\gamma_1\phi_i)''(\gamma_1\phi_j)''.$$

Next we compute standard matrices for the plate. These computations are quite involved and we provide some detail in Appendix C.

$$[M_0^{\Omega}]_{ij} = \int_\Omega \phi_i\phi_j, \quad [M_1^{\Omega}]_{ij} = \int_\Omega \text{grad}\ \phi_i \cdot \text{grad}\ \phi_j \quad \text{and} \quad K_{ij}^{\Omega} = b_\Omega(\phi_i, \phi_j).$$

The bilinear forms are given in Section 3.3. Each basis element is of the form $\tilde{\phi}_i = \langle \phi_i, \gamma_0\phi_i, \gamma_1\phi_i \rangle$. (The forced boundary conditions are satisfied by eliminating certain basis elements.) Now, consider for example

$$c(\tilde{\phi}_i, \tilde{\phi}_j) = c_\Omega(\phi_i, \phi_j) + \beta c_0(\phi_i, \phi_j) + \beta c_1(\phi_i, \phi_j).$$

$c_0(\phi_i, \phi_j)$ involves the restriction of basis functions ϕ_i and ϕ_j to the boundary Γ_0. These restrictions are non-zero only for some of the basis functions associated with nodes on Γ_0. (The restriction of a reduced quintic on Γ_0 is an one-dimensional quintic.) The result is $M_{ij} = M_{ij}^{\Omega} + \beta M_{ij}^{\Gamma_0} + \beta M^{\Gamma_1}$. Consequently $M = M^{\Omega} + \beta M^{\Gamma_0} + \beta M^{\Gamma_1}$,

where

$$M^{\Omega} = M_0^{\Omega} + rM_1^{\Omega}, \ M^{\Gamma_0} = M_0^{\Gamma_0} + r_b M_1^{\Gamma_0} \text{ and } M^{\Gamma_1} = M_0^{\Gamma_1} + r_b M_1^{\Gamma_1}.$$

The computation of the K-matrix is the similar,

$$K = K^{\Omega} + \alpha K^{\Gamma_0} + \alpha K^{\Gamma_1}.$$

7.3 Equilibrium problem

To find the Galerkin approximation for the solutions of the equilibrium problem, we solve a system of linear equations.

Problem BD

$K\bar{u} = \overline{F}$, *where* $F_i = (f, \phi_i)$.

The parameter α gives an indication of the stiffness of the beams in comparison to that of the plate. Increasing the value of α implies an increase in the stiffness of the beams and $\alpha = 0$ corresponds to the case where two sides are free.

For different values of α we compare in Table 7.1 the FEM approximations for the maximum displacement, to the so called exact solution. See [TW]. (Interesting historical remarks are found in [TW].)

Note that the maximum displacement occurs at the centre of the plate as a result of symmetry.

We consider a square plate with the same number of equal intervals per side. We denote this number by n, and use it to distinguish between different meshes.

Denote the maximum displacement obtained from the so called exact solution by $u_{\max}$ and the FEM approximation of the maximum displacement where n subintervals are used, by $u_{\max}^{(n)}$. Choose Poisson's ratio $\nu = 0.3$.

		$(u_{\max} - u_{\max}^{(n)})/u_{\max}$		
α	Exact	$n = 2$	$n = 4$	$n = 8$
100	4.09×10^{-3}	2.0421×10^{-3}	2.9308×10^{-4}	2.3653×10^{-4}
30	4.16×10^{-3}	1.0507×10^{-3}	6.5975×10^{-4}	7.1510×10^{-4}
10	4.34×10^{-3}	1.7896×10^{-3}	1.7460×10^{-4}	1.2220×10^{-4}
6	4.54×10^{-3}	3.5724×10^{-3}	5.0933×10^{-3}	5.1428×10^{-3}
4	4.72×10^{-3}	2.9835×10^{-3}	1.547×10^{-3}	1.5006×10^{-3}
2	5.29×10^{-3}	3.2719×10^{-3}	2.0580×10^{-3}	2.0181×10^{-3}
1	6.24×10^{-3}	1.0228×10^{-3}	9.3231×10^{-5}	6.2112×10^{-5}
0.5	7.56×10^{-3}	2.2617×10^{-3}	1.6195×10^{-3}	1.5973×10^{-3}
0	1.309×10^{-2}	3.2828×10^{-4}	2.8584×10^{-4}	2.8129×10^{-4}

Table 7.1: *Comparison of exact values with FEM approximations of maximum displacement.*

The fact that the relative error originally improves if we double the number of intervals from 2 to 4 and then remains almost the same, suggests that the so called exact solution is not very accurate, as could be expected since only a few significant digits are given.

The relative difference between consecutive FEM approximations strenghtens this observation as can be seen in Table 7.2.

α	$(u_{\max}^{(4)} - u_{\max}^{(2)})/u_{\max}^{(4)}$	$(u_{\max}^{(8)} - u_{\max}^{(4)})/u_{\max}^{(8)}$
100	1.748505×10^{-3}	5.653982×10^{-5}
30	1.711612×10^{-3}	5.539486×10^{-5}
10	1.614713×10^{-3}	5.238781×10^{-5}
6	1.528720×10^{-3}	4.971930×10^{-5}
4	1.433896×10^{-3}	4.677682×10^{-5}
2	1.211367×10^{-3}	3.987090×10^{-5}
1	9.294900×10^{-4}	3.111738×10^{-5}
0.5	6.412267×10^{-4}	2.214781×10^{-5}
0	4.242264×10^{-5}	4.550660×10^{-6}

Table 7.2: *Comparison of FEM approximations for the maximum displacement.*

7.4 Eigenvalue problem

As mentioned before, Section 4.4, the occurence of eigenvalues has a highly irregular pattern in the two-dimensional case. We have an elementary example to illustrate this, and also to show how difficult it can be to identify eigenvalues with multiplicity.

7.4.1 Multiplicity of eigenvalues

Consider the following eigenvalue problem,

$$-\nabla^2 u = \lambda u \text{ on the unit square with } u = 0 \text{ on the boundary.}$$

Clearly,
$u(x, y) = \sin(n\pi x)\sin(m\pi y)$ is an eigenfunction, for n and m integers. The corresponding eigenvalue is $\lambda = n^2 + m^2$.

The popular difference scheme is

$$-h^{-2}\left[u_{i,j+1} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i-1,j}\right] = \lambda u_{i,j}$$

or

$$-h^{-2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] - h^{-2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] = \lambda u_{i,j},$$

where h is the length of a subinterval.

Let $u_{i,j} = \sin(i\omega_k)\sin(j\omega_\ell)$, then

$$u_{i+1,j} + u_{i-1,j} = 2\cos(w_k)u_{i,j}.$$

Hence

$$-h^{-2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] = \lambda_k u_{i,j}, \text{ where } \lambda_k = h^{-2}(2 - 2\cos w_k).$$

$u_{i,j}$ satisfies the boundary conditions if $w_k = (k\pi)/(n+1)$ and $w_\ell = (\ell\pi)/(n+1)$. It satisfies the difference equations if $\lambda_\ell = h^{-2}(2-2\cos w_\ell)$. Hence $u_{i,j}$ is an eigenvector and every eigenvalue is of the form $\lambda_k + \lambda_\ell$.

In Table 7.3 we list the exact eigenvalues for this problem as well as the numerical approximations obtained for different subinterval lengths. We give

i	Exact	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.005$
1	19.7	19.3	19.6	19.7	19.7
2	49.3	45.6	48.2	49.0	49.3
3	49.3	45.6	48.2	49.0	49.3
4	79.0	72	76.8	78.4	78.8
5	98.7	81.6	93.3	97.2	98.3
6	98.7	81.6	93.3	97.2	98.3
7	128	108	122	127	128
8	128	108	122	127	128
9	168	118	151	163	167
10	168	118	151	163	167
11	178	144	167	175	177
12	197	144	180	192	196
13	197	144	180	192	196
14	247	144	217	240	245
15	247	144	217	240	245
16	257	170	225	245	254
17	257	170	225	245	254
18	286	180	246	275	283
19	286	180	246	275	283
20	316	206	283	307	313

Table 7.3: *Comparison of finite difference eigenvalues to the exact eigenvalues.*

only three significant digits as it is sufficient to illustrate difficulties of matching exact eigenvalues and approximate eigenvalues.

Let i denote the number of the eigenvalue and h the length of a subinterval.

Interpreting numerical results with respect to multiplicity of eigenvalues is difficult. Great care should be taken to establish whether approximate eigenvalues that are close together are an indication of multiplicity of exact eigenvalues, or not.

For example, for $h = 0.2$, the eleventh to fifteenth eigenvalues seem to be one eigenvalue with multiplicity five, while it actually approximates three different eigenvalues. Another example is the fifteenth and sixteenth eigenvalues for $h = 0.05$. These eigenvalues seem close and one might expect them to approximate the same eigenvalue with multiplicity more than one.

7.4.2 Plate beam

The generalized eigenvalue problem associated with the plate beam model is given by

Problem CD

$$K\bar{w} = \lambda M\bar{w}.$$

In a joint report [ZVGV1] we consider this eigenvalue problem for the plate beam model excluding rotary inertia. In this subsection we investigate the effect of rotary inertia if included in the model.

The ratio α/β of the dimensionless constants, $\alpha = (E_bI_b)/(aD)$ and $\beta = (\rho_bA)(\rho ah)$, defined in Section 2.6, with the plate of thickness h and the beams of square profile with thickness d, is a measure of the stiffness of the beams in comparison to that of the plate.

In the special case where both the beams and the plate are of the same material, we have

$$\frac{\alpha}{\beta} = \left(\frac{d}{h}\right)^2 (1-\nu^2).$$

As the values of d/h increase, i.e. the stiffness of the beams is increased, the situation approaches the plate problem where all four sides of the plate are rigidly supported. For this problem the eigenvalues and eigenfunctions are known. The eigenvalues are of the form

$$\left((n\pi)^2 + (m\pi)^2\right)^2$$

with corresponding eigenfunctions

$$\sin(n\pi x)\ \sin(m\pi y).$$

Since the exact eigenvalues for the plate beam problem are not available, the FEM approximations for the eigenvalues for large values of d/h can be compared to the eigenvalues of this limiting case, see Table 7.4.

Denote the ith eigenvalue for the case where all four sides are rigidly supported by λ_i. The eigenvalues are ordered according to size. The FEM approximation of the ith eigenvalue is denoted by $\lambda_i^{(n)}$ where n subintervals are used.

Throughout this subsection we use Poisson's ratio as $\nu = 0.3$.

	$\lambda_i^{(8)}$ for different values of d/h				
i	$d/h = 1$	$d/h = 10$	$d/h = 100$	$d/h = 200$	λ_i
1	92.6654	386.3556	389.6361	389.6364	389.6364
2	250.7783	2359.4575	2435.2366	2435.2398	2435.2273
3	1264.1968	2433.5697	2435.2500	2435.2525	2435.2273
4	1514.0745	6221.0210	6234.2338	6234.2346	6234.1818
5	2142.1461	7345.9342	9741.4914	9741.5319	9740.9091
8	7725.6133	11308.1573	16463.7086	16463.7127	16462.1364
10	11599.1799	16455.3398	28158.9627	28159.0563	28151.2273

Table 7.4: *Comparison of FEM eigenvalues for different values of d/h with eigenvalues of a rigidly supported plate. Rotary inertia is excluded.*

As d/h increases, the FEM approximation of the eigenvalues approaches the eigenvalues of the plate rigidly supported on all four sides.

For values of d/h that do not correspond to the limit case, the numerical convergence of the FEM eigenvalues are illustrated in Table 7.5.

	$d/h = 10$		$d/h = 100$	
	$\left(\lambda_i^{(2n)} - \lambda_i^{(n)}\right) / \lambda_i^{(2n)}$		$\left(\lambda_i^{(2n)} - \lambda_i^{(n)}\right) / \lambda_i^{(2n)}$	
i	$n = 2$	$n = 4$	$n = 2$	$n = 4$
1	6.94630×10^{-4}	8.2136×10^{-6}	7.0601×10^{-4}	8.3506×10^{-6}
2	2.1888×10^{-2}	4.0239×10^{-4}	1.2764×10^{-2}	2.9014×10^{-4}
3	4.1506×10^{-2}	4.2474×10^{-4}	5.3144×10^{-2}	5.6098×10^{-4}
4	6.6013×10^{-2}	7.3256×10^{-4}	6.6438×10^{-2}	7.3537×10^{-4}
5	2.3179×10^{-2}	9.5076×10^{-4}	4.8512×10^{-2}	3.1069×10^{-4}
10	2.3327×10^{-1}	5.0927×10^{-3}	5.7506×10^{-1}	1.3180×10^{-2}

Table 7.5: *Numerical convergence of FEM eigenvalues for different values of d/h. Rotary inertia is excluded.*

Remark Choosing $n = 8$, yields (486×486) matrices which are already very time consuming to handle with our available computer hardware and software. Therefore we do not consider more than 8 subintervals.

Including rotary inertia in the model

In addition to the joint report [ZVGV1] we now establish the effect of rotary inertia on the eigenvalues of the plate beam problem.

From Sections 2.5 and 2.6 we have the dimensionless constants $r_b = I_b/(a^2d^2)$ and $r = I/(a^2h)$. In the experimentation we work with a fixed plate, i.e. a and h are fixed, and modify the beams by changing d. Consequently r_b and r depend on the relationship d/h and indicate the effect of rotary inertia.

In Table 7.6 we illustrate numerical convergence of the FEM eigenvalues if rotary inertia is included. For $d/h = 50$ we have $r_b = 2.083 \times 10^{-2}$ and $r = 8.333 \times 10^{-6}$.

i	$\left(\lambda_i^{(4)} - \lambda_i^{(2)}\right)/\lambda_i^{(4)}$	$\left(\lambda_i^{(8)} - \lambda_i^{(4)}\right)/\lambda_i^{(8)}$
1	7.0599×10^{-4}	8.3271×10^{-6}
5	4.8512×10^{-2}	3.1350×10^{-3}
10	5.7479×10^{-1}	1.312×10^{-2}

Table 7.6: *Numerical convergence of FEM eigenvalues for* $d/h = 50$. *Including rotary inertia.*

For the plate supported on all four sides, repeated eigenvalues are expected– and indeed observed. For the plate beam problem the symmetry is partially lost, and the question arises if repeated eigenvalues will occur, and whether those FEM eigenvalues will be observed as repeated eigenvalues?

As with the case excluding rotary inertia, the exact solution is not available. Again, the exact eigenvalues for the plate rigidly supported on all four sides are used to give an indication of what can be expected of the FEM eigenvalues.

As is expected, the presence of rotary inertia decreases the eigenvalues in comparison to the case without rotary inertia. The effect of rotary inertia is illustrated in Table 7.7. For $d/h = 50$ we have $r_b = 2.083 \times 10^{-2}$ and $r = 8.333 \times 10^{-6}$.

i	Excluding rotary inertia $\lambda_i^{(8)}$	Including rotary inertia $\lambda_i^{(8)}$	Exact value λ_i
1	389.6313	389.5673	389.6364
2	2435.1545	2434.1536	2435.2273
3	2435.2439	2434.2429	2435.2273
4	6234.2146	6230.1154	6234.1818
5	9740.7884	9732.7844	9740.9091
6	9741.5344	9733.5289	9740.9091
7	16463.2389	16445.6552	16462.1364
8	16463.6608	16446.0765	16462.1364
9	28154.6617	28115.3573	28151.2273
10	28158.7179	28119.4013	28151.2273
11	31564.1696	31517.5097	31560.5455

Table 7.7: *Effect of rotary inertia on the eigenvalues for $d/h = 10$.*

In Table 7.7 the multiplicity of eigenvalues of the plate rigidly supported on all four sides are observed. These repetitions give reason to expect that the corresponding FEM eigenvalues for the plate beam problem may also be repeated eigenvalues.

The question arises whether the FEM approximation will yield repeated eigenvalues or will the eigenvalues only be close?

Appendix A

Notation

$C^i(I)$	:	The space of functions with continuous derivatives up to order i on I.
$C^i(\bar{I})$	:	The space of functions with continuous derivatives up to order i on $\bar{I}$.
$C_0^\infty(I)$	:	The space of infinitely differentiable functions with compact support contained in I.
$C^m(\bar{\Omega})$	:	The space of functions with continuous derivatives up to order m on $\bar{\Omega}$. (This idea can be made precise by defining a function on an open set containing $\bar{\Omega}$ and taking the restriction of this function using uniform continuity. [Fr, Section 1.1].)
$C^\infty(\bar{\Omega})$	:	Functions in $C^m(\bar{\Omega})$ for all m.
$C_0^\infty(\bar{\Omega})$	:	Functions in $C^\infty(\bar{\Omega})$ with a compact support.
$L^2(\Omega)$	:	The class of square integrable functions on Ω. (Lebesgue integral).

grad u : $(\partial_1 u, \partial_2 u)$ or $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$.

∇u : grad u.

$(u, v)^{\Omega}$: The inner product in $L^2(\Omega)$: $(u, v)^{\Omega} = \int_{\Omega} uv \, dm$.

div u : $\partial_1 u + \partial_2 u$ or $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$.

$\nabla^2 u$: $\nabla^2 u = \partial_1^2 u + \partial_2 u^2$ or $\operatorname{div} \operatorname{grad} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

$\partial_j^k u$: $\frac{\partial^k u}{\partial x_j^k}$.

Appendix B

Sobolev Spaces

B.1 Definitions

For a domain Ω in $\mathbb{R}_n$, the space $C(\Omega)$ or $C(\overline{\Omega})$ is the set $C(\Omega)$ or $C(\overline{\Omega})$ with norm $\|u\|_\infty = \sup_\Omega |u|$.

The space $C^m(\Omega)$ or $C^m(\overline{\Omega})$ is the relevant set of functions with norm $\|u\|_{m,\sup} = \max\{\|v\|_\infty : v \text{ is a derivative of } u \text{ of order at most } m\}$.

For a domain Ω in $\mathbb{R}_n$, $L^2(\Omega)$ is the space of square Lebesgue integrable functions on Ω.

$W^m(\Omega)$ is the subset of $L^2(\Omega)$ of functions for which weak derivatives up to order m exist and are in $L^2(\Omega)$.

For our purposes $H^m(\Omega) = W^m(\Omega)$.

See, for instance, [Fr] or [OR].

Lemma B.1.1 *Sobolev's lemma*

Let $r < m - n/2$. For $u \in H^m(\Omega)$ there exists a function $v \in C^r(\bar{\Omega})$ such that $v = u$ almost everywhere, and there exists a constant C such that

$$\|v\|_{r,\sup} \leq C\|u\|_m \text{ for each } u \in H(\Omega).$$

See [Ag, Section 3].

B.2 Trace operator

For the value of a function at the boundary to make sense, it is necessary to introduce the concept of a trace operator. The following simple result will prove to be useful.

Lemma B.2.1 *Let f be an arbitrary function in $C^1[a,b]$. Then*

$$|f(b)| \leq K\|f\|_1$$

with $K = \max\{\sqrt{b-a}, 1/\sqrt{b-a}\}$, and

$$|f(b) - f(a)| \leq \sqrt{b-a}\|f'\|_0.$$

Proof For any $g \in C^1(a,b)$

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)' = \int_a^b (f'g + fg').$$

From the Schwartz inequality in $L^2(a,b)$ follows that

$$|f(b)g(b) - f(a)g(a)| \leq \|f'\|_0\|g\|_0 + \|f\|_0\|g'\|_0.$$

Now choose $g(x) = \dfrac{x-a}{b-a}$ or $g(x) = 1$ on $[a,b]$. □

Definition B.2.1 *Trace operator for an interval.*

The mapping

$$\gamma : f \in C^1(a,b) \longrightarrow f(b)$$

is called a trace operator.

From Lemma B.2.1 we have $|\gamma f| \leq K\|f\|_1$, hence γ is continuous if $C^1(a,b)$ is regarded as a subspace of $H^1(a,b)$ and this mapping can then be extended by continuity to functions in the Sobolev space $H^1(a,b)$.

Notation We will denote γf by $f(b)$ for simplicity.

The following Poincaré type estimates have many applications.

Lemma B.2.2 *For any function f in $C^1[a,b]$ with a zero in $[a,b]$, we have*

$$\|f\|_{\mathsf{sup}} \leq \sqrt{b-a}\|f'\|_0$$

and

$$\|f\|_0 \leq (b-a)\|f'\|_0.$$

For any function f in $C^1[a,b]$, we have

$$\|f\|_{\mathsf{sup}} \leq K\|f\|_1$$

and

$$\|f\|_0 \leq K\sqrt{b-a}\|f\|_1,$$

where $K = \max\{\sqrt{b-a}, 1/\sqrt{b-a}\ \}$.

Lemma B.2.3 *If $f \in C^2[a,b]$ with $f(0) = f(a) = 0$, then*

$$\|f'\|_0 \leq (b-a)\|f''\|_0$$

and

$$\|f\|_0 \leq (b-a)^2\|f''\|_0.$$

Definition B.2.2 *Trace operator for a rectangle.*

Consider $\Omega = (0,a) \times (0,b)$.

$$\begin{aligned} u \in C'(\Omega) : \gamma_0 \ &: \ \gamma_0 u = u(\cdot, 0) \\ \gamma_1 \ &: \ \gamma_1 u = u(\cdot, b). \end{aligned}$$

From Lemma B.2.1 we have

$$|u(x,0)|^2 \leq K^2 \int_0^b (u(x,y))^2\, dy + K^2 \int_0^b (\partial_y u(x,y))^2\, dy.$$

Hence,

$$\|u(\cdot,0)\|^2_{[0,a]} \leq K^2 \left(\|u\|^2_\Omega + \|\partial_y^2 u\|^2_\Omega\right).$$

We conclude that the operator γ_0 is a bounded operator from $H^1(\Omega)$ to $L^2(0,a)$. Also $C^1(\Omega)$ is dense in $H'(\Omega)$. Hence γ_0 may be extended by continuity to be defined on $H^1(\Omega)$.

B.3 The space $C^m((0,\tau), X)$

Let X be an arbitrary Banach space and u a function with

$$u : (0,\tau) \to X.$$

Definition B.3.1 *If there exists a $w \in X$ such that*

$$\lim_{\varepsilon \to 0} \|\varepsilon^{-1}(u(t+\varepsilon) - u(t)) - w\|_X = 0,$$

then define $u'(t) = w$.

Definition B.3.2 *The function $u' : (0,\tau) \to X$ is defined by $u' : t \to u'(t)$ for each t.*

Higher order derivatives $u^{(k)}$ are defined similarly.

Definition B.3.3 *$C^m((0,\tau), X)$ is defined as the space of functions for which the derivatives up to order m are continuous with the respect to the topology in X on $(0,\tau)$.*

Definition B.3.4 *$C^m([0,\tau), X)$ is defined as the subspace of $C^m((0,\tau), X)$ for which the derivatives up to order m are right continuous with the respect to the topology in X at $t = 0$.*

For $u \in C^1(\Omega \times [0,\tau])$ we associate a function u^* such that

$$u^* : [0,\tau) \to L^2 \text{ with } u^*(t)(x) = u(x_1, x_2, t).$$

Lemma B.3.1 *If $u \in C^1(\Omega \times [0,\tau])$, then $(u^*)'(t) = \partial_t u(., t)$.*

Proof Consider any t in $(0,\tau)$. For each (x_1, x_2) in $\bar{\Omega}$ there exists a $\theta(x_1, x_2)$ between t and $t + h$ such that

$$h^{-1}(u(x_1, x_2, t+h) - u(x_1, x_2, t)) = \partial_t u(x_1, x_2, \theta(x_1, x_2)).$$

Since the derivatives of u are uniformly continuous, $\partial_t u(x_1, x_2, \theta(x))$ will converge uniformly to $\partial_t u(x_1, x_2, t)$ as $h \to 0$. Define the functions $w(t)$ by

$$w(t)(x_1, x_2) = \partial_t u(x_1, x_2, t) \text{ for each } (x_1, x_2) \in \Omega.$$

Then

$$\int_\Omega \left(h^{-1}(u(\cdot,t+h) - u(\cdot,t)) - w(t)\right)^2 \to 0 \text{ if } h \to 0.$$

We have shown that $w(t) = (u^*)'(t)$, the derivative of u^* in the norm of $L^2(\Omega)$. □

Appendix C

Reduced Quintics

C.1 Basis functions on the master element

Reduced quintics are defined in Section 4.3.1 but no explanation given on how to construct them. In this subsection we construct reduced quintics on a so called master element. (Figure C.1.)

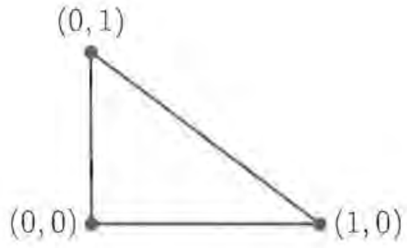

Figure C.1: *The master element.*

With each vertex of the master element we associate six reduced quintics which we will refer to as shape functions. These functions, say P_1 to P_6, associated with $\mathbf{x}$ is described as follows: All their function values and the values of the first and second order derivatives at $\mathbf{x}$, are zero, except one specified value for each function, which is one. The non-zero values are

$$P_1(\mathbf{x}) = \partial_1 P_2(x) = \partial_2 P_3(\mathbf{x}) = \partial_1^2 P_4(\mathbf{x}) = \partial_1\partial_2 P_5(\mathbf{x}) = \partial_2^2 P_6(\mathbf{x}) = 1.$$

The functions values and the values of the first and second order derivatives of these functions at the remaining two vertices, are zero. The shape functions associated with the remaining two vertices are defined in a similar way.

Each one of these shape functions is a polynomial of degree five in x_1 and x_2. Thus there are 21 conditions necessary to determine such a shape function uniquely. Eighteen conditions are obtained from the function values and the values of the first and second order derivatives at the three vertices.

Cowper, Kosko, Lindberg and Olsen, [CKLO1], [CKLO2] and [CKLO3] obtained the remaining three conditions by requiring that the normal derivative of each shape function along each edge reduces to a cubic.

These local shape functions, P, can be expressed in terms of other local basis functions consisting of monomials. We use all monomials of degree five in x_1 and x_2 excluding $x_1x_2^4$ and $x_1^4x_2$. (These two monomials are excluded because of the requirement that the normal derivative of each local shape function along each edge of the master element reduces to a cubic.)

We number the monomials as follows:

$$\begin{array}{ccccc}
Q_1(\mathbf{x}) = 1 & Q_2(\mathbf{x}) = x_1 & Q_3(\mathbf{x}) = x_2 & & \\
Q_4(\mathbf{x}) = x_1^2 & Q_5(\mathbf{x}) = x_1x_2 & Q_6(\mathbf{x}) = x_2^2 & & \\
Q_7(\mathbf{x}) = x_1^3 & Q_8(\mathbf{x}) = x_1^2x_2 & Q_9(\mathbf{x}) = x_1x_2^2 & Q_{10}(\mathbf{x}) = x_2^3 & \\
Q_{11}(\mathbf{x}) = x_1^4 & Q_{12}(\mathbf{x}) = x_1^3x_2 & Q_{13}(\mathbf{x}) = x_1^2x_2^2 & Q_{14}(\mathbf{x}) = x_1x_2^3 & Q_{15}(\mathbf{x}) = x_2^4 \\
Q_{16}(\mathbf{x}) = x_1^5 & Q_{17}(\mathbf{x}) = x_1^3x_2^2 & Q_{18}(\mathbf{x}) = x_1^2x_2^3 & Q_{19}(\mathbf{x}) = x_2^5. &
\end{array}$$

Each monomial, Q_j, can be expressed in terms of the shape functions P_i.

For $j = 1, \ldots, 19$

$$Q_j = \sum_i^{18} T_{ij} P_i,$$

where the matrix T is given at the end of this appendix.

To express P in terms of the local basis Q, the inverse of T has to exist. But T is a 18×19 matrix. By requiring that the normal derivative of each P_i on the hypotenuse has to be a cubic, an additional relationship is obtained. This relationship yields a 19×19 matrix describing the relationship between Q_j and P_i.

The normal derivative of P_i on the hypotenuse is $\operatorname{grad} P_i \cdot (1, 1)$. For this to be a cubic, the following polynomial must be of degree at most three:

$$\begin{array}{llll} b_{16}\partial_1 Q_{16} & +b_{17}\partial_1 Q_{17} & +b_{18}\partial_1 Q_{18} & \\ & b_{17}\partial_2 Q_{17} & +b_{18}\partial_2 Q_{18} & +b_{19}\partial_2 Q_{19} \end{array}$$

$$\begin{array}{llll} = b_{16}5x_1^4 & +3b_{17}x_1^2(1-x_1)^2 & +2b_{18}x_1(1-x_1)^3 & \\ & 2b_{17}x_1^3(1-x_1) & +3b_{18}x_1^2(1-x_1)^2 & +5b_{19}(1-x_1)^4. \end{array}$$

Consequently,

$$(5b_{16} + 3b_{17} - 2b_{18} - 2b_{17} + 3b_{18} + 5b_{19})\, x_1^4 = 0,$$

which implies that

$$5b_{16} + b_{17} + b_{18} + 5b_{19} = 0.$$

The 19th row of T is then

$$\left[\begin{array}{ccccccccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 1 & 5 \end{array}\right].$$

The inverse of the modified 19×19 matrix T exists. By removing the 19th column of T^{-1}, the shape functions can be expressed in terms of the monomials: For $i = 1, \ldots, 18$

$$P_i = \sum_{j=1}^{19} T_{ji}^{-1} Q_j.$$

For example,

$$P_1(x_1, x_2) = 1 - 10x_1^3 - 10x_2^3 + 15x_1^4 - 30x_1^2x_2^2 + 15x_2^4 - 6x_1^5 + 30x_1^3x_2^2 + 30x_1^2x_2^3 - 6x_2^5.$$

The matrix T^{-1} is given at the end of this appendix.

C.2 Computation of matrices on the master element

The integrations necessary to calculate the bending and mass matrices can be simplified using the integrals of local basis functions defined on the master element.

Suppose we intend to compute

$$(\phi_i, \phi_j)_0 = \int_\Omega \phi_i \phi_j.$$

If a corresponding integral is evaluated over the master element, one can obtain the contribution of any given element by a transformation (substitution of variables). Suppose then that we integrate over the master element—denoted by E. If

$$u = \sum_{i=1}^{18} a_i P_i \text{ and } v = \sum_{i=1}^{18} b_i P_i,$$

then

$$\int_E uv = M\bar{a}\cdot\bar{b} \text{ where } M_{ij} = \int_E P_i P_j.$$

Also, if

$$u = \sum_{i=1}^{19} c_i Q_i \text{ and } v = \sum_{i=1}^{19} d_i Q_i,$$

then

$$\int_E uv = N\bar{c}\cdot\bar{d} \text{ where } N_{ij} = \int_E Q_i Q_j.$$

From the definition of T, we have $\bar{d} = T^t\bar{b}$ and $\bar{c} = T^t\bar{a}$. Since

$$\int_E uv = N\bar{c}\cdot\bar{d} = M\bar{a}\cdot\bar{b},$$

the matrices M and N are related by T. It is easy to write computer code to compute N. Using this transformation M can be computed. Using the transformation between the master element and an element in the mesh, we compute the contribution of the relevant element to $\int_\Omega \phi_i\phi_j$.

By adding up the contributions of the different elements, we eventually have $[M_0^\Omega]_{ij} = \int_\Omega \phi_i\phi_j$.

$$T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 6 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 20
\end{bmatrix}.$$

$$T^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-10 & -6 & 0 & -1.5 & 0 & 0 & 10 & -4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & -2 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 \\
-10 & 0 & -6 & 0 & 0 & -1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & -4 & 0 & 0 & 0.5 & 0 \\
15 & 8 & 0 & 1.5 & 0 & 0 & -15 & 7 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-30 & -6 & -6 & -1.5 & 2 & -1.5 & 15 & -7.5 & -1.5 & 1.25 & 0.5 & 0.25 & 15 & -1.5 & -7.5 & 0.25 & 0.5 & 1.25 & -0.5 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\
15 & 0 & 8 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 & 7 & 0 & 0 & -1 & 0 \\
-6 & -3 & 0 & -0.5 & 0 & 0 & 6 & -3 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 9 & 6 & 1.5 & 0 & 1 & -15 & 7.5 & 1.5 & -1.25 & -0.5 & 0.25 & -15 & -1.5 & 7.5 & -0.25 & 0.5 & -1.25 & 0.5 \\
30 & 6 & 9 & 1 & 0 & 1.5 & -15 & 7.5 & -1.5 & -1.25 & 0.5 & -0.25 & -15 & 1.5 & 7.5 & 0.25 & -0.5 & -1.25 & 0.5
\end{bmatrix}$$

Bibliography

[A] J P Aubin, *Behaviour of the error for the approximate solution of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods*, Annali della Scuola Normale di Pisa, Series 3, **21** (599-637), 1967.

[Ag] S Agmon, *Lectures on Elliptic Boundary Value Problems*, D.van Nostrand Company,Inc, Princeton, New Jersey, 1965.

[Ap] Tom M Apostol *Calculus*, Volume I, Second edition, Blaisdell Publishing Company, Waltham, Massachusetts, 1967.

[Ba] G A Baker, *error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal., **13**(4), (564-576), 1976.

[BDSW] G Birkhoff, C de Boor, B Swartz and B Wendroff, *Rayleigh-Ritz approximation by piecewise cubic polynomials*, SIAM J Num Anal, **3** (118-203), 1966.

[BF] G Birkhoff and G Fix, *Accurate eigenvalue computations for elliptic problems*, SIAM-AMS Symposium, Duke University.

[BI] H T Banks and D J Inman, *On damping Mechanisms in Beams*, J. Appl. Mech., **58**, 716, 1991.

[BIt] H T Banks and K Ito, *A unified framework for approximation in inverse problems for distributed paparmeter systems*, Control-Theory and Advanced Technology, **4**(1), (73-90), 1988.

[BK] H T Banks and K Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhäuser, Boston, 1989.

[BST] B M Budak, A A Samarskii and A N Tikhonov, *A collection of problems on mathematical physics*, Pergamon Press, Oxford, 1964.

[C] P G Ciarlet, *The finite element method for elliptic problems*, North Holland, Amsterdam, 1978.

[CDKP] G Chen, M C Delfour, A M Krall and G Payre, *Modeling, stabilization and control of serially connected beams*, J Control and Optimization, **25(3)** (526-546), May 1987.

[CKLO1] G R Cowper, E Kosko, G M Lindberg and M D Olson, *Formulation of a new triangular plate bending element*, Canadian Aeronautics and Space Institute, Trans., **1** (86-90), 1968.

[CKLO2] G R Cowper, E Kosko, G M Lindberg and M D Olson, *Static and dynamic applications of a high-precision triangular plate bending element*, AIAA J, **7** (1957-1965), 1969.

[CKLO3] G R Cowper, E Kosko, G M Lindberg and M D Olson, *A high-precision triangular plate bending element*, Aeronautical Dept. LR-514, National Research Council of Canada, Dec 1986.

[Cl] C Clark, *Elementary Mathematical Analysis*, Second edition, Brooks/Cole Publishing Company, Pacific Grove, California, 1982.

[CR] P G Ciarlet and R A Raviart, *General Lagrange and Hermite interpolation in R^n with application to the finite element method*, Arch. Rat. Mech. Anal., **46** (177-199), 1972.

[D] T DuPont, *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J Num. Anal., **10** (880-889), 1973.

[De] J E Dendy, *Penalty Galerkin methods for partial differential equations*, SIAM J. Numer. Anal., (11), (604-636), 1974.

[FXX] Min-fu Feng, Xiao-ping Xie and Hua-xing Xiong, *Semi-discrete and fully discrete partial projection finite element methods for the vibrating Timoshenko beam*, Journal of Computational Mathematics, **17**(4), (353-368), 1999.

[Fr] A Friedman, *Partial differential equations*, Holt, New York, 1969.

[Fu] Y C Fung, *Foundations of solid mechanics*, Prentice-Hall, New Jersey, 1965.

[GV] M Grobbelaar-van Dalsen and A van der Merwe, *Boundary stabilization for the extensible beam with attached load*, Mathematical Models and Methods in Applied Sciences, **9**(**3**) (379-394), 1999.

[I] D J Inman, *Engineering vibration*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1994.

[JVRV] B R Jooste, H J Viljoen, S L Rohde and N F J van Rensburg, *Experimental and theoretical study of vibrations of a cantilevered beam using a ZnO piezoelectric sensor*, J Vac Sci Tech, **14**(**3**) (714-719), May/June 1996.

[K] S G Krein, *Linear differential equations in Banach space*, Translations of Mathematical Monographs. American mathematical Society , **29**, Providence, R I , 1971.

[Kr] E Kreyzig, *Introductory functional analysis with applications*, John Wiley & Son Inc, 1978.

[LL] J E Lagnese and G Leugering, *Uniform stabilization of a nonlinear beam by nonlinear feedback*, Journal of Differential Equations (355-388), 1991.

[LM] P D Lax and A N Milgram, *Parabolic equations*, Contributions to the theory of partial differential equations, **33**, Princeton University Press, New Jersey, 1954.

[N] J Nitsche, *Ein Kriterium fur die Quasi-Optimalitat dis ritzchen Verfahrens*, Numerische Mathematik, **11** (346-348), 1969.

[OC] J T Oden and F Carey, *Finite elements: Mathematical aspects*, **4**, Prentice-Hall International, Inc, London, 1983.

[OR] J T Oden and J N Reddy, *An introduction to the mathematical theory of finite elements*, John Wiley & Sons, New York-London-Sydney-Toronto, 1976.

[P] A Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[Pr] P M Prenter, *Splines and Variational Methods*, John Wiley & Sons, New York, 1975.

[Ra] J Rauch, *On convergence of the Finite Element Method for the wave equation*, SIAM J. Numer. Anal., **22**(2), (245-249), 1985.

[Re] J N Reddy, *An Introduction to the Finite Element Method*, Second edition, McGraw-Hill, New York, 1993.

[Rei] H Reismann, *Elastic Plates: Theory and Application*, John Wiley & Sons, New York, 1988.

[RM] R D Richtmyer and K W Morton, *Difference methods for initial-value problems*, Interscience Publishers, New York, 1967.

[Sa] N Sauer, *Linear evolution equations in two Banach spaces*, Proc of the Royal Society of Edinburgh, **91A**(1982), 287-303.

[Se] L A Segal, *Mathematics applied to Continuum Mechanics*, Macmillan Publishing Co, Inc, New York, 1977.

[SF] G Strang and G J Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, New Jersey, 1973.

[Sh] R E Showalter, *Hilbert space methods for partial differential equations*, Pitman, 1977.

[TW] S Timoshenko and S Woinowsky-Krieger, *Theory of plates and shells*, Second edition, McGraw-Hill, Kogakusha, Ltd, Tokyo-Johannesburg-London, 1959.

[V1] N F J van Rensburg, *Mathematical model for deflection and vibrations of a rectangular plate with elastic support at the boundary*, Technical Report, UPWT 92/14, University of Pretoria, Pretoria, 1992.

[V2] N F J van Rensburg, *A Mathematical model for transverse vibration of a plate with elastic support at the boundary*, Technical Report, UPWT 2000/02, University of Pretoria, Pretoria, 2000.

[VV] H J Viljoen and N F J van Rensburg, *Damage detection in composites by ZnO sensors*, AIChE Journal, **42**(**4**) (1101-1107), 1996.

[VVZ] A J van der Merwe, N F J van Rensburg and L Zietsman, *Analysis of the solvability of a model for the damped vibrations of a damaged beam*, Technical Report UPTW 99/19, University of Pretoria, Pretoria, 1999.

[W] H F Weinberger, *A First course in Partial Differential Equations*, Xerox College Publishing, Lexington, Massachusetts, 1965.

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

[Ze] E Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer-Verlag, New York, 1995.

[Zi] O C Zienkiewicz, *The Finite Element Method*, Third edition, McGraw-Hill Book Company (UK) Limited, London, 1977.

[ZVGV1] L Zietsman, A J van der Merwe, J J Geldenhuys and N F J van Rensburg, *Application of the finite element method to the vibration of a rectangular plate with elastic support at the boundary*, Technical Report, UPWT 2000/03, University of Pretoria, Pretoria, 2000.

[ZVGV2] L Zietsman, A J van der Merwe, J J Geldenhuys and N F J van Rensburg, *Convergence of the finite element approximation for the natural frequencies and modes of vibration models with interface conditions*, Technical Report, UPWT 2000/04, University of Pretoria, Pretoria, 2000.

[ZVV] L Zietsman, N F J van Rensburg and A J van der Merwe, *A numerical study of the vibrations of a damaged beam* , Mathematical and Computer Modelling, **31**$(6-7)$ (51-60), 2000.