Appendix A

Notation

$C^i(I)$	:	The space of functions with continuous derivatives up to order i on I.
$C^i(\bar{I})$	:	The space of functions with continuous derivatives up to order i on $\bar{I}$.
$C_0^\infty(I)$	:	The space of infinitely differentiable functions with compact support contained in I.
$C^m(\bar{\Omega})$	:	The space of functions with continuous derivatives up to order m on $\bar{\Omega}$. (This idea can be made precise by defining a function on an open set containing $\bar{\Omega}$ and taking the restriction of this function using uniform continuity. [Fr, Section 1.1].)
$C^\infty(\bar{\Omega})$	:	Functions in $C^m(\bar{\Omega})$ for all m.
$C_0^\infty(\bar{\Omega})$	:	Functions in $C^\infty(\bar{\Omega})$ with a compact support.
$L^2(\Omega)$	:	The class of square integrable functions on Ω. (Lebesgue integral).

grad u : $(\partial_1 u, \partial_2 u)$ or $\left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$.

∇u : grad u.

$(u, v)^\Omega$: The inner product in $L^2(\Omega)$: $(u,v)^\Omega = \int_\Omega uv\, dm$.

div u : $\partial_1 u + \partial_2 u$ or $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}$.

$\nabla^2 u$: $\nabla^2 u = \partial_1^2 u + \partial_2 u^2$ or div grad $u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

$\partial_j^k u$: $\dfrac{\partial^k u}{\partial x_j^k}$.

Appendix B

Sobolev Spaces

B.1 Definitions

For a domain Ω in $\mathbb{R}_n$, the space $C(\Omega)$ or $C(\overline{\Omega})$ is the set $C(\Omega)$ or $C(\overline{\Omega})$ with norm $\|u\|_\infty = \sup\limits_{\Omega} |u|$.

The space $C^m(\Omega)$ or $C^m(\overline{\Omega})$ is the relevant set of functions with norm $\|u\|_{m,\text{sup}} = \max\{\|v\|_\infty : v \text{ is a derivative of } u \text{ of order at most } m\}$.

For a domain Ω in $\mathbb{R}_n$, $L^2(\Omega)$ is the space of square Lebesgue integrable functions on Ω.

$W^m(\Omega)$ is the subset of $L^2(\Omega)$ of functions for which weak derivatives up to order m exist and are in $L^2(\Omega)$.

For our purposes $H^m(\Omega) = W^m(\Omega)$.

See, for instance, [Fr] or [OR].

Lemma B.1.1 *Sobolev's lemma*

Let $r < m - n/2$. For $u \in H^m(\Omega)$ there exists a function $v \in C^r(\bar{\Omega})$ such that $v = u$ almost everywhere, and there exists a constant C such that

$$\|v\|_{r,\text{sup}} \leq C\|u\|_m \textit{ for each } u \in H(\Omega).$$

See [Ag, Section 3].

B.2 Trace operator

For the value of a function at the boundary to make sense, it is necessary to introduce the concept of a trace operator. The following simple result will prove to be useful.

Lemma B.2.1 *Let f be an arbitrary function in $C^1[a,b]$. Then*

$$|f(b)| \leq K\|f\|_1$$

with $K = \max\{\sqrt{b-a}, 1/\sqrt{b-a}\}$, and

$$|f(b) - f(a)| \leq \sqrt{b-a}\|f'\|_0.$$

Proof For any $g \in C^1(a,b)$

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)' = \int_a^b (f'g + fg').$$

From the Schwartz inequality in $L^2(a,b)$ follows that

$$|f(b)g(b) - f(a)g(a)| \leq \|f'\|_0\|g\|_0 + \|f\|_0\|g'\|_0.$$

Now choose $g(x) = \dfrac{x-a}{b-a}$ or $g(x) = 1$ on $[a,b]$. □

Definition B.2.1 *Trace operator for an interval.*

The mapping

$$\gamma : f \in C^1(a,b) \longrightarrow f(b)$$

is called a trace operator.

From Lemma B.2.1 we have $|\gamma f| \leq K\|f\|_1$, hence γ is continuous if $C^1(a,b)$ is regarded as a subspace of $H^1(a,b)$ and this mapping can then be extended by continuity to functions in the Sobolev space $H^1(a,b)$.

Notation We will denote γf by $f(b)$ for simplicity.

The following Poincaré type estimates have many applications.

Lemma B.2.2 *For any function f in $C^1[a,b]$ with a zero in $[a,b]$, we have*

$$\|f\|_{\text{sup}} \leq \sqrt{b-a}\|f'\|_0$$

and

$$\|f\|_0 \leq (b-a)\|f'\|_0.$$

For any function f in $C^1[a,b]$, we have

$$\|f\|_{\text{sup}} \leq K\|f\|_1$$

and

$$\|f\|_0 \leq K\sqrt{b-a}\|f\|_1,$$

where $K = \max\{\sqrt{b-a}, 1/\sqrt{b-a}\ \}$.

Lemma B.2.3 *If $f \in C^2[a,b]$ with $f(0) = f(a) = 0$, then*

$$\|f'\|_0 \leq (b-a)\|f''\|_0$$

and

$$\|f\|_0 \leq (b-a)^2\|f''\|_0.$$

Definition B.2.2 *Trace operator for a rectangle.*

Consider $\Omega = (0,a) \times (0,b)$.

$$\begin{aligned} u \in C'(\Omega) : \gamma_0 \quad &: \quad \gamma_0 u = u(\cdot, 0) \\ \gamma_1 \quad &: \quad \gamma_1 u = u(\cdot, b). \end{aligned}$$

From Lemma B.2.1 we have

$$|u(x,0)|^2 \leq K^2 \int_0^b (u(x,y))^2\ dy + K^2 \int_0^b (\partial_y u(x,y))^2\ dy.$$

Hence,

$$\|u(\cdot,0)\|^2_{[0,a]} \leq K^2 \left(\|u\|^2_\Omega + \|\partial^2_y u\|^2_\Omega\right).$$

We conclude that the operator γ_0 is a bounded operator from $H^1(\Omega)$ to $L^2(0,a)$. Also $C^1(\Omega)$ is dense in $H'(\Omega)$. Hence γ_0 may be extended by continuity to be defined on $H^1(\Omega)$.

B.3 The space $C^m((0,\tau),X)$

Let X be an arbitrary Banach space and u a function with

$$u : (0,\tau) \to X.$$

Definition B.3.1 *If there exists a $w \in X$ such that*

$$\lim_{\varepsilon \to 0} \|\varepsilon^{-1}(u(t+\varepsilon) - u(t)) - w\|_X = 0,$$

then define $u'(t) = w$.

Definition B.3.2 *The function $u' : (0,\tau) \to X$ is defined by $u' : t \to u'(t)$ for each t.*

Higher order derivatives $u^{(k)}$ are defined similarly.

Definition B.3.3 *$C^m((0,\tau),X)$ is defined as the space of functions for which the derivatives up to order m are continuous with the respect to the topology in X on $(0,\tau)$.*

Definition B.3.4 *$C^m([0,\tau),X)$ is defined as the subspace of $C^m((0,\tau),X)$ for which the derivatives up to order m are right continuous with the respect to the topology in X at $t = 0$.*

For $u \in C^1(\Omega \times [0,\tau])$ we associate a function u^* such that

$$u^* : [0,\tau) \to L^2 \text{ with } u^*(t)(x) = u(x_1, x_2, t).$$

Lemma B.3.1 *If $u \in C^1(\Omega \times [0,\tau])$, then $(u^*)'(t) = \partial_t u(.,t)$.*

Proof Consider any t in $(0,\tau)$. For each (x_1,x_2) in $\bar{\Omega}$ there exists a $\theta(x_1,x_2)$ between t and $t+h$ such that

$$h^{-1}(u(x_1,x_2,t+h) - u(x_1,x_2,t)) = \partial_t u(x_1,x_2,\theta(x_1,x_2)).$$

Since the derivatives of u are uniformly continuous, $\partial_t u(x_1,x_2,\theta(x))$ will converge uniformly to $\partial_t u(x_1,x_2,t)$ as $h \to 0$. Define the functions $w(t)$ by

$$w(t)(x_1,x_2) = \partial_t u(x_1,x_2,t) \text{ for each } (x_1,x_2) \in \Omega.$$

Then

$$\int_{\Omega} \left(h^{-1}(u(\cdot,t+h) - u(\cdot,t)) - w(t)\right)^2 \to 0 \text{ if } h \to 0.$$

We have shown that $w(t) = (u^*)'(t)$, the derivative of u^* in the norm of $L^2(\Omega)$. □

Appendix C

Reduced Quintics

C.1 Basis functions on the master element

Reduced quintics are defined in Section 4.3.1 but no explanation given on how to construct them. In this subsection we construct reduced quintics on a so called master element. (Figure C.1.)

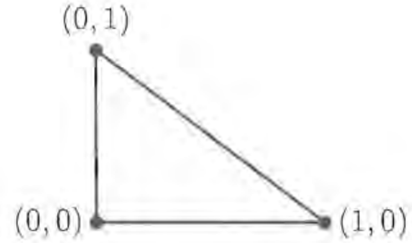

Figure C.1: *The master element.*

With each vertex of the master element we associate six reduced quintics which we will refer to as shape functions. These functions, say P_1 to P_6, associated with $\mathbf{x}$ is described as follows: All their function values and the values of the first and second order derivatives at $\mathbf{x}$, are zero, except one specified value for each function, which is one. The non-zero values are

$$P_1(\mathbf{x}) = \partial_1 P_2(x) = \partial_2 P_3(\mathbf{x}) = \partial_1^2 P_4(\mathbf{x}) = \partial_1\partial_2 P_5(\mathbf{x}) = \partial_2^2 P_6(\mathbf{x}) = 1.$$

The functions values and the values of the first and second order derivatives of these functions at the remaining two vertices, are zero. The shape functions associated with the remaining two vertices are defined in a similar way.

Each one of these shape functions is a polynomial of degree five in x_1 and x_2. Thus there are 21 conditions necessary to determine such a shape function uniquely. Eighteen conditions are obtained from the function values and the values of the first and second order derivatives at the three vertices.

Cowper, Kosko, Lindberg and Olsen, [CKLO1], [CKLO2] and [CKLO3] obtained the remaining three conditions by requiring that the normal derivative of each shape function along each edge reduces to a cubic.

These local shape functions, P, can be expressed in terms of other local basis functions consisting of monomials. We use all monomials of degree five in x_1 and x_2 excluding $x_1x_2^4$ and $x_1^4x_2$. (These two monomials are excluded because of the requirement that the normal derivative of each local shape function along each edge of the master element reduces to a cubic.)

We number the monomials as follows:

$$\begin{array}{lllll}
Q_1(\mathbf{x}) = 1 & Q_2(\mathbf{x}) = x_1 & Q_3(\mathbf{x}) = x_2 & & \\
Q_4(\mathbf{x}) = x_1^2 & Q_5(\mathbf{x}) = x_1x_2 & Q_6(\mathbf{x}) = x_2^2 & & \\
Q_7(\mathbf{x}) = x_1^3 & Q_8(\mathbf{x}) = x_1^2x_2 & Q_9(\mathbf{x}) = x_1x_2^2 & Q_{10}(\mathbf{x}) = x_2^3 & \\
Q_{11}(\mathbf{x}) = x_1^4 & Q_{12}(\mathbf{x}) = x_1^3x_2 & Q_{13}(\mathbf{x}) = x_1^2x_2^2 & Q_{14}(\mathbf{x}) = x_1x_2^3 & Q_{15}(\mathbf{x}) = x_2^4 \\
Q_{16}(\mathbf{x}) = x_1^5 & Q_{17}(\mathbf{x}) = x_1^3x_2^2 & Q_{18}(\mathbf{x}) = x_1^2x_2^3 & Q_{19}(\mathbf{x}) = x_2^5. &
\end{array}$$

Each monomial, Q_j, can be expressed in terms of the shape functions P_i.

For $j = 1, \ldots, 19$

$$Q_j = \sum_i^{18} T_{ij}P_i,$$

where the matrix T is given at the end of this appendix.

To express P in terms of the local basis Q, the inverse of T has to exist. But T is a 18×19 matrix. By requiring that the normal derivative of each P_i on the hypotenuse has to be a cubic, an additional relationship is obtained. This relationship yields a 19×19 matrix describing the relationship between Q_j and P_i.

The normal derivative of P_i on the hypotenuse is $\operatorname{grad} P_i \cdot (1, 1)$. For this to be a cubic, the following polynomial must be of degree at most three:

$$\begin{aligned} & b_{16}\partial_1 Q_{16} \quad +b_{17}\partial_1 Q_{17} \quad +b_{18}\partial_1 Q_{18} \\ & \qquad\qquad b_{17}\partial_2 Q_{17} \quad +b_{18}\partial_2 Q_{18} \quad +b_{19}\partial_2 Q_{19} \\ = \; & b_{16}5x_1^4 \quad +3b_{17}x_1^2(1-x_1)^2 \quad +2b_{18}x_1(1-x_1)^3 \\ & \qquad\qquad 2b_{17}x_1^3(1-x_1) \quad +3b_{18}x_1^2(1-x_1)^2 \quad +5b_{19}(1-x_1)^4. \end{aligned}$$

Consequently,

$$(5b_{16} + 3b_{17} - 2b_{18} - 2b_{17} + 3b_{18} + 5b_{19})\, x_1^4 = 0,$$

which implies that

$$5b_{16} + b_{17} + b_{18} + 5b_{19} = 0.$$

The 19th row of T is then

$$\left[\begin{array}{ccccccccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 1 & 5 \end{array}\right].$$

The inverse of the modified 19×19 matrix T exists. By removing the 19th column of T^{-1}, the shape functions can be expressed in terms of the monomials: For $i = 1, \ldots, 18$

$$P_i = \sum_{j=1}^{19} T_{ji}^{-1} Q_j.$$

For example,

$$P_1(x_1, x_2) = 1 - 10x_1^3 - 10x_2^3 + 15x_1^4 - 30x_1^2x_2^2 + 15x_2^4 - 6x_1^5 + 30x_1^3x_2^2 + 30x_1^2x_2^3 - 6x_2^5.$$

The matrix T^{-1} is given at the end of this appendix.

C.2 Computation of matrices on the master element

The integrations necessary to calculate the bending and mass matrices can be simplified using the integrals of local basis functions defined on the master element.

Suppose we intend to compute

$$(\phi_i, \phi_j)_0 = \int_\Omega \phi_i \phi_j.$$

If a corresponding integral is evaluated over the master element, one can obtain the contribution of any given element by a transformation (substitution of variables). Suppose then that we integrate over the master element—denoted by E. If

$$u = \sum_{i=1}^{18} a_i P_i \text{ and } v = \sum_{i=1}^{18} b_i P_i,$$

then

$$\int_E uv = M\bar{a} \cdot \bar{b} \text{ where } M_{ij} = \int_E P_i P_j.$$

Also, if

$$u = \sum_{i=1}^{19} c_i Q_i \text{ and } v = \sum_{i=1}^{19} d_i Q_i,$$

then

$$\int_E uv = N\bar{c} \cdot \bar{d} \text{ where } N_{ij} = \int_E Q_i Q_j.$$

From the definition of T, we have $\bar{d} = T^t\bar{b}$ and $\bar{c} = T^t\bar{a}$. Since

$$\int_E uv = N\bar{c} \cdot \bar{d} = M\bar{a} \cdot \bar{b},$$

the matrices M and N are related by T. It is easy to write computer code to compute N. Using this transformation M can be computed. Using the transformation between the master element and an element in the mesh, we compute the contribution of the relevant element to $\int_\Omega \phi_i\phi_j$.

By adding up the contributions of the different elements, we eventually have $[M_0^\Omega]_{ij} = \int_\Omega \phi_i\phi_j$.

$$T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 6 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 20
\end{bmatrix}.$$

$$
T^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-10 & -6 & 0 & -1.5 & 0 & 0 & 10 & -4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & -2 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 \\
-10 & 0 & -6 & 0 & 0 & -1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & -4 & 0 & 0 & 0.5 & 0 \\
15 & 8 & 0 & 1.5 & 0 & 0 & -15 & 7 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-30 & -6 & -6 & -1.5 & 2 & -1.5 & 15 & -7.5 & -1.5 & 1.25 & 0.5 & 0.25 & 15 & -1.5 & -7.5 & 0.25 & 0.5 & 1.25 & -0.5 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\
15 & 0 & 8 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 & 7 & 0 & 0 & -1 & 0 \\
-6 & -3 & 0 & -0.5 & 0 & 0 & 6 & -3 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 9 & 6 & 1.5 & 0 & 1 & -15 & 7.5 & 1.5 & -1.25 & -0.5 & 0.25 & -15 & -1.5 & 7.5 & -0.25 & 0.5 & -1.25 & 0.5 \\
30 & 6 & 9 & 1 & 0 & 1.5 & -15 & 7.5 & -1.5 & -1.25 & 0.5 & -0.25 & -15 & 1.5 & 7.5 & 0.25 & -0.5 & -1.25 & 0.5
\end{bmatrix}
$$

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