Chapter 1

Introduction

As the title indicates, this thesis is about the application of the finite element method to vibration problems. We have in mind the implementation of the method (computation of approximations) as well as error analysis.

To be more specific, we are concerned with the vibration of beams and plates. Partial differential equations that model the vibration of beams and plates are classical topics. However, new mathematical problems appear from time to time. One reason is that mathematical models are changed to provide a more accurate description of reality. Another is that new situations arise in industrial applications.

The stabilization and control of beams and plates lead to model problems with non-standard boundary conditions. We give but three examples [BK], [CDKP] and [LL] from a vast literature.

Due to interaction or damage one is confronted by interface conditions or dynamical boundary conditions instead of classical boundary conditions. Our main examples are the vibration of a damaged beam and a plate beam model. These problems are presented in Chapter 2 with the necessary references.

It is necessary to adapt the finite element method to accommodate these problems. The derivation of the variational form is one aspect treated at length in Chapter 3.

For the analysis of the model problems it is necessary to consider product spaces. The basis of the finite dimensional subspace for the Galerkin approximation consists of ordered pairs or triple of functions instead of ordinary functions. As a result finite element interpolation had to be adapted for product spaces.

Our main concern is error analysis. In Chapter 5 we consider the three typical problems: Equilibrium problem, Eigenvalue problem and Vibration problem. In each case we show that the convergence theory can be adapted to product spaces.

To implement the method, it is necessary to adapt the basis functions to avoid the imposition of invalid constraints. To match the theory we rather used the basis elements mentioned earlier. The construction of these basis functions is discussed in Chapter 4.

In Chapters 6 and 7 we have two case studies. Here we consider the damaged beam and plate beam problem and demonstrate the implementation of the finite element method to find approximations for equilibrium problems, eigenvalue problems and vibration problems.

Chapter 2

Model problems

2.1 Introduction

In this chapter we present the one and two dimensional vibration models that form the basis of this study. The common feature in the models is the presence of interface conditions—although these interface conditions may occur for different reasons.

For the analysis as well as implementation of the finite element method, the model problems must be written in variational form. We found the process more manageable if we start with the equations of motion and the constitutive equations. Consequently, we formulate each problem specifying the equations of motion and constitutive equations separately. For the sake of completeness and comparison with the literature, the models are also given in terms of the displacement, i.e. a partial differential equation with boundary and interface conditions. Due to the fact that our problems are not standard, it is necessary to discuss these aspects in some detail.

We also write the model problems in dimensionless form to simplify numerical experimentation. It also facilitates the interpretation of numerical results.

In Sections 2.2 and 2.5 the equations of motion and the constitutive equations for a beam and a plate are discussed. This paves the way for the presentation of the model problems in Sections 2.3, 2.4 and 2.6. The models for a damaged beam in Section 2.3 and a plate beam system in Section 2.6 are the main topics. In Section 2.4 other beam models involving interface conditions are presented.

2.2 Motion of a beam

In this section we focus on the small transverse vibration of a beam modelled as a one dimensional continuum, i.e. the reference configuration is an interval on the real line. The beam has length ℓ, density ρ, cross sectional area A and area moment of inertia I. The position of point x at time t is denoted by $u(x,t)$. The shear force is denoted by F and the bending moment by M. P denotes an external lateral load on the beam, k_1 and k_2 are damping constants and E is Young's modulus. For our approach the equations of motion are important.

The equations of motion are given by

$$\rho A \partial_t^2 u \;=\; \partial_x F - k_1 \partial_t u + P \tag{2.2.1}$$

and

$$\rho I \partial_t^2 \partial_x u \;=\; F + \partial_x M. \tag{2.2.2}$$

A constitutive equation for M is required to complete the model:

$$M \;=\; EI\partial_x^2 u + k_2 I \partial_t \partial_x^2 u. \tag{2.2.3}$$

Viscous damping is included in the equation of motion (2.2.1) by the term $k_1\partial_t u$, and the Kelvin-Voigt damping as the term $k_2 I \partial_t \partial_x^2 u$, in the constitutive equation (2.2.3). The term $\rho I \partial_t^2 \partial_x u$ in the second equation of motion represents the angular momentum density of the cross section relative to the centroid. In the literature it is usually referred to as the rotary inertia term. See, for example, [I], [Fu] or [Se] for background on the modelling procedure.

The mathematical model is given by equations (2.2.1), (2.2.2) and (2.2.3).

Dimensionless form

Choose dimensionless variables $\xi = x/\ell$ and $\tau = t/T$ with T a chosen time which will be specified later. It follows that if $f(x,t) = g(\xi,\tau)$, then

$$\partial_t f = \partial_\tau g \frac{d\tau}{dt} = \frac{1}{T}\partial_\tau g,$$

and, similarly,

$$\partial_x f = \frac{1}{\ell}\partial_\xi g.$$

Introduce the following dimensionless quantities:

$$u^*(\xi,\tau) = \frac{u(x,t)}{\ell}, \quad F^*(\xi,\tau) = \frac{F(x,t)}{EA}, \quad M^*(\xi,\tau) = \frac{\ell M(x,t)}{EI},$$

$$P^*(\xi,\tau) = \frac{\ell^3 P(x,t)}{EI}, \ \lambda = \frac{k_1}{ET}, \ \mu = \frac{k_2}{ET}.$$

In terms of these dimensionless quantities (2.2.1), (2.2.2) and (2.2.3) become

$$\partial_\tau^2 u^* = \frac{ET^2}{\rho\ell^2}\partial_\xi F^* - \frac{ET^2}{\rho A}\lambda\partial_\tau u^* + \frac{EIT^2}{\rho A\ell^4}P^*, \tag{2.2.4}$$

$$\frac{\ell^2\rho}{ET^2}\partial_\tau^2\partial_\xi u^* = \frac{A\ell^2}{I}F^* + \partial_\xi M^*, \tag{2.2.5}$$

$$M^* = \partial_\xi^2 u^* + \mu\partial_\tau\partial_\xi^2 u^*. \tag{2.2.6}$$

We choose

$$T = \ell^2\sqrt{\frac{\rho A}{EI}},$$

and introduce dimensionless constants

$$r = \frac{\rho\ell^2}{ET^2} \text{ and } k = \frac{\lambda\ell^2}{rA}.$$

If we return to the original notation, i.e. use x and t for the spatial and time variables and u, F, M and P for the dimensionless quantities, the dimensionless form of the model is given by

$$\partial_t^2 u = \frac{1}{r}\partial_x F - k\partial_t u + P, \tag{2.2.7}$$

$$r\partial_t^2\partial_x u = \frac{1}{r}F + \partial_x M, \tag{2.2.8}$$

$$M = \partial_x^2 u + \mu\partial_t\partial_x^2 u. \tag{2.2.9}$$

These equations yield the following partial differential equation describing small transverse vibration of a beam in terms of the dimensionless displacement u:

$$\partial_t^2 u - r\partial_t^2\partial_x^2 u = -\partial_x^4 u - \mu\partial_t\partial_x^4 u - k\partial_t u + P.$$

In Section 2.6 we have the situation where a plate is supported by beams. In this case (2.2.2) must be modified to include a couple L to allow for the bending moment density transmitted to the beam by the plate. To obtain the dimensionless form we set

$$L^*(\xi,\tau) = \frac{\ell^2 L(x,t)}{EI}.$$

The second equation of motion (2.2.8) changes to

$$r\partial_t^2\partial_x u = \frac{1}{r}F + \partial_x M + L, \qquad (2.2.10)$$

if we write L for L^*.

2.3 Model for a damaged beam

In this section we consider a model for small transverse vibration of a cantilever beam damaged at a single point. The model was proposed by Viljoen *et al* [VV]. See also [JVRV] for the model that includes Kelvin-Voigt damping. In this model the interface condition is due to the mathematical description of the damage.

We start with the equations, in dimensionless form, that describe the dynamical behaviour of an undamaged cantilever beam. The reference configuration is the interval $I = [0,1]$ and the displacement of x at time t is denoted by $u(x,t)$.

From Section 2.2 the equations of motion are

$$\partial_t^2 u = \frac{1}{r}\partial_x F - k\partial_t u + P \tag{2.3.1}$$

and

$$r\partial_t^2 \partial_x u = \frac{1}{r}F + \partial_x M. \tag{2.3.2}$$

The constitutive equation is

$$M = \partial_x^2 u + \mu\partial_t\partial_x^2 u. \tag{2.3.3}$$

For a cantilever beam the standard boundary conditions at the endpoints are

$$u(0,t) = \partial_x u(0,t) = 0, \tag{2.3.4}$$

$$F(1,t) = M(1,t) = 0. \tag{2.3.5}$$

Suppose now that we have a damaged beam with the damage located at a single point $x = \alpha$. (This is of course impossible but it is a convenient model for approximating the effect of damage.)

At $x = \alpha$ the following interface conditions are prescribed:

$$u(\alpha^+,t) = u(\alpha^-,t), \tag{2.3.6}$$

$$F(\alpha^+,t) = F(\alpha^-,t), \tag{2.3.7}$$

$$M(\alpha^+,t) = M(\alpha^-,t). \tag{2.3.8}$$

Right and left limits are denoted by the superscripts $+$ and $-$.

Condition (2.3.6) specifies the continuity of the beam and (2.3.7) and (2.3.8) follow from the action-reaction principle for the shear force F and the bending moment M at $x = \alpha$.

The effect of the damage is modelled by the jump condition at $x = \alpha$:

$$\begin{aligned} M(\alpha, t) &= \frac{1}{\delta}\left(\partial_x u(\alpha^+, t) - \partial_x u(\alpha^-, t)\right) \\ &\quad + \frac{\mu}{\delta}\left(\partial_t\partial_x u(\alpha^+, t) - \partial_t\partial_x u(\alpha^-, t)\right). \qquad (2.3.9) \end{aligned}$$

Right and left derivatives are denoted by the superscripts $+$ and $-$.

The jump condition (2.3.9) allows for discontinuities in the derivatives $\partial_x u$ and $\partial_t\partial_x u$ at $x = \alpha$. Note that the magnitude of δ indicates the extent of the damage and that $\delta = 0$ corresponds to a beam with no damage. It is clearly impossible to use $\delta = 0$ in this problem. However, our numerical experimentation in [ZVV] showed that solutions for relative small values of δ correspond to solutions of an undamaged beam. If a point force is applied at the free end of the beam it will result in an increase of the gradient. This increase, as a factor of the gradient, is exactly δ. The second term represents internal "friction".

The mathematical model is given by (2.3.1) to (2.3.9). In Section 3.1 we will derive a variational formulation for the problem from these equations.

In terms of the dimensionless displacement u, an equivalent form of the model is given on the next page.

Problem 1

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) \\
&\quad + P(x,t), \text{ for } 0 < x < 1,\ x \neq \alpha,\ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
\partial_x^2 u(1,t) + \mu\partial_t\partial_x^2 u(1,t) &= 0, \\
r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t), \\
u(\alpha^+,t) &= u(\alpha^-,t), \\
\partial_x^2 u(\alpha^+,t) + \mu\partial_t\partial_x^2 u(\alpha^+,t) &= \partial_x^2 u(\alpha^-,t) + \mu\partial_t\partial_x^2 u(\alpha^-,t), \\
r\partial_t^2\partial_x u(\alpha^+,t) - \partial_x^3 u(\alpha^+,t) & \\
-\mu\partial_t\partial_x^3 u(\alpha^+,t) &= r\partial_t^2\partial_x u(\alpha^-,t) - \partial_x^3 u(\alpha^-,t) \\
&\quad -\mu\partial_t\partial_x^3 u(\alpha^-,t), \\
\partial_x^2 u(\alpha,t) + \mu\partial_t\partial_x^2 u(\alpha,t) &= \frac{1}{\delta}\left(\partial_x u(\alpha^+,t) - \partial_x u(\alpha^-,t)\right) \\
&\quad + \frac{\mu}{\delta}\left(\partial_t\partial_x u(\alpha^+,t) - \partial_t\partial_x u(\alpha^-,t)\right).
\end{aligned}$$

Instead of a cantilever beam other boundary conditions can be considered. For example, if the beam is clamped at both ends we have

$$u(1,t) = \partial_x u(1,t) = 0 \tag{2.3.10}$$

instead of (2.3.5).

2.4 Beam models with dynamical boundary conditions

When some object of interest interacts with another object at some part of the boundary, standard boundary conditions are not applicable. The simplest case (which can be found in books on partial differential equations) is probably a rod or spring, executing longitudinal vibrations with a mass attached to one end. See [BST], [GV], [BI] and [V1] for examples of models of this type.

2.4.1 Tip body

In this section we consider a cantilever beam with a body, of mass m_B and moment of inertia I_B, attached to the free end at $x = \ell$. In this case the shear force and bending moment are no longer zero at $x = \ell$, but the following so called dynamical boundary conditions are prescribed.

$$m_B \partial_t^2 u(\ell, t) = -F(\ell, t), \tag{2.4.1}$$
$$I_B \partial_t^2 \partial_x u(\ell, t) = -M(\ell, t). \tag{2.4.2}$$

We assume that the angle θ through which the tip body rotates can be approximated by $\partial_x u(\ell, t)$.

These boundary conditions are also converted into dimensionless form by using the dimensionless quantities introduced in Section 2.2:

$$\frac{\ell m_B}{T^2} \partial_\tau^2 u^*(1, \tau) = -EAF^*(1, \tau),$$
$$\frac{I_B}{T^2} \partial_\tau^2 \partial_\xi u^*(1, \tau) = -\frac{EI}{\ell} M^*(1, \tau).$$

Choosing dimensionless mass m and moment of inertia I_m as

$$m = \frac{m_B \ell}{rEAT^2} = \frac{m_B}{\rho A \ell}$$

and

$$I_m = \frac{I_B \ell}{IET^2} = \frac{rI_B}{\rho \ell I}$$

and returning to the original notation, yield the dimensionless boundary conditions

$$rm\partial_t^2 u(1,t) = -F(1,t), \tag{2.4.3}$$

$$I_m \partial_t^2 \partial_x u(1,t) = -M(1,t). \tag{2.4.4}$$

The mathematical model is given by the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), the standard boundary conditions at $x = 0$,

$$u(0,t) = \partial_x u(0,t) = 0, \tag{2.4.5}$$

and the dynamical boundary conditions (2.4.3) and (2.4.4).

In terms of the dimensionless displacement u, the mathematical model follows as:

Tip body problem

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\qquad \text{for } 0 < x < 1,\ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
m\partial_t^2 u(1,t) &= -r\partial_t^2\partial_x u(1,t) + \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t), \\
I_m\partial_t^2\partial_x u(1,t) &= -\partial_x^2 u(1,t) - \mu\partial_t\partial_x^2 u(1,t).
\end{aligned}$$

2.4.2 Boundary control

For a cantilever beam it is possible to suppress vibration by boundary feedback controls. See [C]. In this case the shear force and bending moment are not zero at $x = \ell$ and the situation is modelled by boundary feedback control conditions:

$$\begin{aligned}
F(\ell,t) &= -k_0\partial_t u(\ell,t), \\
M(\ell,t) &= -k_1\partial_t\partial_x u(\ell,t).
\end{aligned}$$

Choosing the dimensionless quantities

$$\mu_0 = \frac{k_0}{EAT\ell r} \text{ and } \mu_1 = \frac{k_1\ell}{EIT}$$

and returning to the original notation yields the dimensionless boundary conditions:

$$\begin{aligned}
F(1,t) &= -\mu_0\partial_t u(1,t), \\
M(1,t) &= -\mu_1\partial_t\partial_x u(1,t).
\end{aligned}$$

The mathematical model is given by these boundary conditions, the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), and the standard boundary conditions at $x = 0$ (2.4.5).

In terms of the dimensionless displacement u the model follows as:

Boundary damping problem

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\quad \text{for } \ 0 < x < 1, \ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t) - \mu_0\partial_t u(1,t), \\
\partial_x^2 u(1,t) + \mu\partial_t\partial_x^2 u(1,t) &= -\mu_1\partial_t u(1,t).
\end{aligned}$$

2.4.3 General model

For theoretical purposes and without regard for the physical meaning of the model, we will formulate a generalization which contains both the two previous models as special cases. This general model will be used to derive a variational formulation.

The general model is given by the equations of motion (2.2.7) and (2.2.8), the constitutive equation (2.2.9), the standard boundary conditions at $x = 0$ (2.4.5), and the following dynamical boundary conditions

$$\begin{aligned}
F(1,t) &= -\mu_0\partial_t u(1,t) - rm\partial_t^2 u(1,t), & (2.4.6) \\
M(1,t) &= -\mu_1\partial_t\partial_x u(1,t) - I_m\partial_t^2\partial_x u(1,t). & (2.4.7)
\end{aligned}$$

By setting either μ_0 and μ_1 or m and I_m equal to zero, these boundary conditions reduce to the boundary conditions of the appropriate problem.

In terms of the dimensionless displacement u the model follows as:

Problem 2

$$\begin{aligned}
\partial_t^2 u(x,t) - r\partial_t^2\partial_x^2 u(x,t) &= -\partial_x^4 u(x,t) - \mu\partial_t\partial_x^4 u(x,t) - k\partial_t u(x,t) + P(x,t), \\
&\quad \text{for } \ 0 < x < 1, \ t > 0, \\
u(0,t) &= \partial_x u(0,t) = 0, \\
m\partial_t^2 u(1,t) + r\partial_t^2\partial_x u(1,t) &= \partial_x^3 u(1,t) + \mu\partial_t\partial_x^3 u(1,t) - \mu_0\partial_t u(1,t), \\
I_m\partial_t^2\partial_x u(1,t) &= -\partial_x^2 u(1,t) - \mu\partial_t\partial_x^2 u(1,t) - \mu_1\partial_t u(1,t).
\end{aligned}$$

2.5 Motion of a thin plate

When a plate interacts with a beam, one is confronted by more complicated dynamical boundary conditions than those encountered in Section 2.4. To prepare, we discuss the equations of motion and constitutive equation for a plate. We used [Fu], [TW], [Rei], [V1] and [V2].

Consider the transverse motion of a thin plate with density ρ and thickness h. The reference configuration for the plate is a domain Ω in the plane. The transverse displacement of $\mathbf{x} = (x_1, x_2)$ at time t is $u(\mathbf{x}, t)$. This means that the position $\mathbf{r}$ of $\mathbf{x}$ at time t is $\mathbf{r} = (x_1, x_2, u(\mathbf{x}, t))$.

The equations of motion are

$$\rho h \partial_t^2 u = \operatorname{div} \mathbf{T} + q$$

and

$$-R\partial_t \mathbf{H} = \mathbf{T} - R \operatorname{div} M$$

with $\mathbf{T}$ the contact force and q an external lateral load. $\mathbf{H}$ is the angular momentum density relative to the centroid and M the moment or contact couple. For more detail concerning the meaning of $\mathbf{T}$ and M see [Rei], [Fu] or [TW]. R and M are square matrices:

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Note that $R^{-1} = R^T = -R$.

The constitutive equation is given by

$$M = D \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}.$$

(See [Fu, p 461] and [TW, p 81].) D is a measure of stiffness for the plate given by

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

where E is Young's modulus and ν Poisson's ratio.

For small vibrations where shear is ignored, the angular momentum density $\mathbf{H}$ is given by

$$\mathbf{H} = \rho I \partial_t(\partial_2 u, -\partial_1 u)$$

where $I = h^3/12$ is the length moment of inertia.

Dimensionless form

We introduce the dimensionless variables: $\xi_1 = x_1/a$, $\xi_2 = x_2/a$ and $\tau = t/\eta$ where η will be specified later, and a is some typical length dimension of the plate.

We introduce the following dimensionless quantities:

$$u^*(\boldsymbol{\xi},\tau) = \frac{u(\mathbf{x},t)}{a},\ \mathbf{T}^*(\boldsymbol{\xi},\tau) = \frac{I}{hD}\mathbf{T}(\mathbf{x},t),\ \mathbf{H}^*(\boldsymbol{\xi},\tau) = \frac{\eta}{\rho I}\mathbf{H}(\mathbf{x},t),$$

$$q^*(\boldsymbol{\xi},\tau) = \frac{a^3}{D}q(\mathbf{x},t),\ M^*(\boldsymbol{\xi},\tau) = \frac{a}{D}M(\mathbf{x},t).$$

In terms of these quantities the equations of motion are given by

$$\begin{aligned}
\partial_\tau^2 u^* &= \frac{\eta^2 D}{\rho a^2 I}\operatorname{div}\mathbf{T}^* + \frac{\eta^2 D}{\rho h a^4}q^*,\\
-\frac{\rho a^2 I}{D\eta^2}R\partial_\tau \mathbf{H}^* &= \frac{ha^2}{I}\mathbf{T}^* - R\operatorname{div}M^*
\end{aligned}$$

with

$$\mathbf{H}^* = \partial_\tau(\partial_2 u^*, -\partial_1 u^*),$$

and the constitutive equation is given by

$$M^* = \begin{bmatrix} (1-\nu)\partial_1\partial_2 u^* & \partial_2^2 u^* + \nu\partial_1^2 u^* \\ -\partial_1^2 u^* - \nu\partial_2^2 u^* & -(1-\nu)\partial_1\partial_2 u^* \end{bmatrix}.$$

If we choose

$$\eta^2 = \rho a^4 h/D \text{ and } r = I/a^2 h,$$

and return to the original notation, the equations of motion are given by

$$\partial_t^2 u = \frac{1}{r}\operatorname{div}\mathbf{T} + q, \tag{2.5.1}$$

$$-r\,R\partial_t \mathbf{H} = \frac{1}{r}\mathbf{T} - R\operatorname{div}M \tag{2.5.2}$$

with

$$\mathbf{H} = \partial_t(\partial_2 u, -\partial_1 u), \tag{2.5.3}$$

and the constitutive equation by

$$M = \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}. \tag{2.5.4}$$

Substituting (2.5.2) into (2.5.1) yields

$$r \operatorname{div}(R\partial_t \mathbf{H}) + \partial_t^2 u = \operatorname{div}(R \operatorname{div} M) + q.$$

Substituting (2.5.3) here we obtain the partial differential equation

$$\partial_t^2 u - r\,\partial_t^2(\partial_1^2 u + \partial_2^2 u) = -\left(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u\right) + q. \tag{2.5.5}$$

2.6 Plate beam model

We consider a thin plate as in Section 2.5 which interacts with beams at the boundary. See [V1], [V2] and [ZVGV1]. The boundary $\partial\Omega$ consists of two parts, Σ and Γ. The section Σ is rigidly supported and the section Γ elastically supported by a beam. The end points of the beam are also rigidly supported. The orientation of the boundary $\partial\Omega$ is important. The domain Ω is on the "left" of the tangent. To be precise, we require that

$$\mathbf{n} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{\tau} = R\boldsymbol{\tau}$$

where $\mathbf{n}$ is the unit exterior normal and $\boldsymbol{\tau}$ is the unit tangent.

For the mathematical model of the plate we use the dimensionless equations of Section 2.5:

$$\begin{aligned} \partial_t^2 u &= \frac{1}{r}\operatorname{div}\mathbf{T} + q, && (2.6.1)\\ -rR\partial_t\mathbf{H} &= \frac{1}{r}\mathbf{T} - R\operatorname{div}M, && (2.6.2)\\ \mathbf{H} &= \partial_t(\partial_2 u, -\partial_1 u), && (2.6.3)\\ M &= \begin{bmatrix} (1-\nu)\partial_1\partial_2 u & \partial_2^2 u + \nu\partial_1^2 u \\ -\partial_1^2 u - \nu\partial_2^2 u & -(1-\nu)\partial_1\partial_2 u \end{bmatrix}. && (2.6.4)\end{aligned}$$

From Section 2.2 the equations of motion for the supporting beam, in dimensionless form and without damping terms, are given by

$$\partial_t^2 u = \frac{1}{r_b}\partial_s F + P, \tag{2.6.5}$$

and

$$r_b\partial_t^2\partial_s u = \frac{1}{r_b}F + \partial_s M_b + L. \tag{2.6.6}$$

In applications $\partial_s = \pm\partial_1$ or $\partial_s = \pm\partial_2$ depending on the orientation of the beam. A subscript b will be used, where necessary, to indicate quantities associated with the beam.

The length of the beam is a. In terms of this notation the dimensionless quantities chosen in Section 2.2 are:

$$F^* = \frac{1}{E_bA}F,\ P^* = \frac{a^3}{E_bI_b}P,\ L^* = \frac{a^2}{E_bI_b}L,\ M_b^* = \frac{a}{E_bI_b}M_b.$$

For the rotary inertia constant r_b we have

$$r_b = \frac{a^2 \rho_b}{E_b T^2}.$$

In addition, we choose the following dimensionless constants:

$$\alpha = \frac{E_b I_b}{aD} \text{ and } \beta = \frac{\rho_b A}{\rho a h}.$$

It is necessary to adapt (2.6.5) and (2.6.6) to allow for the difference in dimensionless time scale, between the plate model and the beam model. The time derivatives have to be multiplied by a factor T/η. As $T^2/\eta^2 = \beta/\alpha$, (2.6.5) and (2.6.6) change to

$$\beta \partial_t^2 u = \frac{\alpha}{r_b} \partial_s F + \alpha P, \tag{2.6.7}$$

$$\beta r_b \partial_t^2 \partial_s u = \frac{\alpha}{r_b} F + \alpha \partial_s M_b + \alpha L. \tag{2.6.8}$$

The constitutive equation for the beam is:

$$M_b = \partial_s^2 u. \tag{2.6.9}$$

Next we formulate the interface conditions. In this case the force density P is the contact force density that the plate exerts on the beam. It follows that

$$P = -\mathbf{T} \cdot \mathbf{n}. \tag{2.6.10}$$

The moment density L is the moment that the plate exerts on the beam. This implies, for the moment density $-M\mathbf{n}$ of the plate on Γ, that

$$L = -M\mathbf{n} \cdot \mathbf{n}. \tag{2.6.11}$$

Since the plate is merely supported by the beam it also follows that

$$M\mathbf{n} \cdot \boldsymbol{\tau} = 0. \tag{2.6.12}$$

The interface conditions change to the following (again using the original notation to refer to the dimensionless quantities):

$$\alpha P = -\frac{1}{r} \mathbf{T} \cdot \mathbf{n}, \tag{2.6.13}$$

$$M\mathbf{n} \cdot \boldsymbol{\tau} = 0, \tag{2.6.14}$$

$$\alpha L = -M\mathbf{n} \cdot \mathbf{n}. \tag{2.6.15}$$

To complete the model we have to add the boundary conditions for the rigidly supported section Σ:

$$u = 0, \tag{2.6.16}$$
$$M\mathbf{n}\cdot\tau = 0; \tag{2.6.17}$$

and for the rigidly supported end points of the beam:

$$u = 0, \tag{2.6.18}$$
$$M_b = 0. \tag{2.6.19}$$

The mathematical model is given by (2.6.1) to (2.6.4), (2.6.7) to (2.6.9) and (2.6.13) to (2.6.19). Alternatively, the mathematical model can be given in terms of the dimensionless displacement u. We illustrate the procedure for obtaining the boundary conditions for a special domain Ω.

Consider a rectangular plate rigidly supported at two opposing sides and supported by two identical beams at the remaining sides. To find the dimensionless form for the model we choose a as the length of the supporting beams. Then the reference configuration Ω is the rectangle with $0 < x_1 < 1$ and $0 < x_2 < d$. Σ_0 and Σ_1 are those parts of the boundary where $x_1 = 0$ and $x_1 = 1$ respectively, and correspond to the rigidly supported parts of the boundary. Γ_0 and Γ_1 are those parts of the boundary where $x_2 = 0$ and $x_2 = d$ respectively, and correspond to the sections of the boundary supported by beams.

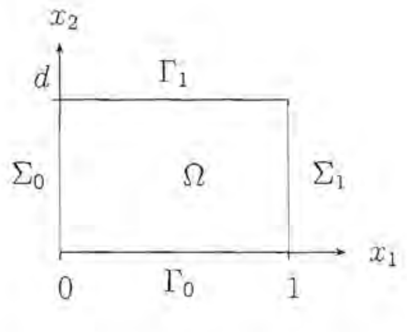

Figure 2.1: *Reference configuration of the plate.*

As in Section 2.5, the partial differential equation (2.5.5) is obtained.

On Σ_0 and Σ_1, (2.6.17) reduces to $M_{21} = 0$. The conditions at the end points of the beams (2.6.18) and (2.6.19) are included by extending the conditions on Σ_0 and Σ_1 to $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$.

On Γ_1, $\mathbf{n} = \mathbf{e}_2$ and $\boldsymbol{\tau} = -\mathbf{e}_1$. Using (2.6.2), we get

$$-\frac{1}{r}\mathbf{T}\cdot\mathbf{n} = -\frac{1}{r}T_2 = \partial_1 M_{11} + \partial_2 M_{12} - r\partial_t^2\partial_2 u$$

and

$$M\mathbf{n}\cdot\boldsymbol{\tau} = M_{12} \text{ and } M\mathbf{n}\cdot\mathbf{n} = M_{22}.$$

Assuming that the beams are merely supporting the plate, we get

$$M_{12} = 0 \text{ and } L = -\frac{1}{\alpha}M_{22} \text{ on } \Gamma_1.$$

Similarly, on Γ_0, $\mathbf{n} = -\mathbf{e}_2$ and $\boldsymbol{\tau} = \mathbf{e}_1$. Hence

$$-\frac{1}{r}\mathbf{T}\cdot\mathbf{n} = \frac{1}{r}T_2 = -\partial_1 M_{11} - \partial_2 M_{12} + r\partial_t^2\partial_2 u$$

and

$$M_{12} = 0 \text{ and } L = -\frac{1}{\alpha}M_{22}.$$

For the two beams, the first equation of motion (2.6.7) reduces to the following two equations. Note that $\partial_s = -\partial_1$ on Γ_1, and $\partial_s = \partial_1$ on Γ_0.

$$\begin{aligned}
\beta\partial_t^2 u &= \frac{\alpha}{r_b}\partial_1 F - \partial_1 M_{11} - \partial_2 M_{12} + r\partial_t^2\partial_2 u \text{ on } \Gamma_0\\
\beta\partial_t^2 u &= -\frac{\alpha}{r_b}\partial_1 F + \partial_1 M_{11} + \partial_2 M_{12} - r\partial_t^2\partial_2 u \text{ on } \Gamma_1.
\end{aligned}$$

The second equation of motion (2.6.8) reduces to

$$\beta r_b\partial_t^2\partial_1 u = \frac{\alpha}{r_b}F + \alpha\partial_1 M_b - M_{22} \text{ on } \Gamma_0$$
$$-\beta r_b\partial_t^2\partial_1 u = \frac{\alpha}{r_b}F - \alpha\partial_1 M_b - M_{22} \text{ on } \Gamma_1.$$

Finally, (2.6.14) implies that

$$M_{12} = 0 \text{ on } \Gamma_0 \text{ and } \Gamma_1.$$

In terms of the dimensionless displacement u the model is given by:

Problem 4

$$
\begin{aligned}
\partial_t^2 u - r\partial_t^2(\partial_1^2 u + \partial_2^2 u) &= -\left(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u\right) + q \text{ in } \Omega, \\
u &= 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1, \\
\partial_1^2 u + \nu\partial_2^2 u &= 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1, \\
\partial_2^2 u + \nu\partial_1^2 u &= 0 \text{ on } \Gamma_0 \text{ and } \Gamma_1, \\
\beta\partial_t^2 u - \beta r_b\partial_t^2\partial_1^2 u - r\partial_t^2\partial_2 u &= -\partial_2^3 u - (2-\nu)\partial_1^2\partial_2 u - \alpha\partial_1^4 u \text{ on } \Gamma_0, \\
\beta\partial_t^2 u - \beta r_b\partial_t^2\partial_1^2 u + r\partial_t^2\partial_2 u &= \partial_2^3 u + (2-\nu)\partial_1^2\partial_2 u - \alpha\partial_1^4 u \text{ on } \Gamma_1.
\end{aligned}
$$

Chapter 3

Variational form and weak solutions

Our main concern is the analysis and implementation of the finite element method, i.e. Chapters 5, 6 and 7. For this we need the model problems in (weak) variational form. Model problems with interface conditions is a relatively new subject, and we could not find adequate derivations of the variational form in standard references. Hence we found it necessary to present rigorous derivations. Much of the material is from [V1], [V2], [VVZ], [ZVGV1] and [ZVGV2].

In each case we find a variational formulation as a first step. This is done by multiplying the equation of motion by an arbitrary smooth function, and integrating over the reference configuration. The variational form is sufficient for the implementation of the finite element method, but for existence theory and analysing the convergence of the finite element method, a weak formulation of the model problem is required. In each case we will define the necessary function spaces and present such a weak form. At this stage a unified approach will become possible as we show that the model problems all have similar weak forms.

In Section 3.4 we discuss existence results for weak solutions. The free response of a system is determined by the natural frequencies and natural modes. These are determined by the eigenvalues and eigenvectors of a bilinear form. This topic is treated in Section 3.5.

3.1 The damaged beam

3.1.1 Variational formulation

Multiplying the equation of motion (2.3.1) by an arbitrary function v and integrating gives

$$\int_0^1 \partial_t^2 u(\cdot,t)v = \frac{1}{r}\int_0^1 \partial_x F(\cdot,t)v - k\int_0^1 \partial_t u(\cdot,t)v + \int_0^1 P(\cdot,t)v. \quad (3.1.1)$$

We use the notation $u(\cdot,t)$ for the function

$$u(\cdot,t) : [0,1] \to \mathbb{R} \text{ with } u(\cdot,t)(x) = u(x,t).$$

As the jump condition allows for discontinuities in $\partial_x u$ and $\partial_t\partial_x u$ at $x = \alpha$, the integration must be performed separately on the subintervals $(0,\alpha)$ and $(\alpha,1)$. Due to the discontinuity of $\partial_x u(\cdot,t)$ at α, the function $\partial_x^2 u(\cdot,t)$ will not exist—not even in a generalized sense. (We exclude δ-functions.) For this reason it is necessary to consider product spaces with pairs of functions as elements. With each function u, we associate a pair $\bar{u} = \langle u_1, u_2\rangle$ with u_1 the restriction of u to the interval $[0,\alpha]$, and u_2 the restriction to $[\alpha,1]$. For simplicity of notation we will write u for $\bar{u}$.

For any open interval $I = (a,b)$ the function spaces $C^i(I)$, $C^i(\bar{I})$, $C_0^\infty(I)$ and $L^2(I)$ are defined in Appendix A.

Let $I = (0,1)$, $I_1 = (0,\alpha)$ and $I_2 = (\alpha,1)$. Define the following product spaces:

$$\begin{aligned} L^2 &:= L^2(I_1) \times L^2(I_2), \\ C^i &:= C^i(\bar{I}_1) \times C^i(\bar{I}_2),\ i = 0,1,\ldots, \\ C_0^\infty &:= C_0^\infty(I_1) \times C_0^\infty(I_2). \end{aligned}$$

In terms of this notation $\partial_x u(\alpha^-,t) = \partial_x u_1(\alpha,t)$ and $\partial_x u(\alpha^+,t) = \partial_x u_2(\alpha,t)$, etcetera.

Notation For any $u = \langle u_1,u_2\rangle \in L^2$ and any $v = \langle v_1,v_2\rangle \in L^2$,

$$(u,v)_0 := \int_0^\alpha u_1v_1 + \int_\alpha^1 u_2v_2.$$

We will also use the notation $u' := \langle u_1', u_2'\rangle$, etcetera.

In terms of the new notation, (3.1.1) can be written as

$$\left(\partial_t^2 u(\cdot,t), v\right)_0 = \left(\frac{1}{r}\partial_x F(\cdot,t), v\right)_0 - (k\partial_t u(\cdot,t), v)_0 + (P(\cdot,t), v)_0. \quad (3.1.2)$$

In the following results the term $\left(\frac{1}{r}\partial_x F(\cdot,t), v\right)_0$ is examined. For simplicity of notation we write F and u for $F(\cdot,t)$ and $u(\cdot,t)$ in the following results.

Lemma 3.1.1 *If the equation of motion* (2.3.2) *is satisfied, then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F, v\right)_0 &= -(M, v'')_0 - (r\partial_t^2\partial_x u, v')_0 + \left[\frac{1}{r}F_1 v_1\right]_0^\alpha + \left[\frac{1}{r}F_2 v_2\right]_\alpha^1 \\ &\quad + [M_1 v_1']_0^\alpha + [M_2 v_2']_\alpha^1 \ \textit{for all } v \in C^2.\end{aligned}$$

Proof The result is obtained by performing integration by parts twice. □

Define the space of test functions T as

$$T = \{v = \langle v_1, v_2\rangle \in C^2 : v_1(0) = v_1'(0) = 0,\ v_1(\alpha) = v_2(\alpha)\}.$$

Corollary 3.1.1 *Assume that the equation of motion* (2.3.2) *is satisfied. If, in addition, F and M satisfy the boundary conditions at $x = 1$,* (2.3.5)*, as well as the interface conditions at $x = \alpha$,* (2.3.6) *to* (2.3.8)*, then*

$$\left(\frac{1}{r}\partial_x F, v\right)_0 = -(M, v'')_0 - (r\partial_t^2\partial_x u, v')_0 - M(\alpha,t)\,(v_2'(\alpha) - v_1'(\alpha))$$

$$\textit{for all } v \in T$$

From the constitutive equation (2.3.3) and the jump condition (2.3.9), the term $\left(\frac{1}{r}\partial_x F, v\right)_0$ can be expressed in terms of u.

Corollary 3.1.2 *If u is a solution of Problem* 1*, then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F, v\right)_0 &= -(\partial_x^2 u, v'')_0 - (\mu\partial_t\partial_x^2 u, v'')_0 - (r\partial_t^2\partial_x u, v')_0 \\ &\quad -\frac{1}{\delta}\left(\partial_x u_2(\alpha,t) - \partial_x u_1(\alpha,t)\right)(v_2'(\alpha) - v_1'(\alpha)) \\ &\quad -\frac{\mu}{\delta}\left(\partial_x\partial_t u_2(\alpha,t) - \partial_x\partial_t u_1(\alpha,t)\right)(v_2'(\alpha) - v_1'(\alpha))\end{aligned}$$

$$\textit{for all } v \in T.$$

We define bilinear forms a, b and c by

$$\begin{aligned} b(u,v) &:= (u'',v'')_0 + \frac{1}{\delta}\left(u_2'(\alpha)-u_1'(\alpha)\right)\left(v_2'(\alpha)-v_1'(\alpha)\right) \text{ for all } u,v\in C^2, \\ a(u,v) &:= \mu b(u,v) + (ku,v)_0 \text{ for all } u,v\in C^2, \\ c(u,v) &:= (u,v)_0 + (ru',v')_0 \text{ for all } u,v\in C^1. \end{aligned}$$

The variational form of Problem 1 can be expressed in terms of these bilinear forms. In the following sections we will show that all the model problems can be reduced to the same abstract form if appropriate bilinear forms are introduced.

Problem 1b: Variational formulation

Find u such that, for all $t>0$, $u(\cdot,t)\in T$ and

$$c(\partial_t^2 u(\cdot,t),v) + a(\partial_t u(\cdot,t),v) + b(u(\cdot,t),v) = (P(\cdot,t),v)_0$$

for all $v\in T$.

Theorem 3.1.1 *If u is a solution of Problem 1, then u is a solution of Problem 1b.*

Proof The proof follows directly from substituting the result in Corollary 3.1.2 into (3.1.2). Note that if u is a solution of Problem 1, it follows from (2.3.4) and (2.3.6) that $u\in T$. □

Theorem 3.1.2 *If u is a solution of Problem 1b and $\partial_t u(\cdot,t)\in C^4$ and $\partial_t^2 u(\cdot,t)\in C^2$, then u is a solution of Problem 1.*

Proof For simplicity of notation we will write u for $u(\cdot,t)$ in this proof. Let $v\in T$ such that $v_1\in C_0^\infty(I_1)$ and $v_2=0$. Performing integration by parts, we find that

$$\int_0^\alpha (\partial_t^2 u_1 - r\partial_t^2\partial_x^2 u_1 + \partial_x^4 u_1 + \mu\partial_t\partial_x^4 u_1 + k\partial_t u_1 - P_1)v_1 = 0. \tag{3.1.3}$$

Since $C_0^\infty(I_1)$ is dense in $L^2(I_1)$, it follows that u satisfies the partial differential equation on $I_1=(0,\alpha)$. The same is obviously true on $I_2=(\alpha,1)$.

A direct consequence is that

$$\begin{aligned} &\int_0^\alpha (-r\partial_t^2\partial_x^2 u_1 + \partial_x^4 u_1 + \mu\partial_t\partial_x^4 u_1)v_1 + \int_\alpha^1 (-r\partial_t^2\partial_x^2 u_2 + \partial_x^4 u_2 + \mu\partial_t\partial_x^4 u_2)v_2 \\ = \ & (r\partial_t^2\partial_x u, v')_0 + b(u,v) + \mu b(\partial_t u, v) \text{ for each } v\in T. \end{aligned}$$

Performing integration by parts once on the terms $r\partial_t^2\partial_x^2 u_1 v_1$ and $r\partial_t^2\partial_x^2 u_2 v_2$ and twice on the terms $\partial_x^4 u_1 v_1$, $\partial_x^4 u_2 v_2$, $\mu\partial_t\partial_x^2 u_1 v_1$, and $\mu\partial_t\partial_x^2 u_2 v_2$ yield that

$$\begin{aligned} & -\left[r\partial_t^2\partial_x u_1 v_1\right]_0^\alpha - \left[r\partial_t^2\partial_x u_2 v_2\right]_\alpha^1 + \left[\partial_x^3(u_1 + \mu\partial_t u_1)v_1\right]_0^\alpha \\ & + \left[\partial_x^3(u_2 + \mu\partial_t u_2)v_2\right]_\alpha^1 - \left[\partial_x^2(u_1 + \mu\partial_t u_1)v_1'\right]_0^\alpha - \left[\partial_x^2(u_2 + \mu\partial_t u_2)v_2'\right]_\alpha^1 = 0 \end{aligned}$$

for each $v \in T$.

Recall that $v_1(0) = v_1'(0) = 0$. Choosing $v_1'(\alpha) = v_2'(\alpha)$ and $v_2(1) = v_2'(1) = 0$, we have

$$\begin{aligned} r\partial_t^2\partial_x u(\alpha^+,t) - \partial_x^3 u(\alpha^+,t) - \mu\partial_t\partial_x^3 u(\alpha^+,t) &= r\partial_t^2\partial_x u(\alpha^-,t) \\ &\quad -\partial_x^3 u(\alpha^-,t) - \mu\partial_t\partial_x^3 u(\alpha^-,t). \end{aligned}$$

All the other conditions follow from suitable choices for the values of v_1', v_2 and v_2' at $x = \alpha$ and $x = 1$. □

3.1.2 Weak formulation

L^2 is a Hilbert space with the inner product $(\cdot,\cdot)_0$. The norms in $L^2(I_j)$ and L^2 will all be denoted by $\|\cdot\|_0$ with the relevant space being clear from the context.

Define the following product spaces:

$$H^i := H^i(I_1) \times H^i(I_2), \ i = 1, 2, \ldots$$

where $H^i(I_j)$ is the Sobolev space of order i on I_j. See Appendix B. The norms in $H^i(I_j)$ and H^i are all denoted by $\|\cdot\|_i$. For the product space H^i we use the usual product space inner product and norms, i.e. if $u = \langle u_1, u_2\rangle \in H^i$, then $\|u\|_i^2 = \|u_1\|_i^2 + \|u_2\|_i^2$.

For the weak formulation of the vibration problem we consider the closure of T in H^2. We denote this closure by V and note that V is a Hilbert space with the inner product of H^2.

The bilinear form c can be extended to H^1 and is an inner product on H^1. We define the space W as the closure of T with respect to the norm induced by c. We refer to this norm as the *inertia norm*. If rotary inertia is ignored (i.e. $r = 0$), it follows that $c(\cdot,\cdot) = (\cdot,\cdot)_0$ and then $W = L^2$.

For $v = \langle v_1, v_2 \rangle \in H^2$, the mappings

$$v_1' \longrightarrow v_1'(\alpha) \text{ and } v_2' \longrightarrow v_2'(\alpha)$$

are well defined in the sense of trace. See Appendix B. We have the following result.

Lemma 3.1.2 *For $i = 1,\ 2$ and $K = \max\{\alpha^{-1},\ (1-\alpha)^{-1}\}$,*

$$|v_i'(\alpha)| \leq K\|v_i\|_2 \text{ for all } v \in V. \tag{3.1.4}$$

Proof From Lemma B.2.1 in Appendix B follows that

$$|v_i'(\alpha)| \leq K\|v_i'\|_1 \leq K\|v_i\|_2 \text{ for all } v \in V.$$

□

It follows that the domains of the bilinear forms a and b can be extended to H^2.

For the definition of $C^m((0,\tau), X)$ see Appendix B.

Let $f(t) = \langle P_1(t), P_2(t) \rangle$.

Problem 1c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_0$$

for all $v \in V$.

The initial conditions $u(0)$ and $u'(0)$ will be discussed later. Note that $u'(0)$ refers to the right derivative of u' at $t = 0$.

Notation With a function u we associate a function u^* such that

$$u^* : [0,\tau] \to L^2 \text{ with } u^*(t)(x) = u(t,x).$$

Theorem 3.1.3 *If u is a solution of Problem 1b, then u^* is a solution of Problem 1c.*

Proof Suppose u is a solution of Problem 1b. Then $(u_1^*)''(t) = \partial_t^2 u_1(\cdot, t)$ and $(u_2^*)''(t) = \partial_t^2 u_2(\cdot, t)$. (See Appendix B.) Obviously the same will be true in the product space. The result now follows from the fact that T is dense in V. □

Notation With $u \in C([0,\tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : [0,\tau] \times (0,1) \to \mathbb{R} \text{ with } \tilde{u}(x,t) = u(t)(x).$$

If the weak solution is smooth enough, it satisfies the variational problem.

Theorem 3.1.4 *If u is a solution of Problem 1c and $u \in C^2([0,\tau], C^2)$, then $\tilde{u}$ is a solution of Problem 1b.*

Proof If $u_1 \in C^2([0,\tau], C^2[0,\alpha])$, then $\partial_t \tilde{u}_1$ exists and $\partial_t^2 \tilde{u}_1(x,t) = u_1''(t)(x)$ for each point (x,t). It is now clear that $\tilde{u}_1 \in C^2([0,\tau) \times [0,\alpha])$. Similarly, $\tilde{u}_2 \in C^2([0,\tau) \times [\alpha, 1])$. □

3.1.3 The energy norm

The following lemma gives some inequalities of Poincaré type for the space V.

Lemma 3.1.3 *For any $u = \langle u_1, u_2 \rangle \in V$,*

$$\|u_1\|_0 \leq \|u_1'\|_0 \leq \|u_1''\|_0, \tag{3.1.5}$$

$$\|u\|_2^2 \leq 14\left(\|u''\|_0^2 + (u_2'(\alpha) - u_1'(\alpha))^2\right). \tag{3.1.6}$$

Proof We assume first that $u \in T$. For $x \in (0,\alpha)$ we choose $[a,b] = [0,x]$ in Lemma B.2.1 and note that $u_1(0) = u_1'(0) = 0$. It follows that

$$|u_1(x)| \leq \|u_1'\|_0 \text{ and } |u_1'(x)| \leq \|u_1''\|_0$$

and the inequalities (3.1.5) are direct consequences.

For $x \in (\alpha, 1)$, we choose $[a,b] = [\alpha, x]$ in Lemma B.2.1, and find that

$$|u_2(x) - u_2(\alpha)| \leq \|u_2'\|_0.$$

As $|u_2(\alpha)| = |u_1(\alpha)| \leq \|u_1'\|_0$ it follows that

$$|u_2(x)| \leq \|u_1'\|_0 + \|u_2'\|_0.$$

Hence

$$\|u_2\|_0^2 \leq \left(\|u_1'\|_0 + \|u_2'\|_0\right)^2.$$

Similarly, from Lemma B.2.1,

$$|u_2'(x) - u_2'(\alpha)| \leq \|u_2''\|_0$$

and as $|u_1'(\alpha)| \leq \|u_1''\|_0$ it follows (using the inverse triangle inequality) that

$$|u_2'(x)| \leq \|u_1''\|_0 + \|u_2''\|_0 + |u_2'(\alpha) - u_1'(\alpha)|.$$

Hence

$$\|u_2'\|_0^2 \leq \left(\|u_1''\|_0 + \|u_2''\|_0 + |u_2'(\alpha) - u_1'(\alpha)|\right)^2.$$

It is now easy to prove that (3.1.6) holds. The inequalities will also hold on V as it is the closure of T in H^2. □

Theorem 3.1.5 *The bilinear form b is bounded and positive definite on V.*

Proof For any $u, v \in V$,

$$\begin{aligned} |b(u,v)| \leq \;& \|u_1\|_2\|v_1\|_2 + \|u_2\|_2\|v_2\|_2 \\ & + \frac{1}{\delta}\left(|u_2'(\alpha)| + |u_1'(\alpha)|\right)\left(|v_2'(\alpha)| + |v_1'(\alpha)|\right). \end{aligned}$$

Using also Lemma 3.1.2, it is easy to see that b is bounded. Clearly, from (3.1.6), there exists a constant C such that

$$b(u,u) \geq C^2\|u\|_2^2 \text{ for all } u \in V.$$ □

Due to the fact that b is symmetric we have the following result.

Corollary 3.1.3 *The bilinear form b defines an inner product on V.*

Define the *energy norm* $\|\cdot\|_E$ in V by

$$\|u\|_E^2 = b(u,u) \text{ for any } u \in V.$$

Corollary 3.1.4 *The energy norm is equivalent to the H^2-norm on V.*

Lemma 3.1.4 *V is dense in W.*

Proof W is the closure of T with respect to the inertia norm and $T \subset V \subset W$. □

Lemma 3.1.5 *V is dense in L^2.*

Proof For $v = \langle v_1, v_2 \rangle \in C_0^\infty = C_0^\infty(I_1) \times C_0^\infty(I_2)$ it is clear that $v_1(\alpha) = v_2(\alpha) = 0$, and hence that $v \in T$. Thus $C_0^\infty \subset T \subset V \subset L^2$, and as C_0^∞ is dense in L^2, it follows that V is dense in L^2. □

The following result is required to prove that bounded subsets of V are precompact in W.

Lemma 3.1.6 *Let $X_1 \subset Y_1$ and $X_2 \subset Y_2$, be four Hilbert spaces. Let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. If bounded sequences in X_1 and X_2, respectively, have convergent subsequences in Y_1 and Y_2, then any bounded subset in X is precompact in Y.*

Proof Suppose the subset A of X is bounded and $\{u^n\} = \{\langle u_1^n, u_2^n \rangle\}$ is a sequence in A. Then $\{u_1^n\}$ and $\{u_2^n\}$ are bounded sequences in X_1 and X_2 respectively. This means that there exists a convergent subsequence $\{u_n^{n_k}\}$ of $\{u_1^n\}$ in Y_1. Consider the sequence $\{u^{n_k}\} = \{\langle u_1^{n^k}, u_2^{n^k} \rangle\}$. It is now obvious that this sequence possesses a convergent subsequence in Y which is a subsequence of $\{u^n\}$. We conclude that A is precompact in Y. □

Lemma 3.1.7 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof Assume that $\{u_1^n\}$ and $\{u_2^n\}$ are bounded sequences in $H^2(I_1)$ and $H^2(I_2)$, respectively. Using the Rellich imbedding theorem (See [Fr, p 31-32]), we can find convergent subsequences in $H^1(I_1)$ and $H^2(I_2)$, respectively. The result follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.2 Beam models with dynamical boundary conditions

3.2.1 Variational formulation

The equation of motion (2.2.7) is multiplied by an arbitrary function v and integrated to get

$$\int_0^1 \partial_t^2 u(\cdot,t)v = \frac{1}{r}\int_0^1 \partial_x F(\cdot,t)v - k\int_0^1 \partial_t u(\cdot,t)v + \int_0^1 P(\cdot,t)v. \quad (3.2.1)$$

Let $I = (0,1)$.

Notation For any $u \in L^2(I)$ and any $v \in L^2(I)$,

$$(u,v)_I := \int_0^1 uv.$$

In the following results the term $(\frac{1}{r}\partial_x F, v)_I$ is examined.

Lemma 3.2.1 *If the equation of motion* (2.2.8) *is satisfied, then*

$$\left(\frac{1}{r}\partial_x F, v\right)_I = -(M,v'')_I - (r\partial_t^2\partial_x u, v')_I + \left[\frac{1}{r}Fv\right]_0^1 + [Mv']_0^1$$

for all $v \in C^2(I)$.

Proof The result is obtained by performing integration by parts twice. □

Define the space of test functions $T(I)$ as

$$T(I) = \{v \in C^2(\bar{I}) : v(0) = v'(0) = 0\}.$$

The following result follows from the constitutive equation (2.2.9) and the boundary conditions (2.4.6) and (2.4.7).

Corollary 3.2.1 *If u is the solution of Problem* 2, *then*

$$\begin{aligned}\left(\frac{1}{r}\partial_x F, v\right)_I &= -(\partial_x^2 u, v'')_I - (\mu\partial_t\partial_x^2 u, v'')_I - (r\partial_t^2\partial_x u, v')_I \\ &\quad - m\partial_t^2 u(1,t)v(1) - I_m\partial_t^2\partial_x u(1,t)v'(1) \\ &\quad - \mu_0\partial_t u(1,t)v(1) - \mu_1\partial_t\partial_x u(1,t)v'(1) \quad \textit{for all } v \in T(I).\end{aligned}$$

We define bilinear forms a, b and c by

$$\begin{aligned}
b(u,v) &:= (u'',v'')_I \text{ for all } u,v \in C^2(\bar{I}),\\
a(u,v) &:= \mu b(u,v) + (ku,v)_I + \mu_0 u(1)v(1) + \mu_1 u'(1)v'(1) \text{ for all } u,v \in C^2(\bar{I}),\\
c(u,v) &:= (u,v)_I + (ru',v')_I + mu(1)v(1) + I_m u'(1)v'(1) \text{ for all } u,v \in C^1(\bar{I}).
\end{aligned}$$

The variational form of Problem 2 can be expressed in terms of these bilinear forms.

Problem 2b: Variational formulation

Find u such that, for all $t > 0$, $u(\cdot,t) \in T(I)$ and

$$c(\partial_t^2 u(\cdot,t),v) + a(\partial_t u(\cdot,t),v) + b(u(\cdot,t),v) = (P(\cdot,t),v)_I$$

for all $v \in T(I)$.

Theorem 3.2.1 *If u is a solution of Problem 2, then u is a solution of Problem 2b.*

Proof The proof follows directly from substituting the result of Corollary 3.2.1 into (3.2.1). Note that if u is a solution of Problem 2, it follows from (2.4.5) that $u \in T(I)$. □

Theorem 3.2.2 *If u is a solution of Problem 2b and $\partial_t u(\cdot,t) \in C^4(\bar{I})$ and $\partial_t^2 u(\cdot,t) \in C^2(\bar{I})$, then u is a solution of Problem 2.*

Proof The proof is virtually the same as that of Theorem 3.1.2. □

3.2.2 Weak formulation

The product spaces $\mathcal{L}^2$ and $\mathcal{H}^m$ are defined by

$$\mathcal{L}^2 := L^2(I) \times \mathbb{R} \times \mathbb{R} = \{v = \langle v_1, v_2, v_3\rangle : v_1 \in L^2(I), v_2 \in \mathbb{R}, v_3 \in \mathbb{R}\}$$

and

$$\mathcal{H}^m := H^m(I) \times \mathbb{R} \times \mathbb{R} = \{v = \langle v_1, v_2, v_3\rangle : v_1 \in H^m(I), v_2 \in \mathbb{R}, v_3 \in \mathbb{R}\}.$$

The inner product in L^2 is given by

$$(u,v)_0 = \left(\int_0^1 u_1 v_1\right) + u_2v_2 + u_3v_3$$

and in H^m by

$$(u,v)_m = \sum_{i=0}^{m} \left(\int_0^1 u_1^{(i)} v_1^{(i)}\right) + u_2v_2 + u_3v_3.$$

The notation $\|\cdot\|_m$ is used for the associated norm in H^m.

The definitions of the bilinear forms a and b are extended to

$$\begin{aligned} b(u,v) &= (u_1'', v_1'')_I \text{ for all } u,v \in H^2, \\ a(u,v) &= \mu b(u_1,v_1) + (ku_1, v_1)_I + \mu_0 u_2 v_2 + \mu_1 u_3 v_3 \text{ for all } u,v \in H^2. \end{aligned}$$

Define the space $\widetilde{T}_2(I)$ as the closure of $T(I)$ in $H^2(I)$. Recall the fact that the boundary values of $v_1^{(i)}$ are defined in the sense of trace, for example $v_1^{(i)}(1)$ is well defined if $v_1 \in H^{i+1}(I)$. (See Appendix B.)

Define the subspace V of H^2 by

$$V = \{v \in H^2 : v_1 \in \widetilde{T}_2(I), v_2 = v_1(1), v_3 = v_1'(1)\}.$$

Lemma 3.2.2 *V is a closed subspace of H^2.*

Proof If $\{v^n\}$ is a sequence in V with limit $v \in H^2$, it follows that

$$\|v_1^n - v_1\|_2 \to 0 \text{ as } n \to \infty.$$

The boundedness of the trace operator yields that

$$|(v_1^n)'(1) - v_1'(1)| \le C\|v_1^n - v_1\|_2$$

and

$$|v_1^n(1) - v_1(1)| \le C\|v_1^n - v_1\|_2.$$

Uniqueness of limits implies that $v \in V$. □

Define $\widetilde{T}_1(I)$ as the closure of $T(I)$ in $H^1(I)$. Define the subspace W of H^1 by

$$W = \{v \in H^1 : v_1 \in \widetilde{T}_1(I), v_2 = v_1(1)\}.$$

Lemma 3.2.3 *W is a closed subspace of H^1.*

Proof The proof is virtually identical to that of Lemma 3.2.2. □

The definition of c is extended to

$$c(u,v) = (u_1, v_1)_I + (ru_1', v_1')_I + mu_2v_2 + I_m u_3 v_3 \text{ for all } u, v \in W.$$

This bilinear form defines an inner product on W, even if $m = I_m = 0$. The *inertia norm* induced by c on W is equivalent to the H^1 norm. If rotary inertia is ignored (i.e. $r = 0$), we again have $W = L^2$.

Choose $f(t) = \langle P(\cdot, t), 0, 0\rangle$.

Problem 2c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_0$$

for all $v \in V$

Notation With the function u we associate a function u^* such that

$$u^* : [0,\tau] \to L^2 \text{ with } u^*(t)(x) = u(t,x).$$

Theorem 3.2.3 *If u is a solution of Problem 2b, then u^* is a solution of Problem 2c.*

Proof The proof is the same as that of Theorem 3.1.3. □

Notation With $u \in C([0,\tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : [0,\tau] \times (0,1) \to \mathbb{R} \text{ with } \tilde{u}(x,t) = u(t)(x).$$

Theorem 3.2.4 *If u is a solution of Problem 2c and $u_1 \in C^2([0,\tau], C^2(\bar{I}))$, then $\tilde{u}$ is a solution of Problem 2b.*

Proof The proof is the same (even simpler) than the proof of Theorem 3.1.4.

□

3.2.3 Energy norm

Lemma 3.2.4 *The bilinear form b is bounded and positive definite on V.*

Proof Clearly, b is bounded in H^2. From Lemma B.2.3, follows that for any $w \in V$,

$$\|w_1\|_0 \leq \|w_1'\|_0 \leq \|w_1''\|_0,$$

as $w_1(0) = w_1'(0) = 0$.

Also, from Lemma B.2.1, $|w_2| \leq \|w_1'\|_0$ and $|w_3| \leq \|w_1''\|_0$. Consequently there exists a constant c, such that

$$\|w\|_2^2 \leq cb(w, w) \text{ for all } w \in V.$$

□

We define the *energy norm* on V by

$$\|w\|_E^2 = b(w, w) \text{ for all } w \in V.$$

It remains to show that V is dense in L^2 and that V is dense in W. This can be done by adapting the proof of [Sa, Prop 8.1], to this situation.

Lemma 3.2.5 *V is dense in L^2.*

Proof Let $f \in C^\infty(0,1)$ and $g \in C^\infty(0,1)$ be such that

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \frac{1}{3}, \\ 1 & \text{for} \quad \frac{2}{3} < x \leq 1, \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \frac{1}{3}, \\ x & \text{for} \quad \frac{2}{3} < x \leq 1. \end{cases}$$

Put $f_n(x) = f(x^n)$ and $g_n(x) = g(x^n)/n$ for $0 \leq x \leq 1$.

Let $w = \langle w_1, w_2, w_3 \rangle \in L^2$. Then there exists a sequence of functions $\{p_n\}$ in $C_0^\infty(0,1)$ such that

$$\|p_n - w_1\|_I \to 0 \text{ as } n \to \infty.$$

Let $v_n = p_n + w_2 f_n + w_3 g_n$. Then $v_n \in T(I)$ and $y_n = \langle v_n, v_n(1), v_n'(1)\rangle \in V$ Now,

$$\|v_n - w_1\|_I \leq \|p_n - w_1\|_I + |w_2|\|f_n\|_I + |w_3|\|g_n\|_I \to 0 \text{ as } n \to \infty.$$

We also have

$$v_n(1) = w_2 + \frac{w_3}{n} \to w_2 \text{ as } n \to \infty$$

and

$$v_n'(1) = w_3 \text{ for all } n.$$

Hence $\|y_n - w\|_0 \to 0$ as $n \to \infty$. □

Lemma 3.2.6 *V is dense in W.*

Proof Consider any $w \in W$. From the definitions of $\widetilde{T}_1(I)$ and $\widetilde{T}_2(I)$ it is clear that there exists a sequence $\{p_n\} \subset \widetilde{T}_2(I)$ such that $\|w_1 - p_n\|_1^I \to 0$ as $n \to \infty$. Using the sequence of functions $\{f_n\}$ defined in the proof of Lemma 3.2.5, we let $v_n = p_n + w_2 f_n$. The rest of the proof in the same as the proof of Lemma 3.2.5. □

Lemma 3.2.7 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof Suppose $\{w^n\}$ is a bounded sequence in V. This implies that $\{w_1^n\}$ is a bounded sequence in $H^2(I)$ and that $\{w_2^n\}$ and $\{w_3^n\}$ are bounded sequences of real numbers. Using the Rellich imbedding theorem (See [Fr, p 31-32]) yields a convergent subsequence of $\{w^n\}$ in $H^1(I)$, and from the Weierstrass theorem we find convergent subsequences of $\{w_2^n\}$ and $\{w_3^n\}$. The result then follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.3 Plate beam model

3.3.1 Variational formulation

The variational form is obtained by multiplying the dimensionless form of the equation of motion (2.6.1) by an arbitrary scalar valued function v and integrating to get

$$\int_\Omega \partial_t^2 uv = \frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v + \int_\Omega qv. \tag{3.3.1}$$

We start by quoting a general Green formula on a domain Ω in the plane:

For any scalar valued function v and any vector valued function $\mathbf{F}$,

$$\int_\Omega (\operatorname{div}\mathbf{F})v = -\int_\Omega \mathbf{F}\cdot\operatorname{grad} v + \int_{\partial\Omega}(\mathbf{F}\cdot\mathbf{n})v\,ds. \tag{3.3.2}$$

We also need a similar result for a matrix valued function A. We define $\operatorname{div} A$ to be a vector with the ith component equal to $\operatorname{div} A^{\text{row } i}$. The trace of the matrix A is denoted by $\operatorname{tr}(A)$.

Lemma 3.3.1 *For any vector valued function* $\mathbf{w}$ *and any matrix valued function* A,

$$\int_\Omega \operatorname{div} A\cdot\mathbf{w} = -\int_\Omega \operatorname{tr}(AW) + \int_{\partial\Omega} A\mathbf{w}\cdot\mathbf{n}\,ds, \tag{3.3.3}$$

where $W = \begin{bmatrix} \partial_1 w_1 & \partial_2 w_1 \\ \partial_1 w_2 & \partial_2 w_2 \end{bmatrix}$.

Proof The proof follows directly if (3.3.2) is applied for each row. □

Lemma 3.3.2 *If the equation of motion* (2.6.2) *is satisfied and* $v \in C^2(\bar\Omega)$, *then*

$$\begin{aligned}\frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v &= \int_\Omega \operatorname{tr}(RMV) - r\int_\Omega \left(\partial_t^2(\operatorname{grad} u)\right)\cdot(\operatorname{grad} v) \\ &\quad - \int_{\partial\Omega}(RM\mathbf{n})\cdot(\operatorname{grad} v)\,ds + \frac{1}{r}\int_{\partial\Omega}(\mathbf{T}\cdot\mathbf{n})v\,ds\end{aligned} \tag{3.3.4}$$

where $V = \begin{bmatrix} \partial_1^2 v & \partial_1\partial_2 v \\ \partial_1\partial_2 v & \partial_2^2 v \end{bmatrix}$.

Proof From (3.3.2), with $\mathbf{F} = \mathbf{T}$, follows that

$$\int_\Omega (\operatorname{div} \mathbf{T})v = -\int_\Omega \mathbf{T} \cdot \operatorname{grad} v + \int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{n})v\, ds.$$

From (2.6.2), $1/r\mathbf{T} = R \operatorname{div} M - rR\partial_t \mathbf{H}$, and hence

$$\begin{aligned} \frac{1}{r}\int_\Omega (\operatorname{div} \mathbf{T})v &= -\int_\Omega (R \operatorname{div} M) \cdot \operatorname{grad} v - r\int_\Omega R\partial_t \mathbf{H} \cdot \operatorname{grad} v \\ &\quad + \frac{1}{r}\int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{n})v\, ds. \end{aligned} \tag{3.3.5}$$

Using (3.3.3) with $\mathbf{w} = \operatorname{grad} v$ and $A = RM$ gives

$$\int_\Omega R \operatorname{div} M \cdot \operatorname{grad} v = -\int_\Omega \operatorname{tr}(RMV) + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds, \tag{3.3.6}$$

where $V = \begin{bmatrix} \partial_1^2 v & \partial_1\partial_2 v \\ \partial_1\partial_2 v & \partial_2^2 v \end{bmatrix}$.

Note that RM is symmetric and $\operatorname{div}(RM) = R \operatorname{div} M$.

From (2.6.3) follows that

$$\int_\Omega R\partial_t \mathbf{H} \cdot \operatorname{grad} v = \int_\Omega \partial_t^2 (\operatorname{grad} u) \cdot \operatorname{grad} v.$$

□

Choose the space of test functions $T(\Omega)$ as

$$T(\Omega) = \{v \in C^2(\bar{\Omega}) : v = 0 \text{ on } \overline{\Sigma}_0 \text{ and } \overline{\Sigma}_1\}.$$

It is necessary to analyze the line integrals in (3.3.4). Let $v \in T(\Omega)$ throughout the discussion that follows.

The boundary $\partial\Omega$ consists of four parts. Consider first the part Σ_1. We have $v = 0$, hence

$$\int_{\Sigma_1} (\mathbf{T} \cdot \mathbf{n})v\, ds = 0. \tag{3.3.7}$$

Since $\mathbf{n} = \mathbf{e}_1$ and $\operatorname{grad} v = \partial_1 v \mathbf{e}_1$,

$$(RM\mathbf{n}) \cdot (\operatorname{grad} v) = -(\partial_1^2 u + \nu\partial_2^2 u)\partial_1 v = 0. \tag{3.3.8}$$

As a consequence the boundary terms vanish. The same will happen on Σ_0.

For the domain Ω the line integrals on Γ_0 and Γ_1 reduce to two one-dimensional integrals on $(0,1)$. Formally, $ds = dx_1$ on Γ_0 and $ds = -dx_1$ on Γ_1, because of the orientation of the line integral. This means that for any function v

$$\int_{\Gamma_0} v\,ds = \int_0^1 v(x_1,0)dx_1 \text{ and } \int_{\Gamma_1} v\,ds = \int_0^1 v(x_1,d)dx_1.$$

We will use subscripts 0 and 1 to differentiate between functions defined on Γ_0 and Γ_1.

Now, consider Γ_0. From (2.6.13),

$$\frac{1}{r}\int_{\Gamma_0}(\mathbf{T}\cdot\mathbf{n})v\,ds = -\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0). \tag{3.3.9}$$

Since $\mathbf{n} = -\mathbf{e}_2$ and $M\mathbf{e}_2\cdot\mathbf{e}_1 = 0$, we have from (2.6.15) that

$$-(RM\mathbf{n})\cdot(\operatorname{grad} v) \;=\; (M\mathbf{e}_2\cdot\mathbf{e}_2)\partial_1 v = -\alpha L_0\partial_1 v. \tag{3.3.10}$$

If (2.6.8) is satisfied and $w \in C^2[0,1]$ with $w(0) = w(1) = 0$, then

$$\begin{aligned}
\frac{\alpha}{r_b}\int_0^1 \partial_x F_0(\cdot,t)w \;&=\; \int_0^1 \left(\alpha\partial_1 M_{b0}(\cdot,t) + \alpha L_0(\cdot,t) - \beta r_b\partial_t^2\partial_1 u(\cdot,0,t)\right)w' \\
&\quad + \left[\frac{\alpha}{r_b}F_0(\cdot,t)w\right]_0^1 \\
&= -\int_0^1 \alpha M_{b0}(\cdot,t)w'' - \beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)w' \\
&\quad + \int_0^1 \alpha L_0(\cdot,t)w' + \left[\frac{\alpha}{r_b}F_0(\cdot,t)w\right]_0^1 + [\alpha M_{b0}(\cdot,t)w']_0^1.
\end{aligned}$$

Clearly, the boundary terms vanish as $M_{b0}(0,t) = M_{b0}(1,t) = 0$ from (2.6.19).

Choosing $w(x_1) = v(x_1,0)$ and combining this result with (3.3.9) and (3.3.10) yield that

$$\begin{aligned}
&\int_{\Gamma_0}\left((-RM\mathbf{n})\cdot\text{grad }v + \frac{1}{r}\mathbf{T}\cdot\mathbf{n}\right)ds \\
&= -\frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 M_{b0}(\cdot,t)\partial_1^2 v(\cdot,0) \\
&\quad -\beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \alpha\int_0^1 P_0(\cdot,t)v(\cdot,0).
\end{aligned} \tag{3.3.11}$$

An analogous result is true for Γ_1.

Lemma 3.3.3 *If u is a solution of Problem 3 and $v \in T(\Omega)$, then*

$$\begin{aligned}
&\frac{1}{r}\int_\Omega (\operatorname{div}\mathbf{T})v + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_1(\cdot,t)v(\cdot,d)\\
= \;& \int_\Omega \operatorname{tr}(RMV) - r\int_\Omega \partial_t^2(\operatorname{grad}u)\cdot \operatorname{grad}v\\
&-\beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \beta r_b\int_0^1 \partial_t^2\partial_1 u(\cdot,d,t)\partial_1 v(\cdot,d)\\
&-\alpha\int_0^1 \partial_1^2 u(\cdot,0,t)\partial_1^2 v(\cdot,0) - \alpha\int_0^1 \partial_1^2 u(\cdot,d,t)\partial_1^2 v(\cdot,d)\\
&-\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 P_1(\cdot,t)v(\cdot,d).
\end{aligned}$$

Proof Substitute (3.3.7), (3.3.8) and (3.3.11) for the boundary integrals in Lemma 3.3.2. □

Notation For any $u \in L^2(\Omega)$ and $v \in L^2(\Omega)$,

$$(u,v)_\Omega = \int_\Omega uv.$$

Define a bilinear form b on $C^2(\bar{\Omega})$ by

$$b(u,v) = b_\Omega(u,v) + \alpha b_0(u,v) + \alpha b_1(u,v)$$

with

$$\begin{aligned}
b_\Omega(u,v) = \int_\Omega \operatorname{tr}(RMV) \;=\;& (\partial_1^2 u, \partial_1^2 v)_\Omega + 2(1-\nu)(\partial_1\partial_2 u, \partial_1\partial_2 v)_\Omega\\
&+(\partial_2^2 u, \partial_2^2 v)_\Omega + \nu(\partial_2^2 u, \partial_1^2 v)_\Omega + \nu(\partial_1^2 u, \partial_2^2 v)_\Omega
\end{aligned}$$

and

$$\begin{aligned}
b_0(u,v) &= \int_0^1 \partial_1^2 u(\cdot,0)\partial_1^2 v(\cdot,0),\\
b_1(u,v) &= \int_0^1 \partial_1^2 u(\cdot,d)\partial_1^2 v(\cdot,d).
\end{aligned}$$

Lemma 3.3.4 *If u is a solution of Problem 3, then*

$$\begin{aligned}
&\frac{1}{r}\int_\Omega (\operatorname{div} \mathrm{T})v + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_0(\cdot,t)v(\cdot,0) + \frac{\alpha}{r_b}\int_0^1 \partial_1 F_1(\cdot,t)v(\cdot,d) \\
= \;& -b(u,v) - r\int_\Omega \partial_t^2(\operatorname{grad} u)\cdot \operatorname{grad} v \\
& -\beta r_b \int_0^1 \partial_t^2\partial_1 u(\cdot,0,t)\partial_1 v(\cdot,0) - \beta r_b\int_0^1 \partial_t^2 \partial_1 u(\cdot,d,t)\partial_1 v(\cdot,d) \\
& -\alpha\int_0^1 P_0(\cdot,t)v(\cdot,0) - \alpha\int_0^1 P_1(\cdot,t)v(\cdot,d)
\end{aligned}$$

for all $v \in T(\Omega)$.

Proof A direct substitution yields that $\int_\Omega \operatorname{tr}(RMV) = b_\Omega(u,v)$. □

Define a bilinear form c on $C^1(\bar{\Omega})$ by

$$c(u,v) = c_\Omega(u,v) + \beta c_0(u,v) + \beta c_1(u,v)$$

with

$$c_\Omega(u,v) = (u,v)_\Omega + r(\partial_1 u, \partial_1 v)_\Omega + r(\partial_2 u, \partial_2 v)_\Omega.$$

and

$$\begin{aligned}
c_0(u,v) &= \int_0^1 u(\cdot,0)v(\cdot,0) + r_b\int_0^1 \partial_1 u(\cdot,0)\partial_1 v(\cdot,0), \\
c_1(u,v) &= \int_0^1 u(\cdot,d)v(\cdot,d) + r_b\int_0^1 \partial_1 u(\cdot,d)\partial_1 v(\cdot,d).
\end{aligned}$$

Problem 3b: Variational formulation

Find u such that for all $t > 0$, $u(\cdot,t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot,t),v) + b(u(\cdot,t),v) = (q(\cdot,t),v)_\Omega$$

for all $v \in T(\Omega)$.

Theorem 3.3.1 *If u is a solution of Problem 3, then u is a solution of Problem 3b.*

Proof If (2.6.7) is multiplied by an arbitrary function $v \in T(\Omega)$ and integrated over Γ_0, it follows that

$$\beta \int_0^1 \partial_t^2 u(\cdot, 0, t) v(\cdot, 0) = \frac{\alpha}{r_b} \int_0^1 \partial_1 F_0(\cdot, t) v(\cdot, 0) + \alpha \int_0^1 P_0(\cdot, t) v(\cdot, 0).$$

A similar result holds on Γ_1.

Combining these results with (3.3.1) and Lemma 3.3.4 completes the proof.

□

Theorem 3.3.2 *If u is a solution of Problem 3b and $\partial_t^2 u(\cdot, t) \in C^2(\bar{\Omega})$, then u is a solution of Problem 3.*

Proof The proof follows the same pattern as in the previous cases and is tedious rather than difficult. We will write u for $u(\cdot, t)$ in this proof.

First let $v \in C_0^\infty(\Omega)$. Using integration by parts and the fact that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we see that u satisfies the partial differential equation in Problem 3.

This in turn implies that, for each $v \in T(\Omega)$,

$$\begin{aligned} &\beta c_0(\partial_t u(\cdot, 0, t), v) + \beta c_1(\partial_t u(\cdot, d, t), v) \\ &+ \int_\Omega r(\partial_t^2 \nabla^2 u) v - \int_\Omega (\partial_1^4 u + 2\partial_1^2 \partial_2^2 u + \partial_2^4 u) v \\ &= -r(\partial_t^2 \partial_1 u, \partial_1 v)_\Omega - r(\partial_t^2 \partial_2 u, \partial_2 v)_\Omega - b(u, v). \end{aligned}$$

Using integration by parts again, we have

$$r \int_\Omega (\nabla^2 \partial_t^2 u) v = -r(\partial_t^2 \partial_1 u, \partial_1 v)_\Omega - r(\partial_t^2 \partial_2 u, \partial_2 v)_\Omega + \int_{\partial\Omega} r v (\operatorname{grad} \partial_t u) \cdot \mathbf{n} \, ds.$$

Hence,

$$\begin{aligned} \beta c_0(\partial_t u(\cdot, 0, t), v) + \beta c_1(\partial_t u(\cdot, d, t), v) - \int_\Omega (\partial_1^4 u + 2\partial_1^2 \partial_2^2 u + \partial_2^4 u) v \\ = -b(u, v) - \int_{\partial\Omega} r v (\operatorname{grad} \partial_t u) \cdot \mathbf{n} \, ds. \end{aligned}$$

For convenience, define a matrix M by (2.5.4) (regardless of any physical interpretation). We have

$$\begin{aligned} b_\Omega(u,v) &= \int_\Omega \operatorname{tr}(RMV) \\ &= -\int_\Omega R\operatorname{div} M \cdot \operatorname{grad} v + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds \\ &= \int_\Omega \operatorname{div}(R\operatorname{div} M) v\, ds - \int_{\partial\Omega} (R\operatorname{div} M \cdot \mathbf{n}) v\, ds \\ &\quad + \int_{\partial\Omega} RM\mathbf{n} \cdot \operatorname{grad} v\, ds. \end{aligned}$$

But $\operatorname{div}(R\operatorname{div} M) = -(\partial_1^4 u + 2\partial_1^2\partial_2^2 u + \partial_2^4 u)$, (see (2.5.4) and (2.5.5)).

We are left with

$$\begin{aligned} \beta c_0(\partial_t u(\cdot,0,t), v) + \beta c_1(\partial_t u(\cdot,d,t), v) &= -\alpha b_0(u(\cdot,0,t),v) - \alpha b_1(u(\cdot,d,t),v) \\ &\quad - \int_{\partial\Omega} rv(\operatorname{grad}\partial_t u)\cdot \mathbf{n}\, ds \\ &\quad + \int_{\partial\Omega} RM\mathbf{n}\cdot \operatorname{grad} v\, ds \\ &\quad - \int_{\partial\Omega} (R\operatorname{div} M \cdot \mathbf{n}) v\, ds \\ &\quad \text{for each } v \in T(\Omega). \end{aligned}$$

Recall that for $v \in T(\Omega)$, $v = 0$ on $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$. All the boundary conditions are obtained by suitable choices of the values of v and $\operatorname{grad} v$ on the boundary $\partial\Omega$. As an example we consider the dynamical boundary condition on Γ_0.

Choose $v \in T(\Omega)$ such that $v = \partial_2 v = 0$ on Γ_1. Then $\partial_1 v = 0$ on Γ_1. In addition, choose $\partial_1 v = 0$ on $\overline{\Sigma}_0$ and $\overline{\Sigma}_1$ and $\partial_2 v = 0$ on Γ_0.

We are left with,

$$\begin{aligned} \beta\int_0^1 \partial_t^2 uv + \beta\int_0^1 r_b\partial_t^2\partial_1 u\partial_1 v &= -\alpha\int_0^1 \partial_1^2 u\partial_1^2 v + \int_0^1 r\partial_t^2\partial_2 uv \\ &\quad + \int_0^1 M_{22}\partial_1 v - \int_0^1 \partial_1 M_{11} v - \int_0^1 \partial_2 M_{12} v. \end{aligned}$$

Applying integration by parts twice to the term $\partial_1^2 u\partial_1^2 v$ and once to the terms $\partial_t^2\partial_1 u\partial_1 v$ and $M_{22}\partial_1 v$ gives the dynamical boundary condition on Γ_0. □

3.3.2 Weak formulation

Define the following product spaces:

$$\begin{aligned} L^2 &:= L^2(\Omega) \times L^2(I) \times L^2(I), \\ H^k &:= H^k(\Omega) \times H^k(I) \times H^k(I). \end{aligned}$$

We use the product space inner products $(\cdot,\cdot)_0$ on L^2 and $(\cdot,\cdot)_k$ on H^k defined by:

$$\begin{aligned} (u,v)_0 &= (u_1,v_1)_\Omega + (u_2,v_2)_I + (u_3,v_3)_I \\ &= \int_\Omega u_1v_1 + \int_0^1 u_2v_2 + \int_0^1 u_3v_3, \\ (u,v)_k &= (u_1,v_1)_k^\Omega + (u_2,v_2)_k^I + (u_3,v_3)_k^I \end{aligned}$$

with $(\cdot,\cdot)_k^\Omega$ and $(\cdot,\cdot)_k^I$ the standard inner products on $H^k(\Omega)$ and $H^k(I)$ respectively.

Define $\widetilde{T}_2(\Omega)$ as the closure of $T(\Omega)$ in $H^2(\Omega)$.

The trace operators γ_0 and γ_1 are defined by

$$\gamma_0 : u \to u(\cdot,0)$$

and

$$\gamma_1 : u \to u(\cdot,d)$$

for any $u \in H^1(\Omega)$. See Appendix B.

The bilinear form b can be extended to H^2

$$b(u,v) = b_\Omega(u_1,v_1) + \alpha b_0(u_2,v_2) + \alpha b_1(u_3,v_3) \text{ for all } u,v \in H^2.$$

The definition of c is extended to

$$c(u,v) = c_\Omega(u_1,v_1) + \beta c_0(u_2,v_2) + \beta c_1(u_2,v_3) \text{ for all } u,v \in H^1.$$

Define a subspace V of H^2 by

$$V = \{v \in H^2 : v_1 \in \widetilde{T}_2(\Omega), v_2 = \gamma_0 v_1, v_3 = \gamma_1 v_1\}.$$

Lemma 3.3.5 *V is a closed subspace of H^2.*

Proof If $\{v^n\}$ is a sequence in V with limit $v \in H^2$, it follows that

$$\|v_1^n - v_1\|_2^\Omega \to 0 \text{ as } n \to \infty.$$

Hence $v_1 \in \widetilde{T}_1(\Omega)$. Also,

$$\|\gamma_0 v_1^n - v_2\|_2^I \to 0 \text{ as } n \to \infty$$

and

$$\|\gamma_1 v_1^n - v_3\|_2^I \to 0 \text{ as } n \to \infty.$$

The boundedness of the trace operators yields that

$$\|\gamma_0 v_1^n - \gamma_0 v_1\|_0^I \to 0 \text{ as } n \to \infty$$

and

$$\|\gamma_1 v_1^n - \gamma_0 v_1\|_0^I \to 0 \text{ as } n \to \infty.$$

Uniqueness of limits implies that $v \in V$. □

Define $\widetilde{T}_1(\Omega)$ as the closure of $\widetilde{T}(\Omega)$ in $H^1(\Omega)$. Define the subspace W of H^1 by

$$W = \{v \in H^1 : v_1 \in \widetilde{T}_1(\omega), v_2 = \gamma_0 v_1, v_3 = \gamma_1 v_1\}.$$

The bilinear form c defines an inner product on W. The *inertia norm* induced by c on W is equivalent to the H^1 norm. If rotary inertia is ignored (i.e. $r = 0$), we again get $W = L^2$.

Lemma 3.3.6 *W is a closed subspace of H^1.*

Proof The proof is virtually identical to that of Lemma 3.3.5. □

Let $f(t) = \langle q(t), 0, 0\rangle$.

Problem 3c: Weak formulation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (f(t), v)_0 \quad \text{for all} \quad v \in V.$$

Notation

With the function $u \in C(\bar{\Omega})$ we associate functions u_1 and u^* such that

$$u_1 : [0, \tau] \to L^2(\Omega) \text{ with } u_1(t)(x) = u(x, t)$$

and

$$u^* : [0, \tau] \to L^2 \text{ with } u^*(t) = (u_1(t), (\gamma_0 u_1)(t), (\gamma_1 u_1)(t)).$$

Theorem 3.3.3 *If u is a solution of Problem 3b, then u^* is a solution of Problem 3c.*

Proof A solution of Problem 3b is in $C^2\left([0, \tau] \times \overline{\Omega}\right)$. In this case the operators γ_0 and γ_1 merely indicate the restriction of a function to Γ_0 or Γ_1. As before (see the proof of Theorem 3.1.3),

$$u_1''(t) = \partial_t^2 u(\cdot, t), \ (\gamma_0 u_1)''(t) = \gamma_0 \partial_t^2 u(\cdot, t) \text{ and } (\gamma_1 u_1)''(t) = \gamma_1 \partial_t^2 u(\cdot, t).$$

We conclude that $(u^*)''(t) = \langle \partial_t^2 u(\cdot, t), \gamma_0 \partial_t^2 u(\cdot, t), \gamma_1 \partial_t^2 u(\cdot, t) \rangle$. □

For $u_1 \in C^2([0, \tau], L^2)$ we associate a function $\tilde{u}$ such that

$$\tilde{u} : \bar{\Omega} \times [0, \tau] \to \mathbb{R} \text{ with } \tilde{u}(x, t) = \begin{cases} u_1(t)(x), & u \in \Omega \\ \gamma_0 u_1(t)(x), & x \in \Gamma_0 \\ \gamma_1 u_1(t)(x), & x \in \Gamma_1. \end{cases}$$

Theorem 3.3.4 *If u is a solution of Problem 3c and $u_1 \in C^2([0, \tau], C^2(\bar{\Omega}))$, then $\tilde{u}$ is a solution of Problem 3b.*

Proof See the proof of Theorem 3.1.4. □

3.3.3 Energy norm

Lemma 3.3.7 *The bilinear form b is positive definite on V.*

Proof We start with a Poincaré type inequality (Lemma B.2.3):

If $f \in C^1[0, a]$ and $f(0) = f(a) = 0$, then

$$\int_0^a f^2 \leq a^4 \int_0^a (f'')^2.$$

Clearly,

$$\|u(\cdot,0)\|_I \leq [b_0(u(\cdot,0),v(.,0))]^{1/2}$$

and

$$\|u(\cdot,d)\|_I \leq [b_1(u(\cdot,d),v(\cdot,d))]^{1/2}.$$

Also

$$\int_0^1 [u(\cdot,y)]^2 \leq \int_0^1 [\partial_x^2 u(\cdot,y)]^2 \text{ for each } y.$$

Evaluating the double integral over Ω, we have

$$\|u\|_\Omega^2 \leq \|\partial_x^2 u\|_\Omega^2.$$

Hence $\|u\|_\Omega \leq a_1^2 b_\Omega(u,u)^{1/2}$. It is now straightforward to derive the desired estimate. □

Lemma 3.3.8 *V is dense in L^2.*

Proof Let $f \in C^\infty(0,d)$ be such that

$$f(y) = \begin{cases} 0 & \text{for} \quad 0 \leq y < \frac{d}{3}, \\ 1 & \text{for} \quad \frac{2d}{3} < y \leq d, \end{cases}$$

let $f_n(y) = f(y^n))$ and let $g_n(y) = f_n(d-y)$ for $0 \leq y \leq d$.

Let $w = \langle w_1, w_2, w_3 \rangle \in L^2$. Then there exists a sequence of functions $\{p_n\}$ in $C_0^\infty(\Omega)$ such that

$$\|p_n - w_1\|_\Omega \to 0 \text{ as } n \to \infty.$$

Let $v_n = p_n + w_2 f_n + w_3 g_n$. Then $v_n \in T(\Omega)$ and $y_n = \langle v_n, \gamma_0 v_n, \gamma_1 v_n \rangle \in V$. Now,

$$\|v_n - w_1\|_\Omega \leq \|p_n - w_1\|_\Omega + \|w_2 f_n\|_\Omega + \|w_3 g_n\|_\Omega \to 0 \text{ as } n \to \infty.$$

We also have $\gamma_0 v_n = w_2$ for each n and $\gamma_1 v_n = w_3$ for each n. Hence $\|y_n - w\|_0 \to 0$ as $n \to \infty$. □

Lemma 3.3.9 *V is dense in W.*

Proof Consider any $w \in W$. From the definitions of $\widetilde{T}_1(I)$ and $\widetilde{T}_2(I)$ it is clear that there exists a sequence $\{p_n\} \subset \widetilde{T}_2(I)$ such that $\|w_1 - p_n\|_1^I \to 0$ as $n \to \infty$. The rest of the proof is the same as the proof of Lemma 3.3.8. □

Lemma 3.3.10 *A bounded subset of V is precompact in W and a bounded subset of W is precompact in L^2.*

Proof $V \subset H^2 = H^2(\Omega) \times H^2(I) \times H^2(I)$ and $W \subset H^1 = H^1(\Omega) \times H^1(I) \times H^1(I)$. The Rellich imbedding theorem (See [Fr, [p 31-32]) yields that bounded sequences in $H^2(\Omega)$ and $H^2(I)$ have convergent subsequences in $H^1(\Omega)$ and $H^1(I)$ respectively. The result follows from Lemma 3.1.6.

The proof of the second part is the same. □

3.4 Abstract differential equation for model problems

All our model problems have now been written in the same weak form. We have two Hilbert spaces V and W with inner products b and c respectively. It will be more convenient to denote $c(\cdot,\cdot)$ by $(\cdot,\cdot)$ and we will reserve the notation $\|\cdot\|$ for the associated norm which is called the *inertia norm*. Recall that the norm associated with the inner product b is called the *energy norm* and denoted by $\|\cdot\|_E$. The inner product in L^2 is denoted by $(\cdot,\cdot)_0$ and the associated norm by $\|\cdot\|_0$.

The properties of the spaces V, W and L^2 are of critical importance in the theory. For convenience we present a summary:

Space	Inner product	Norm
Energy space V	$b(\cdot,\cdot)$	Energy norm $\|\cdot\|_E$
Inertia space W	$c(\cdot,\cdot)=(\cdot,\cdot)$	Inertia norm $\|\cdot\|$
L^2	$(\cdot,\cdot)_0$	$\|\cdot\|_0$

The energy norm is equivalent to the norm of H^2 on V. The inertia norm is equivalent to the norm of H^1 on W if rotary inertia is included.

Estimates

There exist constants C_E and C_I such that

$$\begin{aligned}\|u\|_E &\geq C_E\|u\| \text{ for all } u\in V,\\ \|u\| &\geq C_I\|u\|_0 \text{ for all } u\in W.\end{aligned}$$

Topological properties

V is dense in W with respect to the inertia norm. These spaces are also dense in the underlying Hilbert space L^2.
If a subset of V is bounded with respect to the energy norm $\|\cdot\|_E$, it is precompact with respect to the inertia norm $\|\cdot\|$, and if it is bounded with respect to the inertia norm $\|\cdot\|$, it is precompact with respect to the L^2 norm $\|\cdot\|_0$.

We consider the following problem:

Problem A (Vibration problem)

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ *such that, for all* $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ *and*

$$
\begin{aligned}
(u''(t), v) + a(u'(t), v) + b(u(t), v) &= (f(t), v) \text{ for all } v \in V, \\
u(0) = \alpha,\ u'(0) &= \beta.
\end{aligned}
$$

Remarks

1. In general the damping term a is a non-negative bounded bilinear form on V. In our model problems, we have that

   $$a(u,v) = \mu b(u,v) + k(u,v)_0$$

   and k or μ or both can be zero.

2. For the model problems in Section 3.1 to Section 3.3 the forcing term is $(f(t), v)_0$. In the cases where $W \neq L^2$, it is proved in Lemma 3.4.1 that there exists a function $\tilde{f} : [0,\tau] \to W$ with

   $$(\tilde{f}(t), v) = (f(t), v)_0 \text{ for all } v \in W.$$

   We use the notation f for $\tilde{f}$.

The following results are special cases of the Lax-Milgram Lemma. See [Fr, p 41].

Lemma 3.4.1 *For each* $y \in L^2$ *there exists a unique* $w \in W$ *such that*

$$(w, v) = (y, v)_0 \text{ for all } v \in W.$$

Proof The Riesz Theorem yields that, for any F in the dual of W, there exists a unique $w \in W$ such that

$$(w, v) = F(v) \text{ for all } v \in W.$$

Now define F by $F(v) = (y, v)_0$ for all $v \in W$, then F is a continuous linear functional on W. □

Lemma 3.4.2 *For each $y \in W$ there exists a unique $u \in V$ such that*

$$b(u,v) = (y,v) \text{ for all } v \in V.$$

Proof The proof is exactly the same as that of Lemma 3.4.1. □

It is easy to show that Problem A can have at most one solution for given $u(0)$ and $u'(0)$. If w is a solution of the associated homogeneous problem (i.e. $f(t) = 0$ for $t > 0$, $w(0) = w'(0) = 0$), it follows that

$$(w''(t), v) + a(w'(t), v) + b(w(t), v) = 0 \text{ for all } v \in V.$$

Thus

$$(w''(t), w'(t)) + a(w'(t), w'(t)) + b(w(t), w'(t)) = 0 \text{ for all } t > 0.$$

This means that

$$\frac{d}{dt}\left(\|w'(t)\|^2 + \|w(t)\|_E^2\right) = -2a(w'(t), w'(t)) \leq 0 \text{ for all } t > 0.$$

We conclude that $w(t) = 0$ for all $t > 0$, as $w(0) = 0$ and the uniqueness of solutions for Problem A follows.

Even though Problem A is a typical weak formulation of a vibration model, we were unable to find any directly applicable existence result. Available existence results which follow from standard semigroup theory are all formulated for abstract differential equations. This means that operators associated with the bilinear forms a and b have to be constructed. We present this construction as it will also be used for the analysis of the eigenvalue problem.

Define an operator

$$\Lambda : W \to V \text{ by } b(\Lambda f, v) = (f, v) \text{ for all } v \in V.$$

Lemma 3.4.3 *The operator Λ is a bounded linear operator with trivial null-space and range $R(\Lambda)$ dense in V.*

Proof For any $f \in W$,

$$\|\Lambda f\|_E^2 = b(\Lambda f, \Lambda f) = (f, \Lambda f) \leq \|f\|\|\Lambda f\| \leq C_E^{-1}\|f\|\|\Lambda f\|_E.$$

This implies that Λ is bounded.

Also, if $\Lambda f = 0$, it follows that $0 = b(\Lambda f, v) = (f, v)$ for all $v \in V$. As V is dense in W, this implies that $(f, f) = 0$ and that Λ has a trivial nullspace.

Suppose that the closure $\overline{R}$ of $R(\Lambda)$ with respect to the energy norm is not equal to V. Then there exists a $y \in V$, $y \neq 0$ such that $b(v, y) = 0$ for all $v \in \overline{R}$. As $y \in V$, $(y, y) = b(\Lambda y, y) = 0$ and hence $y = 0$, which is a contradiction. □

Define an operator

$$A : D(A) = R(\Lambda) \subset V \to W \text{ by } A = -\Lambda^{-1}.$$

The idea for the construction of the operator A is due to Lax and Milgram [LM].

Corollary 3.4.1 *The operator A is a closed densely defined symmetric operator with range $R(A) = W$ and*

$$b(u, v) = -(Au, v) \text{ for all } u \in D(A) \text{ and } v \in V.$$

For the damping term a, we can define an associated operator in a similar way. From the model problems in Section 3.1 and Section 3.2 it is clear that there are two special cases to consider.

In the case of Kelvin-Voigt damping, a is positive definite with respect to the energy norm, i.e. there exists a constant c such that

$$a(u, u) \geq c\, b(u, u) \text{ for all } u \in V.$$

Then an operator J can be constructed in exactly the same way as A with $R(J) = W$ and

$$a(u, v) = -(Ju, v) \text{ for all } u \in D(J) \text{ and } v \in V.$$

In this case J will be a closed densely defined symmetric linear operator.

In the case of viscous damping, a is bounded in the inertia norm. The bilinear form a can then be extended to W. From the Riesz Theorem (see Lemma 3.4.1) follows that for any $u \in W$ there exists a unique $w \in W$ with

$$a(u, v) = (w, v) \text{ for all } v \in W.$$

Choose $Ju = -w$. Then

$$a(u, v) = -(Ju, v) \text{ for all } u \in D(J) \text{ and } v \in W.$$

In terms of the operators A and J, the weak formulation of the problem can be represented as follows:

Initial value problem for second order abstract differential equation

Find $u \in C^2((0,\tau), L^2) \cap C^1([0,\tau), L^2)$ such that, for all $t > 0$, $u(t) \in D(A)$, $u'(t) \in D(J)$ and

$$
\begin{aligned}
u''(t) - Ju'(t) - Au(t) &= f(t), \\
u(0) = \alpha,\ u'(0) &= \beta.
\end{aligned}
$$

In a report [VVZ], existence results for Problem A are proved. Some restrictions had to be placed on the initial conditions. A general existence result is proved in [BI] using results of [P] or [Sh]. In the latter references one may also find existence results. See [Sh, Section VI.2, Theorems 2A, 2B and 2C] and also [K, Section III.1, Theorem 1.3]. However, the transition from the abstract existence result to a concrete example is far from trivial. See for instance the treatment of the wave equation in [P, Section 7.4]. We consider this topic to be beyond the scope of this thesis.

As a final remark on the dynamic problem we mention that general existence results do not include satisfactory regularity results. The regularity of the solution is of great importance in convergence theory. (See Section 5.3.) For the one-dimensional wave equation it is a fact that the regularity of the solution depends on the regularity of the initial conditions. This fact is clear from either D'Alemberts method or a Fourier series solution. (See [W].) For two-dimensional problems the shape of the physical domain is also a factor. In Section 5.3 we will allow for different possibilities as far as regularity is concerned.

The following equilibrium problem is associated with Problem A. The solvability of this problem follows from Lemmas 3.4.1 and 3.4.2.

Problem B (Equilibrium problem)

For $f \in L^2$, find $u \in V$ such that, $b(u,v) = (f,v)_0$ for all $v \in V$.

3.5 Eigenvalue problem

Consider the undamped homogeneous problem associated with Problem A:

$$(u''(t), v) + b(u(t), v) = 0 \text{ for all } v \in V. \tag{3.5.1}$$

Applying separation of variables to (3.5.1) (i.e. assuming that $u(t) = \phi(t)w$ with ϕ a real valued function and $w \in V$), yields two problems, namely an eigenvalue problem and an ordinary differential equation.

Problem C (Eigenvalue problem)

Find a complex number λ and $w \in V$, $w \neq 0$, such that

$$b(w, v) = \lambda(w, v) \text{ for all } v \in V. \tag{3.5.2}$$

The differential equation is

$$\phi'' + \lambda\phi = 0.$$

The function u is a solution of (3.5.1) if and only if w is an eigenvector of b, λ is a corresponding eigenvalue and ϕ is a solution of the differential equation.

The constant λ is called an eigenvalue of b and the subspace of solutions w is called the eigenspace E_λ of b corresponding to λ. The elements of E_λ are called eigenvectors. (Recall the fact that b is symmetric.)

$\sqrt{\lambda}$ is called a natural frequency and w a natural mode of vibration. (We will prove that all the eigenvalues are positive.)

Theorem 3.5.1 *Let Λ be the operator defined in Section* 3.4.

1. *λ is an eigenvalue of b if and only if λ^{-1} is an eigenvalue of Λ. The eigenspace of b corresponding to λ is the same as the eigenspace of Λ corresponding to λ^{-1}.*

2. *All the eigenvalues of b are positive.*

3. *Suppose λ and μ are eigenvalues of b with $\lambda \neq \mu$. If $w \in E_\lambda$ and $u \in E_\mu$, then*

$$b(u, w) = (u, w) = 0.$$

Proof

1. Note that b cannot have a zero eigenvalue. Zero is also not an eigenvalue of Λ as the nullspace of Λ is trivial. The definition of Λ implies that

$$b(w,v) = \lambda(w,v) \text{ for all } v \in V$$

if and only if

$$\Lambda w = \lambda^{-1} w.$$

2. The operator Λ is symmetric since b is symmetric. It is well known that the eigenvalues of a symmetric operator are real. Finally, $\lambda > 0$, since $b(w,w) > 0$ and $(w,w) > 0$.

3. From $\lambda(w,u) = b(w,u) = \mu(w,u)$, it follows that $(\lambda - \mu)(w,u) = 0$. As $\lambda \neq \mu$, this yields that $(w,u) = 0$ and as a consequence, $b(w,u) = 0$. □

Lemma 3.5.1 *The operator Λ from W to W is compact.*

Proof The operator Λ maps a bounded subset of W onto a bounded subset of V. But this set is precompact in W. □

Theorem 3.5.2

1. *The set of eigenvalues of b is countable.*

2. *If the sequence of eigenvalues are ordered as a non-decreasing sequence $\lambda_1, \lambda_2, \ldots,$ then $\lambda_n \to \infty$ if $n \to \infty$.*

3. *E_λ is finite dimensional for each λ.*

Proof Due to Theorem 3.5.1, it is sufficient to consider the operator Λ. Since Λ is compact, we have immediately the facts that the eigenvalues are at most countable and the finite dimensionality of the eigenspaces. (See any text on Functional Analysis, for example [Kr, Section 8.3].) For a symmetric compact operator on a Hilbert space we have more: There exists an orthonormal sequence of eigenvectors for which the corresponding sequence of eigenvalues converge to zero. See [Sh, Theorem 7c] or [Ze, Theorem 4A].

Since the Hilbert space V is not finite dimensional, it follows that there must be an infinite number of different eigenvalues. □

Remark The dimension of the eigenspace E_λ is called the multiplicity of the eigenvalue λ.

Definition 3.5.1 *Rayleigh quotient*

The Rayleigh quotient R is defined as

$$R(v) = \frac{b(v,v)}{(v,v)} = \frac{\|v\|_E^2}{\|v\|^2}.$$

The eigenvalues can be characterised in terms of the Rayleigh quotient.

Theorem 3.5.3 *The smallest eigenvalue λ_1, is given by*

$$\lambda_1 = \min\{R(v) : v \in V\}.$$

Proof See [SF, p 220]. □

Remarks

1. Theorem 3.5.3 may be used to order the eigenvalues of b. If u is an eigenvector corresponding to λ_1, we consider the orthogonal complement of u in V, which is again a Hilbert space.
2. For the eigenvalue problem (3.5.2), various bounds for the eigenvector w, associated with an eigenvalue λ, can be obtained. Clearly, in general,
$$\|w\|_E = \lambda^{1/2}\|w\|. \tag{3.5.3}$$
3. We may also consider eigenvalue problems for other bilinear forms, for example
$$(u,v) = c(u,v) = \lambda(u,v)_0.$$
Exactly the same results will be true—this time in the Hilbert space W.

Regularity For beam problems where rotary inertia is ignored (i.e. $W = L^2$), the eigenfunction satisfies the differential equation

$$w^{(4)} = \lambda w.$$

As a consequence we have

$$\|w^{(4)}\|_0 = \lambda\|w\|_0, \tag{3.5.4}$$

Also, from the differential equation,

$$w^{(6)} = \lambda w''.$$

Since the energy norm is equivalent to the H^2-norm, there exists a constant C_b such that

$$\|w^{(6)}\|_0 \leq C_b\lambda^{3/2}\|w\|_0. \tag{3.5.5}$$

For the case where rotary inertia is included,

$$w^{(4)} = \lambda(rw'' + w).$$

In this case it follows that

$$\|w^{(4)}\|_0 = r\lambda\|w''\|_0 + \lambda\|w\|_0 \leq C_b\lambda^{3/2}\|w\|, \tag{3.5.6}$$

if $\lambda > 1$.

Remark For the plate beam problem we do not know if the eigenvectors are in H^k for $k > 2$.

Chapter 4

Discretization

4.1 Galerkin approximation

In Chapter 3 we showed that all the model problems lead to three typical abstract problems. In this section we will formulate a Galerkin approximation for the general vibration problem, Problem A, as well as for the associated equilibrium problem, Problem B, and the eigenvalue problem, Problem C.

To formulate these Galerkin approximations, it is necessary to choose a finite dimensional subspace S^h of V. (At this stage the symbol h is used only to indicate that we are considering approximation in a finite dimensional space.)

Galerkin approximation for the vibration problem

Problem AG

Find $u_h \in C^2((0,\infty), S^h)$, such that for all $t > 0$,

$$\begin{aligned} (u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) &= (f(t), v)_0 \text{ for all } v \in S^h, \\ u_h(0) = \alpha_h,\ u_h'(0) &= \beta_h. \end{aligned}$$

The initial conditions α_h and β_h are approximations in S^h for α and β.

The Galerkin approximation yields a semi-discrete problem which can be written as a system of ordinary differential equations. Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be a basis for S^h. Then there exist functions $u_i(t)$ such that

$$u_h(t) = \sum_{i=1}^{n} u_i(t)\phi_i.$$

Let $\bar{u}$ be a function with values in $\mathbb{R}_n$ such that the vector $\bar{u}(t)$ has components $u_i(t)$.

Problem AD

Find $\bar{u} \in C^2([0,\infty), \mathbb{R}_n)$, such that

$$\begin{aligned} M\bar{u}''(t) + L\bar{u}'(t) + K\bar{u}(t) &= \bar{f}(t),\ t > 0, \\ \bar{u}(0) = \bar{\alpha},\ \bar{u}'(0) &= \bar{\beta}, \end{aligned}$$

with $\bar{\alpha}$ and $\bar{\beta}$ the coefficients of α_h and β_h.

The matrices K, L, M and M_0 are defined as follows

$$K_{ij} = b(\phi_i, \phi_j),\quad L_{ij} = a(\phi_i, \phi_j),\quad M_{ij} = (\phi_i, \phi_j) \text{ and } [M_0]_{ij} = (\phi_i, \phi_j)_0.$$

The vector $\bar{f}(t)$ has components $(f(t), \phi_i)_0$.

It is easy to see that Problem AG is equivalent to Problem AD. For instance, if $v = \sum_1^n v_i\phi_i$, then $(u_h''(t), v) = M\bar{u}''(t) \cdot \bar{v} = \bar{v}^T M\bar{u}''(t)$.

Remark Problem *AD* is an initial value problem for a system of differential equations. It will have a unique solution if $\bar{f}$ is continuous, but the differentiability properties of $\bar{u}$ (and hence u_h) will depend on the differentiability properties of $\bar{f}$.

Galerkin approximation for the equilibrium problem

Problem BG

Find $u_h \in S^h$, such that $b(u_h, v) = (f, v)_0$ for all $v \in S^h$.

Since u_h and v are linear combinations of $\{\phi_1, \phi_2, \ldots, \phi_n\}$, the Galerkin approximation reduces to the system of linear equations:

Problem BD

Find $\bar{u} \in \mathbb{R}_n$, such that $K\bar{u} = \overline{F}$, where $F_i = (f, \phi_i)_0$.

Galerkin approximation for the eigenvalue problem

Problem CG

Find $w_h \in S^h$, $w_h \neq 0$, and a complex number λ^h, such that

$$b(w_h, v) = \lambda^h(w_h, v) \text{ for all } v \in S^h.$$

Since u_h and v are linear combinations of $\{\phi_1,\ \phi_2,\ \ldots,\ \phi_n\}$, the Galerkin approximation reduces to the generalised eigenvalue problem:

Problem CD

Find $\bar{w} \in \mathbb{R}_n$, $\bar{w} \neq 0$, and a complex number λ, such that

$$K\bar{w} = \lambda M\bar{w}.$$

The vector $\bar{w}$ has components w_i where $w_h = \sum_{i=1}^{n} w_i\phi_i$.

More will be said concerning the computation of these matrices in Section 4.4 and Chapters 5 and 6.

4.2 Finite dimensional subspaces. Beam problems

4.2.1 Hermite cubics and Hermite quintics

The well-known Hermite piecewise cubics (see for instance [SF] or [Re]) are used successfully as basis functions for the Galerkin approximation in beam problems. Although cubics are sufficiently accurate for beam problems, we also use Hermite piecewise quintics. The main reason is that cubics will not be compatible with reduced quintics in plate beam models. As a bonus we find that quintics are extremely efficient. (See Chapter 5.)

The interval $I = [a, b]$ is divided into subintervals by nodes x_i, $i = 0, 1, \ldots, n$, with

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Consequently we have elements $\Omega_i = [x_{i-1}, x_i]$ of length h_i.

We proceed to define Hermite piecewise quintic polynomials. For $k = 0,\ 1,\ 2$ and for each element Ω_i there exist six quintic polynomials $\psi_{i-1,i}^{(k)}$ and $\psi_{i,i}^{(k)}$ with the following properties.

For $j = i - 1$ or i:

$$\text{If } m \neq j: \quad \psi_{ji}^{(k)}(x_m) = D\psi_{ji}^{(k)}(x_m) = D^2\psi_{ji}^{(k)} = 0.$$

$$\text{If } m = j: \quad D^\ell\psi_{ji}^{(k)}(x_m) = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k. \end{cases}$$

Next these polynomials are "pieced" together. The basis function $\phi_i^{(k)}$ is defined by

$$\phi_i^{(k)} = \begin{cases} \psi_{i,i+1}^{(k)} & \text{on } \Omega_{i+1}, \\ \psi_{i,i}^{(k)} & \text{on } \Omega_i, \\ 0 & \text{elsewhere.} \end{cases}$$

For piecewise Hermite cubics the construction is virtually the same. The only difference is that $k = 0,\ 1$.

Instead of piecewise Hermite cubics and piecewise Hermite quintics we will refer to cubics and quintics. Note that the cubics are elements of $H^2(a,b)$ and the quintics elements of $H^3(a,b)$.

Definition 4.2.1 *Interpolation operator.*

Let $r = 1$ for cubics and $r = 2$ for quintics. Then we define

$$\Pi u = \sum_{k=0}^{r}\sum_{i=1}^{n} D^k u(x_i)\phi_i^{(k)}.$$

Remark

1. For cubics it is necessary that $u \in H^2(I)$, for then $u \in C^1(\bar{I})$. (See Appendix B.) For quintics it is necessary that $u \in H^3(I)$.
2. Note that if $v = \Pi u$, then $D^k v(x_i) = D^k u(x_i)$.
3. Cubic splines are often used as basis functions. This has advantages over cubics (see [Pr]), but for reasons already mentioned we choose quintics as an alternative to cubics.

4.2.2 The damaged beam

The variational form of the damaged beam is derived in Section 3.1. In this section we construct a finite dimensional subspace for the Galerkin approximation. The interval $I = [0,1]$ is divided in such a way that $x = \alpha$ (location of damage) coincides with an interior node x_p. This means that $I_1 = (0, x_p)$ and $I_2 = (x_p, 1)$.

We construct a finite dimensional subspace of the space V by specifying a basis. The basis elements must be pairs of functions. Suppose $\phi_i^{(k)}$ is a quintic or cubic. Let $\phi_{i1}^{(k)}$ denote the restriction of $\phi_i^{(k)}$ to $I_1 = [0,\alpha]$ and $\phi_{i2}^{(k)}$ the restriction of $\phi_i^{(k)}$ to $I_2 = [\alpha, 1]$. We then define the basis elements $\tilde{\phi}_i^{(k)}$ by $\tilde{\phi}_i^{(k)} = \langle \phi_{i1}^{(k)}, \phi_{i2}^{(k)} \rangle$. These elements are in H^2 since $\phi_{i1}^{(k)} \in H^2(I_1)$ and $\phi_{i2}^{(k)} \in H^2(I_2)$. $\phi_p^{(1)}$ is an exception and may not be used. Instead we have $\tilde{\phi}_{pL}^{(1)} = \langle \phi_{p1}^{(1)}, 0 \rangle$ and $\tilde{\phi}_{pR}^{(1)} = \langle 0, \phi_{p2}^{(1)} \rangle$.

Consequently, $\tilde{\phi}_i^{(k)} \in V$ if $i \neq 0$. Also $\tilde{\phi}_{pL}^{(k)}$ and $\tilde{\phi}_{pR}^{(k)} \in V$. (They all satisfy the forced boundary conditions. For $k = 0$ and $k = 1$, $\tilde{\phi}_0^{(k)}$ are not admissible on account of the forced boundary conditions at $x = 0$.)

Let $h = \max h_i$. The finite dimensional subspace S^h is chosen as the span of all the admissible basis elements. Although the space S^h is determined by the partition of the interval, and not h, the notation S^h is commonly used. Note that S^h is a subspace of V, since the basis of S^h consists of elements of V.

Next we define an interpolation operator Π on H^k. (For any $u \in H^k$, $u_1 \in H^k(I_1)$ and $u_2 \in H^k(I_2)$.) We use the usual interpolation operators for these spaces (defined in the previous subsection) and denote them by Π_1 and Π_2:

Definition 4.2.2 *Interpolation operator*

$$\Pi u := \langle \Pi_1 u_1, \Pi_2 u_2 \rangle \quad \textit{for each} \quad u \in H^k.$$

$k = 3$ *for quintics and* $k = 2$ *for cubics.*

Note that $\Pi u \in V$ for all $u \in V$ as

$$(\Pi_i u_i)(x_j) = u_i(x_j) \quad \text{and} \quad (\Pi_i u_i)'(x_j) = u_i'(x_j).$$

4.2.3 Beam with dynamical boundary conditions

The construction of a finite dimensional subspace is simpler in this case. Suppose $\phi_i^{(k)}$ is either a cubic or a quintic. The basis elements for S^h are

$$\tilde{\phi}_i^{(k)} = \langle \phi_i^{(k)}, \phi_i^{(k)}(1), (\phi_i^{(k)})'(1) \rangle.$$

Note that $\phi_0^{(0)}$ and $\phi_0^{(1)}$ are excluded. Clearly $\tilde{\phi}_i^{(k)} \in V$ for each i and each k.

Definition 4.2.3 *Interpolation operator*

Let Π_1 *denote the usual interpolation operator.*

$$\Pi u := \langle \Pi_1 u_1, (\Pi_1 u_1)(1), (\Pi_1 u_1)'(1) \rangle \textit{ for } u \in H^k.$$

$k = 3$ *for quintics and* $k = 2$ *for cubics.*

Note that $(\Pi_1 u_1)(x_i) = u_1(x_i)$ and $(\Pi_1 u_1)'(x_i) = u_1'(x_i)$. Again, it is easy to see that $\Pi u \in V$ if $u \in V$.

4.3 Finite dimensional subspaces. Plate problems

4.3.1 Reduced quintics

We use only reduced quintics developed by Cowper *et al.* (See for instance [SF], [CKLO1], [CKLO2] and [CKLO3].) These basis functions are highly accurate for the Galerkin approximation in plate problems.

The rectangle Ω is divided into triangles or elements Ω_i. With each node we associate six basis functions. To define these functions the following notation for derivatives is convenient:

$$\begin{aligned} \partial^{(0)}u &= u, & \partial^{(3)}u &= \partial_1^2 u, \\ \partial^{(1)}u &= \partial_1 u, & \partial^{(4)}u &= \partial_1\partial_2 u, \\ \partial^{(2)}u &= \partial_2 u, & \partial^{(5)}u &= \partial_2^2 u. \end{aligned}$$

For each node $\mathbf{x}_j$ and for each element Ω_i (with $\mathbf{x}_j$ as a vertex) there exist six reduced quintics $\psi_{ji}^{(k)}$ for $k = 0,\ 1, \ldots,\ 5$ with the following properties:

$$\text{For } j = m: \qquad \partial^{(\ell)}\psi_{ji}^{(k)}(\mathbf{x}_m) = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k. \end{cases}$$

$$\text{For } j \neq m: \qquad \partial^{(\ell)}\psi_{ji}^{(k)}(\mathbf{x}_m) = 0.$$

Next these polynomials are "pieced" together. We define the basis function $\phi_j^{(k)}$ by:

> The restriction of $\phi_j^{(k)}$ to element Ω_i is $\psi_{ji}^{(k)}$ if $\mathbf{x}_j$ is a vertex of Ω_i.
> If $\mathbf{x}_j$ is not a vertex of Ω_i then $\phi_j^{(k)} = 0$ on Ω_i.

The piecewise polynomial functions $\phi_j^{(k)}$ are continuous and have continuous partial derivatives. The second order partial derivatives are not continuous. However, these functions are elements of $H^2(\Omega)$.

Definition 4.3.1 *Interpolation operator* Π_Ω

$$\Pi_\Omega u := \sum_{j=1}^{n}\sum_{k=0}^{5} \partial^{(k)}u(x_j)\phi_j^{(k)}.$$

Consequently, if $v = \Pi_\Omega u$, then $\partial^{(k)}v(x_j) = \partial^{(k)}u(x_j)$.

4.3.2 Plate beam model

The elements of S^h must be ordered triples of the form $\langle u, \gamma_0 u, \gamma_1 u\rangle$. The basis elements for S^h are

$$\tilde{\phi}_i^{(k)} = \langle \phi_i^{(k)}, \gamma_0\phi_i^{(k)}, \gamma_1\phi_i^{(k)}\rangle.$$

Note that some basis functions must be excluded to satisfy the forced boundary conditions. Clearly $\tilde{\phi}_i^{(k)} \in V$ for each i and each k. The finite dimensional space S^h is the span of all the admissible basis functions.

Definition 4.3.2 *Interpolation operator*

$$\Pi u := \langle \Pi_\Omega u_1, \gamma_0\left(\Pi_\Omega u_1\right), \gamma_1\left(\Pi_\Omega u_1\right)\rangle \textit{ for } u \in H^2.$$

Remark Let Π_Γ denote the interpolation operator defined for quintics in Section 4.2.1. Note that

$$\gamma_0\left(\Pi_\Omega u\right) = \Pi_\Gamma(\gamma_0 u).$$

The same is true for γ_1.

4.4 Implementation

We now reconsider the three types of problems posed in Section 4.1. Having defined finite dimensional subspaces, the matrices K, L, M and M_0 are also defined.

The equilibrium problem, Problem BD, is trivial and needs no further discussion.

The eigenvalue problem, Problem CD, is a generalized eigenvalue problem. For the different model problems, this is solved, after the matrices have been computed, using standard Matlab subroutines. In the one-dimensional case, this is a straightforward procedure. (Chapter 6.) The two-dimensional case is more interesting because of the presence of repeated eigenvalues and the irregular pattern in which they occur. As the multiplicity of eigenvalues for the abstract eigenvalue problem (Problem B) and the Galerkin approximation (Problem BG) do not correspond, it can be difficult to interpret the numerical results. We will elaborate on this in Chapter 7.

The vibration problem, Problem AD, is an initial value problem for a second order system of ordinary differential equations

$$\begin{aligned} M\bar{u}''(t) + L\bar{u}'(t) + K\bar{u}(t) &= \bar{f}(t), \\ \bar{u}(0) = \bar{\alpha},\ \bar{u}'(0) &= \bar{\beta}, \end{aligned}$$

which is to be solved on an interval $[0, \tau]$. The initial conditions, $\bar{\alpha}$ and $\bar{\beta}$, depend on the choice of approximations for α and β. One possibility is to use $\alpha_h = \Pi\alpha$ and $\beta_h = \Pi\beta$. Another possibility is to use projections—to be defined in Section 4.6. This problem is solved using a finite difference method. The interval $[0, \tau]$ is partitioned into subintervals of length δt and $t_k = k\delta t$.

Let $\bar{u}_k$ denote the approximation for $\bar{u}(t_k)$. We use the following scheme:

$$\begin{aligned} &(\delta t)^{-2}M(\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}) \\ &\quad +(2\delta t)^{-1}L(\bar{u}_{k+1} - \bar{u}_{k-1}) \\ &+K(\rho_1\bar{u}_{k+1} + \rho_0\bar{u}_k + \rho_1\bar{u}_{k-1}) = \rho_1\bar{f}_{k+1} + \rho_0\bar{f}_k + \rho_1\bar{f}_{k-1}. \end{aligned} \quad (4.4.1)$$

More detail about the weights, ρ_0 and ρ_1, will be given in Section 5.4 and Chapter 6. The scheme is started with initial conditions $\bar{u}_0 = \bar{\alpha}$ and $(2\delta t)^{-1}(\bar{u}_1 - \bar{u}_{-1}) = \bar{\beta}$.

It is clear that as far as implementation is concerned, the real challenge is the computation of the matrices K, L and M. This is complicated by the interface conditions which result in non-standard basis elements. We write our own code to assemble the matrices.

As an example we illustrate the computation for the simplest case, namely the beam with tip body. The dimensionless model is given in Section 2.4.

The interval $I = [0, 1]$ is divided into subintervals by nodes x_i, $i = 0, 1, \ldots, n$, with

$$0 = x_0 < x_1 < \cdots < x_n = 1.$$

Consequently, we have elements $\Omega_i = [x_{i-1}, x_i]$ of length h_i. Suppose we use Hermite piecewise cubics defined in Section 4.2 and the basis elements are ordered as follows:

$$\tilde{\phi}_i = \begin{cases} \tilde{\phi}_i^{(0)} & \text{for } i = 1,\ 2,\ \ldots,\ n, \\ \tilde{\phi}_{i-n}^{(1)} & \text{for } i = n+1,\ n+2,\ \ldots,\ 2n. \end{cases}$$

First consider the computations for an undamaged beam. We denote the standard cubics by ϕ_i. Also, the matrices M^A, M_1 and K are the usual matrices used for an undamaged beam:

$$M_{ij}^A = \int_0^1 \phi_i \phi_j, \quad (M_1)_{ij} = \int_0^1 \phi_1' \phi_j' \quad \text{and} \quad K_{ij}^A = \int_0^1 \phi_i'' \phi_j''.$$

Next we show how to adapt these matrices for the beam with tip body. (The bilinear forms are defined in Section 3.2 and the matrices in Section 4.1.)

The K-matrix does not change since

$$K_{ij} \;=\; b(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \phi_i'' \phi_j''.$$

Since $\mu_0 = \mu_1 = 0$,

$$L_{ij} \;=\; a(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \left(\mu \phi_i'' \phi_j'' + k \phi_i \phi_j \right)$$

which yields that $L = \mu K + k M_A$. Lastly,

$$\begin{aligned} M_{ij} &= c(\tilde{\phi}_i, \tilde{\phi}_j) = \int_0^1 \left(\phi_i \phi_j + r \phi_i' \phi_j' \right) + m \phi_i(1) \phi_j(1) + I_m \phi_i'(1) \phi_j'(1) \\ &= M_{ij}^A + r \left[M_1 \right]_{ij} + m \phi_i(1) \phi_j(1) + I_m \phi_i'(1) \phi_j'(1). \end{aligned}$$

Thus, $M = M_A + r M_1 + M^*$ where M^* is the $2n \times 2n$ zero matrix, except for two non-zero entries namely $M_{n,n}^*$ and $M_{2n,2n}^*$.

4.5 Interpolation error

4.5.1 Standard estimates

In this subsection we quote standard interpolation estimates, as found in, for instance, [SF], [OR] and [OC]. The standard Sobolev spaces $H^k(I)$ and $H^k(\Omega)$ are used and $\|\cdot\|$ denotes the $L^2(I)$ or $L^2(\Omega)$ norm.

We introduce two parameters for an interpolation operator Π:
$r(\Pi)$ is the highest degree of polynomials left invariant by the operator Π,
$s(\Pi)$ is the highest order derivative used in the definition of Π.

We will use $\widehat{C}$ to denote a generic constant which depends on the constants in Sobolev's lemma and the constants in the Bramble Hilbert lemma.

In the following result $|\cdot|_k$ denotes the seminorm of order k, i.e. $|u|_k = \|u^{(k)}\|$, on an interval I.

Lemma 4.5.1 *Suppose $s(\Pi)+1 \leq k \leq r(\Pi)+1$. Then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \ m = 0, 1, \ldots, k.$$

Notation If $u \in H^k(I)$, let k^* denote the minimum of k and $r(\Pi) + 1$.

Corollary 4.5.1 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$, with $k \geq s(\Pi) + 1$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

Proof $H^k(I) \subset H^{r(\Pi)+1}$ for $k \geq r(\Pi) + 1$. □

Remark For cubics, $k^* = 4$ if $k \geq 4$. For quintics, $k^* = 6$ if $k \geq 6$.

In the following result, for a two-dimensional domain Ω, $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

In the two-dimensional case, the constant of the estimate also depends on the shape of the elements. Care should be taken that the minimum angle of any triangle element does not become too small. See for instance [SF, p 138].

Lemma 4.5.2 *Suppose $s(\Pi)+2 \leq k \leq r(\Pi)+1$. Then there exists a constant $\widehat{C}$ such that, for $u \in H^k(\Omega)$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \ m = 0, 1, \ldots, k.$$

Remark For reduced quintics $r(\Pi) = 4$, hence $k^* = 5$ if $k \geq 5$.

Corollary 4.5.2 *There exists a constant $\widehat{C}$ such that. for all $u \in H^k(\Omega)$. with $k \geq s(\Pi) + 2$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

4.5.2 Damaged beam model

The seminorm of order k for the product space $H^k = H^k(I_1) \times H^k(I_2)$ is defined by $|u|_k^2 = |u_1|_k^2 + |u_2|_k^2$.

Lemma 4.5.3 *There exists a constant $\widehat{C}$ and an s^* such that. for all $u \in H^k$. with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

Proof The result is a direct consequence of the definition of seminorms, norms and the interpolation operator on the product space. For this interpolation operator, $s^* = 2$ and $k^* = \min\{k, 4\}$, if adapted cubic basis functions are used, and $s^* = 3$ and $k^* = \min\{k, 6\}$, if adapted quintic basis functions are used. □

4.5.3 Beam with dynamical boundary conditions

The seminorm of order k for the product space $H^k = H^k(I) \times \mathbb{R} \times \mathbb{R}$ is defined by $|u|_k^2 = |u_1|_k^2$.

Lemma 4.5.4 *There exists a constant $\widehat{C}$ and an s^* such that. for all $u \in H^k$. with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*}, \ m = 0, 1, \ldots, k^*.$$

Proof The result is a direct consequence of the definition of seminorms, norms and the interpolation operator on the product space. Also for this interpolation operator, $s^* = 2$ and $k^* = \min\{k, 4\}$ if adapted cubic basis functions are used, and $s^* = 3$ and $k^* = \min\{k, 6\}$ if adapted quintic basis functions are used. □

4.5.4 Plate beam model

The seminorm of order k on the product space $H^k = H^k(\Omega) \times H^k(I) \times H^k(I)$ is defined by $|u|_k^2 = |u_1|_k^2 + |u_2|_k^2 + |u_3|_k^2$.

Consider the interpolation operator Π defined in Section 4.3.2. In this case k^* is the minimum of k and 5.

Lemma 4.5.5 *There exists a constant $\widehat{C}$ and an s^* such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*},\ m = 0, 1, \ldots, k^*.$$

Proof Note that for any $u \in H^m$, we have that

$$\|u - \Pi u\|_m^2 = \|u_1 - \Pi_\Omega u_1\|_m^2 + \|u_2 - \Pi_\Gamma u_2\|_m^2 + \|u_3 - \Pi_\Gamma u_3\|_m^2.$$

Use the results of Corollaries 4.5.1 and 4.5.2. From Corollary 4.5.2 it follows that $s^* = 4$ as $s(\Pi) = 2$ for reduced quintics. □

4.5.5 Abstract error estimates

At this stage a unified approach is possible. We have a Hilbert space V, a finite dimensional subspace S^h, an interpolation operator Π and require an estimate for the interpolation error $u - \Pi u$.

In Subsections 4.5.2 to 4.5.4 we showed that for all the model problems we can define parameters s^* and k^* for each interpolation operator, such that the following general interpolation estimate holds.

Lemma 4.5.6 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k$, with $k \geq s^*$,*

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k^*-m} |u|_{k^*},\ m = 0, 1, \ldots, k^*.$$

As $V \subset H^2$ for all the model problems, and the energy norm is equivalent to the H^2 norm, the following interpolation estimate holds.

Corollary 4.5.3 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$, with $k \geq s^*$,*

$$\|u - \Pi u\|_E \leq \widehat{C} h^{k^*-2} |u|_{k^*}.$$

For all the model problems $s^* < 4$ and for $k = 4$ it follows that $k^* = 4$. This means that the following result applies to all the interpolation operators that we use.

Corollary 4.5.4 *There exists a constant $\widehat{C}$ such that, for all $u \in H^4 \cap V$,*

$$\|u - \Pi u\|_E \leq \widehat{C} h^2 |u|_4.$$

4.6 Approximation

We have a Hilbert space V, a finite dimensional subspace S^h, an interpolation operator Π and an estimate for the interpolation error $u - \Pi u$. We now introduce a projection of V onto the subspace S^h. This projection will feature in every convergence proof in Chapter 4. For more information on this projection, see any Functional Analysis text, for example [Kr, Section 3.3].

Definition 4.6.1 *Projection P*

P is a projection of V onto S^h with respect to the inner product b.

The definition implies that for any $x \in V$,

$$b(x - Px, v) = 0 \text{ for all } v \in S^h.$$

Due to the important role that P will play in the theory, we display the properties of this projection:

$$\|x - Px\|_E \leq \|x - v\|_E \text{ for all } v \in S^h,$$

$$\|Px - v\|_E \leq \|x - v\|_E \text{ for all } v \in S^h$$

and

$$\|Px\|_E \leq \|x\|_E.$$

Lemma 4.6.1 *There exists a constant $\widehat{C}$ such that, for any $u \in H^k \cap V$ with $k \geq s^*$,*

$$\|Pu - u\|_E \leq \widehat{C}|u|_{k^*} h^{k^*-2} \text{ and } \|\Pi u - Pu\|_E \leq \widehat{C}|u|_{k^*} h^{k^*-2}.$$

Proof

$$\|Pu - u\|_E \leq \|\Pi u - u\|_E$$

and

$$\|\Pi u - Pu\|_E \leq \|u - \Pi u\|_E.$$

Now use the results for the interpolation error in Corollary 4.5.3. □

The next result is convenient due to the fact that it applies to all the interpolation operators that we use.

Corollary 4.6.1 *There exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,*

$$\|Pu - u\|_E \leq \widehat{C}|u|_4 h^2 \quad \text{and} \quad \|\Pi u - Pu\|_E \leq \widehat{C}|u|_4 h^2.$$

Lemma 4.6.2 *For any $\varepsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that*

$$\|u - Pu\|_E < \varepsilon, \quad \text{if } h < \delta.$$

Proof For any $u \in V$ there exists a $w \in H^4 \cap V$ such that

$$\|u - w\|_E \leq \varepsilon.$$

Now,

$$\begin{aligned} \|Pu - u\|_E &\leq \|u - w\|_E + \|w - Pw\|_E + \|Pw - Pu\|_E \\ &\leq \varepsilon + \widehat{C}|w|_4 h^2 + \varepsilon \\ &< 3\varepsilon \text{ for } h \text{ sufficiently small.} \end{aligned}$$

□

In a final result we show that the Aubin-Nitsche trick, [N], can also be applied to find estimates in the inertia norm for the discretization error.

Lemma 4.6.3 *There exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$, with $k \geq s^*$,*

$$\|u - Pu\| \leq \widehat{C} h^{k^*} |u|_{k^*}.$$

Proof Set $e_p = u - Pu$. As b defines an inner product on V it follows from the Riesz theorem that there exists a unique $y \in V$ such that

$$b(y, v) = (e_p, v) \text{ for all } v \in V. \tag{4.6.1}$$

Regularity results yield that $y \in H^4 \cap V$ and that there exists a c_b such that

$$\|y\|_4 \leq c_b \|e_p\|. \tag{4.6.2}$$

Since P is a projection

$$b(e_p, v) = 0 \text{ for all } v \in S^h. \tag{4.6.3}$$

Let $v = e_p$ in (4.6.1) and $v = Py$ in (4.6.3). This yields that

$$\|e_p\|^2 = b(y - Py, e_p) \leq \|y - Py\|_E \|e_p\|_E.$$

From Corollary 4.6.1,

$$\|e_p\|^2 \leq \widehat{C}|y|_4 h^2 \|e_p\|_E.$$

We conclude from (4.6.2) that

$$\|e_p\| \leq \widehat{C}h^2 \|e_p\|_E.$$

The result now follows from Lemma 4.6.1. □

For all our model problems the following result applies.

Corollary 4.6.2 *There exists a constant $\widehat{C}$ such that, for all $u \in H^4 \cap V$,*

$$\|u - Pu\| \leq \widehat{C}h^4 |u|_4.$$