CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

Since time immemorial humanity treasured reliability as a very important human attribute. A reliable person is one who is trustworthy, dependable and consistent. It is undisputable today that industry, commerce and generally society seeks to associate most with persons considered reliable. Humanity craves for things which are consistent and predictable. Much as human beings treasure reliability in human behaviour, it is not easy to define its characteristics or to be able to measure it with precision. In practice no clear line can be drawn between a person who is reliable and one who happens no to be. However, a judgment can be made whether an individual is reliable or not on the basis of a definite human function. For instance the reliability of an individual working with an organisation may be assessed on the basis of punctually of arriving at work or in attending meetings.

Generally, reliability in a wider sense may be considered as a measure of performance. Persons who complete their tasks on schedule are described as reliable because they are able to finish their work on time. Those people who keep time i.e. they are at the right place at the right time may be considered reliable as well because they fulfill their commitments. Reliability of human beings therefore depends on time at which or in which they perform any particular task which may be taken as a measure.

Reliability does not only apply to mans actions but also the objects he uses. We have seen that reliability has been applied to mans activities but when it comes to the objects he has made or invented his expectations of reliability are even higher. This is because it does not only frustrate his/her feelings but is wastes time, costs money and endanger life. According to Green & Bourne (1978) the consequences of unreliability has led to man's greater interest in reliability and more desire to acquire or use more reliable products.

Technological developments have led to an increase in the number of complex systems in addition to the complexity of the systems themselves. Advancements in information, communications technology have made systems even more complicated. These complications have attracted a number of researchers and scientists from various disciplines especially the systems engineers, software engineers and applied probabilists. These developments have seen the emergence of reliability theory another scientific discipline dealing with methods and techniques to ensure maximum effectiveness of systems (from known qualities of their components). Gnedenko et al (1969) indicated that reliability theory assigns quantitative indices to qualities of production which are computed from the design stage through manufacturing process to use and storage of manufactured goods and operating systems. Increased reliability of manufactured goods and operating systems is a challenge to Governments, engineers and scientists. According to Lloyd and Lopow (1962) unreliability costs money, time wested and inconveniences the users, in same cases may joperdize personal and national security. The year 1963 saw the birth of the journal on reliability known as IEEE-Transactions on Reliability.

Mathematical models aid the system designer who is faced with the problems of evaluation of several measures of system performance and methods of improving them. These models describe the various operational and theoretical features of the system under consideration taking into account its essential features. Since unavailability and breakdowns of a system are becoming more and more unacceptable, the demand for systems that perform better but cost less is on the increase. It is common knowledge that repairing failed units and providing redundancy are two important methods of improving the performance of a system.

Reliability theory is multidisciplinary in nature since problem handling requires methods of probability theory and mathematical statistics such as information theory, queuing theory, linear and non-linear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, etc.. Reliability theory has been applied in contemporary medicine, reliable software systems, geoastononmy, irregularities interactions of physiological systems, spontaneous single neon discharge, phase dependence of population growth, fluctuations in business investments,etc. In addition, mathematical models relying on probability theory and stochastic processes are used in making realistic modeling for mobility of of both individuals and industrial labour, advancements in education and diffusion of information. According to Watson and Galton (1874) biological sciences stochastic models were first introduced in the study of extinction of families. This was followed by its application in population genetics, branching process, birth and death processes, recovery, relapse, cell survival after irradiation, the flow of particles through organs, etc. These analytical models have been used in the purchasing behaviour of the individual consumer, credit risk and term structure, income determination under uncertainty etc. The traffic flow studies have also used the theory of stochastic models for traffic of pedestrians, freeways, parking lots, intersections, etc.

Problems have emerged in the design of highly reliable technical systems which include: the creation classes of probability-statistical models which may be used in description of the reliability behaviour of the system, and the development mathematical methods for the assessment of the reliability characteristics of systems.

These problems encouraged studies into the development of high-accuracy methods of reliability analysis. Gnedenko et al (1969), Barlow (1984), Gertsbakh(1989) and Kovalenko et al (1997)considered redundant systems and the classical examples are the models of Markov processes with a finite set of states such as birth and death processes. Cox (1962) studied renewal process method and Cinler (1975) semi-Markov process method and its generalizations, Rubenstein (1981) generalizationed semi-Markov process (GSMP) method while Aven (1996) looked at special models for coherent systems and Ozekici (1996) and Finkelstein (1999a,b,c), Chandrasekhar et al (2005) studied systems with random environment.

Reliability is applicable in many areas in research, a suitable form of reliability form may be introduced. Stochastic analysis are based on good probability models with the ultimate aim of giving numerical estimates of reliability characteristics. Reliability offers its self by providing solutions to a number of problems not handled by the usual standard probability theoretical approach. According to Gertsbakh (1989) reliability of a system depends on the reliability of its components, provides a mathematical expression of aging process, offers well-developed methods of renewal theory, introduces redundant systems to optimize the performance of standby components Gnedenko et al (1969), provides the theory of optimal preventive maintainance and is also a study of inferential statistics often of censored data.

Reliability theory of technical objects and survival analysis of biological entities are similar with the exception of notation. Therefore the term "lifetime" is applicable to engineering systems, components, units etc. and to the disciplines like biological, financial and etc with minor modifications.

1.2 FAILURE

Gertsbakh (1989) defined failure as a result of a joint action of many unpredictable, random processes going on inside the operating system as well as in the environment in which the system its is operating. Failure is stochastic in nature and its operation gets seriously impeded or completely stopped at a certain point in time. Determination of failure may be easily detected in some cases just through observation but in other its very difficult since these units deteriorate continuously and the actual moment of failure is not as easy to determine. We assume that failure is exactly observable in this study and failure is known as a disappointment or a death. When a system fails it enters a down state which may also be called a system breakdown (Finkelstein

(1999a)). According to Zacks (1992) data is of two types: from continuously monitored units for failure and from observations of failure made at discrete points in time.

Villemeur (1992) cited a number of possible failures and their causes, they fall in two categories: random individual independent failures and inter-dependent failures. Failures are either catastrophic or drift depending on whether their parameters fall shapely or gradually as a result of wear and fatigue.

1.3 REPAIRABLE SYSTEMS

Although failed units of a system may be replaced with new ones, repair is always more feasible because of the costs involved in buying new ones. Some systems are repairable while others are not. Repairable (or renewable) systems are those systems (or a units) which may be made operable by a repair facility once its in a down state as a result of a failure. A renewed system has its service time increased as a result of its reliability increased. In case the repair facility is not free then the failed units queue up for repair. In this study the lifetime of a unit while on line, standby or repair are considered as independent variables. We assume that the distributions of this random functions are known with probability density functions. Investigations of repairable systems have been their for ages.

The random variables considered in these researches are as below:

- Availability (or non-availability) and reliability
- Time necessary for repair
- Repairs (numbers) that can be handled
- Switchover time from the repair facility
- Possibility of a vacation time in the repair facility,etc.

The "repairman" (or repair facility) problems have much in common with queuing problems Barlow (1962). The problem of locating an optimum value of an m-out-of-n : G system for maximum reliability was conducted by Rau(1964). Ascher (1968) cited some inconsistencies in modelling of repairable systems using renewal theory. Buzacott (1970), Shooman (1968), Barlow & Proschan (1965), Sandler (1963) and Doyan & Berssenbrugge (1968) and many other authors used continuous time discrete state Markov process models for modelling the behaviour of a repairable systems. Despite the simplicity of these systems conceptually their practicability in large number of states is not feasible. A semi-Markov processes was used for computation of reliability of a system with exponential failures by Gaver (1963), Gnedenko et al (1969), Srinivasan (1966) and Osaki (1970a). Osaki (1969) used signal flow graphs to analyse a two-unit system while Kumagi (1971) applied a semi-Markov processes to determine the impact of different failure distributions on the availability through numerical computations. A semi-Markov process was used by Branson & Shah (1971) to study a repairable systems with arbitrary distributions. Srinivasan & Subramanian

(1980), Venkatakrishnan (1975), Ravichandran (1979), Natarajan (1980) and Sarma (1982) applied regeneration point techniques to study repairable systems using arbitrary distributions. A number of papers have been written in this field and related topics as seen in Subba Rao & Natarajan (1970), Osaki & Nakagawa (1976), Pierskalla & Voelker (1976), Lie et al (1977), Kumar and Agarwal (1980), Birolini (1985) and Yearout et al (1986) and Finkelstein (1993a, 1993b). In order to improve the efficiency of the system Jain and Jain (1994) introduced the regulation of 'up' and 'down' times of repairable systems.

1.4 REDUNDANCY AND DIFFERENT TYPES OF REDUNDANT SYSTEMS

Redundancy is introduced in a system by building into it more units than is actually necessary for the system to properly perform. There are two forms of redundancy namely parallel and standby (sequential) redundancy. Parallel redundancy is when the units form part of the system from the start while in a series redundancy a standby system does not form part of the system until when it is required.

1.4.1 PARALLEL SYSTEMS

A parallel redundant system is defined as one with n units which are all operating simultaneously, despite the fact that system operation needs at least one unit to be in operation. In this case system failure occurs only when all the components have failed. Let k be a non-negative integer, such that $k \leq n$, counting the number of units in an n-unit system. This system is normally referred to as a k-out-of-n system.

k-out-of-n : F-system

If the system only fails when k units fail in a k-out-of-n system, it is known as an F-system. Sfakianakis and Papastavridis (1993) pointed out that the functioning of a minimum number of units ensures that the system is operating and Chao et al (1995) surveyed such systems.

k-out-of-n : G-system

If and only if at least k units out of the n units of the system are operational the system is operational, it is known as a G-system. Zhang and Lam (1998) and Liu (1998) have recently studied such systems, for example a radar network has n radar control stations covering a certain area in which the system can be operable if and only if at least k of these stations are operable. In this case a minimum number of units, k is essential for the functioning of the system.

Attention has shifted to load-sharing $k - out - of - n : G$ systems of late, where serving units share the load and the failure rate of a component is affected by the magnitude of the load it shares.

n-out-of-n : G-system

An n-out-of-n system is basically a series system that consists of n units and failure of any one unit causes the system to fail. This type of system is not really redundant since all the units are in series and have to be operational for the unit to operate however, it is still called a special case of a $k-out-of-n$ system.

Sacheuer (1988) looked at reliability of shared-load in $k - out - of - n : G$ systems and pointed out that there is an increasing failure rate in survivors, assuming i.d. components with constant failure rates. Shao & Laberson (1991) introduced imperfect switching to the same case. A paper by Huamin (1998) considered the influence of work-load sharing in non-identical, non-repairable components, each having an arbitrary failure time distribution. His assumptions were that failure time distribution of the components may be represented by an accelerated failure time model, which happens to be a proportional hazards model when Weibull base-line reliability is used.

1.4.2 STANDBY REDUNDANCY

Standby redundancy comprises of an attachment to an operating unit one or more redundant (standby) units, which can, on failure of the operating unit, be switched on-line (if operable). These units may be classified as cold, warm or hot (Gnedenko et al (1969)).

1. **A cold standby** is not hooked up hence completely inactive, it cannot in (theory) fail until it is put to use by replacing a primary unit. Assume that since it is not in operation it's reliability will not change when it

is put into operation.

2. **A warm standby** is when a unit is partially energized hence has a diminished load. The on-line unit and the standby unit are not subject to the same loading conditions. The failure of a standby unit is attributable to to some extraneous random influence. The probability of failure of the the warm standby unit is smaller than the probability of failure of the on-line unit. This is the most general type of standby due to the high failure rate of the hot standby's and possible lapse before it is operable in the case of a cold standby's.

3. **A hot standby** is fully energized and active in the system although redundant and the possibility of failure of a hot standby is the same as that of an operating unit in the standby state. A hot standby's reliability is independent of the instant at which it takes place in the operable unit.

1.5 MEASURES OF SYSTEM PERFORMANCE

The previous sections presented brief discussion of the various types of redundant systems as cited in the literature. In this section the focus is on important measures of system performance as applicable in different situations. (Barlow & Proschan (1965), Gnedenko et al (1969)).

1.5.1 RELIABILITY

The study of reliability has advanced greatly over the past decades mainly because of the of the development of high risk and complex systems. Reliability is a kind of quantitative measure of operational efficiency. The reliability of a product is therefore a measure of its ability to perform its functions expected, when it is required, for a specific time, in a particular environment. It is measured in terms of probability and comprises of four parts, namely

1. Systems expected function
2. System operating environment (climate, packaging, transportation, storage, installation, pollution etc.)
3. Time, which is often negatively correlated with reliability
4. Probability, which is time dependent

There are two types of reliability namely:

- mission reliability is when a device is made for the performance of one mission only and
- operational reliability is when a system is turned on and off intermittently for the purpose of performing a certain specified function.

The latter case is known as an intermittently used system.

Ordinarily the period of time intended for use is (0,t].

Let $\{\psi(t), t \geq 0\}$ be the performance process of the system.

For fixed t, $\psi(t)$ is a binary random variable which takes on the value 0 if the system operates satisfactorily at time t and takes the value 1 otherwise.

Reliability R(t) is then given as

$$\begin{aligned} \mathrm{R(t)} &= \mathrm{P}\{\text{system is up in } (0,1]\} \\ &= \mathrm{P}\{\psi(u) = 1, \text{ all u } \epsilon \ (0,\mathrm{t}]\} \end{aligned}$$

The performance measure for interval reliability in case the number of system failures in the interval (t,t+x] is considered is

$$R(t,x) = P[\psi(u) = 0 \forall u \in (\mathrm{t,t+x}]$$

When $t = 0$ the interval reliability becomes the reliability R(x). The limiting interval reliability is the limit of R(t,x) as t $\rightarrow \infty$ and it is indicated as $R_{\infty}(x)$.

The mean time to system failure (MTSF) is the expectation of the random variable $\psi(t)$. It represents the duration of the time measured from the point the system commences operation until the instant when it fails for the first time and it can be computed from R(t) as given below

$$MTSF = \int_0^t R(u)du.$$

1.5.2 AVAILABILITY

Availability is a measure of system performance. It is the probability that, the system will be operational at the given time t. It implies that the system is either in active operation or is able to operate if required and consists of aspects of reliability, maintainability and maintainance support.

Availability is applicable only to intermittently used systems or those systems which undergo repair and are restored after failure. In theory availability $A(t)$ should be 100% but in practice, even equipement coming directly out of storage may be defective. Avaialibility is very important and high availability may be obtained either by increasing the average operational time until the next failure, or by improving the maintainability of the system. There are different coefficients of availability for one-unit systems (Gnedenko and Uskakov (1995))

Klassen and van Peppen (1989), Beasley (1991) defined instantaneous or pointwise availability as the 'probability that the system performs satisfactorily at a given instant of time'.

In symbols

$$A(t) = P\{\psi(t) = 1\}$$

According to Barlow Proschan (1965) steady state or asymptotic availability is a limiting availability A_∞ and it is when the expected fraction of time that the system operates satisfactorily in the long run.

$$A_\infty = \lim_{t\to\infty} A(t)$$

The joint availability $A(t,\tau)$ is the probability that the system is operating at t and at $t+\tau$. We have

$$\text{A(t,}\tau) = P\{\psi(t) = 1, \psi(t+\tau) = 1\}$$

Just as reliability and interval reliability are related, availability and joint availability satisfies the following relation $A(t) = A(0,t)$.

The expected number of visits by the repairman is a widely used concept in queuing theory of the server taking vacations and a lot of research has been done on server vacation models [see, for example Doshi (1986), Kella (1989)]. The server takes vacation according to some specified assumptions, whenever a busy period of the service station terminates. We assume that the cost structure whenever the server starts his busy cycle.We consider the idea of server vacations in reliability modeling and compute the expected number of visits by the repairman in an arbitrary interval of time by supposing that the repairman takes vacation whenever the repair facility becomes free and that he returns back only at the epoch of the next failure. In addition to estimating some of the above measures, a few other interesting, important and useful performance measures characteristic to each model are also derived in this thesis.

1.6 COST FUNCTION

There are a number of constraints facing the designer of a system. Some consideration has to be made about the system's reliability and availability, its usefulness and effectiveness. Due to the complexity of the present-day systems, measures such as reliability, availability etc. alone are not sufficient. In addition, cost and profit have become the guiding principles in every industrial and social management endeavour. Hence cost optimisation has become one of the important criteria for system designers.

We have given emphasis, in this thesis to the construction of comprehensive cost function for each of the models considered. Since they are highly non-linear, analytical optimisation of these functions becomes impracticable, if not impossible. Hence we resort to numerical optimisation; assuming that the control parameters are within certain specific intervals, we obtain numerically their optimal values.

1.6.1 MEAN NUMBER OF EVENTS IN (0,t]

Let N(a,t) symbolise the number of a particular type of an event such as a disappointment, system recovery, system down, etc. in (0,t]. The mean number of events in (0,t] is shown below

$$E[N(a,t)] = \int_0^t h_1(u)du$$

where $h_1(u)$ is the first order product density of the events. The product densities will be defined in subsequent sections of this chapter).

The mean stationary rate of occurrence of these events is

$$E[N(a)] = \lim_{t\to\infty} \frac{E[N(a,t)]}{t}$$

1.6.2 CONFIDENCE LIMITS FOR THE STEADY STATE AVAILABILITY

A $100(1-\alpha)\%$ confidence interval for A_∞ is stated as

$$P[a < A_\infty < b] = 1 - \alpha$$

Appropriate statistical tables are used to determine the numbers a and b $(a < b)$. A_∞ is a function of parameters of operating time distribution, repair time, need and no need period distributions etc.

1.7 STOCHASTIC PROCESSES USED IN THE ANALYSIS OF REDUNDANT SYSTEMS

Different types of redundant systems and the various measures of system performance were looked at in the previous sections. This section is devoted to techniques used in the analysis of redundant repairable systems.

1.7.1 RENEWAL THEORY

In renewal theory we are interested in the lifetime of the unit, there exists times, commonly random, from which onward the future of the process is a probabilistic replica of the original process. At the beginning (t = 0) a repairable unit is put into operation and functioning. The unit is replaced by a new one of the same type and subjected to maintenance that completely restores it to an 'as good as new' condition upon failure. This process is repeated upon failure and replacement time is considered negligible. These results in a sequence of lifetimes, and these study is restricted to these renewal points. The number of renewals N_t up to some time t is the probability object in these sums of non-negative i.i.d. random variables.

A number of researchers have studied specific reliability problems using renewal processes. The homogeneous Poisson process has received considerable attention and happens to be the simplest renewal process. The time parameter may be taken as either discrete or continuous. A proper lead for the discrete case was conducted by Feller (1950) followed by a very lucid account of Cox (1962) for the continuous case (he provided an introduction to renewal theory in the case of a repair facility not being available and failed units queuing up for repair). Barlow (1962) applied in his research on repairable systems queuing theory. Some operating characteristics of a one unit system were studied by Srinivasan (1971) while Gnedenko et al (1969) worked out the mean time to system failure of a two-unit standby system. Some priority redundant systems were studied by Buzacott (1971), etc.

In renewal systems the system starts a new cycle after each renewal (which is independent of the previous ones) despite its possibility of taking on different forms. In case repair time is not considered negligible, each cycle comprises of a lifetime and a repair time which are both random variables with individual distributions (repair time may be considered as a fixed time). This process is known as

- An ordinary renewal process if the time origin is the initial installation of the system and the repair time is taken as negligibly small in comparison with the lifetime of the unit - renewal is taken as instantaneous, or
- A general renewal process if the time origin is some point after the initial installation of the system (Cox (1962)). Høyland & Rausand (1994) named this a modified renewal process, while Feller (1957) calls this process considering the residual lifetime of a system at an arbitrary chosen time origin as a delayed renewal process

(a) Ordinary renewal process : instantaneous renewal

This is when a basic model of continuous operation is considered whose unit begins operating at instant $t = 0$ and stays operational for a random time

T_1 and then fails. At this instant the unit is replaced by a new and statistically identical unit, which operates for a length of time T_2 then fails and is again replaced etc. These random component life lengths $T_1, T_2, ...T_r...$ of the identical units are independent, non-negative and identically distributed random variables that constitute ordinary renewal process.

Let

$$P[T_i \leq t] = F(t); t > 0, \text{ i} = 1,2, \, ...$$

be considered as an underlying distribution of the renewal process. The time taken until until the r^{th} renewal is given by

$$t_r = T_1 + T_2 + ... + T_r = \sum_{i=1}^{r} T_i$$

Let the N(t) be a random variable where N(t) $=$ max$\{$r;$R_r \leq t\}$ which denotes the number of times a renewal takes place in the interval (0,t], then the number of renewals in an arbitrary time interval $(t_1, t_2]$ is equal to

$$N(t_2) - N(t_1).$$

A renewal function H(t), which is the expected value of N(t) in the time interval (0,t], can now be defined as

Where $F^{(r)}(.)$ is the r-fold convolution of F.

Furthermore

$$\begin{aligned} \mathrm{H(t)} &= \mathrm{E[N(t)]} \\ &= \sum_{r=1}^{\infty} F^{(r)}(t) \end{aligned}$$

$$H(t) = F(t) + \int_0^t H(t-x)dF(x)$$

The renewal density function is

$$h(t) = \sum_{n=1}^{\infty} f^{(r)}(t)$$

and these renewal density function h(t) satisfies the equation

$$h(t) = f(t) + \int_0^t h(t-x)f(x)dx$$

It indicates that the renewal density $h(t)$ basically differs from the hazard rate $h^o(t)$, as

$$h^o(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)}$$

(b) Random renewal time

In case the time for a renewal is not instantaneous but it is taken as a random variable that is included in the subsequent time-periods, or cycles, of the systems' performance, each cycle will then comprise of a time to failure and a time to repair. The failure and repair time will both be stochastic in nature. The instants of failure and cycles of renewal can be determined.

Let $F(t)$ be the lifetime distribution and $G(x)$ be the repair time function with respective probability density functions f(t) and g(x). Therefore the density function of the cycles C of the lifetime and repair time, say $k(t)$ is estimated using the convolution formula

$$k(t) = \int_0^t f(x)g(t-x)dx$$

Let $N_F(t)$ count the number of failures and $N_R(t)$ the number of repairs in (0,t], define

$$W(t) = E[N_F(t)]$$

and

$$V(t) = E[N_R(t)]$$

and let $Q(t) = W(t) - V(t); \forall t$, assuming that $w(t) = W'(t)$ and $v(t) = V'(t)$.

The failure and repair intensities can then respectively be defined as

$$\lambda(t) = \frac{w(t)}{A(t)}$$

Where $A(t)$ is the availability function

$$\mu(t) = \frac{v(t)}{Q(t)}$$

Where $Q(t) \neq 0$.

(c) Alternating renewal processes

Takács(1957) was the first to study in detail alternating renewal processes and there many text books which have discussed it further (Ross (1970)). A generalization of the ordinary renewal process discussed previously follows where the state of the unit is given by the binary variable

$$X(t) = \begin{cases} 0 & \text{if the unit is functioning at time t} \\ 1 & \text{otherwise} \end{cases}$$

The two alternating states may be taken as 'system up' and 'system down'. If these alternating independent renewal processes are distributed according to F(x) and G(x), there are two renewal processes embedded in them for the different transitions from 'system up' to 'system down'. Usually one-item repairable structures are considered as alternating renewal processes under the assumption that after each repair the item is as good as new.

(d) Age and remaining lifetime of a unit

Let t_r indicate the random component lifetime, i.e. $t_r = \sum\limits_{i=1}^{r} T_i$.

Let $R_r, r \in N$, represent the length of the r^{th} repair time, then the sequence $T_1, R_1, T_2, R_2, ...$ forms an alternating renewal process. Define

$$t_n = T_1 + \sum_{r=1}^{n-1} (R_r + T_{r+1}); n \in N$$

and

$$t_n^o = \sum_{r=1}^{n} (R_r + T_r)$$

and set $t_0 = t_0^o = 0$.

This sequence t_n generates a delayed renewal process.

If $B_1(t)$ demotes the forward recurrence time at time t, then

$$B_1(t) = t_{N_t+1} - t \text{ or } B_1(t) = t_{N_t^o+1} - t$$

Hence,

- $B_1(t)$ equals the time to the next failure time if the system is up at time t, or
- $B_1(t)$ equals the time to complete the repair if the system is down at time t.

Hence,

- $B_2(t)$ equals the age of the unit if the system is up at time t, or
- $B_2(t)$ equals the duration of the repair if the system is down at time t.

Feller (1941) defined the elementary renewal theorem as an ordinary renewal process with underlying exponential distribution (parameter λ and H(t) = λt).

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{\mu}$$

With $\mu = E(T_i) = \frac{1}{\lambda}$ the mean lifetime.

In case the renewals match the component failures, the mean number of failures in (0,t] is approximately (for t large)

$$H(t) = E[N(t)] \approx \frac{1}{\mu} = \frac{1}{MTSF}$$

1.7.2 SEMI-MARKOV AND MARKOV RENEWAL PROCESSES

We shall look at a general description of a process where a system

- Moves from one state to another with random sojourn times in between
- The successive states visited from a Markov chain
- The sojourn times have a distribution which depend both on the present state and the next state.

It is considered a Markov chain if all the sojourn times are equal to one and a Markov process if the distribution of the sojourn times are all exponential and independent of the next state. It is a renewal process if there is only one state allowing an arbitrary distribution of the sojourn times.

The state space may be denoted by the set of non-negative integers {1,2, ...} and transition probabilities by p_{ij}, i,j = 0,1,2, ... If $F_{ij}(t), t > 0$ is the conditional distribution of the sojourn time in state i, given that the next transition will be into state j, let

$$Q_{ij}(t) = p_{ij}F_{ij}(t), \text{ i,j} = 0,1,2 \ldots$$

denote the probability that the process makes a transition into state j in an amount of time less than or equal to t, given that it just entered state i at t = 0. The functions $Q_{ij}(t)$ satisfy the conditions which follow:

$$
\begin{aligned}
Q_{ij}(0) &= 0, \quad Q_{ij}(\infty) = p_{ij} \\
Q_{ij}(t) &\geq 0, \quad ij = 0, 1, 2, ... \\
\sum_{j=0}^{\infty} Q_{ij}(t) &= 1
\end{aligned}
$$

Denote initial state and the state after the n^{th} transition occurs by J_0 and J_n respectively. The embedded Markov chain $\{J_n, n = 0, 1, 2...\}$ then becomes the Markov chain with transition probabilities p_{ij}.

If $N_i(t)$ represents the number of transitions into state i in (0,t] and

$$\text{N(t)} = \sum_{i=0}^{\infty} N_i(t)]$$

A semi-Markov process (SMP)is a stochastic process $\{X(t), t \geq 0\}$ with X(t) = i representing the process in state i at time t and it indicates that $X(t) = J_{N(t)}$. A SMP is a pure jump process and all the states are regeneration states. The subsequent states form a time-homogeneous Markov chain process without memory at the transition point from one state to the next. A Markov renewal process (MRP) is a vector stochastic process $\{N_1(t), N_2(t)...\}$ for $t \geq 0$. A SMP records the state process at each time point while the MRP is a counting process keeping track of the number of visits to each state.

Suppose the time-intervals in which the r.v. X(t) continues to remain in the n-point state are independently distributed such that;

$$
\begin{aligned}
\lim_{\Delta \to 0} P[X(t + x) = j, X(t + u) = i : \forall u \leq x \,|X(t) = i, X(t - \Delta) \neq i] \\
= f_{ij}; i, j = 1, 2, ..., n
\end{aligned}
$$

A Markov chain with a randomly transformed time scale is called a MRP, if the transition of X(t) is characterised by a change of state and the qualities $f_{ii}(.)$ are zero functions.

In order to remove the consequence of $f_{ii}(.) = 0$, another definition of a MRP can be given, namely considering it as a regenerative stochastic process {X(t)} in which the epochs at which X(t) visits any member of a certain countable set of states are regenerative points, the visits become regenerative events.

In order to obtain a more powerful tool than either a Markov chain or a renewal process the two are combined to form a SMP. Lévy (1954) and Smith (1955) introduced SMP independently. Pyke (1961a, 1961b), Cinler (1975) and Ross (1970) have used both SMP and MRP extensively while Barlow and Proschan (1965) applied these processes to determine the MTSF of a two-unit system. In their discussion of certain reliability problems Cinler (1975), Osaki (1970a), Arora (1976), Nakagawa& Osaki (1974, 1976) and Nakagawa (1974) have used the theory of SMP.

1.7.3 REGENERATIVE PROCESSES

A sequence $t_0, t_1, ...$ of stopping times such that $t = \{t_n; n \in N\}$ is a renewal process in a regenerative stochastic process X(t). In case a point of regeneration occurs at $t = t_1$, then the knowledge of the history of the process prior to t_1 loses its predictive value; the future of the process is totally independent of its past. It therefore implies that X(t) regenerates itself repeatedly at these stopping times and the times between consecutive renewals are known regeneration times. Renewal theory is an important tool in elementary probability theory because of its application to regenerative processes.

Delayed renewal process is stated as follows: if $\hat{t} = \{t_n - t_0; n \in N\}$ is a renewal process such that $t_0 \geq 0$ is independent of $\hat{t}$ which implies that the time t_0 of the first renewal is not necessarily the time origin. A delayed renewal process is formed by a delayed regenerative process which is a process with a sequence $t = \{t_n; n \in N\}$ of stopping times. For instance for any initial state i, the times of subsequent entrances to a fixed state j in a Markov process become a delayed process.

In general non-exponentially distributed repair times and/or failure free operating times lead to processes with only a few regeneration states (or even to non-regenerative processes) with the exception of few cases when it may lead to semi-Markov processes. The focus of recent research is on Brownian motion with interest in the random set of all regeneration times and on the excursions of the process between generations.

1.7.4 STOCHASTIC POINT PROCESS

Point processes are widely used in reliability theory to model the appearance of events in time among discrete stochastic processes. A renewal process is used as a mathematical model to describe the flow of failures in time. It is a point process known to be with restricted memory and each event is a regeneration point. In practical applications to reliability problems, the interest is focused on the behaviour of a renewal process in a stationary regime, i.e., when $t \to \infty$, as repairable systems enter an 'almost stationary' regime very quickly. Alternating renewal process is a generalization of a renewal process, which comprises of two types of i.i.d. random variables alternating with each other in turn.

Point processes have been defined differently by different individuals in the different areas of application since recurrent events has had applications in a number of fields including physics, biology, management sciences, cybermetrics and many other areas. Wold (1948) and Bartlett (1954) first studied the properties of stationary point processes to whom we attribute the current terminology. Moyal(1962) provided a formal and well-knit theory of the subject and even extended it to cover non-Euclidean spaces. Srinivasan (1974), Srinivasan & Subramanian (1980) and Finkelstein (1998, 1999c) applied extensively point processes in reliability theory.

Our concern in point processes majors on those applications which, in general, lead to the development of multivariate point processes. In this particular case, a point process can be defined as a stochastic process 'whose realizations are related to a series of point events occurring in a continuous one-dimensional parameter space (such as time, etc.). The time series $\{t_n\}$ are the 'renewal' epochs which generate the point process. The two random variables of concern are the number of points that fall in the interval (t,t+x] and the time that has lapsed since the n^{th} point after (or before)t.

Characterization property of stationarity applies to certain point processes, such as the density function of the number of observed events in a time interval which does not depend on its position on the time axis, but only on the length of the interval (Srinivasan & Subramanian (1980))

(a) Multivariate point processes

Multivariate stationary processes has been applied in in many fields and the properties of these processes have been investigated widely by Cox Lewis (1970). A stationary point process is obtained by relaxing the constraint of independence of the interval in a stationary renewal process; if the same constraint is removed in the case of a Markov renewal process it results in a multivariate stationary point process.

The product density technique as a sophisticated tool for the study of point processes was developed, analysed and perfected by Ramakrishnan (1954). A point process is denoted by the triplet (Φ, B, P), where P is a probability distribution on some σ-field B of subsets of the spaces Φ of all states. A point x of a fixed set of points X describes the state of a set of objects.

Suppose for X is the real number line for this discussion and define A_k as intervals and N(.) as a counting measure which is uniquely associated with a series of points {t} such that:

$$N(A) \quad = \text{the number of points in the sequence } \{t_i : t_i \forall A\}$$

$$N(t,x) \quad = \text{the number of points (events) in the interval } (t : t+x]$$

$$N(t,x) \quad = \text{the number of points (events) in } (t+x : t+x+\Delta]$$

The central quality of interest in the product density technique is this $N^{'}(t,x)$, representing the number of entities with parametric values between x and $x+\Delta$ at time t.

Resulting the factorial moment distribution the product density of order n, which denotes the probability of an event in each of the intervals $(x_1, x_1 + \Delta_1), (x_2, x_2 + \Delta_2), ..., (x_n, x_n + \Delta_n)$, can be defined. It is symbolized by the product of the density of expectation measures at different points as shown below,

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2, ..., \Delta_n \to 0} \frac{E[\prod_{i=1}^{n} N(x_i, \Delta_i)]}{\Delta_1 \Delta_2 ... \Delta_n}; x_1 \neq x_2 \neq ... \neq x_n$$

Or, equivalently

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2, ..., \Delta_n \to 0} \frac{P[N(x_i, \Delta_i) \geq 1, i=1,2,...,n]}{\Delta_1 \Delta_2 ... \Delta_n}; x_1 \neq x_2 \neq ... \neq x_n$$

The density $h_n(...)$ is known as a product density because it is essentially a product of the density of expectation measures at different points. The renewal function H(t) is the expected number of random points in the interval (0,t]. Revise the process by allocation of all integral values to $\{t_i\}$ and suppose a matching sequence of points on the real line. The resultant point process generated by the random variables $\{t_i\}$, the counting process N(t,x) denotes the number of points in the interval (t, t+x] and the product density is

$$h_m(t, t_1, t_2, ..., t_m) = E[N'(t, t_1)N'(t, t_2)...N'(t, t_m)]$$

A product density of degree m is as follows:

$$h_m(t, t_1, t_2, ..., t_m) = h_1(t, t_1)h(t_2 - t_1)h(t_3 - t_2)...h(t_m - t_{m-1})$$
$$(t_1 < t_2 < ... < t_m)$$
$$.$$

CHAPTER 2

APLLICATIONS OF BIVARIATE EXPONENTIAL DISTRIBUTION IN RELIABILITY THEORY

2.1 INTRODUCTION

Analysis of one unit and two unit repairable systems had received considerable attraction and had been extensively studied by several authors in the past. If we assume that the life time density and the repair time density of the unit are arbitrary, we may utmost obtain highly formal expressions for probability distributions and other quantities of interest. These expressions are rarely suitable for numerical computations. In most of the cases, analytically explicit expressions are obtained only under negative exponential assumptions. Recently, Chandrasekhar and Natarajan (2000) have obtained several measures of system performance by assuming that the life time and repair time of one unit system are PH distributions with different representations. Generally speaking, the lifetime and repair time of a unit are assumed to be independent random variables. But in a real life situation, this assumption may not hold good. Hence an attempt is made in this paper to relax this assumption and we obtain several measures of system performance by assuming that the life time and repair time of a one unit system are with dependent structure and the underlying distribution is bivariate exponential. Also, a $100(1-\alpha)\%$ confidence interval for the steady state avaiability of the system is obtained. The system description and assumptions of the model are given in the next section. Further, a two unit cold standby system where the lifetime and repair time of the units have bivariate exponential distribution is studied. System reliability, MTBF and an estimator based on moments for system reliability are obtained.

MODEL - I (One Unit System)

2.2 SYSTEM DESCRIPTION AND ASSUMPTIONS

The system under consideration is a one unit system with a single repair facility. Precisely we have the following assumptions:

1. The system under consideration consists of only one unit and when it fails, it is taken up for repair instantaneously,

2. The life time Y_1 and the repair time Y_2 of the unit are with dependent structure and have bivariate exponential distribution with the survival function given by

$$\bar{F}(y_1, y_2) = e^{-\mu_1 y_1 - \mu_2 y_2 - \mu_3 max(y_1, y_2)} \ ; \ y_1, y_2 > 0 : \mu_1, \mu_2, \mu_3 > 0 \qquad (2.2.1)$$

3. Switch is perfect and the switchover is instantaneous

4. At time t = 0, the unit just begins to operate.

2.3 OPERATING CHARACTERISTICS OF THE SYSTEM

In this section, several measures of system performance are obtained as follows:

(a) SYSTEM RELIABILITY

Since the system reliability is the probability that the unit has not failed in [0,t], it is given by

$$R(t) = e^{(-\mu_1+\mu_2)t} \tag{2.3.1}$$

(b) MEANTIME BEFORE FAILURE (MTBF)

The system mean time before failure is given by

$$MTBF = \frac{1}{(\mu_1+\mu_2)} \tag{2.3.2}$$

(c) SYSTEM AVAILABILITY

To obtain the system availability A(t), we define the following E-event:

E-event: that the system enters the upstate from the down state. Clearly, E-events constitute a renewal process.

By considering the following mutually exclusive and exhaustive cases namely:

1. there is no E-event in [0,t]

2. there is at least one E-event in [0,t], it can be shown that

$$A(t) = \frac{(\mu_2+\mu_3)}{(\mu_1+\mu_2+2\mu_3)} + \frac{(\mu_1+\mu_3)}{(\mu_1+\mu_2+2\mu_3)} e^{-(\mu_1+\mu_2+2\mu_3)t} \tag{2.3.3}$$

(d) STEADY STATE AVAILABILITY

The steady state availability A_∞ of the system is given by

$$A_\infty = \frac{(\mu_2+\mu_3)}{(\mu_1+\mu_2+2\mu_3)} \tag{2.3.4}$$

(e) INTERVAL RELIABILITY

The interval reliability $R(t,x)$ of the system is the probability that at a specified time t, the system is in upstate and will continue to operate for a duration of time x.

i.e. $$R(t,x) = \left[\frac{(\mu_2+\mu_3)}{(\mu_1+\mu_2+2\mu_3)} - \frac{(\mu_1+\mu_3)}{(\mu_1+\mu_2+2\mu_3)} e^{-(\mu_1+\mu_2+2\mu_3)t}\right] e^{-(\mu_1+\mu_3)x} \tag{2.3.5}$$

PERTICULAR CASE

By taking $\mu_3 = 0$ in (2.3.1) - (2.3.5), the following measures of system performance are readily obtained.

$$R(t) = exp(-\mu_1 t) \tag{2.3.6}$$

$$MTBF = \frac{1}{\mu_1} \tag{2.3.7}$$

$$A(t) = \frac{\mu_2}{(\mu_1+\mu_2)} + \frac{\mu_1}{(\mu_1+\mu_2)}e^{-(\mu_1+\mu_2)t} \tag{2.3.8}$$

$$A_\infty = \frac{\mu_2}{(\mu_1+\mu_2)} \tag{2.3.9}$$

$$R(t,x) = \left[\frac{\mu_2}{(\mu_1+\mu_2)} + \frac{\mu_1}{(\mu_1+\mu_2)}e^{-(\mu_1+\mu_2)t}\right] e^{-\mu_1 x} \tag{2.3.10}$$

It may be noted that (2.3.6) - (2.3.10) are in agreement with Rau (1970) and Birolini (1985).

In the next two sections, a CAN estimator and a $100(1-\alpha)\%$ confidence interval for steady state avaiability of the system are obtained.

2.4 CONFIDENCE INTERVAL FOR STEADY STATE AVAILABILITY OF THE SYSTEM

2.4.1 AN ESTIMATOR OF STEADY STATE AVAILABILITY BASED ON MOMENTS

Suppose the life time Y_1 and the repair time Y_2 of the one unit system have bivariate exponential distribution with the survival function given by (2.2.1). Let $(Y_{1i}, Y_{2i}), i = 1, 2, ..., n$ be a random sample of size n drawn from the above bivariate exponential life time and repair time population. It is clear that $\bar{Y_1}$ and $\bar{Y_2}$ are the moment estimators of $\frac{1}{\mu_1+\mu_3}$ and $\frac{1}{\mu_2+\mu_3}$ respectively, where $\bar{Y_1}$ and $\bar{Y_2}$ are the sample means of life times and repair times respectively.

Let $\theta_1 = \frac{1}{\mu_1+\mu_3}$ and $\theta_2 = \frac{1}{\mu_2+\mu_3}$

Clearly, the steady state availability of the system given in (2.3.4) reduces to

$$A_\infty = \frac{\theta_1}{(\theta_1+\theta_2)} \tag{2.4.1}$$

and hence an estimator of A_∞ based on moments is given by

$$\hat{A_\infty} = \frac{\bar{Y_1}}{\bar{Y_1}+\bar{Y_2}} \tag{2.4.2}$$

It may be noted that $\hat{A_\infty}$ given in (2.4.2) is a real valued function in $\bar{Y_1}$

and $\bar{Y}_2$, which is also differentiable. Consider the following application of multivariate central limit theorem. see Rao (1974).

2.4.2 APPLICATION OF MULTIVARIATE CENTRAL LIMIT THEOREM

Suppose $T_1^{'}, T_2^{'}, T_3^{'}, ...$ are independent and identically distributed k - dimensional random variables such that $T_n^{'} = (T_{1n}, T_{2n}, T_{3n}, ..., T_{kn}), n = 1, 2, 3, ...,$ having the first and second order moments $E(T_n) = \mu$ and $D(T_n) = \Sigma$. Define the sequence of random variables.

$\bar{T}_n = (\bar{T}_{1n}, \bar{T}_{2n}, ..., \bar{T}_{kn}), n = 1, 2, 3, ...,$ where

$\bar{T}_{1n} = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, i = 1, 2, ..., k : j = 1, 2, ..., n$

then,

$\sqrt{n}(\bar{T}_n - \mu) \xrightarrow{d} N_k(0, \Sigma)$ as n$\rightarrow \infty$

2.4.3 CAN ESTIMATOR

By applying the multivariate central limit theorem given in section 2.4.2, it readily follows that $\sqrt{n}[(\bar{Y}_1, \bar{Y}_2)..(\theta_1, \theta_2)] \xrightarrow{d} N(0, \Sigma)$ as $n \rightarrow \infty$. The dispersion matrix $\Sigma = (\sigma_{ij})$ is given by (see Barlow and Proschan(1975)).

Again from Rao (1974), we have

$$\Sigma = \begin{array}{c} \bar{Y_1} \\ \bar{Y_2} \end{array} \begin{bmatrix} \theta_1^2 & \frac{\mu_3\theta_1^2\theta_2^2}{\theta_1+\theta_2-\theta_1\theta_2\mu_3} \\ \frac{\mu_3\theta_1^2\theta_2^2}{\theta_1+\theta_2-\theta_1\theta_2\mu_3} & \theta_2^2 \end{bmatrix} \tag{2.4.3}$$

$$\sqrt{n}(\hat{A_\infty} - A_\infty) \xrightarrow{d} N(0, \sigma^2(\theta)) \text{ as } n \to \infty.$$

where $\theta = (\theta_1, \theta_2)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{2}(\tfrac{\partial A_\infty}{\partial \theta_1})^2\sigma_{ii} + 2\tfrac{\partial A_\infty}{\partial \theta_1}\tfrac{\partial A_\infty}{\partial \theta_2} - \tfrac{\mu_3\theta_1^2\theta_2^2}{(\theta_1+\theta_2-\theta_1\theta_2\mu_3)} \tag{2.4.4}$$

Thus $\hat{A_\infty}$ is a CAN estimator of A_∞. There are several methods for generating CAN estimators and the method of moments and the method of maximum likelihood are commonly used to generate such estimators. see Sinha (1986).

2.5 CONFIDENCE INTERVAL FOR THE STEADY STATE AVAILABILITY OF THE SYSTEM

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator namely $(\bar{Y_1}, \bar{Y_2})$. Let $\sigma^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of $\theta, \hat{\sigma^2}$ is a consistent estimator of $\sigma^2(\theta)$, i.e. $\hat{\sigma^2} \xrightarrow{d} \sigma^2(\theta)$ as $n \to \infty$.

By Slutsky theorem, we have

$$\frac{\sqrt{n}(\hat{A}\infty - A\infty)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$$

i.e.,

$$Pr\left[-k_{\alpha/2} < \frac{\sqrt{n}(\hat{A}\infty - A\infty)}{\hat{\sigma}} < k_{\alpha/2}\right] = (1-\alpha)$$

where $k_{\alpha/2}$ is obtained from normal tables. Hence, a $100(1-\alpha)\%$ asymptotic confidence interval for A_∞ is given by

$$\hat{A_\infty} \pm k_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \tag{2.5.1}$$

where $\hat{\sigma}$ is obtained from (2.4.4).

MODEL - II (Two Unit Cold Standby System)

2.6 SYSTEM DESCRIPTION AND ASSUMPTIONS

The system under consideration is a two unit cold standby system with a single repair facility. We have precisely the following assumptions:

(i) The units are similar and statistically not independent. The life time Y_1 and the repair time Y_2 of the units in the system have bivariate exponential distribution with the survival function given by (2.2.1).

(ii) There is only one repair facility

(iii) Each unit is new after repair

(iv) Switch is perfect and the switch over is instantaneous.

2.7 ANALYSIS OF THE SYSTEM

To analyse the behaviour of the system, we note that at any time t, the system will be found in any one of the following mutually exclusive and exhaustive states.

S_0 : Both the components are operable but only one is operating (the other unit is kept in standby)

S_1 : One component has failed and the other components is operating.

S_2 : Both the component have failed.

Since the life time and the repair time of the units are exponential random variables with the parameters $(\mu_1 + \mu_3)$ and $(\mu_2 + \mu_3)$ respectively, the stochastic process describing the behaviour of the system is a Markov process. Let $p_i(t)$ be the probability that the system is in state S_i at time t. Clearly, the infinitesimal generator of the Markov process is given by

$$\mathrm{Q} = \begin{array}{c} S_0 \\ S_1 \\ S_2 \end{array} \begin{bmatrix} -(\mu_1+\mu_3) & (\mu_1+\mu_3) & 0 \\ (\mu_2+\mu_3) & -(\mu_1+\mu_2+2\mu_3) & (\mu_1+\mu_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7.1}$$

We assume that initially both the components are operable and obtain the measures of system performance as follows:

2.7.1 SYSTEM RELIABILITY

The system reliability $R(t)$ is the probability of failure free operation of the system in [0,t] and is obtained as follows:

From the infinitesimal generator given in (2.7.1), we have the following system of differential-difference equations:

$$p_0'(t) = -(\mu_1 + \mu_3)p_0(t) + (\mu_2 + \mu_3)p_1(t) \tag{2.7.2}$$

$$p_1'(t) = (\mu_1 + \mu_3)p_0(t) - (\mu_1 + \mu_2 + 2\mu_3)p_1(t) \tag{2.7.3}$$

$$p_2'(t) = (\mu_1 + \mu_3)p_1(t) \tag{2.7.4}$$

Let $L_i(s)$ be the Laplace transform of $p_i(t), i = 0, 1, 2$. Taking Laplace transform on both the sides of the differential-difference equations given above, solving for $L_i(s), i = 0, 1, 2$ and inverting, we get $p_i(t), i = 0, 1, 2$. Hence the system reliability is given by

$$\begin{aligned} R(t) &= \mathrm{p}_0(t) + p_1(t) \\ &= \tfrac{(s_1 e^{s_2 t} - s_2 e^{s_1 t})}{(s_1 - s_2)} \end{aligned} \tag{2.7.5}$$

where s_1 and s_2 are the roots of $s^2 + (2\mu_1 + \mu_2 + 3\mu_3)s + (\mu_1 + \mu_3)^2 = 0$

2.7.2 MEAN TIME BETWEEN FAILURES (MTBF)

The system mean time between failure is given by

$$MTBF = R*(0) = \frac{2\mu_1+\mu_2+3\mu_3}{(\mu_1+\mu_3)^2} \tag{2.7.6}$$

2.7.3 PARTICULAR CASE

For

$\mu_3 = 0$, we have from (2.7.5) and (2.7.6)

$$R(t) = \frac{(s_1e^{s_2t} - s_2e^{s_1t})}{(s_1 - s_2)}$$

where s_1 and s_2 are the roots of $s^2 + (2\mu_1 + \mu_2)s + \mu_1^2 = 0$ and

$$MTBF = \frac{(2\mu_1+\mu_2)}{\mu_1^2},$$

which are in agreement with Rau (1970).

2.8 AN ESTIMATOR OF SYSTEM RELIABILITY BASED ON MOMENTS

Since $\bar{Y}_1$ and $\bar{Y}_2$ are the moment estimators of $\frac{1}{\mu_1+\mu_3}$ and $\frac{1}{\mu_2+\mu_3}$ respectively, we obtain the moment estimator of system reliability as

$\hat{R}(t) = \frac{(\hat{s}_1 e^{\hat{s}_2 t} - s_2 e^{\hat{s}_1 t})}{(\hat{s}_1 - \hat{s}_2)}$,

where

$\hat{s}_1 = -(a - b)$ and $\hat{s}_2 = -(a + b)$ with

a $= \frac{\bar{Y}_1 + 2\bar{Y}_2}{2\bar{Y}_1 \bar{Y}_2}$

and

b $= \frac{1}{2\bar{Y}_1 \bar{Y}_2} \sqrt{\bar{Y}_1(\bar{Y}_1 + 4\bar{Y}_2)}$.

CHAPTER 3

RELIABILITY ANALYSIS OF A COMPLEX TWO UNIT STANDBY SYSTEM WITH VARYING REPAIR RATE

3.1 INTRODUCTION

Introduction of redundancy, repair maintainance and preventive maintainance are some of the well known methods by which the reliability of a system can be improved. Two unit standby redundant systems have been extensively studied by several authors in the past. A bibliography of the work on two unit systems is given by Osaki and Nakagawa (1976), Kumar and Agarwal (1980). It can be shown that any failure or repair time distribution can be approximated arbitrarily closely by a general Erlang distribution. The most useful of the more general distributions are, however, those that give coeffients of variation that cannot be reasonably approximated by a special Erlangian distribution (see Cox, 1970). An attempt is made in this paper to study a two unit cold standby system with generalised Erlang distribution for the repair time. For the sake of simplicity, we consider a generalised Erlang distribution with two stages. Most of the studies on two unit cold standby systems are confined to obtaining expressions for various measures of system performance and do not consider the associated statistical inference problems. Chandrasekhar and Natarajan (1994) have considered a two unit cold standby system and obtained the exact confidence limits for the steady state availability of the system. Similar results were obtained for a parallel system, with preparation time by Yadavalli et al (2002). Chandrasekhar et al (2004) have studied in detail a complex two unit warm standby system assuming that the repair time distribution is a two stage generalized Erlang distribution. Besides obtaining expressions for the system reliability, mean time before failure (MTBF) and steady state availability, an attempt is made in this paper to obtain a consistent asymptotically normal (CAN) estimator and an asymptotic confidence interval for the steady state availability of a two unit cold standby system in which the failure rate of the unit while on-line is a constant and the repair time distribution is a two stage generalized Erlangian. The model and assumptions are given in the next section.

3.2 THE MODEL AND ASSUMPTIONS

The system under consideration is a two unit cold standby system with a single repair facility. We have precisely the following assumptions:

1. The units are similar and statistically independent. Each unit has a constant failure rate, say λ.

2. There is only one repair facility and the repair time distribution is a two stage generalized Erlang distribution with probability density function (pdf) given by,

$$g(y) = \tfrac{\mu}{k-1}(e^{\frac{-\mu}{k}y} - e^{\mu y}), y > 0, \mu > 0, k \neq 1 \tag{3.2.1}$$

3. Each unit is new after repair.

4. Switch is perfect and the switchover is instantaneous.

Note: The density given in (3.2.1) corresponds to the sum of two independent but not identically distributed exponential variates with the parameters μ and $\frac{\mu}{k}(k \neq 1)$ respectively.

3.3 ANALYSIS OF THE SYSTEM

To analyse the behaviour of the system, we note that at any time t, the system will be found in any one of the following mutually exclusive and exhaustive states.

S_0 : One unit is operating on line and the other is kept in standby

S_1 : One unit is operating online and the other is in the first stage of repair

S_2 : One unit is operating online and the other is in the second stage of repair

S_3 : One unit is in the first stage of repair and the other is waiting for repair

S_4 : One unit is in the second stage of repair and the other is waiting for repair.

Since, a generalized Elang distribution can be considered as the distribution of the sum of two independent but not identically distributed exponential random variables, the underlying stochastic process describing the behaviour of the system is a Markov process. Let $p_i(t), i = 0,1,2,3,4$ be the probability that the system is in the state S_i at time t. Clearly, the infinitesimal generator of the Markov process is given by

$$\mathrm{Q} = \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -(\lambda+\mu) & \mu & \lambda & 0 \\ \frac{\mu}{k} & 0 & -(\lambda+\frac{\mu}{k}) & 0 & \lambda \\ 0 & 0 & 0 & -\mu & \mu \\ 0 & \frac{\mu}{k} & 0 & 0 & -\frac{\mu}{k} \end{bmatrix} \qquad (3.3.1)$$

It may be noted that the states S_0, S_1 and S_2 are system upstates, whereas S_3 and S_4 are system down states. We assume that initially, both the units are operable and obtain the measures of system performance as follows:

3.4 RELIABILITY

The system reliability $R(t)$ is the probability of failure free operation of the system in [0,t]. To derive an expression for the reliability of the system, we restrict the transitions of the Markov process to the system upstates namely S_0, S_1 and S_2. Using the infinitesimal generator given in (3.3.1), pertaining to these upstates and using standard probabilistic arguments, we derive the following system of differential - difference equations:

$$p_0'(t) = -\lambda p_0(t) + \frac{\mu}{k} p_2(t)$$

$$p_1'(t) = \lambda p_0(t) - (\lambda + \mu)p_1(t)$$

$$p_2'(t) = \mu p_1(t) - (\lambda + \frac{\mu}{k})p_2(t)$$

Let $L_i(s)$ be the Laplace transform of $p_i(t), i = 0, 1, 2$. Taking Laplace transform on both sides of the differential - difference equations given above, solving for $L_i(s), i = 0, 1, 2$ and inverting, we get $p_i(t), i = 0, 1, 2$. Hence the system reliability is given by

$$\begin{aligned} R(t) &= \mathrm{p}_0(t) + p_1(t) + p_2(t) \\ &= \sum_{i=1}^{3} \frac{[(\alpha_i+\lambda+\frac{\mu}{k})(\alpha_i+\alpha\lambda+\mu)+\lambda\mu]}{\prod\limits_{i=1, j\neq i}^{3}(\alpha_i-\alpha_j)} e^{\alpha_i t} \end{aligned} \tag{3.4.1}$$

where α_1, α_2 and α_3 are the roots of

$$s^3 + as^2 + bs + c = 0$$

with

$$a = 3\lambda + \mu + \frac{\mu}{k}$$

$$b = 3\lambda^2 + 2\lambda\mu + \frac{2\lambda\mu}{k} + \frac{\mu^2}{k}$$

and

$c = \lambda^3 + \mu\lambda^2 + \frac{\mu\lambda^2}{k}$

3.5 MEAN TIME BEFORE FAILURE (MTBF)

The mean time before failure of the system is given by

$$\begin{aligned} \text{MTBF} &= L_0(0) + L_1(0) + L_2(0) \\ &= \frac{(k\lambda+\mu)(2\lambda+\mu)+k\lambda\mu}{\lambda^2(k\lambda+k\mu+\mu)} \end{aligned} \tag{3.5.1}$$

3.6 STEADY STATE AVAILABILITY

The steady state availability A_∞ is obtained as follows:

Using the infinitesimal generator given in 3.3.1, we obtain the following system of differential - difference equations:

$$p_0'(t) = -\lambda p_0(t) + \frac{\mu}{k}p_2(t) \tag{3.6.1}$$

$$p_1'(t) = \lambda p_0(t) - (\lambda+\mu)p_1(t) + \frac{\mu}{k}p_4(t) \tag{3.6.2}$$

$$p_2'(t) = \mu p_1(t) - (\lambda + \frac{\mu}{k})p_2(t) \tag{3.6.3}$$

$$p_3'(t) = \lambda p_1(t) - \mu p_3(t) \tag{3.6.4}$$

$$p'_4(t) = \lambda p_2(t) + \mu p_3(t) - \tfrac{\mu}{k} p_4(t) \tag{3.6.5}$$

Letting $\lim\limits_{t\to\infty} p_i(t) = p_i$ and solving these equations after taking the limit as $t \to \infty$ and using the condition $\sum\limits_{i=0}^{4} p_i = 1$, we obtain

$$p_0 = \tfrac{\mu}{\triangle} \tag{3.6.6}$$

$$p_1 = \tfrac{\lambda\mu(\lambda k+\mu)}{\triangle} \tag{3.6.7}$$

$$p_2 = \tfrac{k\lambda\mu^2}{\triangle} \tag{3.6.8}$$

$$p_3 = \tfrac{\lambda^2(\lambda k+\mu)}{\triangle} \tag{3.6.9}$$

and

$$p_4 = \tfrac{k\lambda^2[\lambda k+(k+1)\mu]}{\triangle} \tag{3.6.10}$$

where $\triangle = [\mu^3 + \lambda(\lambda+\mu)(k\lambda+\mu) + \lambda k(k\lambda^2+\mu^2) + k(k+1)\lambda^2\mu]$

Since S_3 and S_4 correspond to system down states, the steady state availability of the system is given by

$$\begin{aligned} A_\infty &= 1\text{-}(\mathrm{p}_3 + p_4) \\ &= \tfrac{\mu(\lambda+\mu)(\lambda k+\mu)}{\triangle} \end{aligned} \tag{3.6.11}$$

In the following sections, we obtain a CAN estimator, a $100(1-\alpha)\%$ asysmptotic confidence interval for the steady state availability of the system and an estimator of the system reliability.

3.7 CONFIDENCE INTERVAL FOR STEADY STATE AVAILABILITY OF THE SYSTEM

Let $X_1, X_2, ..., X_n$ be a random sample of size n of times to failure of the unit with pdf given by

$$f(x) = \lambda e^{-\lambda x}, 0 < x < \infty, \lambda > 0. \tag{3.7.1}$$

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n of times to repair of the unit with the pdf as in (3.2.1), where k is known. It is clear that $E(\bar{X}) = \frac{1}{\lambda}$ and $E(\frac{\bar{Y}}{k+1}) = \frac{1}{\mu}$, where $\bar{X}$ and $\bar{Y}$ are the sample means of time to failure and time to repair of the unit respectively. It can be shown that $\bar{X}$ is the maximum likelihood estimator also (moment estimator) of $\frac{1}{\lambda}$ and $\frac{\bar{Y}}{k+1}$ is the moment estimator of $\frac{1}{\mu}$.

Let $\theta_1 = \frac{1}{\lambda}$ and $\theta_2 = \frac{1}{\mu}$, clearly, the steady state availability given in (3.6.11) reduces to

$$A_\infty = \frac{\theta_1(\theta_1+\theta_2)(\theta_1+k\theta_2)}{[\theta_1^3+\theta_2(\theta_1+\theta_2)(k\theta_2+\theta_1)+k\theta_2(\theta_1^2+k\theta_2^2)+k(k+1)\theta_1\theta_2^2)]} \tag{3.7.2}$$

and hence an estimator of A_∞ is given by

$$\hat{A}_{\infty} = \frac{(k+1)\bar{X}[(k+1)\bar{X}+\bar{Y}][(k+1)\bar{X}+k\bar{Y}]}{[(k+1)^3\bar{X}^3+\bar{Y}[(k+1)\bar{X}+\bar{Y}][(k+1)\bar{X}+k\bar{Y}]+k\bar{Y}[(k+1)^2\bar{X}^2+k\bar{Y}^2]+k(k+1)^2\bar{X}\bar{Y}^2]} \quad (3.7.3)$$

It may be noted that $\hat{A}_{\infty}$ is a real valued function in $\bar{X}$ and $\bar{Y}$, which is also differentiable. Now, consider the following application of multivariate central unit theorem (see Rao (1974).

3.7.1 APPLICATION OF MULTIVARIATE CENTRAL LIMIT THEOREM

Suppose $T_1^{'}, T_2^{'}, T_3^{'}, ...$ are independent and identically distributed k - dimensional random variables such that

$$T_n^{'} = (T_{1n}, T_{2n}, ..., T_{kn}), n = 1, 2, 3, ...,$$

having the first and second order moments $E(T_n) = \mu$ and $Var(T_n) = \Sigma$.

Define the sequence of random variables.

$$\bar{T}_n = (\bar{T}_{1n}, \bar{T}_{2n}, ..., \bar{T}_{kn}), n = 1, 2, 3, ...,$$

where

$$\bar{T}_{1n} = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, i = 1, 2, ..., k : j = 1, 2, ...n$$

then, $\sqrt{n}(\bar{T}_n - \mu) \xrightarrow{d} N_k(0, \Sigma)$ as n$\rightarrow \infty$

3.7.2 CAN ESTIMATOR

By applying the multivariate central limit theorem given in section 3.4.1, we get

$\sqrt{n}[(\bar{X}\bar{Y}/(k+1)-(\theta_1,\theta_2)] \xrightarrow{d} N(0,\Sigma)$ as $n\to\infty$, where the dispersion matrix

$$\Sigma = diag(\theta_1^2, \tfrac{(k^2+1)}{(k+1)^2}\theta_2^2).$$

Again from Rao (1974), we have

$$\sqrt{n}(\hat{A_\infty} - A_\infty) \xrightarrow{d} N(0,\sigma^2(\theta)) \text{ as } n\to\infty.$$

where $\theta=(\theta_1,\theta_2)$ and

$$\begin{aligned}\sigma^2(\theta) &= \sum_{i=1}^{2}(\tfrac{\partial A_\infty}{\partial\theta_i})^2\sigma_{ii} \\ &= \theta_1^2(\tfrac{\partial A_\infty}{\partial\theta_1})^2 + \tfrac{(k^2+1)}{(k+1)^2}\theta_2^2(\tfrac{\partial A_\infty}{\partial\theta_2})^2\end{aligned} \tag{3.7.4}$$

Substituting for $(\frac{\partial A_\infty}{\partial\theta_1}), i=1,2$ in (3.4.4), we obtain $\sigma^2(\theta)$. Hence $\hat{A_\infty}$ is a CAN estimator of A_∞. There are several methods for generating CAN estimators and the method of moments and the method of maximum likelihood are commonly used to generate such estimators. see Sinha (1986).

3.7.3 CONFIDENCE INTERVAL FOR THE STEADY STATE AVAILABILITY

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely $\hat{\theta} = (\bar{X}, \frac{\bar{Y}}{(k+1)}$. Let $\hat{\sigma^2} = \sigma^2(\hat{\theta^2})$. Since $\sigma^2(\theta)$ is a continuous function of $\theta, \hat{\sigma^2}$ is a consistent estimator of $\sigma^2(\theta)$, i.e. $\hat{\sigma^2} \xrightarrow{p} \sigma^2(\theta)$ as $n \to \infty$. By Slutsky theorem, we have

$$\frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$$

i.e.,

$$Pr\left[-k_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} < k_{\frac{\alpha}{2}}\right] = (1-\alpha),$$

where $k_{\frac{\alpha}{2}}$ is obtained from normal tables. Hence a $100(1-\alpha)\%$ confidence interval for A_∞ is given by $\hat{A_\infty} \pm k_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$, (3.7.5)

where $\hat{\sigma}$ is obtained from (3.7.4).

3.7.4 AN ESTIMATOR OF SYSTEM RELIABILITY BASED ON MOMENTS

Since $\bar{X}$ and $\frac{\bar{X}}{(k+1)}$ are the moment estimators of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively, we obtain an estimator of system reliability as follows:

$$\hat{R}(t) = \sum_{i=1}^{3} \frac{\frac{1}{k\bar{X}^2\bar{Y}^2}[[k\bar{Y}(\alpha_i\bar{X}+1)+(k+1)\bar{X}][\bar{Y}(\alpha_i\bar{X}+2)+(k+1)\bar{X}]+k(k+1)\bar{X}\bar{Y}]}{\prod_{i=1, j\neq i}^{3}(\hat{\alpha}_i - \hat{\alpha}_j)} e^{\hat{\alpha} t}$$

with

$\hat{a} = \frac{(k+1)^2\bar{X}+3k\bar{Y}}{k\bar{X}\bar{Y}}$

$\hat{b} = \frac{3k\bar{Y}^2+(k+1)^2\bar{X}(\bar{X}+2\bar{Y})}{k\bar{X}^2\bar{Y}^2}$

and

$\hat{c} = \frac{(k+1)^2\bar{X}+k\bar{Y}}{k\bar{X}^3\bar{Y}}$

CHAPTER 4

ASYMPTOTIC CONFIDENCE LIMITS FOR A TWO-UNIT COLD STANDBY SYSTEM WITH ONE REGULAR REPAIRMAN AND EXPERT REPAIRMAN

[1]

[1] A modified version of this chapter was presented at the International Conference in India, which was held in honour of Prof. C.R. Rao

4.1 INTRODUCTION

The object of introducing inspection is two-fold:

(ii) To increase the reliability of the system and

(ii) To avoid failure of the operating systems, which may be costly and dangerous. Weiss (1962) was the first to consider a single unit system with inspection.

Many researchers, Mazumdar (1970), Luss (1977), Keller (1982), investigated various types of maintenance policies with inspection under different sets of assumptions. In all these studies, the time needed for inspection was assumed to be negligible, but in the actual situation there are many cases where the time needed for inspection is not negligible. Another practical aspect, which is generally left out, is that the repairman employed may not be perfect.

In this paper the concept of inspection with a non-negligible time period, together with two repairmen for a two-unit cold standby system - one regular repairman and one expert repairman is introduced. The regular repairer man is always with the system and has a dual role of inspection facility and repair facility, with the known fact that he might not be able to do some complex repairs within some tolerable (patience) time. The patience time is that for which one can wait while the regular repairman tries to repair a failed unit. The expert repairman is called on to do the job on completion of the patience time or on a system failure, which ever is earlier. We also, study the asymptotic confidence limits for the availability of this system [see, Yadavalli et al (2004)].

4.2 SYSTEM DESCRIPTION

1. The system consists of two units. Initially one unit is operating on line and the other one is kept as a cold standby.

2. Failure of a unit is detected by inspection only but system failure is detected instantaneously without inspection.

3. Inspection is carried out periodically. The interval between two successive inspections is a random variable, which is exponentially distributed with parameter d. If by inspection it is revealed that a unit has failed, it is forthwith taken out of the system and repaired. During the time a repair takes place, inspection is held in a state of temporary suspension. The inspections recommences with the same distribution as above, as soon as the repair is complete.

4. Inspection is of instantaneous duration. The probability of discovering a failure by inspection equals one. Inspection does not degrade a unit (if operating).

5. Time to failure of a unit is exponentially distributed with parameter λ.

6. When failure of a unit is detected repair of the failed unit and switching to the standby unit start simultaneously. Switchover is instantaneously.

7. When both units fail, the system becomes inoperable.

8. When the expert repairman is called on to do the job, it takes negligible time to reach the system.

9. Repair times (regular and expert) and patience times are exponentially distributed random variables with parameters c_1 and c_2 and λ , respectively.

10. The expert repairman leaves the system only when both the units are operative.

11. After any repair, a unit works like a new one.

12. All random variables are mutually independent.

NOTATION

0	: Intital state of the system (one unit is operating and the other is kept cold standby)
r,e	: Regular and expert repairman
$A_i(t)$	: P[system is available at t/S_i at t = 0]
S_i	: State i
©	: Convolution sysmbol

State(i)	0	1	2	3	4
Unit 1	0	urr	ure	ure	Ure.r
Unit 2	0s	0	0	qr	qr

0 operable

urr : under repair by regular repairman

ure : under repair by expert repairman

0s : operable standby

qr : queing for repair

$$A_0(t) = e^{-(\lambda+d)t} + \lambda t e^{-(\lambda+d)t} + d e^{-(\lambda+d)t} © \lambda A_0(t) + \lambda t e^{-(\lambda+d)t} © A_1(t) + \lambda^2 t e^{-(\lambda+d)t} © A_3(t) \tag{4.1.1}$$

$$A_1(t) = e^{-(\lambda+c_2+\theta)t} + c_2 e^{-(\lambda+c_2+\theta)t} © A_0(t) + \theta e^{-(\lambda+c_2+\theta)t} © A_2(t) + \lambda e^{-(\lambda+c_2+\theta)t} © A_3(t) \tag{4.1.2}$$

$$A_2(t) = e^{-(\lambda+c_2\theta)t} + c_1 e^{-(\lambda+c_1)t} © A_0(t) + \lambda e^{-(\lambda+c_1)t} © A_2(t) \tag{4.1.3}$$

$$A_3(t) = c_1 e^{-c_1 t} © A_2(t) \tag{4.1.4}$$

Solving the above equations (1) - (4), we get the steady state availability as,

$$A_\infty = \lim_{t\to\infty} A_0(t) \tag{4.1.5}$$

$$A_\infty = \frac{N}{D}$$

For the estimation of failure rates, repair rates (regular and expert), patience

$$\begin{aligned} \mathrm{N} = \quad & 2\lambda d + 3\lambda\theta + c_2 d + d\theta + 2\lambda c_2 + 3\lambda^2 + \lambda^2 c_2 + \lambda^2\theta + \lambda^3 \\ \mathrm{D} = \quad & \lambda d\theta + \lambda^2 d + \lambda^2 c_2 + \lambda^2\theta + \lambda^3 + \lambda^2 dc - \lambda dc + \theta - \lambda^2 c_1 c_2 - \\ & \lambda^3 c_1\theta - \lambda^3 c_1 \end{aligned}$$

rates, let $X_{i1}X_{i2}, \ldots, X_{in}$,(i=1,2, 3, 4) be random samples of size n, drawn from different exponential populations with respective parameters λ , c_1, c_2 and θ .

For this analysis, let

$\alpha_1 = \frac{1}{\lambda}, \alpha_2 = \frac{1}{c_1}, \alpha_3 = \frac{1}{c_2}, \alpha_4 = \frac{1}{\theta}, \alpha_5 = \frac{1}{d}$

The sample means

$\bar{x}_1 = \frac{1}{n}\sum_{j=1}^{n} x_{1j}, \bar{x}_2 = \frac{1}{n}\sum_{j=1}^{n} x_{2j}, \bar{x}_3 = \frac{1}{n}\sum_{j=1}^{n} x_{3j}, \bar{x}_4 = \frac{1}{n}\sum_{j=1}^{n} x_{4j}$

will then be respective MLE'S of the $\alpha_i, i = 1, 2, 3, 4$. Substitution leads to

$$A_\infty = \frac{N_1}{D_1} \tag{4.1.6}$$

and

$$\hat{A}_\infty = \frac{N_2}{D_2} \tag{4.1.7}$$

$$\mathrm{N}_1 = \alpha_2[2\alpha_1^2\alpha_3\alpha_4 + 3\alpha_1^2\alpha_3\alpha_5 + \alpha_1^3\alpha_4 + \alpha_1^3\alpha_3 + 2\alpha_1^2\alpha_4\alpha_5 + 3\alpha_1\alpha_3\alpha_4\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_3\alpha_4\alpha_5]$$

$$\mathrm{D}_1 = \alpha_1^2\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_3\alpha_4\alpha_5 + \alpha_1\alpha_3\alpha_4 - \alpha_1^2\alpha_3 - \alpha_1\alpha_4\alpha_5 - \alpha_3\alpha_5 - \alpha_3\alpha_4\alpha_5$$

$$\mathrm{N}_2 = \bar{x}_2[2\bar{x}_1^2\bar{x}_3\bar{x}_4 + 3\bar{x}_1^2\bar{x}_3\bar{x}_5 + \bar{x}_1^3\bar{x}_4 + \bar{x}_1^3\bar{x}_3 + 2\bar{x}_1^2\bar{x}_4\bar{x}_5 + 3\bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_3\bar{x}_5 + \bar{x}_3\bar{x}_4\bar{x}_5]$$

$$\mathrm{D}_2 = \bar{x}_1^2\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4 + \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_5 + \bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_3\bar{x}_4 - \bar{x}_1^2\bar{x}_3 - \bar{x}_1\bar{x}_4\bar{x}_5 - \bar{x}_3\bar{x}_5 - \bar{x}_3\bar{x}_4\bar{x}_5$$

Application of multivariate central limit theorem (Rao 1974), leads to

$$\sqrt{n}(\bar{x} - \alpha) \xrightarrow{d} N_5(0, \Sigma) \text{ as n} \to \infty \tag{4.1.8}$$

$\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$

$\Sigma = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2)$

where

For n $\rightarrow \infty$

$$\sqrt{n}(\hat{A}\infty - A\infty) \xrightarrow{d} N(0, \sigma^2(\alpha))$$

where

$$\sigma^2(\alpha) = \sum_{i=1}^{5} \left(\frac{\partial A_\infty}{\partial \alpha_i}\right)^2 \sigma_{ii}$$

$$\sigma^2(\alpha) = \sum_{i=1}^{5} \left(\frac{\partial A_\infty}{\partial \alpha_i}\right)^2 \alpha_i^2$$

Replacing by its consistent estimator

$$\hat{\alpha} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$$

it follows that $\hat{\sigma}^2 = \sigma^2(\hat{\alpha})$ is a consistent estimator of $\sigma^2(\alpha)$. Then by Slutzky's theorem $\frac{(\hat{A}_\infty - A_\infty)}{\hat{\sigma}\sqrt{n}} \xrightarrow{d} N(0, 1)$ as n $\rightarrow \infty$

This implies that

$$P\left[-\frac{k_\alpha}{2} \leq \frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \leq \frac{k_\alpha}{2}\right] = 1 - \alpha$$

and the $100(1-\alpha)\%$ confidence limits for A_∞ are therefore

$$\hat{A}_\infty \pm \frac{k_\alpha}{2} . \frac{\hat{\sigma}}{\sqrt{n}}.$$

CHAPTER 5

CONFIDENCE LIMITS FOR A COMPLEX THREE-UNIT PARALLEL SYSTEM WITH ”PREPARATION TIME” FOR THE REPAIR FACILITY

[1]

[1] A modified version of this chapter is submitted to ORiON.

5.1 INTRODUCTION

Multiple unit systems have attracted the attention of many applied probabilists and reliability engineers for their applicability in their respective fields. Kistner and Subramanian (1974) considered an n-unit warm standby redundant system with a single repair facility. In this case, the probability density function of the life time of the online unit was assumed to be arbitrary while all the other distributions are exponential; these results were later extended by Subramanian et al(1976). Gupta et al (1986) studied the cost-benefit analysis of a single server three unit redundant system with inspection, delayed replacement and two types of repair. Kalpakam et al (1987) have considered a multiple component system in which n identical units connected in series are needed for the system to function, the units being supported by m spares and a single repair facility (Keandlin, 2005). Gupta and Bansal (1991)have analysed a cost function for a three unit standby system subject to random shocks and linearly increasing failure rates. The study of n-unit systems, even in the case of cold standbys, appears to be rather complicated. Yadavalli and Parvez (1984) studied a three unit system in which all the distributions are assumed discrete. Muller (2005) studied a three-unit standby system when the lifetime and repair time distributions are assumed to be arbitrary. She obtained expressions for reliability and availability. In all the above models, it is clear that they have assumed that the repair facility is continuously available to attend to the repair of the failed units (see Van der Heijden (1989), Fawzi and Hawkes (1991), Smith and Dekker (1997), Bon and Păltănea (2001), Krishnamoorthy et al (2002), Frostig and Levikson (2002), Ke and Wang (2004), De Smidt-Destombesa et al (2004)).

But it is reasonable to expect that a preparation might be needed to get the repair facility ready before the next repair could be taken up. If this preparation is started only when a unit arrives for repair, it is easy to solve the problem, since the preparation time plus the actual repair time may be taken as the total repair time. But this preparation time starts immediately after each repair completion, so that the facility becomes available at the earliest. Two-unit parallel system with two-dissimilar units and preparation time was studied by Sarma (1982). He assumed that the repair times and preparation time are to be non-markovian. The confidence limits for a two-unit parallel was subsequently studied by Yadavalli et al (2002). In this chapter, a three-unit parallel system is studied in which the repair facility is not available for a random time after each repair completion. This non-available period is called the 'preparation time'.

5.2 SYSTEM DESCRIPTION AND NOTATION

1. The system consists of three identical units connected in parallel.

2. At t = 0, all the units are new and the repair facility is available.

3. There is only one repair facility.

4. The repair facility is not available for a random time after each repair completion. This 'preparation time' is necessary for the repair facility before the next repair could be taken up.

5. The life time, repair time and the preparation time are assumed to be exponential with parameters λ, μ, γ respectively.

6. The life times, the repair times of the units and the preparation time for the repair facility are independent random variables.

5.3 AVALABILITY ANALYSIS

The following states will be used in the solution of the problem (see Table 5.3.1 and Table 5.3.2).

5.3.1 n-UNIT PARALLEL SYSTEM

Let the state of the system be (i,j), where i is the number of failed units, and j is the state of the repair facility (0: available, 1: not available). State transitions are presented in Table 5.3.1

Table 5.3.1

STATE				
From	To	RATE		
(i,0)	(i+1,0)	(n-i)λ	;	i=0,1,2, ... , n-1
(i,0)	(i-1,1)	μ	;	i=1,2, ... , n
(i,1)	(i,0)	γ	;	i=1,2, ... , n
(i,1)	(i+1,1)	(n-i)λ	;	i=0,1,2, ... , n-1

When n=3, the possible transitions are presented in Table 5.3.2

Table 5.3.2

STATE		
From	To	RATE
(0,0)	(1,0)	3λ
(0,1)	(0,0)	γ
(1,0)	(2,0)	2λ
(1,0)	(0,1)	μ
(0,0)	(1,0)	3λ
(1,1)	(1,0)	γ
(2,0)	(3,0)	λ
(2,0)	(1,1)	μ
(1,0)	(2,0)	2λ
(2,1)	(2,0)	γ
(3,0)	(2,1)	3μ
(2,0)	(3,0)	λ

Figure 5.3.1 gives the possible states of the 3-unit system at any time and also the transition intensities. Let us derive the balance equations for the steady-state probability distributions of the number of failed units in the system. Let

N(t) $\equiv$ Number of failed units at time t.

R(t) $\equiv$ The state of the repair facility at time t.

Then {N(t), R(t)} is a continuous time markov process on the state space.

S = {(i,j); i=1,2,3,; j=0,1}

We define $\mathrm{p}_{ij}(t) = P[N(t) = i, R(t) = j]$

$$\mathrm{p}_{00}^{'}(t) = -3\lambda p_{00}(t) + \gamma p_{01}(t) \tag{5.3.1}$$

$$p_{10}^{'}(t) = -(2\lambda + \mu)p_{10}(t) + 3\lambda p_{00}(t) + \gamma p_{11}(t) \tag{5.3.2}$$

$$p_{20}^{'}(t) = -(\lambda + \mu)p_{20}(t) + 2\lambda p_{10}(t) + \gamma p_{21}(t) \tag{5.3.3}$$

$$p_{30}^{'}(t) = -\mu p_{30}(t) + \lambda p_{20}(t) + \gamma p_{31}(t) \tag{5.3.4}$$

$$p_{01}^{'}(t) = -(3\lambda + \gamma)p_{01}(t) + \mu p_{10}(t) \tag{5.3.5}$$

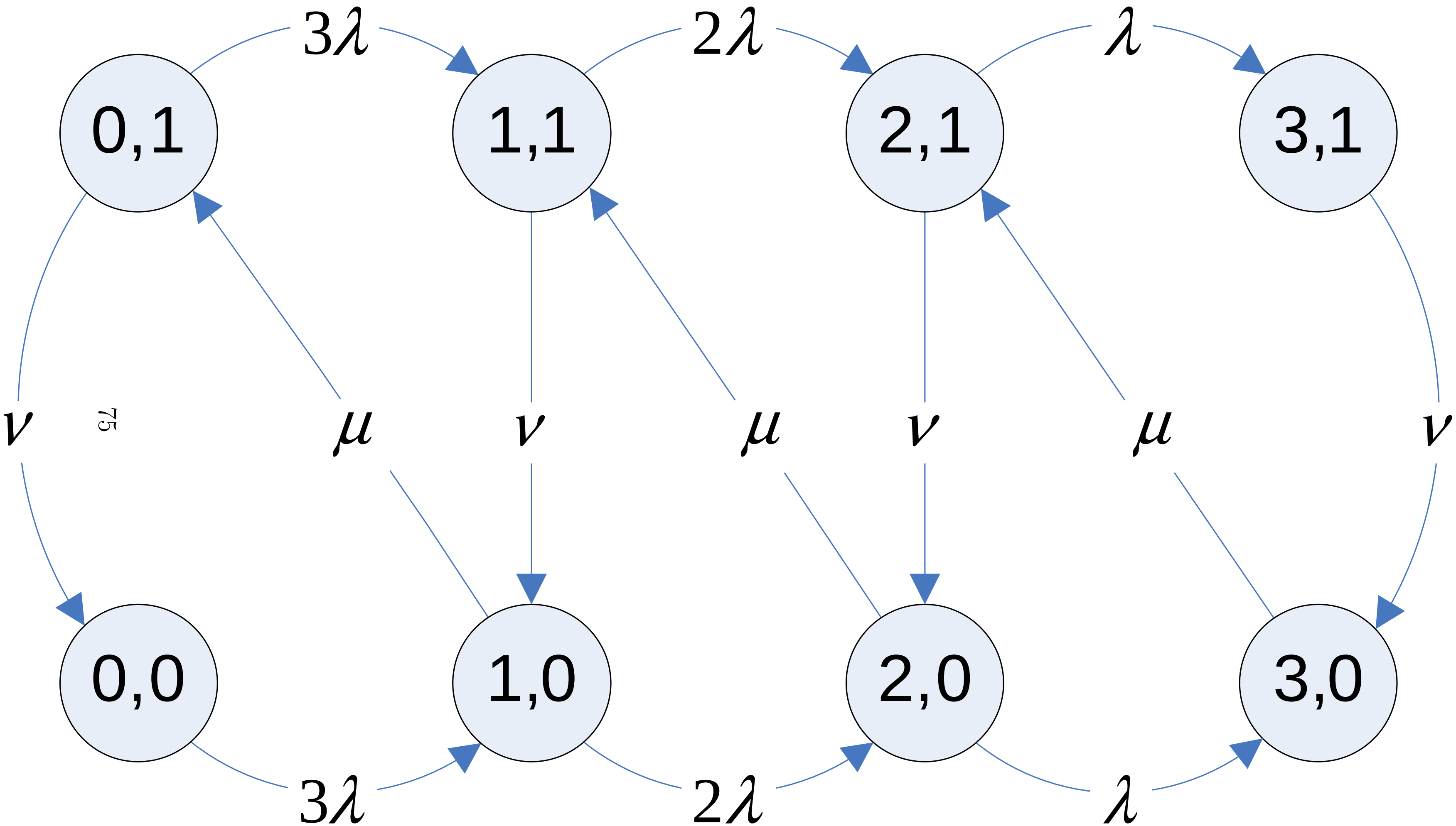


Figure 5.3.1

$$p_{11}^{'}(t) = -(2\lambda + \gamma)p_{11}(t) + 3\lambda p_{01}(t) + \mu p_{20}(t) \tag{5.3.6}$$

$$p_{21}^{'}(t) = -(\lambda + \gamma)p_{21}(t) + 2\lambda p_{11}(t) + \mu p_{30}(t) \tag{5.3.7}$$

$$p_{31}^{'}(t) = -\gamma p_{31}(t) + \lambda p_{21}(t) \tag{5.3.8}$$

In the steady-state

$$\mathrm{p}_{ij} = \lim_{t\to\infty} P[N(t) = i, R(t) = j]$$

From (5.3.1) - (5.3.8), we can easily obtain the steady-state equations.

$$3\lambda p_{00} = \gamma p_{01} \tag{5.3.9}$$

$$(2\lambda + \mu)p_{10} = 3\lambda p_{00} + \gamma p_{11} \tag{5.3.10}$$

$$(\lambda + \mu)p_{20} = 2\lambda p_{10} + \gamma p_{21} \tag{5.3.11}$$

$$\mu p_{30} = \lambda p_{20} + \gamma p_{31} \tag{5.3.12}$$

$$(3\lambda + \gamma)p_{01} = \mu p_{10} \tag{5.3.13}$$

$$(2\lambda + \gamma)p_{11} = 3\lambda p_{01} + \mu p_{20} \tag{5.3.14}$$

$$(\lambda + \gamma)p_{21} = 2\lambda p_{11} + \mu p_{30} \tag{5.3.15}$$

$$\gamma p_{31} = \lambda p_{21} \tag{5.3.16}$$

Since the system is operable in states {(1,0),(0,0),(2,0),(0,1),(1,1),(2,1)}, the steady-state availability of the system is given by

$$A_\infty = \sum_{n=0}^{2} p_{n0} + \sum_{n=0}^{2} p_{n1}$$

5.4 ESTIMATES FOR STEADY-STATE PROBABILITIES AND SYSTEM PERFORMANCE MEASURES

Let $X_1, X_2, \cdots, X_n$ be a sample of failure times for operating units with probability density function (pdf)

$$f_1(x) = \lambda e^{-x}; x > 0; \lambda > 0$$

Let $Y_1, Y_2, \cdots, Y_n$ be a sample of repair times for the failed unit with pdf

$$f_2(y) = \mu e^{-\mu y}; y > 0; \mu > 0$$

Let $Z_1, Z_2, \cdots, Z_n$ be a sample of preparation times of the repair facility with pdf

$f_3(z) = \gamma e^{-\gamma z}; z > 0; \gamma > 0$

Let $\bar{X}, \bar{Y}, \bar{Z}$ be the sample means of the time to failure for operating unit, the time to repair for the failed units, and the time to preparation for the repair facility, respectively.

Then $E(\bar{X}) = \frac{1}{\lambda}, E(\bar{Y}) = \frac{1}{\mu}, E(\bar{Z}) = \frac{1}{\gamma}$

It can be easily shown that $\bar{X}, \bar{Y}, \bar{Z}$ are the maximum likelihood estimates of $\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\gamma}$ respectively.

Furthermore, let $\hat{p}_{ij}$ be estimators of p_{ij}.

We can now obtain the estimator of A_∞

$$\hat{A}_\infty = \sum_{n=0}^{2} \hat{p}_{n0} + \sum_{n=0}^{2} \hat{p}_{n1} \tag{5.3.17}$$

5.5 CONFIDENCE LIMITS FOR AVAILABILITY

From the discussion in the previous section, we know that $\hat{A}_\infty$ is a real-valued function in $\bar{X}, \bar{Y}, \bar{Z}$, which is also differentiable using the application of the multivariate central limit theorem (see Rao, 1973), it follows that

$$\sqrt{n}[(\bar{X}, \bar{Y}, \bar{Z}) - (\theta_1, \theta_2, \theta_3)] \text{ converges to } N_3(0, \Sigma) \text{ in distribution as } n \to \infty$$

Where dispersion matrix

$$\Sigma = [\sigma_{ij}^2]_{3\times 3}$$

is given by

$$\Sigma = diag(\theta_1^2, \theta_2^2, \theta_3^2)$$

using the results by Rao (1973), we have

$$\sqrt{n}[\hat{A}_\infty - A_\infty] \xrightarrow{D} N_3(0, \sigma_1^2(\theta)) \text{ as } n \to \infty$$

with

$$\sigma_1^2(\theta) = \sum_{i=1}^{3} \left[\frac{\partial A_\infty}{\partial \theta_i}\right]^2 \sigma_{ii}$$

where

$$\theta = (\theta_1, \theta_2, \theta_3)$$

Let $\sigma_1^2(\hat{\theta})$ be the estimator for $\sigma_1^2(\theta)$ which is obtained by replacing θ by a consistent estimator $\hat{\theta} = (\bar{X}, \bar{Y}, \bar{Z})$. Since $\sigma_1^2(\theta)$ is a continuous function of θ , we know that $\sigma_1^2(\hat{\theta})$ is consistent estimator of $\sigma_1^2(\theta)$ [see Wackerly et al (2002)]

Therefore $\quad \sigma_1^2(\hat{\theta}) \to \sigma_1^2(\theta)$ as $n \to \infty$.

using Slutzky's theorem, we have

$$\frac{\sqrt{n}[\hat{A}_\infty - A_\infty]}{\sigma_1^2(\hat{\theta})} \xrightarrow{D} N(0,1) \text{ as n} \to \infty$$

Which leads to

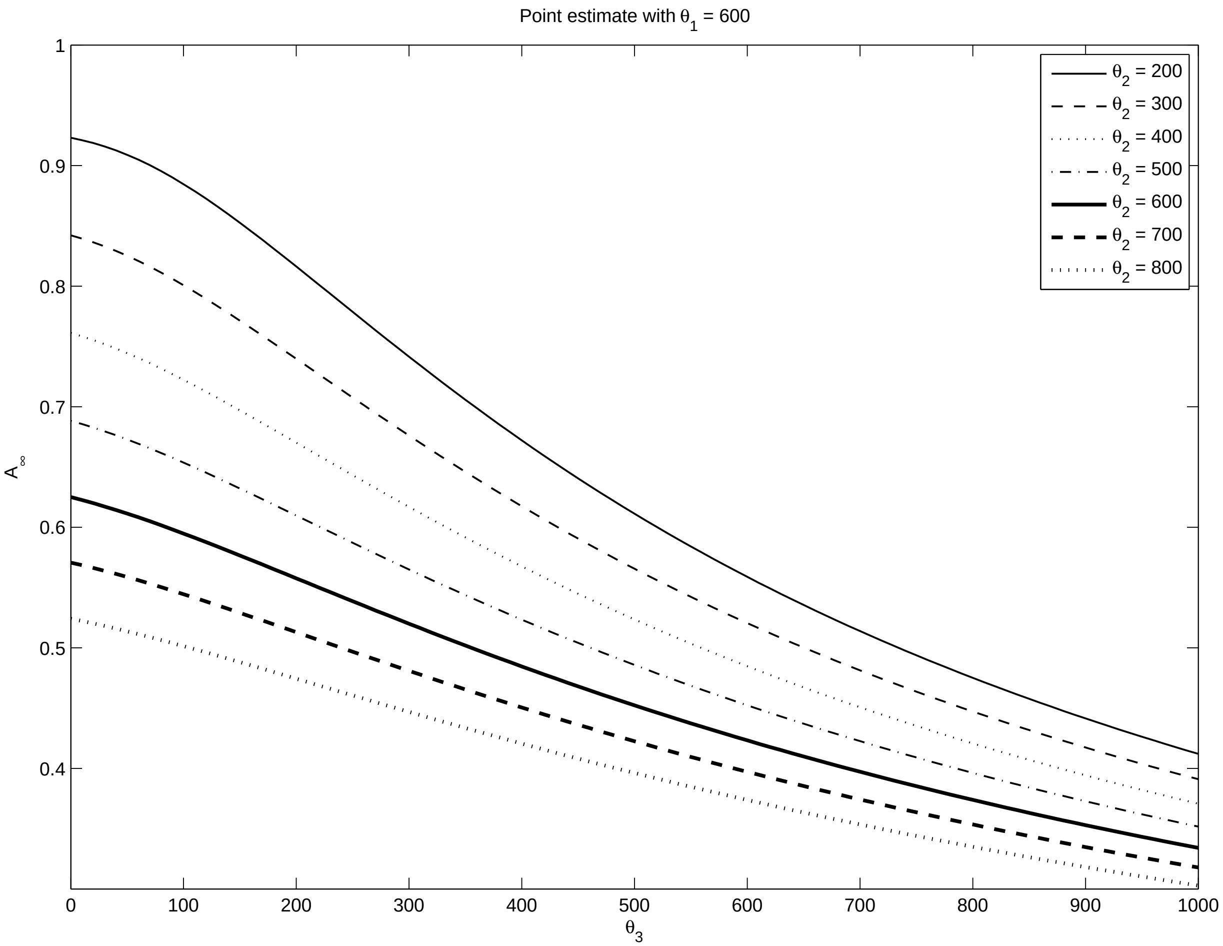
$$P\left[-Z_{\frac{\alpha}{2}} \le \frac{\sqrt{n}[\hat{A}_\infty - A_\infty]}{\sigma_1^2(\hat{\sigma})} \le Z_{\frac{\alpha}{2}}\right] = 1 - \alpha$$

where $Z_{\frac{\alpha}{2}}$ is determined from the standard normal tables or statistical software packages. Hence, the asymptotic $100(1-\alpha)\%$ confidence limits for A_∞ are given by

$$\hat{A}_\infty \pm Z_{\frac{\alpha}{2}} \frac{\sigma_1(\hat{\theta})}{\sqrt{n}}.$$

5.6 NUMERICAL ILLUSTRATION

In this section we provide numerical results of steady-state availability, A_∞. Figure 5.6.1. explains that, for fixed failure times and preparation times, we plotted the repair times vs the A_∞.



Point estimate with θ1 = 600
A∞
θ3
θ2 = 200
θ2 = 300
θ2 = 400
θ2 = 500
θ2 = 600
θ2 = 700
θ2 = 800
0
100
200
300
400
500
600
700
800
900
1000
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 5.6.1

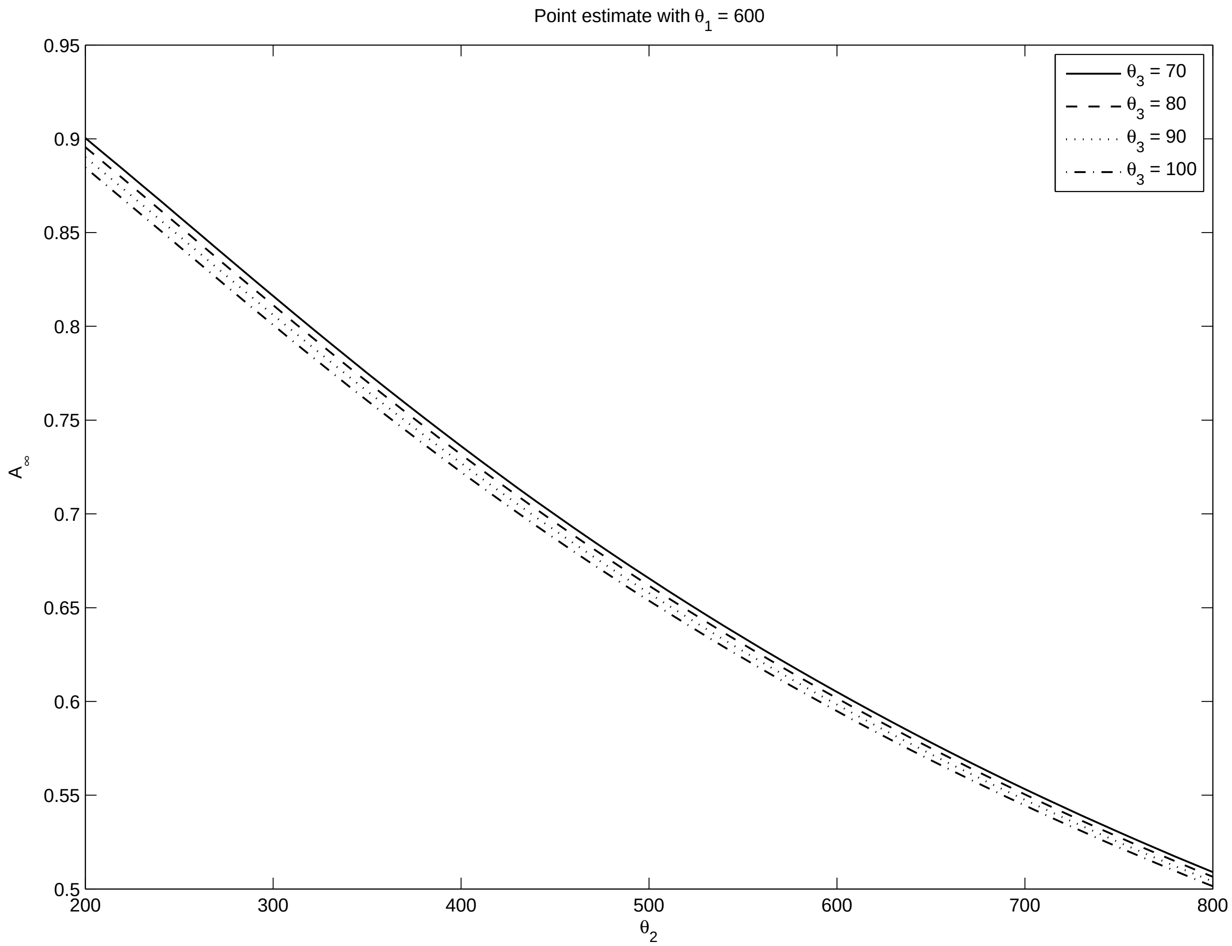

Point estimate with θ_1 = 600
θ_3 = 70
θ_3 = 80
θ_3 = 90
θ_3 = 100
A_∞
θ_2
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
200
300
400
500
600
700
800

Figure 5.6.2

Table 5.6.1a: 95% Confidence Interval for A_∞ for $\theta_1 = 600$

	θ_3	70	80	90	100
n	θ_2				
100	200	0.8499, 0.9509	0.8433, 0.9476	0.8363, 0.9441	0.8290, 0.9402
	300	0.7415, 0.8909	0.7357, 0.8869	0.7297, 0.8826	0.7234,0.8781
	400	0.6478, 0.8245	0.6430, 0.8203	0.6381, 0.8160	0.6331, 0.8115
	500	0.5710, 0.7602	0.5672, 07563	0.5633, 0.7522	0.5593, 0.7481
	600	0.5087, 0.7014	0.5056, 0.6978	0.5024, 0.6942	0.4992, 0.6904
	700	0.4577, 0.6489	0.4552, 0.6457	0.4526, 0.6424	0.4500, 0.6391
	800	0.4155, 0.6024	0.4134, 0.5996	0.4113, 0.5967	0.4091, 0.5938
200	200	0.8646, 0.9361	0.8586, 0.9324	0.8521, 0.9283	0.8453, 0.9239
	300	0.7634, 0.8690	0.7578, 0.8647	0.7521, 0.8602	0.7461, 0.8554
	400	0.6736, 0.7986	0.6690, 0.7944	0.6642, 0.7900	0.6592, 0.7854
	500	0.5987, 0.7325	0.5949, 0.7286	0.5909, 0.7245	0.5869, 0.7204
	600	0.5369, 0.6731	0.5337, 0.6697	0.5305, 0.6661	0.5272, 0.6624
	700	0.4857, 0.6209	0.48831, 0.6178	0.4804, 0.6146	0.4777, 0.6114
	800	0.4429, 0.5750	0.4407, 0.5723	0.4384, 0.5696	0.4362, 0.5667
1000	200	0.8844, 0.9163	0.8790, 0.9120	0.8732, 0.9072	0.8670, 0.9022
	300	0.7926, 0.8398	0.7874, 0.8352	0.7819, 0.8303	0.7763, 0.8252
	400	0.7082, 0.7661	0.7036, 0.7597	0.6989, 0.7552	0.6941, 0.7505
	500	0.6357, 0.6955	0.6318, 0.6916	0.6279, 0.6876	0.6238, 0.6835
	600	0.5745, 0.6355	0.5713, 0.6321	0.5680, 0.6286	0.5646, 0.6250
	700	0.5230, 0.5835	0.5203, 0.5835	0.5203, 0.5805	0.5175, 0.5744
	800	0.4794, 0.5385	0.4771, 0.5359	0.4747, 0.5333	0.4722, 0.5306

Table 5.6.1b: 99% Confidence Interval for A_∞ for $\theta_1 = 600$

	θ_3	70	80	90	100
n	θ_2				
100	200	0.8344, 0.9664	0.8274, 0.9636	0.8199, 0.9606	0.8120, 0.9573
	300	0.7186, 0.9138	0.7126, 0.9100	0.7063, 0.9060	0.6998, 0.9017
	400	0.6207, 0.8516	0.6159, 0.8475	0.6109, 0.8432	0.6058, 0.8388
	500	0.5420, 0.7891	0.5382, 07852	0.56344, 0.7811	0.5304, 0.7769
	600	0.4792, 0.7309	0.4762, 0.7272	0.4731, 0.7235	0.4699, 0.7197
	700	0.4284, 0.6781	0.4260, 0.6748	0.4235, 0.6715	0.4210, 0.6681
	800	0.3869, 0.6310	0.3849, 0.6281	0.3829, 0.6251	0.3808, 0.6220
200	200	0.8537, 0.9470	0.8734, 0.9436	0.8405, 0.9400	0.8320, 0.9360
	300	0.7472, 0.8852	0.7415, 0.8811	0.7355, 0.8767	0.7293, 0.8722
	400	0.6545, 0.8177	0.6498, 0.8136	0.6449, 0.8092	0.6399, 0.8047
	500	0.5782, 0.7530	0.5744, 0.7490	0.5705, 0.7450	0.5665, 0.7408
	600	0.5160, 0.6940	0.5129, 0.6905	0.5098, 0.6868	0.5065, 0.6831
	700	0.4650, 0.6416	0.4624, 0.6384	0.4598, 0.6352	0.5272, 0.6319
	800	0.4226, 0.5953	0.4205, 0.5925	0.4184, 0.5896	0.4162, 0.5867
1000	200	0.8795, 0.9212	0.8739, 0.9170	0.8680, 0.9125	0.8616, 0.9076
	300	0.7853, 0.8471	0.7801, 0.8425	0.7745, 0.8377	0.7688, 0.8327
	400	0.6996, 0.7726	0.6951, 0.7683	0.6903, 0.7638	0.6855, 0.7592
	500	0.6265, 0.7047	0.6227, 0.7008	0.6187, 0.6968	0.6147, 0.6927
	600	0.5652, 0.6448	0.5620, 0.6414	0.5587, 0.6379	0.5553, 0.6343
	700	0.5138, 0.5928	0.5111, 0.5898	0.5083, 0.5867	0.5055, 0.5836
	800	0.4704, 0.5476	0.4681, 0.5450	0.4657, 0.5423	0.4633, 0.5396

CHAPTER 6

AN INTERMITTENTLY USED k OUT OF n : F SYSTEM

6.1 INTRODUCTION

In general, a system can be classified as one of the following two types depending on its usage - one which is used continuously and another which is used intermittently. In this chapter our interest is in the latter i.e. in an intermittently used system. Gaver (1963) studied an intermittently used one unit system. He laid stress on the point event called a disappointment characterised by the entry of the system to either the down state during a need period or the state of the need of the system when the system is already in the down state. Later, Srinivasan (1966), Nakagawa et al (1976), Srinivasan and Bhaskar (1979 a,b,c), Srinivasan and Subramanian (1980), analysed 1-unit and 2-unit redundant intermittently used systems. Yadavalli et al (2000, 2001, 2002), Botha (2000) have studied some estimation problems of the above models. The first attempt of n-unit systems which are used intermittently was due to Kapur and Kapoor (1978, 1980), Subramanian and Sarma (1981), Sarma and Natarajan (1982). They have studied an intermittently used n-unit warm standby system with failure time and repair time distributions are arbitrary. In these models, an expression for the distribution function of the time to the first disappointment. In this chapter an attempt is made to study an intermittently used k out of n : F system with the assumption that the failures will be detected only during the usage period.

The organisation of this chapter is as follows: In section 6.2, the system description is presented, explaining the system characteristics and the required notation. Some auxiliary functions required in the analysis is presented in section 6.3. Important operating characteristics of the system have been derived in section 6.4.

6.2 SYSTEM DESCRIPTION AND NOTATION

1. The system consists of n identical units.The system fails when k-units fail.

2. There is only one repair facility and the repairs are taken in first-in-first-out (FIFO) order.

3. Each unit is new after repair.

4. The failure rate of a unit is a constant and is denoted by 'a'.

5. The repair time of a unit has an arbitrary distribution and its pdf is denoted by g(.).

6. The need and no need periods occur alternately. The pdf of the need period is exponentially distributed with parameter α, and that of no need period is b(.).

7. The failure of a unit is detected only when there is a need for the system and the failure remains undetected until the need occurs. Only if the failed units will be taken up for repair.

8. If the system breakdown when there is a need for the system the need waits indefinitely until the system becomes available again and then the need lasts for a span of time governed by the same exponential distribution.

9. Initially at t=0 there is a system recovery; i.e. the system entering the upstate from the down state.

10. If during a no need period the repair facility becomes free (after completion of a repair) no unit will be taken up for repair until the next need arises.

NOTATION

X(t) : Stochastic process describing the state of the system at any time t, denoting the number of field units at time t;

Z(t) : Two state random process taking values 1 and 0 according as there is need or no need for the system at time t, respectively;

D : Event denoting a disappointment;

'e' : Event denoting a repair commencement;

E_i : Event that the repair for a unit just commences and the number of a failed units is i (i=1,2,$\cdots, n$);

E_0 at t : The state that X(t) = 0 and Z(t) = 1;

E : Event denoting a system recovery i.e. the E_{k-1} event following a 'D' event;

N(η, t) : Number of η events in (0,t], η = E, D, E_i, e;

δ_{ij} : $=\begin{cases} 1 & if \quad i=j \\ 0 & if \quad i\neq j \end{cases}$ Kronecker's delta function.

$Q_{ij}(t)$: $\begin{pmatrix} n-i \\ j-1 \end{pmatrix} (e^{-at})^{n-j}(1-e^{-at})^{j-1}$

i=0,1,2, $\cdots, n-1$

j=0,1,2, $\cdots, n-1, n; \quad j \geq i$

ⓒ : Convolution symbol;

$f^*(s)$: Laplace transform of f(t).

6.3 AUXILIARY FUNCTIONS

To describe the behaviour of the system in an interval between successive e-event, we introduce the following functions:

(i) Functions $\sigma_{ij}(t)$

Let

$$\sigma_{00}(t) = \lim_{\Delta \to 0} \frac{P[Z(t+\Delta)=0 \neq Z(t) | Z(0+)=0 \neq Z(0)]}{\Delta}$$

$$\sigma_{10}(t) = \lim_{\Delta \to 0} \frac{P[Z(t+\Delta)=0 \neq Z(t) | Z(0)=i]}{\Delta}$$

$$\sigma_{01}(t) = P[Z(t) = 1 | Z(0+) = 0 \neq Z(0)]$$

$$\sigma_{11}(t) = P[Z(t) = 1 | Z(0) = i]$$

The above functions $\sigma_{ij}(t)$ will be used to describe the behaviour of the process Z(t) in an interval in which there is no disappointment. We easily see that in such an interval the process Z(t) is an alternating renewal process and hence using renewal theoretic arguments, we have (see Cox, 1962).

$$\sigma_{11}(t) = e^{-\alpha t} + \alpha e^{-\alpha t} © \sigma_{01}(t) \tag{6.3.1}$$

$$\sigma_{01}(t) = b(t) © \sigma_{11}(t) \tag{6.3.2}$$

$$\sigma_{10}(t) = \alpha e^{-\alpha t} © \sigma_{00}(t) \tag{6.3.3}$$

$$\sigma_{11}(t) = \alpha e^{-\alpha t} © b(t) + \alpha e^{-\alpha t} © b(t) © \sigma_{00}(t) \tag{6.3.4}$$

Solving the above equations (6.3.1) - (6.3.4) after taking the Laplace transforms,

$$\sigma^*_{11}(s) = \beta^*(s) \tag{6.3.5}$$

$$\sigma^*_{01}(s) = b^*(s)\beta^*(s)$$

$$\sigma^*_{00}(s) = \alpha\, b^*(s)\beta^*(s)$$

$$\sigma^*_{10}(s) = \alpha\, \beta^*(s)$$

where $\beta^*(s) = [s + \alpha + \alpha b^*(s)]^{-1}$

(ii) FUNCTIONS $d_{ij}(t)$

Let $d_{ij}(t) = \lim_{\Delta \to 0} P[\text{a D event in } (t, t+\Delta), X(t) = j/E_i \text{ at t=0}]\, \Delta$

$$\text{i} = 1,2, \cdots, k-1$$
$$\text{j} = \text{k,k+1}, \cdots, n$$

Further the use of this function will be restricted to a repair time interval. With this restriction imposed on the domain of the function we now desire an expression for it. Since $d_{ij}(t)dt$ is the probability of occurrence of a disappointment in $(t, t + dt)$, we note that a disappointment can occur in the following ways:

(i) a system failure occurs in $(t, t + dt)$ when there is a need for the system;

(ii) a need for the system arises in $(t, t + dt)$ when the number of failed units in the system is $j(> k)$.

Accordingly, we have the following equations:

$$\begin{aligned} d_{ij}(t) = \quad & \delta_{jk}\sigma_{11}(t)Q_{iK-1}(t)(n-k+1)a + \sum_{m=i}^{k-1} \int_0^t \sigma_{10}(u)Q_{im}(u) \\ & b(t-u)Q_{mj}(t-u)du \end{aligned} \tag{6.3.6}$$

Also we define

Let $d_{0j}(t) = \lim_{\Delta \to 0} \frac{P[\text{D event in } (t,t+\Delta), N(e,t)=0, X(t)=j/E_0 \text{ at t=0}]}{\Delta}$

Since the failure of a unit during a need period will lead to the occurrence of an e-event, the only possible way is that the system should fail during a no need period and the disappointment occurs when the next need arises.

$$d_{0j}(t) = \int_0^t Q_{00}(u)\sigma_{10}(u)b(tu)Q_{0j}(t-u)du \tag{6.3.7}$$

(iii) FUNCTIONS $_Dh^1_{ij}(t)$

Let

$_Dh^1_{ij}(t) = \lim\limits_{\Delta \to 0} \frac{P[E_j \text{ in } (t,t+\Delta), N(e,t)=0, N(D,t)=0/E_i \text{ at t=0}]}{\Delta}$

$$\text{i} = 0,1,2,\cdots,k-1$$

$$\text{j} = 0,1,2,\cdots,k-1$$

The functions $_Dh^1_{ij}(t)$ is the pdf of the interval between two successive D avoiding e-events with i failed units in the system at the epoch of commencement of the repair and j failed units at the epoch of the next repair commencement. Further from the definition of these functions it easily follows that both i and j cannot be zero sumultaniously.

Hence we have

Case (i)

$$_Dh^1_{00}(t) = 0 \tag{6.3.8}$$

Case (ii)

For $i = 0, j = 1, 2, \cdots, k-1$, we note that $_Dh^1_{0j}(t)$ is the probability of the first repair commencement in $(t, t+\Delta)$ given to E_0 at $t = 0$. Hence the following possibilities can arise:

(i) the first failure may occur in $(t, t+\Delta)$ when there is a need for the system or

(ii) the first failure may occur before t during a no need period and when the next need occurs in $(t, t+\Delta)$ there may be j failed units in the system.

Hence we have:

$$_{D}h^{1}_{0j}(t) = \delta_{j1} n\, a\, e^{-nat}\sigma_{11}(t) + \int_0^t \sigma_{10}(u) b(t-u) Q_{00}(u) Q_{0j}(t-u) du \quad (6.3.9)$$

Case (iii)

For $j = 0, i = 1, 2, \cdots, k-1$

Since E_0 corresponds to the state in which all the units are operable, we have

$$_{D}h^{1}_{i0}(t) = 0 \text{ for i=2,3, } \cdots, k-1$$

and

$$\begin{aligned} _{D}h^{1}_{10}(t) = \; & g(t)\sigma_{11}(t)Q_{11}(t) + \int_0^t dv \int_0^v \sigma_{10}(u) b(t-u) g(v) \\ & Q_{11}(v) Q_{00}(t-v) du \end{aligned} \quad (6.3.10)$$

Case (iv) $1 \leq i \leq k-1, 1 \leq j \leq k-1$

Since only failures can occur in a repair interval, it follows that

$$_{D}h^{1}_{ij}(t) = 0 \text{ for} j < i-1$$

Case (v) when $i = 1, j = 1, 2, \cdots, k-1$

$$\begin{aligned} {}_Dh^1_{ij}(t) = \quad & \delta_{j1} g(t) Q_{12}(t)\sigma_{11}(t) \quad + \quad g(t)Q_{11}(t)\sigma_{11}(t) \copyright {}_Dh^1_{0j}(t) \quad + \\ & \sum_{n=1}^{j} \int_0^t dv \int_0^v \sigma_{10}(u) b(t-u) g(v) Q_{1m}(v) Q_{m-1j}(t-v) dv + \\ & [\int_0^t dv \int_0^v \sigma_{10}(u) b(t-u) g(v) Q_{00}(t-v) dv] \copyright {}_Dh^1_{0j}(t) \end{aligned} \tag{6.3.11}$$

The above equation is derived by considering the fact that there is a need or no need for the system when the repair is over.

Case (vi)

For $1 \leq i \leq k-1, i-1 \leq j \leq k-1$

Assuming as above

$$\begin{aligned} {}_Dh^1_{ij}(t) = \quad & (1 - \delta_{jk-1}) g(t) Q_{ij+1}(t) + \sum_{m=i, m\neq k}^{j+1} \int_0^t dv \int_0^v \sigma_{10}(u) \\ & b(t-u) Q_{im}(v) g(v) Q_{m-1j}(t-v) du] \end{aligned} \tag{6.3.12}$$

iv) FUNCTION $h^1_{ij}(t)$

Define

$$h^1_{ij}(t) = \lim_{\Delta \to 0} \frac{P[E_j \text{ in } (t,t+\Delta), N(e,t)=0/E_i \text{ at t=0}]}{\Delta}$$

$$\text{i} = 0,1,2, \cdots, k-1$$

$$\text{j} = \text{k,k+1}, \cdots, n$$

To derive an expression for this function we note that a disappointment must occur in $(0,t]$, as $k-i \geq 1$, and $j \geq k$. Also we observe that by assumption 8 whenever a disappointment occurs the need waits indefinetly. Hence we get

$$h^1_{ij}(t) = (1-\delta_{jn})\sum_{m=k}^{j+1} g(t)\int_0^t d_{im}(u)Q_{mj+1}(t-u)du + \sum_{m_1=i}^{j+1}\sum_{m=m_1}^{n}\int_0^t dv\int_0^v \sigma_{10}(u)b(t-u)Q_{im_1}(u)Q_{m_1m}(v-u)du \; g(v)Q_{n-1j}(t-v)du \tag{6.3.13}$$

(v) FUNCTION ${}_Eh^1_{ij}(t)$

Let

$${}_Eh^1_{ij}(t) = \lim_{\Delta\to 0}\frac{P[E_j \text{ in } (t,t+\Delta), N(e,t)=0, N(E,t)=0/E_i \text{ at t=0}]}{\Delta}$$

$$\text{i} = \text{k,k+1}, \cdots, n$$

$$\text{j} = \text{k-1,k,k+1}, \cdots, n$$

This function describes the behaviour of the system in an interval of time in which the disappointment persists.

For $j > i-1$, we have

$${}_Eh^1_{ij}(t) = g(t)Q_{ij+1}(t) \tag{6.3.14}$$

and when $j < i-1$

$$_E h^1_{ij}(t) = 0 \tag{6.3.15}$$

(vi) FUNCTION $H_j(t)$

Let

$$H_j(t) = P[N(D,t) = 0, N(e,t) = 0/E_j \text{ at t=0}]$$

$$\text{j} = 1,2, \cdots, k-1$$

The function $H_j(t)$ is the probability that neither a disappointment nor a repair commencement occurs in (0,t]. Hence to get this probability we note that the repair of the unit which has commenced at t=0 is either completed or not in (0,t]. If $j > 1$, and the repair is completed before t then at the epoch of this repair completion there is no need for the system. This is because if there is a need for the system then the next repair would have commenced leading to the occurance of an e-event before t. When j = 1 at the epoch of repair completion there may be no failed units and there may be a need or no need for the system. If there is a need then E_0 occurs. In case there is 'no need', either the need does not occur up to t or it occurs before t leading to the occurance of an E_0 event. Accordingly we have,

for $j > 1$

for j=1

$$H_j(t) = \sum_{m=j}^{k-1}[\bar{G}(t)Q_{jm}(t)\sigma_{11}(t) + G(t)\int_0^t \sigma_{10}(u)Q_{jm}(u) \bar{B}(t-u)du + \int_0^t \sigma_{10}(u)Q_{jm}(u)\bar{B}(t-u) \{G(t)-G(u)\}du] \tag{6.3.16}$$

$$H_1(t) = \sum_{m=1}^{k-1}\bar{G}(t)Q_{1m}(t)\sigma_{11}(t) + \bar{G}(t)\int_0^t \sigma_{10}(u)Q_{1m}(u) \bar{B}(t-u)du + \int_0^t \sigma_{10}(u)Q_{1m}(u)\{G(t)-G(u)\} \bar{B}(t-u)du + g(t)Q_{11}(t)\sigma_{11}(t)Ⓒ H_0(t) + [\int_0^t dv \int_0^v \sigma_{10}(u)b(t-u)g(v)Q_{11}(v)Q_{00}(t-u)du] Ⓒ H_0(t) \tag{6.3.17}$$

and

$$H_0(t) = e^{-nat}\sigma_{11}(t) + \int_0^t \sigma_{10}(u)e^{-nau}\bar{B}(t-u)du \tag{6.3.18}$$

FUNCTIONS $_DH_{ij}^m(t),_E H_{ij}^m(t),_D H_{ij}(t),_E H_{ij}(t)$

Let

$$_\eta h_{ij}^m(t) = \lim_{\Delta\to 0}\frac{P[E_j \text{ in } (t,t+\Delta), N(e,t)=m-1, N(\eta,t)=0/E_i \text{ at t=0}]}{\Delta}$$

m = 2,3, $\cdots$

η = D,E

when $\eta = D, i = 1, 2, \cdots, k-1$

j=1,2,$\cdots, k-1$

when η = E; i,j=k,k+1, $\cdots, n$

To evaluate ${}_{\eta}h_{ij}^{m}(t)$ we note that the occurance of E_j event in $(t, t+\Delta)$ corresponds to the occurance of the n^{th} repair commencement.

Hence we have

$$ {}_{D}h_{ij}^{m}(t) = \sum_{s=1}^{k-1} {}_{D}h_{1s}^{m-1}(t) \copyright {}_{D}h_{sj}^{1}(t) \tag{6.3.19} $$

$$ {}_{E}h_{ij}^{m}(t) = \sum_{s=k}^{n} {}_{E}h_{is}^{m-1}(t) \copyright {}_{E}h_{sj}^{1}(t) \tag{6.3.20} $$

Let

$$ {}_{\eta}h_{ij}^{m}(t) = \lim_{\Delta \to 0} \frac{P[E_j \text{ in } (t,t+\Delta), N(\eta,t)=0/E_i \text{ at t=0}]}{\Delta} $$

$$ \eta = \text{D,E} $$

$$ \text{when } \eta = \text{D, i,j=1,2,}\cdots, k-1 $$

$$ \text{when } \eta = \text{E; i,j=k,k+1, } \cdots, n $$

We note that the function ${}_{D}h_{ij}(t)$ will be used to describe the behaviour of the system in disappointment free interval and the function ${}_{E}h_{ij}(t)$, in an interval in which the disappointment persists. Also using probabilistic arguments, we have

$$ {}_{\eta}h_{ij}(t) = \sum_{m=1}^{\infty} {}_{\eta}h_{ij}(t), \eta = D, E \tag{6.3.21} $$

6.4 OPERATING CHARACTERISTICS OF THE SYSTEM

6.4.1 TIME TO FIRST DISAPPOINTMENT

With the help of the auxiliary functions described in section 6.3, we now find the survivor function of the time to the first disappointment.

Let

$$_{D}R(t) = P[N(D,t) = 0 \,|\, E \text{ at t=0}]$$

To derive an expression for $_{D}R(t)$ we consider the following mutually exclusive and exhaustive possibilities:

(i) there is no e-event in (0,t] or

(ii) at least one e-event occurs in (0,t].

Accordingly we have

$$_{D}R(t) = H_{k-1}(t) + \sum_{j=1}^{k-1} {}_{D}h_{k-1j}(t) \copyright H_j(t) \tag{6.4.1}$$

Note that the mean time to the first disappointment is given by $_{D}R^{*}(0)$.

6.4.2 EXPECTED NUMBER OF DISAPPOINTMENTS

We observe that the epochs of occurrence of E-events constitute a renewal process. Let $\phi(t)$ be the pdf of the interval between two successive E-events and $\Phi(t)$ is the corresponding survivor function.

$$\bar{\Phi}(t) = P[N(E,t) = 0\,|E \text{ at t=0}]$$

Since an E-event corresponds to a system recovery and $\bar{\Phi}(t)$ is the probability that no E-event occurs in (0,t], we consider the following mutually exclusive and exhaustive possibilities:

(i) there is no D-event in (0,t], or

(ii) a D-event occurs before t.

Under case(i) we also have the following possibilities:

(a) there is no e-event in (0,t] or

(b) at least one e-event occurs in (0,t].

Hence we obtain

We now derive an expression for the renewal density $h_E(t)$ of the renewal process constituted by the E-events. From renewal theory,

$$\bar{\Phi}(t) = {}_{D}R(t) + \sum_{m=k}^{n}\sum_{j=m}^{n} G(t)\int_0^t d_{k-1m}(u)Q_{mj}(t-u)du + \sum_{i=1}^{k-1} {}_{D}h_{k-1i}(t)©[\sum_{m=k}^{n}\sum_{j=m}^{n}\bar{G}(t)\int_0^t d_{im}(u)Q_{mj}(t-u)du] + \sum_{j=k}^{n} h^1_{k-1j}(t)©\bar{G}(t) + \sum_{j=k}^{n}\sum_{m=j}^{n} h^1_{k-1j}(t)©{}_{E}h_{jm}(t)©\bar{G}(t) + \sum_{i=0}^{k-1} {}_{D}h_{k-1i}(t)©(\sum_{j=k}^{n} h^1_{ij}(t))©\bar{G}(t) + \sum_{j=k}^{n}\sum_{m=j}^{n} h^1_{ij}(t)© {}_{E}h_{jm}(t)©\bar{G}(t) \tag{6.4.2}$$

$$h_E(t) = \sum_{n=1}^{\infty}\phi^{(n)}(t) \tag{6.4.3}$$

The expected number of system recoveries in (0,t] is given by

$$\int_0^t h_E(u)du$$

Using now the key renewal theorem we get the stationary rate of occurance of E-events as

$$\frac{1}{\bar{\Phi}^*(0)}$$

6.4.3 EXPECTED NUMBER OF DISAPPOINTMENTS

To derive an expression for the expected number of disappointments in (0,t], we define the following additional auxiliary functions.

Let

$$\mu_j(t) = \lim_{\Delta \to 0} \frac{P[\text{D event in } (t,t+\Delta), X(t)=j, N(E,t)=0/E \text{ at t=0}]}{\Delta}$$

To derive an expression for $\mu_j(t)$ we note that an e-event may or may not occur in (0,t]. Accordingly we have

$$\begin{aligned}\mu_j(t) = \; & \bar{G}(t)d_{k-1j}(t) + \sum_{m=k-1}^{j+1} \int_0^t dv \int_0^t \sigma_{10}(u)b(t-u) \\ & e^{-(n-k+1)au}Q_{k-1m}(v-u)g(v)Q_{m-1j}(t-v)du + \\ & \sum_{i=1}^{k-1} {}_D h_{k-1j}(t) © [\bar{G}(t)d_{ij}(t)] + [\sum_{m_1=i}^{k-1} \sum_{m=m_1}^{j+1} \int_0^t dv \\ & \int_0^v \sigma_{10}(u)b(t-u)Q_{im_1}(u)Q_{m_1m}(v-u)g(v) \\ & Q_{m-1j}(t-v)du] +_D h_{k-10}(t) © d_{0j}(t)\end{aligned} \tag{6.4.4}$$

With the use of this function we now derive an expression for the first order product density (Srinivason, 1974) of the D events which is defined as

$$h_D(t) = \lim_{\Delta \to 0} \frac{P[\text{D event in } (t,t+\Delta)/\text{E at t=0}]}{\Delta}$$

Using the fact that the interval (0,t] is intercepted by an E event or not we have

$$h_D(t) = \sum_{j=k}^{n} \mu_j(t) + h_E(t) © [\sum_{j=k}^{n} \mu_j(t)] \tag{6.4.5}$$

Hence the expected number of disappointment in (0,t] is given by

$$\int_0^t h_D(u)du$$

and the stationary rate of occurance of the D-events is given by

$$\begin{aligned}\lim_{t\to\infty} \tfrac{1}{t}\int_0^t h_D(u)du &= \lim_{s\to 0} s h_D^*(s) \\ &= \tfrac{1}{\Phi^*(0)}\sum_{j=k}^{n}\mu_j^*(0).\end{aligned}$$

CHAPTER 7

APLLICATIONS OF TIME SERIES IN RELIABILITY MODELLING

7.1 INTRODUCTION

Singh and Yadavalli (1997) have discussed an unconventional method for estimating reliability (using sample information) of systems operating under varying operational and environmental conditions. The theme of this chapter is similar to systems composed of components connected in series and/or in parallel. The type of data considered here are assumed to have been observed over a period of time in the field. Such type of data are referred to as the retrospective failure data (RFD) in the literature which is in contrast to life testing data generated by controlled life testing experiments. It is not an uncommon experience in industries that the performance index (or the reliability) of newly manufactured items changes with time due to either engineering design, environmental and operational conditions (see Chandrasekhar et al, 2005) or the maintainance and inspection procedures. Therefore the unwelcome of RFD is that they are contaminated for one reason or the other. Consequently the estimated reliability of a component or the system as a whole are subject to random changes forming either a stationary or non-stationary time series. Further if the inspection and maintainance intervals are periodic, the estimated reliability may exhibit periodicities, a phenomena often overlooked by reliability analysts. Such a time-dependent process is called the reliability decay (or growth) process which can be treated as a stochastic process.

7.2 DEVELOPED MODELS IN RELIABILITY USING TIME SERIES

We summarise, in this section, the univariate time series models and other interesting results studied by Yadavalli et al (2002), Singh (1984), Engel (1984), Singh and Nirmalan (1988) that are applicable to reliability decay (or growth) process of systems constituted of components operating in series and /or parallel under changing conditions.

7.2.1 TIME SERIES MODELS

Let $Z_t, Z_{t-1}, ...$ denote the values of observations collected at equispaced time points $t, t-1, ...$ The observations may be either the times between failures, the actual failure times, the estimated failure rates or the estimated reliability indicies. A suitable model which has achieved a commendable success in application to many commonly occurring non-stationary time series is the 'autoregressive integrated moving average' model of oder p,d,q (ARIMA (p,d,q) due to Box and Jenkins, 1970).

A general form of the model is defined by

$$\phi(B)\nabla^d Z_t = \theta(B)e_t \tag{7.2.1}$$

Where

$\nabla = 1 - B$; B is the backshift operator

$$(B^j X_t = X_{t-j}); \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ..... - \phi_p B^p$$

and

$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ..... - \theta_q B^q$ are AR and MA operators respectively. $\{e_t\}$ is the white noise process such that $E(e_t) = 0$ and $Cov(t, s) = \delta_{ts}\sigma^2$ for all t and s,

$$\delta_{ts} = \begin{cases} 1 & \text{if t = s} \\ 0 & \text{if t} \neq s \end{cases} \quad \delta_{ij} \text{ is the Kronecker's delta function}$$

If $W_t = \nabla^d Z_t$, model (7.2.1) reduces to

$$\phi(B)W_t = \theta(B)e_t \tag{7.2.2}$$

which is called the ARMA (p,q) model. For the process (7.2.2) to be stationary and invertible, the conditions are embodied in the statement that the zeros of polynomials $\phi(B)$ and $\theta(B)$ lie outside the unit circle respectively. If there are physical reasons to believe that a time series consists of a downward or upward trend such as decreasing or increasing reliability indices or failure rate, then that can be reflected in model (7.2.1) by incorporating a deterministic polynomial trend of degree d which can be induced by including a non-zero term δ in model (7.2.1), that is,

$$\phi(B)\nabla^B Z_t = \delta + \theta(B)e_t \tag{7.2.3}$$

When observations s time units apart in the time series display a similar pattern, the time series is said to be seasonal with period s. A general ARIMA model of a seasonal time series is defined by

$$\phi(B)\Phi(B^s)\nabla^d\nabla_s^D Z_t = \delta + \theta(B)\Theta(B^s)e_t \tag{7.2.4}$$

Where B^s is the seasonal backshift operator, $\nabla_s^D = (1 - B^S)^D, \Phi(B^s)$ and $\Theta(B^s)$ are the seasonal AR and MA operators defined by

$$\Phi(B^s) = 1 - \Phi_s B^s - \Phi_{2s} B^{2s} - ... - \Phi_{ps} B^{ps}$$

and

$$\theta(B^s) = 1 - \theta_s B^s - \theta_{2s} B^{2s} - ... - \theta_{qs} B^{qs}$$

respectively. D is the seasonal difference operator, where d, D $\in I^+$, a set of positive integers. Model (7.2.4) is termed as ARIMA(p,d,q)x(P,D,Q)$_s$.

7.2.2 SUMS AND PRODUCTS OF ARMA PROCESSES

In practice for $\rho > 2$ if one fits both an AR(p) and an ARMA(p',q') to a given set of data, he will find the ARMA(p',q') model fitting more satisfactorily and with fewer number of parameters, i.e. $p' + q' < p$. Hence the principle of parsimony suggests that the ARMA should be preferable to its components such as pure AR or MA model.

Although there are good reasons to prefer a model with as few parameters as possible, fitting of a mixed ARMA model is always more difficult and more cumbersome to interpret, comprehend, and explain its occurrence in the real world. For this reason mainly, we discuss in the following a number of ways in which an ARMA model could arise from simpler models or in other words to represent a complex function of ARMA models be simpler models.

7.2.3 SUM OF TWO OR MORE INDEPENDENT ARMA MODELS

Granger and Morris (1976) showed that if X_t and Y_t are two independent, zero-mean stationary ARMA series, namely, $X_t \sim ARMA(p_1, q_1)$ and $Y_t \sim ARMA(p_2, q_2)$, then

$$Z_t = X_t + Y_t \tag{7.2.5}$$

is an ARMA(p,q), where $p \leq p_1 + p_2$ and $q \leq max(p_1 + q_2, p_2 + q_1)$. In general, it is shown that:

$$\sum_{i=1}^{n} ARMA(p_i, q_i) = ARMA(p, q) \tag{7.2.6}$$

where $p \leq \sum_{i=1}^{n} p_i$ and $q \leq max(p - p_j + q_j), j = 1, 2, ..., n$.

In practice there are situations, where series are added together. Examples include macroeconomic series such as GNP, unemployment, exports etc. In other situations, the observed series may be the sum of the true process and the observational error such as 'signals plus noise'.

7.2.4 PRODUCT OF TWO OR MORE INDEPENDENT ARMA PROCESSES

Let us note the following results.

Result 7.2.1

Let Z_t and Y_t be two independent zero-mean stationary processes defined by

$$Z_t = e_t + \theta e_{t-1} \tag{7.2.7}$$

and

$$Y_t = \phi Y_{t-1} + u_t \tag{7.2.8}$$

respectively, where e_t and u_t are independent white noise processes with variances σ_e^2 and σ_u^2 respectively, then the product

$$W_t = Z_t Y_t \tag{7.2.9}$$

can be identifiable by an MA(1) process.

Proof: It is easy to see that $\mathrm{E}(\mathrm{W}_t) = 0$

$$\mathrm{V}(\mathrm{W}_t) = \sigma_e^2(1+\theta^2)[\sigma_u^2 + \tfrac{\phi^2\sigma_e^2}{1-\phi^2}]$$

$$\gamma_j = \begin{cases} \frac{\phi\theta\sigma_\epsilon^2}{1-\phi^2} & ; \quad j = 1 \\ 0 & ; \quad j \geq 2 \end{cases}$$

where $\epsilon = e_t u_t$ and $\sigma_\epsilon^2 = \sigma_e^2\sigma_u^2$

and hence

$$\rho_j = \begin{cases} \frac{\phi\theta}{1+\theta^2} & ; \quad j = 1 \\ 0 & ; \quad j \geq 2 \end{cases} \tag{7.2.9}$$

where γ_j and ρ_j are the autocoverience and autocorrelation functions at lag j respectively. It then follows from (7.2.9) that W_t is an MA(1).

Result 7.2.2

Let Z_t and Y_t be both independent zero-mean stationary AR(1) processes defined by

$$Z_t = \phi Z_{t-1} + e_t; |\phi| \leq 1 \tag{7.2.10}$$

and

$$Y_t = \eta Y_{t-1} + u_t; |\eta| \leq 1 \qquad (7.2.11)$$

respectively, then the product $V_t = Z_t Y_t$ is an AR(1) process.

Proof: Putting $\epsilon_t = e_t u_t$, it is easy to verify that

$$E(V_t) = 0$$

$$V(V_t) = \gamma_0 = \frac{\sigma_\epsilon^2}{(1-\phi^2)(1-\epsilon^2)}$$

$$\begin{aligned}\gamma_j &= \mathrm{E}(\mathrm{V}_t V_{t-j}) \\ &= \frac{\phi^j \eta^j \sigma_\epsilon^2}{(1-\phi^2)(1-\eta^2)}\ ; \mathrm{j} \geq 1\end{aligned}$$

and hence

$$\rho_j = \lambda^{|j|}; j = \pm 1, \pm 2, ... \qquad 7.2.12)$$

where

$$\lambda = \phi\eta; 0 < \lambda < 1 \qquad (7.2.13)$$

from (7.2.12), it follows that V_t is an AR(1).

Using a slightly different approach, Engel (1984) showed that if X_t and Y_t are two independent ARMA processes of order (p_1, q_1) and (p_2, q_2) respectively and if Z_t denotes their product, then ;

$$Z_t \sim ARMA(p,q) \tag{7.2.14}$$

where

$$p \leq p_1 p_2; q \leq p + max(q_1 - p_1, q_2 - p_2).$$

In particular

$$\text{(i) } AR(p_1)AR(p_2) = ARMA(p_1 p_2, p_1 p_2 - min(p_1, p_2)) \tag{7.2.15}$$

$$\text{(ii) } AR(p)AR(p) = ARMA(p^2, p^2 - p) \tag{7.2.16}$$

$$\text{(iii) } ARMA(p_1, q_1)MA(q_2) = MA(q_2) \tag{7.2.17}$$

It may be noted that Results 7.2.1 and 7.2.2 are special cases of (7.2.16) and (7.2.17) respectively.

$$\prod_{i=1}^{n} ARMA(p_i, q_i) = ARMA(p,q) \tag{7.2.18}$$

where

$$p \leq \prod_{i=1}^{n} p_i \text{ and } q \leq p + max(q_i - p_i), i = 1, 2, ..., n$$

7.2.5 SUM OF SUMS AND PRODUCTS OF ARMA PROCESSES

Singh and Nirmalan (1988) have proved the following result in general:

RESULT 7.2.3

Deleting the suffix t for simplicity of notation, let the $X_i, i = 1, 2, ..., n$ be zero mean and independent and let $X_i \sim ARMA(p_i, q_i); i = 1, 2, ..., n$
then

$$Z = \sum_{i=1}^{n} X_i + \prod_{i=1}^{n} X_i \sim ARMA(p_0 q_0) \tag{7.2.19}$$

where

$$p_0 \leq \left[\sum_{i=1}^{n} p_i + \prod_{i=1}^{n} p_i \right]$$

$$q_0 \leq p_0 + max_i(q_i - p_i, i = 1, 2, \cdots, n).$$

RESULT 7.2.4

Let X_t and Y_t be two zero-mean dependent gaussian ARMA processes of oder (p_1, q_1) and (p_2, q_2) respectively and let $Z_t = X_t + Y_t + X_t Y_t$. Further, let there exist a polynomial $\phi(B)$ of degree p_3 with all zeros lying outside the unit circle
such that

$$\phi(B)\{\gamma_{X_tY_t}(k) + \gamma_{Y_tX_t}(k) + \gamma_{X_tY_t}(k)\gamma_{Y_tX_t}(k)\} = 0; k > q_s \quad (7.2.20)$$

Then Z_t is an ARMA(p,q) with $p < p_1 + p_2 + p_3 + p_1p_2, q < p + max(q_i - p_i; i = 1, 2$ where p_3 and q_3 are some positive integers which can always be determined in any specific situation as illustrated in Singh and Nirmalan (1988), $\gamma_{X_tY_t}(k)$ denotes the cross-variance function of X_t and Y_t and lag k. Note that $\gamma_{X_tY_t}(k)\gamma_{X_tY_t}(k) \neq \gamma_{Y_tX_t}(k)$ in general.

7.3 SOME DEFINITIONS AND FAILURE LAWS

In the following, we first define two important characteristics in the failure data analysis, namely (i) the reliability and (ii) the failure rate.

<u>Definition 7.3.1 RELIABILITY</u> The reliability of the system (or a component) at time t denoted by R(t) is given by

$$R(t) = P[x > t]$$

where x denotes the random life length (or failure time) of the system. R(t) is called the reliability function. In terms of probability density function (pdf) of x, the R(t) is expressed as

$$R(t) = \int_t^\infty f(u)du = 1 - p(x \leq t) = \bar{F}(t)$$

$\bar{F}(t)$ is the survivor function.

Definition 7.3.2 INSTANTANEOUS FAILURE RATE

The instantaneous failure rate or simply the failure rate (sometimes called the hazard function) associated with the random variable T is defined by

$$Z(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)} \qquad 7.3.4$$

Note that T is a continuous r.v. and the pdf of T i.e. $f(t)$ uniquely determines the failure rate $Z(t)$. Its converse is also true, that is, $Z(t)$ uniquely determines $f(t)$. It follows that from the solution of the differential equation

$$\frac{d}{dt}R(t) = -Z(t)R(t) \qquad (7.3.5)$$

which, under the initial condition R(0) = 1, is given by

$$f(t) = Z(t)exp\{-\int_0^t Z(u)du\} \qquad (7.3.6)$$

There are several commonly used forms of f(t) that are generally assumed in life testing experiments and reliability problems, we cite only two of them, (i) the exponential distribution and (ii) the weibull distribution, for later one.

7.3.1 EXPONENTIAL LAW

Davis (1952) explained different types of data and found the exponential distribution fitting most of the situations quite well. The simplest form of the exponential distribution is

$$\mathrm{f}(\mathrm{x}/\lambda) = \begin{cases} \lambda e^{-\lambda x} & ; \quad x > 0, \lambda > 0 \\ 0 & ; \quad otherwise \end{cases} \tag{7.3.7}$$

for which

$$R(t) = e^{-\lambda t} \tag{7.3.8}$$

$$Z(t) = \lambda \tag{7.3.9}$$

This means that an exponential failure law is charactised by a constant failure rate.

7.3.2 WEIBULL FAILURE

Another important failure law is the weibull distribution with pdf

$$\mathrm{f}(\mathrm{x}/\lambda, \beta) = \begin{cases} (\lambda\beta)x^{\beta-1}exp(-\lambda x^{\beta}) & , \quad x, \lambda, \beta > 0 \\ 0 & , \quad otherwise \end{cases} \tag{7.3.10}$$

for which

$$R(t) = exp\{-\lambda t^{\beta}\} \tag{7.3.11}$$

and

$$Z(t) = \lambda \beta t^{\beta - 1} \tag{7.3.12}$$

It may be noted from (7.3.12) that for $\beta > 1$, $Z(t)$ is an increasing function of t, for $\beta < 1$,$Z(t)$ is a decreasing function and for $\beta = 1$, $Z(t) = \lambda$, a constant which leads to the exponential distribution. The weibull distribution has been extensively used in life testing and reliability problem. For example, Weibull (1951) found the distribution usefull for 'wear-out' and fatigue failures. Liebetin and Zelen (1956) used it to describe the ball bearing failures. Kao (1959) used it as a model for vacuum tube failures while Mann (1968) considered a variety of situations which could be described well by the weibull distribution.

7.4 ESTIMATION OF RELIABILITY

In this section we discuss the estimation of the reliability function when the distribution of the failure time variable x is (a) unknown and (b) known. Further, we discuss procedures for estimating the reliability when the failure data are (c) uncontaminated, and (d) contaminated.

7.4.1 DISTRIBUTION OF THE FAILURE TIMES UNKNOWN

1(a) When the distribution of the failure times is unknown and the data is uncontaminated, the $R(t)$ may be estimated by

$$R(t)_{t=T} = \frac{\#\text{ items surviving}_{\geq T}}{\#\text{ items initially put to test}} \tag{7.4.1}$$

(b) If the failure times of the component are uncontaminated, that is, are subject to random changes due to unassignable reasons, then the reliability may be estimated following the procedure suggested by Singapurwalla (1978).

2(a) DISTRIBUTION OF THE FAILURE TIMES KNOWN

As mentioned before, there are several laws that are found generally useful in life testing experiments and reliability problems. Examples include the exponential, weibull, Rayleigh, gamma, normal and lognormal distributions. If the data are uncontaminated, the maximum likelihood (ML) and uniformly minimum variance unbiased estimators (MVUE) of parameters involved in a distribution and the corresponding reliability function are discussed in the literature (see Sinha (1986)), Lawless (1981)). Since from a practical point of view, there is insignificant difference between these two types of estimators, we will consider only the ML estimators for R(t) in the case of exponential, Weibull and Raleigh distributions, for later reference.

(b) When the failure data are assumed to have come from a known distribution and if the data are suspected of having undergone random changes, the reliability function R(t) may be estimated following the results mentioned next.

RESULT 7.4.1

If the failure time X of a system follows on exponential distribution with pdf

$$f(x/\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}; x > 0, \theta > 0 \tag{7.4.2}$$

then the ML estimator of the reliability function R(t) of the system is given by the conditional mean of the process.

$$R(t) = exp\{-\frac{1}{\theta}\}R(t-1) + \epsilon(t) \tag{7.4.3}$$

where $\epsilon(t)$ is assumed to follow the truncated normal distribution with mean zero and variance unity $(TN(0,1))$. Truncated distribution of errors is assumed since $R(t)$ lies between 0 and 1. Truncated range for errors can be based on the coefficient in model (7.4.3).

Given a random sample $x_1, x_2, \cdots x_n$ of failure times from an exponential distribution with a single parameter, the ML estimator of the reliability is given by

$$\hat{R}(t) = exp\{-\frac{t}{\bar{x}}\}. \tag{7.4.4}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the ML estimator of θ.

RESULT 7.4.2

Let the failure times of a system follow a two-parameter distribution with pdf

$$\text{f(x}/\sigma, \alpha) = \begin{cases} \frac{\alpha}{\sigma} x^{\alpha-1} exp\{-\frac{x^\alpha}{\sigma}\} & ; \quad x, \alpha, \sigma > 0 \\ 0 & ; \quad otherwise \end{cases} \tag{7.4.5}$$

where α is the shape parameter and σ is the scale parameter of the distribution. Then the ML estimator of the reliability function of the system is given by the conditional mean of the process defined by

$$R(t) = exp\{-\frac{1}{\sigma}[t^\alpha - (t-1)^\alpha]\}R(t-1) + \epsilon(t) \tag{7.4.6}$$

given $R(0) = 1$, where $\epsilon(t) \sim TN(0, 1)$.

The conditional mean of process (7.4.6) is given by

$$\bar{R}(t) = exp\{-\frac{t^\alpha}{\alpha}\} \tag{7.4.7}$$

where α and σ are replaced by their ML estimators $\hat{\alpha}$ and $\hat{\sigma}$, then the ML estimators of $R(t)$ is

$$\hat{R}(t) = exp\{-\frac{t^{\alpha}}{\alpha}\} \tag{7.4.8}$$

(see Sinha, 1986).

RESULT 7.4.3

Let the failure times of a system follow a Rayleigh distribution with pdf

$$\text{f(x/}\sigma) = \begin{cases} \frac{x}{\sigma^2} exp\{-\frac{x^2}{2\sigma^2}\} & ; \quad x, \sigma \geq 0 \\ 0 & ; \quad otherwise \end{cases} \tag{7.4.9}$$

where σ is the scale parameter. The ML estimator of $R(t)$ is then given by the conditional mean of the process defined by

$$R(t) = exp[-\frac{2t-1}{2\sigma^2}]R(t-1) + \epsilon(t), \tag{7.4.10}$$

given $R(0) = 1$, where $\epsilon(t) \sim TN(0, 1)$.

The conditional mean of the process (7.4.7) is given by

$$\bar{R}(t) = exp\{-\frac{t^2}{2\sigma^2}\} \tag{7.4.11}$$

If σ^2 is unknown, it can be replaced by its ML estimator

$$\hat{\sigma^2} = \frac{s^2}{2n}$$

where $s^2 = \sum_{i=1}^{n} x_i^2$. Hence the ML estimator $R(t)$ is

$$\hat{R}(t) = exp\{-\frac{nt^2}{s^2}\} \tag{7.4.12}$$

For illustration, we consider example 1.1 of Sinha(1986) and compare in Figure 7.4.1, the plots of (i) the ML estimate $\hat{R}(t) = exp\{-\frac{t}{\bar{x}}\}$, when the failure data is uncontaminated and $\bar{x} = 346.98$, (ii) the values of $\hat{R}(t)$ generated from $N(0,1)$, lying between -0.05 and 0.05. The closeness of the two functions (random and non-random) may be noticed from Figure (7.4.1). By suitability choosing the range of $\epsilon(t)$, the two curves could be brought closer. This emphasizes the fact that if an observed sample of failure data from an exponential distribution is assumed contaminated due to random changes, then this sample can be taken as if generated by model (7.4.3) with $\theta = \bar{x}$. Hence this model can be used for forecasting into the failure. There may be an alternative approach to this problem.

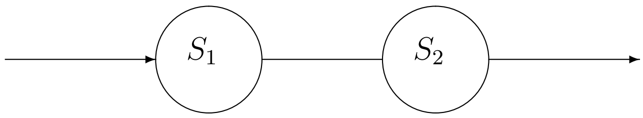
7.5 STOCASTIC MODELLING OF THE ESTIMATED RELIABILITY OF SYSTEMS

This section discusses the time series modeling of the estimated reliability of complex systems consisting of n subsystems (or components) connected in series and/or in parallel, given the reliability of each subsystem (or component) at equidistant points of time. It is assumed that the failure times of each subsystem (or component) are subject to random fluctuations and can be treated as generated by a stochastic process. The complex systems considered are (i) a series system (ii) a parallel system and (iii) a bridge system. Once a suitable time series model is fitted to the estimated reliability of the system over a period of time, it can be used for forecasting its reliability. An example is discussed to illustrate the practical application of the results.

7.5.1 A SERIES SYSTEM

For simplicity, we first consider a system consisting of only two components connected in series as shows in Figure 7.5.1.

Figure 7.5.1



Let $R_i(t), i = 1, 2$ be the reliability of the subsystem i at time t. Then the reliability of the system is given by

$$R(t) = R_1(t)R_2(t) \tag{7.5.1}$$

If the distribution of the failure times of subsystem $i(i = 1, 2)$ is known (see Yadavalli and Hines, 1991), further it is suspected that the observed failure distribution is contaminated by or tampered with the environmental changes, then $R_i(t)$ may be estimated following Singh and Yadavalli (1991). Thus if each of $\hat{R}_1(t)$ and $\hat{R}_2(t)$ is an AR(1) process then following Engel (1984),

$\hat{R}(t)$ is an AR(1) process.

These ideas can easily be extended to a system consisting of k components connected in series. Below we discuss an example for n=2.

Example 7.5.1 Suppose that an electric circuits of a subsystem of silicon transitory (S_1) and another subsystem of silicon diodes (S_2) connected in series (see Figure 7.5.1). Further, suppose that the average failure times of S_1 and S_2 were 30 and 50 days respectively. It was assumed that in each case the failures not only occurred due to their natural wear and tear but also due to the random variation in voltage and various other reasons. Now assuming that the failure times for both subsystems S_1 and S_2 where exponentially distributed, then the reliabilities of S_1 and S_2 can be estimated, given $R(0) = 1$, from

$$R_1(t) = 0.97R_1(t-1) + \epsilon_1(t), t \leq 1 \tag{7.5.2}$$

and

$$R_2(t) = 0.98R_2(t-1) + \epsilon_2(t), t \leq 1 \tag{7.5.3}$$

respectively, where $|\epsilon_1(t)| \leq 0.02$. Then the estimated reliability of the circuit at time $t \geq 1$ is given by

$$\hat{R}(t) = \hat{R}_1(t)\hat{R}_2(t) \tag{7.5.5}$$

Since $\hat{R}_1(t)$ and $\hat{R}_2(t)$ are each an AR(1) process, $\hat{R}(t)$ is also an AR(1) process following result 7.2.2. The coefficient of the process $\hat{R}(t)$ which is AR(1) can be estimated from (7.2.12), that is,

$$\hat{\lambda} = \hat{\phi}\hat{\eta} = 0.97x0.98 = 0.9506 \tag{7.5.6}$$

and hence the $l - step(l = 1, 2, \cdots)$ a head forecast can be obtained from where $\hat{R}(t/l)$ is

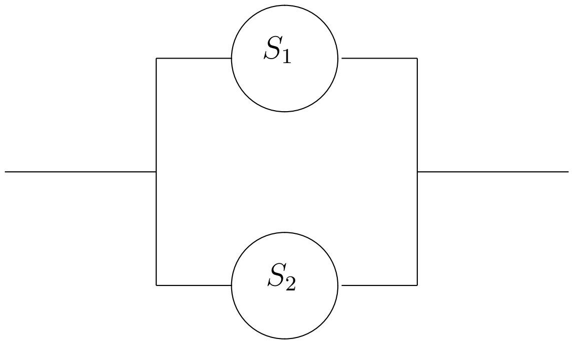
$$\hat{R}(t/l) = (\hat{\lambda})^l \hat{R}(l) \ ; \mathrm{l} = 1,2, \cdots \tag{7.5.7}$$

is the forecast for $\hat{R}(t-1)$. The values for $\hat{R}_1(t), \hat{R}_2(t)$ and $\hat{R}(t)$ are given below (see Table 7.5.1).

7.5.2 A PARALLEL SYSTEM

Consider again a system of two subsystems connected in parallel as shown in Figure 7.5.2.

Figure 7.5.2



Let $R_i(t), i = 1, 2; t \geq 1$ be the estimated reliability of subsystem i at time t. Then the reliability of the system is given by

$$\hat{R}(t) = \hat{R}_1(t) + \hat{R}_2(t) + \hat{R}_1(t)\hat{R}_2(t) \tag{7.5.7}$$

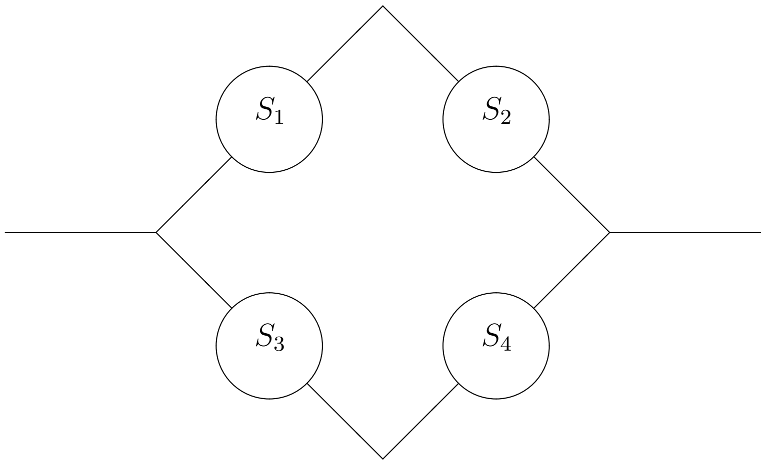
If the distribution of the failure times of the i^{th} subsystem $(i = 1, 2)$ is known and further if it is suspected that the failure times have been contaminated by environmental changes (Yadavalli et al, 2005), then $\hat{R}_1(t)$ and $\hat{R}_2(t)$ is an AR(1) process, it follows from Singh and Nirmalan (1988) that $\hat{R}(t)$ is ARMA (p,q), where $p \leq 3$ and $q \leq p$.

Example 7.5.2 Consider the two subsystems of Example 7.5.1 connected this time in parallel and suppose that their reliabilities are estimated using (7.5.3) and (7.5.4). Then the estimated reliability of the main system can be calculated using (7.5.7). For $t = 1, 2, \cdots, 28$, the values of $\hat{R}(t)$ are plotted in figure 7.5.3 along with the last eight forecasts for comparison with the corresponding estimated values.

7.5.3 A BRIDGE SYSTEM

Consider a set of four subsystems S_1, S_2, S_3 and S_4 and suppose that they are connected as shown in Figure 7.5.4.

Figure 7.5.4



This system is called a Bridge system. Given the estimated reliabilities of S_1, S_2, S_3, S_4, the reliability of the whole system can be calculated from

$$\hat{R}(t) = \hat{R}_1(t)\hat{R}_2(t) + \hat{R}_3(t)\hat{R}_4(t) + \hat{R}_1(t)\hat{R}_2(t)\hat{R}_3(t)\hat{R}_4(t) \tag{7.5.9}$$

The type of the process $\{\hat{R}(t)\}$ can be determined following result 7.3.2 of Singh and Nirmalan (1988), given the type of processes $\{\hat{R}(t); i = 1, 2, 3, 4\}$.

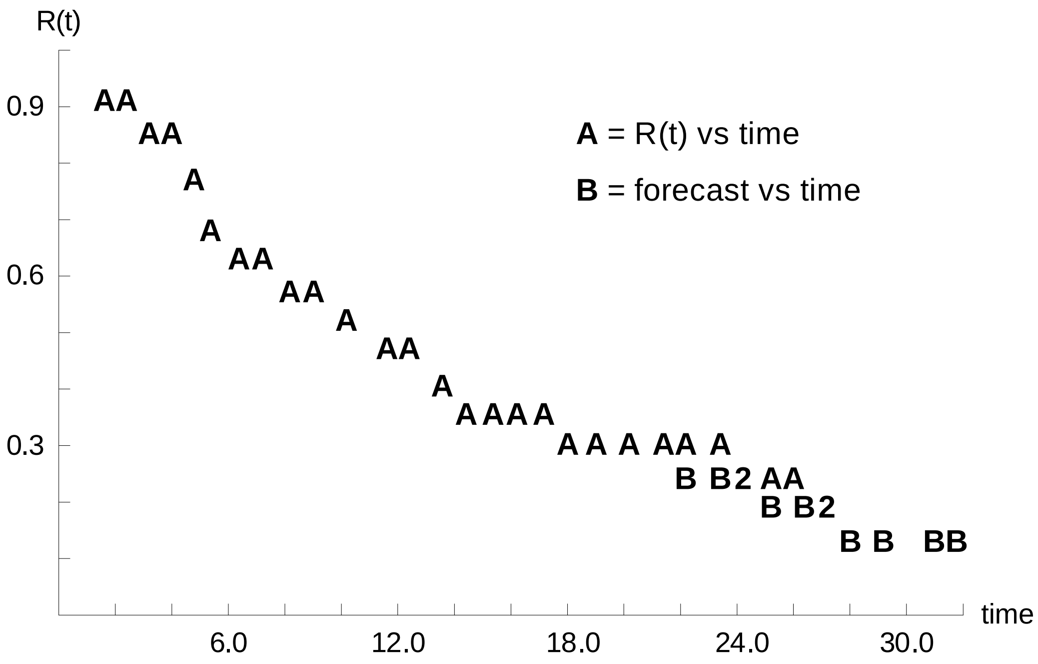

R(t)
0.9
0.6
0.3
A = R(t) vs time
B = forecast vs time
6.0
12.0
18.0
24.0
30.0
time

Figure 7.5.3