Finite element analysis of plate and beam models

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Summary

We consider linear mathematical models for elastic plates and beams. To be specific, we consider the Euler-Bernoulli, Rayleigh and Timoshenko theories for beams and the Kirchhoff and Reissner-Mindlin theories for plates.

The theories mentioned above refer to the partial differential equations that model a beam or plate. The contact with other objects also need to be modelled. The equations that result are referred to as "interface conditions".

We consider three problems concerning interface conditions for plates and beams: A vertical slender structure on a resilient seating, the built in end of a beam and a plate-beam system.

The vertical structure may be modelled as a vertically mounted beam. However, the dynamics of the seating must be included in the model and this increases the complexity of a finite element analysis considerably. We show that the interface conditions and additional equations can be accommodated in the variational form and that the finite element method yields excellent results.

Although the Timoshenko model is considered to be better than the Euler-Bernoulli model, some authors do not agree that it is an improvement for the case of a cantilever beam. In a modal analysis of a two-dimensional beam model, we show that the Timoshenko model is not only better, but it provides good results when the beam is so short that one is reluctant to use beam theory at all.

In applications, structures consisting of linked systems of beams and plates are encountered. We consider a rectangular plate connected to two beams. Combining the Reissner-Mindlin plate model and the Timoshenko beam model can be seen as a first step towards a better model while still avoiding the complexity of a fully three-dimensional model. However, the modelling of

the plate-beam system is more complex than in the case of the classical theory and the mathematical analysis and numerical analysis present additional difficulties.

A weak variational form is derived for all the model problems. This is necessary to apply general existence and uniqueness results. It is also necessary to apply general convergence results and derive error bounds. The setting for the weak variational forms are product spaces. This is due to the complex nature of the model problems.

Contents

Chapter 1

Modelling interface conditions

1.1 Introduction

In this thesis our concern is mathematical models for elastic plates and beams. In real life all objects are three-dimensional. Due to the proportions of a body, it is sometimes justifiable to consider a one-dimensional or two-dimensional model. These models are referred to as beam and plate models respectively.

We restrict our attention to linear models or linear theories for plates and beams. To be specific, we consider the Euler-Bernoulli, Rayleigh and Timoshenko theories for beams and the Kirchhoff and Reissner-Mindlin theories for plates.

The theories mentioned above refer to the partial differential equations that model a beam or plate. The contact − or lack thereof − with other objects also need to be modelled. The equations that result are usually referred to as boundary conditions, but we prefer the more inclusive term "interface conditions".

We consider three problems concerning interface conditions for plates and beams. In this section we present a brief introduction. A detailed discussion will be given in Section 1.5 after a review of the general theory.

The relevant aspects of beam theory and plate theory are presented in Sections 1.2 and 1.3 and a two-dimensional beam model in Section 1.4. We write all the problems in dimensionless form to facilitate numerical experiments.

The model problems to be investigated are presented in Chapter 2.

1.1.1 A vertical slender structure on a resilient seating

Unwanted vibrations often occur in mechanical structures. The following design problem is described in [N1]:

> *"Because of their inherent low damping, free-standing welded steel structures are prone to oscillate in the wind. This may cause the chimney to fail due to metal fatigue. One method of artificially increasing the damping is to mount the chimney on a resilient foundation incorporating bearing pads made of a high-damping material."*

The structure may be modelled as a vertically mounted beam, i.e. a continuum model is used. Engineers often refer to continuum models as distributed parameter system (DPS) models.

In [N2], Newland discusses efforts to compute natural frequencies using DPS models. The results compared poorly with experimental results. Newland pointed out that the models needed to be improved to include the influence of the resilient seating. According to Newland, this increases the complexity of a finite element analysis considerably. As an alternative he proposed lumped parameter system models (LPS).

LPS models are useful for the analysis of vibrating systems when one is primarily interested in the lower order modes (see [CZ]). However, the accuracy is questionable and the theoretical tools for error estimation are not available. We considered it worthwhile to investigate beam models and to compare results.

Our initial objective was to match Newland's results using beam models. In doing so, we demonstrated the flexibility of DPS models in conjunction with the finite element method. We used the Euler-Bernoulli and Rayleigh models for the slender structure since they correspond to Newland's models.

Modelling the behaviour of the resilient seating and foundation leads to a hybrid system. We constructed four mathematical models to match those of Newland and showed that the interface conditions and additional equations can be accommodated in the variational form. Consequently the finite element method can be used. Using a small number of elements, our results compared well with those of Newland (see [N1], [N2] and [LVV]). The numerical results published in [LVV] show clearly the advantage of the finite element method.

In this thesis we investigate aspects not considered in [LVV]. First we use the Timoshenko theory to construct mathematical models and compare the results. We also consider theoretical aspects such as existence and uniqueness of solutions and convergence of finite element method approximations.

1.1.2 Boundary conditions for the clamped end of a beam

The Euler-Bernoulli beam is a popular model for the transverse vibration of a beam which is still used. Although the Timoshenko model is considered to be better (see e.g. [Fu], [I], [N1], [T] and [Wa]), some authors, for instance Duva and Simmonds ([DS]), do not agree that the Timoshenko model is an unqualified improvement. According to [DS], the corrections predicted by the Timoshenko model are in some cases erroneous. The authors claim that for the first eigenfrequency of the cantilever beam, the Timoshenko model provides a correction in the wrong direction and that this is due to "effects at the built in end".

Careful consideration of a clamped end of a beam leads to the conclusion that the boundary conditions for the two models are not compatible. This fact was pointed out in [V3] and an alternative boundary condition was proposed. However, the modified boundary condition worsened the disparities between the two models, i.e. the differences between the natural frequencies were larger. It became clear that further investigation was necessary and in this investigation two-dimensional effects must be taken into account. In order to do this, we consider two-dimensional models for a cantilever beam.

1.1.3 Plate-beam systems

In applications, structures consisting of linked systems of beams and plates are encountered. The reader is referred to [LLS] where a large variety of applications can be found.

We consider a rectangular plate connected to two beams. This problem was also considered in [ZVGV1], [ZVGV3] and [Ziet] using classical plate theory and the Euler-Bernoulli beam theory.

Combining the Reissner-Mindlin plate model and the Timoshenko beam model can be seen as a first step towards a better model while still avoiding the “complications” of a fully three-dimensional model.

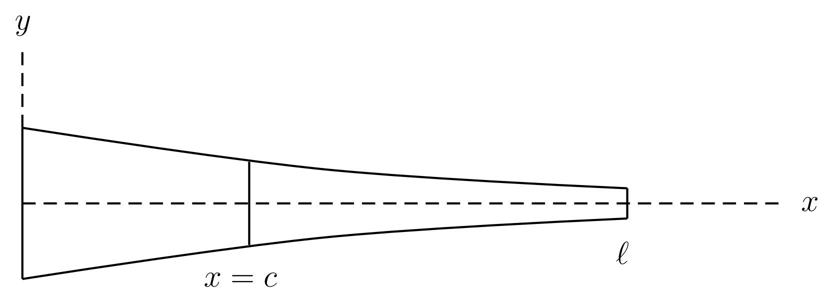
1.2 Beam theory

In this section we consider the transverse motion of a beam. We restrict our attention to a beam that is straight in its undeformed state. We assume that it has a well defined axis of symmetry and that all the cross sections are similar and have their centroids on the axis of symmetry.

The Euler-Bernoulli theory for a beam originated in the 18-th century. An improvement was introduced by Rayleigh in the 19-th century. In 1921, Timoshenko proposed his theory where shear is taken into account.

1.2.1 Equations of motion

Consider a beam as illustrated below. The x-axis is taken to coincide with the line of centroids of the cross sections. We assume that the cross sections and applied loads are symmetric with respect to the xy-plane and consequently the motion of the beam is parallel to the xy-plane.



Consider a cross section at $x = c$. Denote the axial force, shear force and moment by $S(c,t)$, $V(c,t)$ and $M(c,t)$ respectively. We follow the convention that S, V and M denote the forces and moment exerted by the part of the body for which $x > c$ on the rest.

Suppose the beam has constant density ρ, length ℓ and cross sectional area A. We consider a one-dimensional model and the reference configuration is the interval $[0, \ell]$. The transverse displacement (deflection) of the cross section at $x \in [0, \ell]$ at time t is denoted by $w(x, t)$. Assuming that plane cross sections remain plane, the rotation of a cross section is denoted by $\phi(x, t)$. Assume that the load P is in the transverse direction. The equations of motion are then given by

$$\rho A \partial_t^2 w = \partial_x V + P, \tag{1.2.1}$$

$$\rho I \partial_t^2 \phi = V + \partial_x M + L, \tag{1.2.2}$$

where I is the area moment of inertia (see [T, p 331-337] and [I, p 337]).

Remarks

1. The term $\rho I \partial_t^2 \phi$ in Equation (1.2.2) is usually referred to as the rotary inertia term.

2. Note the unusual term L present in Equation (1.2.2). This term represents a moment density term that will be used in some of the mathematical models (see Sections 2.1 and 2.4).

1.2.2 The Timoshenko model

To determine the forces S and V and the moment M, the stresses are integrated over a cross section. For more detail, see [Fu, Sec 7.7], [Co] and [I, p 337-338].

In the linear theory, it is assumed that $\partial_x w$ is small. The following constitutive equations for the moment M and the shear force V are used.

$$M = EI\partial_x \phi, \tag{1.2.3}$$

$$V = AG\kappa^2 \big(\partial_x w - \phi\big). \tag{1.2.4}$$

In these equations, E and G are elastic constants (see Section 1.4) and κ^2 the shear coefficient or shear correction factor. We refer the reader to [T, p 337-338], [Fu, p 323-324], [I, p 337-338] and [N1, p 392-395].

Substituting the constitutive equations (1.2.3) and (1.2.4) into the equations of motion (1.2.1) and (1.2.2), yield the well known Timoshenko model for the free vibration of a beam.

$$\begin{aligned} \rho A\partial_t^2 w &= \partial_x\big(AG\kappa^2(\partial_x w - \phi)\big), \\ \rho I\partial_t^2 \phi &= AG\kappa^2\big(\partial_x w - \phi\big) + \partial_x\big(EI\partial_x\phi\big) + L. \end{aligned}$$

The partial differential equations above can be derived in different ways (see [Fu, p 322-323] and [Co]).

The boundary conditions depend on the configuration and a number of variations are possible (see [I, p 335, 338] and [Fu, p 323-324]).

Note that we will not use the partial differential equations above. When confronted by complex interface conditions, it is advisable to use the equations of motion and constitutive equations (Equations (1.2.1) − (1.2.4)), rather than the partial differential equations.

1.2.3 The Euler-Bernoulli and Rayleigh models

We consider first the **Rayleigh model**. It can be derived formally from the Timoshenko model. Combining Equations (1.2.1) and (1.2.2), we find that

$$\rho A\partial_t^2 w = \rho I\partial_t^2\partial_x\phi - \partial_x^2 M + P - \partial_x L.$$

For this model, it is assumed that a cross section remains perpendicular to the neutral plane. This implies that $\partial_x w = \phi$, and the equation reduces to

$$\rho A\partial_t^2 w = \rho I\partial_t^2\partial_x^2 w - \partial_x^2 M + P - \partial_x L.$$

This is the equation of motion for the Rayleigh model. The constitutive equation for the shear force V is now redundant and the constitutive equation for the bending moment is

$$M = EI\partial_x^2 w.$$

As mentioned before, we do not use the partial differential equations, but we present them for the purpose of comparison. The partial differential equation for the Rayleigh model is

$$\rho A\partial_t^2 w - \rho I\partial_t^2\partial_x^2 w = -EI\partial_x^4 w + P - \partial_x L.$$

The **Euler-Bernoulli model** is a special case of the Rayleigh model where rotary inertia is ignored and the result is

$$\rho A\partial_t^2 w = -EI\partial_x^4 w + P - \partial_x L.$$

1.2.4 Dimensionless form

In this subsection we write the equations of motion and constitutive equations in dimensionless form. Set

$$\tau = \frac{t}{t_0}, \quad \xi = \frac{x}{\ell}, \quad w^*(\xi,\tau) = \frac{w(x,t)}{\ell} \quad \text{and} \quad \phi^*(\xi,\tau) = \phi(x,t).$$

We introduce the dimensionless constants

$$\alpha = \frac{A\ell^2}{I}, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}.$$

The constant γ depends on the elastic constants and the shear correction factor κ^2 that is determined by the shape of the cross section. The values of κ^2 range between $\frac{1}{2}$ and 1 (see [Co] or [BSSS, p 173]). On the other hand, for isotropic materials we assume that $\frac{G}{E} = \frac{1}{2(1+\nu)}$ (see [My, p 174] or [Fu, Sec 7.2]). Realistic values for γ range between $\frac{1}{6}$ and $\frac{1}{2}$. Timoshenko ([T, p 342]) used $\frac{2}{3}$ for κ^2 and $\frac{G}{E} = \frac{3}{8}$.

The constant α is subject to significant variation. With r^2 the radius of gyration we have $\alpha = \frac{A\ell^2}{I} = \frac{\ell^2}{r^2}$.

The forces and moments in dimensionless form are

$$L^*(\xi,\tau) = \frac{L(x,t)}{G\kappa^2 A}, \quad P^*(\xi,\tau) = \frac{\ell P(x,t)}{G\kappa^2 A},$$

$$V^*(\xi,\tau) = \frac{V(x,t)}{G\kappa^2 A} \quad \text{and} \quad M^*(\xi,\tau) = \frac{M(x,t)}{\ell G\kappa^2 A}.$$

A convenient choice for t_0 is

$$t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

Timoshenko model

$$\partial_t^2 w = \partial_x V + P, \tag{1.2.5}$$
$$\frac{1}{\alpha}\,\partial_t^2 \phi = V + \partial_x M + L, \tag{1.2.6}$$
$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{1.2.7}$$
$$V = \partial_x w - \phi. \tag{1.2.8}$$

Rayleigh model

$$\partial_t^2 w = \partial_x V + P, \tag{1.2.9}$$
$$\frac{1}{\alpha}\,\partial_t^2 \partial_x w = V + \partial_x M + L, \tag{1.2.10}$$
$$M = \frac{1}{\beta}\,\partial_x^2 w. \tag{1.2.11}$$

Euler-Bernoulli model

The Euler-Bernoulli model is obtained from the Rayleigh model by omitting the rotary inertia term $\frac{1}{\alpha}\,\partial_t^2 \partial_x w$.

Remark

Note that the rotary inertia term is simply omitted. It is not correct to reason that $\frac{1}{\alpha} \approx 0$, since that would imply that $\frac{1}{\beta} \approx 0$.

Since the Euler-Bernoulli model is a special case of the Rayleigh model, we will not refer to this model again in the theoretical discussions that follow. To obtain results for the Euler-Bernoulli model, one uses the relevant equations for the Rayleigh model with the modification mentioned above.

1.3 Plate theory

In his book *Elastic Plates: Theory and Applications*, Reissman presents an interesting historical note (see [Rei]):

> *"The theory of plates has a colorful history. Classical plate theory was initiated by Mlle. Sophie Germain (1776 – 1831) in direct response to a prize offered by the French Academy (1811) for the explanation of the nodal curves of a vibrating plate, as demonstrated (experimentally) by E. Chladni (1756 – 1829) of Saxony. After two attempts, Mlle. Germain received the prize in 1816 but only after Lagrange, a member of the examination committee, corrected her initially submitted paper. Subsequently, a controversy ensued about the appropriate, associated boundary conditions, and this was settled approximately 34 years after the correct partial differential equations were discovered. No less than the authorities G. R. Kirchhoff (1824 – 1887) and Lord Kelvin (William Thompson) (1824 – 1907) were responsible for this part of the theory."*

From 1945 to 1950 improvements to classical plate theory were made by E. Reissner, H. Hencky, Y. S. Uflyand and R. D. Mindlin (see [Mi] for references).

1.3.1 Equations of motion

We consider small transverse vibration of a thin plate with thickness h and density ρ. The reference configuration for the plate is a domain Ω in the plane.

The transverse displacement of $\boldsymbol{x}$ at time t is denoted by $w(\boldsymbol{x}, t)$. The angle between a "material line" and a perpendicular to the plane is $\psi(\boldsymbol{x}, t)$ and the angle between the projection of the material line in the plane and the unit vector $\boldsymbol{e}_1$ is $\phi(\boldsymbol{x}, t)$ (see [Rei, Sec 3.2, Sec 3.5]). For a linear model $\boldsymbol{\psi}$ is approximated by

$$\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T = [\psi \cos\phi \quad \psi \sin\phi]^T.$$

Then the equations of motion (see [Mi] and [Rei, p 152]) are given by

$$\rho h \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{1.3.1}$$

$$\rho I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \boldsymbol{Q}, \tag{1.3.2}$$

where $I = \dfrac{h^3}{12}$ is the length moment of inertia.

$\boldsymbol{Q}$ represents a force density, $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a moment density and q an external load on the plate.

1.3.2 The Reissner-Mindlin and Kirchhoff models

Constitutive equations

We restrict our attention to the linear theory. The following assumptions are made for small curvature and small partial derivatives (see [Rei, p 61] and [Mi]).

$$\boldsymbol{Q} = \kappa^2 Gh(\nabla w + \boldsymbol{\psi}), \tag{1.3.3}$$

where G is the shear modulus and κ^2 a correction factor.

$$M = \frac{1}{2} D \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix}. \tag{1.3.4}$$

D is a measure of stiffness for the plate and is given by

$$D = \frac{EI}{1-\nu^2},$$

where E is Young's modulus and ν Poisson's ratio.

The correction factor κ^2 is chosen in such a way that the solution of the plate model compares well with the solution of the three-dimensional model. The value of κ^2 depends on Poisson's ratio ν and ranges almost linearly from 0.76 to 0.91 if ν increases from 0 to 0.5 (see [Mi]). Also mentioned in this reference is that Reissner used $\kappa^2 = \dfrac{5}{6}$.

The equations of motion and the constitutive equations above are known as the **Reissner-Mindlin plate model**.

The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [Rei, p 152] and [Mi]). In our approach these partial differential equations are not used.

For classical plate theory, $\boldsymbol{\psi}$ is replaced by $-\nabla w$ and the constitutive equation for $\boldsymbol{Q}$ is no longer necessary. This is sometimes referred to as the **Kirchhoff plate model**.

1.3.3 Dimensionless forms

We introduce the dimensionless variables

$$\tau = \frac{t}{t_0}, \quad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell},$$

where ℓ is a suitable length and t_0 must still be specified.

The dimensionless variables, with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, are

$$w^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell}\right) w(\boldsymbol{x}, t), \quad \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) = \boldsymbol{\psi}(\boldsymbol{x}, t),$$

$$\boldsymbol{Q}^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell G \kappa^2}\right) \boldsymbol{Q}(\boldsymbol{x}, t), \quad M^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell^2 G \kappa^2}\right) M(\boldsymbol{x}, t)$$

$$\text{and} \quad q^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{G\kappa^2}\right) q(\boldsymbol{x}, t).$$

The dimensionless constants that are used are given by

$$h_p = \frac{h}{\ell}, \quad I_p = \frac{h_p^3}{12} \quad \text{and} \quad \beta_p = \frac{\ell^3 G \kappa^2}{EI}.$$

The constant h_p denotes the dimensionless thickness of the plate and I_p the dimensionless length moment of inertia.

We choose $t_0 = \ell \sqrt{\dfrac{\rho}{G\kappa^2}}$ (for convenience) and use the original notation for the corresponding dimensionless quantities. The equations of motion and constitutive equations in dimensionless form are presented below.

Reissner-Mindlin plate model

$$h_p\, \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{1.3.5}$$

$$I_p\, \partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \boldsymbol{Q}, \tag{1.3.6}$$

$$\boldsymbol{Q} = h_p \big(\nabla w + \boldsymbol{\psi}\big), \tag{1.3.7}$$

$$M = \frac{1}{2\beta_p(1-\nu^2)} \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix} \tag{1.3.8}$$

Classical plate model

$$h_p\,\partial_t^2 w = \operatorname{div}\boldsymbol{Q} + q, \tag{1.3.9}$$
$$I_p\,\partial_t^2\left(\boldsymbol{\nabla} w\right) = \boldsymbol{Q} - \operatorname{div} M, \tag{1.3.10}$$

$$M = -\frac{1}{\beta_p(1-\nu^2)}\begin{bmatrix} \left(\partial_1^2 w + \nu\partial_2^2\right)w & (1-\nu)\partial_1\partial_2 w \\ (1-\nu)\partial_1\partial_2 w & \left(\partial_2^2 w + \nu\partial_1^2\right)w \end{bmatrix}. \tag{1.3.11}$$

Generally the rotary inertia term $I_p\partial_t^2\left(\nabla w\right)$ in Equation (1.3.10) is ignored.

1.4 Two-dimensional model for a beam

As mentioned in the introduction, we also consider a two-dimensional model for a beam. To facilitate the discussion, we include a brief review of linear elasticity.

1.4.1 Equation of motion

Consider an elastic body with density ρ. The displacement of a point $\boldsymbol{x}$ in the reference configuration at time t is $\boldsymbol{u}(\boldsymbol{x}, t)$ and the velocity is $\boldsymbol{v} = \partial_t\boldsymbol{u}$.

From the conservation law for momentum, we have the **equation of motion** (see [Fu, Sec 5.5, 5.7]) or [AF, p 125])

$$\rho\partial_t^2\boldsymbol{u} = \operatorname{div} T + \boldsymbol{Q},$$

where T is the first Piola stress tensor and $\boldsymbol{Q}$ an external body force (density force).

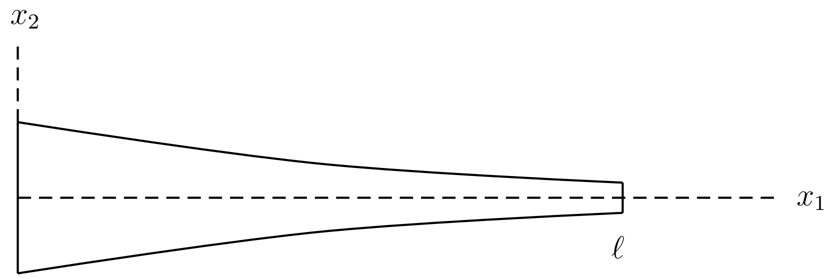
In the case of small local displacements, the **infinitesimal theory of elasticity** or **linear elasticity** may be used. In this case the first Piola stress tensor is approximated by the Cauchy stress tensor (which is symmetric). For an explanation, see [AF, p 45-46, 122, 125]. Another explanation is given in [Fu, Sec 7.1].

In the matrix representation of T the stress components are denoted by σ_{ij} and $\operatorname{div} T$ is a vector with components

$$[\operatorname{div} T]_i = \partial_1\sigma_{i1} + \partial_2\sigma_{i2} + \partial_3\sigma_{i3} \quad \text{for } i = 1,\ 2,\ 3.$$

Simplifying assumptions

Now consider a beam as illustrated below. The x_1-axis is taken to coincide with the line of centroids of the cross sections. We assume that the cross sections and applied loads are symmetric with respect to the x_1x_2-plane and consequently the motion of the beam is parallel to the x_1x_2-plane.



For beam problems it is reasonable to assume that the body or beam is in a state of plane stress. To be specific, we assume that $\sigma_{3i} = \sigma_{i3} = 0$. However, this does not imply that the problem is two-dimensional since $\partial_3\sigma_{ij}$ need not be zero. This is an assumption that we make. The interpretation is that the stresses are averages across the width of the beam. This approach is in line with Cowper's ([Co]) derivation of the Timoshenko model. It is reasonable to assume that the two-dimensional model is more accurate than beam models (but obviously less accurate than three-dimensional models).

Constitutive equations

The **infinitesimal strain** $\mathcal{E}$ is given by

$$e_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right).$$

(See [AF, p 25] or [Fu, p 155].)

Constitutive equations are required to express the relationship between the stress T and the strain $\mathcal{E}$. These depend on the elastic properties of the material under consideration. An **isotropic** material exhibits no preferred direction in its response to a given state of stress. For a **homogeneous** material the elastic properties are the same at all points of the reference configuration.

We use **Hooke's law for homogeneous isotropic materials** ([Fu, Sec 9.1] or [My, p 173, 182]) for the special case of plane stress.

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(e_{11} + \nu e_{22}\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(e_{22} + \nu e_{11}\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{1+\nu}\; e_{12}, \end{aligned}$$

where E is Young's modulus and ν Poisson's ratio.

The constitutive equation in terms of the components of $\boldsymbol{u}$ follow as

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(\partial_1 u_1 + \nu\partial_2 u_2\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(\partial_2 u_2 + \nu\partial_1 u_1\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{2(1+\nu)}\left(\partial_1 u_2 + \partial_2 u_1\right). \end{aligned}$$

Substitution of the constitutive equation into the equation of motion yields a system of partial differential equations for the components of the displacement. We will not make use of this system of partial differential equations.

1.4.2 Dimensionless form

The dimensionless variables and constants must be the same or compatible with those in Section 1.2. Set

$$\tau = \frac{t}{t_0}, \quad \xi_i = \frac{x_i}{\ell} \quad \boldsymbol{u}^*(\boldsymbol{\xi},\tau) = \frac{1}{\ell}\,\boldsymbol{u}(\boldsymbol{x},t),$$

$$\text{and} \quad \sigma^*_{ij}(\boldsymbol{\xi},\tau) = \left(\frac{1}{G\kappa^2}\right)\sigma_{ij}(\boldsymbol{x},t).$$

Recall that $t_0 = \ell\sqrt{\dfrac{\rho}{G\kappa^2}}$ and $\gamma = \dfrac{G\kappa^2}{E}$.

Returning to the original notation we present the equations of motion and constitutive equations in **dimensionless form**. In the problems under consideration $\boldsymbol{Q} = \boldsymbol{0}$.

Equation of motion

$$\partial_t^2 \boldsymbol{u} = \operatorname{div} T, \quad \text{where} \tag{1.4.1}$$
$$\operatorname{div} T = \begin{bmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} \\ 0 \end{bmatrix}.$$

Constitutive equations

$$\begin{aligned} \sigma_{11} &= \frac{1}{\gamma(1-\nu^2)}\big(\partial_1 u_1 + \nu\partial_2 u_2\big), \\ \sigma_{22} &= \frac{1}{\gamma(1-\nu^2)}\big(\partial_2 u_2 + \nu\partial_1 u_1\big), \\ \sigma_{12} = \sigma_{21} &= \frac{1}{2\gamma(1+\nu)}\big(\partial_1 u_2 + \partial_2 u_1\big). \end{aligned} \tag{1.4.2}$$

1.5 Interface conditions

It is now possible to provide more detail concerning the problems that we investigate.

1.5.1 Vertical slender structure

The vertical slender structure (for example a chimney), is modelled as a vertical beam with $x = 0$ at the ground level. The boundary conditions at the top present no problem and we have that

$$M(1,t) = V(1,t) = 0.$$

For a built in beam the conventional boundary conditions at the bottom are given by $w(0,t) = \partial_x w(0,t) = 0$. However, the conventional boundary conditions yielded poor results (as Newland mentioned in [N2]).

Modelling the behaviour of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations. This was done in [LVV] with satisfactory results – as mentioned before. In this thesis we adapt the interface conditions for the Timoshenko theory.

1.5.2 Boundary conditions for the clamped end of a beam

First we show that the boundary conditions used for the Euler-Bernoulli and Timoshenko models are incompatible. Consider a beam in equilibrium clamped at $x = 0$ and an external vertical force F at the endpoint $x = 1$.

The usual boundary conditions at $x = 0$ for an Euler-Bernoulli beam are

$$w(0) = w'(0) = 0\,.$$

For a Timoshenko beam the boundary conditions at $x = 0$ are

$$w(0) = \phi(0) = 0\,.$$

When an external force F is applied at $x = 1$, the implication is that the shear force throughout the beam is constant and equal to F, hence $V(0) = F$. Since $\phi(0) = 0$, it follows from the constitutive equation (1.2.8) that

$$w'(0) = V(0) = F\,.$$

However, for the Euler-Bernoulli and the Rayleigh models it is assumed that $w'(0) = 0$. Clearly $\phi(0)$ and $w'(0)$ can not both be zero.

The boundary condition $w'(0) = 0$ is realistic from a modelling perspective. This suggests that the boundary conditions at a built in end for the Timoshenko theory deserves closer examination.

One possibility is the boundary condition proposed in [V3], which we consider in this thesis. However, as mentioned in Section 1.1.2, this boundary condition creates larger disparities and it is logical to consider other possibilities.

We consider the possibility that the constitutive equation (1.2.8) does not reflect reality at the built in end. The quantity $w' - \phi$ represents the average shear for a cross section. As x tends to zero, both w' and ϕ become small, but the shear force V remains constant. These facts suggests that $\dfrac{V}{w' - \phi}$ is not constant. The Timoshenko theory implies that a cross section remains plane and that the shearing strain $w' - \phi$ is constant on a cross section. In reality, the strain is zero at both the bottom and the top of a horizontal beam. In the Timoshenko theory, the quantity $w' - \phi$ represents the average strain of a cross section. It is possible that this is not realistic at the clamped end.

To investigate the difficulties mentioned, we consider a prismatic beam with the simplifying assumptions mentioned in Section 1.4. Usually the boundary condition for the "fixed end" is to set the displacement $\boldsymbol{u} = \mathbf{0}$. This will not do if the objective is to determine the strain at the clamped end, hence we also consider configurations where part of the beam is embedded (see Section 2.3).

Finally, there is another aspect that needs to be mentioned. The first two or three eigenfrequencies of the Euler-Bernoulli and the Timoshenko models for a cantilever beam differ very little, unless the beam is short (relative to its thickness) – to be precise, when the parameter α is small. The first eigenfrequency differs appreciably when the beam is so short that one is reluctant to use beam theory at all. Comparisons are given in Section 7.1.

1.5.3 Plate-beam system

When a plate and a beam are connected, numerous aspects need to be considered. These aspects may be classified under geometrical constraints and mechanical interaction. A Reissner-Mindlin-Timoshenko plate-beam system is extremely complex due to the presence of five equations of motion. One could say that the boundary conditions are partial differential equations themselves.

Another complication is the fact that the angles ψ (for the plate) and ϕ (for the beam) do not present a physically reality but convenient averages. Consequently it is not clear what the geometrical constraints should be.

Not only is the modelling for a Reissner-Mindlin-Timoshenko plate-beam system more complex, but the mathematical analysis and numerical analysis present additional difficulties. Finally, the numerical algorithms also present nontrivial difficulties not present in the plate-beam system using classical plate and beam theory.

In this thesis we consider a Reissner-Mindlin plate supported by two Timoshenko beams. The case where the plate is connected rigidly to the beam, can be found in [LLS].

One expects that in some cases the Reissner-Mindlin-Timoshenko model will compare well with the Kirchhoff-Euler-Bernoulli model that is investigated in [ZVGV1], [ZVGV3] and [Ziet]. In Chapter 8 we present some results on this comparison.

Chapter 2

Model problems

2.1 Vertical slender structure

In this section we present DPS models that correspond to Newland's LPS models [N1, p 129-132] and [N2]. The slender structure (e.g. a steel chimney) is modelled as a Euler-Bernoulli, a Rayleigh or a Timoshenko beam mounted vertically and gravity is taken into account. (The reason for including gravity in the model, is to match Newland's models.)

From Section 1.2 we have the relevant equations of motion for the Rayleigh and Timoshenko theories in dimensionless form. In this case we have free vibration and therefore $P = 0$.

The relevant constitutive equations are also given in Section 1.2. The term $L = -S\partial_x w$ is a moment density (measured in Newton) due to gravity. The axial force due to gravity is given by

$$S(x) = -\rho A g(\ell - x).$$

With $\mu = \dfrac{\rho g \ell}{G\kappa^2}$ and using the original notation, the dimensionless moment density (see Section 1.2) is given by

$$L(x,t) = \mu(1-x)\partial_x w(x,t).$$

2.1.1 Simplistic Models

Initially we considered the Rayleigh theory as this corresponds to the Newland models.

Boundary conditions at $x = 0$

There are a number of possibilities for the boundary conditions at $x = 0$. Following Newland ([N1], [N2]), four models are considered in [LVV]. The first two are rather simplistic. In Model 1, the foundation is completely rigid and the boundary conditions at the base are given by

$$w(0,t) \;=\; \partial_x w(0,t) \;=\; 0.$$

In the second model that corresponds to the model in [N1, p 133], the effect of the resilient seating is taken into account. The foundation is modelled to be elastic with damping. Hence the moment $M(0,t)$ is determined by the elasticity and damping of the foundation. In this case the boundary conditions at the base for the Rayleigh model are given by

$$\begin{aligned} w(0,t) &= 0, \\ M(0,t) &= k\,\partial_x w(0,t) + c\,\partial_t\partial_x w(0,t), \end{aligned}$$

where the constants k and c are nonnegative.

Our results for these models were compared to Newland's results and was published in [LVV]. Models 1 and 2 are not considered in this thesis.

2.1.2 The dynamics of the foundation block and resilient seating

The mathematical models presented later in this Section as Problem VR 3 and Problem VR 4, were published in [LVV]. For these models the dynamics of the resilient seating and foundation block is taken into account.

As our point of departure, we consider the physical model in [N2]. Figure 1 corresponds to Figure 2 in [N2].

Figure 1: Simplified sketch of the system

The springs and damping mechanisms in the sketch are schematic.

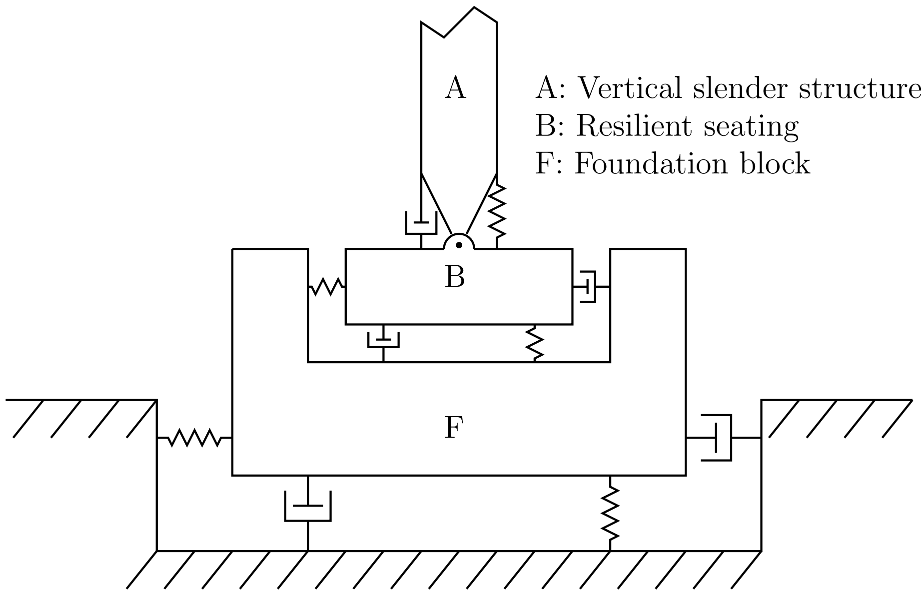
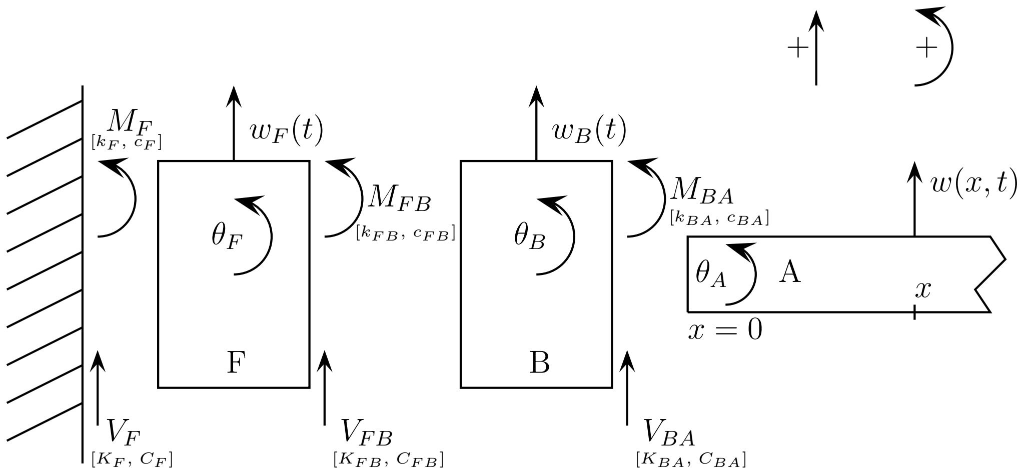


Figure 2: Displacements, angles of rotation, moments and forces

Convention: Moments and forces are denoted by the action of right on left. For instance, M_{FB} denotes the moment exerted by B on F.



To formulate the boundary conditions at the base, it is necessary to consider the equations of motion for the resilient seating and foundation block. Both are modelled as rigid bodies connected to linear elastic springs and linear damping mechanisms.

Equations of motion

$$\begin{aligned} m_F \ddot{w}_F &= V_{FB} - V_F, \\ m_B \ddot{w}_B &= V_{BA} - V_{FB}, \\ I_F \ddot{\theta}_F &= M_{FB} - M_F, \\ I_B \ddot{\theta}_B &= M_{BA} - M_{FB}. \end{aligned}$$

Constitutive equations

$$\begin{aligned} V_F &= K_F w_F + C_F \dot{w}_F, \\ V_{FB} &= K_{FB}(w_B - w_F) + C_{FB}(\dot{w}_B - \dot{w}_F), \\ M_F &= k_F \theta_F + c_F \dot{\theta}_F, \\ M_{FB} &= k_{FB}(\theta_B - \theta_F) + c_{FB}(\dot{\theta}_B - \dot{\theta}_F). \end{aligned}$$

The reader must take note of the use of upper case and lower case letters for the constants.

Interface conditions

Let $\theta_A(t)$ denote the rotation of the end point of the vertical structure.

$$\begin{aligned} M_{BA}(t) &= k_{BA}\Big(\theta_A(t) - \theta_B(t)\Big) + c_{BA}\Big(\dot{\theta}_A(t) - \dot{\theta}_B(t)\Big), \\ M_{BA}(t) &= M(0,t), \\ V_{BA}(t) &= V(0,t), \\ w_B(t) &= w(0,t), \\ \theta_B(t) &\neq \theta_A(t) \quad \text{(in general)}. \end{aligned}$$

We make the following assumptions for $\theta_A(t)$:

- $\theta_A(t) = \partial_x w(0,t)$ for the Rayleigh models and
- $\theta_A(t) = \phi(0,t)$ for the Timoshenko models.

Dimensionless constants

The dimensionless constants for the foundation block and the resilient seating are

$$m^* = \frac{m}{\ell \rho A}\,, \quad \text{and} \quad I^* = \frac{I}{\ell^3 \rho A}\,.$$

The different elastic and damping constants are

$$K^* = \frac{K\ell}{AG\kappa^2}\,, \quad k^* = \frac{k}{AG\kappa^2 \ell}\,, \quad C^* = \frac{C\ell}{AG\kappa^2 t_0} \quad \text{and} \quad c^* = \frac{c}{AG\kappa^2 t_0 \ell}\,.$$

The following equalities hold for the the scaling factors of m and I:

$$\ell \rho A = \frac{t_0^2 AG\kappa^2}{\ell} \quad \text{and} \quad \ell^3 \rho A = t_0^2 AG\kappa^2 \ell^2$$

All the constants in the equations of motion for the foundation block and resilient seating and the equations for the interface conditions, must be replaced by the corresponding dimensionless constants.

The diagrams and equations in this subsection are from [LVV].

2.1.3 Rayleigh models

The Rayleigh theory applied to Models 3 and 4 yields the same equations of motion, constitutive equations and boundary conditions at the top.

Equations of motion

$$\partial_t^2 w = \partial_x V, \tag{2.1.1}$$

$$\frac{1}{\alpha}\,\partial_t^2 \partial_x w = V + \partial_x M + L, \tag{2.1.2}$$

Constitutive equations

$$M = \frac{1}{\beta}\,\partial_x^2 w, \tag{2.1.3}$$

$$L(x,t) = \mu(1-x)\,\partial_x w(x,t). \tag{2.1.4}$$

Boundary conditions at $x = 1$

$$M(1,t) = V(1,t) = 0.$$

The interface conditions for these two problems differ. We will refer to the two problems as Problem VR 3 and Problem VR 4 (corresponding to models 3 and 4 in [LVV]).

Problem VR 3

Equations of motion: (2.1.1) and (2.1.2).

Constitutive equations: (2.1.3) and (2.1.4).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

The motion of B is neglected and B is considered to be rigidly connected to the foundation block F and this case corresponds to Model 2 in [N2]. The conditions are

$$w_F(t) = w_B(t) = w(0,t), \quad \theta_F = \theta_B, \quad V_{FB}(t) = V_{BA}(t) = V(0,t)$$

$$\text{and} \quad M_{FB}(t) = M_{BA}(t) = M(0,t).$$

The constant k_{BA} is replaced by k and c_{BA} by c.

The interface conditions and the equations of motion of the foundation block and resilient seating reduce to the following three equations:

$$m_F \partial_t^2 w(0,t) = V(0,t) - K_F w(0,t) - C_F \partial_t w(0,t), \tag{2.1.5}$$

$$I_F \ddot{\theta}_F(t) = k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t \partial_x w(0,t) - \dot{\theta}_F(t)\Big) - k_F \theta_F(t) - c_F \dot{\theta}_F(t), \tag{2.1.6}$$

$$M(0,t) = k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t \partial_x w(0,t) - \dot{\theta}_F(t)\Big). \tag{2.1.7}$$

Problem VR 4

Equations of motion: (2.1.1) and (2.1.2).

Constitutive equations: (2.1.3) and (2.1.4).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

In Model 4, the interface conditions and the equations of motion of the foundation block and resilient seating follow directly from the discussion in Section 2.1.2.

The following five equations formulate the interface conditions.

$$\begin{aligned} m_B\partial_t^2 w(0,t) &= V(0,t) - K_{FB}\Big(w(0,t) - w_F(t)\Big) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big), \end{aligned} \tag{2.1.8}$$

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{2.1.9}$$

$$\begin{aligned} M(0,t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) \\ &\quad +c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big), \end{aligned} \tag{2.1.10}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{2.1.11}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{2.1.12}$$

Remarks

1. The stiffness and damping in the mounting are modelled to be due to linear springs and linear dashpots. The limitations of these assumptions are discussed in [N2].

2. Problems VR 3 and VR 4 are from [LVV].

2.1.4 Timoshenko models

As mentioned before, results for Model 3 and Model 4 using the Rayleigh and Euler-Bernoulli theory were published in [LVV]. In this thesis our main objective is to use the Timoshenko beam theory in the models and compare the results to the results where the Rayleigh theory is used. We refer to these problems as Problem VT 3 and Problem VT 4.

The equations of motion, constitutive equations and the boundary conditions at the top are the same for both problems.

Equations of motion

$$\partial_t^2 w = \partial_x V, \tag{2.1.13}$$

$$\frac{1}{\alpha}\,\partial_t^2 \phi = \partial_x M + V + L. \tag{2.1.14}$$

Constitutive equations

$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{2.1.15}$$

$$V = \partial_x w - \phi, \tag{2.1.16}$$

$$L(x,t) = \mu(1-x)\partial_x w(x,t). \tag{2.1.17}$$

Boundary conditions at $x = 1$

$$M(1,t) = V(1,t) = 0.$$

Modifications on some of the interface conditions are necessary for the Timoshenko theory and we state the full set of interface conditions. Note that the first and last interface condition differ from those for the Rayleigh theory.

Interface conditions

$$\begin{aligned}
M_{BA}(t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big),\\
M_{BA}(t) &= M(0,t),\\
V_{BA}(t) &= V(0,t),\\
w_B(t) &= w(0,t),\\
\theta_B(t) &\neq \phi(0,t) \quad \text{(in general)}.
\end{aligned}$$

Problem VT 3

Equations of motion: (2.1.13) and (2.1.14).

Constitutive equations: (2.1.15), (2.1.16) and (2.1.17).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

As in the Rayleigh models, the motion of B is neglected and B is considered to be rigidly connected to F. Hence

$$w_F(t) = w_B(t) = w(0,t), \quad \theta_F = \theta_B, \quad V_{FB}(t) = V_{BA}(t) = V(0,t)$$
$$\text{and} \quad M_{FB}(t) = M_{BA}(t) = M(0,t).$$

The constant k_{BA} is replaced by k and c_{BA} by c.

The interface conditions and the equations of motion of the foundation block and resilient seating reduce to the following three equations:

$$m_F\partial_t^2 w(0,t) = V(0,t) - K_F w(0,t) - C_F\partial_t w(0,t), \tag{2.1.18}$$

$$\begin{aligned}
I_F\ddot{\theta}_F(t) &= k\Big(\phi(0,t) - \theta_F(t)\Big) + c\Big(\partial_t\phi(0,t) - \dot{\theta}_F(t)\Big)\\
&\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t),
\end{aligned} \tag{2.1.19}$$

$$M(0,t) = k\Big(\phi(0,t) - \theta_F(t)\Big) + c\Big(\partial_t\phi(0,t) - \dot{\theta}_F(t)\Big). \tag{2.1.20}$$

Problem VT 4

Equations of motion: (2.1.13) and (2.1.14).

Constitutive equations: (2.1.15), (2.1.16) and (2.1.17).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

The interface conditions are given by

$$\begin{aligned} m_B\partial_t^2 w(0,t) &= V(0,t) - K_{FB}\Big(w(0,t) - w_F(t)\Big) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big), \qquad (2.1.21) \end{aligned}$$

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \qquad (2.1.22) \end{aligned}$$

$$M(0,t) = k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big), \qquad (2.1.23)$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \qquad (2.1.24) \end{aligned}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \qquad (2.1.25) \end{aligned}$$

2.2 The cantilever beam

In Chapter 7 we compare the natural frequencies of the Euler-Bernoulli and Timoshenko models for the free vibration of a cantilever beam. For reference purposes we state the equations of motion, constitutive equations and the standard boundary conditions.

Timoshenko theory

$$\begin{aligned}
\partial_t^2 w &= \partial_x V, \\
\frac{1}{\alpha}\,\partial_t^2 \phi &= V + \partial_x M, \\
M &= \frac{1}{\beta}\,\partial_x \phi, \\
V &= \partial_x w - \phi, \\
M(1,t) &= V(1,t) \;=\; 0, \\
w(0,t) &= \phi(0,t) \;=\; 0.
\end{aligned}$$

Euler-Bernoulli theory

$$\begin{aligned}
\partial_t^2 w &= \partial_x V, \\
0 &= V + \partial_x M, \\
M &= \frac{1}{\beta}\,\partial_x^2 w, \\
M(1,t) &= V(1,t) \;=\; 0, \\
w(0,t) &= \partial_x w(0,t) \;=\; 0.
\end{aligned}$$

A modification of the boundary conditions for the Timoshenko model (suggested in [V3]) is

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} V(0,t) \\ M(0,t) \end{bmatrix} = \begin{bmatrix} w(0,t) \\ \phi(0,t) \end{bmatrix}.$$

The standard boundary conditions for the Timoshenko model is a special case of the modified boundary conditions, where

$$c_{11} = c_{12} = c_{21} = c_{22} = 0.$$

2.3 Two-dimensional model for a cantilever beam

We consider a prismatic beam built in at one end. In Section 1.4 a two-dimensional model is proposed. The equation of motion and the constitutive equation are given by Equations (1.4.1) and (1.4.2).

It is not obvious how to model the built in end of a cantilever beam. Therefore we consider different configurations and discuss them briefly. A detailed discussion is given in Section 7.2.1.

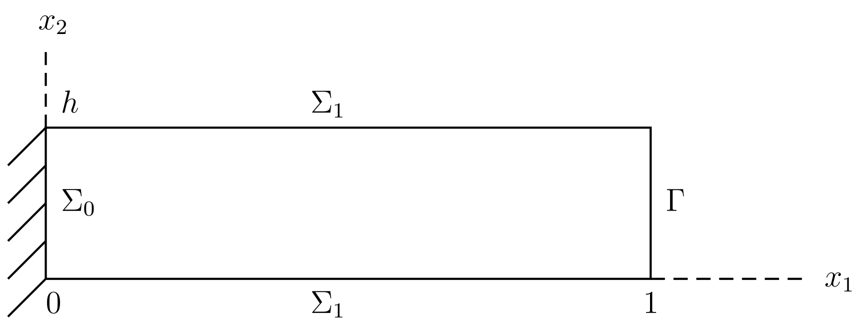
Rigidly attached beam

We consider a rigidly attached beam as in Figure 1. For this case the reference configuration Ω is the rectangle given by

$$0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq h$$

and the beam is attached at $x_1 = 0$.

Figure 1: Rigidly attached beam

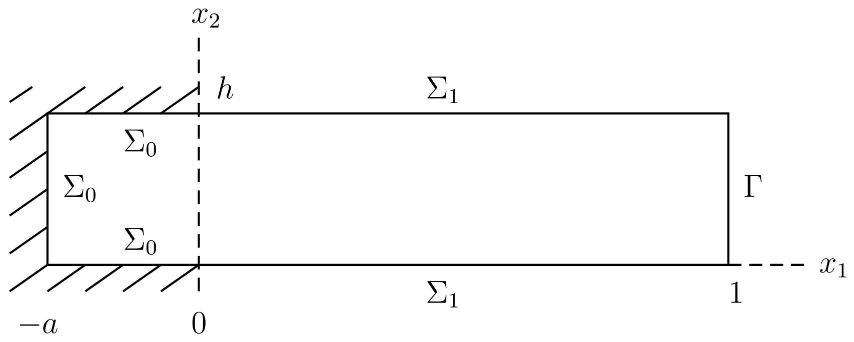


Built in beam

In this case we consider a beam that is built in at $x_1 = 0$ as in Figure 2. The reference configuration Ω is the rectangle given by

$$-a \leq x_1 \leq 1, \quad 0 \leq x_2 \leq h.$$

Figure 2: Built in beam



To apply the theory, it is preferable to formulate the model problems for a general domain. Let Ω be an open convex subset in the plane. The boundary of Ω consists of smooth curves, $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ and Γ.

Boundary conditions

The traction $\boldsymbol{t} = T\boldsymbol{n}$ is specified on Γ and on Σ_i we have $T\boldsymbol{n} \cdot \boldsymbol{u} = 0$ for each i, with the additional restriction that $u_1 = 0$ on at least one of the sets Σ_i and $u_2 = 0$ on at least one of the sets Σ_j.

Equilibrium problem

For the equilibrium problem a transverse force is applied at Γ. However, for the boundary value problem it is necessary to prescribe the traction on Γ.

Problem CTD 1

$$\begin{aligned} \operatorname{div} T &= \mathbf{0} \quad \text{in} \ \Omega, \\ T\boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on} \ \Sigma, \\ T\boldsymbol{n} &= \boldsymbol{t} \quad \text{on} \ \Gamma, \end{aligned}$$

with the constitutive equation given by Equation (1.4.2).

Free vibration

Problem CTD 2

$$\begin{aligned} \partial_t^2 \boldsymbol{u} &= \operatorname{div} T \quad \text{in} \ \ \Omega, \\ T\boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on} \ \ \Sigma, \\ T\boldsymbol{n} &= \boldsymbol{0} \quad \text{on} \ \ \Gamma, \end{aligned}$$

with the constitutive equation given by Equation (1.4.2).

Remark

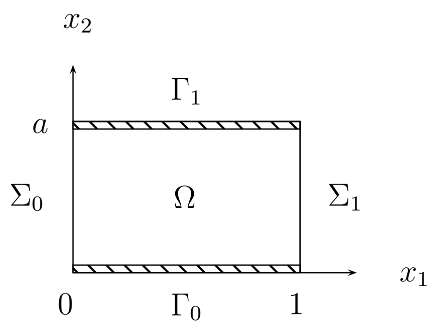
The condition $T\boldsymbol{n} \cdot \boldsymbol{u} = 0$ represents a number of possibilities, e. g. $\boldsymbol{u} = \boldsymbol{0}$ or $T\boldsymbol{n} = \boldsymbol{0}$ or various different combinations. The different configurations are given in Chapter 7.

2.4 A plate-beam system

Consider small transverse vibration of a thin rectangular plate supported by identical beams at two opposing sides and rigidly supported at the remaining sides. The beams are supported at their endpoints. Assume furthermore the case of free vibration, i.e. $q = 0$. The displacement for the system is measured with respect to the equilibrium state. (Due to gravity, the equilibrium state is not the same as the undeformed state.) It is assumed that the plate remains in contact with the beams and supporting structure at all times. This mathematical model is considered in [V4].

The reference configuration for the plate is the rectangle Ω, where $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq a$. The plate is rigidly supported at $x_1 = 0$ and $x_1 = 1$. These sections of the boundary of Ω are denoted by Σ_0 and Σ_1 respectively. The plate is supported by beams at $x_2 = 0$ and $x_2 = a$ and these sections are denoted by Γ_0 and Γ_1 respectively. Figure 1 depicts the reference configuration. The shaded areas represent the beams.

Figure 1: Reference configuration of the plate-beam system.



Notation

To avoid confusion, we adapt (if necessary) the symbols used for quantities related to the beams by using the subscript "b".

2.4.1 The Reissner-Mindlin-Timoshenko model

For the mathematical model we use the Reissner-Mindlin plate theory and the Timoshenko beam theory. On the rectangle Ω, the equations of motion (1.3.1) and (1.3.2) are satisfied and on Γ_0 and Γ_1, the two sets of equations of motion are given by (1.2.1) and (1.2.2). In Equation (1.2.2), L represents a moment density transmitted from the plate to the beam and P a force desity transmitted from the plate to the beam.

Boundary conditions on Σ_0 and Σ_1

On these sections of the boundary, the conventional homogeneous boundary conditions for a rigidly supported plate are used, i.e.

$$w = 0, \quad \psi_2 = 0 \quad \text{and} \quad M\boldsymbol{n} \cdot \boldsymbol{n} = 0, \tag{2.4.1}$$

where $\boldsymbol{n}$ is the unit exterior normal (see [Rei, p 66]). The third condition reduces to $M_{11} = 0$.

Interface conditions on Γ_0 and Γ_1

On Γ_0 and Γ_1 the interaction between the plate and the beams is considered. The interface conditions are given in [V4] for a general case. For this special case they reduce to

$$w_b(x_1,t) = w(x_1,0,t) \text{ on } \Gamma_0\,, \quad w_b(x_1,t) = w(x_1,a,t) \text{ on } \Gamma_1\,, \qquad (2.4.2)$$
$$\phi_b(x_1,t) = -\psi_1(x_1,0,t) \text{ on } \Gamma_0\,, \quad \phi_b(x_1,t) = -\psi_1(x_1,a,t) \text{ on } \Gamma_1\,. \qquad (2.4.3)$$

The interface conditions for the force densities and moment densities on Γ_0 and Γ_1 are given by

$$\boldsymbol{Q}\cdot\boldsymbol{n} = -P, \qquad (2.4.4)$$
$$M\boldsymbol{n}\cdot\boldsymbol{\tau} = L, \qquad (2.4.5)$$
$$M\boldsymbol{n}\cdot\boldsymbol{n} = 0, \qquad (2.4.6)$$

where $\boldsymbol{\tau}$ is the unit tangent oriented in such a way that Ω is on the left hand side of $\boldsymbol{\tau}$. For a detailed explanation of the moments $M\boldsymbol{n}\cdot\boldsymbol{n}$ and $M\boldsymbol{n}\cdot\boldsymbol{\tau}$, see [Rei, p 66].

Remarks

1. Note the difference in sign convention for measuring the angles $\boldsymbol{\psi}$ and ϕ_b in the plate and beam models.

2. Care should be taken to also incorporate the difference between sign conventions for moments in the plate and beam models. The beam equations for Γ_1 is derived for a beam oriented from left to right. When applying the interface condition (2.4.5) on Γ_1, the moment L has to be replaced by $-L$.

Conditions at the endpoints of Γ_0 and Γ_1

At the endpoints of Γ_0 and Γ_1 we have the obvious boundary conditions for the beams, namely

$$w_b = 0 \quad \text{and} \quad M_b = 0. \qquad (2.4.7)$$

Dimensionless form

The dimensionless form for the plate model has been derived in Section 1.3.3. For the beam equations it has to be recalculated using the scaling of the plate model and

$$\tau = \frac{t}{t_0} \quad \text{and} \quad \xi_1 = \frac{x_1}{\ell}\,.$$

Also set

$$w_b^* = \left(\frac{1}{\ell}\right) w_b\,, \quad \phi_b^* = \phi_b\,,$$

$$P^* = \left(\frac{1}{\ell G \kappa^2}\right) P, \quad V^* = \left(\frac{1}{\ell^2 G \kappa^2}\right) V,$$

$$M_b^* = \left(\frac{1}{\ell^3 G \kappa^2}\right) M_b \quad \text{and} \quad L^* = \left(\frac{1}{\ell^2 G \kappa^2}\right) L.$$

Note that the parameters of the plate are used for the scaling. Choosing $t_0 = \ell \sqrt{\dfrac{\rho}{G\kappa^2}}$ as in Section 1.3.3 and using the original notation for the corresponding dimensionless quantities, the dimensionless beam model is given by

$$\eta_1\, \partial_t^2 w_b = \partial_1 V + P, \tag{2.4.8}$$

$$\eta_1\, \partial_t^2 \phi_b = \alpha_b\,(\partial_1 M_b + V + L), \tag{2.4.9}$$

$$V = \eta_2\,(\partial_1 w_b - \phi_b), \tag{2.4.10}$$

$$\beta_b M_b = \eta_2\, \partial_1 \phi_b. \tag{2.4.11}$$

The dimensionless constants α_b and β_b are as in Section 1.2.4, i.e.

$$\alpha_b = \frac{A_b \ell^2}{I_b}, \quad \beta_b = \frac{A_b G_b \kappa_b^2 \ell^2}{E_b I_b}\,.$$

The two additional dimensionless constants η_1 and η_2 express ratios for the material properties and the geometrical properties of the plate and the beams:

$$\eta_1 = \left(\frac{\rho_b}{\rho}\right)\left(\frac{A_b}{\ell^2}\right) \quad \text{and} \quad \eta_2 = \left(\frac{G_b}{G}\right)\left(\frac{\kappa_b^2}{\kappa^2}\right)\left(\frac{A_b}{\ell^2}\right).$$

The interface conditions remain unchanged.

The mathematical model

The vibration problem for the plate-beam system is given by the following equations.

Problem RMT

Equations of motion for the plate: (1.3.5) and (1.3.6) on Ω.

Constitutive equations for the plate: (1.3.7) and (1.3.8) on Ω.

Equations of motion for the beams: (2.4.8) and (2.4.9) on Γ_0 and Γ_1 .

Constitutive equations for the beams: (2.4.10) and (2.4.11) on Γ_0 and Γ_1 .

Interface conditions: (2.4.2) to (2.4.6) on Γ_0 and Γ_1 .

Boundary conditions: (2.4.1) on Σ_0 and Σ_1 .

Endpoint conditions: (2.4.7) at the endpoints of Γ_0 and Γ_1 .

2.4.2 Other models

A simplified model is obtained if the Kirchhoff plate model (with rotary inertia) and the Rayleigh beam model is used. Formally, this model problem can be derived from Problem RMT. We consider the model problems referred to for the purpose of comparison. It should be noted that the scaling for the dimensionless form differs from the scaling used in [ZVGV3] and [Ziet].

In this case the vibration problem for the plate-beam system is given by the following equations.

Problem KR

Equations of motion for the plate: (1.3.9) and (1.3.10) on Ω.

Constitutive equation for the plate: (1.3.11) on Ω.

Equations of motion for the beams:

$$\eta_1 \partial_t^2 w_b = \partial_1 V + P \text{ on } \Gamma_0 \text{ and } \Gamma_1 \, ,$$

$$\eta_1 \partial_t^2 \partial_x w_b = \alpha_b(\partial_1 M_b + V + L) \text{ on } \Gamma_0 \text{ and } \Gamma_1 \, .$$

Constitutive equation for the beams:

$$\beta_b M_b = \eta_2 \partial_1^2 w_b \text{ on } \Gamma_0 \text{ and } \Gamma_1 \, .$$

Interface conditions: (2.4.2) to (2.4.6) on Γ_0 and Γ_1 .

Boundary conditions: (2.4.1) on Σ_0 and Σ_1 .

Endpoint conditions: (2.4.7) at the endpoints of Γ_0 and Γ_1 .

Problem KEB

An even simpler model is obtained if rotary inertia is ignored in the plate and the beams.

Chapter 3

Variational forms

3.1 Introduction

In this chapter we consider the variational and weak variational forms for the problems under consideration. The variational form is used when we approximate solutions with the finite element method and the weak variational form is necessary for theoretical considerations.

In this section we consider free vibrations of a Timoshenko cantilever beam as an example. The equations of motion are then given by Equations (1.2.5) and (1.2.6). For this model $P = L = 0$.

To find the variational form of this problem, multiply the two equations of motion with functions v and ψ respectively and integrate.

$$\begin{aligned}\int_0^1 \big(\partial_t^2 w(x,t)\big)v(x)dx &= \int_0^1 \big(\partial_x V(x,t)\big)v(x)dx,\\ \int_0^1 \frac{1}{\alpha}\big(\partial_t^2 \phi(x,t)\big)\psi(x)dx &= \int_0^1 \big(\partial_x M(x,t)\big)\psi(x)dx + \int_0^1 V(x,t)\psi(x)dx.\end{aligned}$$

We use the notation

$$(f,g) = \int_0^1 f(x)g(x)dx$$

for convenience. (The fact that this is the inner product for $\mathcal{L}^2(0,1)$, is not relevant at this stage.)

Use integration by parts to find that

$$\begin{aligned}\left(\partial_t^2 w(\cdot,t), v\right) &= -(V(\cdot,t), v') + [V(\cdot,t)v]_0^1, \\ \frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) &= -(M(\cdot,t), \psi') + (V(\cdot,t), \psi) + [M(\cdot,t)\psi]_0^1.\end{aligned}$$

Since $V(1,t) = M(1,t) = 0$,

$$\left(\partial_t^2 w(\cdot,t), v\right) = -(V(\cdot,t), v') - V(0,t)v(0), \tag{3.1.1}$$

$$\frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) = -(M(\cdot,t), \psi') + (V(\cdot,t), \psi) - M(0,t)\psi(0). \tag{3.1.2}$$

The test functions are defined as

$$T(0,1) = \left\{v \in C^1(0,1) \;\middle|\; v(0) = 0\right\}.$$

We substitute the constitutive equations into the equations above to find the variational form of the problem.

Variational form

Find w and ϕ such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, $\phi(\cdot,t) \in T(0,1)$,

$$\left(\partial_t^2 w(\cdot,t), v\right) = -(\partial_x w(\cdot,t) - \phi(\cdot,t), v') \tag{3.1.3}$$

for each $v \in T(0,1)$,

$$\frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) = -\frac{1}{\beta}\left(\partial_x \phi(\cdot,t), \psi'\right) + (\partial_x w(\cdot,t) - \phi(\cdot,t), \psi) \tag{3.1.4}$$

for each $\psi \in T(0,1)$.

Remark

The variational form can be used to compute approximations for the solutions of the vibration problem as well as the eigenvalue problem. The variational form can also be used to investigate the solvability of the problem. This is done by showing that the results for a general linear vibration problem may be applied to this specific problem.

General linear vibration problem

Let H be a Hilbert space and u a function mapping the interval $[0, T]$ into H. The derivatives of u are defined in the usual way (see Appendix 4). Every linear vibration problem can be written in the form below for suitable bilinear forms a, b and c defined on H.

For each $t \in (0, T)$,

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = \big(f(t), v\big),$$

for each $v \in V$, where V is some subspace of H.

The existence theory for the general problem is discussed in Section 3.8 and the theory of eigenvalue problems in Sections 3.9 and 3.10.

To apply the theory to the problem we are considering, the problem must be written in the appropriate form and the necessary estimates derived. The first step is to add Equations (3.1.3) and (3.1.4). We find that

$$\begin{aligned} & \Big(\partial_t^2 w(\cdot, t), v\Big) + \frac{1}{\alpha}\,\Big(\partial_t^2 \phi(\cdot, t), \psi\Big) \\ = \;& -\frac{1}{\beta}\,(\partial_x \phi(\cdot, t), \psi') - (\partial_x w(\cdot, t) - \phi(\cdot, t), v' - \psi). \end{aligned}$$

Next we need a suitable Hilbert space and subspace to relate our problem to the general vibration problem. We use the Sobolev space $H^1(0, 1)$ discussed in Appendix 1 to define suitable product spaces.

Product spaces

Consider the product spaces

$$X = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1) \quad \text{and} \quad H^1 = H^1(0, 1) \times H^1(0, 1).$$

Let $V(0, 1)$ be the closure of $T(0, 1)$ in the Sobolev space $H^1(0, 1)$ and let $V = V(0, 1) \times V(0, 1)$. (Note that V is a subspace of the Hilbert space H^1.)

Bilinear forms

For u and v in $\mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1)$,

$$c(u, v) \;=\; (u_1, v_1) + \frac{1}{\alpha}\,(u_2, v_2).$$

For u and v in $H^1(0,1) \times H^1(0,1)$,

$$b(u,v) = \frac{1}{\beta}(u_2', v_2') + (u_1' - u_2, v_1' - v_2).$$

Note that both b and c are symmetric.

Weak variational form

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) \quad \text{for each } v \in V.$$

Estimates

For the product spaces X and H^1, we may use the obvious inner products and as a consequence we have the respective norms

$$\|u\|_{0,2} = \sqrt{\|u_1\|^2 + \|u_2\|^2}$$
$$\text{and} \quad \|u\|_{1,2} = \sqrt{\|u_1\|_1^2 + \|u_2\|_1^2}.$$

However, other equivalent norms are more convenient.

Theorem 1

Assume that $\alpha \geq 1$. Then

(a) $\|u\|_{0,2}^2 \leq \alpha\, c(u,u) \leq \alpha\, \|u\|_{0,2}^2$ for each $u \in X$.

(b) $\|u\|_{0,2}^2 \leq \|u\|_{1,2}^2 \leq 6\beta\, b(u,u) \leq 12\beta\, \|u\|_{1,2}^2$ for each $u \in V$.

Proof

(a) The proof is trivial.

(b) For $u \in V$, we have that u_1 and u_2 are in $V(0,1)$. Since $V(0,1)$ is the closure of $T(0,1)$ in $H^1(0,1)$, it follows from Theorem 1 Appendix 2 that

$$\|u_1\| \leq \|u_1'\| \quad \text{and} \quad \|u_2\| \leq \|u_2'\|$$

Therefore
$$\|u\|_{0,2}^2 \le \|u_1'\|^2 + \|u_2'\|^2 \le \|u\|_{1,2}^2 \,.$$
This proves the first inequality.

We use $\|u_1'\| \le \|u_1' - u_2\| + \|u_2\|$ and $(a+b)^2 \le 2a^2 + 2b^2$ to find
$$\|u_1'\|^2 \le 2\|u_1' - u_2\|^2 + 2\|u_2\|^2 \,.$$

It follows that
$$\|u_1'\|^2 + \|u_2'\|^2 \le 2\|u_1' - u_2\|^2 + 3\|u_2'\|^2 \le 3\beta\, b(u,u).$$

The second inequality follows from the inequality above and the inequalities
$$\|u_1\| \le \|u_1'\| \quad \text{and} \quad \|u_2\| \le \|u_2'\|.$$

The last inequality is trivial since $\|u_1' - u_2\|^2 \le 2\|u_1'\|^2 + 2\|u_2\|^2$. □

Conclusion

The bilinear form c is an inner product for the space X and b is an inner product for the space V. Theorem 1 shows that for the space X, the norm associated with c is equivalent to $\|\cdot\|_{0,2}$. Similarly, for the space V, the norm associated with b is equivalent to $\|\cdot\|_{1,2}$.

Notation

$$\|u\|_X = \sqrt{c(u,u)} \quad \text{and} \quad \|u\|_V = \sqrt{b(u,u)} \,.$$

We call the space X with inner product c the **inertia space** and the space V with inner product b the **energy space**.

Theorem 2

Assume that $\alpha \ge 1$. The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

From Theorem 2 Appendix 1 it follows that $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ is separable. Furthermore, $\|\cdot\|_X$ and $\|\cdot\|_{0,2}$ are equivalent norms in X and it follows that X is separable.

$T(0,1)$ is dense in $\mathcal{L}^2(0,1)$, since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ (Theorem 3 Appendix 1). Clearly $V = V(0,1) \times V(0,1)$ is dense in $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ and hence V is dense in X. □

Remark

The assumption that $\alpha \geq 1$ is not necessary and the result is true for $\alpha > 0$. However, in applications α is large compared to one.

Theorem 3

The embedding of the space V into X is compact.

Proof

The embedding of $H^1(0,1)$ into $\mathcal{L}^2(0,1)$ is compact (Theorem 7 Appendix 1). Consequently the embedding of H^1 into $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ is compact. The result follows since the relevant norms are equivalent. □

The assumptions in Sections 3.8, 3.9 and 3.10 are valid for the cantilever Timoshenko beam and hence the theory can be applied to this model problem.

3.2 Vertical slender structure: Rayleigh models

3.2.1 Variational forms

To obtain the variational form of Problems VR 3 and VR 4, Equation (2.1.1) is multiplied by a function v and integration by parts (as in Section 3.1) yields

$$(\partial_t^2 w(\cdot,t), v') = -(V(\cdot,t), v') - V(0,t)v(0).$$

Multiply Equation (2.1.2) by v' to find

$$\frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = (V(\cdot,t), v') + (\partial_x M(\cdot,t), v') + (L(\cdot,t), v').$$

Adding the two equations we have

$$(\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = (\partial_x M(\cdot,t), v') + (L(\cdot,t), v') - V(0,t)v(0).$$

Integration by parts on the first term on the right and substitution of Equation (2.1.3) yield

$$\begin{aligned}(\partial_x M(\cdot,t), v') &= -(M(\cdot,t), v'') - M(0,t)v'(0)\\ &= -\frac{1}{\beta}\,(\partial_x^2 w(\cdot,t), v'') - M(0,t)v'(0).\end{aligned}$$

From Equation (2.1.4) we have

$$(L(\cdot,t), v') = \mu\int_0^1 (1-x)\partial_x w(x,t)v'(x)dx.$$

Combining the results above, we obtain a general variational form.

$$(\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = -\frac{1}{\beta}\,(\partial_x^2 w(\cdot,t), v'')$$

$$+\mu\int_0^1 (1-x)\partial_x w(x,t)v'(x)dx - V(0,t)v(0) - M(0,t)v'(0). \qquad (3.2.1)$$

The variational form of each model depends on how we treat the terms containing $V(0,t)$ and $M(0,t)$. In all the models the solution w must satisfy Equation (3.2.1) for all test functions v.

For Problems VR 3 and VR 4 there are no restrictions on the space of test functions $T(0,1)$. Consequently, there are no forced boundary conditions for the solution w and it must satisfy Equation (3.2.1) for an arbitrary function $v \in T(0,1) = C^2[0,1]$.

We define the the following **bilinear forms**.

$$\begin{aligned}c_A(u,v) &= (u,v) + \frac{1}{\alpha}\,(u',v') + m_F\,u(0)v(0)\\ b_A(u,v) &= \frac{1}{\beta}\,(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)dx\\ &\quad + K_F u(0)v(0) + ku'(0)v'(0)\end{aligned}$$

Equations (3.2.1), (2.1.5) and (2.1.7) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v) &= -b_A(w(\cdot,t),v) - C_F\partial_t w(0,t)v(0) + k\theta_F(t)v'(0) \\ &\quad -c\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_F(t)\Big)v'(0) \end{aligned} \tag{3.2.2}$$

Together with Equation (2.1.6) given by

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t), \end{aligned} \tag{2.1.6}$$

we find the variational form of the problem.

Variational form of Problem VR 3

Find w and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, Equation (3.2.2) holds for each $v \in T(0,1)$ and Equation (2.1.6) holds. □

For Problem VR 4 we define the following **bilinear forms**.

$$\begin{aligned} c_A(u,v) &= (u,v) + \frac{1}{\alpha}\,(u',v') + m_B\,u(0)v(0)\,, \\ b_A(u,v) &= \frac{1}{\beta}\,(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)dx \\ &\quad +K_{FB}u(0)v(0) + k_{BA}u'(0)v'(0)\,. \end{aligned}$$

Equations (3.2.1), (2.1.8) and (2.1.10) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v) &= -b_A(w(\cdot,t),v) + K_{FB}w_F(t)v(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v(0) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big)v'(0) \\ &\quad +k_{BA}\theta_B(t)v'(0) \end{aligned} \tag{3.2.3}$$

Together with Equations (2.1.9), (2.1.11) and (2.1.12) which are given again for convenience, we are able to formulate the variational form of Problem

VR 4.

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{2.1.9}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{2.1.11}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{2.1.12}$$

Variational form of Problem VR 4

Find w, θ_B, w_F and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, Equation (3.2.3) holds for each $v \in T(0,1)$ and Equations (2.1.9), (2.1.11) and (2.1.12) hold. □

The variational forms above are used for finite element approximations (see Chapter 6).

3.2.2 Weak variational forms

For the analysis of the vibration problems we consider the weak variational forms. We consider only Problem VR 4, since Problem VR 3 is similar to Problem VR 4 but simpler.

For the weak variational form we redefine c_A and b_A in Subsection 3.2.1.

Bilinear forms

For u and v in $H^1(0,1)$,

$$c_A(u,v) = (u,v) + \frac{1}{\alpha}\,(u',v') + m_B u(0)v(0).$$

For u and v in $H^2(0,1)$,

$$b_A(u,v) = \frac{1}{\beta}(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)\,dx.$$

With the new notation and setting $v = v_1$, Equation (3.2.3) becomes

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v_1) &= -b_A(w(\cdot,t),v_1) - K_{FB}\Big(w(0,t) - w_F(t)\Big)v_1(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v_1(0) \\ &\quad -k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big)v_1'(0) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big)v_1'(0). \end{aligned} \tag{3.2.4}$$

Multiplying Equation (2.1.9) by an arbitrary real number v_2 and adding this to Equation (3.2.4) results in

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v_1) + I_B\ddot{\theta}_B(t)v_2 &= -b_A(w(\cdot,t),v_1) \\ &\quad -K_{FB}\Big(w(0,t) - w_F(t)\Big)v_1(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v_1(0) \\ &\quad -k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big)\Big(v_1'(0) - v_2\Big) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big)\Big(v_1'(0) - v_2\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)v_2 \\ &\quad -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)v_2. \end{aligned} \tag{3.2.5}$$

Define a function y with values in $\mathcal{L}^2(0,1)$ by $y(t) = w(\cdot,t)$.

For the definition of the derivatives $y'(t)$ and $y''(t)$, see Appendix 4. In this subsection, we will use the notation $\dot{y}(t)$ and $\ddot{y}(t)$ instead of $y'(t)$ and $y''(t)$ to distinguish between time and spatial derivatives.

Finally we need the **trace operator** γ which is defined in Appendix 3. Here

$\gamma u = u(0)$ for $u \in \mathcal{L}^2(0,1)$. With the new notation, Equation (3.2.5) becomes

$$\begin{aligned} c_A(\ddot{y}(t), v_1) + I_B\ddot{\theta}_B(t)v_2 &= -b_A(y(t), v_1) \\ &\quad -K_{FB}\Big(\gamma y(t) - w_F(t)\Big)\gamma v_1 \\ &\quad -C_{FB}\Big(\gamma\big(\dot{y}(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\ &\quad -k_{BA}\Big(\gamma\left[\big(y(t)\big)'\right] - \theta_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\ &\quad -c_{BA}\Big(\gamma\left[\big(\dot{y}(t)\big)'\right] - \dot{\theta}_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)v_2 \\ &\quad -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)v_2 \end{aligned} \tag{3.2.6}$$

Remark

Note that $\partial_t w(0,t)$ is replaced by $\gamma\big(\dot{y}(t)\big)$ and not $d_t(\gamma y)(t)$. This is necessary for the weak variational form of the problem. Fortunately, the choice is not a problem. This fact is discussed at the end of the section.

Multiply Equation (2.1.11) by v_3 and Equation (2.1.12) by v_4.

$$\begin{aligned} m_F\ddot{w}_F(t)v_3 &= K_{FB}\Big(w_B(t) - w_F(t)\Big)v_3 + C_{FB}\Big(\dot{w}_B(t) - \dot{w}_F(t)\Big)v_3 \\ &\quad -K_F w_F(t)v_3 - C_F\dot{w}_F(t)v_3 \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t)v_4 &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)v_4 + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)v_4 \\ &\quad -k_F\theta_F(t)v_4 - c_F\dot{\theta}_F(t)v_4 \end{aligned} \tag{3.2.8}$$

Add Equations (3.2.6), (3.2.7) and (3.2.8) to find

$$\begin{aligned}
& c_A(\ddot{y}(t),v_1) + I_B\ddot{\theta}_B(t)v_2 + m_F\ddot{w}_F(t)v_3 + I_F\ddot{\theta}_F(t)v_4 \\
= \; & -b_A(y(t),v_1) - K_{FB}\Big(\gamma y(t) - w_F(t)\Big)\Big(\gamma v_1 - v_3\Big) \\
& -C_{FB}\Big(\gamma\big(\dot{y}(t)\big) - \dot{w}_F(t)\Big)\Big(\gamma v_1 - v_3\Big) \\
& -k_{BA}\Big(\gamma[\big(y(t)\big)'] - \theta_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\
& -c_{BA}\Big(\gamma[\big(\dot{y}(t)\big)'] - \dot{\theta}_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\
& -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)\Big(v_2 - v_4\Big) \\
& -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big(v_2 - v_4\Big) \\
& -K_F w_F(t)v_3 - C_F\dot{w}_F(t)v_3 - k_F\theta_F(t)v_4 - c_F\dot{\theta}_F(t)v_4\,. \qquad (3.2.9)
\end{aligned}$$

To formulate the weak form of the variational problem, the following product spaces and bilinear forms are necessary.

Product spaces

Define the product spaces

$$X = H^1(0,1)\times I\!R^3 \quad \text{and} \quad V = H^2(0,1)\times I\!R^3.$$

Bilinear forms

For u and v in $H^1(0,1)$,

$$\begin{aligned}
c(u,v) &= c_A(u_1,v_1) + I_Bu_2v_2 + m_Fu_3v_3 + I_Fu_4v_4\,, \\
a(u,v) &= C_{FB}\,(\gamma u_1 - u_3)\,(\gamma v_1 - v_3) + c_{BA}(\gamma u_1' - u_2)(\gamma v_1' - v_2) \\
&\quad + c_{FB}(u_2 - u_4)(v_2 - v_4) + C_Fu_3v_3 + c_Fu_4v_4\,.
\end{aligned}$$

For u and v in $H^2(0,1)$,

$$\begin{aligned}
b(u,v) &= b_A(u_1,v_1) + K_{FB}\,(\gamma u_1 - u_3)\,(\gamma v_1 - v_3) \\
&\quad + k_{BA}(\gamma u_1' - u_2)(\gamma v_1' - v_2) + k_{FB}(u_2 - u_4)(v_2 - v_4) \\
&\quad + K_Fu_3v_3 + k_Fu_4v_4\,.
\end{aligned}$$

Note that a, b and c are all symmetric.

We are now ready to formulate the weak variational form of Problem VR 4 in terms of the defined bilinear forms. The table below shows the relationship between the components of u and the variables in Equation (3.2.9).

$u_1(t)$	u_2	u_3	u_4
$w(\cdot,t)$	$\theta_B(t)$	$w_F(t)$	$\theta_F(t)$

Weak variational form of Problem VR 4

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) - a(u'(t), v) \quad \text{for each } v \in V. \qquad \square$$

The existence theorem for problems of this type is presented in Section 3.8. If the initial conditions are chosen properly, $u_1 \in C^2\big((0,T);\ H^1(0,1)\big)$ and we find for example that $(\dot{y}(t))' = d_t\,(y'(t))$ and $d_t\,[\gamma y(t)] = \gamma \dot{y}(t)$.

The inertia space $\boldsymbol{X}$

The bilinear form c is an inner product for the space X and consequently we may define a norm for $u \in X$ by

$$\|u\|_X = \sqrt{c(u,u)}\ .$$

The space X with norm $\|\cdot\|_X$ is called the **inertia space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

The proof is similar to the proof of Theorem 2 in Section 3.1. $\square$

It is obvious that the inner products of $H^2(0,1)$ and $I\!R^3$ can be used to define an inner product for the space V. We will show that the symmetric bilinear form b is also an inner product for V which is convenient for the theory.

Remark

In the following theorems, we assume throughout that the inequalities below hold for the physical constants.

$$1 > 2\mu\beta, \quad k_{BA} > 4\mu, \quad k_{FB} > 8\mu \;\text{ and }\; k_F > 8\mu.$$

These assumptions are physically realistic, as can be seen in Section 6.5.

Theorem 2

There exists a constant K_{bc} such that

$$\|u\|_X^2 \leq K_{bc}\, b(u,u)$$

for each $u \in V$.

Proof

In the proof we use the elementary inequalities

$$\|x\| \leq \|x-y\| + \|y\| \quad \text{and} \quad (a+b)^2 \leq 2(a^2+b^2)$$

and the fact that

$$\|u_1\| \leq \|u_1'\| + |\gamma u_1|.$$

(See Theorem 3 Appendix 3.)

This implies that

$$\begin{aligned} \|u_1\|^2 &\leq 2\|u_1'\|^2 + 2(\gamma u_1)^2 \\ \text{and} \quad \|u_1'\|^2 &\leq 2\|u_1''\|^2 + 2(\gamma u_1')^2. \end{aligned}$$

Therefore

$$c_A(u_1,u_1) \leq 2\left(2+\frac{1}{\alpha}\right)\left(\|u_1''\|^2 + (\gamma u_1')^2\right) + (2+m_B)\,(\gamma u_1)^2.$$

With

$$\begin{aligned}(\gamma u_1)^2 &\leq 2(\gamma u_1 - u_3)^2 + 2u_3^2, \\ (\gamma u_1')^2 &\leq 2(\gamma u_1' - u_2)^2 + 4(u_2 - u_4)^2 + 4u_4^2 \\ u_2^2 &\leq (u_2 - u_4)^2 + u_4^2 \\ \text{and} \quad \|u\|_X^2 &= c_A(u_1, u_1) + I_B u_2^2 + m_F u_3^2 + I_F u_4^2,\end{aligned}$$

it follows that

$$\|u\|_X^2 \leq K_c\Big(\|u_1''\|^2 + (\gamma u_1' - u_2)^2 + (\gamma u_1 - u_3)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\Big)$$

where

$$K_c = \max\left\{2\,(2 + m_B) + m_F\,,\ 8\left(2 + \frac{1}{\alpha}\right) + 2I_B + I_F\right\}.$$

From the fact that

$$\int_0^1 (1-x)\big(u_1'(x)\big)^2 dx \leq \|u_1'\|^2,$$

and using the inequality for $(\gamma u_1')^2$ above, it follows that

$$\begin{aligned}b_A(u_1, u_1) &= \frac{1}{\beta}\|u_1''\|^2 - \mu\int_0^1 (1-x)\big(u_1'(x)\big)^2 dx \\ &\geq \frac{1}{\beta}\|u_1''\|^2 - \mu\|u_1'\|^2 \\ &\geq \left(\frac{1}{\beta} - 2\mu\right)\|u_1''\|^2 - \mu\left(4\,(\gamma u_1' - u_2)^2 + 8\,(u_2 - u_4)^2 + 8u_4^2\right)\end{aligned}$$

Therefore

$$\begin{aligned}b(u,u) &\geq \left(\frac{1}{\beta} - 2\mu\right)\|u_1''\|^2 + K_{FB}\big(\gamma u_1 - u_3\big)^2 + (k_{BA} - 4\mu)\,(\gamma u_1' - u_2)^2 \\ &\quad + (k_{FB} - 8\mu)\,(u_2 - u_4)^2 + K_F u_3^2 + (k_F - 8\mu)\,u_4^2 \\ &\geq K_b\left(\|u_1''\|^2 + (\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right)\end{aligned}$$

where

$$K_b = \min\left\{\frac{1}{\beta} - 2\mu,\ K_{FB},\ K_F,\ k_{BA} - 4\mu,\ k_{FB} - 8\mu,\ k_F - 8\mu\right\}.$$

With $K_{bc} = \dfrac{K_c}{K_b}$, it follows that

$$K_{bc}\, b(u,u) \geq \|u\|_X^2.$$

□

Remark

If $b(u,u) = 0$, it follows from Theorem 2 that $u = 0$.

The energy space *V*

The bilinear form b is an inner product for the space V and for $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\ .$$

The space V with the norm $\|\cdot\|_V$ is called the **energy space**.

Theorem 3

There exists a constant K_{ba} such that for any $u \in V$ and $v \in V$,

$$|a(u,v)| \leq K_{ba}\|u\|_V\|v\|_V.$$

Proof

We can prove that

$$|a(u,v)| \leq \sqrt{a(u,u)}\ \sqrt{a(v,v)}$$

in a similar way as the proof for the Cauchy-Schwartz inequality.

From the proof in Theorem 2 it follows that a constant $K_b > 0$ exists such that

$$\begin{aligned} b(u,u) &\geq K_b\left(\|u_1''\|^2 + (\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right) \\ &\geq K_b\left((\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right). \end{aligned}$$

Furthermore

$$\begin{aligned} |a(u,u)| &= C_{FB}\,(\gamma u_1 - u_3)^2 + c_{BA}\,(\gamma u_1' - u_2)^2 + c_{FB}\,(u_2 - u_4)^2 \\ &\quad + C_F u_3^2 + c_F u_4^2 \\ &\leq K_a\left((\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right) \end{aligned}$$

where

$$K_a = \max\left\{ C_{FB},\ C_F,\ \ c_{BA},\ c_{FB},\ c_F \right\}.$$

Hence, with $K_{ba} = \dfrac{K_a}{K_b}$,

$$|a(u,u)| \ \leq\ K_{ba}\ b(u,u)\ =\ K_{ba}\|u\|_V^2.$$

Consequently

$$|a(u,v)| \leq \sqrt{a(u,u)}\sqrt{a(v,v)} \leq K_{ba}\|u\|_V\|v\|_V.$$

3.3 Vertical slender structure: Timoshenko models

3.3.1 Variational forms

The variational forms are found by multiplying Equation (2.1.13) with a test function v and Equation (2.1.14) with a test function ψ and applying integration by parts. This results in

$$\begin{aligned}
\big(\partial_t^2 w(\cdot,t),v\big) &= -\big(V(\cdot,t),v'\big) - V(0,t)v(0),\\
\frac{1}{\alpha}\,\big(\partial_t^2\phi(\cdot,t),\psi\big) &= -\big(M(\cdot,t),\psi'\big) + \big(V(\cdot,t),\psi\big) + \big(L(\cdot,t),\psi\big) - M(0,t)\psi(0)
\end{aligned}$$

Substituting the constitutive equations (2.1.15), (2.1.16) and (2.1.17) into the equations above, we find that

$$\big(\partial_t^2 w(\cdot,t),v\big) = -\big(\partial_x w(\cdot,t)-\phi(\cdot,t),v'\big) - V(0,t)v(0), \tag{3.3.1}$$

$$\begin{aligned}
\frac{1}{\alpha}\,\big(\partial_t^2\phi(\cdot,t),\psi\big) &= -\frac{1}{\beta}\,\big(\partial_x\phi(\cdot,t),\psi'\big) + \big(\partial_x w(\cdot,t)-\phi(\cdot,t),\psi\big)\\
&\quad +\mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)dx - M(0,t)\psi(0). \qquad (3.3.2)
\end{aligned}$$

First consider **Problem VT 3**. Equations (3.3.1) and (2.1.18) result in

$$\begin{aligned}
\big(\partial_t^2 w(\cdot,t),v\big) + m_F\partial_t^2 w(0,t)v(0) &= -\big(\partial_x w(\cdot,t)-\phi(\cdot,t),v'\big) - K_F w(0,t)v(0)\\
&\quad -C_F\partial_t w(0,t)v(0). \qquad (3.3.3)
\end{aligned}$$

Equations (3.3.2) and (2.1.20) result in

$$\begin{aligned}\frac{1}{\alpha}\left(\partial_t^2\phi(\cdot,t),\psi\right) &= -\frac{1}{\beta}\left(\partial_x\phi(\cdot,t),\psi'\right)+\left(\partial_x w(\cdot,t)-\phi(\cdot,t),\psi\right)\\ &\quad +\mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)\,dx\\ &\quad -k\big(\phi(0,t)-\theta_F(t)\big)\psi(0)\\ &\quad -c\big(\partial_t\phi(0,t)-\dot\theta_F(t)\big)\psi(0) \qquad (3.3.4)\end{aligned}$$

Equation (2.1.19) is presented again for convenience.

$$\begin{aligned}I_F\ddot\theta_F(t) &= k\Big(\phi(0,t)-\theta_F(t)\Big)+c\Big(\partial_t\phi(0,t)-\dot\theta_F(t)\Big)\\ &\quad -k_F\theta_F(t)-c_F\dot\theta_F(t). \qquad (2.1.19)\end{aligned}$$

As for the Rayleigh models, there are no forced boundary condtions on the test functions. Therefore, for both variational forms of Problems VT 3 and VT 4, both v and ψ are in $T(0,1)=C^1[0,1]$.

Variational form of Problem VT 3

Find w, ϕ and θ_F such that for each $t>0$, $w(\cdot,t)\in T(0,1)$, $\phi(\cdot,t)\in T(0,1)$ and $\theta_F(t)\in I\!R$ and Equations (3.3.3) and (3.3.4) hold for each $v\in T(0,1)$ and $\psi\in T(0,1)$ respectively and Equation (2.1.19) holds. □

Now consider **Problem VT 4**. Equations (3.3.1) and (2.1.21) result in

$$\begin{aligned}\left(\partial_t^2 w(\cdot,t),v\right)+m_B\,\partial_t^2 w(0,t)v(0) &= -\left(\partial_x w(\cdot,t)-\phi(\cdot,t),v'\right)\\ &\quad -K_{FB}\Big(w(0,t)-w_F(t)\Big)v(0)\\ &\quad -C_{FB}\Big(\partial_t w(0,t)-\dot w_F(t)\Big)v(0). \qquad (3.3.5)\end{aligned}$$

Equations (3.3.2) and (2.1.23) result in

$$\begin{aligned}\frac{1}{\alpha}\left(\partial_t^2\phi(\cdot,t),\psi\right) &= -\frac{1}{\beta}\left(\partial_x\phi(\cdot,t),\psi'\right)+\left(\partial_x w(\cdot,t)-\phi(\cdot,t),\psi\right)\\ &\quad +\mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)\,dx\\ &\quad -k_{BA}\Big(\phi(0,t)-\theta_B(t)\Big)\psi(0)\\ &\quad -c_{BA}\Big(\partial_t\phi(0,t)-\dot\theta_B(t)\Big)\psi(0). \qquad (3.3.6)\end{aligned}$$

There are three additional equations in the system, presented again for convenience.

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \qquad (2.1.22) \\ m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \qquad (2.1.24) \\ I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \qquad (2.1.25) \end{aligned}$$

Variational form of Problem VT 4

Find w, ϕ, θ_B, w_F and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, $\phi(\cdot,t) \in T(0,1)$ and Equations (3.3.5) and (3.3.6) hold for each $v \in T(0,1)$ and $\psi \in T(0,1)$ respectively and Equations (2.1.22), (2.1.24) and (2.1.25) hold. □

The variational forms of the problems above are used for computational purposes (see Chapter 6), but for theoretical purposes we consider the weak variational form.

3.3.2 Weak variational forms

Problems VT 3 and VT 4 are similar and we consider only Problem VT 4. We omit the "gravity" term for a reason to be given later.

First we add Equations (3.3.5) and (3.3.6) to find

$$\begin{aligned} &\big(\partial_t^2 w(\cdot,t), v\big) + \frac{1}{\alpha}\,\big(\partial_t^2\phi(\cdot,t),\psi\big) + m_B\,\partial_t^2 w(0,t)v(0) \\ = \;& -\frac{1}{\beta}\,\big(\partial_x\phi(\cdot,t),\psi'\big) - \big(\partial_x w(\cdot,t) - \phi(\cdot,t), v' - \psi\big) \\ & -K_{FB}\Big(w(0,t) - w_F(t)\Big)v(0) - C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v(0) \\ & -k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big)\psi(0) - c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big)\psi(0). \qquad (3.3.7) \end{aligned}$$

Next we introduce "time derivatives" and the trace operator γ as in Section 3.2.2.

Define a function y with values $y(t) = \langle w(\cdot, t), \phi(\cdot, t)\rangle$ in $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$. Furthermore, set $v = v_1$ and $\psi = v_2$. Equation (3.3.7) then becomes

$$\begin{aligned}
&(\ddot{y}_1(t), v_1) + \frac{1}{\alpha}\,(\ddot{y}_2(t), v_2) + m_B\gamma\big(\ddot{y}_1(t)\big)\gamma v_1 \\
&= -\frac{1}{\beta}\,\big((y_2(t))', v_2'\big) - \big((y_1(t))' - y_2(t), v_1' - v_2\big) \\
&-K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\gamma v_1 - C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\
&-k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\gamma v_2 - c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\gamma v_2. \qquad (3.3.8)
\end{aligned}$$

Bilinear forms

For $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$ in $H^1(0,1) \times H^1(0,1)$,

$$\begin{aligned}
c_A(u,v) &= (u_1, v_1) + \frac{1}{\alpha}\,(u_2, v_2) + m_B\gamma u_1 \gamma v_1, \\
b_A(u,v) &= \frac{1}{\beta}\,(u_2', v_2') + (u_1' - u_2, v_1' - v_2).
\end{aligned}$$

Write Equation (3.3.8) in terms of the bilinear forms:

$$\begin{aligned}
c_A(\ddot{y}(t), v) &= -b_A(y(t), v) - K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\gamma v_1 \\
&\quad -C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\
&\quad -k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\gamma v_2 \\
&\quad -c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\gamma v_2 \qquad (3.3.9)
\end{aligned}$$

for any $v \in H^1(0,1) \times H^1(0,1)$.

Multiply Equations (2.1.22), (2.1.24) and (2.1.25) with v_3, v_4 and v_5 respectively to find

$$\begin{aligned}
I_B\ddot{\theta}_B(t)v_3 &= \Big[k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big) + c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big) \\
&\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big]\, v_3 \qquad (3.3.10) \\
m_F\ddot{w}_F(t)v_4 &= \Big[K_{FB}\Big(w_B(t) - w_F(t)\Big) + C_{FB}\Big(\dot{w}_B(t) - \dot{w}_F(t)\Big) \\
&\quad -K_F w_F(t) - C_F\dot{w}_F(t)\big]\, v_4 \qquad (3.3.11) \\
I_F\ddot{\theta}_F(t)v_5 &= \Big[k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\
&\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t)\Big]\, v_5 \qquad (3.3.12)
\end{aligned}$$

Adding Equations (3.3.9), (3.3.10), (3.3.11) and (3.3.12) results in

$$\begin{aligned}
& c_A(\ddot{y}(t), v) + I_B\ddot{\theta}_B(t)v_3 + m_F\ddot{w}_F(t)v_4 + I_F\ddot{\theta}_F(t)v_5 \\
= \; & -b_A(y(t), v) - K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\Big(\gamma v_1 - v_4\Big) \\
& -C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\Big(\gamma v_1 - v_4\Big) \\
& -k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\Big(\gamma v_2 - v_3\Big) \\
& -c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\Big(\gamma v_2 - v_3\Big) \\
& -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)\Big(v_3 - v_5\Big) \\
& -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big(v_3 - v_5\Big) \\
& -K_F w_F(t)v_4 - C_F\dot{w}_F(t)v_4 \\
& -k_F\theta_F(t)v_5 - c_F\dot{\theta}_F(t)v_5. \qquad (3.3.13)
\end{aligned}$$

To formulate the weak form of the variational problem, the following product spaces and bilinear forms are necessary.

Product spaces

$$X = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1) \times I\!R^3 \quad \text{and} \quad V = H^1(0,1) \times H^1(0,1) \times I\!R^3\,.$$

Bilinear forms

For u and v in X,

$$\begin{aligned}
c_A(u,v) &= (u_1,v_1)+\frac{1}{\alpha}\,(u_2,v_2)+m_B\gamma u_1\gamma v_1\,,\\
c(u,v) &= c_A(u,v)+I_Bu_3v_3+m_Fu_4v_4+I_Fu_5v_5\,,\\
a(u,v) &= C_{FB}(\gamma u_1-u_4)(\gamma v_1-v_4)+c_{BA}(\gamma u_2-u_3)(\gamma v_2-v_3)\\
&\quad +c_{FB}(u_3-u_5)(v_3-v_5)+C_Fu_4v_4+c_Fu_5v_5\,.
\end{aligned}$$

For u and v in V,

$$\begin{aligned}
b_A(u,v) &= \frac{1}{\beta}\,(u_2',v_2')+(u_1'-u_2,v_1'-v_2)\,,\\
b(u,v) &= b_A(u,v)+K_{FB}(\gamma u_1-u_4)(\gamma v_1-v_4)+k_{BA}(\gamma u_2-u_3)(\gamma v_2-v_3)\\
&\quad +k_{FB}(u_3-u_5)(v_3-v_5)+K_Fu_4v_4+k_Fu_5v_5\,.
\end{aligned}$$

The relationship between the variables u_1 to u_5 and the variables in Problem VT 4 is shown in the next table.

$u_1(t)$	u_2	u_3	u_4	u_5
$w(\cdot,t)$	$\phi(\cdot,t)$	$\theta_B(t)$	$w_F(t)$	$\theta_F(t)$

Weak variational form of Problem VT 4

Find u such that for each $t>0$, $u(t)\in V$ and

$$c(u''(t),v)=-b(u(t),v)-a(u'(t),v)\quad\text{for each } v\in V.$$

Remark

Inclusion of the "gravity"-term in the definition of b_A will result in an unsymmetrical form. Consequently the bilinear form b will be unsymmetrical and symmetry is crucial in the theory – see Section 3.8.

The inertia space X

Note that c is an inner product for this space and consequently we may define a norm for $u \in X$ by

$$\|u\|_X = \sqrt{c(u,u)}\ .$$

The space X with norm $\|\cdot\|_X$ is the **inertia space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

The proof is similar to the proof of Theorem 2 in Section 3.1. □

We will show that the bilinear form b is an inner product for the space V.

Theorem 2

There exists a constant K_{bc} such that

$$\|u\|_X^2 \leq K_{bc}\ b(u,u)$$

for each $u \in V$.

Proof

As before we use the elementary inequalities

$$\|x\| \leq \|x-y\| + \|y\| \quad \text{and} \quad (a+b)^2 \leq 2(a^2+b^2).$$

It follows from Theorem 3 Appendix 3 that

$$\|u_i\| \leq \|u_i'\| + |\gamma u_i| \quad \text{for } i = 1,\ 2.$$

Together with $\|u_1'\|^2 \leq 2\|u_1' - u_2\|^2 + 2\|u_2\|^2$, we find that

$$\|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 \leq 4\|u_1' - u_2\|^2 + \left(8 + \frac{2}{\alpha}\right)\left(\|u_2'\|^2 + (\gamma u_2)^2\right) + 2\left(\gamma u_1\right)^2.$$

Furthermore,

$$(\gamma u_i)^2 \leq 2(\gamma u_i - u_j)^2 + 2u_j^2, \quad u_3^2 \leq 2(u_3 - u_5)^2 + 2u_5^2$$

and

$$c(u,u) = \|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 + m_B\left(\gamma u_1\right)^2 + I_B u_3^2 + m_F u_4^2 + I_F u_5^2.$$

It follows that

$$\begin{aligned} c(u,u) \leq\ & K_c\Big\{\|u_1' - u_2\|^2 + \|u_2'\|^2 + (\gamma u_1 - u_4)^2 + (\gamma u_2 - u_3)^2 \\ & + (u_3 - u_5)^2 + u_4^2 + u_5^2\Big\} \end{aligned}$$

with

$$K_c = \max\left\{4 + 2m_B + m_F,\ I_F + 2I_B + 32 + \frac{8}{\alpha}\right\}.$$

$$\begin{aligned} b(u,u) =\ & \frac{1}{\beta}\|u_2'\|^2 + \|u_1' - u_2\|^2 + K_{FB}\left(\gamma u_1 - u_4\right)^2 + k_{BA}\left(\gamma u_2 - u_3\right)^2 \\ & + k_{FB}\left(u_3 - u_5\right)^2 + K_F u_4^2 + k_F u_5^2 \\ \geq\ & K_b\Big\{\|u_2'\|^2 + \|u_1' - u_2\|^2 + (\gamma u_1 - u_4)^2 + (\gamma u_2 - u_3)^2 \\ & + (u_3 - u_5)^2 + u_4^2 + u_5^2\Big\} \end{aligned}$$

with

$$K_b = \min\left\{\frac{1}{\beta},\ K_{FB},\ K_F,\ k_{FB},\ k_F,\ k_{BA}\right\}.$$

Let $K_{bc} = \dfrac{K_c}{K_b}$, then $c(u,u) \leq K_{bc}\, b(u,u)$ and the result follows. □

The energy space V

For $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\ .$$

The space V with the norm $\|\cdot\|_V$ is called the **energy space**.

Theorem 3

For any $u \in V$ and $v \in V$,

$$|a(u,v)| \leq \|u\|_V \|v\|_V \, .$$

Proof

This proof is similar to the proof of Theorem 3 Section 3.2. □

3.4 The cantilever beam

The variational form of the cantilever Timoshenko beam is derived in Section 3.1. Recall that our main concern is the choice of boundary conditions at the clamped end. Returning to Equations (3.1.1) and (3.1.2), we are now ready to comment on this choice.

Choosing test functions v and ψ such that $v(0) = \psi(0) = 0$, the terms $V(0,t)v(0)$ and $M(0,t)\psi(0)$ vanish. Therefore the boundary conditions $w(0,t) = \phi(0,t) = 0$ is a convenient choice from a variational point of view. Note that $w(0,t) = 0$ is realistic as far as modelling is concerned, but **not** $\phi(0,t) = 0$ (see Section 1.5.2).

The following alternative boundary conditions may be considered (see Section 2.2).

$$\begin{bmatrix} V(0,t) \\ M(0,t) \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} w(0) \\ \phi(0) \end{bmatrix}$$

Note that $D = C^{-1}$.

We now have no restriction on the test functions and the energy space $V = H^1(0,1) \times H(0,1)$. The bilinear form b must be redefined and

$$b(u,v) = \frac{1}{\beta}(u_2', v_2') + (u_1' - u_2, v_1' - v_2) + \begin{bmatrix} \gamma u_1 & \gamma u_2 \end{bmatrix} D \begin{bmatrix} \gamma v_1 & \gamma v_2 \end{bmatrix}^T .$$

The matrix D must be nonnegative for the bilinear form b to be an inner product. For a discussion of the results, see Chapter 7.

3.5 Two-dimensional model for the cantilever beam

3.5.1 Variational forms

Consider the equation of motion (1.4.1). Multiply both sides by an arbitrary vector valued function $\boldsymbol{\phi}$ and integrate over the reference configuration Ω.

$$\iint_\Omega (\partial_t^2 \boldsymbol{u}) \cdot \boldsymbol{\phi}\, dA = \iint_\Omega (\operatorname{div} T) \cdot \boldsymbol{\phi}\, dA.$$

If T is symmetric, $\operatorname{div}(T\boldsymbol{\phi}) = (\operatorname{div} T)\cdot\boldsymbol{\phi} + \operatorname{tr}(T\Phi)$, where

$$\Phi = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}.$$

Application of the divergence theorem and the symmetry of T yield

$$\iint_\Omega \operatorname{div}(T\boldsymbol{\phi})\, dA = \int_{\partial\Omega} T\boldsymbol{\phi}\cdot\boldsymbol{n}\, ds = \int_{\partial\Omega} T\boldsymbol{n}\cdot\boldsymbol{\phi}\, ds.$$

Combining the results above, we have the **Green formula**

$$\iint_\Omega (\operatorname{div} T)\cdot\boldsymbol{\phi}\, dA = -\iint_\Omega \operatorname{tr}(T\Phi)\, dA + \int_{\partial\Omega} T\boldsymbol{n}\cdot\boldsymbol{\phi}\, ds.$$

Consequently,

$$\iint_\Omega (\partial_t^2\boldsymbol{u})\cdot\boldsymbol{\phi}\, dA = -\iint_\Omega \operatorname{tr}(T\Phi)\, dA + \int_{\partial\Omega} T\boldsymbol{n}\cdot\boldsymbol{\phi}\, ds \tag{3.5.1}$$

for any vector field $\boldsymbol{\phi}$ that is sufficiently smooth.

The bilinear form $b(\boldsymbol{u},\boldsymbol{\phi})$ is defined by

$$b(\boldsymbol{u},\boldsymbol{\phi}) = \iint_\Omega \operatorname{tr}\,(T\Phi)\, dA.$$

If Hooke's law, Equation (1.4.2), is substituted into the definition of the bilinear form, we obtain

$$\begin{aligned} b(\boldsymbol{u},\boldsymbol{\phi}) &= \iint_\Omega \big(\sigma_{11}\partial_1\phi_1 + \sigma_{12}\partial_1\phi_2 + \sigma_{21}\partial_2\phi_1 + \sigma_{22}\partial_2\phi_2\big)\, dA \\ &= \frac{1}{\gamma(1-\nu^2)} \iint_\Omega \big(\partial_1 u_1\partial_1\phi_1 + \partial_2 u_2\partial_2\phi_2 + \nu(\partial_1 u_1\partial_2\phi_2 + \partial_2 u_2\partial_1\phi_1)\big)\, dA \\ &\quad + \frac{1}{2\gamma(1+\nu)} \iint_\Omega \big(\partial_1 u_2 + \partial_2 u_1\big)\big(\partial_1\phi_2 + \partial_2\phi_1\big)\, dA. \end{aligned}$$

To define the space of test functions $T(\Omega)$ for Problems CTD 1 and CTD 2, take note that the boundary of Ω consists of the the two parts Σ and Γ. The test functions must satisfy the forced boundary conditions on Σ, i.e. ϕ_1 and ϕ_2 must be zero when it is required that u_1 and u_2 are zero. All that matters at this stage is that $T\boldsymbol{n} \cdot \boldsymbol{\phi} = \boldsymbol{0}$ on Σ.

Consequently

$$\int_{\partial\Omega} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds = \int_{\Gamma} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds$$

for each $\boldsymbol{\phi} \in T$.

For the equilibrium problem the traction is prescribed on Γ.

Variational form of Problem CTD 1

Given the traction $\boldsymbol{t}$ on Γ, find $\boldsymbol{u} \in T(\Omega)$ such that

$$b(\boldsymbol{u}, \boldsymbol{\phi}) = \int_{\Gamma} \boldsymbol{t} \cdot \boldsymbol{\phi}\, ds$$

for each $\boldsymbol{\phi} \in T(\Omega)$. □

In the second problem we consider free vibration and there is no traction on Γ.

Variational form of Problem CTD 2

Find $\boldsymbol{u}$ such that for $t > 0$, $\boldsymbol{u}(\cdot, t) \in T(\Omega)$ and

$$\iint_{\Omega} \partial_t^2 \boldsymbol{u} \cdot \boldsymbol{\phi}\, dA = -b(\boldsymbol{u}, \boldsymbol{\phi})$$

for each $\boldsymbol{\phi} \in T(\Omega)$. □

Remark

Our main concern is to study the natural frequencies and modes. The relevant eigenvalue problems are considered in Section 3.9 and Chapter 7.

3.5.2 Weak variational forms

For the theory it is necessary to place some restrictions on the sets Ω and Γ. These assumptions are listed below and more detail is given in Appendix 1.

1. The set Ω is open, bounded and convex.
2. The boundary of Ω consists of a finite number of smooth curves.
3. The set Γ is a smooth part of the boundary of Ω.

Remark

In our application, Ω is a rectangle and Γ one of the sides.

The function spaces $\mathcal{L}^2(\Omega)^2$, $\mathcal{L}^2(\Gamma)^2$, $H^k(\Omega)^2$ and $H^k(\Gamma)^2$ are relevant for the theory. The detail and notation are discussed in Appendix 1.

The trace operator γ is now a mapping of a function "onto its value" on Γ. The definition is given in Appendix 3.

We follow the same line of reasoning for the weak formulation as before. Let V be the closure of $T(\Omega)$ in $H^1(\Omega)^2$. We may consider the following weak variational form of Problem CTD 1. Given $t \in \mathcal{L}^2(\Gamma)^2$, find $u \in V$ such that

$$b(u,v) = (t, \gamma v)_{0,2}^{\Gamma} \quad \text{for each } v \in V.$$

However, to apply the theory we consider another form. We define a functional f corresponding to the traction on Γ. Given $t \in \mathcal{L}^2(\Gamma)^2$, let

$$f(v) = (t, \gamma v)_{0,2}^{\Gamma} \quad \text{for each } v \in V,$$

This leads to the following form.

Weak variational form of Problem CTD 1

Given f in the dual of V, find $u \in V$ such that

$$b(u,v) = f(v) \quad \text{for each } v \in V.$$

Weak variational form of Problem CTD 2

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) \quad \text{for each } v \in V,$$

where $c(\cdot\,,\,\cdot) = (\cdot\,,\,\cdot)$ is the inner product of $\mathcal{L}^2(\Omega)^2$.

Theorem 1

There exists a constant K such that

$$|f(v)| \leq K\, \|\gamma v\|^{\Omega}_{1,2} \quad \text{for each } v \in V.$$

Proof

It follows from Theorem 4 Appendix 3 that

$$\|\gamma v\|^{\Gamma}_{0,2} \leq K_{\Gamma}\, \|v\|^{\Omega}_{1,2} \quad \text{for each } v \in H^1(\Omega)^2.$$

Consequently,

$$|f(v)| \leq \|t\|^{\Gamma}_{0,2}\, \|\gamma v\|^{\Gamma}_{0,2} \leq K_{\Gamma}\, \|t\|^{\Gamma}_{0,2}\, \|v\|^{\Omega}_{1,2}.$$

Theorem 2 (Poincare-Friedrichs)

There exists a constant c_F such that,

$$\|u\|_{0,2} \leq c_F |u|_{1,2} \quad \text{for each } u \in V.$$

Proof

The inequality holds for each $u \in T(\Omega)$ (see the corollary to Theorem 2 Appendix 2). Clearly the same is true for $u \in V$.

Theorem 3 (Korn)

There exists a constant c_K such that,

$$|u|^2_{1,2} \;\leq\; c_K\, b(u,u) \quad \text{for each } u \in H^1(\Omega)^2.$$

Proof

[Br, p 288-289]

Theorem 4

There exists a constant c_1 such that,

$$\|u\|_{1,2} \leq c_1\sqrt{b(u,u)} \quad \text{for each } u \in V.$$

Proof

Combine Korn's inequality with the Poincare-Friedrichs inequality.

The energy space *V*

For $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\,.$$

The space V with norm $\|\cdot\|_V$ is called the **energy space**. Due to Theorem 4 the norms $\|\cdot\|_V$ and $\|\cdot\|_{1,2}$ are equivalent on V.

Theorem 5

The space $\mathcal{L}^2(\Omega)^2$ is a separable Hilbert space and V is a dense subset of $\mathcal{L}^2(\Omega)^2$.

Proof

The space $\mathcal{L}^2(\Omega)^2$ is a separable Hilbert space and $C_0^\infty(\Omega)^2$ is a dense subset of $\mathcal{L}^2(\Omega)^2$ (from Theorems 2 and 3 Appendix 1). Since $C_0^\infty(\Omega)^2 \subset V$, the result follows.

Theorem 6

The embedding of the space V into $\mathcal{L}^2(\Omega)^2$ is compact.

Proof

The embedding of the space $H^1(\Omega)^2$ into $\mathcal{L}^2(\Omega)^2$ is compact (from Theorem 7 Appendix 1). The result follows from the equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_{1,2}$.

Theorem 7

For any $t \in \mathcal{L}^2(\Gamma)^2$, there exists a unique $u \in V$ such that

$$b(u,v) = (t,v)^{\Gamma}_{0,2} \quad \text{for each } v \in V.$$

Proof

See Section 3.7.

3.6 Plate-beam system

3.6.1 Variational form of problem RMT

For any function v,

$$\iint_{\Omega} (\operatorname{div} \boldsymbol{Q}) v \, dA \quad = \quad - \iint_{\Omega} \boldsymbol{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds. \qquad (3.6.1)$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T$, using the Green formula from Section 3.5, we have

$$\iint_{\Omega} \operatorname{div} M \cdot \boldsymbol{\phi} \, dA = - \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA + \int_{\partial\Omega} M\boldsymbol{n} \cdot \boldsymbol{\phi} \, ds, \qquad (3.6.2)$$

where $\Phi = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$ and "tr" denotes the trace of the matrix.

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$, with

$$T_1(\Omega) = \left\{ v \in C^1(\bar{\Omega}) \,\middle|\, v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\},$$

$$T_2(\Omega) = \left\{ \boldsymbol{\phi} = [\phi_1 \ \phi_2]^T \,\middle|\, \phi_1, \phi_2 \in C^1(\bar{\Omega}), \ \phi_2 = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\}.$$

Combining Equation (1.3.5) (first equation of motion for the plate) with Equation (3.6.1) yield

$$h_p \iint_\Omega \partial_t^2 w v \, dA + \iint_\Omega \boldsymbol{Q} \cdot \nabla v \, dA - \int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds = 0 \tag{3.6.3}$$

for each $v \in T_1(\Omega)$.

It follows from Equation (2.4.8) (first equation of motion for the beams), using integration by parts, that

$$\eta_1 \int_0^1 \partial_t^2 w_{b0} v_0 \, dx + \int_0^1 V_0 \partial_x v_0 \, dx = \int_0^1 P_0 v_0 \, dx \tag{3.6.4}$$

for each v_0 in $C^1[0,1]$ with $v_0(0) = v_0(1) = 0$ and

$$\eta_1 \int_0^1 \partial_t^2 w_{b1} v_1 \, dx + \int_0^1 V_1 \partial_x v_1 \, dx = \int_0^1 P_1 v_1 \, dx \tag{3.6.5}$$

for each v_1 in $C^1[0,1]$ with $v_1(0) = v_1(1) = 0$. The subscripts "0" and "1" are used to distinguish between quantities associated with the two different beams.

To accommodate Equation (2.4.2) (interface condition for w_{b0} and w_{b1}), choose $v_0(x_1) = v(x_1, 0)$ and $v_1(x_1) = v(x_1, a)$, where a denotes the dimensionless width of the plate.

The fact that $v = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and that $\boldsymbol{Q} \cdot \boldsymbol{n} = -P$ on both Γ_0 and Γ_1 (interface condition (2.4.4)), result in some cancellations when adding Equations (3.6.3), (3.6.4) and (3.6.5). We have

$$\begin{aligned}
\int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds &= \int_{\Gamma_0} (\boldsymbol{Q}_0 \cdot \boldsymbol{n}) v \, ds - \int_{\Gamma_1} (\boldsymbol{Q}_1 \cdot \boldsymbol{n}) v \, ds \\
&= -\int_{\Gamma_0} P_0 v \, ds - \int_{\Gamma_1} P_1 v \, ds \\
&= -\left[\int_0^1 P_0 v \, dx_1 \right]_{x_2=0} - \left[\int_0^1 P_1 v \, dx_1 \right]_{x_2=a}
\end{aligned}$$

for each $v \in T_1(\Omega)$.

From Equation (2.4.2) (interface condition), the remaining integrals on Γ_0 and Γ_1 can be expressed in terms of w. Therefore,

$$\begin{aligned} & h_p \iint_\Omega \partial_t^2 w v \, dA + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\ & + \iint_\Omega \boldsymbol{Q} \cdot \nabla v \, dA + \left[\int_0^1 V_0 \partial_1 v \, dx_1 \right]_{x_2=0} + \left[\int_0^1 V_1 \partial_1 v \, dx_1 \right]_{x_2=a} \\ & = 0 \end{aligned}$$

for each $v \in T_1(\Omega)$.

Equations (1.3.7) (constitutive equation for $\boldsymbol{Q}$) and (2.4.10) (constitutive equations for V_0 and V_1) are expressed in terms of w and ψ_1 (found from the interface conditions (2.4.2) and (2.4.3)). They are used to obtain the final form of this variational equation. This leads to

$$\begin{aligned} & h_p \iint_\Omega \partial_t^2 w v \, dA + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\ & + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \nabla v \, dA \\ & + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \partial_1 v \, dx_1 \right]_{x_2=0} + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \partial_1 v \, dx_1 \right]_{x_2=a} \\ & = 0 \end{aligned} \tag{3.6.6}$$

for each $v \in T_1(\Omega)$.

A similar calculation is performed for the remaining equations of motion. Combining Equation (1.3.6) (second equation of motion for the plate) with the Green formula (3.6.2) yields

$$\begin{aligned} I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_\Omega \operatorname{tr}(M\Phi) \, dA - \int_{\partial\Omega} M\boldsymbol{n} \cdot \boldsymbol{\phi} \, ds & \\ + \iint_\Omega \boldsymbol{Q} \cdot \boldsymbol{\phi} \, dA = 0 & \end{aligned} \tag{3.6.7}$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

It follows from Equation (2.4.9) (second equation of motion for the beams) and using integration by parts, that

$$\frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b0} \chi_0 \, dx + \int_0^1 M_{b0} \partial_x \chi_0 \, dx - \int_0^1 (V_0 + L_0) \chi_0 \, dx = 0 \tag{3.6.8}$$

for each $\chi_0 \in C^1[0,1]$ and

$$\frac{\eta_1}{\alpha_b}\int_0^1 \partial_t^2\phi_{b1}\chi_1\,dx + \int_0^1 M_{b1}\partial_x\chi_1\,dx - \int_0^1 (V_1 + L_1)\chi_1\,dx = 0 \qquad (3.6.9)$$

for each $\chi_1 \in C^1[0,1]$. (M_{b0} and M_{b1} are zero at the endpoints of the beams.)

The functions χ_0 and χ_1 in Equations (3.6.8) and (3.6.9) must satisfy the conditions $\chi_0(x_1) = -\phi_1(x_1,0)$ and $\chi_1(x_1) = -\phi_1(x_1,a)$ in order to accommodate the interface condition (2.4.3) for ϕ_{b0} and ϕ_{b1}.

Hence Equation (3.6.8) becomes

$$\frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\psi_1\phi_1\,dx_1\right]_{x_2=0} - \left[\int_0^1 M_{b0}\partial_1\phi_1\,dx_1\right]_{x_2=0} + \left[\int_0^1 (V_0 + L_0)\phi_1\,dx_1\right]_{x_2=0} = 0 \qquad (3.6.10)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$, and Equation (3.6.9) becomes

$$\frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\psi_1\phi_1\,dx_1\right]_{x_2=a} - \left[\int_0^1 M_{b1}\partial_1\phi_1\,dx_1\right]_{x_2=a} + \left[\int_0^1 (V_1 + L_1)\phi_1\,dx_1\right]_{x_2=a} = 0 \qquad (3.6.11)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

As before, adding Equations (3.6.7), (3.6.10) and (3.6.11), some cancellation of terms occur. Note that $\boldsymbol{\phi} = (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{\tau})\boldsymbol{\tau}$ and consequently,

$$\int_{\partial\Omega} M\boldsymbol{n}\cdot\boldsymbol{\phi}\,ds = \int_{\partial\Omega}\Big((\boldsymbol{\phi}\cdot\boldsymbol{n})M\boldsymbol{n}\cdot\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{\tau})M\boldsymbol{n}\cdot\boldsymbol{\tau}\Big).ds$$

The natural boundary condition on Ω is $M\boldsymbol{n}\cdot\boldsymbol{n} = 0$.

From the definition of the test functions, $\phi_2 = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and therefore $\boldsymbol{\phi}\cdot\boldsymbol{\tau} = 0$ on Σ_0 and Σ_1.

On Γ_0 and Γ_1 the interface conditions (2.4.5) and (2.4.6) are used. It follows that

$$\begin{aligned}\int_{\partial\Omega} M\boldsymbol{n}\cdot\boldsymbol{\phi}\,ds &= \int_{\Gamma_0} L_0\phi_1\,ds + \int_{\Gamma_1}(-L_1)(-\phi_1)\,ds \\ &= \left[\int_0^1 L_0\phi_1\,dx_1\right]_{x_2=0} + \left[\int_0^1 L_1\phi_1\,dx_1\right]_{x_2=a}.\end{aligned}$$

Consequently,

$$
\begin{aligned}
& I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_\Omega \operatorname{tr}(M\Phi) \, dA + \iint_\Omega \boldsymbol{Q} \cdot \boldsymbol{\phi} \, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& - \left[\int_0^1 M_{b0} \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} - \left[\int_0^1 M_{b1} \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \left[\int_0^1 V_0 \phi_1 \, dx_1 \right]_{x_2=0} + \left[\int_0^1 V_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& = 0
\end{aligned}
$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Equations (1.3.7) and (1.3.8) are the constitutive equations for $\boldsymbol{Q}$ and M for the plate. These equations are expressed in terms of w and ψ_1 (using the interface conditions (2.4.2)). Similarly, Equation (2.4.10) (constitutive equations for V_1 and V_2) and Equation (2.4.11) (constitutive equations for M_{b0} and M_{b1}) are expressed in terms of w and ψ_1. They are used to obtain the final form the second variational equation.

We define a bilinear form b_B by

$$
\begin{aligned}
b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_\Omega \operatorname{tr}(M\Phi) \, dA \\
&= \frac{1}{\beta_p(1-\nu^2)} \iint_\Omega \Big((\partial_1\psi_1 + \nu\partial_2\psi_2)\partial_1\phi_1 + (\partial_2\psi_2 + \nu\partial_1\psi_1)\partial_2\phi_2 \Big) \, dA \\
&\quad + \frac{1}{2\beta_p(1+\nu)} \iint_\Omega (\partial_1\psi_2 + \partial_2\psi_1)(\partial_1\phi_2 + \partial_2\phi_1) \, dA.
\end{aligned}
$$

for each $\boldsymbol{\psi}$, $\boldsymbol{\phi}$ in $H^1(\Omega)^2$.

Finally, the second variational equation is given by

$$
\begin{aligned}
& I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA \,+\, b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) \,+\, h_p \iint_\Omega (\nabla w \,+\, \boldsymbol{\psi}) \cdot \boldsymbol{\phi} \, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \eta_2 \left[\int_0^1 (\partial_1 w \,+\, \psi_1) \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \eta_2 \left[\int_0^1 (\partial_1 w \,+\, \psi_1) \phi_1 \, dx_1 \right]_{x_2=a} \\
& = \; 0 \qquad (3.6.12)
\end{aligned}
$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Variational form of Problem RMT

Find w and $\boldsymbol{\psi}$ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\boldsymbol{\psi}(\cdot, t) \in T_2(\Omega)$ and Equations (3.6.6) and (3.6.12) hold for each $v \in T_1(\Omega)$ and each $\boldsymbol{\phi} \in T_2(\Omega)$. □

The variational form above is used for computational purposes (see Chapter 8), but for theoretical purposes we consider the weak form of the variational problem.

3.6.2 Variational form of Problems KR and KEB

The variational form of Problem KR can be obtained by setting $\boldsymbol{\psi} = -\nabla w$ and choosing $\boldsymbol{\phi} = -\nabla v$ in Equations (3.6.6) and (3.6.12). In this case the test functions are defined by

$$T(\Omega) = \left\{ v \in C^2(\bar{\Omega}) \,\middle|\, v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1. \right\}$$

The variational equations reduce to

$$
\begin{aligned}
& h_p \iint_\Omega \partial_t^2 w v \, dA \,+\, \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} \,+\, \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\
& = 0 \qquad (3.6.13)
\end{aligned}
$$

and

$$\begin{aligned}
& I_p \iint_\Omega \partial_t^2(\nabla w) \cdot \nabla v \, dA \;+\; b_B(\nabla w, \nabla v) \\
& + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2 \partial_1 w v \, dx_1\right]_{x_2=0} \;+\; \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2 \partial_1 w v \, dx_1\right]_{x_2=a} \\
& + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w \partial_1^2 v \, dx_1\right]_{x_2=0} \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w \partial_1^2 v \, dx_1\right]_{x_2=a} \\
& = \; 0 \qquad (3.6.14)
\end{aligned}$$

for each $v \in T(\Omega)$.

Redefine the bilinear form b_B in Equation (3.6.14) by

$$\begin{aligned}
b_B(w, v) \;=\; & \frac{1}{\beta_p(1-\nu^2)} \iint_\Omega \Big((\partial_1^2 w + \nu \partial_2^2 w)\partial_1^2 v + (\partial_2^2 w + \nu \partial_1^2 w)\partial_2^2 v \Big) \, dA \\
& + \frac{2}{\beta_p(1+\nu)} \iint_\Omega \partial_1 \partial_2 w \partial_1 \partial_2 v \, dA.
\end{aligned}$$

for each w, v in $H^2(\Omega)$.

For Problem KR the variational form is reduced to a single equation by adding Equations (3.6.13) and (3.6.14).

Variational form of Problem KR

Find w such that, for $t > 0$, $w(.,t) \in T(\Omega)$,

$$\begin{aligned}
& h_p \iint_\Omega \partial_t^2 w v \, dA \;+\; I_p \iint_\Omega \partial_t^2(\nabla w) \cdot \nabla v \, dA \\
& + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1\right]_{x_2=0} \;+\; \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1\right]_{x_2=a} \\
& + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2(\partial_1 w) v \, dx_1\right]_{x_2=0} \;+\; \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2(\partial_1 w) v \, dx_1\right]_{x_2=a} \\
& + b_B(w, v) \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w \partial_1^2 v \, dx_1\right]_{x_2=0} \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w \partial_1^2 v \, dx_1\right]_{x_2=a} \\
& = 0 \qquad (3.6.15)
\end{aligned}$$

for each v in $T(\Omega)$. □

Variational form of Problem KEB

The variational form of the case where rotary inertia is ignored, is obtained by ignoring the terms containing I_p and $\dfrac{\eta_1}{\alpha_b}$ in Equation (3.6.15).

3.6.3 Weak variational form of Problem RMT

For $I = (0,1)$, the space $T(I)$ is defined as

$$T(I) = \left\{ v \in C^1(\bar{I}) \,\middle|\, v(0) = v(1) = 0 \right\}.$$

The trace operators γ_0 and γ_1 are defined in Appendix 3. At this stage we are dealing with smooth functions and γ_0 and γ_1 simply map a function onto its value at the boundary. Therefore

$$\gamma_0 v = v(\cdot\,,0) \quad \text{and} \quad \gamma_1 v = v(\cdot\,,a).$$

In order to formulate the weak variational form of Problem RMT, we start by rewriting Equations (3.6.6) and (3.6.12) in terms of inner products. The notation is explained in Appendix 1.

$$\begin{aligned}
&h_p\Big(\partial_t^2 w(\cdot,t)\,,\,v\Big)_\Omega + \eta_1\Big(\gamma_0(\partial_t^2 w(\cdot,t))\,,\,\gamma_0 v\Big)_I + \eta_1\Big(\gamma_1(\partial_t^2 w(\cdot,t))\,,\,\gamma_1 v\Big)_I \\
&+ h_p\Big(\nabla w(\cdot,t) + \boldsymbol{\psi}(\cdot,t)\,,\,\nabla v\Big)^\Omega_{0,2} + \eta_2\Big(\gamma_0(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_0(\partial_1 v)\Big)_I \\
&+ \eta_2\Big(\gamma_1(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_1(\partial_1 v)\Big)_I = 0. \qquad (3.6.16)
\end{aligned}$$

$$\begin{aligned}
&I_p\Big(\partial_t^2 \boldsymbol{\psi}(\cdot,t)\,,\,\boldsymbol{\phi}\Big)^\Omega_{0,2} + b_B(\boldsymbol{\psi}(\cdot,t),\boldsymbol{\phi}) + h_p\Big(\nabla w(\cdot,t) + \boldsymbol{\psi}(\cdot,t)\,,\,\boldsymbol{\phi}\Big)^\Omega_{0,2} \\
&+ \frac{\eta_1}{\alpha_b}\Big(\gamma_0(\partial_t^2 \psi_1(\cdot,t))\,,\,\gamma_0\phi_1\Big)_I + \frac{\eta_1}{\alpha_b}\Big(\gamma_1(\partial_t^2 \psi_1(\cdot,t))\,,\,\gamma_1\phi_1\Big)_I \\
&+ \frac{\eta_2}{\beta_b}\Big(\gamma_0(\partial_1 \psi_1(\cdot,t))\,,\,\gamma_0(\partial_1\phi_1)\Big)_I + \frac{\eta_2}{\beta_b}\Big(\gamma_1(\partial_1 \psi_1(\cdot,t))\,,\,\gamma_1(\partial_1\phi_1)\Big)_I \\
&+ \eta_2\Big(\gamma_0(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_0\phi_1\Big)_I \\
&+ \eta_2\Big(\gamma_1(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_1\phi_1\Big)_I = 0. \qquad (3.6.17)
\end{aligned}$$

The next step is to define product spaces.

Product spaces

$$X = L^2(\Omega) \times L^2(\Omega)^2 \mathop{\times}_{n=1}^{4} L^2(I)\,,$$

$$H^1 = H^1(\Omega) \times H^1(\Omega)^2 \mathop{\times}_{n=1}^{4} H^1(\Omega)$$

$$S = T_1(\Omega) \times T_2(\Omega)^2 \mathop{\times}_{n=1}^{2} \big(T(I) \times C^1(\bar{I})\big)$$

$$T = \big\{v \in S \,\big|\, \gamma_0 v_1 = v_3, \gamma_1 v_1 = v_5, \gamma_0\,(v_2 \cdot \boldsymbol{e}_1) = -v_4, \gamma_1\,(v_2 \cdot \boldsymbol{e}_1) = -v_6\big\}$$

The following table explains the relationship between the functions used in Equations (3.6.16) and (3.6.17) and Equations (3.6.18) and (3.6.19) to follow. Note that for $v \in C^1(\bar{\Omega})$ and $i = 0,\, 1,\;\; \gamma_i\,(\partial_1 v) = (\gamma_i v)\,'$ – the derivative with respect to the variable x_1.

$u_1(t)$	$w(\cdot, t)$	v_1	v
$u_2(t)$	$\boldsymbol{\psi}(\cdot, t)$	v_2	$\boldsymbol{\phi}$
$u_3(t)$	$\gamma_0\big(w(\cdot, t)\big) = \gamma_0 u_1(t)$	v_3	$\gamma_0 v = \gamma_0 v_1$
$u_4(t)$	$-\gamma_0\psi_1(\cdot, t) = -\gamma_0(u_2(t) \cdot \boldsymbol{e}_1)$	v_4	$-\gamma_0\phi_1 = -\gamma_0(v_2 \cdot \boldsymbol{e}_1)$
$u_5(t)$	$\gamma_1\big(w(\cdot, t)\big) = \gamma_1 u_1(t)$	v_5	$\gamma_1 v = \gamma_1 v_1$
$u_6(t)$	$-\gamma_1\psi_1(\cdot, t) = -\gamma_1(u_2(t) \cdot \boldsymbol{e}_1)$	v_6	$-\gamma_1\phi_1 = -\gamma_1(v_2 \cdot \boldsymbol{e}_1)$

In the new notation, Equations (3.6.16) and (3.6.17) become

$$\begin{aligned} & h_p\Big(\ddot{u}_1(t)\,,\, v_1\Big)_{\Omega} + \eta_1\Big(\ddot{u}_3(t)\,,\, v_3\Big)_I + \eta_1\Big(\ddot{u}_5(t)\,,\, v_5\Big)_I \\ & + h_p\Big(\nabla u_1(t) + u_2(t)\,,\, \nabla v_1\Big)^{\Omega}_{0,2} + \eta_2\Big(u_3'(t) - u_4(t)\,,\, v_3'\Big)_I \\ & + \eta_2\Big(u_5'(t) - u_6(t)\,,\, v_5'\Big)_I \;=\; 0 \end{aligned} \tag{3.6.18}$$

and

$$\begin{aligned} & I_p\Big(\ddot{u}_2(t)\,,\, v_2\Big)^{\Omega}_{0,2} + b_B(u_2(t), v_2) + h_p\Big(\nabla u_1(t) + u_2(t)\,,\, v_2\Big)^{\Omega}_{0,2} \\ & + \frac{\eta_1}{\alpha_b}\Big(-\ddot{u}_4(t)\,,\, -v_4\Big)_I + \frac{\eta_1}{\alpha_b}\Big(-\ddot{u}_6(t)\,,\, -v_6\Big)_I \\ & + \frac{\eta_2}{\beta_b}\Big(-u_4'(t)\,,\, -v_4'\Big)_I + \frac{\eta_2}{\beta_b}\Big(-u_6'(t)\,,\, -v_6'\Big)_I \\ & + \eta_2\Big(u_3'(t) - u_4(t)\,,\, -v_4\Big)_I + \eta_2\Big(u_5'(t) - u_6(t)\,,\, -v_6\Big)_I \\ & = \; 0 \end{aligned} \tag{3.6.19}$$

Bilinear forms

For u and v in T, define

$$\begin{aligned} c(u,v) &= h_p(u_1,v_1)_\Omega + I_P(u_2,v_2)^\Omega_{0,2} + \eta_1(u_3,v_3)_I \\ &\quad + \frac{\eta_1}{\alpha_b}(u_4,v_4)_I + \eta_1(u_5,v_5)_I + \frac{\eta_1}{\alpha_b}\eta_1(u_6,v_6)_I\,, \\ b_\Gamma(u,v) &= \eta_2\Big(u_3'-u_4, v_3'-v_4\Big)_I + \eta_2\Big(u_5'-u_6, v_5'-v_6\Big)_I \\ &\quad + \frac{\eta_2}{\beta_b}(u_4',v_4')_I + \frac{\eta_2}{\beta_b}(u_6',v_6')_I\,, \\ b_\Omega(u,v) &= b_B(u_2,v_2) + h_p\Big(\nabla u_1 + u_2, \nabla v_1 + v_2\Big)^\Omega_{0,2}\,, \\ b(u,v) &= b_\Omega(u,v) + b_\Gamma(u,v)\,. \end{aligned}$$

By adding Equations (3.6.18) and (3.6.19), we arrive at the following variational problem.

Find $u(t) \in T$ such that $c(\ddot{u}(t),v) = -b(u(t),v)$ for each $v \in T$.

We are now ready to consider the weak variational form. We define V as the closure of T in H^1. Note that all the bilinear forms are defined for elements of V, except for b_Γ. For u and v in V, define

$$b_\Gamma(u,v) = \lim_{n\to\infty} b_\Gamma(u_n,v_n)\,,$$

with $\{u_n\}$ and $\{v_n\}$ sequences in T such that $u_n \to u$ and $v_n \to v$.

As a consequence, the bilinear forms b and b_Γ are now defined on V.

Weak variational form of Problem RMT

Find $u \in C^1([0,\infty),V) \cap C^2((0,\infty),X)$ such that for each $t > 0$, $u'(t) \in V$, $u''(t) \in X$ and

$$c(u''(t),v) = -b(u(t),v) \quad \text{for each } v \in V.$$

Inertia space

The space X with the norm induced by the inner product c is the **inertia space**.

Energy space

The closure of T in H^1 is denoted by V. A norm on V is defined by $\|u\|_V = \sqrt{b(u,u)}$ and is called the **energy norm**. The space V with norm $\|\cdot\|_V$ called the **energy space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is dense in X.

Proof

Appendix 5.

Theorem 2

There exist constants c_1 and c_2 such that

$$\|u\|_X \leq c_1\|u\|_{H^1} \leq c_2\|u\|_V$$

for each $u \in V$.

Proof

Appendix 5.

3.7 Equilibrium problems

In the rest of Chapter 3, X and V denote spaces with the following properties:

X is a Hilbert space with inner product c and norm $\|\cdot\|_X$;

V is a Hilbert space with inner product b and norm $\|\cdot\|_V$;

V is a subspace of X.

Theorem (Riesz)

For any f in the dual of V, there exists a unique $u \in V$ such that

$$b(u, v) = f(v) \quad \text{for each } v \in V.$$

Corollary

Suppose $\|u\|_X \leq \|u\|_V$ for each $u \in V$. For any $f \in X$, there exists a unique $u \in V$ such that

$$b(u, v) = c(f, v) \quad \text{for each } v \in V.$$

Proof

Let $g(v) = c(f, v)$ for each $v \in V$, then $|g(v)| \leq \|f\|_X \|v\|_V$ proving that g is in the dual of V. Applying the theorem yields the desired result.

Application

The theorem above yields the existence of a weak solution for Problem CTD 1.

Proof of Theorem 7 Section 3.5

The result follows from the theorem above and Theorem 1 Section 3.5. Recall that $f(v) = (t, \gamma v)_{0,2}^{\Gamma}$.

3.8 Vibration problems

In this section we consider the general linear vibration problem. Consider the Hilbert spaces X and V introduced in Section 3.7. Consider also a bilinear form a defined on V.

For any Banach space Y the spaces $C^k\big([0,\infty),Y\big)$ and $C^k\big((0,\infty),Y\big)$ are defined in Appendix 4.

Problem D

Find $u \in C^1\big([0,\infty),V\big) \cap C^2\big((0,\infty),X\big)$ such that for each $t > 0$,

$$c\big(u''(t),v\big) + a\big(u'(t),v\big) + b\big(u(t),v\big) = 0 \quad \text{for each } v \in V,$$

$$u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Theorem

Suppose

(a) V is dense in X,

(b) $\|u\|_X \leq K\|u\|_V$ for each $u \in V$,

(c) the bilinear form a is symmetric, nonnegative and $|a(u,v)| \leq C\|u\|_V\|v\|_V$ for each u and v in V,

(d) $u_0 \in V$, $u_1 \in V$ and for some $y \in X$,

$$b(u_0,v) + a(u_1,v) = c(y,v) \quad \text{for each } v \in V.$$

Then Problem D has a unique solution.

Proof

See [VV].

Remark

It is possible to define linear operators M, C and K and arrive at an abstract differential equation $Mu'' + Cu' + Ku = 0$. It is then possible to prove an equivalent existence result, see e.g. [Sho, p 131].

Applications

For Problems VR 4, VT 4, CTD 2 and RMT the first three conditions in the Theorem are met. This is proven in each section where the weak variational forms of the problems are discussed.

3.9 Modal analysis

In this section we consider the modal analysis of the general linear vibration problem. Consider the Hilbert spaces X and V introduced in Section 3.7. Consider also a bilinear form a defined on V.

The fact that a solution of the (general) vibration problem exists is not enough. To determine the response of a system to excitation, knowledge of the vibration spectrum is required. We need to know whether the solution may be written as the superposition of modes.

First consider the case of **no damping**, i.e. $a = 0$. For the modal analysis of the system, a function $\tilde{u}(x,t) = T(t)u(x)$ is considered as a possible solution. This requires consideration of the following eigenvalue problem.

Problem E1

Find a complex number λ and $u \in V$ such that

$$b(u, v) = \lambda c(u, v) \quad \text{for each } v \in V.$$

Natural frequencies and modes

The function T_n satisfies $T_n'' = -\lambda_n T$ and hence the **natural angular frequencies** are equal to $\omega_n = \sqrt{\lambda_n}$. The formal solution of Problem D (general vibration problem) is given by

$$u(t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \sin \omega_n t)\ e_n,$$

where each e_n is an eigenvector. For the series above to converge, a necessary condition is that it must be possible to write the the initial values u_0 and u_1 as a series using the sequence of eigenvectors. This implies that the existence of a complete orthonormal sequence of eigenvectors is required. We present two well-known results, slightly modified.

Theorem 1

(a) The eigenvalues are (real and) positive.

(b) The eigenfunctions are orthogonal in the inertia space X with respect to the inner product c.

Proof

The bilinear forms on both sides of the equation are inner products. Consequently, the eigenvalues must be real and positive. Furthermore, for different eigenvalues λ and μ it follows that $\lambda c(u, v) = b(u, v) = \mu c(u, v)$. Therefore that $(\lambda - \mu)c(u, v) = 0$ and consequently $c(u, v) = 0$.

Theorem 2

Suppose the embedding of V into X is compact.

(a) The set of eigenvalues can be ordered as a sequence $\{\lambda_n\}$ converging to ∞ as $n \to \infty$.

(b) The set of eigenvectors can be ordered as a sequence and this sequence is complete (or total) in X.

Proof

For each $f \in X$, there exists a unique $u \in V$ such that

$$b(u, v) = c(f, v) \quad \text{for each } v \in V$$

by the corollary in Section 3.7. (The embedding of V into X is bounded.) Define a mapping K by $u = Kf$, then

$$b(Kf, v) = c(f, v) \quad \text{for each } v \in V.$$

The mapping K is defined on X and it is clearly linear. Note that

$$b(u, v) = \lambda c(u, v) \quad \text{for each } v \in V$$

if and only if $\lambda K u = u$ or $Ku = \lambda^{-1} u$.

The operator K is symmetric due to the fact that b and c are symmetric. The inequality

$$\|Kf\|_V^2 \leq \|f\|_X \|Kf\|_X \leq k_{bc} \|f\|_X \|Kf\|_V$$

implies that K is a bounded operator from the inertia space X into the energy space V. If a set A is bounded in the inertia space, then the set KA is bounded in the energy space and consequently pre-compact in the inertia space (due to the compactness of the embedding). Therefore the operator K is compact.

Both conclusions of the theorem now follow from the theory of compact symmetric linear operators on a separable Hilbert space, see e.g. [Ze, p 232].

Modal damping

We now consider the case where the bilinear form a is not zero but we assume that

$$a = k_1 c + k_2 b.$$

Consider a function $\tilde{u}(x, t) = T(t)u(x)$ as a possible solution. The eigenvalue problem is the same as for the undamped case but

$$T'' c(u, v) + T' \big(k_1 c(u, v) + k_2 b(u, v)\big) + T b(u, v) = 0.$$

This leads to the following ordinary differential equation

$$T'' + (k_1 + k_2 \lambda) T' + \lambda T = 0.$$

Again it is possible to present the solution in series form.

3.10 Nonmodal damping

In this section we consider the the general linear vibration problem (Section 3.8) with nonmodal damping, i.e.

$$a \neq k_1 c + k_2 b.$$

Nonmodal damping is often a consequence of boundary damping. It also features in hybrid systems such as the models for the vertical structure presented in Section 2.1. Computation of the natural frequencies leads to a quadratic eigenvalue problem with complex eigenvalues and eigenvectors (see Chapter 6).

The quadratic eigenvalue problem

Consider the Hilbert spaces X and V introduced in Section 3.8 and the general linear vibration problem, Problem D. In general, consideration of a solution of the form $e^{\lambda t}u$ leads to a **quadratic eigenvalue problem**.

$$\lambda^2 c(u,v) + \lambda a(u,v) + b(u,v) = 0 \quad \text{for each } v \in V.$$

This problem is a generalization of the eigenvalue problems in Chapter 6.

It is clear that imaginary eigenvalues and eigenvectors are possible and X is a real Hilbert space. However, we may consider the space X to be embedded in complex space $\widetilde{X}$. This can be done in a rigorous manner, see e.g. [Sch, p 154]. Elements of $\widetilde{X}$ are of the form $x = x_1 + ix_2$, where x_1 and x_2 are in X. We also have a subspace $\widetilde{V}$ with elements of the form $x = x_1 + ix_2 \in \widetilde{V}$ where x_1 and x_2 are in V.

The bilinear forms a, b and c must be extended to $\widetilde{X}$ and $\widetilde{V}$. Consider for example the bilinear form c:

$$\tilde{c}(x,y) = c(x_1,y_1) + ic(x_2,y_1) - ic(x_1,y_2) + c(x_2,y_2).$$

It is easily checked that the bilinear form $\tilde{c}$ is an inner product for $\widetilde{X}$ and that $\widetilde{X}$ is a separable Hilbert space. Similarly, we find that $\widetilde{V}$ is a Hilbert space with inner product $\tilde{b}$. Furthermore, $\widetilde{V}$ is dense in $\widetilde{X}$ and the relevant estimates remain valid. We have for example

$$\begin{aligned}
\tilde{c}(x,x) &= c(x_1,x_1) + c(x_2,x_2) \\
&\leq K_{bc}\big(b(x_1,x_1) + b(x_2,x_2)\big) \\
&= K_{bc}\,\tilde{b}(x,x).
\end{aligned}$$

We now return to the **original notation** and consider the quadratic eigenvalue problem.

Problem QE

Find a complex number λ and $u \in V$ such that

$$\lambda^2 c(u,v) + \lambda a(u,v) + b(u,v) = 0 \quad \text{for each } v \in V.$$

To apply the theory on convergence, we need an alternative formulation.

Non selfadjoint eigenvalue problem

The quadratic eigenvalue problem is equivalent to a conventional abstract eigenvalue problem in a product space. Let $H = V \times X$ and

$$(x,y)_H = b(x_1,y_1) + c(x_2,y_2) \quad \text{for} \quad x,\, y \in H.$$

It is easy to see that $(\cdot,\cdot)_H$ is an **inner product** for H and that H is complete.

Problem E2

Find a complex number λ and $x \in H$ such that

$$\begin{aligned} x_2 &= \lambda x_1 \\ b(x_1,v) + a(x_2,v) &= -\lambda c(x_2,v) \quad \text{for each } v \in V. \end{aligned}$$

If λ is an eigenvalue and u an eigenvector of Problem QE, then λ is an eigenvalue and $\langle u, \lambda u\rangle$ an eigenvector of Problem E2. Conversely, if λ is an eigenvalue and x an eigenvector of Problem E2, then λ is an eigenvalue and x_1 an eigenvector of Problem QE.

If the sequence of eigenvectors is complete in the complex Hilbert space X, the solution of Problem D can be written in series form. The abstract form of the quadratic eigenvalue problem is considered in Section 5.4. It is of the same type as the abstract form of the eigenvalue problem for a Timoshenko beam with boundary damping considered in a recent paper [Shu]. Shubov proved that the sequence of eigenvectors is complete but it should be noted that the problems are not the same.

Chapter 4

Interpolation

4.1 Hermite cubics

The well-known Hermite piecewise cubics (see [SF] or [Re]) are successfully used as basis functions for the Galerkin approximation in beam problems.

The construction and properties of Hermite cubics are treated in detail in the book of Strang and Fix ([SF, p 55-59]). Divide the interval $[a, b]$ into n subintervals by a partitioning

$$a \;=\; x_0 \;< x_1 \;<\; \cdots \;<\; x_n \;=\; b.$$

This yields n elements, $\Omega_i = [x_{i-1}, x_i]$, each of length h_i, for $i = 1, 2, \ldots, n$.

For $i = 0, 1, \ldots, n$, we have two piecewise cubics denoted by $\delta_i^{(j)}$ with $j = 0$ or $j = 1$ with the following properties:

1. For $k = 0, 1, \ldots, n, \quad i = 0, 1, \ldots, n$ and $j = 0, 1,$ the restriction of $\delta_k^{(j)}$ to any Ω_i is either a cubic polynomial or zero.
2. $\delta_i^{(j)} \in C^1[a, b]$ and $D^2\delta_i^{(j)}$ is piecewise continuous with possible discontinuities at the nodes.
3. $\delta_i^{(0)}(x_i) = 1, \quad D\delta_i^{(0)}(x_i) = 0, \quad \delta_i^{(1)}(x_i) = 0, \quad D\delta_i^{(1)}(x_i) = 1.$
4. $\delta_i^{(0)}(x_k) = 0, \quad D\delta_i^{(0)}(x_k) = 0, \quad \delta_i^{(1)}(x_k) = 0, \quad D\delta_i^{(1)}(x_k) = 0$ if $k \neq i$.
5. $\delta_i^{(j)}$ is zero on any element Ω_k with $k \neq i$ or $i + 1$.

We refer to these two types of functions as Type 1 ($j = 0$) or Type 2 ($j = 1$) functions. Typical graphs of $\delta_i^{(0)}$ and $\delta_i^{(1)}$ are shown in Figures 1 and 2.

Figure 1: Type 1 Hermite piecewise cubic

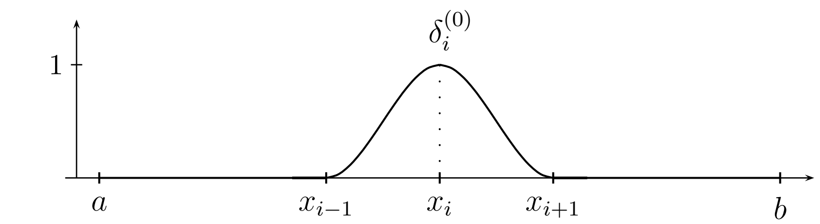
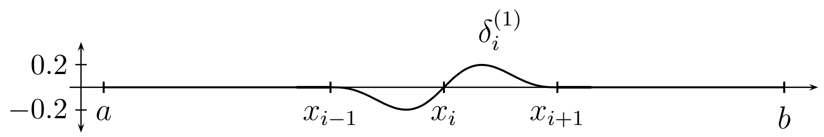


Figure 2: Type 2 Hermite piecewise cubic



Remarks

1. The graphs in Figures 1 and 2 must be adapted for the functions $\delta_0^{(0)}$, $\delta_n^{(0)}$, $\delta_0^{(1)}$ and $\delta_n^{(1)}$.
2. We will refer to the Hermite piecewise cubic functions as **Hermite cubics**.
3. $\delta_i^{(j)} \in H^2[a, b] \quad \forall \;\; i = 0, 1, \ldots, n \;$ and $\; j = 0, 1$.

Cubic interpolation operator

For $w \in H^2(a, b)$, we define the cubic interpolation operator Π_c as

$$\Pi_c\, w = \sum_{j=0}^{1} \sum_{i=0}^{n} (D^j w)(x_i)\, \delta_i^{(j)}.$$

Note that $\Pi_c\, \delta_i^{(j)} = \delta_i^{(j)}$ for $i = 0, 1, \ldots, n$ and $j = 0, 1$.

4.2 Hermite bicubic functions

The Hermite piecewise bicubic functions are constructed by using a product of the Hermite piecewise cubic functions in Section 4.1, hence the name **bicubics**. (For a fixed x or y, a piecewise bicubic reduces to a piecewise cubic.) See [SF, p 88-89] for detail. It is also mentioned there that bicubics rank amongst the best provided that rectangular elements are used.

The rectangle $\bar{\Omega} = [a, b] \times [c, d]$ is divided in rs elements as follows. Partition $[a, b]$ and $[c, d]$ by

$$a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad c = y_0 < y_1 < \cdots < y_s = d,$$

and set

$$h_i = x_i - x_{i-1} \quad \text{and} \quad k_j = y_j - y_{j-1}.$$

This defines a grid on $\bar{\Omega}$ with the grid lines $x = x_i$ and $y = y_j$. A general element is given by

$$\bar{\Omega}_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

For $i = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$, we have four piecewise bicubics denoted by $\delta_{ij}^{(k)}$ with $k = 0, 1, 2, 3,$ with the following properties:

1. The restriction of $\delta_{ij}^{(k)}$ to any $\bar{\Omega}_{IJ}$ is either a bicubic polynomial or zero for i and $I = 0, 1, \ldots, r,$ j and $J = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3$.

2. $\delta_{ij}^{(k)} \in C^1(\bar{\Omega})$ and all second order partial derivatives are piecewise continuous with possible discontinuities on the edges of the elements.

3. $$\delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

   $$\partial_x \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise,} \end{cases}$$

   $$\partial_y \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 2 \\ 0 & \text{otherwise,} \end{cases}$$

   $$\partial_x \partial_y \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

4. $\delta_{ij}^{(k)}(x_I, y_J) = 0,$ $\partial_x \delta_{ij}^{(k)}(x_I, y_J) = 0,$ $\partial_y \delta_{ij}^{(k)}(x_I, y_J) = 0$ and $\partial_x \partial_y \delta_{ij}^{(k)}(x_I, y_J) = 0$ if $(i, j) \neq (I, J)$.

5. $\delta_{ij}^{(k)}$ is zero on any element Ω_{IJ} not adjacent to Ω_{ij} .

We refer to these four types of functions as Type 1 ($k = 0$), Type 2 ($k = 1$), Type 3 ($k = 2$) and Type 4 ($k = 3$) functions.

Remarks

1. As mentioned, for a fixed x or y, a piecewise bicubic reduces to a piecewise cubic. This compatibility is needed for the plate-beam problems.
2. $\delta_{ij}^{(k)} \in H^2(\Omega) \quad \forall \;\; i = 0, 1, \ldots, r, \;\; j = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3$.

We use the following notation for the partial derivatives that play a role in construction of the bicubics.

$$\partial^{(k)} w = \begin{cases} w & \text{for } k = 0 \\ \partial_x w & \text{for } k = 1 \\ \partial_y w & \text{for } k = 2 \\ \partial_x \partial_y w & \text{for } k = 3 \end{cases}$$

Bicubic interpolation operator

For $w \in H^4(\Omega)$, we define the bicubic interpolation operator Π_b as

$$\Pi_b \, w \;\; = \;\; \sum_{k=0}^{3} \sum_{i=0}^{r} \sum_{j=0}^{s} (\partial^{(k)} w)(x_i, y_j) \, \delta_{ij}^{(k)} \, .$$

Note that $\Pi_b \, \delta_{ij}^{(k)} = \delta_{ij}^{(k)}$ for $i = 0, 1, \ldots, r, \; j = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3$.

4.3 Standard estimates for the interpolation error

Standard interpolation estimates can be found in, for instance, [SF], [OR] and [OC]. The following two parameters for an interpolation operator are used in the interpolation estimates:

$r(\Pi)$ is the highest degree of polynomials left invariant by Π.

$s(\Pi)$ is the highest order derivative used in the definition of Π.

We will use $\widehat{C}$ to denote a generic constant which depends on the constants in Sobolev's lemma and the constants in the Bramble-Hilbert lemma.

Theorems 1 and 2 below are formulated as a special case of a general result. This result may be found in [SF, p 144], [OC, p 76] and [OR, p 279].

4.3.1 One-dimensional domain

We consider a one-dimensional domain $\Omega = (a, b)$. Here $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k = \|u^{(k)}\|\,.$$

(See Appendix 1.)

Theorem 1

Suppose $s(\Pi) + 1 \leq k \leq r(\Pi) + 1$. Then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)$,

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

Corollary

Consider the Hermite piecewise cubic functions and the interpolation operator Π_c.

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,

$$\|u - \Pi_c u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,

$$\|u - \Pi_c u\|_m \leq \widehat{C} h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof

$r(\Pi_c) = 3$ and $s(\Pi_c) = 1$.

a) The result follows directly from Theorem 1.

b) If $k > 4$, $H^k(0,1) \subset H^4(0,1)$. The result follows from Theorem 1.

4.3.2 Two-dimensional domain

For a two-dimensional convex domain Ω, $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix 1.)

In the following theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e.

Theorem 2

Suppose $s(\Pi) + 2 \leq k \leq r(\Pi) + 1$. Then there exists a constant $\widehat{C}$ such that, for $u \in H^k(\Omega)$,

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

Corollary

Consider the piecewise Hermite bicubic functions and the interpolation operator Π_b. For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$

$$\|u - \Pi_b u\|_m \leq \widehat{C} h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof

$r(\Pi_b) = 3$ and $s(\Pi_b) = 2$. If $k > 4$, $H^k(0,1) \subset H^4(0,1)$ and the result follows from Theorem 2.

Remark

The constant $\widehat{C}$ depends on the ratio length versus width for the elements. Care should be taken that these ratios remain within specific bounds.

4.3.3 Vector-valued functions

Definition

For $u = \langle u_1,\, u_2\rangle \in H^k(\Omega)^2$, we define

$$\Pi_B\, u = \langle \Pi_b\, u_1,\, \Pi_b\, u_2\rangle\,.$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2\,.$$

(See Appendix 1.)

Theorem 3

There exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2$ with $k \geq 4$,

$$\|u - \Pi_B\, u\|_{m,2} \leq \widehat{C}h^{4-m}|u|_{4,2}\,, \quad m = 0, 1, \ldots . 4\,.$$

Proof

The proof follows directly from the definition of the interpolation operator Π_B, the norm and seminorm on the product space and the corollary in Subsection 4.3.2.

$$\begin{aligned} \|u - \Pi_B\, u\|_{m,2}^2 &= \|u_1 - \Pi_b\, u_1\|_m^2 + \|u_2 - \Pi_b\, u_2\|_m^2 \\ &\leq \left[\widehat{C}h^{4-m}\,|u_1|_4\right]^2 + \left[\widehat{C}h^{4-m}\,|u_2|_4\right]^2 \\ &= \left[\widehat{C}h^{4-m}|u|_{4,2}\right]^2. \end{aligned}$$

Corollary

There exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2 \cap V$ with $k \geq 4$,

$$\|u - \Pi_B\, u\|_V \leq \widehat{C} h^3 |u|_{4,2}\,.$$

Proof

The norms $\|\cdot\|_{1,2}$ and $\|\cdot\|_V$ are equivalent (Theorem 4 Sec 3.5).

4.4 Interpolation estimates for the one-dimensional hybrid models

Consider Problem VT 4 (Section 3.3). Let $\Omega = (a, b)$ and define H^k as $H^k = H^k(\Omega) \times H^k(\Omega) \times I\!R^3$. An interpolation operator on the product spaces H^k can now be defined.

Definition

$$\Pi u = \langle \Pi_c\, u_1,\, \Pi_c\, u_2,\, u_3,\, u_4,\, u_5 \rangle \quad \text{for } u \in H^k\,.$$

An **inner product** for H^k is defined by

$$\big(u, v\big)_{H^k} \;=\; \big(u_1, v_1\big)_k + \big(u_2, v_2\big)_k + u_3 v_3 + u_4 v_4 + u_5 v_5\,.$$

The corresponding **norm** is

$$\|u\|_{H^k} = \sqrt{\big(u, u\big)_{H^k}}\,.$$

A **seminorm** for H^k is defined by

$$|u|_{k,H^k} \;=\; \sqrt{|u_1|_k^2 + |u_2|_k^2}\,,$$

with $|\cdot|_k$ the seminorm in $H^k(\Omega)$.

Theorem

Consider the piecewise Hermite cubic functions and the interpolation operator Π.

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{m,H^k} \leq \widehat{C} h^{k-m} |u|_{k,H^k}, \quad m = 0, 1, \ldots . k.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{m,H^k} \leq \widehat{C} h^{k-m} |u|_{4,H^k}, \quad m = 0, 1, \ldots . 4.$$

Proof

In this proof, we use the result in Subsection 4.3.1.

$$\begin{aligned}
\|u - \Pi u\|^2_{m,H^k} &= \|\langle u_1 - \Pi_c u_1 \,,\, u_2 - \Pi_c u_2 \,,\, 0 \,,\, 0 \,,\, 0\rangle\|^2_{m,H^k} \\
&= \sum_{j=1}^{\ell} \|u_j - \Pi_c u_j\|^2_m \\
&\leq \begin{cases} \displaystyle\sum_{j=1}^{2} \left[\widehat{C} h^{k-m} \, |u_j|_k\right]^2 & \text{if } \ 2 \leq k \leq 4, \\ \displaystyle\sum_{j=1}^{2} \left[\widehat{C} h^{4-m} \, |u_j|_4\right]^2 & \text{if } \ k > 4, \end{cases} \\
&= \begin{cases} \left[\widehat{C} h^{k-m} |u|_k\right]^2 & \text{if } \ 2 \leq k \leq 4, \\ \left[\widehat{C} h^{4-m} |u|_4\right]^2 & \text{if } \ k > 4. \end{cases}
\end{aligned}$$

Remark

It is easy to see that similar results can be found for the product spaces $H^k(\Omega) \times H^k(\Omega) \times I\!R$, $H^k(\Omega) \times I\!R^3$ and $H^k(\Omega) \times I\!R$.

Corollary 1 (Problems VR 3 and VR 4)

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^{k-2} |u|_{k,H^k} .$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^2 |u|_{4,H^k} .$$

Proof

The results follow from the theorem, the fact that $V \subset H^2$ and the equivalence of the energy norm $\|\cdot\|_V$ and the H^2–norm.

Corollary 2 (Problems VT 3 and VT 4)

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^{k-1} |u|_{k,H^k} .$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^3 |u|_{4,H^k} .$$

Proof

The energy norm $\|\cdot\|_V$ and the H^1–norm are equivalent.

4.5 Interpolation estimates for the plate-beam system

We consider an interval $I = (a, b)$ and a rectangle $\Omega = (a, b) \times (c, d)$. Define

$$H^k = H^k(\Omega) \times H^k(\Omega)^2 \bigtimes_{n=1}^{4} H^1(\Omega) .$$

The other relevant product spaces are defined in Section 3.6.

Definition

For $u \in H^k$ we define the interpolation operator

$$\Pi u = \langle \Pi_b \, u_1, \; \Pi_B \, u_2, \; \Pi_c \, u_3, \; \Pi_c \, u_4, \; \Pi_c \, u_5, \; \Pi_c \, u_6 \rangle \, .$$

An **inner product** for H^k is defined by

$$\big(u, v\big)_{H^k} = (u_1, v_1)_k^\Omega + (u_2, v_2)_{k,2}^\Omega + \sum_{j=3}^{6} \big(u_j, v_j\big)_k^I \, ,$$

The corresponding **norm** is given by

$$\|u\|_{H^k} = \sqrt{\big(u, u\big)_{H^k}}$$

and the **seminorm** $| \cdot |_{H^k}$ of order k is defined by

$$|u|_{H^k}^2 = \big(|u_1|_k^\Omega\big)^2 + \big(|u_2|_{k,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(|u_j|_k^I\big)^2$$

Theorem

Consider the interpolation operator Π defined above. For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{H^m} \leq \widehat{C} h^{4-m} |u|_{H^4} \, , \quad m = 0, 1, \ldots . 4.$$

Proof

We use the results in Section 4.3.

$$\begin{aligned}
\|u - \Pi u\|_{H^m}^2 &= \big(\|u_1 - \Pi_b \, u_1\|_m^\Omega\big)^2 + \big(\|u_2 - \Pi_B \, u_2\|_{m,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(\|u_j - \Pi_b \, u_j\|_m^I\big)^2 \\
&\leq \Big(\widehat{C}_1 h^{4-m} |u_1|_m^\Omega\Big)^2 + \Big(\widehat{C}_2 h^{4-m} |u_2|_{m,2}^\Omega\Big)^2 + \sum_{j=3}^{6} \Big(\widehat{C}_j h^{4-m} |u_j|_m^I\Big)^2 \\
&\leq \Big(\widehat{C} h^{4-m}\Big)^2 \Big[\big(|u_1|_m^\Omega\big)^2 + \big(|u_2|_{m,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(|u_j|_m^I\big)^2 \Big] \\
&= \Big[\widehat{C} h^{4-m} |u|_{H^4}\Big]^2
\end{aligned}$$

Corollary

For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in V \cap H^k$,

$$\|u - \Pi u\|_V \leq \widehat{C} h^3 |u|_{H^4} \, .$$

Proof

The norms $\| \cdot \|_{H^1}$ and $\| \cdot \|_V$ are equivalent (Theorem 2 Section 3.6).

Chapter 5

Approximation

5.1 Projections

For the spaces H^k and V as defined Sections 4.3, 4.4 and 4.5, we have the situation that for all our model problems a finite dimensional subspace S^h of V is constructed in such a way that the forced boundary conditions are met. At this stage an estimate for the interpolation error $u - \Pi u$ is available.

All the convergence results in this chapter are based on projection methods.

Definition (Projection P_h)

For each $x \in V$, we define $P_h x$ to be the unique element of S^h such that

$$b(x - P_h x, v) = 0 \quad \text{for all } v \in S^h.$$

It is well known and easy to prove that

$$b(x - P_h x, v) = 0 \quad \text{for all } v \in S^h$$

if and only if

$$\|x - P_h x\|_V \leq \|x - v\|_V \quad \text{for all } v \in S^h\,.$$

Since S^h is a finite dimensional subspace of the space V, the projection exists. This is a result from linear algebra (see e.g. [Ap, Chapter 15]). The result is also true for an infinite dimensional subspace (see e.g. [Kr, Sec 3.3]).

We display for convenience the elementary yet important properties of the projection P_h.

$$\begin{aligned} \|x - P_h x\|_V &\leq \|x - v\|_V \quad \text{for all } v \in S^h, \\ \|P_h x - v\|_V &\leq \|x - v\|_V \quad \text{for all } v \in S^h, \\ \text{and} \quad \|P_h x\|_V &\leq \|x\|_V. \end{aligned}$$

5.1.1 One-dimensional models

For the one-dimensional models, we consider only eigenvalue problems. The solutions of the differential equations are in $C^\infty(\bar{\Omega})$ and hence in $H^4(\Omega)$. This implies that the eigenvectors of the weak problem are in the product space H^4.

Theorem 1

Suppose the energy norm is equivalent to the norm of H^m on V. Then there exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_{4,H^4}$ and $\|\Pi u - P_h u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_{4,H^4}$.

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^{2(4-m)}\, |u|_{4,H^4}$.

Remark

Problems VRE 3, VRE 4, VTE 3 and VTE 4 are defined in Section 6.2. For Problems VRE 3 and VRE 4 we have that $m = 2$ and for Problems VTE 3 and VTE 4 we have $m = 1$.

Proof

(a) It follows from the properties of the projection operator P_h that

$$\|P_h u - u\|_V \leq \|\Pi u - u\|_V \quad \text{and} \quad \|\Pi u - P_h u\|_V \leq \|\Pi u - u\|_V .$$

The estimates are found from Corollaries 1 and 2 in Section 4.4.

(b) Set $e_p = u - P_h u$. As b defines an inner product on V, it follows from Riesz's theorem that there exists a unique $u \in V$ such that

$$b(u, v) \quad = \quad c(e_p\,, v) \quad \text{for all } v \in V. \tag{5.1.1}$$

Regularity results yield that $u \in H^4 \cap V$ and that there exists a c_b such that

$$\|u\|_4 \quad \leq \quad c_b \|e_p\|_X\,. \tag{5.1.2}$$

Since P_h is a projection,

$$b(e_p, v) \quad = \quad 0 \quad \text{for all } v \in S. \tag{5.1.3}$$

Let $v = e_p$ in Equation (5.1.1) and $v = P_h u$ in Equation (5.1.3). This yields

$$\|e_p\|_X^2 = b(u - P_h u, e_p) \leq \|u - P_h u\|_V \|e_p\|_V\,.$$

From part (a) of the Theorem, it follows that

$$\|e_p\|_X^2 \leq \widehat{C}\, h^2\, |u|_{4,H^4}\, \|e_p\|_V\,.$$

We conclude from Inequality (5.1.2) that

$$\|e_p\|_X \leq c_b\, \widehat{C} h^{4-m} \|e_p\|_V\,.$$

The result now follows from part (a) of the Theorem. □

Remark

The proof of part (b) of the Theorem is known as the Aubin-Nitsche trick ([Au] and [N]. This version is from the book of Strang and Fix ([SF, p 166]).

5.1.2 Two-dimensional models

The first result concerns Problems CTD 1 and CTD 2.

Theorem 2

There exists a constant $\widehat{C}$ such that, for any $u \in H^4(\Omega)^2 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^3\, |u|_{4,2}$ and $\|\Pi_B u - P_h u\|_V \leq \widehat{C}\, h^3\, |u|_{4,2}\,.$

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^6\, |u|_{4,2}\,.$

Proof

The proof is similar to the proof of Theorem 1. $\square$

The next result applies to Problems RMT and KEB.

Theorem 3

Suppose the energy norm is equivalent to the norm of H^m on V. Then there exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_4$ and $\|\Pi u - P_h u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_4\,.$

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^{2(4-m)}\, |u|_4\,.$

Proof

The proof is similar to the proof of Theorem 1. $\square$

For the two-dimensional problems, regularity can not be guaranteed, i.e. a solution may be in the space V but not in H^4. The following theorem is applicable in the case that u is not an element of one of the H^4-spaces as defined above.

Theorem 4

For any $\epsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that

$$\|u - P_h u\|_V < \epsilon \quad \text{if } h < \delta.$$

Proof

For any $u \in V$ there exists a $w \in H^4 \cap V$ such that $\|u - w\|_V \leq \epsilon$. Then

$$\begin{aligned} \|P_h u - u\|_V &\leq \|u - w\|_V + \|w - P_h w\|_V + \|P_h w - P_h u\|_V \\ &\leq \epsilon + \widehat{C}\, h^2\, |w|_4 + \epsilon \\ &< 3\epsilon \quad \text{for } h \text{ sufficiently small.} \end{aligned}$$

$\square$

5.2 Equilibrium problems

We consider the convergence of the Galerkin approximation of Problem CTD 1 to the solution of Problem CTD 1.

Assume that $u^h \in S^h$ is the solution of

$$b(u^h, v) \quad = \quad f(v) \quad \text{for all } v \in S^h \tag{5.2.1}$$

and that $u \in V$ is the solution of

$$b(u, v) = f(v) \quad \text{for all } v \in V. \tag{5.2.2}$$

Theorem

(a) If $u \in V$, then $\|u - u^h\|_V \longrightarrow 0$ as $h \longrightarrow 0$.

(b) If $u \in H^4 \cap V$, then

$$\|u - u^h\|_V \leq \widehat{C}\, h^3\, |u|_{4,2} \quad \text{and} \quad \|u - u^h\|_X \leq \widehat{C}\, h^6\, |u|_{4,2}\,.$$

Proof

Subtracting Equation (5.2.1) from Equation (5.2.2), we find that

$$b(u - u^h, v) = 0 \quad \text{for all } v \in S^h\,.$$

Hence $u^h = P_h u$. Therefore $\|u - u^h\|_V = \|u - P_h u\|_V$ and the result follows from Theorem 4 Section 5.1.

5.3 Symmetrical eigenvalue problems

We consider the eigenvalue problem E1 in Section 3.9. The seminorm $|\cdot|_4$ used in this paragraph is general and used for a unified formulation of the theory. When applying the theory to Problem CDT 2, this seminorm is substituted by $|\cdot|_{4,2}$ and for Problem RMT by $|\cdot|_{4,H^4}$. A similar situation holds for the use of H^4.

Regularity assumption

The eigenvectors are in H^4 and there exists a constant C_b depending on the bilinear forms b and c, such that for each eigenvector y,

$$|y|_k \leq C_b \, \lambda \, \|y\|_X .$$

The Rayleigh quotient can be used to order the sequence of eigenvalues. Assume the eigenvalues are ordered as

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

Consider the eigenvalues $\lambda_1 , \lambda_2 , \ldots , \lambda_m$ for some m with the corresponding normalized eigenvectors $y_1 , y_2 , \ldots , y_m$. Assume furthermore that $\lambda_j \neq \lambda_m$ if $j > m$ ($\lambda_i = \lambda_j$ is possible for $i \leq m$ and $j \leq m$).

Corresponding to this situation, we have the eigenvalues $\lambda_1^h , \lambda_2^h , \ldots , \lambda_m^h$ (also ordered) and the corresponding eigenvectors $y_1^h , y_2^h , \ldots , y_m^h$ in S^h. In the case of a multiple eigenvalue, the eigenvector is not uniquely determined. The following three theorems are from [SF]. In [ZVGV2] and [Ziet] it was shown that the results are applicable in the general abstract case.

Theorem 1

$\lambda_i^h \geq \lambda_i$ for each i.

Theorem 2

(a) $\lambda_m^h \longrightarrow \lambda_m$ as $h \longrightarrow 0$.

(b) If the regularity assumption holds, then $\lambda_m^h - \lambda_m \leq \widehat{C}\, C_b \, \lambda_m^2 \, h^{2(4-m)}$.

We assume that the sequence of eigenvector approximations $\left\{ y_j^h \right\}$ is normalized.

Theorem 3

Suppose that the dimension of the eigenspace E_m corresponding to λ_m is r.

(a) Let $\epsilon > 0$. For h sufficiently small, there exists a $y \in E_m$ with $\|y\| = 1$ such that

$$\|y - y^h_{m-r+j}\| \leq \epsilon$$

for $j = 1, 2, \ldots, r$.

(b) Suppose Problem E1 satisfies the regularity assumption. If h is sufficiently small, there exists a $y \in E_m$ with $\|y\| = 1$ such that

$$\|y - y^h_{m-r+j}\|_V \leq \widehat{C}\, C_b\, \lambda_m\, h^{4-m}$$

for $j = 1, 2, \ldots, r$.

5.4 Non selfadjoint eigenvalue problem

In this section we consider Problem E2 formulated in Section 3.10.

5.4.1 Abstract eigenvalue problem

Following [VV], we introduce a linear operator Λ on H with the property that the eigenvalues of Λ are the reciprocals of the eigenvalues of Problem E2 and the eigenvectors are the same.

Recall that X and V are complex Hilbert spaces with V dense in X. Also, H is the product space $V \times X$ with inner product

$$(x, y)_H = b(x_1, y_1) + c(x_2, y_2).$$

Theorem 1

Suppose

(a) V is dense in X,

(b) $\|u\|_X \leq K\|u\|_V$ for each $u \in V$,

(c) the bilinear form a is symmetric, nonnegative and $|a(u, v)| \leq C\|u\|_V\|v\|_V$ for each u and v in V.

Then, for each $y \in H$, there exists a unique $x \in H$ such that

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

Proof

Appendix 5.

Definition Operator (Λ)

$\Lambda y = x$ if

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

It is easy to see that Λ is linear.

Theorem 2

Λ is bounded.

Proof

Appendix 5.

Theorem 3

λ is an eigenvalue and x an eigenvector of Problem E2 if and only if $\lambda \Lambda x = x$.

Proof

Simply substitute $y = \lambda x$ in the definition of Λ.

Theorem

Λ is invertible and its range is dense in H.

Proof

See [VV].

Remark

We may define a linear operator $T = \Lambda^{-1}$. It is clear that T is a closed linear operator with domain $\mathcal{D}(T)$ which is dense in H. As a consequence one may study the eigenvalue problem $Tx = \lambda x$. This problem is equivalent to the problem considered in [Shu].

5.4.2 Galerkin approximation

Consider a finite dimensional subspace S^h of the complex Hilbert space V. The following problem yields the approximations for the quadratic eigenvalue problem QE.

Problem QED

Find a complex number λ_h and $u^h \in S^h$ such that

$$\lambda_h^2 c(u^h, v) + \lambda_h a(u^h, v) + b(u^h, v) = 0 \quad \text{for each } v \in S^h.$$

This is the type of problem solved in Chapter 6.

Definition (Subspace H^h)

$$H^h = S^h \times S^h.$$

Problem E2D

Find a complex number λ_h and $x^h \in H^h$ such that

$$\begin{aligned} x_2^h &= \lambda^h x_1^h \\ b(x_1^h, v) + a(x_2^h, v) &= -\lambda^h c(x_2^h, v) \quad \text{for each } v \in S^h \, . \end{aligned}$$

If λ_h is an eigenvalue and u^h an eigenvector of Problem QED, then λ_h is an eigenvalue and $\langle u^h, \lambda u^h \rangle$ an eigenvector of Problem E2D. Conversely, if λ_h is an eigenvalue and x^h an eigenvector of Problem E2D, then λ_h is an eigenvalue and x_1^h an eigenvector of Problem QED.

Projection

Recall the projection P^h defined in Section 4.1. Without changing the notation, we define a projection for the complex space V by $P^h x = P^h x_1 + i P^h x_2$. It is clear that we still have the following properties.

$$\begin{aligned} b(x - P^h x, v) = 0 \quad &\text{for each } v \in S^h \, , \\ \|x - P^h x\|_V \leq \|x - v\|_V \quad &\text{for each } v \in S^h \, . \end{aligned}$$

5.4.3 Operator approximations

Let $y \in H$ and consider the problem to find $u^h \in S^h$ such that

$$b(u^h, v) + a(y_1, v) = -c(y_2, v) \quad \text{for each } v \in S^h.$$

It is clear that a unique solution exists (see Theorem 1).

Definition (Operator Λ^h)

$\Lambda^h y = x$ if $x_1 \in S^h$ and

$$\begin{aligned} x_2 &= y_1, \\ b(x_1, v) + a(y_1, v) &= -c(y_2, v) \quad \text{for each } v \in S^h \, . \end{aligned}$$

It is easy to see that Λ^h is linear.

Theorem 5

Λ^h is bounded and the restriction of Λ^h to $S^h \times S^h$ is a bijection.

Proof

The same as the proof of Theorem 2.

Theorem 6

λ^h is an eigenvalue and x^h an eigenvector of Problem E2D if and only if $\lambda^h \Lambda^h x^h = x^h$.

Proof

Simply substitute $y = \lambda x^h$ in the definition of Λ^h.

Remark

It is clear that Λ^h has a zero eigenvalue since $N(\Lambda^h) = \left(S^h \times S^h\right)^\perp$.

Notation

$$\delta^h(x) = \inf\left\{\|x_1 - v\|_V \;\middle|\; v \in S^h\right\}.$$

Remark

In general, $\delta^h(x) \to 0$ as $h \to 0$ for each $x \in H$.

Theorem 7

If $\Lambda y = x$, then
$$\|\Lambda^h y - \Lambda y\|_H \leq \delta^h(x).$$

Proof

If $\Lambda^h y = x^h$, then

$$b(x_1 - x_1^h, v) = 0 \quad \text{for each } v \in S^h.$$

5.4.4 Convergence

Consider a sequence of operators $\Lambda_n = \Lambda^{h_n}$ where $h_n \to 0$.

Notation

Let λ be an isolated eigenvalue of Λ, P the spectral projection and $M = PH$ the invariant subspace associated with λ. Assume that $\dim M = m < \infty$. There exists a $\rho > 0$ such that λ is the only eigenvalue in $B_\rho(\lambda)$. M_n denotes the invariant subspace of Λ_n associated with the m eigenvalues (counting multiplicity) contained in $B_\rho(\lambda)$.

Theorem 8

Suppose that $\{\Lambda_n\}$ is a strongly stable approximation of Λ in $B_\rho(\lambda)$. Then, for n sufficiently large, Λ_n has m eigenvalues in $B_\rho(\lambda)$, counting their multiplicities. All these eigenvalues converge to λ as $n \longrightarrow \infty$.

Proof

See [Ch, p 234].

Definition (Gap between subspaces)

P is an orthogonal projection on M,

Q is an orthogonal projection on M_n,

$\alpha = \sup \left\{ \|x - Qx\|_H \;\middle|\; x \in M;\; \|x\|_H = 1 \right\}$,

$$\beta = \sup \left\{ \|x - Px\|_H \;\middle|\; x \in M_n;\ \|x\|_H = 1 \right\},$$

$$\Theta(M, M_n) = \max \{\alpha, \beta\}$$

Remark

If M and M_n are one-dimensional (as is mostly the case in our applications), then $\Theta(M, M_n) = \sin\theta$ where θ is the angle between M and M_n.

Theorem 9

If Λ_n is an approximation of Λ and strongly stable on $\overline{B_\rho(\lambda)}$, then $\Theta(M, M_n) \longrightarrow 0$ as $n \to \infty$.

Proof

[Ch, p 235-236].

5.4.5 Application

We apply the theory to the one-dimensional hybrid models in Sections 3.2 and 3.3. Consider for example Problem VTE 4 with weak variational form in Section 3.3.2. In this case, the quadratic eigenvalue problem Problem QE and its equivalent form Problem E2 involves ordinary differential equations. Any eigenvector for Problem QE is in $C^\infty[0,1] \times C^\infty[0,1] \times I\!R^3$. The error bounds for the projection P^h in Section 5.1 are valid. Also, the operator Λ associated with Problem $E2$ is compact.

Convergence

Theorem 10

For $\mu \in B_\rho(\lambda)$, $\mu \neq 0$ and $\mu \neq \lambda$, $\mu I - \Lambda_n$ is a strongly stable approximation for $\mu I - \Lambda$.

Proof

Since $(\mu I - \Lambda)^{-1}$ exists and $\mu I - \Lambda_n$ converges pointwise to $\mu I - \Lambda$, it follows that $(\mu I - \Lambda_n)^{-1}$ converges pointwise to $(\mu I - \Lambda)^{-1}$. But Λ_n converges compactly to Λ ([Ch, p 122]). Consequently, $\mu I - \Lambda_n$ is a strongly stable approximation of $\mu I - \Lambda$ for $\mu \neq 0$ (Lemma 5.24 and Theorem 5.26 ([Ch, p 247-248])). Finally, Proposition 5.27 ([Ch, p 248-249]) implies the result.

Remark

Theorems 8 and 9 may now be applied.

Error bounds

The theory in [Ch, Sec 6.2] on projection methods, is applicable to our situation.

Notation

$$\widehat{\lambda}_n = \frac{1}{m}\sum_{j=1}^{m} \lambda_j, \text{ where } \lambda_j \in B_\delta(\lambda).$$

$$\delta_n(x) = \delta^h(x) \text{ where } h = h_n \text{ and } H_n = H^{h_n}.$$

$$\delta\left(M, H_n\right) = \sup\left\{\delta_n(x) \;\middle|\; x \in M;\ \|x\|_H = 1\right\}.$$

Theorem 11

Consider Problem E2 for the system in Problem VTE 4. Then

$$\begin{aligned} \left|\lambda - \widehat{\lambda}_n\right| &\leq K\delta\left(M, H_n\right), \\ \Theta\left(M, M_n\right) &\leq K\delta\left(M, H_n\right). \end{aligned}$$

Proof

See Lemmas 6.9 and 6.10 in [Ch, p 284].

Theorem 12

There exists a constant C_λ such that $\delta\,(M, H_n) \leq C_\lambda h_n\,.$

Proof

Note that for each $u \in H$, we have (Theorem 1, Section 5.1)

$$\delta_n(u) \leq \widehat{C}\, h_n\, |u|_2\,.$$

But,

$$\begin{aligned} -u_1'' + u_2' &= \lambda u_1, \\ -\frac{1}{\gamma}u_2'' - \alpha u_1' + \alpha u_2 &= \lambda u_2\,. \end{aligned}$$

Consequently,

$$|u|_2 \leq K_\lambda \|u\|_V \leq K_\lambda \|u\|_H$$

for some constant K_λ.

Consequently, there exists a constant C_λ such that

$$\delta(M, H_n) \leq C_\lambda\, \widehat{C}\, h_n\,.$$

Chapter 6

Vertical slender structure

6.1 Introduction

Newland claims in [N2] that the inclusion of a resilient seating increases the complexity of a finite element analysis. However, in the way that we model the resilient seating, the application of the finite element method doesn't become more complicated. It involves only the introduction of some extra variables. The effect is, that in comparison with the models for a rigid base, the inertia, bending and damping matrices slightly increase in size and the entries of the mentioned matrices change at only a few entries.

We show in this chapter how to implement the finite element method for approximating the eigenvalues for Problems VR 4 and VT 4. Results are compared to results in [LVV].

We also study the effect of the "gravity"-term described in Chapter 2 by Equations (2.1.4) and (2.1.17), i.e. the constitutive equation

$$L(x,t) = \mu(1-x)\partial_x w(x,t).$$

6.2 The eigenvalue problem

6.2.1 The Rayleigh model

For the modal analysis of the system, $\tilde{w}(x,t) = e^{\lambda t}w(x)$ is considered as a possible solution. For the two different models under consideration (models 3 and 4), the additional variables are handled in a similar way as for w, as can be seen below. This requires consideration of the corresponding eigenvalue problems.

For Problem VR 3, we consider $\tilde{\theta}_F(t) = e^{\lambda t}\theta_F$ as a possible solution.

Variational form for Problem VRE 3

Find w and θ_F such that $w \in T(0,1)$,

$$\begin{aligned}\lambda^2 c_A(w,v) + b_A(w,v) + \lambda C_F w(0)v(0) - k\theta_F v'(0) \\ +\lambda c\Big(w'(0) - \theta_F\Big)v'(0) &= 0 \end{aligned} \tag{6.2.1}$$

holds for each $v \in T(0,1)$ and

$$\begin{aligned}\lambda^2 I_F\theta_F - k\Big(w'(0) - \theta_F\Big) - \lambda c\Big(w'(0) - \theta_F\Big) \\ +k_F\theta_F + \lambda c_F\theta_F &= 0. \end{aligned} \tag{6.2.2}$$

For Problem VR 4 consider the following possible solutions.

$$\tilde{w}_F(t) = e^{\lambda t}w_F, \quad \tilde{\theta}_B(t) = e^{\lambda t}\theta_B \quad \text{and} \quad \tilde{\theta}_F(t) = e^{\lambda t}\theta_F.$$

Variational form for Problem VRE 4

Find w, w_F, θ_B and θ_F such that $w \in T(0,1)$,

$$\begin{aligned}\lambda^2 c_A(w,v) + b_A(w,v) - K_{FB}w_F v(0) + \lambda C_{FB}\Big(w(0) - w_F\Big)v(0) \\ +\lambda c_{BA}\Big(w'(0) - \theta_B\Big)v'(0) - k_{BA}\theta_B v'(0) &= 0 \end{aligned} \tag{6.2.3}$$

holds for each $v \in T(0,1)$ and

$$\lambda^2 I_B\theta_B - k_{BA}\Big(w'(0) - \theta_B\Big) - \lambda c_{BA}\Big(w'(0) - \theta_B\Big)$$
$$+k_{FB}\Big(\theta_B - \theta_F\Big) + \lambda c_{FB}\Big(\theta_B - \theta_F\Big) \;=\; 0, \tag{6.2.4}$$

$$\lambda^2 m_F w_F - K_{FB}\Big(w(0) - w_F\Big) - \lambda C_{FB}\Big(w(0) - w_F\Big)$$
$$+K_F w_F + \lambda C_F w_F \;=\; 0, \tag{6.2.5}$$

$$\lambda^2 I_F\theta_F - k_{FB}\Big(\theta_B - \theta_F\Big) - \lambda c_{FB}\Big(\theta_B - \theta_F\Big)$$
$$+k_F\theta_F + \lambda c_F\theta_F \;=\; 0. \tag{6.2.6}$$

Remark

Note that the bilinear forms used in the formulation above, are the bilinear forms defined for the variational form and not the weak variational form.

6.2.2 The Timoshenko model

Here we consider $\tilde{w}(x,t) = e^{\lambda t}w(x)$ and $\tilde{\phi}(x,t) = e^{\lambda t}\phi(x)$ as possible solutions and follow the same approach as in the Rayleigh models to formulate Problems VTE 3 and VTE 4.

We consider $\tilde{\theta}_F(t) = e^{\lambda t}\theta_F$ as a possible solution for Problem VT 3.

Problem VTE 3

Find w, ϕ and θ_F such that $w \in T(0,1)$ and $\phi \in T(0,1)$,

$$\lambda^2(w,v) + \lambda^2 m_F w(0)v(0) + (w' - \phi, v') + K_F w(0)v(0)$$
$$+\lambda C_F w(0)v(0) \;=\; 0 \tag{6.2.7}$$

holds for each $v \in T(0,1)$,

$$\frac{\lambda^2}{\alpha}\,(\phi,\psi) + \frac{1}{\beta}\,(\phi',\psi') - (w' - \phi,\psi) - \mu\int_0^1 (1-x)w'(x)\psi(x)\,dx$$
$$+k\Big(\phi(0) - \theta_F\Big)\psi(0) + \lambda c\Big(\phi(0) - \theta_F\Big)\psi(0) \;=\; 0, \tag{6.2.8}$$

holds for each $\psi \in T(0,1)$ and

$$\lambda^2 I_F \theta_F - k\Big(\phi(0) - \theta_F\Big) - \lambda c\Big(\phi(0) - \theta_F\Big) + k_F\theta_F + \lambda c_F \theta_F = 0. \tag{6.2.9}$$

For Problem VT 4, consider the following possible solutions.

$$\tilde{w}_F(t) = e^{\lambda t} w_F, \quad \tilde{\theta}_B(t) = e^{\lambda t}\theta_B \;\text{ and }\; \tilde{\theta}_F(t) = e^{\lambda t}\theta_F.$$

Variational form for Problem VTE 4

Find w, ϕ, w_F, θ_B and θ_F such that $w \in T(0,1)$ and $\phi \in T(0,1)$,

$$\lambda^2(w,v) + \lambda^2 m_B w(0)v(0) + (w' - \phi, v') + K_{FB}\Big(w(0) - w_F\Big)v(0) + \lambda C_{FB}\Big(w(0) - w_F\Big)v(0) = 0 \tag{6.2.10}$$

holds for each $v \in T(0,1)$,

$$\frac{\lambda^2}{\alpha}\,(\phi,\psi) + \frac{1}{\beta}\,(\phi',\psi') - (w' - \phi, \psi) - \mu\int_0^1 (1-x)w'(x)\psi(x)\,dx + k_{BA}\Big(\phi(0) - \theta_B\Big)\psi(0) + \lambda c_{BA}\Big(\phi(0) - \theta_B\Big)\psi(0) = 0 \tag{6.2.11}$$

holds for each $\psi \in T(0,1)$ and

$$\lambda^2 I_B\theta_B - k_{BA}\Big(\phi(0) - \theta_B\Big) - \lambda c_{BA}\Big(\phi(0) - \theta_B\Big) + k_{FB}\Big(\theta_B - \theta_F\Big) + \lambda c_{FB}\Big(\theta_B - \theta_F\Big) = 0, \tag{6.2.12}$$

$$\lambda^2 m_F w_F - K_{FB}\Big(w(0) - w_F\Big) - \lambda C_{FB}\Big(w(0) - w_F\Big) + K_F w_F + \lambda C_F w_F = 0, \tag{6.2.13}$$

$$\lambda^2 I_F\theta_F - k_{FB}\Big(\theta_B - \theta_F\Big) - \lambda c_{FB}\Big(\theta_B - \theta_F\Big) + k_F\theta_F + \lambda c_F\theta_F = 0. \tag{6.2.14}$$

6.3 Galerkin approximations for the eigenvalue problem

6.3.1 Rayleigh models

The interval $[0, 1]$ is divided in n subintervals of the same length. The approximate solution is denoted by w^h and written in terms of cubic basis functions δ_j as

$$w^h(x) = \sum_{j=1}^{2n+2} \delta_j(x)\, w_j\,.$$

For Problem VRE 3, substitute w^h into the variational form given by Equations (6.2.1) and (6.2.2) and take $v = \delta_i$ for $i = 1, 2, \ldots, 2n+2$. This results in the following eigenvalue problem, with θ_F as an additional unknown.

Galerkin approximation for Problem VRE 3

$$\lambda^2 \sum_{j=1}^{2n+2} c_A(\delta_j, \delta_i) w_j + \sum_{j=1}^{2n+2} b_A(\delta_j, \delta_i) w_j + \lambda C_F w_1 \delta_i(0) - k\,\theta_F\,\delta_i'(0) + \lambda c\Big(w_{n+2} - \theta_F\Big)\delta_i'(0) = 0, \tag{6.3.1}$$

$$\lambda^2 I_F \theta_F - k\Big(w_{n+2} - \theta_F\Big) - \lambda c\Big(w_{n+2} - \theta_F\Big) + k_F\theta_F + \lambda c_F \theta_F = 0. \tag{6.3.2}$$

Equations (6.3.1) and (6.3.2) describe an eigenvalue problem for which the relevant matrices are $(2n + 3) \times (2n + 3)$ matrices.

The explicit appearance of w_1 and w_{n+2} in these equations are due to the fact that $\delta_i(0) = 0$ unless $i = 1$ and $\delta_i'(0) = 0$ unless $i = n + 2$.

The same procedure is followed as for Problem VRE 3, using Equations (6.2.3) – (6.2.6). This yields a $(2n + 5) \times (2n + 5)$ eigenvalue problem, with θ_B, w_F and θ_F as three extra unknowns.

Galerkin approximation for Problem VRE 4

$$\lambda^2 \sum_{j=1}^{2n+2} c_A(\delta_j,\delta_i)w_j + \sum_{j=1}^{2n+2} b_A(\delta_j,\delta_i)w_j - K_{FB}w_F\delta_i(0)$$
$$+\lambda C_{FB}\Big(w_1 - w_F\Big)\delta_i(0) + \lambda c_{BA}\Big(w_{n+2} - \theta_B\Big)\delta_i'(0)$$
$$-k_{BA}\theta_B\delta_i'(0) = 0, \qquad (6.3.3)$$

$$\lambda^2 I_B\theta_B - k_{BA}\Big(w_{n+2} - \theta_B\Big) - \lambda c_{BA}\Big(w_{n+2} - \theta_B\Big)$$
$$+k_{FB}\Big(\theta_B - \theta_F\Big) + \lambda c_{FB}\Big(\theta_B - \theta_F\Big) = 0, \qquad (6.3.4)$$

$$\lambda^2 m_F w_F - K_{FB}\Big(w_1 - w_F\Big) - \lambda C_{FB}\Big(w_1 - w_F\Big)$$
$$+K_F w_F + \lambda C_F w_F = 0, \qquad (6.3.5)$$

$$\lambda^2 I_F\theta_F - k_{FB}\Big(\theta_B - \theta_F\Big) - \lambda c_{FB}\Big(\theta_B - \theta_F\Big)$$
$$+k_F\theta_F + \lambda c_F\theta_F = 0. \qquad (6.3.6)$$

6.3.2 Timoshenko models

The interval $[0,1]$ is divided in n subintervals of the same length. The approximate solutions are denoted by w^h and ϕ^h. Written in terms of the basis functions we have

$$w^h(x) = \sum_{j=1}^{2n+2} \delta_j(x)\, w_j \quad \text{and} \quad \phi^h(x) = \sum_{j=1}^{2n+2} \delta_j(x)\, \phi_j\,.$$

Following the same line of reasoning as in the Rayleigh models, we substitute w^h and ϕ^h into the variational form equations, Equations (6.2.7) $-$ (6.2.9). Furthermore, we let $v = \delta_i$ and $\psi = \delta_i$ for $i = 1, 2, \ldots, 2n+2$. This yields the following $(4n+5)\times(4n+5)$ eigenvalue problem.

Galerkin approximation for Problem VTE 3

$$\lambda^2 \sum_{j=1}^{2n+2} (\delta_j, \delta_i) w_j + \lambda^2 m_F w_1 \delta_i(0) + \sum_{j=1}^{2n+2} (\delta_j', \delta_i') w_j$$
$$- \sum_{j=1}^{2n+2} (\delta_j, \delta_i') \phi_j + K_F w_1 \delta_i(0) + \lambda C_F w_1 \delta_i(0) = 0, \tag{6.3.7}$$

$$\frac{\lambda^2}{\alpha} \sum_{j=1}^{2n+2} (\delta_j, \delta_i) \phi_j + \frac{1}{\beta} \sum_{j=1}^{2n+2} (\delta_j', \delta_i') \phi_j - \sum_{j=1}^{2n+2} (\delta_j', \delta_i) w_j$$
$$+ \sum_{j=1}^{2n+2} (\delta_j, \delta_i) \phi_j - \mu \sum_{j=1}^{2n+2} \Big(\int_0^1 (1-x) \delta_j'(x) \delta_i(x)\, dx \Big) w_j$$
$$+ k \Big(\phi_1 - \theta_F \Big) \delta_i(0) + \lambda c \Big(\phi_1 - \theta_F \Big) \delta_i(0) = 0, \tag{6.3.8}$$

$$\lambda^2 I_F \theta_F - k \Big(\phi_1 - \theta_F \Big) - \lambda c \Big(\phi_1 - \theta_F \Big) + k_F \theta_F + \lambda c_F \theta_F = 0. \tag{6.3.9}$$

Substituting w^h and ϕ^h into the variational form (6.2.10) – (6.2.14) and taking $v = \delta_i$ and $\psi = \delta_i$ for $i = 1, 2, \ldots, 2n+2$, the following eigenvalue problem is found.

Galerkin approximation for Problem VTE 4

$$\lambda^2 \sum_{j=1}^{2n+2} (\delta_j, \delta_i) w_j + \lambda^2 m_B w_1 \delta_i(0) + \sum_{j=1}^{2n+2} (\delta_j', \delta_i') w_j - \sum_{j=1}^{2n+2} (\delta_j, \delta_i') \phi_j$$
$$+ K_{FB}\Big(w_1 - w_F\Big)\delta_i(0) + \lambda C_{FB}\Big(w_1 - w_F\Big)\delta_i(0) = 0, \tag{6.3.10}$$

$$\frac{\lambda^2}{\alpha} \sum_{j=1}^{2n+2} (\delta_j, \delta_i) \phi_j + \frac{1}{\beta} \sum_{j=1}^{2n+2} (\delta_j', \delta_i') \phi_j - \sum_{j=1}^{2n+2} (\delta_j', \delta_i) w_j + \sum_{j=1}^{2n+2} (\delta_j, \delta_i) \phi_j$$
$$- \mu \sum_{j=1}^{2n+2} \Big(\int_0^1 (1-x) \delta_j'(x) \delta_i(x)\, dx \Big) w_j + k_{BA}\Big(\phi_1 - \theta_B\Big)\delta_i(0)$$
$$+ \lambda c_{BA}\Big(\phi_1 - \theta_B\Big)\delta_i(0) = 0, \tag{6.3.11}$$

$$\lambda^2 I_B \theta_B - k_{BA}\Big(\phi_1 - \theta_B\Big) - \lambda c_{BA}\Big(\phi_1 - \theta_B\Big)$$
$$+ k_{FB}\Big(\theta_B - \theta_F\Big) + \lambda c_{FB}\Big(\theta_B - \theta_F\Big) = 0, \tag{6.3.12}$$

$$\lambda^2 m_F w_F - K_{FB}\Big(w_1 - w_F\Big) - \lambda C_{FB}\Big(w_1 - w_F\Big)$$
$$+ K_F w_F + \lambda C_F w_F = 0, \tag{6.3.13}$$

$$\lambda^2 I_F \theta_F - k_{FB}\Big(\theta_B - \theta_F\Big) - \lambda c_{FB}\Big(\theta_B - \theta_F\Big)$$
$$+ k_F \theta_F + \lambda c_F \theta_F = 0. \tag{6.3.14}$$

Equations (6.3.10) – (6.3.14) form a $(4n + 7) \times (4n + 7)$ system of linear equations.

6.4 Matrix form of the semi-discrete problem

All four eigenvalue problems result in a quadratic eigenvalue problem of the form

$$\left(\lambda^2\mathcal{M} + \lambda\mathcal{D} + \mathcal{K}\right)\boldsymbol{w} = \mathbf{0}.$$

The inertia matrix $\mathcal{M}$, the bending matrix $\mathcal{K}$ and the matrix $\mathcal{D}$ due to damping are found from the variational forms for the problems. The construction of the matrices is described below.

6.4.1 The Rayleigh models

Define the $(2n+2)\times(2n+2)$ matrices M and K by

$$M_{ij} = c_A(\delta_j, \delta_i) \quad \text{and} \quad K_{ij} = b_A(\delta_j, \delta_i)\,.$$

Problem VRE 3

Let $\theta_F = w_{2n+3}$ and define

$$\boldsymbol{w} = [w_1\ w_2\ \cdots\ w_{2n+2}\ w_{2n+3}]^{\,T}$$

where "T" denotes the transpose of a matrix.

Let O be the $1\times(2n+2)$ zero matrix and V a $1\times(2n+2)$ matrix with all zero entries except for $V_{1,n+2} = -k$. Then

$$\mathcal{M} = \begin{bmatrix} M & O^T \\ O & I_F \end{bmatrix} \quad \text{and} \quad \mathcal{K} = \begin{bmatrix} K & V^T \\ V & (k+k_F) \end{bmatrix}.$$

Define the matrix $D^{(1)}$ as the $(n+1)\times(n+1)$ matrix with zeros everywhere except for entry $(1,1)$, for which $D^{(1)}_{11} = C_F$.

Define $D^{(2)}$ as the $(n+2)\times(n+2)$ matrix with zero entries except for $D^{(2)}_{11} = c$, $D^{(2)}_{1,n+2} = D^{(2)}_{n+2,1} = -c$ and $D^{(2)}_{n+2,n+2} = c + c_F$.

Let O be the zero matrix of size $(n+2)\times(n+1)$. Then

$$\mathcal{D} = \begin{bmatrix} D^{(1)} & O^T \\ O & D^{(2)} \end{bmatrix}.$$

Problem VRE 4

Let $\theta_B = w_{2n+3},\ \ w_F = w_{2n+4},\ \ \theta_F = w_{2n+5}$ and define

$$\boldsymbol{w} = [w_1\ w_2\ \cdots\ w_{2n+4}\ w_{2n+5}]^T .$$

O is the zero $3 \times (2n+2)$ matrix and $K^{(1)}$ a $3 \times (2n+2)$ matrix with zeros entries except for $K^{(1)}_{1,n+2} = -k_{BA}$ and $K^{(1)}_{21} = -K_{FB}$.

The following matrices are defined for the damping matrix:

A $(2n+2) \times (2n+2)$ matrix $D^{(1)}$ and a $3 \times (2n+2)$ matrix $D^{(2)}$ with zero entries except for the following values:

$$D^{(1)}_{11} = C_{FB}, \quad D^{(1)}_{n+2,n+2} = c_{BA}, \quad D^{(2)}_{1,n+2} = -c_{BA} \quad \text{and} \quad D^{(2)}_{21} = -C_{FB} .$$

Let

$$M^{(1)} = \begin{bmatrix} I_B & 0 & \\ 0 & m_F & 0 \\ 0 & 0 & I_F \end{bmatrix}, \quad K^{(2)} = \begin{bmatrix} (k_{BA}+k_{FB}) & 0 & -k_{FB} \\ 0 & (K_{FB}+K_F) & 0 \\ -k_{FB} & 0 & (k_{FB}+k_F) \end{bmatrix}$$

$$\text{and} \quad D^{(3)} = \begin{bmatrix} (c_{BA}+c_{FB}) & 0 & -c_{FB} \\ 0 & (C_{FB}+C_F) & 0 \\ -c_{FB} & 0 & (c_{FB}+c_F) \end{bmatrix}.$$

Then

$$\mathcal{M} = \begin{bmatrix} M & O^T \\ O & M^{(1)} \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K & \left(K^{(1)}\right)^T \\ K^{(1)} & K^{(2)} \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \begin{bmatrix} D^{(1)} & \left(D^{(2)}\right)^T \\ D^{(2)} & D^{(3)} \end{bmatrix}.$$

6.4.2 The Timoshenko models

The ij-th entry for the $(2n+2) \times (2n+2)$ matrices K, L, M and P are defined by

$$K_{ij} = (\delta_j', \delta_i'), \quad L_{ij} = (\delta_j, \delta_i'), \quad M_{ij} = (\delta_j, \delta_i) \quad \text{and}$$

$$P_{ij} = \int_0^1 (1-x)\delta_j'(x)\delta_i(x)\,dx.$$

Problem VTE 3

Define

$$\boldsymbol{w} = [w_1 \ w_2 \ \cdots \ w_{2n+2}]^T \quad \text{and} \quad \boldsymbol{\phi} = [\phi_1 \ \phi_2 \ \cdots \ \phi_{2n+2}]^T.$$

Define $\boldsymbol{z}$ in terms of the unknowns $\boldsymbol{w}$, $\boldsymbol{\phi}$ and w_F such that

$$\boldsymbol{z} = [\boldsymbol{w} \ \ \boldsymbol{\phi} \ \ w_F]^T.$$

The matrices $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{D}$ are all partitioned in the same way and we describe the partitioning for $\mathcal{M}$:

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix},$$

where $\mathcal{M}_{11}$, $\mathcal{M}_{12}$, $\mathcal{M}_{21}$, and $\mathcal{M}_{22}$ are all $(2n+2) \times (2n+2)$ matrices.

$\mathcal{M}_{31}$, $\mathcal{M}_{32}$, $(\mathcal{M}_{13})^T$ and $(\mathcal{M}_{23})^T$ are all $1 \times (2n+2)$ matrices and $\mathcal{M}_{33}$ is a 1×1 matrix.

Denoting the entries for the partitioned matrices with superscripts in brackets, we find the following results.

$$\mathcal{M}_{11}^{(11)} = M_{11} + m_F \quad \text{and} \quad \mathcal{M}_{11}^{(ij)} = M_{ij} \quad \text{otherwise},$$

$$\mathcal{M}_{22} = \frac{1}{\alpha} M \quad \text{and} \quad \mathcal{M}_{33} = I_F.$$

All the other partitioned matrices in $\mathcal{M}$ are zero matrices.

$$\mathcal{K}_{11}^{(11)} = K_{11} + K_F \quad \text{and} \quad \mathcal{K}_{11}^{(ij)} = K_{ij} \quad \text{otherwise},$$

$$\mathcal{K}_{12} = -L \quad \text{and} \quad \mathcal{K}_{21} = -(L^T + \mu P),$$

$$\mathcal{K}_{22}^{(11)} = \frac{1}{\beta} K_{11} + M_{11} + k \quad \text{and} \quad \mathcal{K}_{22}^{(ij)} = \frac{1}{\beta} K_{ij} + M_{ij} \quad \text{otherwise},$$

$$\mathcal{K}_{32}^{(11)} = -k \quad \text{and} \quad \mathcal{K}_{32}^{(1j)} = 0 \quad \text{otherwise},$$

$$\mathcal{K}_{23} = \mathcal{K}_{32}^T \quad \text{and} \quad \mathcal{K}_{33} = k + k_F.$$

All the other partitioned matrices in $\mathcal{K}$ are zero matrices.

$$\mathcal{D}_{11}^{(11)} = C_F \quad \text{and} \quad \mathcal{D}_{11}^{(ij)} = 0 \quad \text{otherwise},$$

$$\mathcal{D}_{22}^{(11)} = c \text{ and } \mathcal{D}_{22}^{(ij)} = 0 \text{ otherwise,}$$
$$\mathcal{D}_{32}^{(11)} = -c \text{ and } \mathcal{D}_{32}^{(1j)} = 0 \text{ otherwise,}$$
$$\mathcal{D}_{23} = (\mathcal{D}_{32})^T \text{ and } \mathcal{D}_{33}^{(11)} = c + c_F\,.$$

All the other partitioned matrices in $\mathcal{D}$ are zero matrices.

Problem VTE 4

We define $\boldsymbol{w}$ and $\boldsymbol{\phi}$ as the $4n+7$ column vector $\boldsymbol{z}$

$$\boldsymbol{z} = [\boldsymbol{w} \;\; \boldsymbol{\phi} \;\; \theta_B \;\; w_F \;\; \theta_F]^T.$$

The matrices $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{D}$ are partitioned in the same way and a description for the matrix $\mathcal{M}$ follows.

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix},$$

where $\mathcal{M}_{11},\ \mathcal{M}_{12},\ \mathcal{M}_{21},$ and $\mathcal{M}_{22}$ are all $(2n+2)\times(2n+2)$ matrices.

$\mathcal{M}_{31},\ \mathcal{M}_{32},\ (\mathcal{M}_{13})^T$ and $(\mathcal{M}_{23})^T$ are all $3\times(2n+2)$ matrices and $\mathcal{M}_{33}$ is a 3×3 matrix.

We have that

$$\mathcal{M}_{11}^{(11)} = M_{11} + m_B \text{ and } \mathcal{M}_{11}^{(ij)} = M_{ij} \text{ otherwise,}$$

$$\mathcal{M}_{22} = \frac{1}{\alpha}\, M \text{ and } \mathcal{M}_{33} = \begin{bmatrix} I_B & 0 & 0 \\ 0 & m_F & 0 \\ 0 & 0 & I_F \end{bmatrix}.$$

All the other partitioned matrices in $\mathcal{M}$ are zero matrices.

$$\mathcal{K}_{11}^{(11)} = K_{11} + K_{FB} \text{ and } \mathcal{K}_{11}^{(ij)} = K_{ij} \text{ otherwise,}$$
$$\mathcal{K}_{12} = -L \text{ and } \mathcal{K}_{21} = -(L^T + \mu P)\,,$$
$$\mathcal{K}_{22}^{(11)} = \frac{1}{\beta}\, K_{11} + M_{11} + k_{BA} \text{ and } \mathcal{K}_{22}^{(ij)} = \frac{1}{\beta}\, K_{ij} + M_{ij} \text{ otherwise,}$$
$$\mathcal{K}_{31}^{(21)} = -K_{FB} \text{ and } \mathcal{K}_{31}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{K}_{32}^{(11)} = -k_{BA} \text{ and } \mathcal{K}_{32}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{K}_{13} = \mathcal{K}_{31}^T \text{ and } \mathcal{K}_{23} = \mathcal{K}_{32}^T.$$

$$\mathcal{K}_{33} = \begin{bmatrix} (k_{BA} + k_{FB}) & 0 & -k_{FB} \\ 0 & (K_{FB} + K_F) & 0 \\ -k_{FB} & 0 & (k_{FB} + k_F) \end{bmatrix}.$$

All the other partitioned matrices in $\mathcal{K}$ are zero matrices.

$$\mathcal{D}_{11}^{(11)} = C_{FB} \text{ and } \mathcal{D}_{11}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{D}_{22}^{(11)} = c_{BA} \text{ and } \mathcal{D}_{22}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{D}_{31}^{(21)} = -C_{FB} \text{ and } \mathcal{D}_{31}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{D}_{32}^{(11)} = -c_{BA} \text{ and } \mathcal{D}_{32}^{(ij)} = 0 \text{ otherwise,}$$

$$\mathcal{D}_{13} = \mathcal{D}_{31}^T \text{ and } \mathcal{D}_{23} = \mathcal{D}_{32}^T,$$

$$\mathcal{D}_{33} = \begin{bmatrix} (c_{BA} + c_{FB}) & 0 & -c_{FB} \\ 0 & (C_{FB} + C_F) & 0 \\ -c_{FB} & 0 & (c_{FB} + c_F) \end{bmatrix}.$$

6.5 Numerical results

In [LVV], the Rayleigh and Euler-Bernoulli models were used to find the first four frequencies for the Newland chimney. The first two models were discussed in detail in [LVV] and will not be discussed here. The contribution due to gravity was approximated in [LVV], whereas we use the exact value for the integral $\int_0^1 (1-x)\partial_x w(x,t)dx$ in the finite element approximation. The effect of the approximation of the above mentioned integral to the exact value is minimal. We found that results differ with less than 1%.

Approximations for the eigenvalue problems VRE 3, VRE 4, VTE 3 and VTE 4 are found with the finite element method. MATLAB codes were written for calculating the $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{K}$ matrices for these problems and the standard MATLAB routines were used for solving the quadratic eigenvalue problems.

6.5.1 Physical constants

For the purpose of comparing our results with those of Newland, the values for the physical constants that we use are displayed in Table 1.

The results are found for a typical steel chimney of height $\ell = 42\,m$, mass $21\,000\,kg$, diameter $D = 2.25\,m$ and wall thickness $t = 6.8\,mm$. Young's modulus E is taken as $E = 2.1 \times 10^{11}$ and $\rho A = 500$. Approximations for I and A are used, with $I \approx \dfrac{\pi D^3 t}{8}$ and $A \approx \pi D t$. We choose $\dfrac{G}{E} = \dfrac{3}{8}$ and $\kappa^2 = \dfrac{2}{3}$.

Table 1: Constants

	Model 3		Model 4	
	(Physical)	(Dimensionless)	(Physical)	(Dimensionless)
m_B			500	2.3810×10^{-2}
m_F	3×10^5	1.4286×10^1	3×10^5	1.4286×10^1
I_B			300	8.0985×10^{-6}
I_F	1.5×10^6	4.0492×10^{-2}	1.5×10^6	4.0492×10^{-2}
K_{FB}			1×10^{10}	1.6644×10^2
K_F	2×10^{10}	3.3287×10^2	2×10^{10}	3.3287×10^2
C_{FB}			1×10^7	8.9026×10^0
C_F	1×10^7	8.9026×10^0	1×10^7	8.9026×10^0
k_{FB}			1×10^{10}	9.4352×10^{-2}
k_F	6×10^{10}	5.6611×10^{-1}	6×10^{10}	5.6611×10^{-1}
c_{FB}			2×10^7	1.0094×10^{-2}
c_F	2×10^7	1.0094×10^{-2}	2×10^7	1.0094×10^{-2}
k_{BA}			2×10^9	1.8870×10^{-2}
c_{BA}			1×10^6	5.0468×10^{-4}

We find that $\mu = 8.1637 \times 10^{-5}$ and $\beta = 6.9689 \times 10^2$.

For Problems VRE 3 and VRE 4, the constants must satisfy the inequalities

$$1 > 2\mu\beta, \quad k_{BA} > 4\mu, \quad k_{FB} > 8\mu \quad \text{and} \quad k_F > 8\mu$$

in order to assure a unique solution (see Section 3.2). It is clear that these conditions are met with our choice of constants.

6.5.2 Convergence

The convergence of the first four eigenvalues is established empirically by increasing the number of elements. Note that the eigenvalues are complex. We consider the imaginary part of the eigenvalues. Convergence on the real parts of the eigenvalues is also established, but not displayed.

Experiments on the convergence of the eigenvalues were done for the cases $\mu = 0$ and $\mu \neq 0$. Results for for the case that $\mu = 0$ for Problems VRE 4 and VTE 4 are given in Tables 2 and 3 respectively. The other cases yield similar results. The eigenvalues occur in complex conjugate pairs and we only list the imaginary parts. From the tables we see that the displayed eigenvalues are accurate to 5 significant digits.

Table 2: Imaginary parts of the eigenvalues Problem VRE 4 ($\mu = 0$)

j	$Im(\lambda_1^{(j)})$	$Im(\lambda_2^{(j)})$	$Im(\lambda_3^{(j)})$	$Im(\lambda_4^{(j)})$
10	6.063598992319	39.20328150732	111.2984099726	200.5292536731
20	6.063595477419	39.20234445416	111.2772664612	200.5237167363
40	6.063595255872	39.20228511054	111.2759080178	200.5233527121
80	6.063595480677	39.20228147057	111.2758225750	200.5233296975
160	6.063597520788	39.20228156442	111.2758174750	200.5233097736

Table 3: Imaginary parts of the eigenvalues Problem VTE 4 ($\mu = 0$)

j	$Im(\lambda_1^{(j)})$	$Im(\lambda_2^{(j)})$	$Im(\lambda_3^{(j)})$	$Im(\lambda_4^{(j)})$
10	6.048681000262	38.57139344125	107.1242256458	199.6387452859
20	6.048680557234	38.57127388156	107.1215429222	199.6359153764
40	6.048680548363	38.57127156531	107.1214873202	199.6358505940
80	6.048680543847	38.57127152495	107.1214863566	199.6358494317
160	6.048680553394	38.57127152813	107.1214863407	199.6358494379

6.5.3 Effect of gravity, rotary inertia and shear

Recall that the inclusion of gravity in Problems VRE 3 and VRE 4 yield symmetrical bilinear forms, which is desirable from a theoretical point of view (see Section 3.2.) However, gravity is excluded in the formulation of Problems VTE 3 and VTE 4, since inclusion results in a non-symmetric bilinear form b. The existence and uniqueness of the solution in this case has not been proved (see Section 3.3). Adapting Problems VTE 3 and VTE 4 to include gravity, "solutions" for the Timoshenko models with gravity are simulated. This is done to compare results to Problems VRE 3 and VRE 4 with the gravity term included and omitted. A comparison on the imaginary part of the eigenvalues for all these cases are displayed in Table 4. We list only the positive values for the imaginary part (since the eigenvalues occur in complex conjugate pairs).

Table 4: Effect of gravity

	Euler-Bernoulli		Rayleigh		Timoshenko	
	$\mu = 0$	$\mu \neq 0$	$\mu = 0$	$\mu \neq 0$	$\mu = 0$	$\mu \neq 0$
$Im(\lambda_1)$	6.0680	6.0584	6.0636	6.0540	6.0487	6.0418
$Im(\lambda_2)$	39.408	39.407	39.202	39.201	38.571	38.592
$Im(\lambda_3)$	112.70	112.70	111.28	111.28	107.12	107.14
$Im(\lambda_4)$	200.72	200.72	200.52	200.52	199.64	199.64

Denoting the eigenvalues by λ_k^{EB}, λ_k^R and λ_k^T for $k = 1, 2, 3$ and 4, we observe that

$$Im(\lambda_k^{EB}) > Im(\lambda_{kc}^R) > Im(\lambda_{kc}^T) \, ,$$

which is to be expected.

The effect of rotary inertia is negligible (Rayleigh model versus Euler-Bernoulli model). The maximum relative error for comparable eigenvalues is less than 1%. The effect of shear, although slightly larger than the effect of rotary inertia, is also negligible (Timoshenko model versus Euler-Bernoulli model).

Comparing results for the three models with respect to the gravity term, shows that the influence of gravity is minimal. The maximum relative error (with respect to the case $\mu = 0$) for all three models is less than 0.2% and this occurs for the first eigenvalue.

6.5.4 Conclusion

From the results we see that the effect of rotary inertia, shear and gravity is minimal.

We conclude that using more complex models for calculating eigenvalues is not justified. Implementation of the finite element method is more complex for the models that include the above mentioned factors. It is therefore sufficient to use the Euler-Bernoulli model for finding the first four eigenvalues.

Chapter 7

Cantilever beam

7.1 Scope of the investigation

As indicated in Section 1.5, we are concerned with the Euler-Bernoulli and Timoshenko models for a cantilever beam. In this section we provide more detail.

Eigenvalues

We start with a comparison of eigenvalues for the two models. Depending on the parameter α, a number of small eigenvalues do not differ significantly. For a beam with square cross sectional area $h \times h$,

$$\alpha = 12\left(\frac{\ell}{h}\right)^2$$

and a value of $\alpha = 1200$ represents a beam of length to height ratio $10 : 1$, whereas $\alpha = 300$ is associated with a beam of length to height ratio $5 : 1$.

The first four eigenvalues for $\alpha = 4800$, $\alpha = 1200$ and $\alpha = 300$ are presented in Table 1.

Table 1: Comparison of eigenvalues

$\alpha = 4800$

	Euler-Bernoulli	Timoshenko
λ_1	1.030×10^{-2}	1.025×10^{-2}
λ_2	4.046×10^{-1}	3.914×10^{-1}
λ_3	3.172×10^{0}	2.937×10^{0}
λ_4	1.218×10^{1}	1.062×10^{1}

$\alpha = 1200$

	Euler-Bernoulli	Timoshenko
λ_1	3.214×10^{-2}	3.164×10^{-2}
λ_2	1.262×10^{0}	1.136×10^{0}
λ_3	9.897×10^{0}	7.862×10^{0}
λ_4	3.800×10^{1}	2.587×10^{1}

$\alpha = 300$

	Euler-Bernoulli	Timoshenko
λ_1	1.286×10^{-1}	1.209×10^{-1}
λ_2	5.049×10^{0}	3.507×10^{0}
λ_3	3.959×10^{1}	1.987×10^{1}
λ_4	1.520×10^{2}	5.477×10^{1}

The first three eigenvalues differ slightly for the case $\alpha = 4800$. It is doubtful if this is of practical importance. If $\alpha = 1200$, the first eigenvalues are close. For $\alpha = 300$, the first eigenvalue differs significantly and the others differ dramatically. However, one may question the applicability of beam theory in this case.

The two-dimensional model for a cantilever beam should be closer to reality than a one-dimensional model. In Section 7.7 we compute the eigenvalues and corresponding eigenfunctions for the two-dimensional beam and compare the results to those of the Timoshenko and Euler-Bernoulli models. We also consider the case $\alpha = 300$ to determine whether beam theory is still applicable.

Alternative boundary condition

Consider the alternative boundary conditions in Section 2.2. Results for this model will differ little from those obtained with the conventional boundary conditions if $c_{ij} \approx 0$, i.e. d_{ij} large. (Recall that $D = C^{-1}$.)

The weak variational form for the cantilever Timoshenko beam is presented in Section 3.1. The bilinear form b for the alternative boundary condition is presented in Section 3.4. If λ_1 is the smallest eigenvalue, then

$$\lambda_1 = R(v) = \min \left\{ b(v,v) \;\middle|\; \|v\|_X = 1 \right\}$$

where R is the Rayleigh quotient. Since

$$b(v,v) \quad = \quad \frac{1}{\beta}\|v_2'\|^2 + \|v_1' - v_2\|^2 + [\gamma v_1 \; \gamma v_2] \; D \; [\gamma v_1 \; \gamma v_2]^T ,$$

the eigenvalue λ_1 increases as the elements d_{ij} of D increase. This implies that the first eigenvalue is always less than the first eigenvalue for the conventional boundary conditions. This is why the alternative model will amplify the difference between the two models. It serves no purpose to investigate the alternative boundary condition any further and more can be achieved by consideration of two-dimensional or three-dimensional models.

Equilibrium problem

Since the shear stress is a multiple of the shear strain and we are interested in qualitative results, it is irrelevant whether we consider the stress distribution or the strain distribution. Our main concern is the shear at the built in end. We use solutions of the equilibrium problem, Problem CTD 1, to determine the shear strain in the two-dimensional cantilever beam.

The solution of Problem CTD 1 also yields the deflection and we compare the results to the deflection for the Euler-Bernoulli and Timoshenko models.

7.2 Boundary conditions and test functions

7.2.1 Boundary conditions

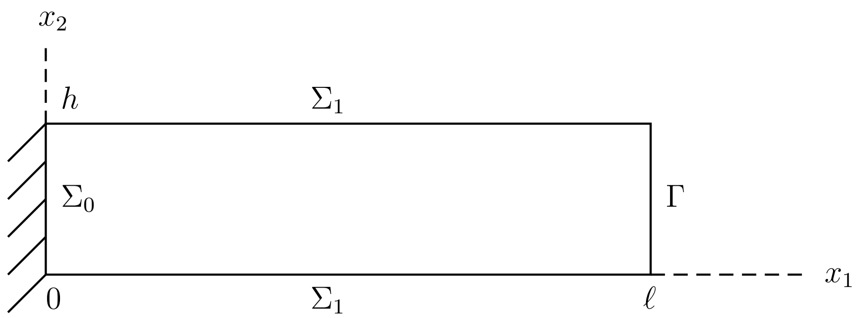
The boundary conditions given in Section 2.3 were of a general nature. In this section we provide more detail. In an effort to model the built in end of a beam, we consider three configurations, which are described below. The first configuration is commonly used but it can not be used to investigate shear at the cross section where we have the transition from clamped to free.

Note that for all three configurations the boundary consists of parts Σ and Γ, and that the boundary Σ is made up by the parts as shown below in the description for the different configurations. For the equilibrium problem and the eigenvalue problem, the conditions on Σ remain the same for both Problems CTD 1 and CTD 2, and are listed in the tables. However, the conditions on Γ differ for the two problems: For Problem CTD 1 the traction $\boldsymbol{t} = t\boldsymbol{e}_2$ with t a positive function and Γ is stress free for Problem CTD 2.

Configuration 1: Fixed beam

For this problem we assume that the beam is fixed rigidly to the support at $x_1 = 0$. The reference configuration is the rectangle $0 \leq x_1 \leq \ell$ and $0 \leq x_2 \leq h$. In this case the displacements are zero on Σ_0 and the two parts of Σ_1 are stress free.

Reference Configuration 1



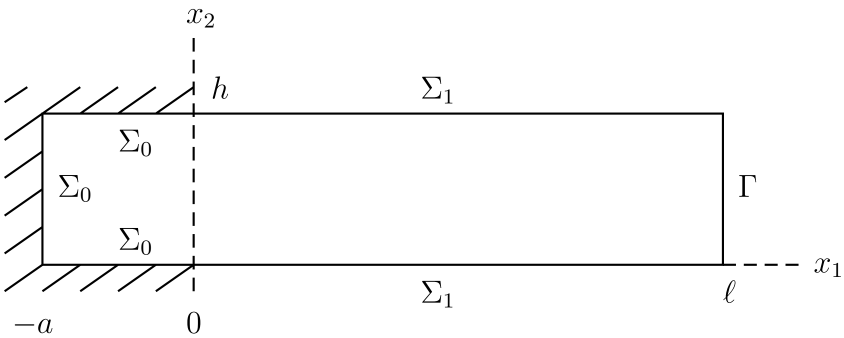
Boundary conditions for Configuration 1

Section	Coordinates	Conditions
Σ_0	$x_1 = 0, \quad 0 < x_2 < h$	$u_1 = u_2 = 0$
Σ_1	$0 < x_1 < \ell, \quad x_2 = 0$	$T\boldsymbol{e}_2 = \boldsymbol{0}$
	$0 < x_1 < \ell, \quad x_2 = h$	$T\boldsymbol{e}_2 = \boldsymbol{0}$
Γ	$x_1 = \ell, \quad 0 < x_2 < h$	$T\boldsymbol{e}_1 = t\boldsymbol{e}_2$ (Equilibrium problem) $T\boldsymbol{e}_1 = \boldsymbol{0}$ (Eigenvalue problem)

Configuration 2: Built in beam - case I

In this case we assume that a section of the beam is embedded in an inelastic support as in the diagram below. The reference configuration is the rectangle $-a \leq x_1 \leq \ell$ and $0 \leq x_2 \leq h$. The boundary Σ_1 is stress free.

Reference Configuration 2



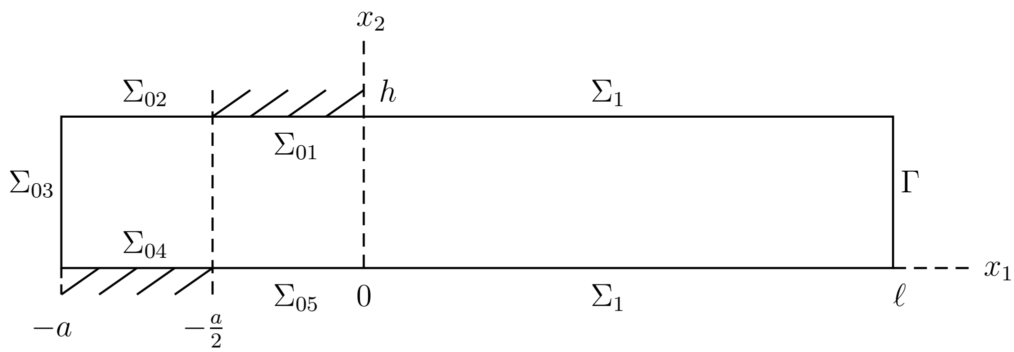
Boundary conditions for Configuration 2

Section	Coordinates	Conditions
Σ_0	$-a < x_1 < 0, \quad x_2 = 0$	$u_1 = u_2 = 0$
	$-a < x_1 < 0, \quad x_2 = h$	$u_1 = u_2 = 0$
	$x_1 = -a, \quad 0 < x_2 < h$	$u_1 = u_2 = 0$
Σ_1	$0 < x_1 < \ell, \quad x_2 = 0$	$T\boldsymbol{e}_2 = \boldsymbol{0}$
	$0 < x_1 < \ell, \quad x_2 = h$	$T\boldsymbol{e}_2 = \boldsymbol{0}$
Γ	$x_1 = \ell, \quad 0 < x_2 < h$	$T\boldsymbol{e}_1 = t\boldsymbol{e}_2$ (Equilibrium problem) $T\boldsymbol{e}_1 = \boldsymbol{0}$ (Eigenvalue problem)

Configuration 3: Built in beam - case II

In this case we still assume that a section of the beam is embedded in an inelastic support. The forced boundary conditions in Configuration 2 are mathematically convenient but not completely realistic, since "negative pressures" on the beam are possible. To avoid this, we consider Configuration 3 for Problem CTD 1. The reference configuration is the rectangle $-a \leq x_1 \leq \ell$ and $0 \leq x_2 \leq h$, x_2 being vertical. The results from Configuration 2 were used to specify the boundary conditions for Configuration 3 to make them more realistic.

Reference Configuration 3



Boundary conditions for Configuration 3

Section	Coordinates	Conditions
Σ_{01}	$-\frac{a}{2} < x_1 < 0, \quad x_2 = h$	$u_2 = 0 \;\;\&\;\; \sigma_{12} = 0$
Σ_{02}	$-a < x_1 < -\frac{a}{2}, \quad x_2 = h$	$T\boldsymbol{e}_2 = \mathbf{0}$
Σ_{03}	$x_1 = -a, \quad 0 < x_2 < h$	$u_1 = 0 \;\;\&\;\; \sigma_{12} = 0$
Σ_{04}	$-a < x_1 < -\frac{a}{2}, \quad x_2 = 0$	$u_2 = 0 \;\;\&\;\; \sigma_{12} = 0$
Σ_{05}	$-\frac{a}{2} < x_1 < 0, \quad x_2 = 0$	$T\boldsymbol{e}_2 = \mathbf{0}$
Σ_1	$0 < x_1 < \ell, \quad x_2 = 0$	$T\boldsymbol{e}_2 = \mathbf{0}$
	$0 < x_1 < \ell, \quad x_2 = h$	$T\boldsymbol{e}_2 = \mathbf{0}$
Γ	$x_1 = \ell, \quad 0 < x_2 < h$	$T\boldsymbol{e}_1 = \mathbf{0}$

7.2.2 Test functions

The test functions must satisfy the forced boundary conditions as specified in Section 2.3. A vector valued function $\boldsymbol{\phi}$ is a test function if each component $\phi_i \in C^1(\bar{\Omega})$ and $\phi_i = 0$ on some part of Σ.

For the three Configurations the set of test functions are as follows.

Configuration 1

$$T(\Omega) = \left\{ \boldsymbol{\phi} \in C^1(\bar{\Omega})^2 \;\middle|\; \boldsymbol{\phi} = \mathbf{0} \text{ on } \Sigma_0 \right\}$$

Configuration 2

$$T(\Omega) = \left\{ \boldsymbol{\phi} \in C^1(\bar{\Omega})^2 \;\middle|\; \boldsymbol{\phi} = \mathbf{0} \text{ on } \Sigma_0 \right\}$$

Configuration 3

$$T(\Omega) = \left\{ \boldsymbol{\phi} \in C^1(\bar{\Omega})^2 \;\middle|\; \phi_1 = 0 \text{ on } \Sigma_{03} \text{ and } \phi_2 = 0 \text{ on } \Sigma_{01} \text{ and } \Sigma_{04} \right\}$$

7.3 Galerkin approximation

We consider Problems CTD 1 and CTD 2 with the three configurations as discussed in Section 7.2.1 but in a finite dimensional subspace of $T(\Omega)$.

Consider a set of basis functions

$$\{\delta_1,\ \delta_2,\ \ldots\ \delta_p\}$$

and set

$$\boldsymbol{u}^h \ = \ \left[u_1^h \ \ u_2^h\right]^T \ = \ \left[\sum_{j=1}^{p} \delta_j u_{1j} \quad \sum_{j=1}^{p} \delta_j u_{2j}\right]^T .$$

The set with elements

$$[\delta_1 \ 0]^T, \quad [\delta_2 \ 0]^T, \quad \ldots \quad [\delta_p \ 0]^T,$$

$$[0,\ \delta_1]^T, \quad [0,\ \delta_2]^T, \quad \ldots \quad [0,\ \delta_p]^T$$

now is a basis for

$$S^h = \left\{ \left[\sum_{j=1}^{p} \delta_j u_{1j} \quad \sum_{j=1}^{p} \delta_j u_{2j}\right]^T \;\middle|\; u_{1j} \ \text{ and } \ u_{2j} \ \in I\!R \right\}$$

7.3.1 Equilibrium problem

For the equilibrium problem we consider the case that a (dimensionless) vertical force F is applied at $x_1 = 1$. This leads to

$$\boldsymbol{t} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} . \tag{7.3.1}$$

Note that t is a function of x_2 and $F = \int_0^h t(x_2)\, dx_2$. An obvious possibility is to choose t constant, but it is important to realize that such a choice is arbitrary and that it is advisable to consider other possibilities.

Galerkin approximation

The Galerkin approximation for the equilibrium problem is formulated.

Find $\boldsymbol{u}^h \in S^h$ so that

$$b(\boldsymbol{u}^h, \boldsymbol{\phi}) = \int_\Gamma \boldsymbol{t} \cdot \boldsymbol{\phi}\, ds \quad \forall \;\; \boldsymbol{\phi} \in \; S^h.$$

To find the Galerkin approximation for the problem in variational form, we substitute $\boldsymbol{\phi} = [\phi_1, \;\phi_2]^T$ with

$$[\,\delta_1\,, \; 0\,]^T, \; [\,\delta_2\,, \; 0\,]^T, \; \ldots \; [\,\delta_p\,, \; 0\,]^T,$$

and

$$[\,0, \;\delta_1\,]^T, \; [\,0, \;\delta_2\,]^T, \; \ldots \; [0, \;\delta_p\,]^T.$$

in the variational form.

For the remainder of this chapter, we use the notation

$$(f, g) = \iint_\Omega f g\,,$$

where Ω denotes the reference configuration.

We obtain the following system of linear equations:

$$\frac{\left(\partial_1 u_1^h + \nu \partial_2 u_2^h,\ \partial_1 \delta_i\right)}{\gamma(1-\nu^2)} + \frac{\left(\partial_1 u_2^h + \partial_2 u_1^h,\ \partial_2 \delta_i\right)}{2\gamma(1+\nu)} = 0 \quad \text{for } i = 1, 2, \ldots, p \tag{7.3.2}$$

$$\frac{\left(\partial_2 u_2^h + \nu \partial_1 u_1^h,\ \partial_2 \delta_i\right)}{\gamma(1-\nu^2)} + \frac{\left(\partial_1 u_2^h + \partial_2 u_1^h,\ \partial_1 \delta_i\right)}{2\gamma(1+\nu)} = \int_0^h t(x_2)\delta_i(1,x_2)dx_2, \quad \text{for } i = 1, 2, \ldots, p \tag{7.3.3}$$

7.3.2 The eigenvalue problem

Galerkin approximation

Find $\boldsymbol{u}^h \in S^h$ so that

$$b(\boldsymbol{u}^h, \boldsymbol{\phi}) = \lambda \iint_\Omega \boldsymbol{u}^h \cdot \boldsymbol{\phi}\, dA \quad \forall \ \boldsymbol{\phi} \in \ S^h .$$

Following the same procedure as in Section 7.3.1, we find the Galerkin approximation by solving the following system of linear equations:

$$\frac{\left(\partial_1 u_1^h + \nu \partial_2 u_2^h,\ \partial_1 \delta_i\right)}{\gamma(1-\nu^2)} + \frac{\left(\partial_1 u_2^h + \partial_2 u_1^h,\ \partial_2 \delta_i\right)}{2\gamma(1+\nu)} = \lambda\left(u_1, \delta_i\right) \quad \text{for } i = 1, 2, \ldots, p \tag{7.3.4}$$

$$\frac{\left(\partial_2 u_2^h + \nu \partial_1 u_1^h,\ \partial_2 \delta_i\right)}{\gamma(1-\nu^2)} + \frac{\left(\partial_1 u_2^h + \partial_2 u_1^h,\ \partial_1 \delta_i\right)}{2\gamma(1+\nu)} = \lambda\left(u_2, \delta_i\right) \quad \text{for } i = 1, 2, \ldots, p \tag{7.3.5}$$

7.4 Matrix formulation

We use the bicubic basis functions described in Section 4.2. We divide Ω in rs rectangular elements, where r denotes the number of intervals on the

x_1-axis and s the number of elements on the x_2-axis. The number of nodes for this grid is $N = (r+1)(s+1)$ and hence the number of bicubic basis functions is $4N$. Hence $p = 4N$ in the description in Sections 7.3.1 and 7.3.2.

The approximate solution is denoted by $\boldsymbol{u}^h$ and the components u_1 and u_2 are expressed as a linear combination of bicubic basis functions δ_j as

$$u_i^h(x) = \sum_{j=1}^{4N} \delta_j(x)\, u_{ij}\,.$$

For the equilibrium problem we define a load vector $\boldsymbol{c}$ with the two components $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. In this case $\boldsymbol{c}_1 = \mathbf{0}$ and the i-th component of $\boldsymbol{c}_2$ is

$$c_{2i} = \int_0^h t(x_2)\delta_i(1, x_2)dx_2\,.$$

The Galerkin approximations for both Problems CTD 1 and CTD 2 for the different configurations can now be be written in matrix form. The different configurations determine which of the coefficients u_{1j} and u_{2j} for $j = 1,\ 2,\ \ldots\ 4N$ are zero.

For the equilibrium problem the matrix form is given by

$$\mathcal{K}\boldsymbol{u} = \boldsymbol{c},$$

and for the eigenvalue problem the matrix form is

$$\mathcal{K}\boldsymbol{u} = \lambda\mathcal{M}\boldsymbol{u}.$$

The matrices $\mathcal{K}$ and $\mathcal{M}$ will differ for the different configurations.

7.4.1 Construction of the matrices $\mathcal{K}$ and $\mathcal{M}$

To construct $\mathcal{K}$ and $\mathcal{M}$, the following matrices are needed:

$$K_{pq} = \Big((\partial_p\delta_j,\ \partial_q\delta_i) \Big)_{4N\times 4N} \qquad \text{where} \quad p = 1,\ 2 \quad \text{and} \quad q = 1,\ 2$$

$$\text{and} \quad M = \Big((\delta_j,\ \delta_i) \Big)_{4N\times 4N}$$

Note that $K_{12} = K_{21}$.

Define the following matrices:

$$K_{11}^{\Omega} = K_{11} + \frac{(1-\nu)}{2} K_{22}, \qquad K_{12}^{\Omega} = \nu K_{21} + \frac{(1-\nu)}{2} K_{12}$$

$$K_{21}^{\Omega} = \nu K_{12} + \frac{(1-\nu)}{2} K_{21}, \qquad K_{22}^{\Omega} = K_{22} + \frac{(1-\nu)}{2} K_{11}$$

$$M_{11}^{\Omega} = M, \qquad M_{12}^{\Omega} = O, \qquad M_{21}^{\Omega} = O, \qquad M_{22}^{\Omega} = M$$

Define the $8N \times 8N$ matrices

$$K^{\Omega} = \begin{bmatrix} K_{11}^{\Omega} & K_{12}^{\Omega} \\ K_{21}^{\Omega} & K_{22}^{\Omega} \end{bmatrix} \quad \text{and} \quad M^{\Omega} = \begin{bmatrix} M_{11}^{\Omega} & M_{12}^{\Omega} \\ M_{21}^{\Omega} & M_{22}^{\Omega} \end{bmatrix}.$$

Let $\boldsymbol{u}_i = [\, u_{i1} \;\; u_{i2} \;\; \cdots \;\; u_{i,4N} \,]^T$ for $i = 1$ and $i = 2$ and $\boldsymbol{u}^{\Omega} = [\, \boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \,]^T$.

Define $\boldsymbol{c}^{\Omega} = [\, \boldsymbol{0} \;\; \boldsymbol{b} \,]^T$ with $\boldsymbol{0}$ a $4N \times 1$ zero matrix.

The vector $\boldsymbol{b}$ is a $4N \times 1$ matrix that results from the line integral $\int_{\Gamma} \boldsymbol{t} \cdot \boldsymbol{\phi} \, ds$.

The matrices $\mathcal{K}$, $\mathcal{M}$, $\boldsymbol{u}$ and $\boldsymbol{c}$ are found from K^{Ω}, M^{Ω}, $\boldsymbol{u}^{\Omega}$ and $\boldsymbol{c}^{\Omega}$ respectively by omitting appropriate rows and columns, according to the restrictions on the test functions.

The test functions must satisfy the forced boundary conditions, and when we use the bicubic basis functions, care must be taken that the "not so obvious" basis functions are also omitted. As an example, consider Configuration 1 for which $u_1 = u_2 = 0$ on Σ_0. Then the tangential derivatives $\partial_2 u_1 = \partial_2 u_2 = 0$ on Σ_0.

Remark

For the mixed derivatives equal to zero in Configuration 3, recall that $\sigma_{21} = 0$ and therefore $\partial_1 u_2 + \partial_2 u_1 = 0$.

7.5 Shear strain distribution

In this section we determine the shear strain profiles for a built in beam. For the reason given earlier, we consider Problem CTD 1 with Configuration 3.

To be more specific, we determine the shear strain distribution in the region where there is a transition from contact to free surface.

A beam of length 1.1, width 0.1 and height $h = 0.1$ is considered. The built in part of the beam has length $a = 0.1$. For experimental results, we used the constants (see Section 1.2.4)

$$\nu = 0.3 \quad \text{and} \quad \kappa^2 = \frac{5}{6}.$$

In the initial numerical experiments, we computed the shear strain for the entire beam with a constant stress t on the right hand side of the beam, namely $t(x_2) = 0.001$. Experiments show that to obtain accurate results, refining the grid in the x_2-direction is essential.

The stress distribution in the middle of the beam (for x_1-values roughly between 0.15 and 0.85), follows a "parabolic profile" which varies little. This is in line with the theory ([Fu, Sec 7.7] and [My, Sec 9.2]). From these results we find, with the given physical constants, an approximation for the stress distribution

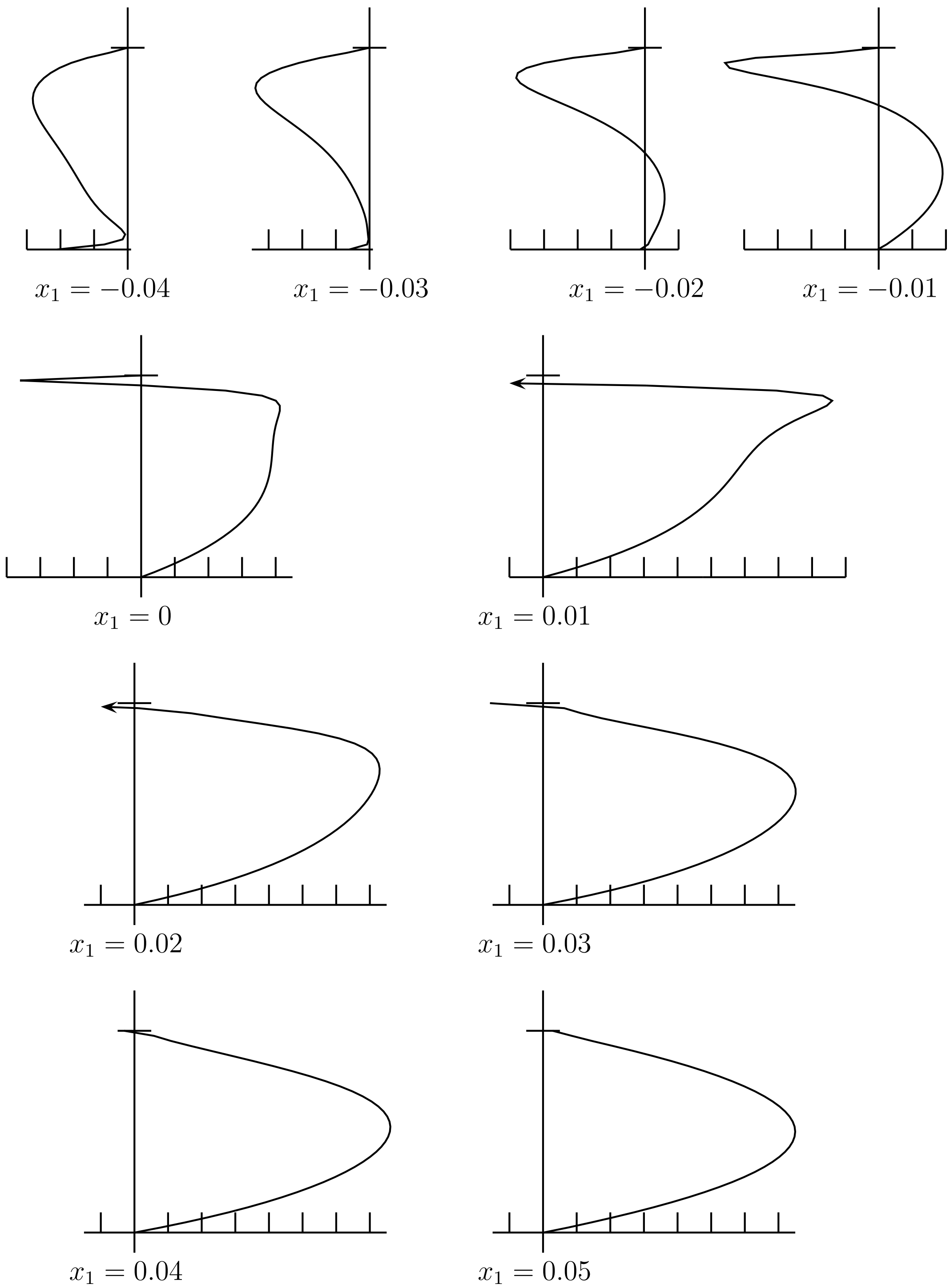
$$\sigma_{21}(x_1, x_2) \approx -6x_2^2 + 0.6x_2.$$

To obtain accurate results, the stress profile in the middle of the beam is used as an input on a part of the original beam. We consider the part of the beam of length 0.25. The first part of length 0.1 coincides with the built in end of the original beam and the part of length 0.15 with the free part closest to the built in end.

The results are interpreted graphically in Figure 1, where the stress profiles are plotted at specified x_1-values. In each graph, the vertical axis denotes the x_2-axis, whereas the horizontal axis is the σ_{21}-axis for the specified x_1-value that is shown at the bottom of each graph. The scale on the axes stay throughout the same for all the graphs, in order to compare stress profiles at different x_1-values. Each interval shown on the horizontal axis has a length of 0.002 and the length of the interval shown on the vertical axis is 0.1.

For the profiles at $x_1 = 0.1$ and $x_1 = 0.2$, the values for σ_{21} have magnitudes that are quite large in magnitude close to $x_2 = 0.1$. This explains the arrowheads in these two graphs. It is clear that the shear stress distribution exhibits enormous variation in a small interval containing $x_1 = 0$. We conclude that the constitutive equation (1.2.8) is not valid in this interval. (Recall the remarks in Section 1.2.2.) However, the phenomenon observed is not sufficient to reject the Timoshenko model.

Figure 1: Stress profiles at built in end



7.6 Deflection

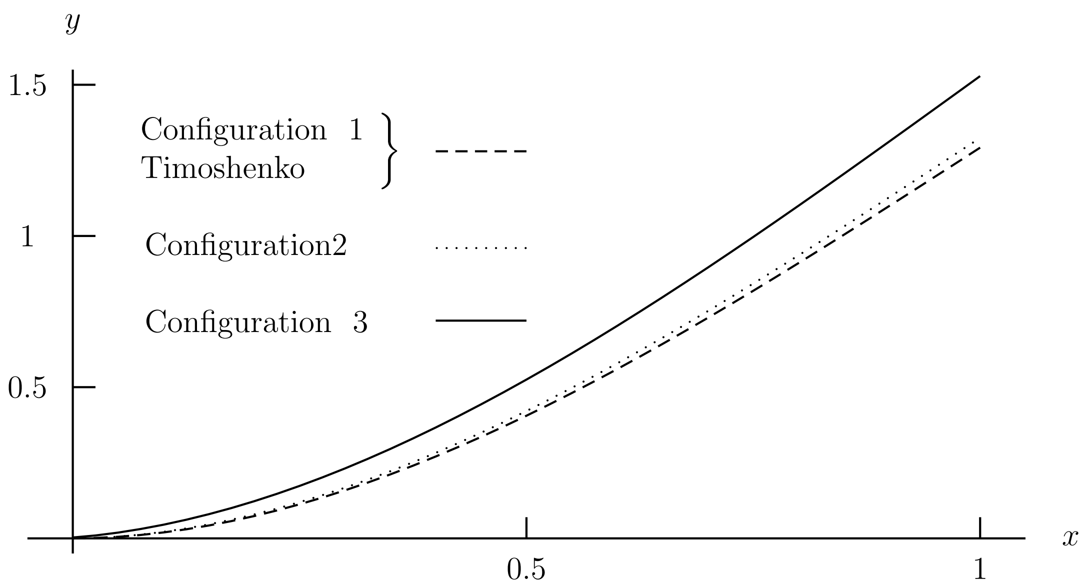
Although the main objective is to determine the stress distribution, it is of interest to compare the deflection for the two-dimensional beam with a one-dimensional model. We consider all three configurations. For both Configurations 2 and 3 we consider a built in end of length 0.1 and the free part of length 1. We take $\nu = 0.3$, $\kappa^2 = 5/6$ and $\alpha = 1200$. The deflections at $x_1 = 1$ are displayed in Table 2 for the three configurations as well as for the Timoshenko model and an accuracy of three significant digits are guaranteed for all three configurations.

Table 2: Deflections

Configuration 1	1.29
Configuration 2	1.32
Configuration 3	1.53
Timoshenko	1.29

Graphs representing these results are shown in Figure 2. Since the deflections for Configuration 1 and the Timoshenko model are the same when rounded to three significant digits, only one graph is shown.

Figure 2: Deflection comparison



The deflections of the neutral plane for Problem CTD 1 for Configurations 1 and 2 do not differ much from the deflection for the Timoshenko model, but for Configuration 3 the deflection at the endpoint is almost 20 % higher than for the Timoshenko model.

It is interesting that the Timoshenko model yields results that are so close to those obtained for Configuration 1, which is a configuration mostly used in the literature.

We are reluctant to draw any conclusion from the result. Clearly there is a need for further research and more attention should be paid to the modelling of the way the beam is built in or welded to a structure.

7.7 Eigenvalues and eigenfunctions

Eigenvalues

For the eigenvalue problem only the first two configurations are used as the third configuration gives rise to a nonlinear problem. In Table 3, the first 8 eigenvalues are compared to the corresponding eigenvalues of the cantilever Timoshenko beam and Euler-Bernoulli beam.

All the eigenvalues are given accurately to three significant digits and shown in the next table.

Table 3: Eigenvalues ($\alpha = 1200$)

Euler-Bernoulli	Timoshenko	Configuration 1	Configuration 2
$\chi_1 = 3.21 \times 10^{-2}$	$\lambda_1 = 3.16 \times 10^{-2}$	$\mu_1 = 3.17 \times 10^{-2}$	$\eta_1 = 3.06 \times 10^{-2}$
$\chi_2 = 1.26 \times 10^{0}$	$\lambda_2 = 1.14 \times 10^{0}$	$\mu_2 = 1.14 \times 10^{0}$	$\eta_2 = 1.11 \times 10^{0}$
$\chi_3 = 9.90 \times 10^{0}$	$\lambda_3 = 7.86 \times 10^{0}$	$\mu_3 = 7.72 \times 10^{0}$	$\eta_3 = 7.31 \times 10^{0}$
		$\mu_4 = 7.92 \times 10^{0}$	$\eta_4 = 7.76 \times 10^{0}$
$\chi_4 = 3.80 \times 10^{1}$	$\lambda_4 = 2.59 \times 10^{1}$	$\mu_5 = 2.62 \times 10^{1}$	$\eta_5 = 2.57 \times 10^{1}$
$\chi_5 = 1.04 \times 10^{2}$	$\lambda_5 = 5.99 \times 10^{1}$	$\mu_6 = 6.08 \times 10^{1}$	$\eta_6 = 5.99 \times 10^{1}$
		$\mu_7 = 6.93 \times 10^{1}$	$\eta_7 = 6.57 \times 10^{1}$
$\chi_6 = 2.32 \times 10^{2}$	$\lambda_6 = 1.13 \times 10^{2}$	$\mu_8 = 1.15 \times 10^{2}$	$\eta_8 = 1.14 \times 10^{2}$

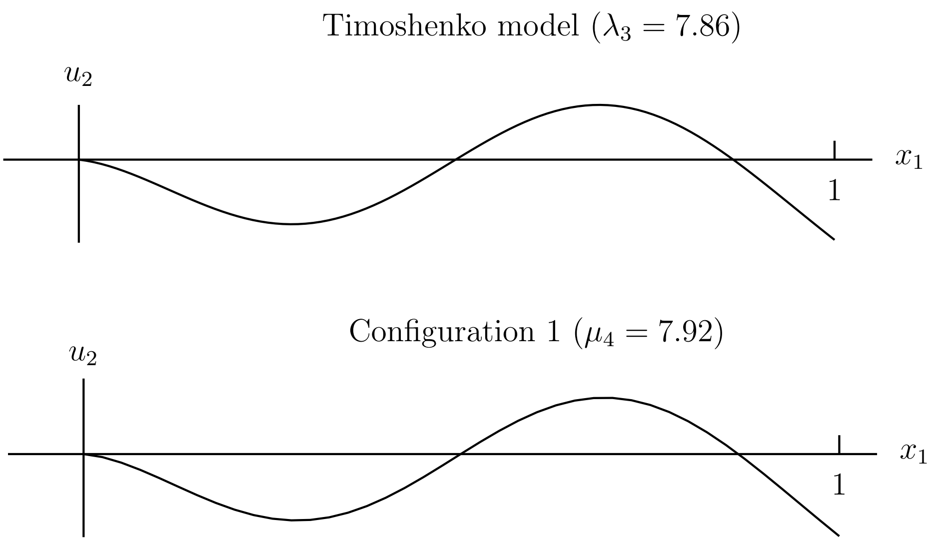
The eigenvalues for the Timoshenko beam compare well with those for the two-dimensional beam except for μ_3, η_3 μ_7 and η_7. It is notable that none of the eigenvalues for the Timoshenko model are related to them. To find an explanation, we turn to the eigenfunctions.

Eigenfunctions

We used only Configuration 1 to approximate eigenfunctions for the two-dimensional model. The mode shapes for the first five eigenvalues for the Timoshenko model compare well with the mode shapes for the two-dimensional models, where the comparison is made with the vertical displacements of the neutral plane ($x_2 = 0.05$). We present one example. In Figure 3 the mode shape of the deflection of the Timoshenko model for $\lambda_3 = 7.86$ is shown, as well as the mode shape for Configuration 1 for $\mu_4 = 7.92$. These two mode shapes are the same. It is clear that μ_4 correspond to λ_3.

Now, consider the mode shape for $\mu_3 = 7.72$. The vertical displacement of the neutral plane turns out to be the zero function. We conclude that this eigenvalue corresponds to a two-dimensional effect.

Figure 3: Mode shape comparison



Two-dimensional effects

The two-dimensional effects become visible when the displacement of the line $x_1 = 0.5$ (in the reference configuration) is examined. The Timoshenko model suggests straight lines when u_1 is plotted versus x_2 (at fixed x_1-values). The eigenvalue $\mu_4 = 7.92$ yields this result. We have a completely different result for $\mu_3 = 7.72$. The values of u_1 are almost constant (but not zero) and the line remains vertical. This implies a horizontal shift and it is clear that we have a two-dimensional effect that is not related to the one-dimensional beam theory.

The results indicate that the Timoshenko model is remarkably accurate compared to the two-dimensional model, provided that the application is one for which beam theory is intended. However, comparison to a three-dimensional model is preferable to establish the accuracy of the Timoshenko model. The conclusion is that further research needs to be done.

Chapter 8

Plate-beam system

8.1 Introduction

The differences between the Euler-Bernoulli, Rayleigh and Timoshenko beam models can be investigated by comparing the natural frequencies predicted by the different models. It is well known that in general, the shear corrections introduced by the Timoshenko model are larger than the rotary inertia corrections of the Rayleigh model. For the first (smallest) eigenvalue these corrections are small, but for the higher eigenvalues they are of significance. See Section 8.1.1 for a numerical example.

The same tendency is seen when we compare the eigenvalues for the classical plate models, i.e. the Kirchhoff model with and without rotary inertia, with those of the Reissner-Mindlin plate model. See Section 8.1.2 for a numerical example.

8.1.1 Pinned-pinned beam

For a pinned-pinned beam the eigenvalues and eigenfunctions for the Euler-Bernoulli model, the Rayleigh model and the Timoshenko model can be obtained in closed form.

Euler-Bernoulli model

The eigenvalues are

$$\lambda = \frac{k^4\pi^4}{\beta_b}, \quad k = 1,\, 2,\, \ldots,$$

with associated eigenfunctions

$$w(x) = \sin k\pi x.$$

Rayleigh model

The eigenvalues are

$$\lambda = \frac{k^4\pi^4}{\beta_b(1+\alpha_b^{-1}k^2\pi^2)}, \quad k = 1,\, 2,\, \ldots,$$

with associated eigenfunctions

$$w(x) = \sin k\pi x.$$

Timoshenko model

The eigenvalues are the roots of

$$\lambda^2 - \left(\alpha_b + \left(1 + \frac{\alpha_b}{\beta_b}\right) k^2\pi^2\right)\lambda + \frac{\alpha_b}{\beta_b}\, k^4\pi^4 = 0 \quad \text{for } k = 1,\, 2,\, \ldots$$

For each k, two eigenvalues λ_k and λ_k^* are obtained. If λ_k^* denotes the larger one of the two, it is known that $\lambda_k^* > \alpha_b$ for all k. In the numerical examples the first few eigenvalues are considered and the λ_k^* will not feature. The associated eigenfunction pairs are

$$w_k(x) = \sin k\pi x, \quad \phi_k(x) = \frac{k^2\pi^2 - \lambda_k}{k\pi}\, \cos k\pi x \quad \text{and}$$

$$w_k^*(x) = \sin k\pi x, \quad \phi_k^*(x) = \frac{k^2\pi^2 - \lambda_k^*}{k\pi}\, \cos k\pi x.$$

Comparison of eigenvalues

As an example we present some numerical results for a pinned-pinned beam with a length to depth ratio of 20:1 and a square profile, i.e. $\alpha_b = 4800$. We choose $\beta_b = 0.25$. The percentage differences for the first five eigenvalues are shown in Table 1, where $\lambda_i^{(EB)}$, $\lambda_i^{(R)}$ and $\lambda_i^{(T)}$ denote the i-th eigenvalue for the Euler-Bernoulli, Rayleigh and Timoshenko models respectively. The percentage differences are calculated with respect to the Euler-Bernoulli eigenvalues. Clearly, the shear corrections are larger than the corrections due to rotary inertia and the corrections for larger eigenvalues are significant. For a "shorter" beam (smaller α_b) these corrections are even larger.

Table 1: Corrections for a pinned-pinned beam

i	Rotary inertia $\frac{\lambda_i^{(EB)}-\lambda_i^{(R)}}{\lambda_i^{(EB)}}$	Shear $\frac{\lambda_i^{(EB)}-\lambda_i^{(T)}}{\lambda_i^{(EB)}}$
1	0.21 %	1.02%
2	0.82 %	3.93%
3	1.82 %	8.36%
4	3.19 %	13.85%
5	4.89 %	19.01%

8.1.2 Rigidly supported plate

For a plate supported rigidly on all four sides, the eigenvalues and eigenfunctions can be determined in closed form for all the different plate models, i.e. the Kirchhoff model with and without rotary inertia and the Reissner-Mindlin model.

Kirchhoff model without rotary inertia

The eigenvalues are

$$\lambda = \frac{\pi^4(n^2+m^2)^2}{\beta_p(1-\nu_p^2)h_p}, \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots$$

with associated eigenfunctions

$$w(x_1, x_2) = \sin(n\pi x_1)\ \sin(m\pi x_2).$$

Kirchhoff model with rotary inertia

The eigenvalues are

$$\lambda \ = \ \frac{\pi^4(n^2+m^2)^2}{\beta_p(1-\nu_p^2)(h_p + I_p\pi^2(n^2+m^2))}\,, \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots$$

with associated eigenfunctions

$$w(x_1, x_2) = \sin(n\pi x_1)\ \sin(m\pi x_2).$$

Reissner-Mindlin model

The eigenvalues are the solutions of the quadratic equation

$$r\lambda^2 - (1 + (r+\gamma)f)\lambda + \gamma f^2 = 0.$$

In this equation

$$r \ = \ \frac{h_p^2}{12} \quad \text{and} \quad \gamma \ = \ \frac{1}{2\beta_p(1-\nu_p^2)h_p}\,.$$

A sequence of values for f are used, each yielding two eigenvalues.

$$f \ = \ \pi^2(n^2+m^2) \quad \text{for} \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots.$$

The associated eigenfunction pairs are of the form

$$\begin{aligned} w(x_1,x_2) &= \sin(n\pi x_1)\ \sin(m\pi x_2), \\ \psi_1(x_1,x_2) &= A_{nm}\cos(n\pi x_1)\ \sin(m\pi x_2), \\ \psi_2(x_1,x_2) &= B_{nm}\sin(n\pi x_1)\ \cos(m\pi x_2). \end{aligned}$$

Since these formulae will not be used in our calculations, we do not display the closed form expressions for A_{nm} and B_{nm}.

Comparison of eigenvalues

For the numerical calculations we use a square plate with dimensionless thickness $h_p = 0.05$, Poisson's ratio $\nu_p = 0.3$ and shear correction factor $\kappa_p^2 = 5/6$. The first six eigenvalues for the three different models are given in Table 2. Note that due to the spatial symmetry of the problem, repeated eigenvalues occur, e.g. Eigenvalues 2 and 3, and also, 5 and 6.

Table 2: Eigenvalues for rigidly supported plate

i	Kirchhoff without rotary inertia	Kirchhoff with rotary inertia	Reissner-Mindlin
1	0.2783	0.2772	0.2733
2	1.7394	1.7217	1.6643
3	1.7394	1.7217	1.6643
4	4.4530	4.3809	4.1540
5	6.9578	6.8176	6.3849
6	6.9578	6.8176	6.3849

The percentage differences for the first six eigenvalues are shown in Table 3, where $\lambda_i^{(K)}$, $\lambda_i^{(KR)}$ and $\lambda_i^{(RM)}$ denote the i-th eigenvalue for the Kirchhoff model without rotary inertia, the Kirchhoff model with rotary inertia and the Reissner-Mindlin model respectively. The percentage differences are calculated with respect to the Kirchhoff eigenvalues. It is clear that the corrections due to shear are larger than the corrections due to rotary inertia.

Table 3: Corrections for rigidly supported plate

i	Rotary inertia $\frac{\lambda_i^{(K)}-\lambda_i^{(KR)}}{\lambda_i^{(K)}}$	Shear $\frac{\lambda_i^{(K)}-\lambda_i^{(RM)}}{\lambda_i^{(K)}}$
1	0.41 %	1.78 %
2	1.02 %	4.32 %
3	1.02 %	4.32 %
4	1.62 %	6.71 %
5	2.01 %	8.23 %
6	2.01 %	8.23 %

8.1.3 Plate-beam system

In [ZVGV3] a plate-beam system consisting of the classical plate model and the Euler-Bernoulli beam model is investigated. It is shown that introducing rotary inertia into the model does not causes a significant change in the eigenvalues. It is also shown that when the ratio d_b/h_p is increased, the eigenvalues of the plate-beam system tend to those of the rigidly supported plate.

An initial aim is to compare the eigenvalues of the Reissner-Mindlin-Timoshenko (RMT) plate-beam system with those of the Kirchhoff-Euler-Bernoulli (KEB) plate-beam system. We will also consider the asymptotic behaviour of the eigenvalues of the RMT system when ratio d_b/h_p is increased. Some interesting phenomena present themselves and will be discussed in Section 8.5.5.

8.2 The eigenvalue problems

In this section the variational forms in Chapter 3 for the different plate-beam systems are used to derive the associated eigenvalue problems.

8.2.1 Reissner-Mindlin-Timoshenko plate-beam system

As explained in Section 3.9, if $\tilde{w}(\boldsymbol{x},t) = T(t)w(\boldsymbol{x})$ and $\tilde{\boldsymbol{\psi}}(\boldsymbol{x},t) = T(t)\boldsymbol{\psi}(\boldsymbol{x})$ is considered as a possible solution for Equations (3.6.6) and (3.6.12), the following eigenvalue problem is obtained.

Problem RMT

$$\begin{aligned}
& \lambda \left\{ h_p \iint_\Omega wv\,dA + \eta_1 \left[\int_0^1 wv\,dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 wv\,dx_1 \right]_{x_2=a} \right\} \\
= \; & h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \nabla v\,dA + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\,dx_1 \right]_{x_2=0} \\
& + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\,dx_1 \right]_{x_2=a} \qquad (8.2.1)
\end{aligned}$$

for all v in $T_1(\Omega)$ and

$$\begin{aligned} &\lambda\left\{I_p \iint_\Omega \boldsymbol{\psi}\cdot\boldsymbol{\phi}\,dA + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \psi_1\phi_1\,dx_1\right]_{x_2=0} + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \psi_1\phi_1\,dx_1\right]_{x_2=a}\right\} \\ = &\; b_B(\boldsymbol{\psi},\boldsymbol{\phi}) + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi})\cdot\boldsymbol{\phi}\,dA \\ &+\frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1\psi_1\partial_1\phi_1\,dx_1\right]_{x_2=0} + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1\psi_1\partial_1\phi_1\,dx_1\right]_{x_2=a} \\ &+\eta_2\left[\int_0^1 (\partial_1 w + \psi_1)\phi_1\,dx_1\right]_{x_2=0} + \eta_2\left[\int_0^1 (\partial_1 w + \psi_1)\phi_1\,dx_1\right]_{x_2=a} \end{aligned} \tag{8.2.2}$$

for all $\boldsymbol{\phi}$ in $T_2(\Omega)$.

($T_1(\Omega)$, $T_2(\Omega)$ and b_B are defined in Section 3.6.1.)

8.2.2 Kirchhoff-Rayleigh plate-beam system

If $\tilde{w}(\boldsymbol{x},t) = T(t)w(\boldsymbol{x})$ is considered as a possible solution for Equation (3.6.15), the following eigenvalue problem is obtained.

Problem KR

$$\begin{aligned} &\lambda\left\{h_p \iint_\Omega wv\,dA + I_p \iint_\Omega (\nabla w)\cdot\nabla v\,dA\right\} \\ &+\lambda\left\{\eta_1\left[\int_0^1 wv\,dx_1\right]_{x_2=0} + \eta_1\left[\int_0^1 wv\,dx_1\right]_{x_2=a}\right\} \\ &+\lambda\left\{\frac{\eta_1}{\alpha_b}\left[\int_0^1 (\partial_1 w)v\,dx_1\right]_{x_2=0} + \frac{\eta_1}{\alpha_b}\left[\int_0^1 (\partial_1 w)v\,dx_1\right]_{x_2=a}\right\} \\ = &\; b_B(w,v) + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\,dx_1\right]_{x_2=0} + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\,dx_1\right]_{x_2=a} \end{aligned} \tag{8.2.3}$$

for all $v \in T(\Omega)$.

($T(\Omega)$ and b_B are defined in Section 3.6.2.)

8.2.3 Kirchhoff-Euler-Bernoulli plate-beam system

The eigenvalue problem for the case where rotary inertia is ignored, is obtained by ignoring the terms containing I_p and η_1/α_b in (8.2.3). We refer to the corresponding problem as **Problem KEB**.

8.3 Galerkin approximations for the eigenvalue problems

For all three eigenvalue problems we consider an approximate solution

$$w^h(\boldsymbol{x}) = \sum_{i=1}^{N} w_i\gamma_i(\boldsymbol{x}), \quad \psi_1^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{1i}\gamma_i(\boldsymbol{x}) \quad \text{and} \quad \psi_2^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{2i}\gamma_i(\boldsymbol{x})$$

in terms of the bicubic basis functions

$$\gamma_i, \quad i = 1, 2, \ldots N,$$

where the functions ψ_1^h and ψ_2^h are only applicable for Problem RMT.

As $w^h \in T_1(\Omega)$ and $\boldsymbol{\psi}^h = [\psi_1^h \ \psi_2^h]^T \in T_2(\Omega)$, some of these coefficients will be equal to zero.

8.3.1 Galerkin approximation for Problem RMT

If $\boldsymbol{w} = [w_1\, w_2\, \ldots\, w_N]^T$, $\boldsymbol{\psi}_1 = [\psi_{11}\, \psi_{12}\, \ldots\, \psi_{1N}]^T$ and $\boldsymbol{\psi}_2 = [\psi_{21}\, \psi_{22}\, \ldots\, \psi_{2N}]^T$, the Galerkin approximation for Problem RMT is given by three matrix equations. These equations are obtained by choosing $v = \gamma_j$ in (8.2.1) and $\boldsymbol{\phi} = [\gamma_j\ 0]^T$ and $\boldsymbol{\phi} = [0\ \gamma_j]^T$ in (8.2.2). Recall that $v \in T_1(\Omega)$ and $\boldsymbol{\phi} \in T_2(\Omega)$ and that only admissible basis functions should be used.

8.3.2 Galerkin approximation for Problem KEB

If $\boldsymbol{w} = [w_1\ w_2\ \ldots\ w_N]^T$, the Galerkin approximation for Problem KEB is given by a matrix equation. This equation is obtained by choosing $v = \gamma_j$ in (8.2.3). Recall that $v \in T(\Omega)$ and that only admissible basis functions should be used.

8.4 Matrix formulation of Galerkin approximations

The eigenvalue problem for both Problem RMT and KEB can be represented in matrix notation as

$$\mathcal{K}\boldsymbol{z} = \lambda \mathcal{M}\boldsymbol{z}.$$

The following matrices are required for defining the matrices $\mathcal{K}$ and $\mathcal{M}$ for the different eigenvalue problems.

$$I_{ij}^{\Omega 12} = \iint_\Omega \partial_1\partial_2\gamma_j \partial_1\partial_2\gamma_i \, dA \,,$$

$$J_{ij}^{\Omega 11} = \iint_\Omega \partial_1^2\gamma_j \partial_1^2\gamma_i \, dA \,, \quad J_{ij}^{\Omega 22} = \iint_\Omega \partial_2^2\gamma_j \partial_2^2\gamma_i \, dA \,, \quad J_{ij}^{\Omega 12} = \iint_\Omega \partial_1^2\gamma_j \partial_2^2\gamma_i \, dA \,,$$

$$J_{ij}^{0} = \int_0^1 \partial_1^2\gamma_j(x_1,0) \partial_1^2\gamma_i(x_1,0) \, dx_1 \,, \quad J_{ij}^{1} = \int_0^1 \partial_1^2\gamma_j(x_1,a) \partial_1^2\gamma_i(x_1,a) \, dx_1 \,,$$

$$K_{ij}^{\Omega 11} = \iint_\Omega \partial_1\gamma_j \partial_1\gamma_i \, dA \,, \quad K_{ij}^{\Omega 22} = \iint_\Omega \partial_2\gamma_j \partial_2\gamma_i \, dA \,, \quad K_{ij}^{\Omega 12} = \iint_\Omega \partial_1\gamma_j \partial_2\gamma_i \, dA \,,$$

$$K_{ij}^{0} = \int_0^1 \partial_1\gamma_j(x_1,0) \partial_1\gamma_i(x_1,0) \, dx_1 \,, \quad K_{ij}^{1} = \int_0^1 \partial_1\gamma_j(x_1,a) \partial_1\gamma_i(x_1,a) \, dx_1 \,,$$

$$L_{ij}^{\Omega 1} = \iint_\Omega \gamma_j \partial_1\gamma_i \, dA \,, \quad L_{ij}^{\Omega 2} = \iint_\Omega \gamma_j \partial_2\gamma_i \, dA \,,$$

$$L_{ij}^{0} = \int_0^1 \gamma_j(x_1,0) \partial_1\gamma_i(x_1,0) \, dx_1 \,, \quad L_{ij}^{1} = \int_0^1 \gamma_j(x_1,a) \partial_1\gamma_i(x_1,a) \, dx_1 \,,$$

$$M_{ij}^{\Omega} = \iint_\Omega \gamma_j \gamma_i \, dA \,,$$

$$M_{ij}^{0} = \int_0^1 \gamma_j(x_1,0) \gamma_i(x_1,0) \, dx_1 \,, \quad M_{ij}^{1} = \int_0^1 \gamma_j(x_1,a) \gamma_i(x_1,a) \, dx_1 .$$

8.4.1 Construction of $\mathcal{K}$ and $\mathcal{M}$ for Problem RMT

We define the following matrices which are needed to construct $\mathcal{K}$ and $\mathcal{M}$.

$$\begin{aligned}
K_w &= h_p\left(K^{\Omega 11} + K^{\Omega 22}\right) + \eta_2\left(K^0 + K^1\right), \\
L_1 &= h_p\, L^{\Omega 1} + \eta_2\left(L^0 + L^1\right), \\
L_2 &= h_p\, L^{\Omega 2}, \\
K_1 &= \frac{1}{\beta_p\,(1-\nu_p^2)}\left(K^{\Omega 11} + \frac{1-\nu_p}{2}\,K^{\Omega 22}\right) + \frac{\eta_2}{\beta_b}\left(K^0 + K^1\right) \\
&\quad + h_p\, M^{\Omega} + \eta_2\left(M^0 + M^1\right), \\
K_\nu &= \frac{1}{\beta_p\,(1-\nu_p^2)}\left(\nu_p\left(K^{\Omega 12}\right)^T + \frac{1-\nu_p}{2}\,K^{\Omega 12}\right), \\
K_2 &= \frac{1}{\beta_p\,(1-\nu_p^2)}\left(\frac{1-\nu_p}{2}\,K^{\Omega 11} + K^{\Omega 22}\right) + h_p\, M^{\Omega}, \\
M_w &= h_p\, M^{\Omega} + \eta_1\left(M^0 + M^1\right), \\
M_1 &= I_p\, M^{\Omega} + \frac{\eta_1}{\alpha_b}\left(M^0 + M^1\right), \\
M_2 &= I_p\, M^{\Omega}.
\end{aligned}$$

We define the matrices $\mathcal{K}^{RMT}$ and $\mathcal{M}^{RMT}$ by

$$\mathcal{K}^{RMT} = \begin{bmatrix} K_w & L_1 & L_2 \\ L_1^T & K_1 & K_\nu \\ L_2^T & K_\nu^T & K_2 \end{bmatrix} \quad \text{and} \quad \mathcal{M}^{RMT} = \begin{bmatrix} M_w & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{bmatrix}.$$

The matrices $\mathcal{K}$ and $\mathcal{M}$ that are needed for Problem RMT are found from the matrices above by omitting rows and columns according to the restrictions on the test functions.

8.4.2 Construction of $\mathcal{K}$ and $\mathcal{M}$ for Problem KEB

We define the matrices

$$
\begin{aligned}
\mathcal{K}^{KEB} &= \frac{1}{\beta_p(1-\nu_p^2)}\left(J^{\Omega 11} + J^{\Omega 22} + \nu_p\left(J^{\Omega 12} + J^{\Omega 21}\right)\right) \\
&\quad + \frac{2}{\beta_p(1+\nu_p)}\, I^{\Omega 12} + \frac{\eta_2}{\beta_b}\left(J^0 + J^1\right), \\
\mathcal{M}^{KEB} &= h_p M^{\Omega} + \eta_1(M^0 + M^1).
\end{aligned}
$$

The matrices $\mathcal{K}$ and $\mathcal{M}$ that are needed for Problem KEB are constructed from the matrices above by omitting rows and columns in accordance to the restrictions on the test functions.

8.5 Numerical results

8.5.1 Parameters

For the numerical results we consider consider a square plate and beams with a rectangular profile of thickness d and height $5d$. The dimensionless thickness d_b of the beams is denoted by $d_b = d/\ell$. For both the plate and the beams, we choose Poisson's ratio $\nu_p = \nu_b = 0.3$ and the shear correction factors $\kappa_p^2 = \kappa_b^2 = 5/6$. We also assume that the plate and the beams are made of the same isotropic material and therefore we use $G = \dfrac{E}{2(1+\nu)}$.

For this special case the dimensionless constants reduce to

$$
\begin{aligned}
\eta_1 &= 5d_b^2\,, \\
\eta_2 &= 5\left(\frac{\kappa_b^2}{\kappa_p^2}\right) d_b^2\,, \\
I_p &= \frac{h_p^3}{12}\,, \\
\frac{1}{\alpha_b} &= \frac{25d_b^2}{12}\,, \\
\frac{1}{\beta_p} &= \frac{(1+\nu_p)h_p^3}{6\kappa_p^2}\,, \\
\frac{1}{\beta_b} &= \frac{25(1+\nu_b)d_b^2}{6\kappa_b^2}\,.
\end{aligned}
$$

In all the numerical experiments a square plate is considered (i.e. $a = 1$) and the value of h_p is fixed at $h_p = 0.05$, while the value of d_b is varied to allow for different values of the ratio d_b/h_p.

8.5.2 Convergence

MATLAB programs have been written for calculating the eigenvalues of the RMT and KEB plate-beam systems, using the finite element method. The results of convergence tests are discussed briefly for Problem RMT. In this case $h_p = d_b = 0.05$.

In Table 4 the first ten eigenvalues of the RMT plate-beam system are listed for a 2×2, 4×4, 8×8 and a 16×16 grid. The value $\lambda_i^{(k)}$ denotes the approximation for eigenvalue i when using a $k \times k$ grid for the finite element calculations. When the grid is refined, the eigenvalues form a decreasing sequence, which is in line with the theory.

Table 4: Convergence

i	$\lambda_i^{(2)}$	$\lambda_i^{(4)}$	$\lambda_i^{(8)}$	$\lambda_i^{(16)}$
1	2.3517×10^{-1}	2.3412×10^{-1}	2.3401×10^{-1}	2.3400×10^{-1}
2	7.9829×10^{-1}	7.7665×10^{-1}	7.7474×10^{-1}	7.7443×10^{-1}
3	1.1934×10^{0}	1.1822×10^{0}	1.1790×10^{0}	1.1785×10^{0}
4	1.9352×10^{0}	1.6459×10^{0}	1.6408×10^{0}	1.6406×10^{0}
5	2.8759×10^{0}	2.4348×10^{0}	2.4271×10^{0}	2.4266×10^{0}
6	4.4995×10^{0}	3.9430×10^{0}	3.9317×10^{0}	3.9311×10^{0}
7	8.6576×10^{0}	6.4642×10^{0}	6.3653×10^{0}	6.3615×10^{0}
8	9.8681×10^{0}	7.4870×10^{0}	7.3860×10^{0}	7.3816×10^{0}
9	1.0396×10^{1}	8.7615×10^{0}	8.6805×10^{0}	8.6743×10^{0}
10	1.3119×10^{1}	1.0499×10^{1}	1.0391×10^{1}	1.0386×10^{1}

The relative errors for the first 10 eigenvalues are displayed in Table 5.

Table 5: Convergence

i	$\dfrac{\lambda_i^{(4)} - \lambda_i^{(2)}}{\lambda_i^{(4)}}$	$\dfrac{\lambda_i^{(8)} - \lambda_i^{(4)}}{\lambda_i^{(8)}}$	$\dfrac{\lambda_i^{(16)} - \lambda_i^{(8)}}{\lambda_i^{(16)}}$
1	4.4676×10^{-3}	4.5354×10^{-4}	6.5675×10^{-5}
2	2.7862×10^{-1}	2.4688×10^{-3}	3.9173×10^{-4}
3	9.4811×10^{-3}	2.6543×10^{-3}	4.4116×10^{-4}
4	1.7577×10^{-1}	3.1193×10^{-3}	1.1919×10^{-4}
5	1.8115×10^{-1}	3.1752×10^{-3}	2.2742×10^{-4}
6	1.4113×10^{-1}	2.8553×10^{-3}	1.6457×10^{-4}
7	3.3931×10^{-1}	1.5547×10^{-1}	5.9373×10^{-4}
8	3.1804×10^{-1}	1.3668×10^{-1}	6.0509×10^{-4}
9	1.8660×10^{-1}	9.3339×10^{-3}	7.1587×10^{-4}
10	2.4949×10^{-1}	1.0448×10^{-1}	4.1081×10^{-4}

We find that the first six eigenvalues, which we will consider in the following experiments, are accurate to three significant digits for a 16×16 grid.

8.5.3 Comparison of Reissner-Mindlin-Timoshenko system with Kirchhoff-Euler-Bernoulli system

In [ZVGV3], a numerical investigation of a similar plate-beam system is done for a combination of the classical plate model and the Euler-Bernoulli beam model. It was found that the inclusion of rotary inertia in the plate and beam models had little effect on the eigenvalues.

We now compare the eigenvalues for the RMT system to those of the KEB system for $d_b/h_p = 1$ and show that the shear corrections on the higher eigenvalues are of more significance than the corrections due to rotary inertia.

Table 6: Eigenvalues for plate-beam system

i	KEB	RMT	Shear correction
1	0.2413	0.2340	3.03 %
2	0.8765	0.7744	11.65 %
3	1.3715	1.1785	14.07 %
4	1.7197	1.6406	4.60 %
5	2.6642	2.4266	8.92 %
6	4.2835	3.9311	8.23 %

8.5.4 Comparison of Kirchhoff-Euler-Bernoulli system with a rigidly supported Kirchhoff plate

[ZVGV3] contains a numerical experiment where the eigenvalues of the KEB plate-beam system is compared to the eigenvalues of a rigidly supported Kirchhoff plate, for different values of the ratio d_b/h_p. The experiment is repeated for the sake of completeness, as well as the fact that a different time scaling is used in [ZVGV3]. The exact eigenvalues for the supported Kirchhoff plate appear in the last column. From Table 7 it is clear that the eigenvalues of the KEB plate-beam system tend to the eigenvalues of the rigidly supported plate as the ratio d_b/h_p is increased.

Table 7: Kirchhoff-Euler-Bernoulli

	λ_i Plate-beam system $h_p = 0.05$				λ_i
i	$d_b/h_p = 1$	$d_b/h_p = 2$	$d_b/h_p = 4$	$d_b/h_p = 8$	Supported plate
1	0.2413	0.2760	0.2782	0.2783	0.2783
2	0.8765	1.6853	1.7368	1.7393	1.7394
3	1.3715	1.7383	1.7394	1.7395	1.7394
4	1.7197	4.4436	4.4525	4.4530	4.4530
5	2.6642	5.2472	6.9312	6.9574	6.9578
6	4.2835	6.1048	6.9587	6.9687	6.9578

8.5.5 Comparison of Reissner-Mindlin-Timoshenko system with a rigidly supported Reissner-Mindlin plate

In Table 8 the eigenvalues of the Reissner-Mindlin-Timoshenko plate-beam system are compared to the eigenvalues of a Reissner-Mindlin plate that is rigidly supported on all four sides. The exact eigenvalues for the rigidly supported plate is presented in the last column.

It is clear that, as expected, the eigenvalues of the Reissner-Mindlin-Timoshenko plate-beam system tend to the eigenvalues of the rigidly supported Reissner-Mindlin plate as the ratio d_b/h_p is increased.

Table 8: Reissner-Mindlin-Timoshenko

	λ_i Plate-beam system $h_p = 0.05$				λ_i
i	$d_b/h_p = 1$	$d_b/h_p = 2$	$d_b/h_p = 4$	$d_b/h_p = 8$	Supported plate
1	0.2340	0.2702	0.2730	0.2733	0.2733
2	0.7744	1.5695	1.6552	1.6627	1.6643
3	1.1785	1.6619	1.6639	1.6642	1.6643
				3.0030	
				3.0030	
4	1.6406	3.2510	4.1503	4.1532	4.1540
5	2.4266	3.5914	5.8931	6.3471	6.3849
6	3.9311	4.1320	6.3844	6.3849	6.3849

Two interesting phenomena in this table warrant some further comment. The first is that the eigenvalues of the plate-beam system corresponding to the the double eigenvalues of the supported plate remain further apart for the RMT system than for the KEB system.

Secondly, for large values of the ratio d_b/h_p, an “extra” pair of eigenvalues appear for the Reissner-Mindlin-Timoshenko system. For $d_b/h_p = 8$ in Table 8, the double eigenvalue $\lambda \approx 3$ does not correspond to an eigenvalue of the supported plate. These eigenvalues did not appear in numerical experimentation with the Kirchhoff-Euler-Bernoulli system. The explanation for these extra eigenvalues for the RMT system lies in the fact that “pure shear" modes exist for the RMT system under consideration. These modes are discussed in the next section.

8.5.6 Pure shear modes

A fact that is often overlooked is that for certain configurations, "pure shear" modes exist for the Timoshenko beam model and for the Reissner-Mindlin plate model.

Timoshenko model

For a pinned-pinned Timoshenko beam it is easy to see that $\lambda = \alpha_b$ is an eigenvalue with the associated pair of eigenfunctions

$$w(x) = 0, \quad \phi(x) = 1.$$

Reissner-Mindlin-Timoshenko plate-beam system

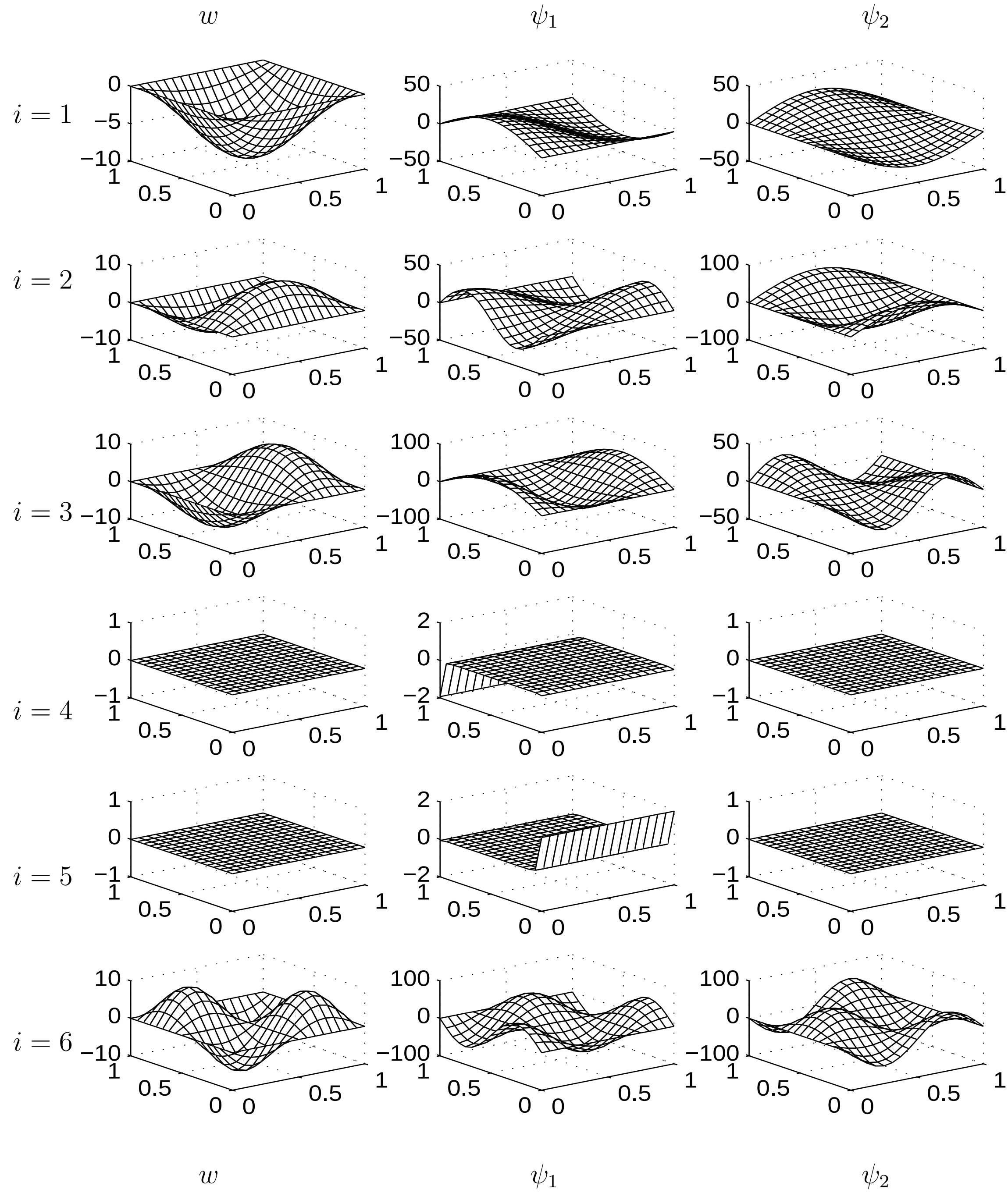
Returning to the numerical results for the RMT plate-beam system in Table 8, note that if $d_b/h_p = 8$ and $h_p = 0.05$, then $d_b = 0.4$ and hence $\alpha_b = 3$. It seems likely that the pair of "extra eigenvalues" in Table 8 is a consequence of the pure shear mode of the Timoshenko beam model. This conjecture is supported by the graphs of the eigenfunction pairs of the system in Figure 1.

Remark

Note that as the height of the beam is $5d_b$, it means that in this case, the length to height ratio for the beam is 1 : 2. One would not expect the Timoshenko beam model to yield realistic results and consequently the RMT plate-beam system will also not be a reasonable model to use. Hence this phenomenon is only of theoretical significance.

Figure 8: Eigenfunctions for the RMT plate-beam system

(Note the differences in scaling.)



Appendix 1. Sobolev Spaces

The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω. The first result is well known.

Theorem 1

The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with **inner product**

$$(f, g) = \int_\Omega fg = \int_\Omega fg \; d\mu$$

where μ is the n-dimensional Lebesgue measure.

Theorem 2

The space $\mathcal{L}^2(\Omega)$ is separable (See [Ad, Th 2.15, p 28]).

Theorem 3

$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ (See [Ad, Th 2.13, p 28]).

The one-dimensional case

Suppose Ω is a **bounded open** interval. The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition

For f and g in $H^m(\Omega)$,

$$[f, g]_m = (f^{(m)}, g^{(m)}) \quad \text{for} \quad m = 0,\ 1,\ \ldots$$

For $m \geq 1$, the bilinear form $[\cdot, \cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f, f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f, f]_m} \quad \text{for} \quad m = 0,\ 1,\ \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

The two-dimensional case

Suppose Ω is a **bounded open convex** subset of $I\!R^2$. The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Remark

It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Definition

For f and g in $H^m(\Omega)$,

$$[f,g]_m = \sum_{i+j=m} (\partial_1^i \partial_2^j f, \partial_1^i \partial_2^j g) \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

For $m \geq 1$ the bilinear form $[\cdot,\cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f,f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f,f]_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

The boundary

Recall that a curve is called **smooth** if its parametrization has a continuous derivative. The boundary of Ω is called **piecewise smooth** if it consists of a finite number of smooth curves.

For a vector valued function r such that $r_i \in C^1[a,b]$ for $i = 1,\, 2$, the range $\boldsymbol{C}$ of r defines a smooth curve in the plane.

Suppose that $\boldsymbol{C}$ is a part of the boundary of Ω. A function f is Lebesgue integrable on $\boldsymbol{C}$ if $f \circ r \sqrt{(r_1')^2 + (r_2')^2}$ is Lebesgue integrable on the interval $[a,b]$.

A function f is in $\mathcal{L}^2(\boldsymbol{C})$ if f^2 is Lebesgue integrable over $\boldsymbol{C}$. The inner product for $\mathcal{L}^2(\boldsymbol{C})$ is defined by

$$(f,g)_{\boldsymbol{C}} = \int_{\boldsymbol{C}} fg\, ds = \int_a^b (f \circ r)\,(g \circ r)\, \sqrt{(r_1')^2 + (r_2')^2}\,.$$

When necessary, we use the **notation** $(f,g)_\Omega$ and $(f,g)_\Gamma$ to avoid confusion.

Sobolev spaces of vector valued functions

Definition

$u \in \mathcal{L}^2(\Omega)^2$ if $u_i \in \mathcal{L}^2(\Omega)$ for $i = 1, 2$.

$u \in \mathcal{L}^2(\Gamma)^2$ if $u_i \in \mathcal{L}^2(\Gamma)$ for $i = 1, 2$.

$u \in H^k\Omega)^2$ if $u_i \in H^k(\Omega)$ for $i = 1, 2$.

$[u, v]_{m,2} = [u_1, v_1]_m + [u_2, v_2]_m$ for $u \in \mathcal{L}^2(\Omega)^2$ and $v \in \mathcal{L}^2(\Omega)^2$.

$|u|_{m,2} = \sqrt{[u, u]_{m,2}}$ for $u \in \mathcal{L}^2(\Omega)^2$.

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|^{\Omega}_{m,2}$ and $\|\cdot\|^{\Gamma}_{m,2}$.

General definitions and results

Suppose Ω is a **bounded open interval** or a **bounded open convex** subset of $\mathbb{R}^2$.

Notation

$H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

Definition

The inner product for $H^m(\Omega)$ is defined by

$$(f, g)_m = \sum_{k=0}^{m} [f, g]_k \quad \text{for} \quad m = 0, 1, \dots$$

Definition

The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f,g)_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The inner product for $H^m(\Omega)^2$ is defined by

$$(f,g)_{m,2} = \sum_{k=0}^{m} [f,g]_{k,2} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f,g)_{m,2}} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Theorem 4

The space $H^m(\Omega)$ is complete (See [Ad, Th 3.2, p 45]).

Theorem 5

$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.
(See [OR, Th 2.10, p 53].)

Theorem 6

The space $H^m(\Omega)$ is separable (See [Ad, Th 3.5, p 47]).

Theorem 7 (Rellich)

For m any nonnegative integer, the embedding of $H^{m+1}(\Omega)$ into $H^m(\Omega)$ is compact (See [Ad, Th 6.2, p 144]).

Notation

$$\begin{aligned} \partial^\alpha &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}, \quad \text{where} \\ |\alpha| &= \alpha_1 + \alpha_2 \cdots \alpha_n \end{aligned}$$

and $|\alpha|$ denotes the order of the derivative.

Theorem 8 (Sobolev's lemma)

Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \leq \|u\|_p \quad \text{for} \quad |\alpha| \leq m.$$

(See [OR, Th 3.10, p 80].)

Remarks

1. Theorems 4 to 8 are also true for vector valued functions. The proofs are all trivial.

2. When we need to distinguish between different domains, say Ω and Γ, they will appear as superscripts, for instance $\|\cdot\|_k^\Omega$ and $(f,g)_{m,2}^\Gamma$.

Appendix 2. Inequalities

The one-dimensional case

Proposition 1

Consider any $u \in C^1[0,1]$. For any two points x and y in $[0,1]$,

$$|u(x)| \leq \|u'\| + |u(y)|.$$

Proof

Assuming that $x > y$ (without loss of generality), we have

$$u(x) = \int_y^x u' + u(y).$$

But $|\int_y^x f| \leq \|f\|$ for any $f \in \mathcal{L}^2(0,1)$. This follows from the Cauchy-Schwartz inequality

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right)$$

by choosing $g = 1$. The rest is obvious.

Theorem 1

For any $u \in C^1[0,1]$ with a zero in $[0,1]$ we have

$$\|u\| \leq \|u'\|.$$

Proof

Suppose $u(y) = 0$, then $|u(x)| \leq \|u'\|$ by Proposition 1.

Hence $\|u\|_{sup} \leq \|u'\|$. The rest is obvious since $\|u\| \leq \|u\|_{sup}$.

Proposition 2

For any $u \in C^1[0,1]$, $\quad |u(0)| \leq \sqrt{2}\, \|u\|_1$.

Proof

Let $g(x) = 1 - x$ and $v = gu$ and consider the fact that

$$u(0) = v(0) = -\int_0^1 v' + v(1)\,.$$

Since $v(1) = 0$,

$$|u(0)| = \left| \int_0^1 (u'g + ug') \right| \leq \|u'\|\|g\| + \|u\|\|g'\| \leq \|u'\| + \|u\|.$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, it follows that

$$|u(0)|^2 \leq 2\|u'\|^2 + 2\|u\|^2\,.$$

The two-dimensional case

Suppose Ω is a bounded open convex subset of $\mathbb{R}^2$ with a piecewise smooth boundary. The following result is referred to as the Poincare-Friedrichs inequality or Friedrichs's inequality or Poincare's inequality.

Theorem 2

Suppose Σ is a part of the boundary of Ω with nonzero length. Denote the set

$$\{u \in C^1(\bar{\Omega}) \,\big|\, u = 0 \text{ on } \Sigma\}$$

by $F(\Omega)$. There exists a constant c_F such that, for each $u \in F(\Omega)$,

$$\|u\| \leq c_F |u|_1.$$

Proof

See e.g. [Br, p 30].

Corollary

Suppose Σ_1 and Σ_2 are parts of the boundary of Ω with nonzero length. Denote the set

$$\{u \in C^1(\bar{\Omega})^2 \,\big|\, u_1 = 0 \text{ on } \Sigma_1 \text{ and } u_2 = 0 \text{ on } \Sigma_2\}$$

by $F(\Omega)^2$. There exists a constant c_F such that for each $u \in F(\Omega)^2$,

$$\|u\|_{0,2} \leq c_F |u|_{1,2}\,.$$

Note that Σ_1 and Σ_2 may overlap and even be equal.

Theorem 3 (Korn's inequality)

Suppose b_B is the bilinear form for the Reissner-Mindlin plate. There exists a constant c_Ω such that

$$|u|_{1,2}^2 \leq c_\Omega \, b_B(u,u)$$

for each $u \in V$.

Proof

See e.g. [Br, p 288-289].

Appendix 3. Trace

The one-dimensional case

Recall that for each $u \in H^1(0,1)$, there exists a sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

Theorem 1

For each $u \in H^1(0,1)$, there exists a unique real number γu with the following property: For each sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} u_n(0) = \gamma u.$$

Proof

Due to Proposition 2 in Appendix 2, $\lim_{n\to\infty} u_n(0)$ exists for each sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Also due to this proposition, the limit is independent of the choice of the sequence $\{u_n\}$.

Theorem 2

The mapping γ is linear and bounded. In fact,

$$|\gamma u| \leq \sqrt{2}\, \|u\|_1 .$$

Proof

The linearity follows from the properties of limits and the estimate from Proposition 2 in Appendix 2 by considering the limits.

Remark

The mapping γ is a bounded linear functional.

Theorem 3

For any $u \in H^1(0,1)$,

$$\|u\| \leq \|u'\| + |\gamma u|.$$

Proof

Consider any $u \in C^1[0,1]$. Proposition 1 in Appendix 2 implies that

$$\|u\|_{sup} \leq \|u'\| + |u(0)|.$$

Consequently,

$$\|u\| \leq \|u'\| + |\gamma u|.$$

The same inequality holds for each $u \in H^1(0,1)$ since $C^1[0,1]$ is dense in $H^1(0,1)$.

The two-dimensional case

Definition (Trace operator γ)

For $u \in C(\bar{\Omega})$, the function γu is the restriction of the function u to Γ.

Theorem 4

The trace operator γ can be extended to a bounded linear operator mapping $H^1(\Omega)$ onto $L^2(\Gamma)$ and $\|\gamma u\|_\Gamma \leq K\|u\|_1^\Omega$.

Proof

This result is a special case of results in [OR, p 141-142].

Definition

For $u \in H^1(\Omega)^2$, we define γu by

$$\gamma u = \langle \gamma u_1, \gamma u_2 \rangle.$$

Theorem 5

Suppose Ω is the open rectangle $0 < x_1 < 1,\ 0 < x_2 < a$. Γ is the side where $x_2 = 0$ and $\gamma_0 u = u(\cdot, 0)$. Then there exists a constant K such that

$$\|u\|_\Omega \le K|u|_1^\Omega + K\|\gamma_0 u\|_\Gamma$$

for all $u \in H^1(\Omega)$.

Proof

Proposition 1 Appendix 2 implies that for each $x_2 \in [0, a]$,

$$|u(x)|^2 \le 2a^2 \int_0^a \big[\partial_2 u(x_1, \cdot)\big]^2 + 2\big[u(x_1, 0)\big]^2 .$$

Therefore

$$\int_0^a \big[u(x_1, \cdot)\big]^2 \le 2a^2 \int_0^a \big[\partial_2 u(x_1, \cdot)\big]^2 + 2\big[u(x_1, 0)\big]^2 .$$

Integration with respect to x_1 yields

$$\|u\|_\Omega^2 \le 2a^2\|\partial_2 u\|_\Omega^2 + 2\|\gamma_0 u\|_\Gamma^2 .$$

The result follows.

Appendix 4. The spaces $C^k(J;Y)$

Consider $J=(a,b)$ or $J=[a,b)$. Let Y be any Banach space and consider a function u with values in Y. Let t be any interior point of J.

Definition (Derivative)

Suppose there exists a $v \in Y$ such that

$$\lim_{h\to 0} \left\|h^{-1}\big(u(t+h)-u(t)\big)-v\right\|_Y = 0,$$

then v is the derivative of u at t. We write $u'(t)$ for the derivative.

It is obvious how to adapt the definition for the case $t=a$. The derivative (function) u' and the second order derivative u'' are defined in the usual way.

Notation

$C^k\big([0,\infty);Y\big)$ and $C^k\big((0,\infty);Y\big)$

Appendix 5. Proofs

All the results and proofs are from [V4] and presented here for completeness.

Plate-beam system

Proposition 1

There exists a constant K_T such that

$$\|w\|^2 + \|\phi\|^2 \leq K_T \left(\|\phi'\|^2 + \|w' - \phi\|^2\right)$$

for each $(w, \phi) \in T(0,1) \times C^1[0,1]$

Proof

Suppose it is not true. Then there exists a sequence $\{(w_n, \phi_n)\}$ such that

$$\|w_n\|^2 + \|\phi_n\|^2 = 1,$$

while

$$\|\phi_n'\|^2 + \|w_n' - \phi_n\|^2 \to 0 \quad \text{as } n \to \infty.$$

- We prove first that for n sufficiently large, ϕ_n does not have a zero.

Suppose ϕ_n does have a zero. Then $\|\phi_n\| \leq \|\phi_n'\|$. We also have $\|w_n\| \leq \|w_n'\|$ since $w_n(0) = 0$. Consequently

$$\|w_n\| \leq \|w_n'\| \leq \|w_n' - \phi_n\| + \|\phi_n\| \leq \|w_n' - \phi_n\| + \|\phi_n'\|.$$

This implies that

$$\|w_n\| + \|\phi_n\| \leq \|w_n' - \phi_n\| + \|\phi_n'\| \quad \text{for each } n.$$

This is a contradiction.

• We now show that $\|\phi_n\| > 1/2$ for n sufficiently large.

If it is not true, then

$$\|w_n\| \leq \|w_n'\| \leq \|w_n' - \phi_n\| + \|\phi_n\| \leq 1/4 + 1/2 < 3/4$$

for n sufficiently large. Consequently

$$\|w_n\|^2 + \|\phi_n\|^2 < 9/16 + 1/4 = 13/16 < 1.$$

This is a contradiction.

• Next we show that $\int_0^1 \phi_n > 1/10$ for n sufficiently large.

We may assume without loss of generality that $\phi_n > 0$. Writing ϕ for ϕ_n, we have

$$\phi_{\max} - \phi_{\min} \leq \|\phi'\| \leq 1/20 \quad \text{and} \quad \phi_{\max} \leq 21/20$$

for n sufficiently large. Consequently

$$\phi_{\min}^2 = \int_0^1 \phi_{\min}^2 = \int_0^1 [\phi_{\min}^2 - \phi^2] + \int_0^1 \phi^2$$

$$= \int_0^1 \phi^2 - \int_0^1 [\phi^2 - \phi_{\min}^2] \geq 1/4 - 1/10 > 1/10.$$

Therefore

$$\int_0^1 \phi \geq \int_0^1 \phi_{\min} > 1/10.$$

• $\int_0^1 w_n' > 0$ for n sufficiently large.

$$\left| \int_0^1 w_n' - \int_0^1 \phi_n \right| \leq \int_0^1 |w_n' - \phi_n| \leq \|w' - \phi\| \leq 1/20.$$

• Finally we obtain a contradiction.

Since $\int_0^1 w_n' > 0$ and $w_n(0) = 0$, we have $w_n(1) > 0$ which is a contradiction. $\square$

Corollary

There exists a constant K such that

$$\|w\|_1^2 + \|\phi\|_1^2 \leq K \left(\|\phi'\|^2 + \|w' - \phi\|^2 \right) \,.$$

Proposition 2

$$\|w\|_\Omega^2 + \left(\|\boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 \leq K \left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 + K \left(|\boldsymbol{\psi}|_{1,2}^\Omega \right)^2 + K \|\gamma_0 \psi_1\|_I^2$$

for each $(w, \boldsymbol{\psi}) \in T_1(\Omega) \times T_2(\Omega)$.

Proof

Since ψ_2 is zero on a part of the boundary, we may use the Friedrichs inequality $\|\psi_2\|_\Omega \leq c_F |\psi_2|_1^\Omega$. We also use

$$\|\psi_1\|_\Omega \leq c_1 |\psi_1|_1^\Omega + c_1 \|\gamma \psi_1\|_I$$

(see Theorem 5 Appendix 3). Combining the two inequalities, we have

$$\left(\|\boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 \leq c_2 \left(|\boldsymbol{\psi}|_{1,2}^\Omega \right)^2 + c_2 \|\gamma \psi_1\|_I^2.$$

Since w is zero on a part of the boundary, $\|w\|_\Omega \leq c_F |w|_1^\Omega$, using the Friedrichs inequality. Therefore

$$\|w\|_\Omega \leq c_F |w|_1^\Omega = c_F \|\nabla w\|_{0,2}^\Omega \leq c_F \|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega + c_F \|\boldsymbol{\psi}\|_{0,2}^\Omega \,.$$

Consequently

$$\begin{aligned} \|w\|_\Omega^2 + \left(\|\boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 &\leq K \left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 + K \left(\|\boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 \\ &\leq K \left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 + K \left(|\boldsymbol{\psi}|_{1,2}^\Omega \right)^2 + K \|\gamma_0 \psi_1\|_I^2 \,, \end{aligned}$$

where K is a generic constant depending on c_1 and c_F .

Corollary

$$\left(\|w\|_1^\Omega \right)^2 + \left(\|\boldsymbol{\psi}\|_{1,2}^\Omega \right)^2 \leq K \left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega \right)^2 + K \left(|\boldsymbol{\psi}|_{1,2}^\Omega \right)^2 + K \|\gamma_0 \psi_1\|_I^2 \,.$$

Theorem 1

The inertia space X is a separable Hilbert space and V is dense in X.

Proof

Since $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$, we have that $T_1(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ and $T_2(\Omega)$ is dense in $\mathcal{L}^2(\Omega)^2$. Also $T(I)$ is dense in $\mathcal{L}^2(I)$. We conclude that the space T is dense in the space X equipped with the inner product

$$(u,v)_{\mathcal{L}^2} = (u_1,v_1)_\Omega + (u_2,v_2)_{0,2}^\Omega + \sum_{j=3}^{6}(u_j,v_j)_I\,.$$

The norms $\|u\|_X$ and $\|u\|_{\mathcal{L}^2}$, where

$$\|u\|_X^2 = c(u,u) \quad \text{and} \quad \|u\|_{\mathcal{L}^2}^2 = (u,u)_{\mathcal{L}^2}\,,$$

are equivalent. Therefor T is a dense subset of X with respect to the inertia norm and $T \subset V \subset X$.

Theorem 2 (Korn's inequality)

$$b_B(u_2,u_2) \geq K|u_2|_{1,2}^2 \quad \text{for each } u \in T.$$

Theorem 3

There exist constants c_1 and c_2 such that

$$\|u\|_X \leq c_1\|u\|_{H^1} \leq c_2\|u\|_V$$

for each $u \in T$.

Proof

From the corollary to Proposition 1, we have

$$\|u_3\|_1^2 + \|u_4\|_1^2 + \|u_5\|_1^2 + \|u_6\|_1^2 \leq C_\Gamma b_\Gamma(u,u).$$

Rewrite the corollary to Proposition 2.

$$\left(\|u_1\|_1^\Omega\right)^2 + \left(\|u_2\|_{1,2}^\Omega\right)^2 \leq K\left(\|\nabla u_1 + u_2\|_{0,2}^\Omega\right)^2 + K\left(|u_2|_{1,2}^\Omega\right)^2 + K\|\gamma_0 u_{21}\|_I^2 .$$

Combining the results, we have

$$\|u\|_{H^1}^2 \leq C\left(\|\nabla u_1 + u_2\|_{0,2}^\Omega\right)^2 + C\left(|u_2|_{1,2}^\Omega\right)^2 + Cb_\Gamma(u,u) ,$$

using Proposition 1 again. Now use Korn's inequality.

Nonmodal damping

Theorem 1

For each $y \in H$ there exists a unique $x \in H$ such that

$$\begin{aligned} x_2 &= y_1 \\ b(x_1,v) + a(x_2,v) &= -c(y_2,v) \quad \text{for each } v \in V. \end{aligned}$$

Proof

Let

$$g(v) = -a(y_1,v) - c(y_2,v) \quad \text{for each } v \in V,$$

then g is clearly a linear functional on V. Furthermore

$$|g(v)| \leq K\|y_1\|_V\|v\|_V + c\|y_2\|_X\|v\|_X \quad \text{for each } v \in V,$$

showing that g is bounded. The result follows from the well known theorem of Riesz.

Theorem 2

Λ is bounded.

Proof

Consider any $y \in H$ and suppose $x = \Lambda y$.

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

It follows that

$$\|x_2\|_X \leq K\|x_2\|_V = \|y_1\|_V$$

and

$$\begin{aligned} \|x_1\|_V^2 &= b(x_1, x_1) \\ &\leq |a(x_2, x_1)| + |c(y_2, x_1)| \\ &\leq K\|x_2\|_V \, \|x_1\|_V + K\|y_2\|_X \, \|x_1\|_V \end{aligned}$$

Consequently

$$\|x_1\|_V \leq K\|y_1\|_V + K\|y_2\|_X \ \leq K\|y\|_H.$$

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