Chapter 4

Interpolation

4.1 Hermite cubics

The well-known Hermite piecewise cubics (see [SF] or [Re]) are successfully used as basis functions for the Galerkin approximation in beam problems.

The construction and properties of Hermite cubics are treated in detail in the book of Strang and Fix ([SF, p 55-59]). Divide the interval $[a, b]$ into n subintervals by a partitioning

$$a \; = \; x_0 \; < x_1 \; < \; \cdots \; < \; x_n \; = \; b.$$

This yields n elements, $\Omega_i = [x_{i-1}, x_i]$, each of length h_i, for $i = 1, 2, \ldots, n$.

For $i = 0, 1, \ldots, n$, we have two piecewise cubics denoted by $\delta_i^{(j)}$ with $j = 0$ or $j = 1$ with the following properties:

1. For $k = 0, 1, \ldots, n$, $\quad i = 0, 1, \ldots, n$ and $j = 0, 1$, the restriction of $\delta_k^{(j)}$ to any Ω_i is either a cubic polynomial or zero.
2. $\delta_i^{(j)} \in C^1[a, b]$ and $D^2\delta_i^{(j)}$ is piecewise continuous with possible discontinuities at the nodes.
3. $\delta_i^{(0)}(x_i) = 1, \quad D\delta_i^{(0)}(x_i) = 0, \quad \delta_i^{(1)}(x_i) = 0, \quad D\delta_i^{(1)}(x_i) = 1.$
4. $\delta_i^{(0)}(x_k) = 0, \quad D\delta_i^{(0)}(x_k) = 0, \quad \delta_i^{(1)}(x_k) = 0, \quad D\delta_i^{(1)}(x_k) = 0$ if $k \neq i$.
5. $\delta_i^{(j)}$ is zero on any element Ω_k with $k \neq i$ or $i + 1$.

We refer to these two types of functions as Type 1 ($j = 0$) or Type 2 ($j = 1$) functions. Typical graphs of $\delta_i^{(0)}$ and $\delta_i^{(1)}$ are shown in Figures 1 and 2.

Figure 1: Type 1 Hermite piecewise cubic

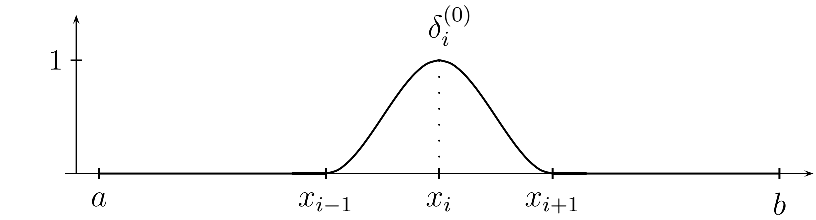
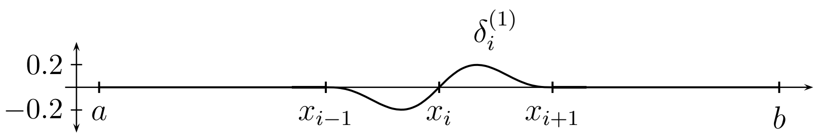


Figure 2: Type 2 Hermite piecewise cubic



Remarks

1. The graphs in Figures 1 and 2 must be adapted for the functions $\delta_0^{(0)}$, $\delta_n^{(0)}$, $\delta_0^{(1)}$ and $\delta_n^{(1)}$.
2. We will refer to the Hermite piecewise cubic functions as **Hermite cubics**.
3. $\delta_i^{(j)} \in H^2[a, b] \quad \forall \;\; i = 0, 1, \ldots, n \;$ and $\; j = 0, 1$.

Cubic interpolation operator

For $w \in H^2(a, b)$, we define the cubic interpolation operator Π_c as

$$\Pi_c\, w = \sum_{j=0}^{1} \sum_{i=0}^{n} (D^j w)(x_i)\, \delta_i^{(j)}.$$

Note that $\Pi_c\, \delta_i^{(j)} = \delta_i^{(j)}$ for $i = 0, 1, \ldots, n$ and $j = 0, 1$.

4.2 Hermite bicubic functions

The Hermite piecewise bicubic functions are constructed by using a product of the Hermite piecewise cubic functions in Section 4.1, hence the name **bicubics**. (For a fixed x or y, a piecewise bicubic reduces to a piecewise cubic.) See [SF, p 88-89] for detail. It is also mentioned there that bicubics rank amongst the best provided that rectangular elements are used.

The rectangle $\bar{\Omega} = [a, b] \times [c, d]$ is divided in rs elements as follows. Partition $[a, b]$ and $[c, d]$ by

$$a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad c = y_0 < y_1 < \cdots < y_s = d,$$

and set

$$h_i = x_i - x_{i-1} \quad \text{and} \quad k_j = y_j - y_{j-1}.$$

This defines a grid on $\bar{\Omega}$ with the grid lines $x = x_i$ and $y = y_j$. A general element is given by

$$\bar{\Omega}_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

For $i = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$, we have four piecewise bicubics denoted by $\delta_{ij}^{(k)}$ with $k = 0, 1, 2, 3,$ with the following properties:

1. The restriction of $\delta_{ij}^{(k)}$ to any $\bar{\Omega}_{IJ}$ is either a bicubic polynomial or zero for i and $I = 0, 1, \ldots, r,$ j and $J = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3.$

2. $\delta_{ij}^{(k)} \in C^1(\bar{\Omega})$ and all second order partial derivatives are piecewise continuous with possible discontinuities on the edges of the elements.

3. $\delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise,} \end{cases}$

   $\partial_x \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise,} \end{cases}$

   $\partial_y \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 2 \\ 0 & \text{otherwise,} \end{cases}$

   $\partial_x \partial_y \delta_{ij}^{(k)}(x_i, y_j) = \begin{cases} 1 & \text{if } k = 3 \\ 0 & \text{otherwise.} \end{cases}$

4. $\delta_{ij}^{(k)}(x_I, y_J) = 0, \quad \partial_x \delta_{ij}^{(k)}(x_I, y_J) = 0, \quad \partial_y \delta_{ij}^{(k)}(x_I, y_J) = 0$ and $\partial_x \partial_y \delta_{ij}^{(k)}(x_I, y_J) = 0$ if $(i, j) \neq (I, J).$

5. $\delta_{ij}^{(k)}$ is zero on any element Ω_{IJ} not adjacent to Ω_{ij}.

We refer to these four types of functions as Type 1 ($k = 0$), Type 2 ($k = 1$), Type 3 ($k = 2$) and Type 4 ($k = 3$) functions.

Remarks

1. As mentioned, for a fixed x or y, a piecewise bicubic reduces to a piecewise cubic. This compatibility is needed for the plate-beam problems.
2. $\delta_{ij}^{(k)} \in H^2(\Omega) \quad \forall \ i = 0, 1, \ldots, r, \quad j = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3$.

We use the following notation for the partial derivatives that play a role in construction of the bicubics.

$$\partial^{(k)} w = \begin{cases} w & \text{for } k = 0 \\ \partial_x w & \text{for } k = 1 \\ \partial_y w & \text{for } k = 2 \\ \partial_x \partial_y w & \text{for } k = 3 \end{cases}$$

Bicubic interpolation operator

For $w \in H^4(\Omega)$, we define the bicubic interpolation operator Π_b as

$$\Pi_b \, w = \sum_{k=0}^{3} \sum_{i=0}^{r} \sum_{j=0}^{s} (\partial^{(k)} w)(x_i, y_j) \, \delta_{ij}^{(k)} .$$

Note that $\Pi_b \, \delta_{ij}^{(k)} = \delta_{ij}^{(k)}$ for $i = 0, 1, \ldots, r, \ j = 0, 1, \ldots, s$ and $k = 0, 1, 2, 3$.

4.3 Standard estimates for the interpolation error

Standard interpolation estimates can be found in, for instance, [SF], [OR] and [OC]. The following two parameters for an interpolation operator are used in the interpolation estimates:

$r(\Pi)$ is the highest degree of polynomials left invariant by Π.

$s(\Pi)$ is the highest order derivative used in the definition of Π.

We will use $\widehat{C}$ to denote a generic constant which depends on the constants in Sobolev's lemma and the constants in the Bramble-Hilbert lemma.

Theorems 1 and 2 below are formulated as a special case of a general result. This result may be found in [SF, p 144], [OC, p 76] and [OR, p 279].

4.3.1 One-dimensional domain

We consider a one-dimensional domain $\Omega = (a, b)$. Here $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k = \|u^{(k)}\|\,.$$

(See Appendix 1.)

Theorem 1

Suppose $s(\Pi) + 1 \leq k \leq r(\Pi) + 1$. Then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)$,

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

Corollary

Consider the Hermite piecewise cubic functions and the interpolation operator Π_c.

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,

$$\|u - \Pi_c u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$,

$$\|u - \Pi_c u\|_m \leq \widehat{C} h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof

$r(\Pi_c) = 3$ and $s(\Pi_c) = 1$.

a) The result follows directly from Theorem 1.

b) If $k > 4$, $H^k(0,1) \subset H^4(0,1)$. The result follows from Theorem 1.

4.3.2 Two-dimensional domain

For a two-dimensional convex domain Ω, $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix 1.)

In the following theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e.

Theorem 2

Suppose $s(\Pi) + 2 \leq k \leq r(\Pi) + 1$. Then there exists a constant $\widehat{C}$ such that, for $u \in H^k(\Omega)$,

$$\|u - \Pi u\|_m \leq \widehat{C} h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

Corollary

Consider the piecewise Hermite bicubic functions and the interpolation operator Π_b. For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k(I)$

$$\|u - \Pi_b u\|_m \leq \widehat{C} h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof

$r(\Pi_b) = 3$ and $s(\Pi_b) = 2$. If $k > 4$, $H^k(0,1) \subset H^4(0,1)$ and the result follows from Theorem 2.

Remark

The constant $\widehat{C}$ depends on the ratio length versus width for the elements. Care should be taken that these ratios remain within specific bounds.

4.3.3 Vector-valued functions

Definition

For $u = \langle u_1,\, u_2\rangle \in H^k(\Omega)^2$, we define

$$\Pi_B\, u = \langle \Pi_b\, u_1,\, \Pi_b\, u_2\rangle\,.$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2\,.$$

(See Appendix 1.)

Theorem 3

There exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2$ with $k \geq 4$,

$$\|u - \Pi_B\, u\|_{m,2} \leq \widehat{C}h^{4-m}|u|_{4,2}\,, \quad m = 0, 1, \ldots .4\,.$$

Proof

The proof follows directly from the definition of the interpolation operator Π_B, the norm and seminorm on the product space and the corollary in Subsection 4.3.2.

$$\begin{aligned}
\|u - \Pi_B\, u\|_{m,2}^2 &= \|u_1 - \Pi_b\, u_1\|_m^2 + \|u_2 - \Pi_b\, u_2\|_m^2 \\
&\leq \left[\widehat{C}h^{4-m}\,|u_1|_4\right]^2 + \left[\widehat{C}h^{4-m}\,|u_2|_4\right]^2 \\
&= \left[\widehat{C}h^{4-m}|u|_{4,2}\right]^2 .
\end{aligned}$$

Corollary

There exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2 \cap V$ with $k \geq 4$,

$$\|u - \Pi_B\, u\|_V \leq \widehat{C} h^3 |u|_{4,2}\,.$$

Proof

The norms $\|\cdot\|_{1,2}$ and $\|\cdot\|_V$ are equivalent (Theorem 4 Sec 3.5).

4.4 Interpolation estimates for the one-dimensional hybrid models

Consider Problem VT 4 (Section 3.3). Let $\Omega = (a, b)$ and define H^k as $H^k = H^k(\Omega) \times H^k(\Omega) \times I\!R^3$. An interpolation operator on the product spaces H^k can now be defined.

Definition

$$\Pi u = \langle \Pi_c\, u_1,\, \Pi_c\, u_2,\, u_3,\, u_4,\, u_5 \rangle \quad \text{for } u \in H^k\,.$$

An **inner product** for H^k is defined by

$$\big(u, v\big)_{H^k} \quad = \quad \big(u_1, v_1\big)_k + \big(u_2, v_2\big)_k + u_3 v_3 + u_4 v_4 + u_5 v_5\,.$$

The corresponding **norm** is

$$\|u\|_{H^k} = \sqrt{\big(u, u\big)_{H^k}}\,.$$

A **seminorm** for H^k is defined by

$$|u|_{k,H^k} \quad = \quad \sqrt{|u_1|_k^2 + |u_2|_k^2}\,,$$

with $|\cdot|_k$ the seminorm in $H^k(\Omega)$.

Theorem

Consider the piecewise Hermite cubic functions and the interpolation operator Π.

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{m,H^k} \leq \widehat{C} h^{k-m} |u|_{k,H^k}\,, \quad m = 0, 1, \ldots . k.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{m,H^k} \leq \widehat{C} h^{k-m} |u|_{4,H^k}\,, \quad m = 0, 1, \ldots . 4.$$

Proof

In this proof, we use the result in Subsection 4.3.1.

$$\begin{aligned}
\|u - \Pi u\|_{m,H^k}^2 &= \|\langle u_1 - \Pi_c u_1\,,\, u_2 - \Pi_c u_2\,,\, 0\,,\, 0\,,\, 0\rangle\|_{m,H^k}^2 \\
&= \sum_{j=1}^{\ell} \|u_j - \Pi_c u_j\|_m^2 \\
&\leq \begin{cases} \displaystyle\sum_{j=1}^{2} \left[\widehat{C} h^{k-m}\, |u_j|_k\right]^2 & \text{if } \ 2 \leq k \leq 4, \\ \displaystyle\sum_{j=1}^{2} \left[\widehat{C} h^{4-m}\, |u_j|_4\right]^2 & \text{if } \ k > 4, \end{cases} \\
&= \begin{cases} \left[\widehat{C} h^{k-m} \big|u\big|_k\right]^2 & \text{if } \ 2 \leq k \leq 4, \\ \left[\widehat{C} h^{4-m} \big|u\big|_4\right]^2 & \text{if } \ k > 4. \end{cases}
\end{aligned}$$

Remark

It is easy to see that similar results can be found for the product spaces $H^k(\Omega) \times H^k(\Omega) \times I\!R$, $H^k(\Omega) \times I\!R^3$ and $H^k(\Omega) \times I\!R$.

Corollary 1 (Problems VR 3 and VR 4)

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^{k-2} |u|_{k,H^k} \,.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^2 |u|_{4,H^k} \,.$$

Proof

The results follow from the theorem, the fact that $V \subset H^2$ and the equivalence of the energy norm $\| \cdot \|_V$ and the H^2–norm.

Corollary 2 (Problems VT 3 and VT 4)

a) If $2 \leq k \leq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^{k-1} |u|_{k,H^k} \,.$$

b) If $k > 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k \cap V$,

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^3 |u|_{4,H^k} \,.$$

Proof

The energy norm $\| \cdot \|_V$ and the H^1–norm are equivalent.

4.5 Interpolation estimates for the plate-beam system

We consider an interval $I = (a, b)$ and a rectangle $\Omega = (a, b) \times (c, d)$. Define

$$\mathcal{H}^k = H^k(\Omega) \times H^k(\Omega)^2 \mathop{\times}_{n=1}^{4} H^1(\Omega) \,.$$

The other relevant product spaces are defined in Section 3.6.

Definition

For $u \in H^k$ we define the interpolation operator

$$\Pi u = \langle \Pi_b\, u_1,\ \Pi_B\, u_2,\ \Pi_c\, u_3,\ \Pi_c\, u_4,\ \Pi_c\, u_5,\ \Pi_c\, u_6 \rangle\,.$$

An **inner product** for H^k is defined by

$$\big(u, v\big)_{H^k} = (u_1, v_1)_k^\Omega + (u_2, v_2)_{k,2}^\Omega + \sum_{j=3}^{6} \big(u_j, v_j\big)_k^I\,,$$

The corresponding **norm** is given by

$$\|u\|_{H^k} = \sqrt{\big(u, u\big)_{H^k}}$$

and the **seminorm** $|\cdot|_{H^k}$ of order k is defined by

$$|u|_{H^k}^2 = \big(|u_1|_k^\Omega\big)^2 + \big(|u_2|_{k,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(|u_j|_k^I\big)^2$$

Theorem

Consider the interpolation operator Π defined above. For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in H^k$,

$$\|u - \Pi u\|_{H^m} \leq \widehat{C} h^{4-m} |u|_{H^4}\,, \quad m = 0, 1, \ldots . 4.$$

Proof

We use the results in Section 4.3.

$$\begin{aligned}
\|u - \Pi u\|_{H^m}^2 &= \big(\|u_1 - \Pi_b\, u_1\|_m^\Omega\big)^2 + \big(\|u_2 - \Pi_B\, u_2\|_{m,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(\|u_j - \Pi_b\, u_j\|_m^I\big)^2 \\
&\leq \Big(\widehat{C}_1 h^{4-m} |u_1|_m^\Omega\Big)^2 + \Big(\widehat{C}_2 h^{4-m} |u_2|_{m,2}^\Omega\Big)^2 + \sum_{j=3}^{6} \Big(\widehat{C}_j h^{4-m} |u_j|_m^I\Big)^2 \\
&\leq \Big(\widehat{C} h^{4-m}\Big)^2 \Big[\, \big(|u_1|_m^\Omega\big)^2 + \big(|u_2|_{m,2}^\Omega\big)^2 + \sum_{j=3}^{6} \big(|u_j|_m^I\big)^2 \Big] \\
&= \Big[\widehat{C} h^{4-m} |u|_{H^4}\Big]^2
\end{aligned}$$

Corollary

For $k \geq 4$, there exists a constant $\widehat{C}$ such that, for all $u \in V \cap H^k$,

$$\|u - \Pi u\|_V \leq \widehat{C} h^3 |u|_{H^4}\,.$$

Proof

The norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_V$ are equivalent (Theorem 2 Section 3.6).

Chapter 5

Approximation

5.1 Projections

For the spaces H^k and V as defined Sections 4.3, 4.4 and 4.5, we have the situation that for all our model problems a finite dimensional subspace S^h of V is constructed in such a way that the forced boundary conditions are met. At this stage an estimate for the interpolation error $u - \Pi u$ is available.

All the convergence results in this chapter are based on projection methods.

Definition (Projection P_h)

For each $x \in V$, we define $P_h x$ to be the unique element of S^h such that

$$b(x - P_h x, v) = 0 \quad \text{for all } v \in S^h.$$

It is well known and easy to prove that

$$b(x - P_h x, v) = 0 \quad \text{for all } v \in S^h$$

if and only if

$$\|x - P_h x\|_V \leq \|x - v\|_V \quad \text{for all } v \in S^h\,.$$

Since S^h is a finite dimensional subspace of the space V, the projection exists. This is a result from linear algebra (see e.g. [Ap, Chapter 15]). The result is also true for an infinite dimensional subspace (see e.g. [Kr, Sec 3.3]).

We display for convenience the elementary yet important properties of the projection P_h.

$$\begin{aligned} \|x - P_h x\|_V &\leq \|x - v\|_V \quad \text{for all } v \in S^h, \\ \|P_h x - v\|_V &\leq \|x - v\|_V \quad \text{for all } v \in S^h, \\ \text{and} \quad \|P_h x\|_V &\leq \|x\|_V . \end{aligned}$$

5.1.1 One-dimensional models

For the one-dimensional models, we consider only eigenvalue problems. The solutions of the differential equations are in $C^\infty(\bar{\Omega})$ and hence in $H^4(\Omega)$. This implies that the eigenvectors of the weak problem are in the product space H^4 .

Theorem 1

Suppose the energy norm is equivalent to the norm of H^m on V. Then there exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_{4,H^4}$ and $\|\Pi u - P_h u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_{4,H^4}$.

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^{2(4-m)}\, |u|_{4,H^4}$.

Remark

Problems VRE 3, VRE 4, VTE 3 and VTE 4 are defined in Section 6.2. For Problems VRE 3 and VRE 4 we have that $m = 2$ and for Problems VTE 3 and VTE 4 we have $m = 1$.

Proof

(a) It follows from the properties of the projection operator P_h that

$$\|P_h u - u\|_V \leq \|\Pi u - u\|_V \quad \text{and} \quad \|\Pi u - P_h u\|_V \leq \|\Pi u - u\|_V .$$

The estimates are found from Corollaries 1 and 2 in Section 4.4.

(b) Set $e_p = u - P_h u$. As b defines an inner product on V, it follows from Riesz's theorem that there exists a unique $u \in V$ such that

$$b(u, v) = c(e_p, v) \quad \text{for all } v \in V. \tag{5.1.1}$$

Regularity results yield that $u \in H^4 \cap V$ and that there exists a c_b such that

$$\|u\|_4 \leq c_b \|e_p\|_X . \tag{5.1.2}$$

Since P_h is a projection,

$$b(e_p, v) = 0 \quad \text{for all } v \in S. \tag{5.1.3}$$

Let $v = e_p$ in Equation (5.1.1) and $v = P_h u$ in Equation (5.1.3). This yields

$$\|e_p\|_X^2 = b(u - P_h u, e_p) \leq \|u - P_h u\|_V \|e_p\|_V .$$

From part (a) of the Theorem, it follows that

$$\|e_p\|_X^2 \leq \widehat{C}\, h^2 \, |u|_{4,H^4} \, \|e_p\|_V .$$

We conclude from Inequality (5.1.2) that

$$\|e_p\|_X \leq c_b \, \widehat{C} h^{4-m} \|e_p\|_V .$$

The result now follows from part (a) of the Theorem. □

Remark

The proof of part (b) of the Theorem is known as the Aubin-Nitsche trick ([Au] and [N]. This version is from the book of Strang and Fix ([SF, p 166]).

5.1.2 Two-dimensional models

The first result concerns Problems CTD 1 and CTD 2.

Theorem 2

There exists a constant $\widehat{C}$ such that, for any $u \in H^4(\Omega)^2 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^3 \, |u|_{4,2}$ and $\|\Pi_B u - P_h u\|_V \leq \widehat{C}\, h^3 \, |u|_{4,2}$.

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^6 \, |u|_{4,2}$.

Proof

The proof is similar to the proof of Theorem 1. $\square$

The next result applies to Problems RMT and KEB.

Theorem 3

Suppose the energy norm is equivalent to the norm of H^m on V. Then there exists a constant $\widehat{C}$ such that, for any $u \in H^4 \cap V$,

(a) $\|P_h u - u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_4$ and $\|\Pi u - P_h u\|_V \leq \widehat{C}\, h^{4-m}\, |u|_4$.

(b) $\|P_h u - u\|_X \leq \widehat{C}\, h^{2(4-m)}\, |u|_4$.

Proof

The proof is similar to the proof of Theorem 1. $\square$

For the two-dimensional problems, regularity can not be guaranteed, i.e. a solution may be in the space V but not in H^4. The following theorem is applicable in the case that u is not an element of one of the H^4-spaces as defined above.

Theorem 4

For any $\epsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that

$$\|u - P_h u\|_V < \epsilon \quad \text{if } h < \delta.$$

Proof

For any $u \in V$ there exists a $w \in H^4 \cap V$ such that $\|u - w\|_V \leq \epsilon$. Then

$$\begin{aligned}
\|P_h u - u\|_V &\leq \|u - w\|_V + \|w - P_h w\|_V + \|P_h w - P_h u\|_V \\
&\leq \epsilon + \widehat{C}\, h^2\, |w|_4 + \epsilon \\
&< 3\epsilon \quad \text{for } h \text{ sufficiently small.}
\end{aligned}$$

$\square$

5.2 Equilibrium problems

We consider the convergence of the Galerkin approximation of Problem CTD 1 to the solution of Problem CTD 1.

Assume that $u^h \in S^h$ is the solution of

$$b(u^h, v) \quad = \quad f(v) \;\; \text{for all } v \in S^h \tag{5.2.1}$$

and that $u \in V$ is the solution of

$$b(u, v) = f(v) \;\; \text{for all } v \in V. \tag{5.2.2}$$

Theorem

(a) If $u \in V$, then $\|u - u^h\|_V \longrightarrow 0$ as $h \longrightarrow 0$.

(b) If $u \in H^4 \cap V$, then

$$\|u - u^h\|_V \leq \widehat{C}\, h^3 \, |u|_{4,2} \quad \text{and} \quad \|u - u^h\|_X \leq \widehat{C}\, h^6 \, |u|_{4,2} \, .$$

Proof

Subtracting Equation (5.2.1) from Equation (5.2.2), we find that

$$b(u - u^h, v) = 0 \;\; \text{for all } v \in S^h \, .$$

Hence $u^h = P_h u$. Therefore $\|u - u^h\|_V = \|u - P_h u\|_V$ and the result follows from Theorem 4 Section 5.1.

5.3 Symmetrical eigenvalue problems

We consider the eigenvalue problem E1 in Section 3.9. The seminorm $|\cdot|_4$ used in this paragraph is general and used for a unified formulation of the theory. When applying the theory to Problem CDT 2, this seminorm is substituted by $|\cdot|_{4,2}$ and for Problem RMT by $|\cdot|_{4,H^4}$. A similar situation holds for the use of H^4.

Regularity assumption

The eigenvectors are in H^4 and there exists a constant C_b depending on the bilinear forms b and c, such that for each eigenvector y,

$$|y|_k \leq C_b \, \lambda \, \|y\|_X .$$

The Rayleigh quotient can be used to order the sequence of eigenvalues. Assume the eigenvalues are ordered as

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

Consider the eigenvalues $\lambda_1 , \lambda_2 , \ldots , \lambda_m$ for some m with the corresponding normalized eigenvectors $y_1 , y_2 , \ldots , y_m$. Assume furthermore that $\lambda_j \neq \lambda_m$ if $j > m$ ($\lambda_i = \lambda_j$ is possible for $i \leq m$ and $j \leq m$).

Corresponding to this situation, we have the eigenvalues $\lambda_1^h , \lambda_2^h , \ldots , \lambda_m^h$ (also ordered) and the corresponding eigenvectors $y_1^h , y_2^h , \ldots , y_m^h$ in S^h. In the case of a multiple eigenvalue, the eigenvector is not uniquely determined. The following three theorems are from [SF]. In [ZVGV2] and [Ziet] it was shown that the results are applicable in the general abstract case.

Theorem 1

$\lambda_i^h \geq \lambda_i$ for each i.

Theorem 2

(a) $\lambda_m^h \longrightarrow \lambda_m$ as $h \longrightarrow 0$.

(b) If the regularity assumption holds, then $\lambda_m^h - \lambda_m \leq \widehat{C} \, C_b \, \lambda_m^2 \, h^{2(4-m)}$.

We assume that the sequence of eigenvector approximations $\left\{ y_j^h \right\}$ is normalized.

Theorem 3

Suppose that the dimension of the eigenspace E_m corresponding to λ_m is r.

(a) Let $\epsilon > 0$. For h sufficiently small, there exists a $y \in E_m$ with $\|y\| = 1$ such that

$$\|y - y^h_{m-r+j}\| \leq \epsilon$$

for $j = 1, 2, \ldots, r$.

(b) Suppose Problem E1 satisfies the regularity assumption. If h is sufficiently small, there exists a $y \in E_m$ with $\|y\| = 1$ such that

$$\|y - y^h_{m-r+j}\|_V \leq \widehat{C}\, C_b\, \lambda_m\, h^{4-m}$$

for $j = 1, 2, \ldots, r$.

5.4 Non selfadjoint eigenvalue problem

In this section we consider Problem E2 formulated in Section 3.10.

5.4.1 Abstract eigenvalue problem

Following [VV], we introduce a linear operator Λ on H with the property that the eigenvalues of Λ are the reciprocals of the eigenvalues of Problem E2 and the eigenvectors are the same.

Recall that X and V are complex Hilbert spaces with V dense in X. Also, H is the product space $V \times X$ with inner product

$$(x, y)_H = b(x_1, y_1) + c(x_2, y_2).$$

Theorem 1

Suppose

(a) V is dense in X,

(b) $\|u\|_X \leq K\|u\|_V$ for each $u \in V$,

(c) the bilinear form a is symmetric, nonnegative and $|a(u, v)| \leq C\|u\|_V\|v\|_V$ for each u and v in V.

Then, for each $y \in H$, there exists a unique $x \in H$ such that

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

Proof

Appendix 5.

Definition Operator (Λ)

$\Lambda y = x$ if

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

It is easy to see that Λ is linear.

Theorem 2

Λ is bounded.

Proof

Appendix 5.

Theorem 3

λ is an eigenvalue and x an eigenvector of Problem E2 if and only if $\lambda \Lambda x = x$.

Proof

Simply substitute $y = \lambda x$ in the definition of Λ.

Theorem

Λ is invertible and its range is dense in H.

Proof

See [VV].

Remark

We may define a linear operator $T = \Lambda^{-1}$. It is clear that T is a closed linear operator with domain $\mathcal{D}(T)$ which is dense in H. As a consequence one may study the eigenvalue problem $Tx = \lambda x$. This problem is equivalent to the problem considered in [Shu].

5.4.2 Galerkin approximation

Consider a finite dimensional subspace S^h of the complex Hilbert space V. The following problem yields the approximations for the quadratic eigenvalue problem QE.

Problem QED

Find a complex number λ_h and $u^h \in S^h$ such that

$$\lambda_h^2 c(u^h, v) + \lambda_h a(u^h, v) + b(u^h, v) = 0 \quad \text{for each } v \in S^h.$$

This is the type of problem solved in Chapter 6.

Definition (Subspace $\boldsymbol{H^h}$)

$$H^h = S^h \times S^h.$$

Problem E2D

Find a complex number λ_h and $x^h \in H^h$ such that

$$\begin{aligned} x_2^h &= \lambda^h x_1^h \\ b(x_1^h, v) + a(x_2^h, v) &= -\lambda^h c(x_2^h, v) \quad \text{for each } v \in S^h \,. \end{aligned}$$

If λ_h is an eigenvalue and u^h an eigenvector of Problem QED, then λ_h is an eigenvalue and $\langle u^h, \lambda u^h \rangle$ an eigenvector of Problem E2D. Conversely, if λ_h is an eigenvalue and x^h an eigenvector of Problem E2D, then λ_h is an eigenvalue and x_1^h an eigenvector of Problem QED.

Projection

Recall the projection P^h defined in Section 4.1. Without changing the notation, we define a projection for the complex space V by $P^h x = P^h x_1 + i P^h x_2$. It is clear that we still have the following properties.

$$\begin{aligned} b(x - P^h x, v) = 0 \quad &\text{for each } v \in S^h \,, \\ \|x - P^h x\|_V \leq \|x - v\|_V \quad &\text{for each } v \in S^h \,. \end{aligned}$$

5.4.3 Operator approximations

Let $y \in H$ and consider the problem to find $u^h \in S^h$ such that

$$b(u^h, v) + a(y_1, v) = -c(y_2, v) \quad \text{for each } v \in S^h.$$

It is clear that a unique solution exists (see Theorem 1).

Definition (Operator Λ^h)

$\Lambda^h y = x$ if $x_1 \in S^h$ and

$$\begin{aligned} x_2 &= y_1, \\ b(x_1, v) + a(y_1, v) &= -c(y_2, v) \quad \text{for each } v \in S^h \,. \end{aligned}$$

It is easy to see that Λ^h is linear.

Theorem 5

Λ^h is bounded and the restriction of Λ^h to $S^h \times S^h$ is a bijection.

Proof

The same as the proof of Theorem 2.

Theorem 6

λ^h is an eigenvalue and x^h an eigenvector of Problem E2D if and only if $\lambda^h \Lambda^h x^h = x^h$.

Proof

Simply substitute $y = \lambda x^h$ in the definition of Λ^h.

Remark

It is clear that Λ^h has a zero eigenvalue since $N(\Lambda^h) = \left(S^h \times S^h\right)^\perp$.

Notation

$$\delta^h(x) = \inf\left\{\|x_1 - v\|_V \;\middle|\; v \in S^h\right\}.$$

Remark

In general, $\delta^h(x) \to 0$ as $h \to 0$ for each $x \in H$.

Theorem 7

If $\Lambda y = x$, then
$$\|\Lambda^h y - \Lambda y\|_H \leq \delta^h(x).$$

Proof

If $\Lambda^h y = x^h$, then

$$b(x_1 - x_1^h, v) = 0 \quad \text{for each } v \in S^h.$$

5.4.4 Convergence

Consider a sequence of operators $\Lambda_n = \Lambda^{h_n}$ where $h_n \to 0$.

Notation

Let λ be an isolated eigenvalue of Λ, P the spectral projection and $M = PH$ the invariant subspace associated with λ. Assume that $\dim M = m < \infty$. There exists a $\rho > 0$ such that λ is the only eigenvalue in $B_\rho(\lambda)$. M_n denotes the invariant subspace of Λ_n associated with the m eigenvalues (counting multiplicity) contained in $B_\rho(\lambda)$.

Theorem 8

Suppose that $\{\Lambda_n\}$ is a strongly stable approximation of Λ in $B_\rho(\lambda)$. Then, for n sufficiently large, Λ_n has m eigenvalues in $B_\rho(\lambda)$, counting their multiplicities. All these eigenvalues converge to λ as $n \longrightarrow \infty$.

Proof

See [Ch, p 234].

Definition (Gap between subspaces)

P is an orthogonal projection on M,

Q is an orthogonal projection on M_n,

$\alpha = \sup \left\{ \|x - Qx\|_H \;\middle|\; x \in M;\ \|x\|_H = 1 \right\}$,

$$\beta = \sup \left\{ \|x - Px\|_H \;\middle|\; x \in M_n;\ \|x\|_H = 1 \right\},$$

$$\Theta(M, M_n) = \max \{\alpha, \beta\}$$

Remark

If M and M_n are one-dimensional (as is mostly the case in our applications), then $\Theta(M, M_n) = \sin\theta$ where θ is the angle between M and M_n.

Theorem 9

If Λ_n is an approximation of Λ and strongly stable on $\overline{B_\rho(\lambda)}$, then $\Theta(M, M_n) \longrightarrow 0$ as $n \to \infty$.

Proof

[Ch, p 235-236].

5.4.5 Application

We apply the theory to the one-dimensional hybrid models in Sections 3.2 and 3.3. Consider for example Problem VTE 4 with weak variational form in Section 3.3.2. In this case, the quadratic eigenvalue problem Problem QE and its equivalent form Problem E2 involves ordinary differential equations. Any eigenvector for Problem QE is in $C^\infty[0,1] \times C^\infty[0,1] \times \mathbb{R}^3$. The error bounds for the projection P^h in Section 5.1 are valid. Also, the operator Λ associated with Problem $E2$ is compact.

Convergence

Theorem 10

For $\mu \in B_\rho(\lambda)$, $\mu \neq 0$ and $\mu \neq \lambda$, $\mu I - \Lambda_n$ is a strongly stable approximation for $\mu I - \Lambda$.

Proof

Since $(\mu I - \Lambda)^{-1}$ exists and $\mu I - \Lambda_n$ converges pointwise to $\mu I - \Lambda$, it follows that $(\mu I - \Lambda_n)^{-1}$ converges pointwise to $(\mu I - \Lambda)^{-1}$. But Λ_n converges compactly to Λ ([Ch, p 122]). Consequently, $\mu I - \Lambda_n$ is a strongly stable approximation of $\mu I - \Lambda$ for $\mu \neq 0$ (Lemma 5.24 and Theorem 5.26 ([Ch, p 247-248])). Finally, Proposition 5.27 ([Ch, p 248-249]) implies the result.

Remark

Theorems 8 and 9 may now be applied.

Error bounds

The theory in [Ch, Sec 6.2] on projection methods, is applicable to our situation.

Notation

$\widehat{\lambda}_n = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \lambda_j$, where $\lambda_j \in B_\delta(\lambda)$.

$\delta_n(x) = \delta^h(x)$ where $h = h_n$ and $H_n = H^{h_n}$.

$\delta\left(M, H_n\right) = \sup\left\{\delta_n(x) \;\middle|\; x \in M;\ \|x\|_H = 1\right\}$.

Theorem 11

Consider Problem E2 for the system in Problem VTE 4. Then

$$\begin{aligned} \left|\lambda - \widehat{\lambda}_n\right| &\leq K\delta\left(M, H_n\right), \\ \Theta\left(M, M_n\right) &\leq K\delta\left(M, H_n\right). \end{aligned}$$

Proof

See Lemmas 6.9 and 6.10 in [Ch, p 284].

Theorem 12

There exists a constant C_λ such that $\delta\left(M, H_n\right) \leq C_\lambda h_n$.

Proof

Note that for each $u \in H$, we have (Theorem 1, Section 5.1)

$$\delta_n(u) \leq \widehat{C}\, h_n\, |u|_2\,.$$

But,

$$\begin{aligned} -u_1'' + u_2' &= \lambda u_1, \\ -\frac{1}{\gamma} u_2'' - \alpha u_1' + \alpha u_2 &= \lambda u_2\,. \end{aligned}$$

Consequently,

$$|u|_2 \leq K_\lambda \|u\|_V \leq K_\lambda \|u\|_H$$

for some constant K_λ.

Consequently, there exists a constant C_λ such that

$$\delta(M, H_n) \leq C_\lambda\, \widehat{C}\, h_n\,.$$