Chapter 3

Variational forms

3.1 Introduction

In this chapter we consider the variational and weak variational forms for the problems under consideration. The variational form is used when we approximate solutions with the finite element method and the weak variational form is necessary for theoretical considerations.

In this section we consider free vibrations of a Timoshenko cantilever beam as an example. The equations of motion are then given by Equations (1.2.5) and (1.2.6). For this model $P = L = 0$.

To find the variational form of this problem, multiply the two equations of motion with functions v and ψ respectively and integrate.

$$\begin{aligned}
\int_0^1 \big(\partial_t^2 w(x,t)\big)v(x)dx &= \int_0^1 \big(\partial_x V(x,t)\big)v(x)dx, \\
\int_0^1 \frac{1}{\alpha}\big(\partial_t^2 \phi(x,t)\big)\psi(x)dx &= \int_0^1 \big(\partial_x M(x,t)\big)\psi(x)dx + \int_0^1 V(x,t)\psi(x)dx.
\end{aligned}$$

We use the notation

$$(f,g) = \int_0^1 f(x)g(x)dx$$

for convenience. (The fact that this is the inner product for $\mathcal{L}^2(0,1)$, is not relevant at this stage.)

Use integration by parts to find that

$$\begin{aligned}\left(\partial_t^2 w(\cdot,t), v\right) &= -(V(\cdot,t), v') + [V(\cdot,t)v]_0^1,\\ \frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) &= -(M(\cdot,t), \psi') + (V(\cdot,t), \psi) + [M(\cdot,t)\psi]_0^1.\end{aligned}$$

Since $V(1,t) = M(1,t) = 0$,

$$\left(\partial_t^2 w(\cdot,t), v\right) = -(V(\cdot,t), v') - V(0,t)v(0), \tag{3.1.1}$$

$$\frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) = -(M(\cdot,t), \psi') + (V(\cdot,t), \psi) - M(0,t)\psi(0). \tag{3.1.2}$$

The test functions are defined as

$$T(0,1) = \left\{v \in C^1(0,1) \;\middle|\; v(0) = 0\right\}.$$

We substitute the constitutive equations into the equations above to find the variational form of the problem.

Variational form

Find w and ϕ such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, $\phi(\cdot,t) \in T(0,1)$,

$$\left(\partial_t^2 w(\cdot,t), v\right) = -(\partial_x w(\cdot,t) - \phi(\cdot,t), v') \tag{3.1.3}$$

for each $v \in T(0,1)$,

$$\frac{1}{\alpha}\left(\partial_t^2 \phi(\cdot,t), \psi\right) = -\frac{1}{\beta}\left(\partial_x \phi(\cdot,t), \psi'\right) + (\partial_x w(\cdot,t) - \phi(\cdot,t), \psi) \tag{3.1.4}$$

for each $\psi \in T(0,1)$.

Remark

The variational form can be used to compute approximations for the solutions of the vibration problem as well as the eigenvalue problem. The variational form can also be used to investigate the solvability of the problem. This is done by showing that the results for a general linear vibration problem may be applied to this specific problem.

General linear vibration problem

Let H be a Hilbert space and u a function mapping the interval $[0, T]$ into H. The derivatives of u are defined in the usual way (see Appendix 4). Every linear vibration problem can be written in the form below for suitable bilinear forms a, b and c defined on H.

For each $t \in (0, T)$,

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = \big(f(t), v\big),$$

for each $v \in V$, where V is some subspace of H.

The existence theory for the general problem is discussed in Section 3.8 and the theory of eigenvalue problems in Sections 3.9 and 3.10.

To apply the theory to the problem we are considering, the problem must be written in the appropriate form and the necessary estimates derived. The first step is to add Equations (3.1.3) and (3.1.4). We find that

$$\begin{aligned} &\Big(\partial_t^2 w(\cdot, t), v\Big) + \frac{1}{\alpha}\Big(\partial_t^2 \phi(\cdot, t), \psi\Big) \\ = \quad &-\frac{1}{\beta}\left(\partial_x \phi(\cdot, t), \psi'\right) - \left(\partial_x w(\cdot, t) - \phi(\cdot, t), v' - \psi\right). \end{aligned}$$

Next we need a suitable Hilbert space and subspace to relate our problem to the general vibration problem. We use the Sobolev space $H^1(0, 1)$ discussed in Appendix 1 to define suitable product spaces.

Product spaces

Consider the product spaces

$$X = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1) \quad \text{and} \quad H^1 = H^1(0, 1) \times H^1(0, 1).$$

Let $V(0, 1)$ be the closure of $T(0, 1)$ in the Sobolev space $H^1(0, 1)$ and let $V = V(0, 1) \times V(0, 1)$. (Note that V is a subspace of the Hilbert space H^1.)

Bilinear forms

For u and v in $\mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1)$,

$$c(u, v) \quad = \quad (u_1, v_1) + \frac{1}{\alpha}(u_2, v_2).$$

For u and v in $H^1(0,1) \times H^1(0,1)$,

$$b(u,v) = \frac{1}{\beta}(u_2', v_2') + (u_1' - u_2, v_1' - v_2).$$

Note that both b and c are symmetric.

Weak variational form

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) \text{ for each } v \in V.$$

Estimates

For the product spaces X and H^1, we may use the obvious inner products and as a consequence we have the respective norms

$$\|u\|_{0,2} = \sqrt{\|u_1\|^2 + \|u_2\|^2}$$
$$\text{and} \quad \|u\|_{1,2} = \sqrt{\|u_1\|_1^2 + \|u_2\|_1^2}.$$

However, other equivalent norms are more convenient.

Theorem 1

Assume that $\alpha \geq 1$. Then

(a) $\|u\|_{0,2}^2 \leq \alpha\, c(u,u) \leq \alpha\, \|u\|_{0,2}^2$ for each $u \in X$.

(b) $\|u\|_{0,2}^2 \leq \|u\|_{1,2}^2 \leq 6\beta\, b(u,u) \leq 12\beta\, \|u\|_{1,2}^2$ for each $u \in V$.

Proof

(a) The proof is trivial.

(b) For $u \in V$, we have that u_1 and u_2 are in $V(0,1)$. Since $V(0,1)$ is the closure of $T(0,1)$ in $H^1(0,1)$, it follows from Theorem 1 Appendix 2 that

$$\|u_1\| \leq \|u_1'\| \quad \text{and} \quad \|u_2\| \leq \|u_2'\|$$

Therefore

$$\|u\|_{0,2}^2 \leq \|u_1'\|^2 + \|u_2'\|^2 \leq \|u\|_{1,2}^2 .$$

This proves the first inequality.

We use $\|u_1'\| \leq \|u_1' - u_2\| + \|u_2\|$ and $(a+b)^2 \leq 2a^2 + 2b^2$ to find

$$\|u_1'\|^2 \leq 2\|u_1' - u_2\|^2 + 2\|u_2\|^2 .$$

It follows that

$$\|u_1'\|^2 + \|u_2'\|^2 \leq 2\|u_1' - u_2\|^2 + 3\|u_2'\|^2 \leq 3\beta\, b(u,u).$$

The second inequality follows from the inequality above and the inequalities

$$\|u_1\| \leq \|u_1'\| \quad \text{and} \quad \|u_2\| \leq \|u_2'\|.$$

The last inequality is trivial since $\|u_1' - u_2\|^2 \leq 2\|u_1'\|^2 + 2\|u_2\|^2$. □

Conclusion

The bilinear form c is an inner product for the space X and b is an inner product for the space V. Theorem 1 shows that for the space X, the norm associated with c is equivalent to $\|\cdot\|_{0,2}$. Similarly, for the space V, the norm associated with b is equivalent to $\|\cdot\|_{1,2}$.

Notation

$$\|u\|_X = \sqrt{c(u,u)} \quad \text{and} \quad \|u\|_V = \sqrt{b(u,u)} \; .$$

We call the space X with inner product c the **inertia space** and the space V with inner product b the **energy space**.

Theorem 2

Assume that $\alpha \geq 1$. The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

From Theorem 2 Appendix 1 it follows that $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ is separable. Furthermore, $\|\cdot\|_X$ and $\|\cdot\|_{0,2}$ are equivalent norms in X and it follows that X is separable.

$T(0,1)$ is dense in $\mathcal{L}^2(0,1)$, since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ (Theorem 3 Appendix 1). Clearly $V = V(0,1) \times V(0,1)$ is dense in $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ and hence V is dense in X. □

Remark

The assumption that $\alpha \geq 1$ is not necessary and the result is true for $\alpha > 0$. However, in applications α is large compared to one.

Theorem 3

The embedding of the space V into X is compact.

Proof

The embedding of $H^1(0,1)$ into $\mathcal{L}^2(0,1)$ is compact (Theorem 7 Appendix 1). Consequently the embedding of H^1 into $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ is compact. The result follows since the relevant norms are equivalent. □

The assumptions in Sections 3.8, 3.9 and 3.10 are valid for the cantilever Timoshenko beam and hence the theory can be applied to this model problem.

3.2 Vertical slender structure: Rayleigh models

3.2.1 Variational forms

To obtain the variational form of Problems VR 3 and VR 4, Equation (2.1.1) is multiplied by a function v and integration by parts (as in Section 3.1) yields

$$(\partial_t^2 w(\cdot,t), v') = -(V(\cdot,t), v') - V(0,t)v(0).$$

Multiply Equation (2.1.2) by v' to find

$$\frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = (V(\cdot,t), v') + (\partial_x M(\cdot,t), v') + (L(\cdot,t), v').$$

Adding the two equations we have

$$(\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = (\partial_x M(\cdot,t), v') + (L(\cdot,t), v') - V(0,t)v(0).$$

Integration by parts on the first term on the right and substitution of Equation (2.1.3) yield

$$\begin{aligned}(\partial_x M(\cdot,t), v') &= -(M(\cdot,t), v'') - M(0,t)v'(0)\\ &= -\frac{1}{\beta}\,(\partial_x^2 w(\cdot,t), v'') - M(0,t)v'(0).\end{aligned}$$

From Equation (2.1.4) we have

$$(L(\cdot,t), v') = \mu\int_0^1 (1-x)\partial_x w(x,t)v'(x)dx.$$

Combining the results above, we obtain a general variational form.

$$(\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}\,(\partial_t^2\partial_x w(\cdot,t), v') = -\frac{1}{\beta}\,(\partial_x^2 w(\cdot,t), v'')$$

$$+\mu\int_0^1 (1-x)\partial_x w(x,t)v'(x)dx - V(0,t)v(0) - M(0,t)v'(0). \qquad (3.2.1)$$

The variational form of each model depends on how we treat the terms containing $V(0,t)$ and $M(0,t)$. In all the models the solution w must satisfy Equation (3.2.1) for all test functions v.

For Problems VR 3 and VR 4 there are no restrictions on the space of test functions $T(0,1)$. Consequently, there are no forced boundary conditions for the solution w and it must satisfy Equation (3.2.1) for an arbitrary function $v \in T(0,1) = C^2[0,1]$.

We define the the following **bilinear forms**.

$$\begin{aligned}c_A(u,v) &= (u,v) + \frac{1}{\alpha}\,(u',v') + m_F\,u(0)v(0)\\ b_A(u,v) &= \frac{1}{\beta}\,(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)dx\\ &\quad +K_F u(0)v(0) + ku'(0)v'(0)\end{aligned}$$

Equations (3.2.1), (2.1.5) and (2.1.7) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v) &= -b_A(w(\cdot,t),v) - C_F\partial_t w(0,t)v(0) + k\theta_F(t)v'(0) \\ &\quad -c\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_F(t)\Big)v'(0) \end{aligned} \tag{3.2.2}$$

Together with Equation (2.1.6) given by

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t), \end{aligned} \tag{2.1.6}$$

we find the variational form of the problem.

Variational form of Problem VR 3

Find w and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, Equation (3.2.2) holds for each $v \in T(0,1)$ and Equation (2.1.6) holds. □

For Problem VR 4 we define the following **bilinear forms**.

$$\begin{aligned} c_A(u,v) &= (u,v) + \frac{1}{\alpha}\,(u',v') + m_B\,u(0)v(0)\,, \\ b_A(u,v) &= \frac{1}{\beta}\,(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)dx \\ &\quad +K_{FB}u(0)v(0) + k_{BA}u'(0)v'(0)\,. \end{aligned}$$

Equations (3.2.1), (2.1.8) and (2.1.10) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v) &= -b_A(w(\cdot,t),v) + K_{FB}w_F(t)v(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v(0) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big)v'(0) \\ &\quad +k_{BA}\theta_B(t)v'(0) \end{aligned} \tag{3.2.3}$$

Together with Equations (2.1.9), (2.1.11) and (2.1.12) which are given again for convenience, we are able to formulate the variational form of Problem

VR 4.

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{2.1.9}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{2.1.11}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{2.1.12}$$

Variational form of Problem VR 4

Find w, θ_B, w_F and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, Equation (3.2.3) holds for each $v \in T(0,1)$ and Equations (2.1.9), (2.1.11) and (2.1.12) hold. □

The variational forms above are used for finite element approximations (see Chapter 6).

3.2.2 Weak variational forms

For the analysis of the vibration problems we consider the weak variational forms. We consider only Problem VR 4, since Problem VR 3 is similar to Problem VR 4 but simpler.

For the weak variational form we redefine c_A and b_A in Subsection 3.2.1.

Bilinear forms

For u and v in $H^1(0,1)$,

$$c_A(u,v) = (u,v) + \frac{1}{\alpha}(u',v') + m_B u(0)v(0).$$

For u and v in $H^2(0,1)$,

$$b_A(u,v) = \frac{1}{\beta}(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)\,dx.$$

With the new notation and setting $v = v_1$, Equation (3.2.3) becomes

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v_1) &= -b_A(w(\cdot,t),v_1) - K_{FB}\Big(w(0,t)-w_F(t)\Big)v_1(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t)-\dot{w}_F(t)\Big)v_1(0) \\ &\quad -k_{BA}\Big(\partial_x w(0,t)-\theta_B(t)\Big)v_1'(0) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t)-\dot{\theta}_B(t)\Big)v_1'(0). \end{aligned} \tag{3.2.4}$$

Multiplying Equation (2.1.9) by an arbitrary real number v_2 and adding this to Equation (3.2.4) results in

$$\begin{aligned} c_A(\partial_t^2 w(\cdot,t),v_1) + I_B\ddot{\theta}_B(t)v_2 &= -b_A(w(\cdot,t),v_1) \\ &\quad -K_{FB}\Big(w(0,t)-w_F(t)\Big)v_1(0) \\ &\quad -C_{FB}\Big(\partial_t w(0,t)-\dot{w}_F(t)\Big)v_1(0) \\ &\quad -k_{BA}\Big(\partial_x w(0,t)-\theta_B(t)\Big)\Big(v_1'(0)-v_2\Big) \\ &\quad -c_{BA}\Big(\partial_t\partial_x w(0,t)-\dot{\theta}_B(t)\Big)\Big(v_1'(0)-v_2\Big) \\ &\quad -k_{FB}\Big(\theta_B(t)-\theta_F(t)\Big)v_2 \\ &\quad -c_{FB}\Big(\dot{\theta}_B(t)-\dot{\theta}_F(t)\Big)v_2. \end{aligned} \tag{3.2.5}$$

Define a function y with values in $\mathcal{L}^2(0,1)$ by $y(t) = w(\cdot,t)$.

For the definition of the derivatives $y'(t)$ and $y''(t)$, see Appendix 4. In this subsection, we will use the notation $\dot{y}(t)$ and $\ddot{y}(t)$ instead of $y'(t)$ and $y''(t)$ to distinguish between time and spatial derivatives.

Finally we need the **trace operator** γ which is defined in Appendix 3. Here

$\gamma u = u(0)$ for $u \in \mathcal{L}^2(0,1)$. With the new notation, Equation (3.2.5) becomes

$$\begin{aligned} c_A(\ddot{y}(t), v_1) + I_B \ddot{\theta}_B(t) v_2 &= -b_A(y(t), v_1) \\ &\quad -K_{FB}\Big(\gamma y(t) - w_F(t)\Big)\gamma v_1 \\ &\quad -C_{FB}\Big(\gamma\big(\dot{y}(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\ &\quad -k_{BA}\Big(\gamma\left[\big(y(t)\big)'\right] - \theta_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\ &\quad -c_{BA}\Big(\gamma\left[\big(\dot{y}(t)\big)'\right] - \dot{\theta}_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)v_2 \\ &\quad -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)v_2 \end{aligned} \tag{3.2.6}$$

Remark

Note that $\partial_t w(0,t)$ is replaced by $\gamma\big(\dot{y}(t)\big)$ and not $d_t(\gamma y)(t)$. This is necessary for the weak variational form of the problem. Fortunately, the choice is not a problem. This fact is discussed at the end of the section.

Multiply Equation (2.1.11) by v_3 and Equation (2.1.12) by v_4.

$$\begin{aligned} m_F \ddot{w}_F(t) v_3 &= K_{FB}\Big(w_B(t) - w_F(t)\Big)v_3 + C_{FB}\Big(\dot{w}_B(t) - \dot{w}_F(t)\Big)v_3 \\ &\quad -K_F w_F(t) v_3 - C_F \dot{w}_F(t) v_3 \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} I_F \ddot{\theta}_F(t) v_4 &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)v_4 + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)v_4 \\ &\quad -k_F \theta_F(t) v_4 - c_F \dot{\theta}_F(t) v_4 \end{aligned} \tag{3.2.8}$$

Add Equations (3.2.6), (3.2.7) and (3.2.8) to find

$$\begin{aligned}
& c_A(\ddot{y}(t), v_1) + I_B\ddot{\theta}_B(t)v_2 + m_F\ddot{w}_F(t)v_3 + I_F\ddot{\theta}_F(t)v_4 \\
= & -b_A(y(t), v_1) - K_{FB}\Big(\gamma y(t) - w_F(t)\Big)\Big(\gamma v_1 - v_3\Big) \\
& -C_{FB}\Big(\gamma\big(\dot{y}(t)\big) - \dot{w}_F(t)\Big)\Big(\gamma v_1 - v_3\Big) \\
& -k_{BA}\Big(\gamma[\big(y(t)\big)'] - \theta_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\
& -c_{BA}\Big(\gamma[\big(\dot{y}(t)\big)'] - \dot{\theta}_B(t)\Big)\Big(\gamma v_1' - v_2\Big) \\
& -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)\Big(v_2 - v_4\Big) \\
& -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big(v_2 - v_4\Big) \\
& -K_F w_F(t)v_3 - C_F\dot{w}_F(t)v_3 - k_F\theta_F(t)v_4 - c_F\dot{\theta}_F(t)v_4 \,. \qquad (3.2.9)
\end{aligned}$$

To formulate the weak form of the variational problem, the following product spaces and bilinear forms are necessary.

Product spaces

Define the product spaces

$$X = H^1(0,1) \times I\!R^3 \quad \text{and} \quad V = H^2(0,1) \times I\!R^3.$$

Bilinear forms

For u and v in $H^1(0,1)$,

$$\begin{aligned}
c(u,v) &= c_A(u_1, v_1) + I_B u_2 v_2 + m_F u_3 v_3 + I_F u_4 v_4 \,, \\
a(u,v) &= C_{FB}\,(\gamma u_1 - u_3)\,(\gamma v_1 - v_3) + c_{BA}(\gamma u_1' - u_2)(\gamma v_1' - v_2) \\
&\quad + c_{FB}(u_2 - u_4)(v_2 - v_4) + C_F u_3 v_3 + c_F u_4 v_4 \,.
\end{aligned}$$

For u and v in $H^2(0,1)$,

$$\begin{aligned}
b(u,v) &= b_A(u_1, v_1) + K_{FB}\,(\gamma u_1 - u_3)\,(\gamma v_1 - v_3) \\
&\quad + k_{BA}(\gamma u_1' - u_2)(\gamma v_1' - v_2) + k_{FB}(u_2 - u_4)(v_2 - v_4) \\
&\quad + K_F u_3 v_3 + k_F u_4 v_4 \,.
\end{aligned}$$

Note that a, b and c are all symmetric.

We are now ready to formulate the weak variational form of Problem VR 4 in terms of the defined bilinear forms. The table below shows the relationship between the components of u and the variables in Equation (3.2.9).

$u_1(t)$	u_2	u_3	u_4
$w(\cdot,t)$	$\theta_B(t)$	$w_F(t)$	$\theta_F(t)$

Weak variational form of Problem VR 4

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) - a(u'(t), v) \quad \text{for each } v \in V. \qquad \square$$

The existence theorem for problems of this type is presented in Section 3.8. If the initial conditions are chosen properly, $u_1 \in C^2\big((0,T);\ H^1(0,1)\big)$ and we find for example that $(\dot{y}(t))' = d_t\,(y'(t))$ and $d_t\,[\gamma y(t)] = \gamma \dot{y}(t)$.

The inertia space *X*

The bilinear form c is an inner product for the space X and consequently we may define a norm for $u \in X$ by

$$\|u\|_X = \sqrt{c(u,u)}\ .$$

The space X with norm $\|\cdot\|_X$ is called the **inertia space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

The proof is similar to the proof of Theorem 2 in Section 3.1. $\square$

It is obvious that the inner products of $H^2(0,1)$ and $I\!R^3$ can be used to define an inner product for the space V. We will show that the symmetric bilinear form b is also an inner product for V which is convenient for the theory.

Remark

In the following theorems, we assume throughout that the inequalities below hold for the physical constants.

$$1 > 2\mu\beta, \quad k_{BA} > 4\mu, \quad k_{FB} > 8\mu \ \text{ and } \ k_F > 8\mu.$$

These assumptions are physically realistic, as can be seen in Section 6.5.

Theorem 2

There exists a constant K_{bc} such that

$$\|u\|_X^2 \le K_{bc}\, b(u,u)$$

for each $u \in V$.

Proof

In the proof we use the elementary inequalities

$$\|x\| \le \|x-y\| + \|y\| \quad \text{and} \quad (a+b)^2 \le 2(a^2+b^2)$$

and the fact that

$$\|u_1\| \le \|u_1'\| + |\gamma u_1|.$$

(See Theorem 3 Appendix 3.)

This implies that

$$\begin{aligned} \|u_1\|^2 &\le 2\|u_1'\|^2 + 2(\gamma u_1)^2 \\ \text{and} \ \ \|u_1'\|^2 &\le 2\|u_1''\|^2 + 2(\gamma u_1')^2. \end{aligned}$$

Therefore

$$c_A(u_1,u_1) \ \le \ 2\left(2+\frac{1}{\alpha}\right)\left(\|u_1''\|^2 + (\gamma u_1')^2\right) + (2+m_B)\,(\gamma u_1)^2.$$

With

$$\begin{aligned} (\gamma u_1)^2 &\leq 2(\gamma u_1 - u_3)^2 + 2u_3^2, \\ (\gamma u_1')^2 &\leq 2(\gamma u_1' - u_2)^2 + 4(u_2 - u_4)^2 + 4u_4^2 \\ u_2^2 &\leq (u_2 - u_4)^2 + u_4^2 \\ \text{and} \quad \|u\|_X^2 &= c_A(u_1, u_1) + I_B u_2^2 + m_F u_3^2 + I_F u_4^2, \end{aligned}$$

it follows that

$$\|u\|_X^2 \leq K_c\Big(\|u_1''\|^2 + (\gamma u_1' - u_2)^2 + (\gamma u_1 - u_3)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\Big)$$

where

$$K_c = \max\left\{2\,(2 + m_B) + m_F\,,\ 8\left(2 + \frac{1}{\alpha}\right) + 2I_B + I_F\right\}.$$

From the fact that

$$\int_0^1 (1-x)\big(u_1'(x)\big)^2 dx \leq \|u_1'\|^2,$$

and using the inequality for $(\gamma u_1')^2$ above, it follows that

$$\begin{aligned} b_A(u_1, u_1) &= \frac{1}{\beta}\|u_1''\|^2 - \mu \int_0^1 (1-x)\big(u_1'(x)\big)^2 dx \\ &\geq \frac{1}{\beta}\|u_1''\|^2 - \mu\|u_1'\|^2 \\ &\geq \left(\frac{1}{\beta} - 2\mu\right)\|u_1''\|^2 - \mu\left(4\left(\gamma u_1' - u_2\right)^2 + 8\left(u_2 - u_4\right)^2 + 8u_4^2\right) \end{aligned}$$

Therefore

$$\begin{aligned} b(u,u) &\geq \left(\frac{1}{\beta} - 2\mu\right)\|u_1''\|^2 + K_{FB}\big(\gamma u_1 - u_3\big)^2 + (k_{BA} - 4\mu)\left(\gamma u_1' - u_2\right)^2 \\ &\quad + (k_{FB} - 8\mu)\left(u_2 - u_4\right)^2 + K_F u_3^2 + (k_F - 8\mu)\,u_4^2 \\ &\geq K_b\left(\|u_1''\|^2 + (\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right) \end{aligned}$$

where

$$K_b = \min\left\{\frac{1}{\beta} - 2\mu,\ K_{FB},\ K_F,\ k_{BA} - 4\mu,\ k_{FB} - 8\mu,\ k_F - 8\mu\right\}.$$

With $K_{bc} = \dfrac{K_c}{K_b}$, it follows that

$$K_{bc}\, b(u,u) \geq \|u\|_X^2.$$

□

Remark

If $b(u,u) = 0$, it follows from Theorem 2 that $u = 0$.

The energy space V

The bilinear form b is an inner product for the space V and for $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\ .$$

The space V with the norm $\|\cdot\|_V$ is called the **energy space**.

Theorem 3

There exists a constant K_{ba} such that for any $u \in V$ and $v \in V$,

$$|a(u,v)| \leq K_{ba}\|u\|_V\|v\|_V.$$

Proof

We can prove that

$$|a(u,v)| \leq \sqrt{a(u,u)}\ \sqrt{a(v,v)}$$

in a similar way as the proof for the Cauchy-Schwartz inequality.

From the proof in Theorem 2 it follows that a constant $K_b > 0$ exists such that

$$\begin{aligned} b(u,u) &\geq K_b\left(\|u_1''\|^2 + (\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right) \\ &\geq K_b\left((\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right). \end{aligned}$$

Furthermore

$$\begin{aligned} |a(u,u)| &= C_{FB}\,(\gamma u_1 - u_3)^2 + c_{BA}\,(\gamma u_1' - u_2)^2 + c_{FB}\,(u_2 - u_4)^2 \\ &\quad + C_F u_3^2 + c_F u_4^2 \\ &\leq K_a\left((\gamma u_1 - u_3)^2 + (\gamma u_1' - u_2)^2 + (u_2 - u_4)^2 + u_3^2 + u_4^2\right) \end{aligned}$$

where

$$K_a = \max\left\{ C_{FB},\ C_F,\ \ c_{BA},\ c_{FB},\ c_F \right\}.$$

Hence, with $K_{ba} = \dfrac{K_a}{K_b}$,

$$|a(u,u)| \ \leq\ K_{ba}\ b(u,u) \ =\ K_{ba}\|u\|_V^2.$$

Consequently

$$|a(u,v)| \leq \sqrt{a(u,u)}\sqrt{a(v,v)} \leq K_{ba}\|u\|_V\|v\|_V.$$

3.3 Vertical slender structure: Timoshenko models

3.3.1 Variational forms

The variational forms are found by multiplying Equation (2.1.13) with a test function v and Equation (2.1.14) with a test function ψ and applying integration by parts. This results in

$$\begin{aligned}
\big(\partial_t^2 w(\cdot,t), v\big) &= -\big(V(\cdot,t), v'\big) - V(0,t)v(0),\\
\frac{1}{\alpha}\ \big(\partial_t^2 \phi(\cdot,t), \psi\big) &= -\big(M(\cdot,t), \psi'\big) + \big(V(\cdot,t), \psi\big) + \big(L(\cdot,t), \psi\big) - M(0,t)\psi(0)
\end{aligned}$$

Substituting the constitutive equations (2.1.15), (2.1.16) and (2.1.17) into the equations above, we find that

$$\big(\partial_t^2 w(\cdot,t), v\big) = -\big(\partial_x w(\cdot,t) - \phi(\cdot,t), v'\big) - V(0,t)v(0), \tag{3.3.1}$$

$$\begin{aligned}
\frac{1}{\alpha}\ \big(\partial_t^2 \phi(\cdot,t), \psi\big) &= -\frac{1}{\beta}\ \big(\partial_x \phi(\cdot,t), \psi'\big) + \big(\partial_x w(\cdot,t) - \phi(\cdot,t), \psi\big)\\
&\quad + \mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)dx - M(0,t)\psi(0).
\end{aligned} \tag{3.3.2}$$

First consider **Problem VT 3**. Equations (3.3.1) and (2.1.18) result in

$$\begin{aligned}
\big(\partial_t^2 w(\cdot,t), v\big) + m_F\partial_t^2 w(0,t)v(0) &= -\big(\partial_x w(\cdot,t) - \phi(\cdot,t), v'\big) - K_F w(0,t)v(0)\\
&\quad - C_F\partial_t w(0,t)v(0).
\end{aligned} \tag{3.3.3}$$

Equations (3.3.2) and (2.1.20) result in

$$\begin{aligned}\frac{1}{\alpha}\left(\partial_t^2\phi(\cdot,t),\psi\right) &= -\frac{1}{\beta}\left(\partial_x\phi(\cdot,t),\psi'\right)+\left(\partial_x w(\cdot,t)-\phi(\cdot,t),\psi\right)\\ &\quad +\mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)\,dx\\ &\quad -k\big(\phi(0,t)-\theta_F(t)\big)\psi(0)\\ &\quad -c\big(\partial_t\phi(0,t)-\dot\theta_F(t)\big)\psi(0) \qquad (3.3.4)\end{aligned}$$

Equation (2.1.19) is presented again for convenience.

$$\begin{aligned}I_F\ddot\theta_F(t) &= k\Big(\phi(0,t)-\theta_F(t)\Big)+c\Big(\partial_t\phi(0,t)-\dot\theta_F(t)\Big)\\ &\quad -k_F\theta_F(t)-c_F\dot\theta_F(t). \qquad (2.1.19)\end{aligned}$$

As for the Rayleigh models, there are no forced boundary condtions on the test functions. Therefore, for both variational forms of Problems VT 3 and VT 4, both v and ψ are in $T(0,1)=C^1[0,1]$.

Variational form of Problem VT 3

Find w, ϕ and θ_F such that for each $t>0$, $w(\cdot,t)\in T(0,1)$, $\phi(\cdot,t)\in T(0,1)$ and $\theta_F(t)\in I\!R$ and Equations (3.3.3) and (3.3.4) hold for each $v\in T(0,1)$ and $\psi\in T(0,1)$ respectively and Equation (2.1.19) holds. □

Now consider **Problem VT 4**. Equations (3.3.1) and (2.1.21) result in

$$\begin{aligned}\left(\partial_t^2 w(\cdot,t),v\right)+m_B\,\partial_t^2 w(0,t)v(0) &= -\left(\partial_x w(\cdot,t)-\phi(\cdot,t),v'\right)\\ &\quad -K_{FB}\Big(w(0,t)-w_F(t)\Big)v(0)\\ &\quad -C_{FB}\Big(\partial_t w(0,t)-\dot w_F(t)\Big)v(0). \qquad (3.3.5)\end{aligned}$$

Equations (3.3.2) and (2.1.23) result in

$$\begin{aligned}\frac{1}{\alpha}\left(\partial_t^2\phi(\cdot,t),\psi\right) &= -\frac{1}{\beta}\left(\partial_x\phi(\cdot,t),\psi'\right)+\left(\partial_x w(\cdot,t)-\phi(\cdot,t),\psi\right)\\ &\quad +\mu\int_0^1 (1-x)\partial_x w(x,t)\psi(x)\,dx\\ &\quad -k_{BA}\Big(\phi(0,t)-\theta_B(t)\Big)\psi(0)\\ &\quad -c_{BA}\Big(\partial_t\phi(0,t)-\dot\theta_B(t)\Big)\psi(0). \qquad (3.3.6)\end{aligned}$$

There are three additional equations in the system, presented again for convenience.

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\phi(0,t)-\theta_B(t)\Big)+c_{BA}\Big(\partial_t\phi(0,t)-\dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t)-\theta_F(t)\Big)-c_{FB}\Big(\dot{\theta}_B(t)-\dot{\theta}_F(t)\Big), \qquad (2.1.22) \\ m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t)-w_F(t)\Big)+C_{FB}\Big(\partial_t w(0,t)-\dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t)-C_F\dot{w}_F(t), \qquad (2.1.24) \\ I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t)-\theta_F(t)\Big)+c_{FB}\Big(\dot{\theta}_B(t)-\dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t)-c_F\dot{\theta}_F(t). \qquad (2.1.25) \end{aligned}$$

Variational form of Problem VT 4

Find w, ϕ, θ_B, w_F and θ_F such that for each $t>0$, $w(\cdot,t)\in T(0,1)$, $\phi(\cdot,t)\in T(0,1)$ and Equations (3.3.5) and (3.3.6) hold for each $v\in T(0,1)$ and $\psi\in T(0,1)$ respectively and Equations (2.1.22), (2.1.24) and (2.1.25) hold. □

The variational forms of the problems above are used for computational purposes (see Chapter 6), but for theoretical purposes we consider the weak variational form.

3.3.2 Weak variational forms

Problems VT 3 and VT 4 are similar and we consider only Problem VT 4. We omit the "gravity" term for a reason to be given later.

First we add Equations (3.3.5) and (3.3.6) to find

$$\begin{aligned} &\big(\partial_t^2 w(\cdot,t),v\big)+\frac{1}{\alpha}\,\big(\partial_t^2\phi(\cdot,t),\psi\big)+m_B\,\partial_t^2 w(0,t)v(0) \\ = &-\frac{1}{\beta}\,\big(\partial_x\phi(\cdot,t),\psi'\big)-\big(\partial_x w(\cdot,t)-\phi(\cdot,t),v'-\psi\big) \\ &-K_{FB}\Big(w(0,t)-w_F(t)\Big)v(0)-C_{FB}\Big(\partial_t w(0,t)-\dot{w}_F(t)\Big)v(0) \\ &-k_{BA}\Big(\phi(0,t)-\theta_B(t)\Big)\psi(0)-c_{BA}\Big(\partial_t\phi(0,t)-\dot{\theta}_B(t)\Big)\psi(0). \qquad (3.3.7) \end{aligned}$$

Next we introduce "time derivatives" and the trace operator γ as in Section 3.2.2.

Define a function y with values $y(t) = \langle w(\cdot,t), \phi(\cdot,t)\rangle$ in $\mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$. Furthermore, set $v = v_1$ and $\psi = v_2$. Equation (3.3.7) then becomes

$$\begin{aligned}
&(\ddot{y}_1(t), v_1) + \frac{1}{\alpha}\,(\ddot{y}_2(t), v_2) + m_B\gamma\big(\ddot{y}_1(t)\big)\gamma v_1 \\
&= -\frac{1}{\beta}\,\big((y_2(t))', v_2'\big) - \big((y_1(t))' - y_2(t), v_1' - v_2\big) \\
&-K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\gamma v_1 - C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\
&-k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\gamma v_2 - c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\gamma v_2. \qquad (3.3.8)
\end{aligned}$$

Bilinear forms

For $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$ in $H^1(0,1) \times H^1(0,1)$,

$$\begin{aligned}
c_A(u,v) &= (u_1, v_1) + \frac{1}{\alpha}\,(u_2, v_2) + m_B\gamma u_1\gamma v_1, \\
b_A(u,v) &= \frac{1}{\beta}\,(u_2', v_2') + (u_1' - u_2, v_1' - v_2).
\end{aligned}$$

Write Equation (3.3.8) in terms of the bilinear forms:

$$\begin{aligned}
c_A(\ddot{y}(t), v) &= -b_A(y(t), v) - K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\gamma v_1 \\
&\quad -C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\gamma v_1 \\
&\quad -k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\gamma v_2 \\
&\quad -c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\gamma v_2 \qquad (3.3.9)
\end{aligned}$$

for any $v \in H^1(0,1) \times H^1(0,1)$.

Multiply Equations (2.1.22), (2.1.24) and (2.1.25) with v_3, v_4 and v_5 respectively to find

$$\begin{aligned}
I_B\ddot{\theta}_B(t)v_3 &= \Big[k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big) + c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big) \\
&\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big]\, v_3 \qquad (3.3.10) \\
m_F\ddot{w}_F(t)v_4 &= \Big[K_{FB}\Big(w_B(t) - w_F(t)\Big) + C_{FB}\Big(\dot{w}_B(t) - \dot{w}_F(t)\Big) \\
&\quad -K_F w_F(t) - C_F\dot{w}_F(t)\big]\, v_4 \qquad (3.3.11) \\
I_F\ddot{\theta}_F(t)v_5 &= \Big[k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\
&\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t)\Big]\, v_5 \qquad (3.3.12)
\end{aligned}$$

Adding Equations (3.3.9), (3.3.10), (3.3.11) and (3.3.12) results in

$$\begin{aligned}
& c_A(\ddot{y}(t), v) + I_B\ddot{\theta}_B(t)v_3 + m_F\ddot{w}_F(t)v_4 + I_F\ddot{\theta}_F(t)v_5 \\
= \; & -b_A(y(t), v) - K_{FB}\Big(\gamma\big(y_1(t)\big) - w_F(t)\Big)\Big(\gamma v_1 - v_4\Big) \\
& -C_{FB}\Big(\gamma\big(\dot{y}_1(t)\big) - \dot{w}_F(t)\Big)\Big(\gamma v_1 - v_4\Big) \\
& -k_{BA}\Big(\gamma\big(y_2(t)\big) - \theta_B(t)\Big)\Big(\gamma v_2 - v_3\Big) \\
& -c_{BA}\Big(\gamma\big(\dot{y}_2(t)\big) - \dot{\theta}_B(t)\Big)\Big(\gamma v_2 - v_3\Big) \\
& -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)\Big(v_3 - v_5\Big) \\
& -c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big(v_3 - v_5\Big) \\
& -K_F w_F(t)v_4 - C_F\dot{w}_F(t)v_4 \\
& -k_F\theta_F(t)v_5 - c_F\dot{\theta}_F(t)v_5. \qquad (3.3.13)
\end{aligned}$$

To formulate the weak form of the variational problem, the following product spaces and bilinear forms are necessary.

Product spaces

$$X = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1) \times I\!R^3 \quad \text{and} \quad V = H^1(0,1) \times H^1(0,1) \times I\!R^3 \,.$$

Bilinear forms

For u and v in X,

$$\begin{aligned} c_A(u,v) &= (u_1,v_1) + \frac{1}{\alpha}\,(u_2,v_2) + m_B\gamma u_1\gamma v_1\,, \\ c(u,v) &= c_A(u,v) + I_B u_3 v_3 + m_F u_4 v_4 + I_F u_5 v_5\,, \\ a(u,v) &= C_{FB}(\gamma u_1 - u_4)(\gamma v_1 - v_4) + c_{BA}(\gamma u_2 - u_3)(\gamma v_2 - v_3) \\ &\quad + c_{FB}(u_3 - u_5)(v_3 - v_5) + C_F u_4 v_4 + c_F u_5 v_5\,. \end{aligned}$$

For u and v in V,

$$\begin{aligned} b_A(u,v) &= \frac{1}{\beta}\,(u_2',v_2') + (u_1' - u_2, v_1' - v_2)\,, \\ b(u,v) &= b_A(u,v) + K_{FB}(\gamma u_1 - u_4)(\gamma v_1 - v_4) + k_{BA}(\gamma u_2 - u_3)(\gamma v_2 - v_3) \\ &\quad + k_{FB}(u_3 - u_5)(v_3 - v_5) + K_F u_4 v_4 + k_F u_5 v_5\,. \end{aligned}$$

The relationship between the variables u_1 to u_5 and the variables in Problem VT 4 is shown in the next table.

$u_1(t)$	u_2	u_3	u_4	u_5
$w(\cdot,t)$	$\phi(\cdot,t)$	$\theta_B(t)$	$w_F(t)$	$\theta_F(t)$

Weak variational form of Problem VT 4

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t),v) = -b(u(t),v) - a(u'(t),v) \quad \text{for each } v \in V.$$

Remark

Inclusion of the "gravity"-term in the definition of b_A will result in an unsymmetrical form. Consequently the bilinear form b will be unsymmetrical and symmetry is crucial in the theory – see Section 3.8.

The inertia space X

Note that c is an inner product for this space and consequently we may define a norm for $u \in X$ by

$$\|u\|_X = \sqrt{c(u,u)}\ .$$

The space X with norm $\|\cdot\|_X$ is the **inertia space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is a dense subset of X.

Proof

The proof is similar to the proof of Theorem 2 in Section 3.1. □

We will show that the bilinear form b is an inner product for the space V.

Theorem 2

There exists a constant K_{bc} such that

$$\|u\|_X^2 \leq K_{bc}\ b(u,u)$$

for each $u \in V$.

Proof

As before we use the elementary inequalities

$$\|x\| \leq \|x-y\| + \|y\| \quad \text{and} \quad (a+b)^2 \leq 2(a^2+b^2).$$

It follows from Theorem 3 Appendix 3 that

$$\|u_i\| \leq \|u_i'\| + |\gamma u_i| \quad \text{for } i = 1,\ 2.$$

Together with $\|u_1'\|^2 \leq 2\|u_1' - u_2\|^2 + 2\|u_2\|^2$, we find that

$$\|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 \;\leq\; 4\|u_1' - u_2\|^2 + \left(8 + \frac{2}{\alpha}\right)\left(\|u_2'\|^2 + (\gamma u_2)^2\right) + 2(\gamma u_1)^2 .$$

Furthermore,

$$(\gamma u_i)^2 \leq 2(\gamma u_i - u_j)^2 + 2u_j^2, \quad u_3^2 \leq 2(u_3 - u_5)^2 + 2u_5^2$$

and

$$c(u,u) \;=\; \|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 + m_B(\gamma u_1)^2 + I_B u_3^2 + m_F u_4^2 + I_F u_5^2 .$$

It follows that

$$\begin{aligned} c(u,u) \;\leq\; & K_c\Big\{\|u_1' - u_2\|^2 + \|u_2'\|^2 + (\gamma u_1 - u_4)^2 + (\gamma u_2 - u_3)^2 \\ & + (u_3 - u_5)^2 + u_4^2 + u_5^2\Big\} \end{aligned}$$

with

$$K_c = \max\left\{4 + 2m_B + m_F,\; I_F + 2I_B + 32 + \frac{8}{\alpha}\right\}.$$

$$\begin{aligned} b(u,u) \;=\; & \frac{1}{\beta}\,\|u_2'\|^2 + \|u_1' - u_2\|^2 + K_{FB}(\gamma u_1 - u_4)^2 + k_{BA}(\gamma u_2 - u_3)^2 \\ & + k_{FB}(u_3 - u_5)^2 + K_F u_4^2 + k_F u_5^2 \\ \geq\; & K_b\Big\{\|u_2'\|^2 + \|u_1' - u_2\|^2 + (\gamma u_1 - u_4)^2 + (\gamma u_2 - u_3)^2 \\ & + (u_3 - u_5)^2 + u_4^2 + u_5^2\Big\} \end{aligned}$$

with

$$K_b = \min\left\{\frac{1}{\beta},\; K_{FB},\; K_F,\; k_{FB},\; k_F,\; k_{BA}\right\}.$$

Let $K_{bc} = \dfrac{K_c}{K_b}$, then $c(u,u) \leq K_{bc}\, b(u,u)$ and the result follows. □

The energy space V

For $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\ .$$

The space V with the norm $\|\cdot\|_V$ is called the **energy space**.

Theorem 3

For any $u \in V$ and $v \in V$,

$$|a(u,v)| \leq \|u\|_V \|v\|_V \,.$$

Proof

This proof is similar to the proof of Theorem 3 Section 3.2. □

3.4 The cantilever beam

The variational form of the cantilever Timoshenko beam is derived in Section 3.1. Recall that our main concern is the choice of boundary conditions at the clamped end. Returning to Equations (3.1.1) and (3.1.2), we are now ready to comment on this choice.

Choosing test functions v and ψ such that $v(0) = \psi(0) = 0$, the terms $V(0,t)v(0)$ and $M(0,t)\psi(0)$ vanish. Therefore the boundary conditions $w(0,t) = \phi(0,t) = 0$ is a convenient choice from a variational point of view. Note that $w(0,t) = 0$ is realistic as far as modelling is concerned, but **not** $\phi(0,t) = 0$ (see Section 1.5.2).

The following alternative boundary conditions may be considered (see Section 2.2).

$$\begin{bmatrix} V(0,t) \\ M(0,t) \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} w(0) \\ \phi(0) \end{bmatrix}$$

Note that $D = C^{-1}$.

We now have no restriction on the test functions and the energy space $V = H^1(0,1) \times H(0,1)$. The bilinear form b must be redefined and

$$b(u,v) \;=\; \frac{1}{\beta}\,(u_2', v_2') + (u_1' - u_2, v_1' - v_2) + \begin{bmatrix} \gamma u_1 & \gamma u_2 \end{bmatrix} D \begin{bmatrix} \gamma v_1 & \gamma v_2 \end{bmatrix}^T.$$

The matrix D must be nonnegative for the bilinear form b to be an inner product. For a discussion of the results, see Chapter 7.

3.5 Two-dimensional model for the cantilever beam

3.5.1 Variational forms

Consider the equation of motion (1.4.1). Multiply both sides by an arbitrary vector valued function $\boldsymbol{\phi}$ and integrate over the reference configuration Ω.

$$\iint_\Omega (\partial_t^2 \boldsymbol{u}) \cdot \boldsymbol{\phi}\, dA = \iint_\Omega (\operatorname{div} T) \cdot \boldsymbol{\phi}\, dA.$$

If T is symmetric, $\operatorname{div}(T\boldsymbol{\phi}) = (\operatorname{div} T) \cdot \boldsymbol{\phi} + \operatorname{tr}(T\Phi)$, where

$$\Phi = \begin{bmatrix} \partial_1 \phi_1 & \partial_2 \phi_1 \\ \partial_1 \phi_2 & \partial_2 \phi_2 \end{bmatrix}.$$

Application of the divergence theorem and the symmetry of T yield

$$\iint_\Omega \operatorname{div}(T\boldsymbol{\phi})\, dA = \int_{\partial\Omega} T\boldsymbol{\phi} \cdot \boldsymbol{n}\, ds = \int_{\partial\Omega} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds.$$

Combining the results above, we have the **Green formula**

$$\iint_\Omega (\operatorname{div} T) \cdot \boldsymbol{\phi}\, dA = -\iint_\Omega \operatorname{tr}(T\Phi)\, dA + \int_{\partial\Omega} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds.$$

Consequently,

$$\iint_\Omega (\partial_t^2 \boldsymbol{u}) \cdot \boldsymbol{\phi}\, dA = -\iint_\Omega \operatorname{tr}(T\Phi)\, dA + \int_{\partial\Omega} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds \tag{3.5.1}$$

for any vector field $\boldsymbol{\phi}$ that is sufficiently smooth.

The bilinear form $b(\boldsymbol{u}, \boldsymbol{\phi})$ is defined by

$$b(\boldsymbol{u}, \boldsymbol{\phi}) = \iint_\Omega \operatorname{tr}(T\Phi)\, dA.$$

If Hooke's law, Equation (1.4.2), is substituted into the definition of the bilinear form, we obtain

$$\begin{aligned} b(\boldsymbol{u}, \boldsymbol{\phi}) &= \iint_\Omega \big(\sigma_{11}\partial_1\phi_1 + \sigma_{12}\partial_1\phi_2 + \sigma_{21}\partial_2\phi_1 + \sigma_{22}\partial_2\phi_2\big)\, dA \\ &= \frac{1}{\gamma(1-\nu^2)} \iint_\Omega \big(\partial_1 u_1 \partial_1 \phi_1 + \partial_2 u_2 \partial_2 \phi_2 + \nu(\partial_1 u_1 \partial_2 \phi_2 + \partial_2 u_2 \partial_1 \phi_1)\big)\, dA \\ &\quad + \frac{1}{2\gamma(1+\nu)} \iint_\Omega \big(\partial_1 u_2 + \partial_2 u_1\big)\big(\partial_1 \phi_2 + \partial_2 \phi_1\big)\, dA. \end{aligned}$$

To define the space of test functions $T(\Omega)$ for Problems CTD 1 and CTD 2, take note that the boundary of Ω consists of the the two parts Σ and Γ. The test functions must satisfy the forced boundary conditions on Σ, i.e. ϕ_1 and ϕ_2 must be zero when it is required that u_1 and u_2 are zero. All that matters at this stage is that $T\boldsymbol{n} \cdot \boldsymbol{\phi} = \boldsymbol{0}$ on Σ.

Consequently

$$\int_{\partial\Omega} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds = \int_{\Gamma} T\boldsymbol{n} \cdot \boldsymbol{\phi}\, ds$$

for each $\boldsymbol{\phi} \in T$.

For the equilibrium problem the traction is prescribed on Γ.

Variational form of Problem CTD 1

Given the traction $\boldsymbol{t}$ on Γ, find $\boldsymbol{u} \in T(\Omega)$ such that

$$b(\boldsymbol{u}, \boldsymbol{\phi}) = \int_{\Gamma} \boldsymbol{t} \cdot \boldsymbol{\phi}\, ds$$

for each $\boldsymbol{\phi} \in T(\Omega)$. □

In the second problem we consider free vibration and there is no traction on Γ.

Variational form of Problem CTD 2

Find $\boldsymbol{u}$ such that for $t > 0$, $\boldsymbol{u}(\cdot, t) \in T(\Omega)$ and

$$\iint_{\Omega} \partial_t^2 \boldsymbol{u} \cdot \boldsymbol{\phi}\, dA = -b(\boldsymbol{u}, \boldsymbol{\phi})$$

for each $\boldsymbol{\phi} \in T(\Omega)$. □

Remark

Our main concern is to study the natural frequencies and modes. The relevant eigenvalue problems are considered in Section 3.9 and Chapter 7.

3.5.2 Weak variational forms

For the theory it is necessary to place some restrictions on the sets Ω and Γ. These assumptions are listed below and more detail is given in Appendix 1.

1. The set Ω is open, bounded and convex.
2. The boundary of Ω consists of a finite number of smooth curves.
3. The set Γ is a smooth part of the boundary of Ω.

Remark

In our application, Ω is a rectangle and Γ one of the sides.

The function spaces $\mathcal{L}^2(\Omega)^2$, $\mathcal{L}^2(\Gamma)^2$, $H^k(\Omega)^2$ and $H^k(\Gamma)^2$ are relevant for the theory. The detail and notation are discussed in Appendix 1.

The trace operator γ is now a mapping of a function "onto its value" on Γ. The definition is given in Appendix 3.

We follow the same line of reasoning for the weak formulation as before. Let V be the closure of $T(\Omega)$ in $H^1(\Omega)^2$. We may consider the following weak variational form of Problem CTD 1. Given $t \in \mathcal{L}^2(\Gamma)^2$, find $u \in V$ such that

$$b(u, v) = (t, \gamma v)_{0,2}^{\Gamma} \quad \text{for each } v \in V.$$

However, to apply the theory we consider another form. We define a functional f corresponding to the traction on Γ. Given $t \in \mathcal{L}^2(\Gamma)^2$, let

$$f(v) = (t, \gamma v)_{0,2}^{\Gamma} \quad \text{for each } v \in V,$$

This leads to the following form.

Weak variational form of Problem CTD 1

Given f in the dual of V, find $u \in V$ such that

$$b(u, v) = f(v) \quad \text{for each } v \in V.$$

Weak variational form of Problem CTD 2

Find u such that for each $t > 0$, $u(t) \in V$ and

$$c(u''(t), v) = -b(u(t), v) \quad \text{for each } v \in V,$$

where $c(\cdot\,,\,\cdot) = (\cdot\,,\,\cdot)$ is the inner product of $\mathcal{L}^2(\Omega)^2$.

Theorem 1

There exists a constant K such that

$$|f(v)| \leq K\, \|\gamma v\|^{\Omega}_{1,2} \quad \text{for each } v \in V.$$

Proof

It follows from Theorem 4 Appendix 3 that

$$\|\gamma v\|^{\Gamma}_{0,2} \leq K_{\Gamma}\, \|v\|^{\Omega}_{1,2} \quad \text{for each } v \in H^1(\Omega)^2.$$

Consequently,

$$|f(v)| \leq \|t\|^{\Gamma}_{0,2}\, \|\gamma v\|^{\Gamma}_{0,2} \leq K_{\Gamma}\, \|t\|^{\Gamma}_{0,2}\, \|v\|^{\Omega}_{1,2}.$$

Theorem 2 (Poincare-Friedrichs)

There exists a constant c_F such that,

$$\|u\|_{0,2} \leq c_F |u|_{1,2} \quad \text{for each } u \in V.$$

Proof

The inequality holds for each $u \in T(\Omega)$ (see the corollary to Theorem 2 Appendix 2). Clearly the same is true for $u \in V$.

Theorem 3 (Korn)

There exists a constant c_K such that,

$$|u|^2_{1,2} \;\leq\; c_K\, b(u, u) \quad \text{for each } u \in H^1(\Omega)^2.$$

Proof

[Br, p 288-289]

Theorem 4

There exists a constant c_1 such that,

$$\|u\|_{1,2} \leq c_1 \sqrt{b(u,u)} \quad \text{for each } u \in V.$$

Proof

Combine Korn's inequality with the Poincare-Friedrichs inequality.

The energy space *V*

For $u \in V$ we define

$$\|u\|_V = \sqrt{b(u,u)}\,.$$

The space V with norm $\|\cdot\|_V$ is called the **energy space**. Due to Theorem 4 the norms $\|\cdot\|_V$ and $\|\cdot\|_{1,2}$ are equivalent on V.

Theorem 5

The space $\mathcal{L}^2(\Omega)^2$ is a separable Hilbert space and V is a dense subset of $\mathcal{L}^2(\Omega)^2$.

Proof

The space $\mathcal{L}^2(\Omega)^2$ is a separable Hilbert space and $C_0^\infty(\Omega)^2$ is a dense subset of $\mathcal{L}^2(\Omega)^2$ (from Theorems 2 and 3 Appendix 1). Since $C_0^\infty(\Omega)^2 \subset V$, the result follows.

Theorem 6

The embedding of the space V into $\mathcal{L}^2(\Omega)^2$ is compact.

Proof

The embedding of the space $H^1(\Omega)^2$ into $\mathcal{L}^2(\Omega)^2$ is compact (from Theorem 7 Appendix 1). The result follows from the equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_{1,2}$.

Theorem 7

For any $t \in \mathcal{L}^2(\Gamma)^2$, there exists a unique $u \in V$ such that

$$b(u,v) = (t,v)^{\Gamma}_{0,2} \quad \text{for each } v \in V.$$

Proof

See Section 3.7.

3.6 Plate-beam system

3.6.1 Variational form of problem RMT

For any function v,

$$\iint_{\Omega} (\operatorname{div} \boldsymbol{Q}) v \, dA \;=\; -\iint_{\Omega} \boldsymbol{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds. \qquad (3.6.1)$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T$, using the Green formula from Section 3.5, we have

$$\iint_{\Omega} \operatorname{div} M \cdot \boldsymbol{\phi} \, dA = -\iint_{\Omega} \operatorname{tr}(M\Phi) \, dA + \int_{\partial\Omega} M\boldsymbol{n} \cdot \boldsymbol{\phi} \, ds, \qquad (3.6.2)$$

where $\Phi = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$ and "tr" denotes the trace of the matrix.

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$, with

$$T_1(\Omega) = \left\{ v \in C^1(\bar{\Omega}) \;\middle|\; v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\},$$

$$T_2(\Omega) = \left\{ \boldsymbol{\phi} = [\phi_1 \ \phi_2]^T \;\middle|\; \phi_1, \phi_2 \in C^1(\bar{\Omega}), \ \ \phi_2 = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\}.$$

Combining Equation (1.3.5) (first equation of motion for the plate) with Equation (3.6.1) yield

$$h_p \iint_\Omega \partial_t^2 w v \, dA + \iint_\Omega \boldsymbol{Q} \cdot \nabla v \, dA - \int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds = 0 \tag{3.6.3}$$

for each $v \in T_1(\Omega)$.

It follows from Equation (2.4.8) (first equation of motion for the beams), using integration by parts, that

$$\eta_1 \int_0^1 \partial_t^2 w_{b0} v_0 \, dx + \int_0^1 V_0 \partial_x v_0 \, dx = \int_0^1 P_0 v_0 \, dx \tag{3.6.4}$$

for each v_0 in $C^1[0,1]$ with $v_0(0) = v_0(1) = 0$ and

$$\eta_1 \int_0^1 \partial_t^2 w_{b1} v_1 \, dx + \int_0^1 V_1 \partial_x v_1 \, dx = \int_0^1 P_1 v_1 \, dx \tag{3.6.5}$$

for each v_1 in $C^1[0,1]$ with $v_1(0) = v_1(1) = 0$. The subscripts "0" and "1" are used to distinguish between quantities associated with the two different beams.

To accommodate Equation (2.4.2) (interface condition for w_{b0} and w_{b1}), choose $v_0(x_1) = v(x_1, 0)$ and $v_1(x_1) = v(x_1, a)$, where a denotes the dimensionless width of the plate.

The fact that $v = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and that $\boldsymbol{Q} \cdot \boldsymbol{n} = -P$ on both Γ_0 and Γ_1 (interface condition (2.4.4)), result in some cancellations when adding Equations (3.6.3), (3.6.4) and (3.6.5). We have

$$\begin{aligned}
\int_{\partial\Omega} (\boldsymbol{Q} \cdot \boldsymbol{n}) v \, ds &= \int_{\Gamma_0} (\boldsymbol{Q}_0 \cdot \boldsymbol{n}) v \, ds - \int_{\Gamma_1} (\boldsymbol{Q}_1 \cdot \boldsymbol{n}) v \, ds \\
&= -\int_{\Gamma_0} P_0 v \, ds - \int_{\Gamma_1} P_1 v \, ds \\
&= -\left[\int_0^1 P_0 v \, dx_1 \right]_{x_2=0} - \left[\int_0^1 P_1 v \, dx_1 \right]_{x_2=a}
\end{aligned}$$

for each $v \in T_1(\Omega)$.

From Equation (2.4.2) (interface condition), the remaining integrals on Γ_0 and Γ_1 can be expressed in terms of w. Therefore,

$$h_p \iint_\Omega \partial_t^2 wv\,dA + \eta_1 \left[\int_0^1 \partial_t^2 wv\,dx_1\right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 wv\,dx_1\right]_{x_2=a}$$
$$+ \iint_\Omega \boldsymbol{Q}\cdot\nabla v\,dA + \left[\int_0^1 V_0\partial_1 v\,dx_1\right]_{x_2=0} + \left[\int_0^1 V_1\partial_1 v\,dx_1\right]_{x_2=a}$$
$$= 0$$

for each $v \in T_1(\Omega)$.

Equations (1.3.7) (constitutive equation for $\boldsymbol{Q}$) and (2.4.10) (constitutive equations for V_0 and V_1) are expressed in terms of w and ψ_1 (found from the interface conditions (2.4.2) and (2.4.3)). They are used to obtain the final form of this variational equation. This leads to

$$h_p \iint_\Omega \partial_t^2 wv\,dA + \eta_1 \left[\int_0^1 \partial_t^2 wv\,dx_1\right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 wv\,dx_1\right]_{x_2=a}$$
$$+ h_p \iint_\Omega (\nabla w + \boldsymbol{\psi})\cdot\nabla v\,dA$$
$$+ \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\,dx_1\right]_{x_2=0} + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\,dx_1\right]_{x_2=a}$$
$$= 0 \tag{3.6.6}$$

for each $v \in T_1(\Omega)$.

A similar calculation is performed for the remaining equations of motion. Combining Equation (1.3.6) (second equation of motion for the plate) with the Green formula (3.6.2) yields

$$I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi}\cdot\boldsymbol{\phi}\,dA + \iint_\Omega \mathrm{tr}(M\Phi)\,dA - \int_{\partial\Omega} M\boldsymbol{n}\cdot\boldsymbol{\phi}\,ds$$
$$+ \iint_\Omega \boldsymbol{Q}\cdot\boldsymbol{\phi}\,dA = 0 \tag{3.6.7}$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

It follows from Equation (2.4.9) (second equation of motion for the beams) and using integration by parts, that

$$\frac{\eta_1}{\alpha_b}\int_0^1 \partial_t^2 \phi_{b0}\chi_0\,dx + \int_0^1 M_{b0}\partial_x\chi_0\,dx - \int_0^1 (V_0 + L_0)\chi_0\,dx = 0 \tag{3.6.8}$$

for each $\chi_0 \in C^1[0,1]$ and

$$\frac{\eta_1}{\alpha_b}\int_0^1 \partial_t^2\phi_{b1}\chi_1\,dx + \int_0^1 M_{b1}\partial_x\chi_1\,dx - \int_0^1 (V_1+L_1)\chi_1\,dx = 0 \qquad (3.6.9)$$

for each $\chi_1 \in C^1[0,1]$. (M_{b0} and M_{b1} are zero at the endpoints of the beams.)

The functions χ_0 and χ_1 in Equations (3.6.8) and (3.6.9) must satisfy the conditions $\chi_0(x_1) = -\phi_1(x_1,0)$ and $\chi_1(x_1) = -\phi_1(x_1,a)$ in order to accommodate the interface condition (2.4.3) for ϕ_{b0} and ϕ_{b1}.

Hence Equation (3.6.8) becomes

$$\frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\psi_1\phi_1\,dx_1\right]_{x_2=0} - \left[\int_0^1 M_{b0}\partial_1\phi_1\,dx_1\right]_{x_2=0} + \left[\int_0^1 (V_0+L_0)\phi_1\,dx_1\right]_{x_2=0} = 0 \qquad (3.6.10)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$, and Equation (3.6.9) becomes

$$\frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\psi_1\phi_1\,dx_1\right]_{x_2=a} - \left[\int_0^1 M_{b1}\partial_1\phi_1\,dx_1\right]_{x_2=a} + \left[\int_0^1 (V_1+L_1)\phi_1\,dx_1\right]_{x_2=a} = 0 \qquad (3.6.11)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

As before, adding Equations (3.6.7), (3.6.10) and (3.6.11), some cancellation of terms occur. Note that $\boldsymbol{\phi} = (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{\tau})\boldsymbol{\tau}$ and consequently,

$$\int_{\partial\Omega} M\boldsymbol{n}\cdot\boldsymbol{\phi}\,ds = \int_{\partial\Omega}\Big((\boldsymbol{\phi}\cdot\boldsymbol{n})M\boldsymbol{n}\cdot\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{\tau})M\boldsymbol{n}\cdot\boldsymbol{\tau}\Big).ds$$

The natural boundary condition on Ω is $M\boldsymbol{n}\cdot\boldsymbol{n} = 0$.

From the definition of the test functions, $\phi_2 = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and therefore $\boldsymbol{\phi}\cdot\boldsymbol{\tau} = 0$ on Σ_0 and Σ_1.

On Γ_0 and Γ_1 the interface conditions (2.4.5) and (2.4.6) are used. It follows that

$$\begin{aligned}\int_{\partial\Omega} M\boldsymbol{n}\cdot\boldsymbol{\phi}\,ds &= \int_{\Gamma_0} L_0\phi_1\,ds + \int_{\Gamma_1}(-L_1)(-\phi_1)\,ds \\ &= \left[\int_0^1 L_0\phi_1\,dx_1\right]_{x_2=0} + \left[\int_0^1 L_1\phi_1\,dx_1\right]_{x_2=a}.\end{aligned}$$

Consequently,

$$
\begin{aligned}
& I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA + \iint_\Omega \operatorname{tr}(M\Phi)\, dA + \iint_\Omega \boldsymbol{Q} \cdot \boldsymbol{\phi}\, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& - \left[\int_0^1 M_{b0} \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} - \left[\int_0^1 M_{b1} \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \left[\int_0^1 V_0 \phi_1 \, dx_1 \right]_{x_2=0} + \left[\int_0^1 V_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& = 0
\end{aligned}
$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Equations (1.3.7) and (1.3.8) are the constitutive equations for $\boldsymbol{Q}$ and M for the plate. These equations are expressed in terms of w and ψ_1 (using the interface conditions (2.4.2)). Similarly, Equation (2.4.10) (constitutive equations for V_1 and V_2) and Equation (2.4.11) (constitutive equations for M_{b0} and M_{b1}) are expressed in terms of w and ψ_1. They are used to obtain the final form the second variational equation.

We define a bilinear form b_B by

$$
\begin{aligned}
b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_\Omega \operatorname{tr}(M\Phi)\, dA \\
&= \frac{1}{\beta_p(1-\nu^2)} \iint_\Omega \Big((\partial_1\psi_1 + \nu\partial_2\psi_2)\partial_1\phi_1 + (\partial_2\psi_2 + \nu\partial_1\psi_1)\partial_2\phi_2 \Big)\, dA \\
&\quad + \frac{1}{2\beta_p(1+\nu)} \iint_\Omega (\partial_1\psi_2 + \partial_2\psi_1)(\partial_1\phi_2 + \partial_2\phi_1)\, dA.
\end{aligned}
$$

for each $\boldsymbol{\psi}$, $\boldsymbol{\phi}$ in $H^1(\Omega)^2$.

Finally, the second variational equation is given by

$$
\begin{aligned}
& I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA \,+\, b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) \,+\, h_p \iint_\Omega (\nabla w \,+\, \boldsymbol{\psi}) \cdot \boldsymbol{\phi}\, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \eta_2 \left[\int_0^1 (\partial_1 w \,+\, \psi_1) \phi_1 \, dx_1 \right]_{x_2=0} \,+\, \eta_2 \left[\int_0^1 (\partial_1 w \,+\, \psi_1) \phi_1 \, dx_1 \right]_{x_2=a} \\
& = \, 0 \qquad (3.6.12)
\end{aligned}
$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Variational form of Problem RMT

Find w and $\boldsymbol{\psi}$ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\boldsymbol{\psi}(\cdot, t) \in T_2(\Omega)$ and Equations (3.6.6) and (3.6.12) hold for each $v \in T_1(\Omega)$ and each $\boldsymbol{\phi} \in T_2(\Omega)$. □

The variational form above is used for computational purposes (see Chapter 8), but for theoretical purposes we consider the weak form of the variational problem.

3.6.2 Variational form of Problems KR and KEB

The variational form of Problem KR can be obtained by setting $\boldsymbol{\psi} = -\nabla w$ and choosing $\boldsymbol{\phi} = -\nabla v$ in Equations (3.6.6) and (3.6.12). In this case the test functions are defined by

$$
T(\Omega) = \left\{ v \in C^2(\bar{\Omega}) \;\middle|\; v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1. \right\}
$$

The variational equations reduce to

$$
\begin{aligned}
& h_p \iint_\Omega \partial_t^2 w v \, dA \,+\, \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} \,+\, \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\
& = 0 \qquad (3.6.13)
\end{aligned}
$$

and

$$\begin{aligned}
& I_p \iint_\Omega \partial_t^2(\nabla w)\cdot\nabla v\, dA \;+\; b_B(\nabla w, \nabla v) \\
& + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\partial_1 wv\, dx_1\right]_{x_2=0} \;+\; \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2\partial_1 wv\, dx_1\right]_{x_2=a} \\
& + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\, dx_1\right]_{x_2=0} \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\, dx_1\right]_{x_2=a} \\
& = \; 0 \qquad (3.6.14)
\end{aligned}$$

for each $v \in T(\Omega)$.

Redefine the bilinear form b_B in Equation (3.6.14) by

$$\begin{aligned}
b_B(w,v) \;=\; & \frac{1}{\beta_p(1-\nu^2)} \iint_\Omega \Big((\partial_1^2 w + \nu\partial_2^2 w)\partial_1^2 v + (\partial_2^2 w + \nu\partial_1^2 w)\partial_2^2 v \Big)\, dA \\
& + \frac{2}{\beta_p(1+\nu)} \iint_\Omega \partial_1\partial_2 w \partial_1\partial_2 v\, dA.
\end{aligned}$$

for each w, v in $H^2(\Omega)$.

For Problem KR the variational form is reduced to a single equation by adding Equations (3.6.13) and (3.6.14).

Variational form of Problem KR

Find w such that, for $t > 0$, $w(.,t) \in T(\Omega)$,

$$\begin{aligned}
& h_p \iint_\Omega \partial_t^2 wv\, dA \;+\; I_p \iint_\Omega \partial_t^2(\nabla w)\cdot\nabla v\, dA \\
& + \eta_1 \left[\int_0^1 \partial_t^2 wv\, dx_1\right]_{x_2=0} \;+\; \eta_1 \left[\int_0^1 \partial_t^2 wv\, dx_1\right]_{x_2=a} \\
& + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2(\partial_1 w)v\, dx_1\right]_{x_2=0} \;+\; \frac{\eta_1}{\alpha_b}\left[\int_0^1 \partial_t^2(\partial_1 w)v\, dx_1\right]_{x_2=a} \\
& + b_B(w,v) \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\, dx_1\right]_{x_2=0} \;+\; \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\, dx_1\right]_{x_2=a} \\
& = 0 \qquad (3.6.15)
\end{aligned}$$

for each v in $T(\Omega)$. □

Variational form of Problem KEB

The variational form of the case where rotary inertia is ignored, is obtained by ignoring the terms containing I_p and $\frac{\eta_1}{\alpha_b}$ in Equation (3.6.15).

3.6.3 Weak variational form of Problem RMT

For $I = (0,1)$, the space $T(I)$ is defined as

$$T(I) = \left\{ v \in C^1(\bar{I}) \,\middle|\, v(0) = v(1) = 0 \right\}.$$

The trace operators γ_0 and γ_1 are defined in Appendix 3. At this stage we are dealing with smooth functions and γ_0 and γ_1 simply map a function onto its value at the boundary. Therefore

$$\gamma_0 v = v(\cdot\,,0) \quad \text{and} \quad \gamma_1 v = v(\cdot\,,a).$$

In order to formulate the weak variational form of Problem RMT, we start by rewriting Equations (3.6.6) and (3.6.12) in terms of inner products. The notation is explained in Appendix 1.

$$\begin{aligned}
&h_p\Big(\partial_t^2 w(\cdot,t)\,,\, v\Big)_\Omega + \eta_1\Big(\gamma_0(\partial_t^2 w(\cdot,t))\,,\,\gamma_0 v\Big)_I + \eta_1\Big(\gamma_1(\partial_t^2 w(\cdot,t))\,,\,\gamma_1 v\Big)_I \\
&+ h_p\Big(\nabla w(\cdot,t) + \boldsymbol{\psi}(\cdot,t)\,,\,\nabla v\Big)_{0,2}^{\Omega} + \eta_2\Big(\gamma_0(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_0(\partial_1 v)\Big)_I \\
&+ \eta_2\Big(\gamma_1(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_1(\partial_1 v)\Big)_I = 0. \qquad (3.6.16)
\end{aligned}$$

$$\begin{aligned}
&I_p\Big(\partial_t^2 \boldsymbol{\psi}(\cdot,t)\,,\,\boldsymbol{\phi}\Big)_{0,2}^{\Omega} + b_B(\boldsymbol{\psi}(\cdot,t),\boldsymbol{\phi}) + h_p\Big(\nabla w(\cdot,t) + \boldsymbol{\psi}(\cdot,t)\,,\,\boldsymbol{\phi}\Big)_{0,2}^{\Omega} \\
&+ \frac{\eta_1}{\alpha_b}\Big(\gamma_0(\partial_t^2 \psi_1(\cdot,t))\,,\,\gamma_0\phi_1\Big)_I + \frac{\eta_1}{\alpha_b}\Big(\gamma_1(\partial_t^2 \psi_1(\cdot,t))\,,\,\gamma_1\phi_1\Big)_I \\
&+ \frac{\eta_2}{\beta_b}\Big(\gamma_0(\partial_1 \psi_1(\cdot,t))\,,\,\gamma_0(\partial_1\phi_1)\Big)_I + \frac{\eta_2}{\beta_b}\Big(\gamma_1(\partial_1 \psi_1(\cdot,t))\,,\,\gamma_1(\partial_1\phi_1)\Big)_I \\
&+ \eta_2\Big(\gamma_0(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_0\phi_1\Big)_I \\
&+ \eta_2\Big(\gamma_1(\partial_1 w(\cdot,t) + \psi_1(\cdot,t))\,,\,\gamma_1\phi_1\Big)_I = 0. \qquad (3.6.17)
\end{aligned}$$

The next step is to define product spaces.

Product spaces

$$X = L^2(\Omega) \times L^2(\Omega)^2 \mathop{\times}_{n=1}^{4} L^2(I)\,,$$

$$H^1 = H^1(\Omega) \times H^1(\Omega)^2 \mathop{\times}_{n=1}^{4} H^1(\Omega)$$

$$S = T_1(\Omega) \times T_2(\Omega)^2 \mathop{\times}_{n=1}^{2} \big(T(I) \times C^1(\bar{I})\big)$$

$$T = \big\{v \in S \,\big|\, \gamma_0 v_1 = v_3, \gamma_1 v_1 = v_5, \gamma_0\,(v_2 \cdot \boldsymbol{e}_1) = -v_4, \gamma_1\,(v_2 \cdot \boldsymbol{e}_1) = -v_6\big\}$$

The following table explains the relationship between the functions used in Equations (3.6.16) and (3.6.17) and Equations (3.6.18) and (3.6.19) to follow. Note that for $v \in C^1(\bar{\Omega})$ and $i = 0,\, 1,\;\; \gamma_i\,(\partial_1 v) = (\gamma_i v)\,'$ – the derivative with respect to the variable x_1.

$u_1(t)$	$w(\cdot,t)$	v_1	v
$u_2(t)$	$\boldsymbol{\psi}(\cdot,t)$	v_2	$\boldsymbol{\phi}$
$u_3(t)$	$\gamma_0\big(w(\cdot,t)\big) = \gamma_0 u_1(t)$	v_3	$\gamma_0 v = \gamma_0 v_1$
$u_4(t)$	$-\gamma_0\psi_1(\cdot,t) = -\gamma_0(u_2(t)\cdot\boldsymbol{e}_1)$	v_4	$-\gamma_0\phi_1 = -\gamma_0(v_2\cdot\boldsymbol{e}_1)$
$u_5(t)$	$\gamma_1\big(w(\cdot,t)\big) = \gamma_1 u_1(t)$	v_5	$\gamma_1 v = \gamma_1 v_1$
$u_6(t)$	$-\gamma_1\psi_1(\cdot,t) = -\gamma_1(u_2(t)\cdot\boldsymbol{e}_1)$	v_6	$-\gamma_1\phi_1 = -\gamma_1(v_2\cdot\boldsymbol{e}_1)$

In the new notation, Equations (3.6.16) and (3.6.17) become

$$\begin{aligned} & h_p\Big(\ddot{u}_1(t)\,,\,v_1\Big)_\Omega + \eta_1\Big(\ddot{u}_3(t)\,,\,v_3\Big)_I + \eta_1\Big(\ddot{u}_5(t)\,,\,v_5\Big)_I \\ & + h_p\Big(\nabla u_1(t) + u_2(t)\,,\,\nabla v_1\Big)^\Omega_{0,2} + \eta_2\Big(u_3'(t) - u_4(t)\,,\,v_3'\Big)_I \\ & + \eta_2\Big(u_5'(t) - u_6(t)\,,\,v_5'\Big)_I \;=\; 0 \end{aligned} \tag{3.6.18}$$

and
$$\begin{aligned} & I_p\Big(\ddot{u}_2(t)\,,\,v_2\Big)^\Omega_{0,2} + b_B(u_2(t),v_2) + h_p\Big(\nabla u_1(t) + u_2(t)\,,\,v_2\Big)^\Omega_{0,2} \\ & + \frac{\eta_1}{\alpha_b}\Big(-\ddot{u}_4(t)\,,\,-v_4\Big)_I + \frac{\eta_1}{\alpha_b}\Big(-\ddot{u}_6(t)\,,\,-v_6\Big)_I \\ & + \frac{\eta_2}{\beta_b}\Big(-u_4'(t)\,,\,-v_4'\Big)_I + \frac{\eta_2}{\beta_b}\Big(-u_6'(t)\,,\,-v_6'\Big)_I \\ & + \eta_2\Big(u_3'(t) - u_4(t)\,,\,-v_4\Big)_I + \eta_2\Big(u_5'(t) - u_6(t)\,,\,-v_6\Big)_I \\ & = \;0 \end{aligned} \tag{3.6.19}$$

Bilinear forms

For u and v in T, define

$$\begin{aligned} c(u,v) &= h_p(u_1,v_1)_\Omega + I_P(u_2,v_2)^\Omega_{0,2} + \eta_1(u_3,v_3)_I \\ &\quad + \frac{\eta_1}{\alpha_b}(u_4,v_4)_I + \eta_1(u_5,v_5)_I + \frac{\eta_1}{\alpha_b}\eta_1(u_6,v_6)_I\,, \\ b_\Gamma(u,v) &= \eta_2\Big(u_3'-u_4, v_3'-v_4\Big)_I + \eta_2\Big(u_5'-u_6, v_5'-v_6\Big)_I \\ &\quad + \frac{\eta_2}{\beta_b}(u_4',v_4')_I + \frac{\eta_2}{\beta_b}(u_6',v_6')_I\,, \\ b_\Omega(u,v) &= b_B(u_2,v_2) + h_p\Big(\nabla u_1 + u_2, \nabla v_1 + v_2\Big)^\Omega_{0,2}\,, \\ b(u,v) &= b_\Omega(u,v) + b_\Gamma(u,v)\,. \end{aligned}$$

By adding Equations (3.6.18) and (3.6.19), we arrive at the following variational problem.

Find $u(t) \in T$ such that $c(\ddot{u}(t),v) = -b(u(t),v)$ for each $v \in T$.

We are now ready to consider the weak variational form. We define V as the closure of T in H^1. Note that all the bilinear forms are defined for elements of V, except for b_Γ. For u and v in V, define

$$b_\Gamma(u,v) = \lim_{n\to\infty} b_\Gamma(u_n,v_n)\,,$$

with $\{u_n\}$ and $\{v_n\}$ sequences in T such that $u_n \to u$ and $v_n \to v$.

As a consequence, the bilinear forms b and b_Γ are now defined on V.

Weak variational form of Problem RMT

Find $u \in C^1([0,\infty),V) \cap C^2((0,\infty),X)$ such that for each $t > 0$, $u'(t) \in V$, $u''(t) \in X$ and

$$c(u''(t),v) = -b(u(t),v) \quad \text{for each } v \in V.$$

Inertia space

The space X with the norm induced by the inner product c is the **inertia space**.

Energy space

The closure of T in H^1 is denoted by V. A norm on V is defined by $\|u\|_V = \sqrt{b(u,u)}$ and is called the **energy norm**. The space V with norm $\|\cdot\|_V$ called the **energy space**.

Theorem 1

The inertia space X is a separable Hilbert space and V is dense in X.

Proof

Appendix 5.

Theorem 2

There exist constants c_1 and c_2 such that

$$\|u\|_X \leq c_1\|u\|_{H^1} \leq c_2\|u\|_V$$

for each $u \in V$.

Proof

Appendix 5.

3.7 Equilibrium problems

In the rest of Chapter 3, X and V denote spaces with the following properties:

X is a Hilbert space with inner product c and norm $\|\cdot\|_X$;

V is a Hilbert space with inner product b and norm $\|\cdot\|_V$;

V is a subspace of X.

Theorem (Riesz)

For any f in the dual of V, there exists a unique $u \in V$ such that

$$b(u, v) = f(v) \quad \text{for each } v \in V.$$

Corollary

Suppose $\|u\|_X \leq \|u\|_V$ for each $u \in V$. For any $f \in X$, there exists a unique $u \in V$ such that

$$b(u, v) = c(f, v) \quad \text{for each } v \in V.$$

Proof

Let $g(v) = c(f, v)$ for each $v \in V$, then $|g(v)| \leq \|f\|_X \|v\|_V$ proving that g is in the dual of V. Applying the theorem yields the desired result.

Application

The theorem above yields the existence of a weak solution for Problem CTD 1.

Proof of Theorem 7 Section 3.5

The result follows from the theorem above and Theorem 1 Section 3.5. Recall that $f(v) = (t, \gamma v)^{\Gamma}_{0,2}$.

3.8 Vibration problems

In this section we consider the general linear vibration problem. Consider the Hilbert spaces X and V introduced in Section 3.7. Consider also a bilinear form a defined on V.

For any Banach space Y the spaces $C^k\big([0,\infty),Y\big)$ and $C^k\big((0,\infty),Y\big)$ are defined in Appendix 4.

Problem D

Find $u \in C^1\big([0,\infty),V\big) \cap C^2\big((0,\infty),X\big)$ such that for each $t>0$,

$$c\big(u''(t),v\big) + a\big(u'(t),v\big) + b\big(u(t),v\big) = 0 \quad \text{for each } v \in V,$$

$$u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Theorem

Suppose

(a) V is dense in X,

(b) $\|u\|_X \leq K\|u\|_V$ for each $u \in V$,

(c) the bilinear form a is symmetric, nonnegative and $|a(u,v)| \leq C\|u\|_V\|v\|_V$ for each u and v in V,

(d) $u_0 \in V$, $u_1 \in V$ and for some $y \in X$,

$$b(u_0,v) + a(u_1,v) = c(y,v) \quad \text{for each } v \in V.$$

Then Problem D has a unique solution.

Proof

See [VV].

Remark

It is possible to define linear operators M, C and K and arrive at an abstract differential equation $Mu'' + Cu' + Ku = 0$. It is then possible to prove an equivalent existence result, see e.g. [Sho, p 131].

Applications

For Problems VR 4, VT 4, CTD 2 and RMT the first three conditions in the Theorem are met. This is proven in each section where the weak variational forms of the problems are discussed.

3.9 Modal analysis

In this section we consider the modal analysis of the general linear vibration problem. Consider the Hilbert spaces X and V introduced in Section 3.7. Consider also a bilinear form a defined on V.

The fact that a solution of the (general) vibration problem exists is not enough. To determine the response of a system to excitation, knowledge of the vibration spectrum is required. We need to know whether the solution may be written as the superposition of modes.

First consider the case of **no damping**, i.e. $a = 0$. For the modal analysis of the system, a function $\tilde{u}(x,t) = T(t)u(x)$ is considered as a possible solution. This requires consideration of the following eigenvalue problem.

Problem E1

Find a complex number λ and $u \in V$ such that

$$b(u,v) = \lambda c(u,v) \quad \text{for each } v \in V.$$

Natural frequencies and modes

The function T_n satisfies $T_n'' = -\lambda_n T$ and hence the **natural angular frequencies** are equal to $\omega_n = \sqrt{\lambda_n}$. The formal solution of Problem D (general vibration problem) is given by

$$u(t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \sin \omega_n t)\; e_n,$$

where each e_n is an eigenvector. For the series above to converge, a necessary condition is that it must be possible to write the the initial values u_0 and u_1 as a series using the sequence of eigenvectors. This implies that the existence of a complete orthonormal sequence of eigenvectors is required. We present two well-known results, slightly modified.

Theorem 1

(a) The eigenvalues are (real and) positive.

(b) The eigenfunctions are orthogonal in the inertia space X with respect to the inner product c.

Proof

The bilinear forms on both sides of the equation are inner products. Consequently, the eigenvalues must be real and positive. Furthermore, for different eigenvalues λ and μ it follows that $\lambda c(u,v) = b(u,v) = \mu c(u,v)$. Therefore that $(\lambda - \mu)c(u,v) = 0$ and consequently $c(u,v) = 0$.

Theorem 2

Suppose the embedding of V into X is compact.

(a) The set of eigenvalues can be ordered as a sequence $\{\lambda_n\}$ converging to ∞ as $n \to \infty$.

(b) The set of eigenvectors can be ordered as a sequence and this sequence is complete (or total) in X.

Proof

For each $f \in X$, there exists a unique $u \in V$ such that

$$b(u, v) = c(f, v) \quad \text{for each } v \in V$$

by the corollary in Section 3.7. (The embedding of V into X is bounded.) Define a mapping K by $u = Kf$, then

$$b(Kf, v) = c(f, v) \quad \text{for each } v \in V.$$

The mapping K is defined on X and it is clearly linear. Note that

$$b(u, v) = \lambda c(u, v) \quad \text{for each } v \in V$$

if and only if $\lambda Ku = u$ or $Ku = \lambda^{-1}u$.

The operator K is symmetric due to the fact that b and c are symmetric. The inequality

$$\|Kf\|_V^2 \leq \|f\|_X \|Kf\|_X \leq k_{bc} \|f\|_X \|Kf\|_V$$

implies that K is a bounded operator from the inertia space X into the energy space V. If a set A is bounded in the inertia space, then the set KA is bounded in the energy space and consequently pre-compact in the inertia space (due to the compactness of the embedding). Therefore the operator K is compact.

Both conclusions of the theorem now follow from the theory of compact symmetric linear operators on a separable Hilbert space, see e.g. [Ze, p 232].

Modal damping

We now consider the case where the bilinear form a is not zero but we assume that

$$a = k_1 c + k_2 b.$$

Consider a function $\tilde{u}(x, t) = T(t)u(x)$ as a possible solution. The eigenvalue problem is the same as for the undamped case but

$$T''c(u, v) + T'\big(k_1 c(u, v) + k_2 b(u, v)\big) + Tb(u, v) = 0.$$

This leads to the following ordinary differential equation

$$T'' + (k_1 + k_2\lambda)T' + \lambda T = 0.$$

Again it is possible to present the solution in series form.

3.10 Nonmodal damping

In this section we consider the the general linear vibration problem (Section 3.8) with nonmodal damping, i.e.

$$a \neq k_1 c + k_2 b.$$

Nonmodal damping is often a consequence of boundary damping. It also features in hybrid systems such as the models for the vertical structure presented in Section 2.1. Computation of the natural frequencies leads to a quadratic eigenvalue problem with complex eigenvalues and eigenvectors (see Chapter 6).

The quadratic eigenvalue problem

Consider the Hilbert spaces X and V introduced in Section 3.8 and the general linear vibration problem, Problem D. In general, consideration of a solution of the form $e^{\lambda t}u$ leads to a **quadratic eigenvalue problem**.

$$\lambda^2 c(u,v) + \lambda a(u,v) + b(u,v) = 0 \quad \text{for each } v \in V.$$

This problem is a generalization of the eigenvalue problems in Chapter 6.

It is clear that imaginary eigenvalues and eigenvectors are possible and X is a real Hilbert space. However, we may consider the space X to be embedded in complex space $\widetilde{X}$. This can be done in a rigorous manner, see e.g. [Sch, p 154]. Elements of $\widetilde{X}$ are of the form $x = x_1 + ix_2$, where x_1 and x_2 are in X. We also have a subspace $\widetilde{V}$ with elements of the form $x = x_1 + ix_2 \in \widetilde{V}$ where x_1 and x_2 are in V.

The bilinear forms a, b and c must be extended to $\widetilde{X}$ and $\widetilde{V}$. Consider for example the bilinear form c:

$$\tilde{c}(x,y) = c(x_1,y_1) + ic(x_2,y_1) - ic(x_1,y_2) + c(x_2,y_2).$$

It is easily checked that the bilinear form $\tilde{c}$ is an inner product for $\widetilde{X}$ and that $\widetilde{X}$ is a separable Hilbert space. Similarly, we find that $\widetilde{V}$ is a Hilbert space with inner product $\tilde{b}$. Furthermore, $\widetilde{V}$ is dense in $\widetilde{X}$ and the relevant estimates remain valid. We have for example

$$\begin{aligned}
\tilde{c}(x,x) &= c(x_1,x_1) + c(x_2,x_2) \\
&\leq K_{bc}\left(b(x_1,x_1) + b(x_2,x_2)\right) \\
&= K_{bc}\,\tilde{b}(x,x).
\end{aligned}$$

We now return to the **original notation** and consider the quadratic eigenvalue problem.

Problem QE

Find a complex number λ and $u \in V$ such that

$$\lambda^2 c(u,v) + \lambda a(u,v) + b(u,v) = 0 \quad \text{for each } v \in V.$$

To apply the theory on convergence, we need an alternative formulation.

Non selfadjoint eigenvalue problem

The quadratic eigenvalue problem is equivalent to a conventional abstract eigenvalue problem in a product space. Let $H = V \times X$ and

$$(x,y)_H = b(x_1,y_1) + c(x_2,y_2) \quad \text{for} \quad x,\, y \in H.$$

It is easy to see that $(\cdot,\cdot)_H$ is an **inner product** for H and that H is complete.

Problem E2

Find a complex number λ and $x \in H$ such that

$$\begin{aligned} x_2 &= \lambda x_1 \\ b(x_1,v) + a(x_2,v) &= -\lambda c(x_2,v) \quad \text{for each } v \in V. \end{aligned}$$

If λ is an eigenvalue and u an eigenvector of Problem QE, then λ is an eigenvalue and $\langle u, \lambda u\rangle$ an eigenvector of Problem E2. Conversely, if λ is an eigenvalue and x an eigenvector of Problem E2, then λ is an eigenvalue and x_1 an eigenvector of Problem QE.

If the sequence of eigenvectors is complete in the complex Hilbert space X, the solution of Problem D can be written in series form. The abstract form of the quadratic eigenvalue problem is considered in Section 5.4. It is of the same type as the abstract form of the eigenvalue problem for a Timoshenko beam with boundary damping considered in a recent paper [Shu]. Shubov proved that the sequence of eigenvectors is complete but it should be noted that the problems are not the same.