Appendix A

Reliability Statistics Preliminaries

A.1 Laplace's trend test

De Laplace (1773) makes use of the fact that under the HPP assumption, the first $m-1$ arrival times, $T_1, T_2, ..., T_{m-1}$ are the order statistics from a uniform distribution on $(0, T_m]$ and hence is,

$$U = \frac{\frac{\sum_{i=1}^{m-1} T_i}{m-1} - \frac{T_m}{2}}{T_m \sqrt{\frac{1}{12(m-1)}}} \tag{A.1}$$

U approximates a standardized normal variate at a 5% level of significance as soon as $m \geq 4$.

In the case where $U \geq 2$ there is strong evidence of reliability degradation while $U \leq -2$ indicates reliability improvement. If $1 \geq U \geq -1$, there is no evidence of an underlying trend and it is referred to as a non-committal data set.

A.2 Renewal theory

A.2.1 Basic concepts

Only IID data sets can be used meaningfully in renewal theory. Data sets of this types are very often, but not necessarily, generated by parts (as defined in Section 1.2.1).

Suppose the interarrival times are part of a distribution $f_X(x)$ with cumulative distribution $F_X(x)$. $F_X(x)$ is referred to as the *unreliability* function since it gives the probability of failure up to a certain age x, i.e. $F_X(x) = \Pr[X \leq x]$. Similarly, is the *reliability* function, $R_X(x)$, defined as $R_X(x) = \Pr[X \geq x]$ or $R_X(x) = 1 - F_X(x)$, i.e. the probability of survival up to age x. From this it is possible to define the force of mortality (FOM) or hazard rate of an item that gives the probability of failure within a short time, provided that the item lived up to that time, i.e. $h_X(x) = \Pr[x < X \leq x + dx | X > x]$. The FOM can also be expressed as,

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)} \tag{A.2}$$

The FOM is often described as a conditional probability density function. This is not true because,

$$R_X(x) = e^{-\int_0^x h_X(\tau)d\tau} \tag{A.3}$$

and since $R_X(\infty) = 0$ it implies that

$$\lim_{x \to \infty} \int_0^x h_X(\tau)d\tau = \infty \tag{A.4}$$

A.2.2 Distributions

Some distributions often used to model renewal situations are summarized in Table A.1 below.

Table A.1: Distributions often used in renewal theory

Distribution	**Probability Density Function** ($f_X(x)$)	**FOM** ($h_X(x)$)
Exponential $\lambda > 0, x \geq 0$	$\lambda \exp(-\lambda x)$	λ
Weibull $\beta, \eta > 0, x \geq 0$	$\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}}$	$\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}$
Log-normal $\sigma > 0, x \geq 0$	$\frac{\exp\left[-\frac{1}{2}(\ln x - \mu)^2\right]}{\sqrt{2\pi}x\sigma}$	$f_X(x)/\left[1 - \int_0^x f_T(\tau)d\tau\right]$
Log-logistic $\alpha, \lambda > 0, x \geq 0$	$\frac{\alpha x^{\alpha-1}\lambda}{[1+\lambda x^{\alpha}]^2}$	$\frac{\alpha x^{\alpha-1}\lambda}{1+\lambda x^{\alpha}}$
Normal $\sigma > 0, x \geq 0$	$\frac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]}{\sqrt{2\pi}\sigma}$	$f_X(x)/\left[1 - \int_0^x f_T(\tau)d\tau\right]$

A.2.3 Incomplete observations in renewal situations

Very often only partial information is available on an item's survival time. These are referred to as *censored* or *truncated* information. In many cases, this type of information is the only type available in reliability modeling.

A.2.3.1 Censoring

Type I Right Censoring occurs where an event is observed only if it happens prior to some prespecified time. *Progressive Type I Right Censoring* occurs where specimens have different, fixed-sacrifice censoring times, predetermined by the observer. This has the advantage that the sacrificed specimens give information on the natural history of nonlethal events.

Type II Right Censoring occurs where a study continues until the failure of the first r individuals, with $r < n$ and n the total number of individuals. This type of censoring scheme may save time and money if equipment is tested. *Progressive Type II Right Censoring* is a natural extension of *Progressive Type I Right Censoring*.

Left censored observations occur when the event of interest has occurred to the specimen before the period of observation. A good example is a study on the time to first use of marijuana by boys, where the question was asked: "When did you first use marijuana?" and the response "I have used it but I cannot recall just when the time was".

A data set contains *doubly censored* observations where some are left censored and some right censored. If an event is only known to have occurred within a certain interval, the observation is called *interval censored*.

A.2.3.2 Truncation

A *truncation* is defined as a condition where certain subjects are screened so that the investigator is not aware of their existence. If Y is the time of the event which truncates individuals, then, for left-truncated samples, only individuals with $X \geq Y$ are observed. For example, if survival times in an old age home are studied where the age of 60 is a prerequisite ($Y = 60$).

It is also possible to define *right truncations*. This situation is encountered where an event has to occur first before a specimen is included in the sample. A good example is a mortality study on AIDS infected people.

A.2.3.3 Contribution of incomplete observations to the likelihood

The maximum likelihood method is most often used to estimate model parameters in survival analysis and it is thus important to note incomplete observations' respective contributions to the likelihood.

Table A.2: The contributions of incomplete observations to the likelihood

Observation type	Contribution to likelihood
Exact lifetimes	$f_X(x)$
Right-censored	$R_X(r_i)$
Left-censored	$1 - R_X(l_i)$
Left-truncations	$f_X(x)/R_X(Y)$
Right-truncations	$f_X(Y)/[1 - R_X(Y)]$
Interval-censored	$[R_X(l_i) - R_X(r_i)]$

In Table A.2, l_i and r_i refer to the left and right margin of an observation interval respectively. Klein and Moeschberger (1990) discuss incomplete information in survival analysis in detail.

A.3 Point Process Theory

A *point process* is a mathematical model that describes a physical phenomenon occurring as highly localized events, distributed randomly in a continuum. In this case, the events are failures and the continuum is time. Brillinger (1978) gives a formal definition.

A.3.1 Basic concepts

Counting process. A counting process, $N(t)$, counts the number of events that have occurred up to time t, where $N(t) \in \mathbb{Z}^+$ and $t \in \mathbb{R}^+$.

Independent increments. A counting process $N(t), t \geq 0$, has independent increments if $N(t_1) - N(0), ..., N(t_k) - N(t_{k-1})$ for $0 < t_1 < ... < t_k$, $k = 2, 3, ..$, are independent random variables.

Stationary increments. A counting process $N(t), t \geq 0$, has stationary increments if for any two points $t > s \geq 0$ and any $\Delta > 0$, the random variables $(N(t) - N(s))$ and $(N(t + \Delta) - N(s + \Delta))$ are identically distributed.

Stationarity of a point process. If a point process has stationary increments, it is said to be stationary.

Intensity. The intensity of a counting process is defined as:

$$\iota(t) \equiv \lim_{\Delta t \to 0} \frac{\Pr\{N(t+\Delta t) - N(t) \geq 1 | H_t\}}{\Delta t} \tag{A.5}$$

where $N(t)$ is the observed number of failures in $(0, t]$ and H_t is the history up to, but not including, time t. Thus, $\iota(t)\Delta t$ is, for a small Δt, the approximate probability of an event in $[t, t + \Delta t)$, given the process history.

When simultaneous failures cannot occur (when the process is orderly) and also stationary, then $\iota(t) = v(t)$, where $v(t)$ is the so called ROCOF, i.e.

$$v(t) = \frac{d}{dt} E\{N(t)\} \tag{A.6}$$

The ROCOF of an NHPP is referred to as the *peril* rate and is denoted by $\rho(t)$.

A.3.2 Homogeneous Poisson Process (HPP)

The HPP is a non-terminating sequence of independent and identically exponentially distributed X_i's. A counting process, $N(t)$, is said to be an HPP if:

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments, i.e. $N(t_2) - N(t_1) \perp N(t_1)$.

(iii) The number of events in any interval of length $t_2 - t_1$ has a Poisson distribution with mean $\rho(t_2 - t_1)$. This implies that for $t_2 > t_1 \geq 0$,

$$\Pr\{N(t_2) - N(t_1) = j\} = \frac{e^{-\rho(t_2 - t_1)}\{\rho(t_2 - t_1)\}^j}{j!} \tag{A.7}$$

for $j \geq 0$

A.3.3 Non-homogeneous Poisson Process (NHPP)

The NHPP is a non-terminating sequence of independent and identically exponentially distributed X_i's. A counting process, $N(t)$, is said to be an NHPP if:

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments, i.e. $N(t_2) - N(t_1) \perp N(t_1)$

(iii) The number of events in any interval of length $t_2 - t_1$ has a Poisson distribution with mean $\int_{t_1}^{t_2} \rho(t)dt$. This implies that for $t_2 > t_1 \geq 0$,

$$\Pr\{N(t_2) - N(t_1) = j\} = \frac{e^{-\rho(t_2-t_1)}\{\int_{t_1}^{t_2} \rho(t)dt\}^j}{j!} \tag{A.8}$$

for $j \geq 0$.

Bain, Engelhardt, and Wright (1985) proposed some methods to test for the validity of either the NHPP or HPP assumption. Two popular parametric forms for the peril rate of an NHPP are (1) $\rho_1(t) = \alpha e^{\gamma t}$ (log-linear); and (2) $\rho_2(t) = \alpha\gamma t^{\gamma-1}$ (power-law). The latter is often referred to as a Weibull process because the distribution of times to first failure of processes of this kind will be Weibull. To avoid confusion, this term will not be used.

A.3.4 Branching Poisson Process (BPP)

The BPP is discussed in detail in Cox and Lewis (1966) and a summary of their discussion in the present notation is given here. For this process a series of primary events is generated by an HPP and each primary event has positive probability of generating a series of subsidiary events according to a finite renewal process. It is also assumed that the two series of events are indistinguishable. As before, the interarrival times to events (primary or subsidiary) are denoted by X_i, while the interarrival times between primary events are described by Z_i. The interarrival time between a primary and subsidiary event or between two subsidiary events is called Y_i.

Let q be the probability that a primary event triggers a series of a subsidiary events. From this it follows that the expected number of subsidiary events, given that at least 1 subsidiary event occurs, is a/q. Also, if it is assumed that times between subsidiary events will tend to be small relative to Z_i's, it is possible to calculate $\hat{E}[Z]$ with,

$$\hat{E}[Z] = \frac{\sum_{j=1}^{l}(G_j - y)}{l} \tag{A.9}$$

where G_j is the j^{th} excess time over j and l is the total number of observed intervals. (y should be interpreted in the same way as x, defined in Figure 1.2).

A.3.5 Likelihood construction for PMIM applied on Poisson Process data

Define the PMIM as,

$$\iota_u(t, \boldsymbol{z}) = \iota_{u_0}(t) \cdot \exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}) \tag{A.10}$$

The corresponding cumulative intensity function is:

$$I_u(t, \boldsymbol{z}) = I_{u0}(t) \cdot \exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}) \tag{A.11}$$

where $I_{u_0}(t) = \int_0^t \iota_0(u)\mathrm{d}u$.

Suppose m individuals are under observation. Individual i is observed over the time interval (S_i, T_i) and n_i events are observed at times $t_{i1} < ... < t_{in_i}$. For simplicity suppose S_i is equal to zero. Now, let $\iota_{u0}(t)$ be specified by parameters in the vector $\boldsymbol{\theta}$. The likelihood function is then,

$$L(\boldsymbol{\theta}, \boldsymbol{\gamma}) = \prod_{i=1}^{m} \left\{ \prod_{j=1}^{n_i} \iota_u(t_{ij}) \right\} \exp\{-I_u(T_i)\} \tag{A.12}$$

which can be decomposed as,

$$\begin{aligned} L(\boldsymbol{\theta}, \boldsymbol{\gamma}) &= \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{\iota_{u0}(t_{ij}; \boldsymbol{\theta})}{I_{u0}(T_i; \boldsymbol{\theta})} \cdot \prod_{i=1}^{m} \exp\left[-I_{u0}(T_i; \boldsymbol{\theta}) e^{\boldsymbol{\gamma} \cdot \boldsymbol{z}_i}\right] \left[I_{u0}(T_i; \boldsymbol{\theta}) e^{\boldsymbol{\gamma} \cdot \boldsymbol{z}_i}\right]^{n_i} \\ &= L_1(\boldsymbol{\theta}) \cdot L_2(\boldsymbol{\theta}, \boldsymbol{\gamma}) \end{aligned} \tag{A.13}$$

The likelihood kernel $L_2(\boldsymbol{\theta}, \boldsymbol{\gamma})$ arises from the Poisson distribution of the counts $n_1, n_2, ..., n_m$, and the kernel $L_1(\boldsymbol{\theta})$ arises from the conditional distribution of the event times, given the counts. Lawless (1987) has shown that if the failure times are not too different, the two kernels can be solved individually to obtain a result fairly close to the full maximum likelihood estimate.

If it assumed that all the T_i's are equal to T, then $L_2(\boldsymbol{\theta}, \boldsymbol{\gamma})$ can be decomposed as,

$$L_2(\boldsymbol{\theta}, \boldsymbol{\gamma}) \propto \Pr(n_1, ..., n_m | \sum_{i=1}^{m} n_i = n) \cdot \Pr(\sum_{i=1}^{m} n_i = n) \tag{A.14}$$

or,

$$L_2(\boldsymbol{\theta}, \boldsymbol{\gamma}) = L_3(\boldsymbol{\gamma}) \cdot L_4(\boldsymbol{\theta}, \boldsymbol{\gamma}) \tag{A.15}$$

where

$$L_3(\boldsymbol{\gamma}) = \prod_{i=1}^{m} \left[\frac{\exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}_i)}{\sum_{l=1}^{m} \exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}_l)} \right]^{n_i} \tag{A.16}$$

and

$$L_4(\boldsymbol{\theta}, \boldsymbol{\gamma}) = \exp\left[-I_{u0}(T; \boldsymbol{\theta}) \sum_{l=1}^{m} \exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}_l) \right] \cdot \left[I_{u0}(T; \boldsymbol{\theta}) \sum_{l=1}^{m} \exp(\boldsymbol{\gamma} \cdot \boldsymbol{z}_l) \right]^{n} \tag{A.17}$$

Williams (1981) indicated that $L_3(\boldsymbol{\gamma})$ is precisely Cox's partial likelihood. Lawless (1987) used data from Gail, Santner, and Brown (1980) to illustrate the convenience of the theory above.

Appendix B

Simplifications of the Combined Models

B.1 The non-repairable case

In this section, it is shown that equation (3.8) (repeated below as equation (B.1) for convenience) can be reduced to the majority of models considered in the literature survey on advanced intensity models (Section 2.3).

$$h(x,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(g_s^{k_l}(x, \tau_s^{k_l}, \psi_s^{k_l}) \cdot \lambda(\boldsymbol{\gamma}_s^{k_l} \cdot \boldsymbol{z}_s^{k_l}) + \nu(\boldsymbol{\alpha}_s^{k_l} \cdot \boldsymbol{z}_s^{k_l}) \right) \tag{B.1}$$

It is assumed that covariate values are always positive.

B.1.1 Proportional Hazards Model

Restrictions are summarized in Table B.1.

Table B.1: Parameter restrictions for equation (B.1) to obtain a Proportional Hazards Model

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	γ_j, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.1) reduces to,

$$h(x,\boldsymbol{\theta}) = g(x) \cdot \lambda(\boldsymbol{\gamma} \cdot \boldsymbol{z}(x)) \tag{B.2}$$

which is similar to (2.8) of Section 2.3.1.1, if λ is chosen to be exponential. For the fully parametric Weibull PHM, $g(x)$ should be substituted with the FOM of a Weibull distribution. To obtain a stratified PHM, s should not be fixed to 1 but, for example, used to denote the previous number of failures, i.e. $s^k = 1$ if $x \leq X_1^k$, $s^k = 2$ if $X_1^k < x \leq X_2^k$, etc. This leads to,

$$h(x,\boldsymbol{\theta}) = g_s(x) \cdot \lambda(\gamma_s \cdot \boldsymbol{z}(x)) \tag{B.3}$$

which was introduced in (2.11).

B.1.2 Proportional Odds Model for Non-repairable Systems

Equation (B.1) should be reduced to $g(x)$ only. Restrictions are summarized in Table B.2.

Table B.2: Parameter restrictions for equation (B.1) to obtain a Proportional Odds Model for non-repairable systems

Parameter	Restriction		
n:	n	=	1, thus $l = 1$
k:	k	=	$1, 2, ..., w$
s:	s^l	=	1, for all values of i^{k_l}
$\varsigma_s^{k_l}$:	$\varsigma_s^{k_l}$	=	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	=	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	=	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	=	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	=	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

The restrictions in Table B.2 lead to $h(x,\boldsymbol{\theta}) = g(x)$. To obtain the effect of diminishing covariates, $g(x)$ should be substituted with the FOM of a log-logistic distribution, i.e.

$$h(x;\boldsymbol{\theta}) = \frac{\delta}{x \cdot (1 + x^{-\delta} \cdot \exp(-\boldsymbol{\gamma} \cdot \boldsymbol{z}(x)))} \tag{B.4}$$

as explained in Section 2.3.1.3.

B.1.3 Additive Hazards Model

Restrictions are summarized in Table B.3.

Table B.3: Parameter restrictions for equation (B.1) to obtain an Additive Hazards Model

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	α_j, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.1) reduces to,

$$h(x, \boldsymbol{\theta}) = g(x) + \nu(\boldsymbol{\alpha} \cdot \boldsymbol{z}(x)) \tag{B.5}$$

If s is not fixed to 1, the model can be stratified as Pijnenburg (1991) suggested.

B.1.4 PWP Model 2

Restrictions are summarized in Table B.4.

Table B.4: Parameter restrictions for equation (B.1) to obtain a PWP Model 2

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	i^{k_l}, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	γ_{s_j}, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.1) reduces to,

$$h(x, \boldsymbol{\theta}) = g_s(x) \cdot \lambda(\boldsymbol{\gamma}_s \cdot \boldsymbol{z}(x)) \tag{B.6}$$

which is similar to the PWP Model 2 presented in (2.37). The combined model is not able to reduce to the model proposed by Prentice et al. in (2.42). To have (2.42) as a special case of (3.1), a second stratification variable would be required.

B.1.5 Accelerated Failure Time Model for Non-repairable Systems

Restrictions are summarized in Table B.5.

Table B.5: Parameter restrictions for equation (B.1) to obtain an Accelerated Failure Time Model for non-repairable systems

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	ψ_s^k	$=$	$\phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(x))$, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.1) reduces to,

$$h(x, \boldsymbol{\theta}) = g(x \cdot \phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(x)) \tag{B.7}$$

which allows for implementation of (2.44) to (2.47). By lifting the restriction that $\gamma_{s_j}^{k_l} = -\infty$, for all values of s, k and $j \in \{1, 2, ..., m\}$, the Extended Hazard Regression Model of Ciampi and Etezadi-Amoli (1985) and Etezadi-Amoli and Ciampi (1987) can be obtained, i.e.

$$h(x, \boldsymbol{\theta}) = g(x \cdot \phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(x)) \cdot \lambda(\boldsymbol{\gamma} \cdot \boldsymbol{z}(x)) \tag{B.8}$$

as presented in (2.48).

B.1.6 Proportional Age Reduction

Restrictions are summarized in Table B.6.

Table B.6: Parameter restrictions for equation (B.1) to obtain an Proportional Age Reduction Model

Parameter	Restriction
n:	$n = 1$, thus $l = 1$
k:	$k = 1, 2, ..., w$
s:	$s^l = 1$, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l} = 1$, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k} = 1$, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l} = \tau$, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l} = -\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l} = -\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.1) reduces to,

$$h(x, \boldsymbol{\theta}) = g(x, \tau) \tag{B.9}$$

The FOM in (B.9) is only a function of x and the factor τ that allows for a jump or setback in time. This model can be used to formulate any PAR model discussed in Section 2.3.3.4.

B.1.7 The model of Lawless and Thiagarajah (1996)

For this model the baseline function g is chosen to be 1. Further restrictions are summarized in Table B.7.

Table B.7: Parameter restrictions for equation (B.1) to obtain an Proportional Age Reduction Model

Parameter	Restriction
n:	$n = 1$, thus $l = 1$
k:	$k = 1, 2, ..., w$
s:	$s^l = 1$, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l} = 1$, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l} = -\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\boldsymbol{\gamma}_s^{k} = [\ln \frac{\beta}{\eta^\beta} \quad \beta - 1]$, for $j = 1, 2, ..., m$ and all values of s, k and l
$z_s^{k_l}$:	$\boldsymbol{z}_s^{k_l} = [1 \quad \ln x]$, for all values of s, k and l

Equation (B.1) reduces to,

$$\begin{aligned} h(x,\boldsymbol{\theta}) &= \frac{\beta}{\eta^\beta}x^{\beta-1} \\ &= \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} \end{aligned} \tag{B.10}$$

which is a simple Weibull FOM. Following the same argument, it is also possible to obtain the models proposed by Calabria and Pulcini (2000), which are special cases of the model by Lawless and Thiagarajah (1996).

B.2 The repairable case

It is shown in this section that equation (3.25) (repeated below as equation (B.11) for convenience) can be reduced to the majority of models considered in the literature survey on advanced intensity models (Section 2.3).

$$v(t,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(g_s^{k_l}(t,\tau_s^{k_l},\psi_s^{k_l}) \cdot \lambda(\boldsymbol{\gamma}_s^{k_l} \cdot \boldsymbol{z}_i^{k_l}) + \nu(\boldsymbol{\alpha}_s^{k_l} \cdot \boldsymbol{z}_i^{k_l}) \right) \tag{B.11}$$

It is assumed that covariate values are always positive.

B.2.1 Proportional Mean Intensity Model

Restrictions are summarized in Table B.8.

Table B.8: Parameter restrictions for equation (B.11) to obtain a Proportional Mean Intensity Model

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	γ_j, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.11) reduces to,

$$v(t, \boldsymbol{\theta}) = g(t) \cdot \lambda(\boldsymbol{\gamma} \cdot \boldsymbol{z}(t)) \tag{B.12}$$

which is similar to (2.19) of Section 2.3.1.2, if λ is chosen to be exponential. If the PMIM in (B.12) is parameterized with a log-linear representation of a NHPP, i.e. $g(t)$ is chosen to be log-linear, the model in (2.23) is obtained. Equation (B.12) can also be stratified as described in Section 2.3.1.2.

B.2.2 Proportional Odds Model

No reference was found where the POM was applied on repairable systems, but a similar approach as in Section B.1.2 can be followed where (B.11) is reduced to $g(t)$ only. The restrictions are summarized in Table B.9.

Table B.9: Parameter restrictions for equation (B.11) to obtain a Proportional Odds Model for repairable systems

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Following the argument of B.1.2, the restrictions in Table B.9 lead to $v(t, \boldsymbol{\theta}) = g(t)$. To obtain the effect of diminishing covariates, $g(t)$ could be substituted with a function where if $t \to \infty$, $\gamma \to 0$. One such function is,

$$v(t; \boldsymbol{\theta}) = \frac{\delta}{t \cdot (1 + t^{-\delta} \cdot \exp(-\boldsymbol{\gamma} \cdot \boldsymbol{z}(t)))} \tag{B.13}$$

where δ is a measure of precision as before.

B.2.3 Additive Mean Intensity Model (Additive ROCOF Model)

Restrictions are summarized in Table B.10.

Table B.10: Parameter restrictions for equation (B.11) to obtain an Additive Mean Intensity Model (Additive ROCOF Model)

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	α_j, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.11) reduces to,

$$v(t, \boldsymbol{\theta}) = g(t) + \nu(\boldsymbol{\alpha} \cdot \boldsymbol{z}(t)) \tag{B.14}$$

If s is not fixed to 1, the model can be stratified.

B.2.4 PWP Model 1

Restrictions are summarized in Table B.11.

Table B.11: Parameter restrictions for equation (B.11) to obtain a PWP Model 1

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	i^{k_l}, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	$\psi_s^{k_l}$	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	γ_{s_j}, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.11) reduces to,

$$v(x, \boldsymbol{\theta}) = g_s(t) \cdot \lambda(\boldsymbol{\gamma}_s \cdot \boldsymbol{z}(t)) \tag{B.15}$$

which is similar to the PWP Model 1 presented in (2.36). The combined model is not able to reduce to the model proposed by Prentice et al. in (2.41). To have (2.41) as a special case of (B.11), a second stratification variable would be required.

B.2.5 Accelerated Failure Time Model for Repairable Systems

Restrictions are summarized in Table B.12.

Table B.12: Parameter restrictions for equation (B.11) to obtain an Accelerated Failure Time Model for repairable systems

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	1, for all values of s, k and l
$\psi_s^{k_l}$:	ψ_s^{k}	$=$	$\phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(x))$, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l

Equation (B.11) reduces to,

$$v(t, \boldsymbol{\theta}) = g(t \cdot \phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(t)) \tag{B.16}$$

which allows for implementation of (2.44) to (2.47). By lifting the restriction that $\gamma_{s_j}^{k_l} = -\infty$, for all values of s, k and $j \in \{1, 2, ..., m\}$, the Extended Hazard Regression Model of Ciampi and Etezadi-Amoli (1985) and Etezadi-Amoli and Ciampi (1987) can be obtained, i.e.

$$h(x, \boldsymbol{\theta}) = g(x \cdot \phi(\boldsymbol{\omega} \cdot \boldsymbol{z}(x)) \cdot \lambda(\boldsymbol{\gamma} \cdot \boldsymbol{z}(x)) \tag{B.17}$$

as presented in (2.48).

B.2.6 Proportional Age Reduction Model

Restrictions are summarized in Table B.13.

Table B.13: Parameter restrictions for equation (B.11) to obtain an Proportional Age Reduction Model

Parameter	Restriction		
n:	n	$=$	1, thus $l = 1$
k:	k	$=$	$1, 2, ..., w$
s:	s^l	$=$	1, for all values of i^{k_l}
$\zeta_s^{k_l}$:	$\zeta_s^{k_l}$	$=$	$(1 - \epsilon_k)$, for all values of s, k and l
$\psi_s^{k_l}$:	ψ_s^k	$=$	1, for all values of s, k and l
$\tau_s^{k_l}$:	$\tau_s^{k_l}$	$=$	0, for all values of s, k and l
$\alpha_s^{k_l}$:	$\alpha_{s_j}^{k_l}$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s, k and l
$\gamma_s^{k_l}$:	$\gamma_{s_j}^{k_l}$	$=$	γ_j, for $j = 1, 2, ..., m$ and all values of s, k and l

In this case $g(t)$ should be selected such that $g(t) = t$. Equation (B.11) reduces to,

$$v(t, \boldsymbol{\theta}) = (1 - \epsilon_k) \cdot \lambda(\boldsymbol{\gamma} \cdot \boldsymbol{z}(t)) \cdot t \tag{B.18}$$

which is similar to the model proposed in (2.56).

Appendix C

Numerical Optimization Techniques

C.1 Introduction

Four optimization techniques were implemented successfully to solve the objective functions described in Chapter 3, i.e. converged to the point where all the objective function's partial derivatives were zero, namely:

(i) A Nelder-Mead type simplex search method. (See Buchanan and Turner (1992)).

(ii) A Standard Broyden-Fletcher-Goldfarb-Shanno (BFGS) Quasi-Newton method with a mixed quadratic and cubic line search procedure. (See Wismer and Chattergy (1978)).

(iii) Snyman's dynamic trajectory optimization method. (See Snyman (1982) and Snyman (1983)).

(iv) A modified Newton-Raphson procedure. (See Klein and Moeschberger (1990) and Press, Teukolsky, Vetterling, and Flannery (1993)).

The performance of each one of the methods was measured according to their economy (number of iterations needed before convergence, number of objective function evaluations and number of partial derivative evaluations) and robustness (the accuracy of initial values required for convergence and its ability to handle steep valleys and discontinuities in the objective function). Methods (i) and (ii) maximized the objective functions successfully but performed mediocre. Snyman's method was found to be expensive but extremely robust which is a very valuable attribute. The modified Newton-Raphson method proved to be the most economical and fairly robust as well. For the above mentioned reasons, only Snyman's method and the modified Newton-Raphson method are considered in this discussion on numerical optimization procedures.

C.2 Snyman's Dynamic Trajectory Optimization Method

Snyman's method models a conservative force field in m-dimensions (the number of variables in the objective function) with the objective function and then monitors the trajectory of a particle of unit mass (released from rest) as it 'rolls' down the objective function to the point of least potential energy, which is the minimum of the objective function. In this brief presentation of Snyman's technique, the objective function is $l(x, \boldsymbol{\theta})$, the log-likelihood function as presented in (3.15).

The attributes of Snyman's technique can be summarized as follows:

(i) It uses only gradient information, i.e. $\nabla[l(x, \boldsymbol{\theta})]$.
(ii) No explicit line searches are performed.
(iii) It is extremely robust and handles steep valleys and discontinuities in the objective function or gradient with ease.
(iv) This algorithm seeks a low local minimum and it can be used as a basic component in a methodology for global optimization.
(v) The method is not as efficient on smooth and near quadratic functions as classical methods.

The basic dynamic model assumes a particle of unit mass in a m-dimensional conservative force field with potential energy at $\boldsymbol{\theta}$ given by $l(x, \boldsymbol{\theta})$, then the force experienced by the particle at $\boldsymbol{\theta}$ is given by $m\boldsymbol{a} = \ddot{\boldsymbol{\theta}} = -\nabla[l(x, \boldsymbol{\theta})]$. From this it follows that for the time interval $[0, x]$,

$$\frac{1}{2}\left\|\dot{\boldsymbol{\theta}}|_{x=x}\right\|^2 - \frac{1}{2}\left\|\dot{\boldsymbol{\theta}}|_{x=0}\right\|^2 = l(0, \boldsymbol{\theta}) - l(x, \boldsymbol{\theta}) \tag{C.1}$$

Equation (C.1) can be simplified by expressing it in terms of kinetic energy, T, as $T(x) - T(0) = l(0, \boldsymbol{\theta}) - l(x, \boldsymbol{\theta})$. It is clear that $l(x, \boldsymbol{\theta}) + T(x)$ is constant, which indicates conservation of energy in the conservative force field. It should also be noted that $\Delta l = -\Delta T$, therefore as long as T increases, l decreases, which is the basis of the dynamic algorithm.

Suppose $l(x, \boldsymbol{\theta})$ has to be minimized from a starting point $\boldsymbol{\theta}|_{x=0} = \boldsymbol{\theta}_0$, then the dynamic algorithm is as follows:

(i) Compute the dynamic trajectory by solving the initial value problem, $\ddot{\boldsymbol{\theta}}|_{x=x} = -\nabla[l(x, \boldsymbol{\theta})]$, $\dot{\boldsymbol{\theta}}|_{x=0} = 0$ and $\boldsymbol{\theta}|_{x=0} = \boldsymbol{\theta}_0$. In practice the numerical integration of the initial value problem is often done by the "leap-frog" method. Compute for $k = 1, 2, ..$ and time step Δx, the following: $\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k + \dot{\boldsymbol{\theta}}^k \Delta x$ and $\dot{\boldsymbol{\theta}}^{k+1} = \dot{\boldsymbol{\theta}}^k + \ddot{\boldsymbol{\theta}}^k \Delta x$, where $\ddot{\boldsymbol{\theta}}^k = -\nabla[l(x, \boldsymbol{\theta}^k)]$ and $\dot{\boldsymbol{\theta}}_0 = {}^1\!/_2 \ddot{\boldsymbol{\theta}}_0 \Delta x$.

(ii) Monitor $\dot{\boldsymbol{\theta}}|_{x=x}$, the velocity of the particle. As long as the kinetic energy $T = {}^1\!/_2 \left\|\dot{\boldsymbol{\theta}}|_{x=x}\right\|$ increases, the potential energy decreases, i.e. $l(x, \boldsymbol{\theta})$ decreases.

(iii) As soon as T decreases, the particle is moving uphill and the objective function is increasing, i.e. $||\dot{\theta}^{k+1}|| \leqslant ||\dot{\theta}^{k}||$. Some interfering strategy should be applied to extract energy from the particle to increase the likelihood of decent. A typical interfering strategy is to let $\dot{\boldsymbol{\theta}}^k = {}^1\!/_4(\dot{\boldsymbol{\theta}}^{k+1} + \dot{\boldsymbol{\theta}}^k)$ and $\boldsymbol{\theta}^{k+1} = {}^1\!/_2(\boldsymbol{\theta}^{k+1} + \boldsymbol{\theta}^k)$ after which a new $\boldsymbol{\theta}^{k+1}$ is calculated and the algorithm is continued.

(iv) To accelerate convergence of the method, the algorithm should allow for magnification and reduction of the step size, Δx , depending on the particle's position.

The method is extremely robust and particularly useful when variables in the objective function is totally unknown.

C.3 Modified Newton-Raphson Optimization Method

The objective of the numerical procedure is to find the value of $\boldsymbol{\theta}$ where all the partial derivatives of $l(x, \boldsymbol{\theta})$ are zero. Suppose $(F(x))$ and $(G(x))$ are matrices containing the first and second partial derivatives of $l(x, \boldsymbol{\theta})$, respectively. An approximation often used for $(F(x))$ is $(F(\boldsymbol{\theta})) \approx (F(\boldsymbol{\theta})) + (G(\boldsymbol{\theta}_0)) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ where $\boldsymbol{\theta}_0$ is an initial estimate. It is required to solve $(F(\boldsymbol{\theta}_0)) + (G(\boldsymbol{\theta}_0)) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0) = 0$ to determine the optimal value of $\boldsymbol{\theta}$.

The conventional Newton-Raphson procedure would solve for $\boldsymbol{\theta}$ as follows:

(i) Estimate a meaningful initial value for $\boldsymbol{\theta}_0$, i.e. $\boldsymbol{\theta}$.

(ii) Calculate $(F(x))$ and $(G(x))$.

(iii) Solve for Δ_0 in the system $(G(\boldsymbol{\theta}_0))\Delta_0 = -(F(\boldsymbol{\theta}_0))$.

(iv) Set $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_0 + \Delta_0$ and repeat the procedure until convergence.

Instead of the conventional Newton-Raphson method, a variable metric method (quasi-Newton method) can be used to overcome some numerical difficulties. In this modified Newton-Raphson method, $(G(x))$ is not calculated directly but an approximation of $(G(x))$ is used that is chosen to be always positive definite, thereby eliminating the possibility of singular matrices. The approximation of $(G(x))$ is explained in detail in Press, Teukolsky, Vetterling, and Flannery (1993). Press et al. also describe methods to vary step sizes in the procedure as well as stopping rule procedures. Vlok (1999) discusses methods to accelerate convergence and increase the accuracy of the procedure by transforming the data before iterations start.

Appendix D

SASOL Data

D.1 Inspection data for Bearing 3

The inspection data for Bearing 3 is presented in Table D.1 on the next page, where the columns have the following meanings:

Pump ID: Pump identification number.

Age: Global age of the pump measured in days.

Date: Actual date of inspection.

A: RF043H, i.e 0.4× rotational frequency amplitude, measured on horizontally on Bearing 3 in mm/s, indicative of a bearing defect.

B: RF13H, i.e. 1× rotational frequency amplitude, measured horizontally on Bearing 3 in mm/s, indicative of unbalance in the pump.

C: RF23H, i.e. 2× rotational frequency amplitude, measured horizontally on Bearing 3 in mm/s, indicative of misalignment in the pump.

D: RF53H, i.e. 5× rotational frequency amplitude, measured horizontally on Bearing 3 in mm/s, indicative of cavitation in the pump.

E: HFD3H, i.e. high frequency domain components between 1200-2400 Hz, measured on Bearing 3, indicative of a bearing defect. This is a subjective covariate where 1 indicates a presence and 0 an absence of the mentioned components.

F: LNF3H, i.e. lifted noise floor in 600-1200 Hz range, measured on Bearing 3, indicative of a lack of lubrication where 1 indicates a presence and 0 an absence of the mentioned components.

Table D.1: Inspection data for Bearing 3

Pump ID	Age (Days)	Date	A [mm/s]	B [mm/s]	C [mm/s]	D [mm/s]	E [0/1]	F [0/1]
PC1131	159	02/07/97	0	0.7	0.3	0.8	1	0
PC1131	295	06/23/97	0.15	0.3	0.25	0.55	0	1
PC1131	387	09/23/97	0.3	3	0.9	8	1	0
PC1131	394	09/30/97	0.8	2.4	1	12.3	1	0
PC1131	397	10/03/97	250	175	20	17	1	0
PC1131	530	02/13/98	0.1	11.5	3.2	11	0	0
PC1131	533	02/16/98	0.3	8.8	3.5	13	1	0
PC1131	554	03/09/98	0.5	7	3.8	16	0	0
PC1131	578	04/02/98	1	19.5	1.5	2	1	0
PC1131	597	04/21/98	0.3	27.5	1.5	1.6	1	0
PC1131	639	06/02/98	0.5	31	6	4	1	0
PC1131	689	07/22/98	0	9	2	0.8	0	0
PC1131	690	07/23/98	0	8.27	1.82	0.67	0	0
PC1131	703	08/05/98	0.05	1.2	0.95	0.2	1	0
PC1131	712	08/14/98	0.05	0.5	0.8	1.4	1	0
PC1131	765	10/06/98	0.05	0.4	0.7	2.7	1	0
PC1131	791	11/01/98	0.5	9	2	12	0	0
PC1132	239	04/28/97	0	0.9	0.3	1.5	0	0
PC1132	386	09/22/97	0.1	7	0.6	2.1	1	0
PC1132	394	09/30/97	0.2	8	0.5	11	1	0
PC1132	397	10/03/97	0.1	6.2	0.2	3	0	0
PC1132	491	01/05/98	0.1	5	0.5	1	0	0
PC1132	499	01/13/98	0.1	27.5	2	2.5	0	0
PC1132	533	02/16/98	0.1	35	2.5	12	0	0
PC1132	543	02/26/98	5	19	26	9	0	0
PC1132	544	02/27/98	5.61	16.94	28.93	8.56	0	0
PC1132	557	03/12/98	3	43	9	2	0	0
PC1132	558	03/13/98	1	41	14	3	0	0
PC1132	597	04/21/98	4	29	3.7	2.6	0	1
PC1132	689	07/22/98	0.1	5.6	1.7	0.3	0	1
PC1132	712	08/14/98	0.1	3.4	0.6	0.9	0	1
PC1132	751	09/22/98	0.99	3.01	0.3	2.99	0	1
PC1132	791	11/01/98	0.08	4.65	0.17	2.01	0	0
PC1231	239	04/28/97	0.3	5.5	1.9	1	0	0
PC1231	295	06/23/97	1.3	10.4	2.2	1	0	0
PC1231	390	09/26/97	1	56	12	3	0	0

PC1231	530	02/13/98	0.3	18.1	6.1	8.5	1	0
PC1231	563	03/18/98	0.09	12	1.18	10.24	1	0
PC1231	578	04/02/98	1	33	18	6	1	1
PC1231	653	06/16/98	0.22	3.57	0.98	0.57	0	0
PC1231	698	07/31/98	0.68	8.11	1.47	0.61	0	0
PC1231	791	11/01/98	0.73	38.64	7.68	1.86	0	0
PC1232	583	04/07/98	0.5	56	9	4	0	0
PC1232	592	04/16/98	0.4	54	4	6.5	0	0
PC1232	597	04/21/98	0.6	48	9	3.5	0	0
PC1232	599	04/23/98	0.05	7	2.1	0.6	1	1
PC1232	699	08/01/98	0.33	34.16	5.76	2.48	0	0
PC1232	791	11/01/98	0.24	32.4	2.44	4.09	0	0
PC2131	156	02/04/97	0	9	1.2	0.4	0	0
PC2131	159	02/07/97	0.1	5.8	2.2	0.6	0	1
PC2131	178	02/26/97	0.2	4	3.3	1.35	0	1
PC2131	179	02/27/97	0	8.3	2	0.9	0	0
PC2131	184	03/04/97	0	36.39	2	1	0	1
PC2131	239	04/28/97	0.09	3.65	1.6	1.55	1	0
PC2131	241	04/30/97	0.05	3.1	0.75	1.7	1	0
PC2131	295	06/23/97	0.1	2.55	2.2	1.4	1	0
PC2131	386	09/22/97	0.4	5.6	7.5	0.7	1	0
PC2131	470	12/15/97	1200	120	30	10	0	0
PC2131	535	02/18/98	0.2	20.9	1.6	4.8	0	0
PC2131	583	04/07/98	2	77	46	11	0	0
PC2131	597	04/21/98	2	66	43	6	0	0
PC2131	604	04/28/98	1	74	37.5	5	1	0
PC2131	611	05/05/98	0.01	20	4.1	11.6	1	0
PC2131	631	05/25/98	0.1	18	10	72.33	1	0
PC2131	640	06/03/98	0.6	10.5	2.8	5.9	1	0
PC2131	689	07/22/98	0.09	1.7	0.4	0.5	1	0
PC2131	768	10/09/98	0.1	1.92	0.55	0.66	1	0
PC2131	774	10/15/98	0.14	2.66	0.76	1.12	1	0
PC2131	791	11/01/98	0.16	13.37	1.08	3.69	0	0
PC3131	241	04/30/97	0.1	6.8	3.9	1.3	1	0
PC3131	295	06/23/97	0.8	29	17	14	1	0
PC3131	386	09/22/97	0.5	37	6.5	4	1	0
PC3131	450	11/25/97	0.2	20.52	6	3	1	0
PC3131	550	03/05/98	0.09	7.2	3.74	1.27	1	0
PC3131	651	06/14/98	0.96	33.06	17.34	16.8	1	0

PC3131	750	09/21/98	0.59	40.33	6.43	4.16	1	0
PC3131	791	11/01/98	0.2	19.48	5.82	3.39	1	0
PC3132	239	04/28/97	0.1	2.4	0.15	0.39	1	0
PC3132	295	06/23/97	0.2	9.6	1.8	1.6	1	1
PC3132	386	09/22/97	0.2	24	3	3.5	1	1
PC3132	450	11/25/97	0.5	32	21	13	0	0
PC3132	506	01/20/98	0.97	37.56	48.37	26.84	0	0
PC3132	566	03/21/98	0.12	2.44	0.16	0.45	1	1
PC3132	711	08/13/98	0.19	11.04	1.92	1.82	1	1
PC3132	791	11/01/98	0.2	27.6	3.27	3.39	1	1
PC3232	239	04/28/97	0.3	11.5	3.8	0.6	1	0
PC3232	295	06/23/97	1	43	8	6	1	0
PC3232	386	09/22/97	2	39	6	6	1	0
PC3232	535	02/18/98	0	66	44	7	0	0
PC3232	563	03/18/98	0	75.72	56.86	7.33	1	0
PC3232	591	04/15/98	0	235	22	10	0	0
PC3232	604	04/28/98	2	175	18	7	0	0
PC3232	639	06/02/98	3	74	9	3	0	0
PC3232	722	08/24/98	0	20.5	14.8	1.9	1	1
PC3232	723	08/25/98	0	21.45	15.1	1.96	1	1
PC3232	748	09/19/98	0.18	7.59	2.96	0.39	1	0
PC3232	783	10/24/98	0.62	26.66	5.44	4.5	1	0
PC3232	791	11/01/98	1.28	28.08	3.72	4.08	1	0

In Chapter 5 it is shown that RF53H is a good predictor of failure and plays an significant role in the maximum likelihood. For the sake of completeness, the data for this covariate is also displayed graphically in Figures D.1 to D.8 for each lifetime.

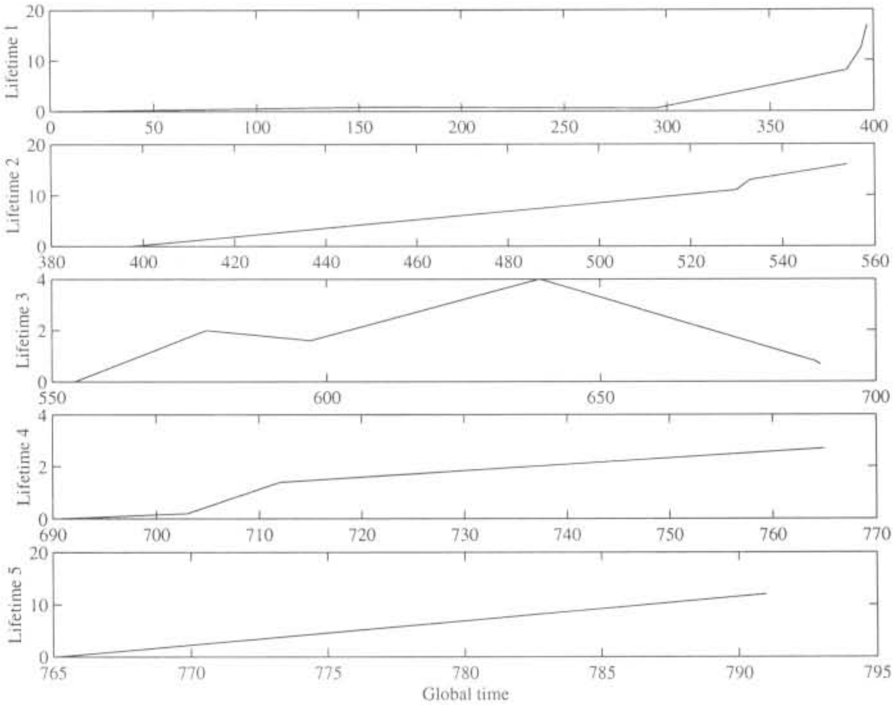

Figure D.1: Observed values of RF53H for PC1131

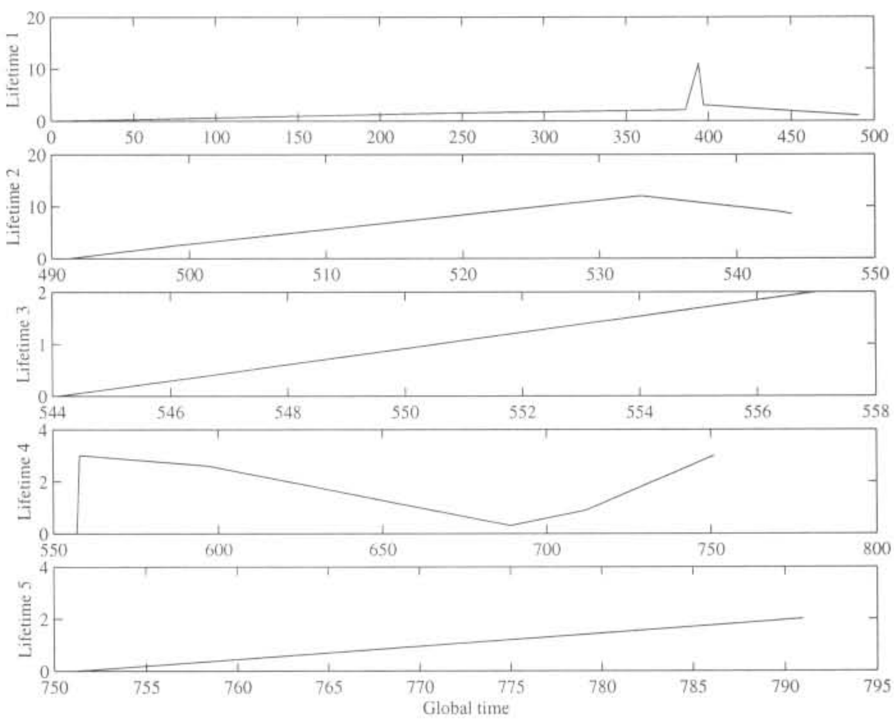

Figure D.2: Observed values of RF53H for PC1132

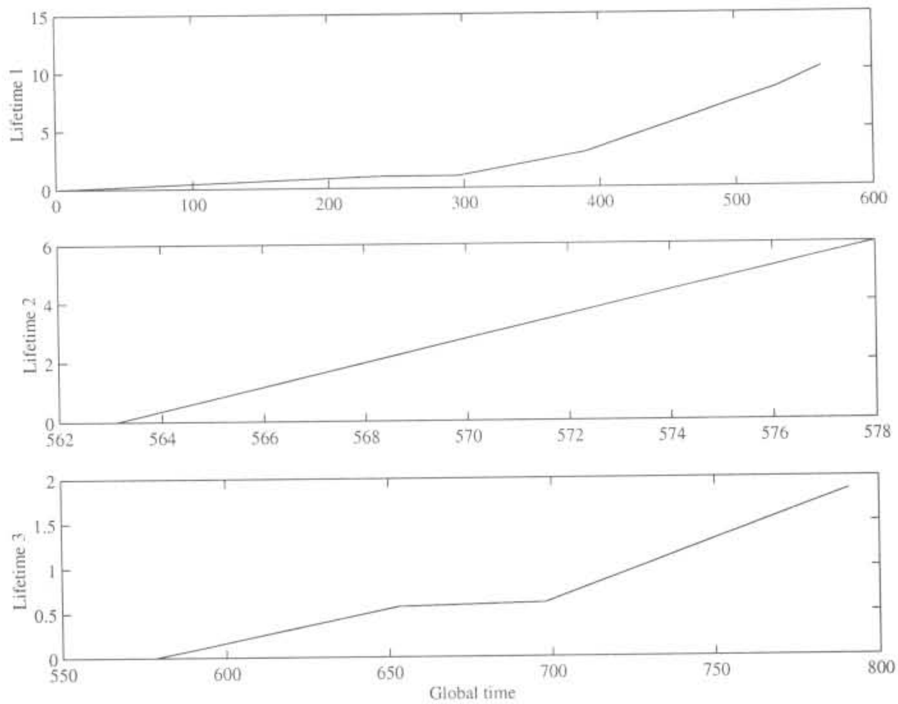

Figure D.3: Observed values of RF53H for PC1231

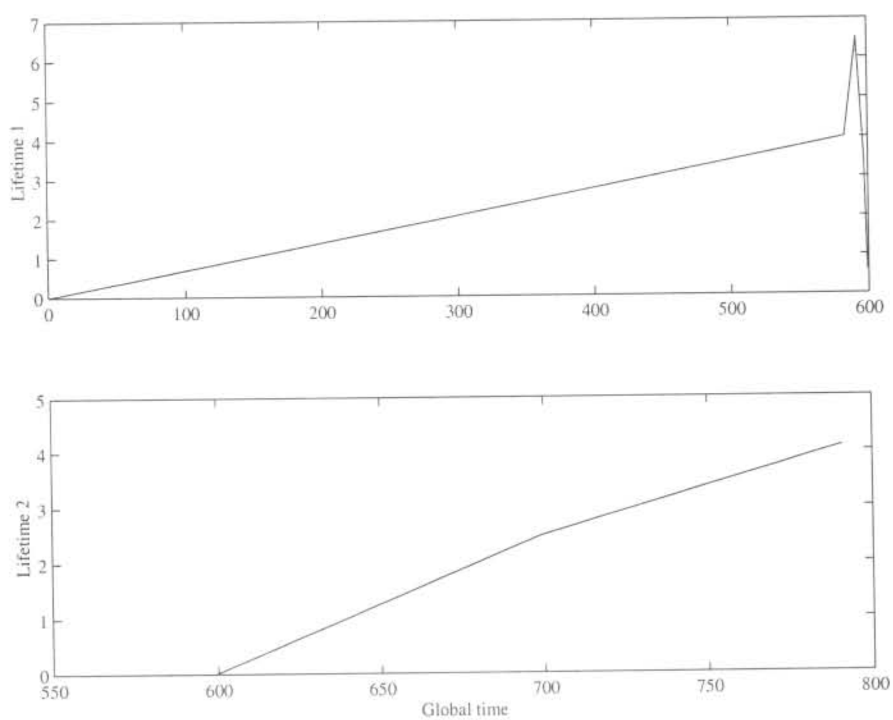

Figure D.4: Observed values of RF53H for PC1232

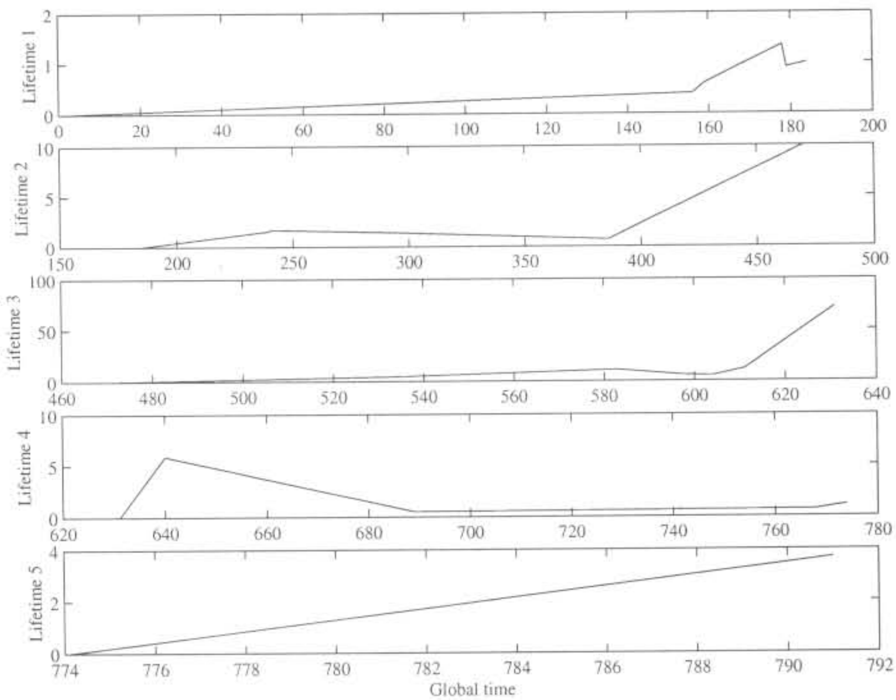

Figure D.5: Observed values of RF53H for PC2131

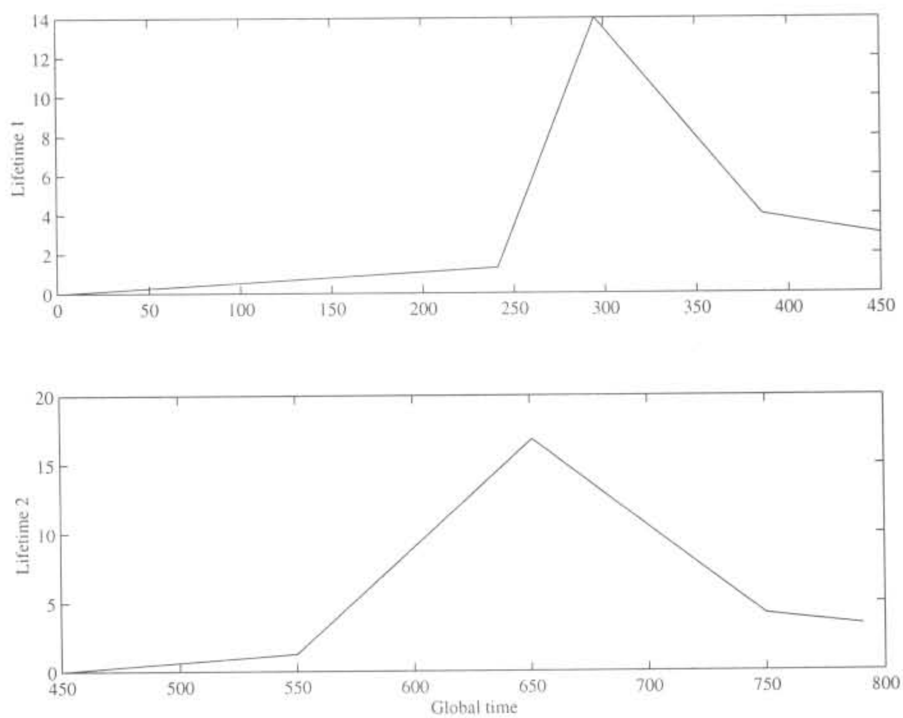

Figure D.6: Observed values of RF53H for PC3131

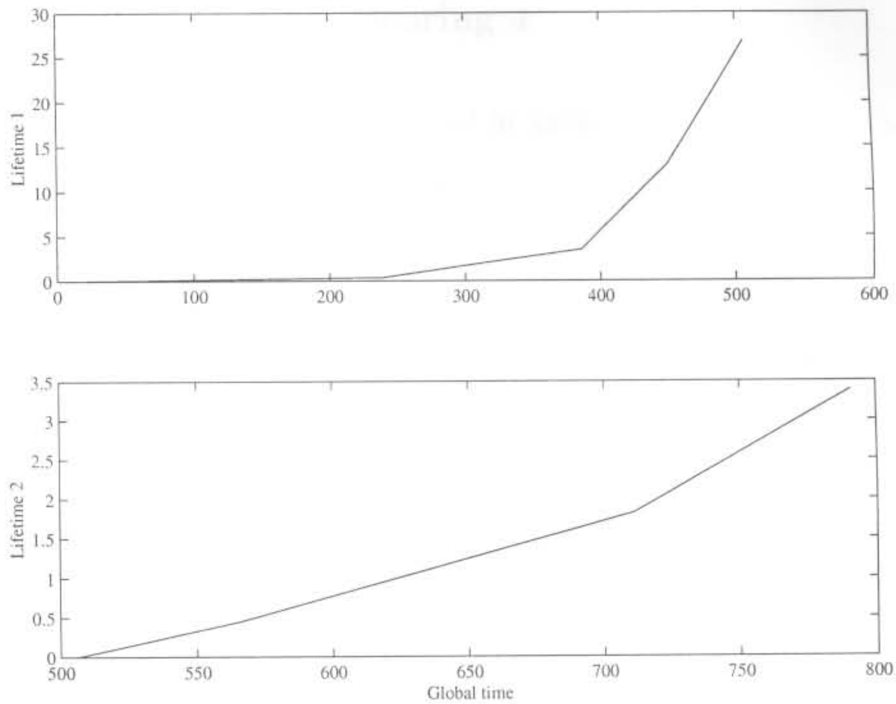

Figure D.7: Observed values of RF53H for PC3132

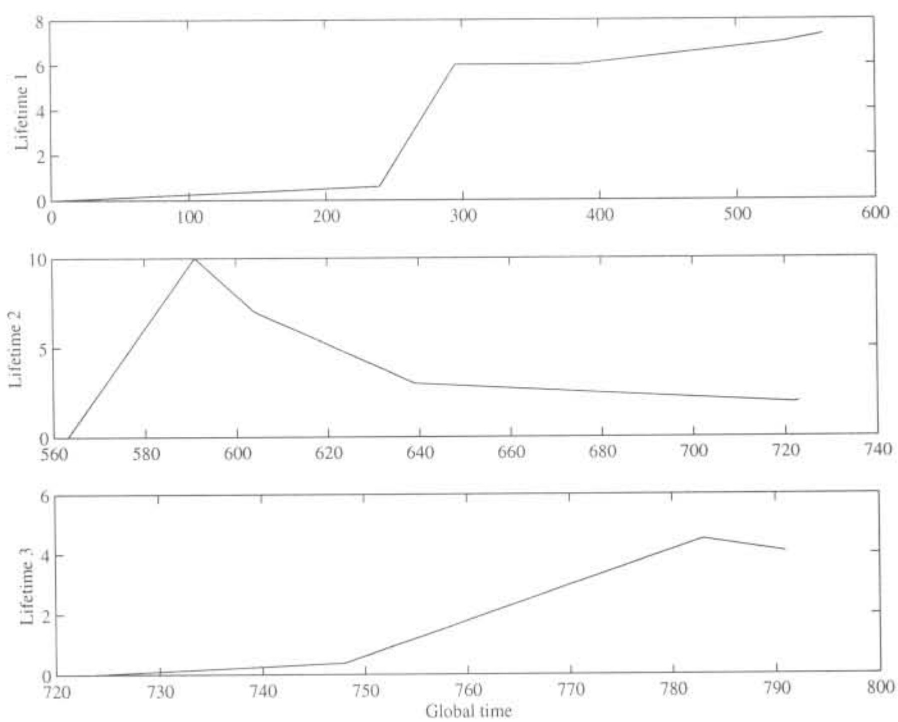

Figure D.8: Observed values of RF53H for PC3232

D.2 Inspection data for Bearing 4

The inspection data for Bearing 4 is presented in Table D.2 on the next page, where the columns have the following meanings:

Pump ID: Pump identification number.
Age: Global age of the pump measured in days.
Date: Actual date of inspection.
A: RF044H, i.e 0.4× rotational frequency amplitude, measured on horizontally on Bearing 4 in mm/s, indicative of a bearing defect.

B: RF14H, i.e. 1× rotational frequency amplitude, measured horizontally on Bearing 4 in mm/s, indicative of unbalance in the pump.
C: RF24H, i.e. 2× rotational frequency amplitude, measured horizontally on Bearing 4 in mm/s, indicative of misalignment in the pump.
D: RF54H, i.e. 5× rotational frequency amplitude, measured horizontally on Bearing 4 in mm/s, indicative of cavitation in the pump.
E: HFD4H, i.e. high frequency domain components between 1200-2400 Hz, measured on Bearing 4, indicative of a bearing defect. This is a subjective covariate where 1 indicates a presence and 0 an absence of the mentioned components.
F: LNF4H, i.e. lifted noise floor in 600-1200 Hz range, measured on Bearing 4, indicative of a lack of lubrication where 1 indicates a presence and 0 an absence of the mentioned components.

Table D.2: Inspection data for Bearing 4

Pump ID	Age (Days)	Date	A [mm/s]	B [mm/s]	C [mm/s]	D [mm/s]	E [0/1]	F [0/1]
PC1131	159	02/07/97	0.05	0.85	0.3	0.1	1	0
PC1131	295	06/23/97	0.2	0.45	0.25	0.12	0	1
PC1131	387	09/23/97	0.1	4	1.7	6.2	1	0
PC1131	394	09/30/97	2.3	4	2.1	5	0	0
PC1131	397	10/03/97	4	4.6	2.8	6	1	0
PC1131	530	02/13/98	0.1	13.2	3.5	5.5	0	0
PC1131	533	02/16/98	0.2	10	3.8	7	1	0
PC1131	554	03/09/98	0.3	5	4.2	10	0	0
PC1131	578	04/02/98	0.7	42	3	3	1	0
PC1131	597	04/21/98	0.5	52	2	5	1	0
PC1131	639	06/02/98	0.5	47	8	5	1	0
PC1131	689	07/22/98	0	14	2	1.2	0	0
PC1131	690	07/23/98	0	13.04	1.73	1.08	0	0

PC1131	703	08/05/98	0.2	2.25	0.9	0.4	1	0
PC1131	712	08/14/98	0.05	0.58	1.3	0.41	1	1
PC1131	765	10/06/98	0.05	0.4	2.1	0.6	1	1
PC1131	791	11/01/98	0.2	12	2	7	0	0
PC1132	239	04/28/97	0	1.65	0.3	0.72	0	1
PC1132	386	09/22/97	0.1	12.2	0.7	7.8	1	0
PC1132	394	09/30/97	0.1	14	0.9	8.2	1	0
PC1132	397	10/03/97	0.2	12	0.9	12	1	0
PC1132	491	01/05/98	1	10	0.8	30	1	0
PC1132	499	01/13/98	0.1	66	4	12	0	0
PC1132	533	02/16/98	0	65	3	10	0	0
PC1132	543	02/26/98	1	120	38	7	0	0
PC1132	544	02/27/98	1.13	126.88	42.38	6.64	0	0
PC1132	557	03/12/98	1	34	5	2.5	1	0
PC1132	558	03/13/98	2	27.5	6.5	1	0	0
PC1132	597	04/21/98	1	24	4.2	5.4	0	1
PC1132	689	07/22/98	0.1	4.8	0.7	0.4	0	0
PC1132	712	08/14/98	0.05	2.7	0.3	0.4	0	0
PC1132	751	09/22/98	0.13	1.61	0.06	1.54	0	1
PC1132	791	11/01/98	0.15	7.8	0.56	7.68	1	0
PC1231	239	04/28/97	0	9	0.6	0.4	0	0
PC1231	295	06/23/97	0.3	16.5	2.3	0.3	0	0
PC1231	390	09/26/97	0	67	6	4	0	0
PC1231	530	02/13/98	0	21	6	6	1	1
PC1231	563	03/18/98	0.08	10	5.05	5.87	1	1
PC1231	578	04/02/98	2	51	16	9	1	1
PC1231	653	06/16/98	0	6.75	0.41	0.27	0	0
PC1231	698	07/31/98	0.22	10.72	1.35	0.15	0	0
PC1231	791	11/01/98	0	46.9	4.14	2.64	0	0
PC1232	583	04/07/98	0	71	8	3	0	0
PC1232	592	04/16/98	0.05	53	3	2	0	0
PC1232	597	04/21/98	1	57	6	3	0	0
PC1232	599	04/23/98	0.15	7.9	3.5	0.9	0	1
PC1232	699	08/01/98	0	49.7	5.28	1.92	0	0
PC1232	791	11/01/98	0.03	36.57	2.04	1.24	0	0
PC2131	156	02/04/97	0	15.5	2.1	0.5	0	1
PC2131	159	02/07/97	0	7	1.8	0.4	0	1
PC2131	178	02/26/97	0.05	6.7	2.3	0.4	0	0
PC2131	179	02/27/97	0	12.2	2.2	0.4	0	0

PC2131	184	03/04/97	0	47.97	1.51	0.4	0	1
PC2131	239	04/28/97	0.05	9.6	1.1	0.7	0	0
PC2131	241	04/30/97	0.1	8.1	1	0.7	1	0
PC2131	295	06/23/97	0.2	6.1	1.5	0.4	1	0
PC2131	386	09/22/97	1.7	21	1.4	3.7	1	0
PC2131	470	12/15/97	78	48	12	9	0	0
PC2131	535	02/18/98	0.5	27	7.4	7	0	0
PC2131	583	04/07/98	2	62	39	6	0	0
PC2131	597	04/21/98	2	64	38	4	0	0
PC2131	604	04/28/98	2	61	37	5	1	0
PC2131	611	05/05/98	0.01	24	6	1.4	1	0
PC2131	631	05/25/98	0.01	10	10	1	1	0
PC2131	640	06/03/98	0.2	26	1	4	1	0
PC2131	689	07/22/98	0.05	4.6	0.25	0.33	1	0
PC2131	768	10/09/98	0.05	4.2	0.3	0.2	1	0
PC2131	774	10/15/98	0.06	5.89	0.37	0.48	1	0
PC2131	791	11/01/98	0.34	17.55	4.66	5.6	0	0
PC3131	241	04/30/97	0.1	8	1.7	1	1	0
PC3131	295	06/23/97	0.7	35	10	7	1	0
PC3131	386	09/22/97	2	33	5	7	1	0
PC3131	450	11/25/97	3.13	20	4	2	1	0
PC3131	550	03/05/98	0.1	8.08	1.81	1.2	1	0
PC3131	651	06/14/98	0.71	39.2	9.8	7.7	1	0
PC3131	750	09/21/98	2.4	36.3	4.9	6.58	1	0
PC3131	791	11/01/98	3.47	21.4	4.08	1.8	1	0
PC3132	239	04/28/97	0.2	3.6	0.25	0.55	1	0
PC3132	295	06/23/97	0.3	12.2	0.9	2.2	1	1
PC3132	386	09/22/97	0.05	35	2.5	2.4	1	1
PC3132	450	11/25/97	0	81	8	6.5	0	0
PC3132	506	01/20/98	0.04	141.55	15.78	12.77	0	0
PC3132	566	03/21/98	0.23	4.32	0.25	0.59	1	0
PC3132	711	08/13/98	0.37	15.61	1.06	2.35	1	1
PC3132	791	11/01/98	0.06	39.9	3.25	2.61	1	1
PC3232	239	04/28/97	0.01	16	2.3	0.3	1	0
PC3232	295	06/23/97	1	48	9	4	1	0
PC3232	386	09/22/97	1	52	4	3	1	0
PC3232	535	02/18/98	0	91	26	8	0	0
PC3232	563	03/18/98	0	102.83	34.32	9.86	0	0
PC3232	591	04/15/98	0	280	10	15	0	0

PC3232	604	04/28/98	0	150	9	8	0	0
PC3232	639	06/02/98	5	73	6	6	0	0
PC3232	722	08/24/98	0	27	10	0.8	0	0
PC3232	723	08/25/98	0	27.62	10.14	0.73	0	0
PC3232	748	09/19/98	0	12	1.84	0.23	1	0
PC3232	783	10/24/98	0.73	30.72	5.85	3.2	1	0
PC3232	791	11/01/98	0.72	31.2	2.96	1.95	1	0

In Chapter 5 it is shown that RF54H is a good predictor of failure and plays an significant role in the maximum likelihood. For the sake of completeness, the data for this covariate is also displayed graphically in Figures D.9 to D.16 for each lifetime.

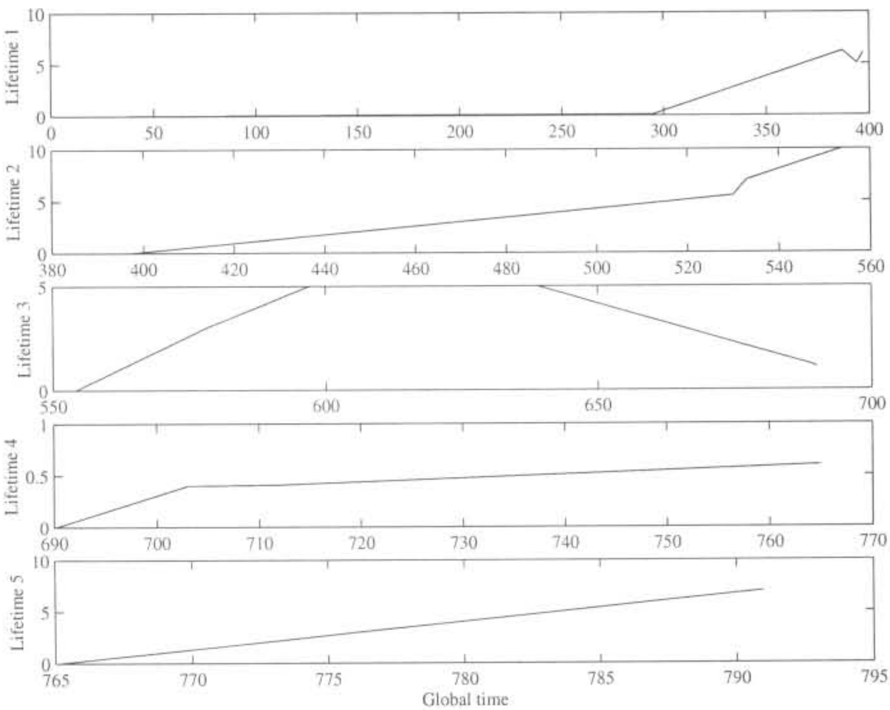

Figure D.9: Observed values of RF54H for PC1131

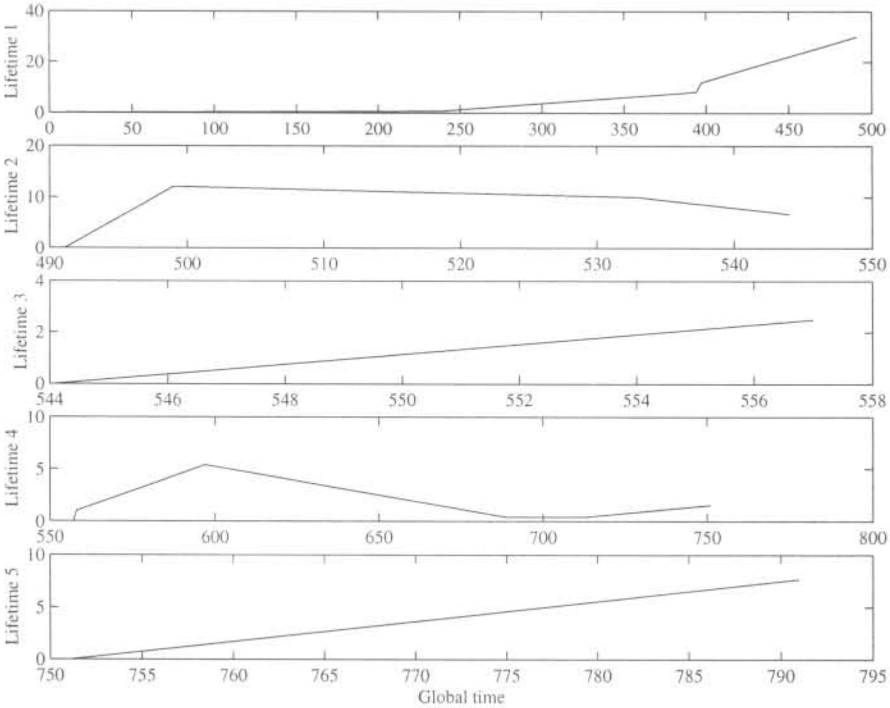

Figure D.10: Observed values of RF54H for PC1132

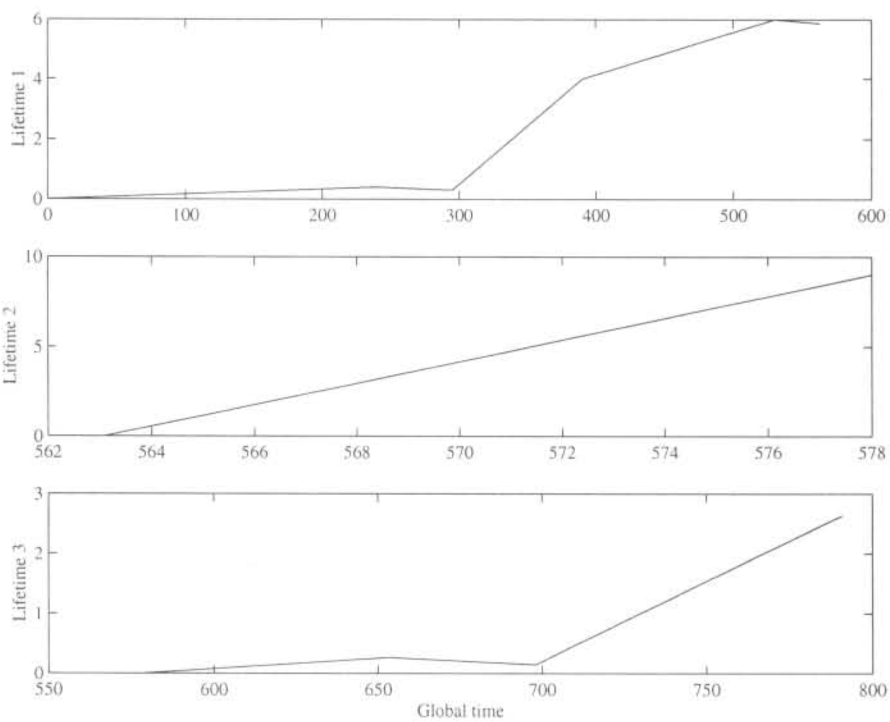

Figure D.11: Observed values of RF53H for PC1231

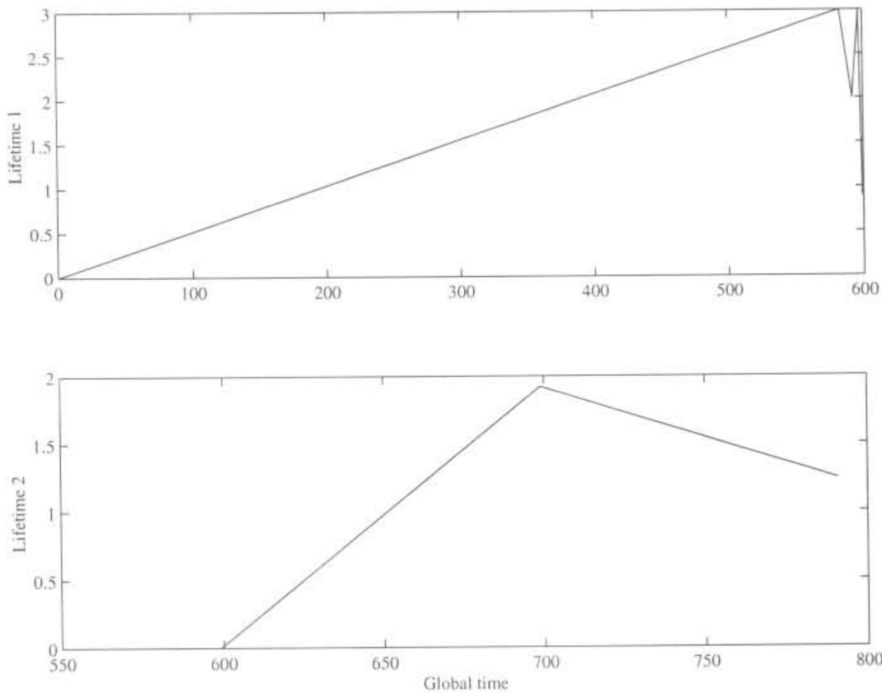

Figure D.12: Observed values of RF54H for PC1232

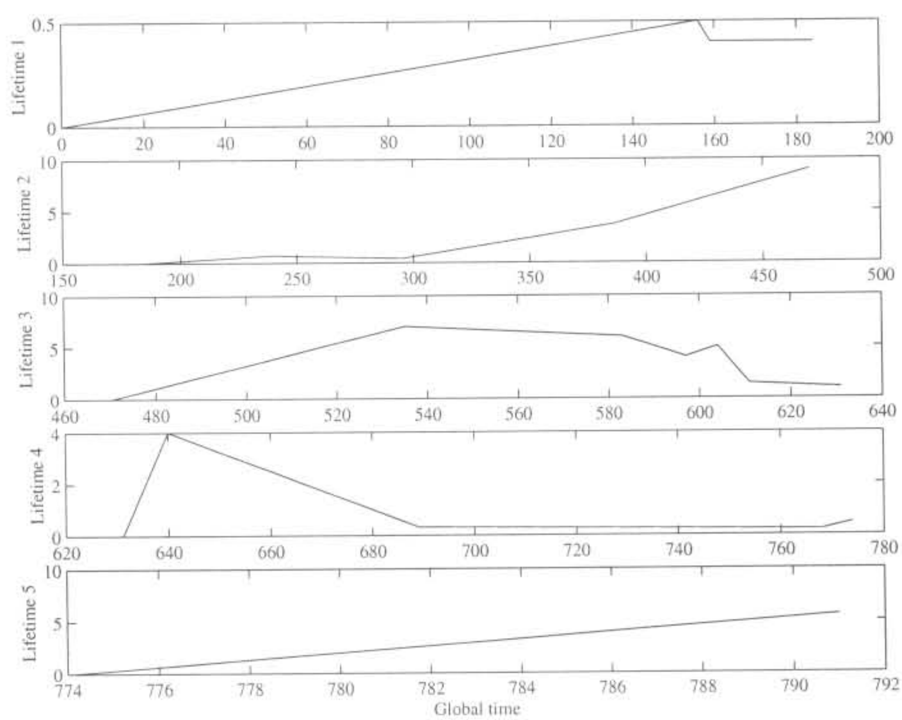

Figure D.13: Observed values of RF54H for PC2131

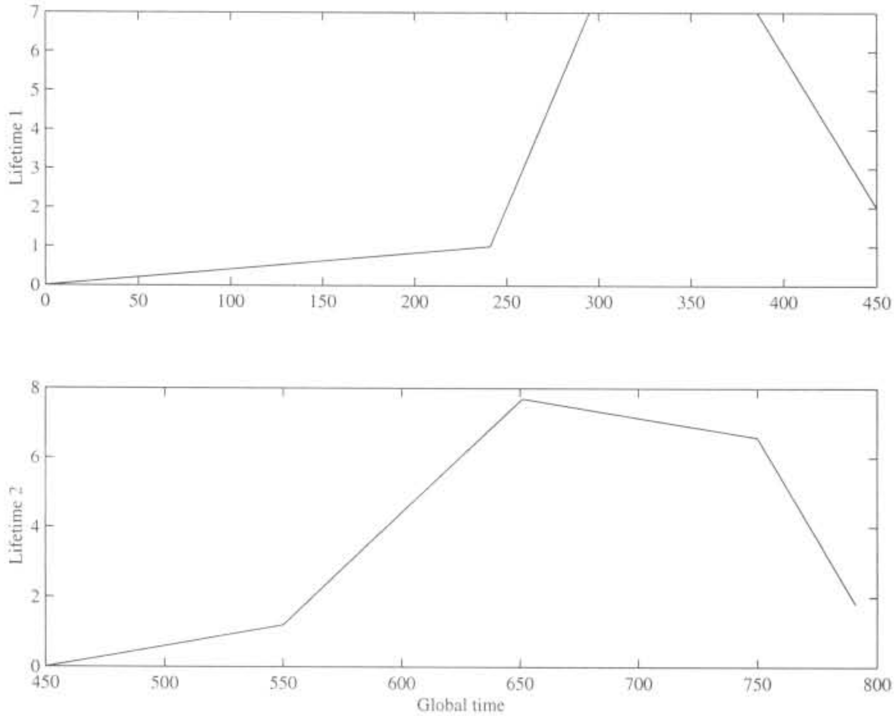

Figure D.14: Observed values of RF54H for PC3131

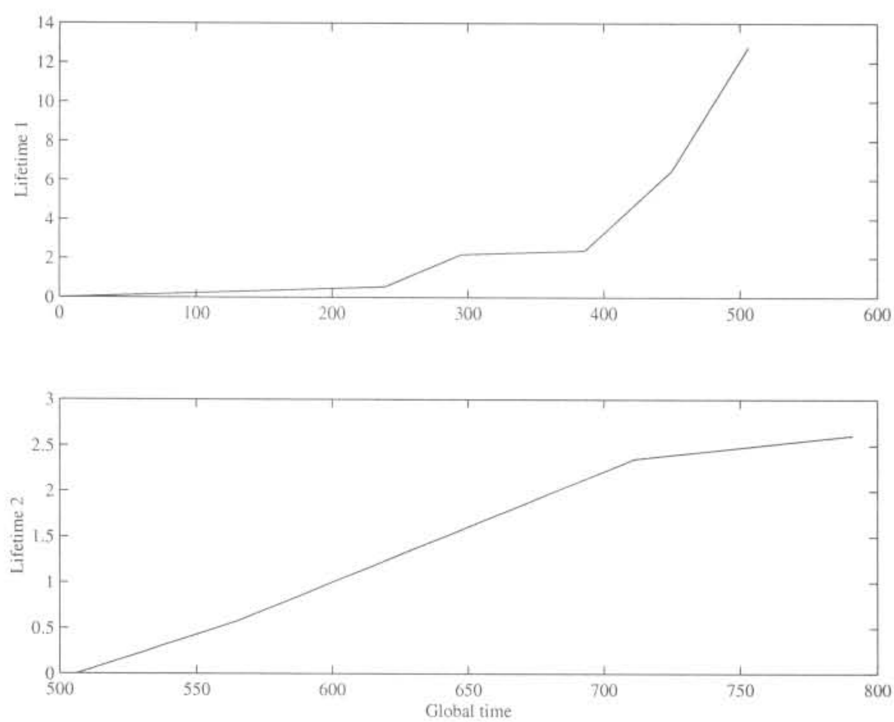

Figure D.15: Observed values of RF54H for PC3132

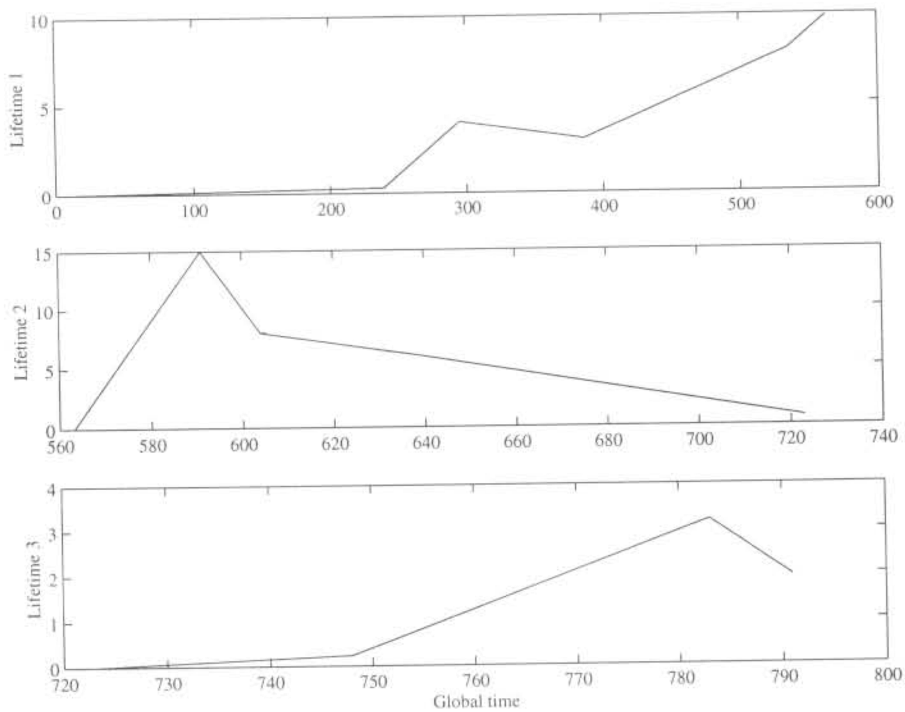

Figure D.16: Observed values of RF54H for PC3232

Appendix E

Approximations for Covariates RF53H and RF54H

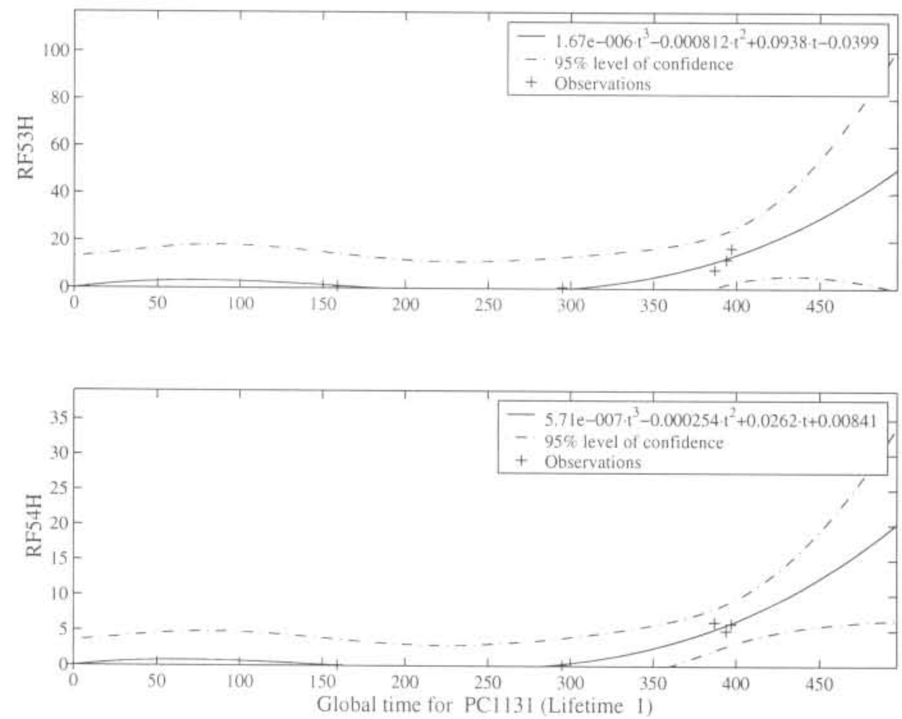

Figure E.1: Approximation of RF53H and RF54H measured on PC1131 during Lifetime 1

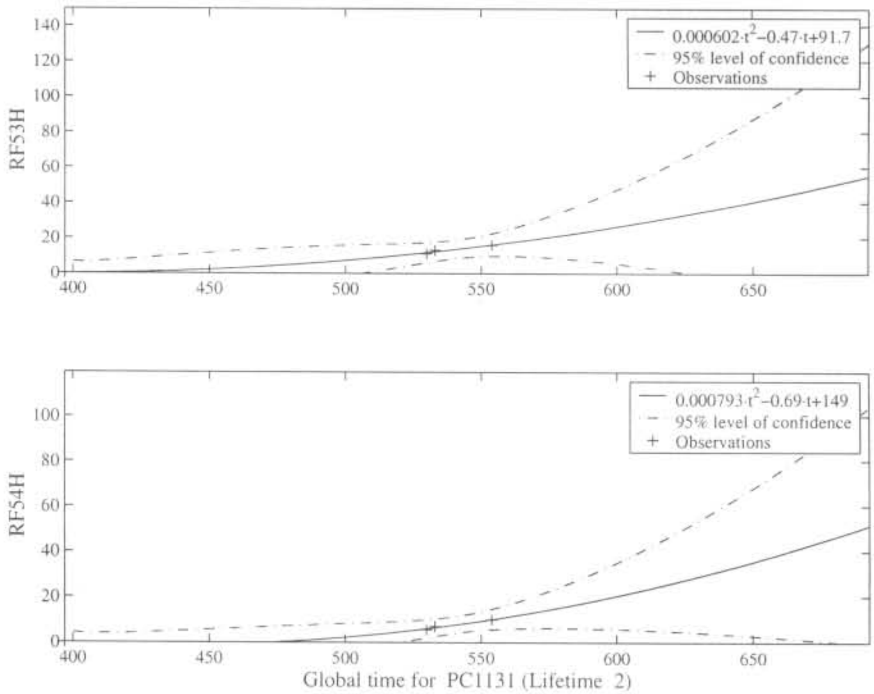

Figure E.2: Approximation of RF53H and RF54H measured on PC1131 during Lifetime 2

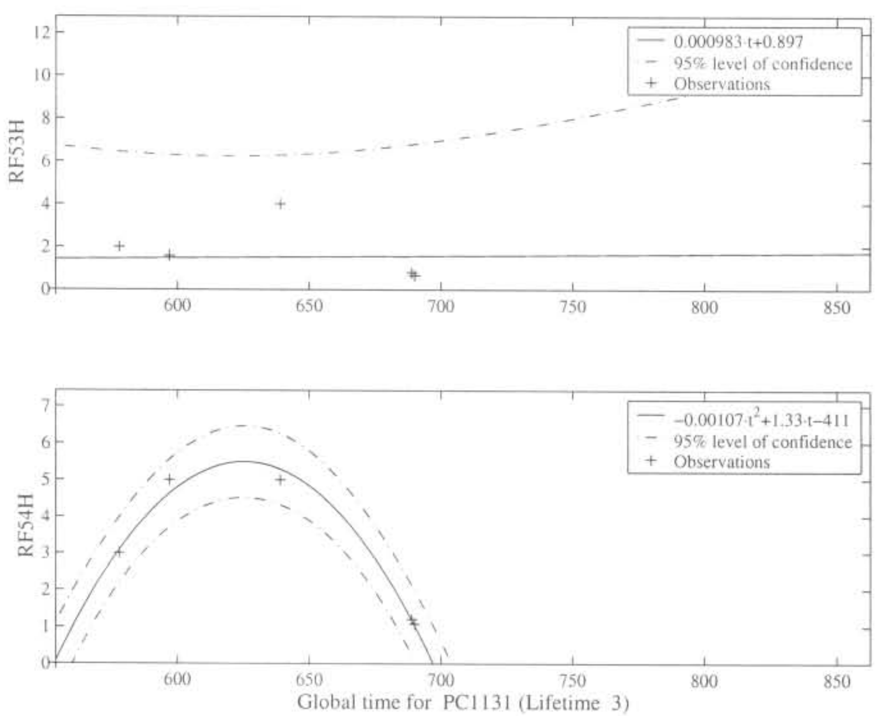

Figure E.3: Approximation of RF53H and RF54H measured on PC1131 during Lifetime 3

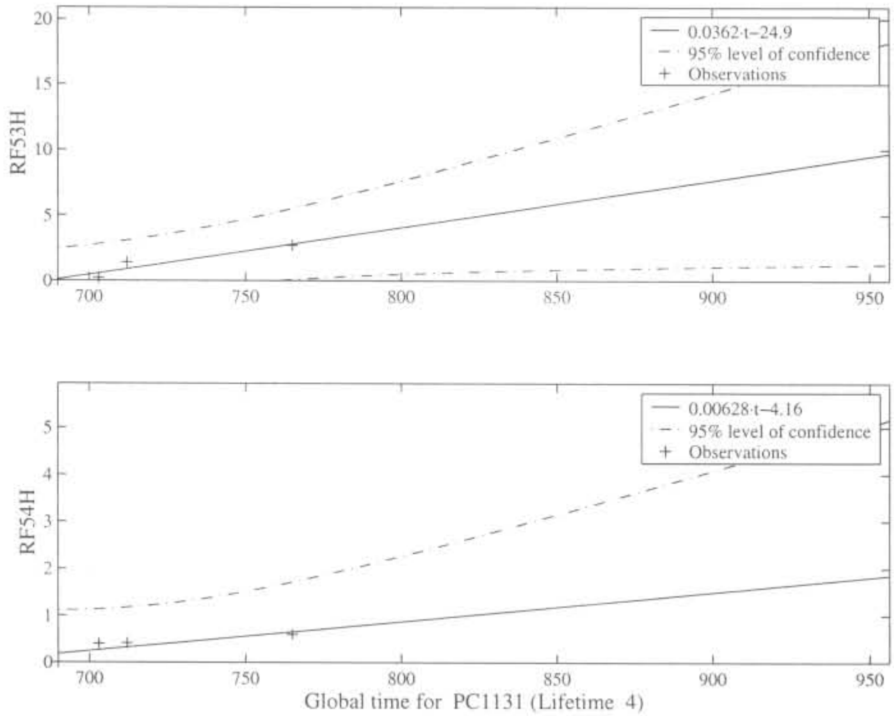

Figure E.4: Approximation of RF53H and RF54H measured on PC1131 during Lifetime 4

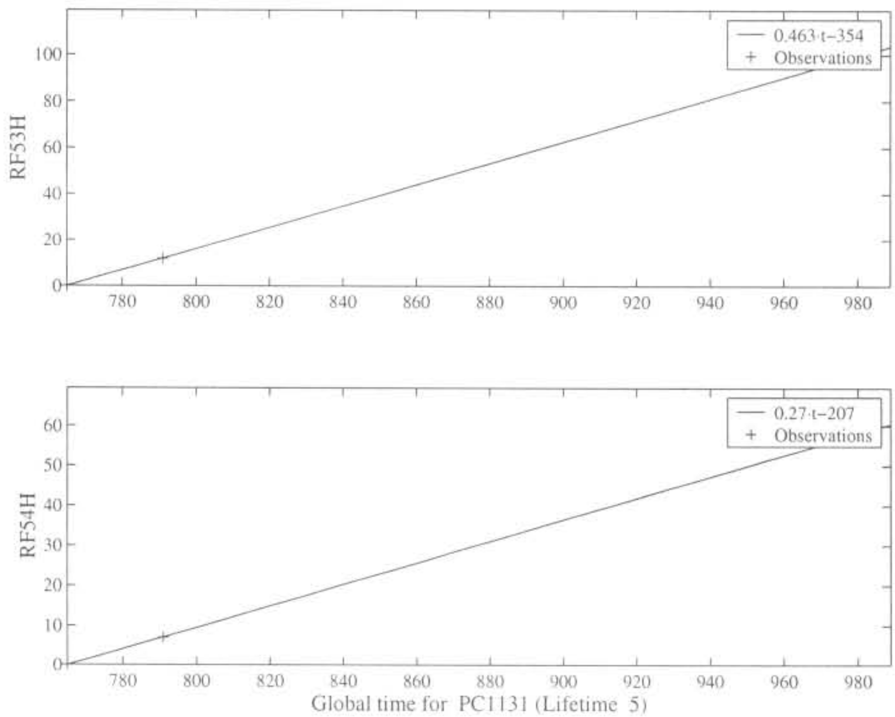

Figure E.5: Approximation of RF53H and RF54H measured on PC1131 during Lifetime 5

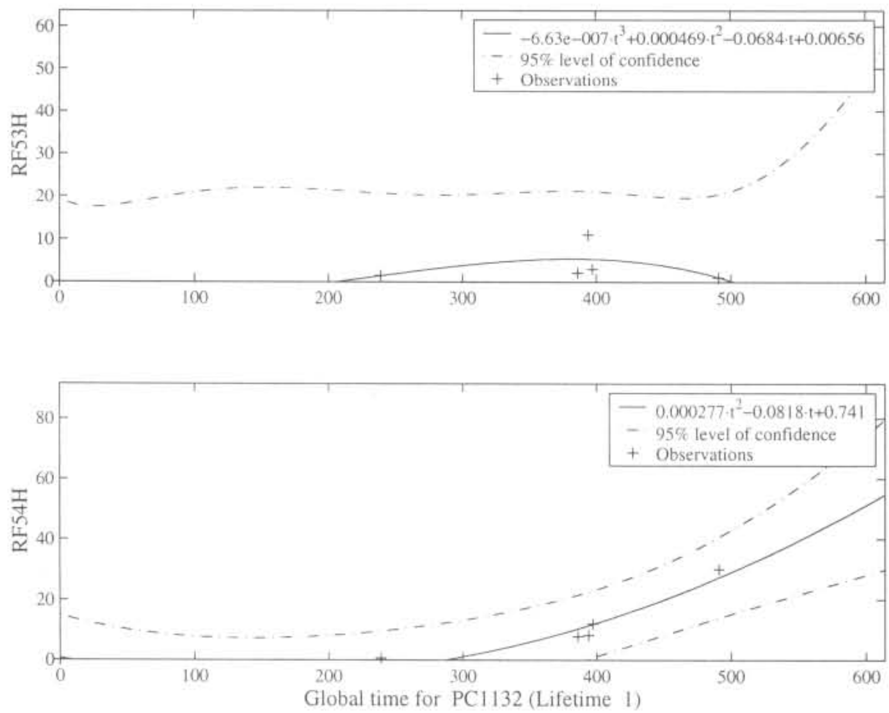

Figure E.6: Approximation of RF53H and RF54H measured on PC1132 during Lifetime 1

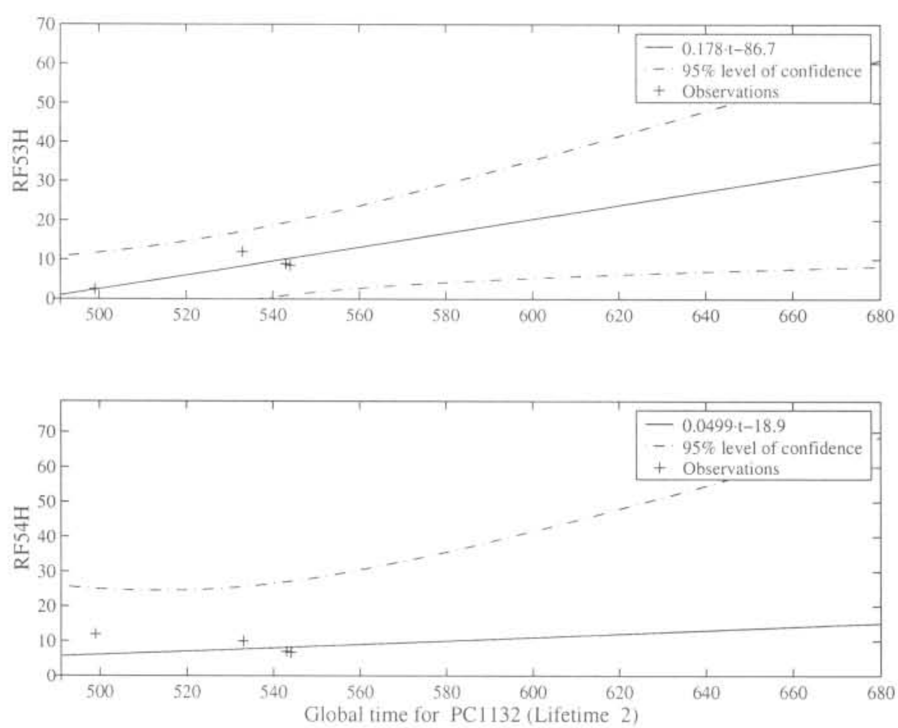

Figure E.7: Approximation of RF53H and RF54H measured on PC1132 during Lifetime 2

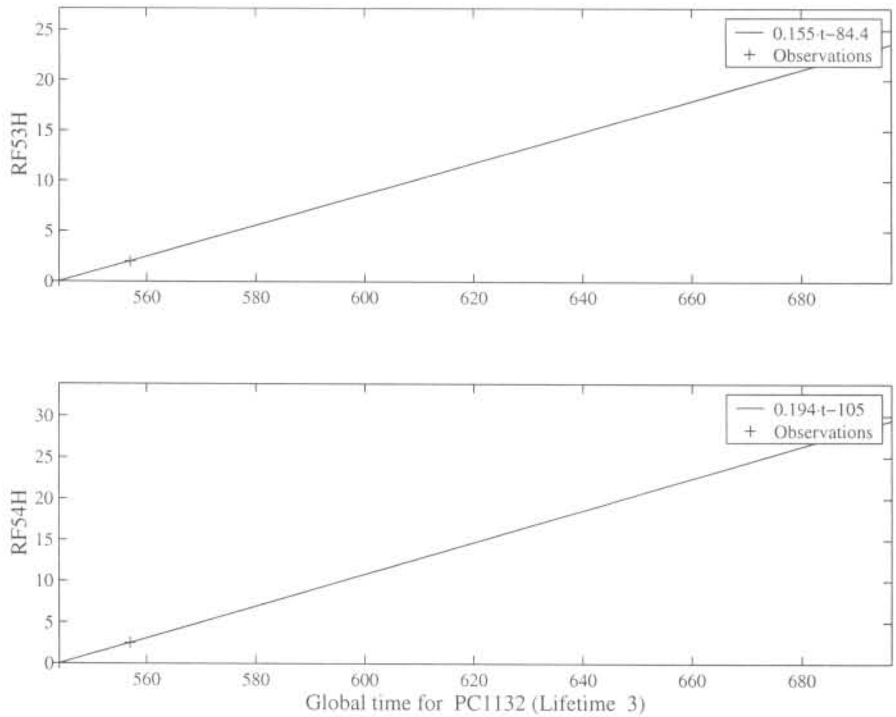

Figure E.8: Approximation of RF53H and RF54H measured on PC1132 during Lifetime 3

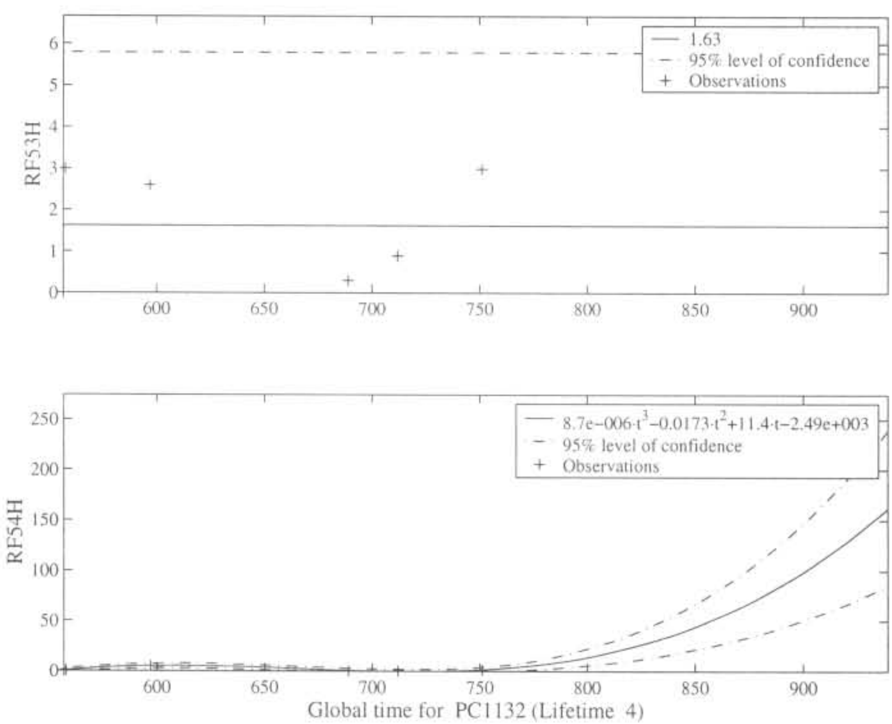

Figure E.9: Approximation of RF53H and RF54H measured on PC1132 during Lifetime 4

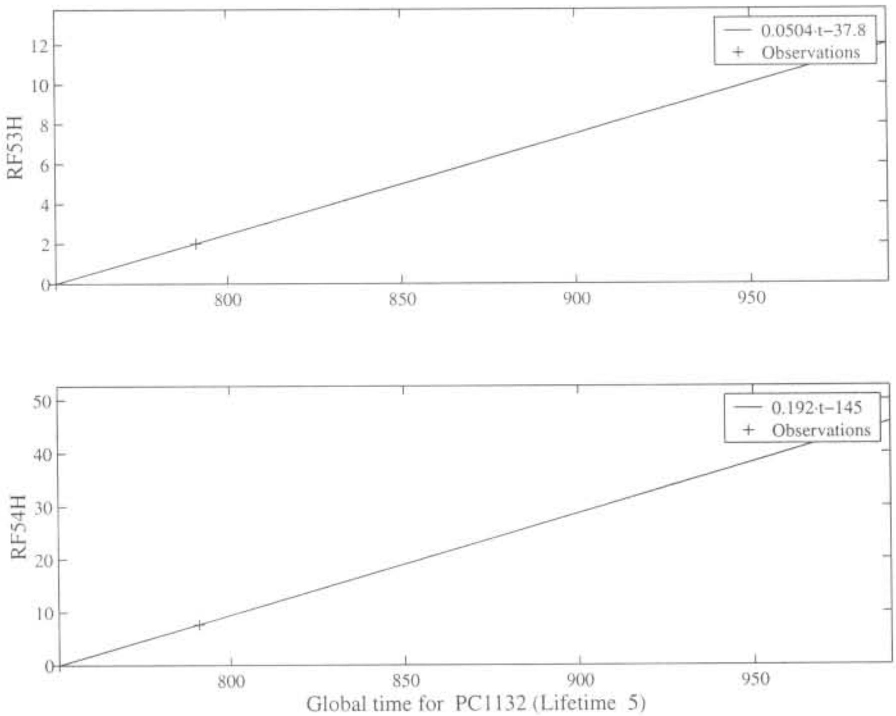

Figure E.10: Approximation of RF53H and RF54H measured on PC1132 during Lifetime 5

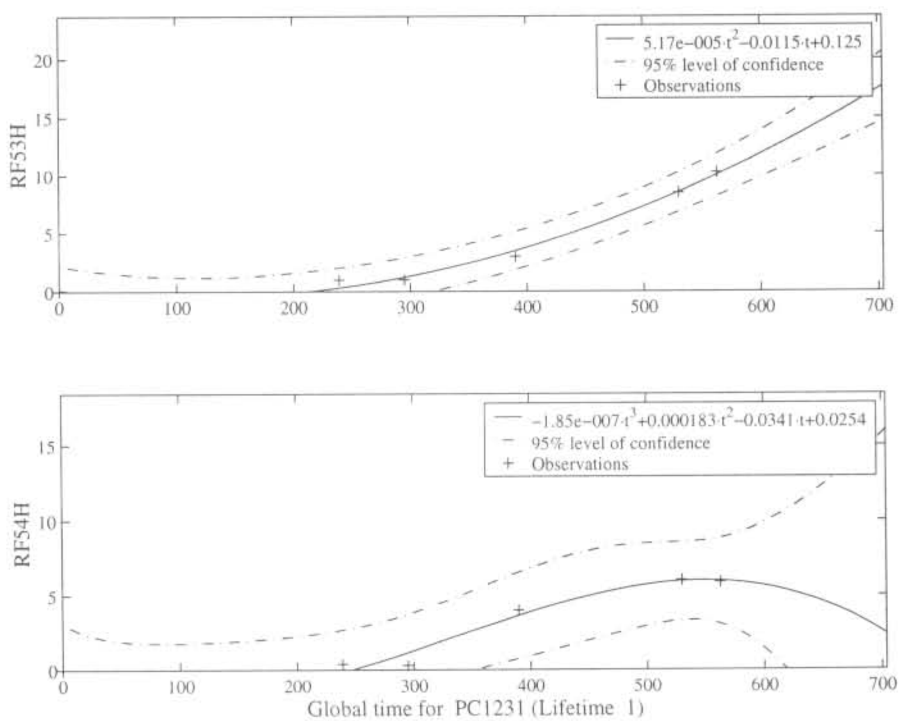

Figure E.11: Approximation of RF53H and RF54H measured on PC1231 during Lifetime 1

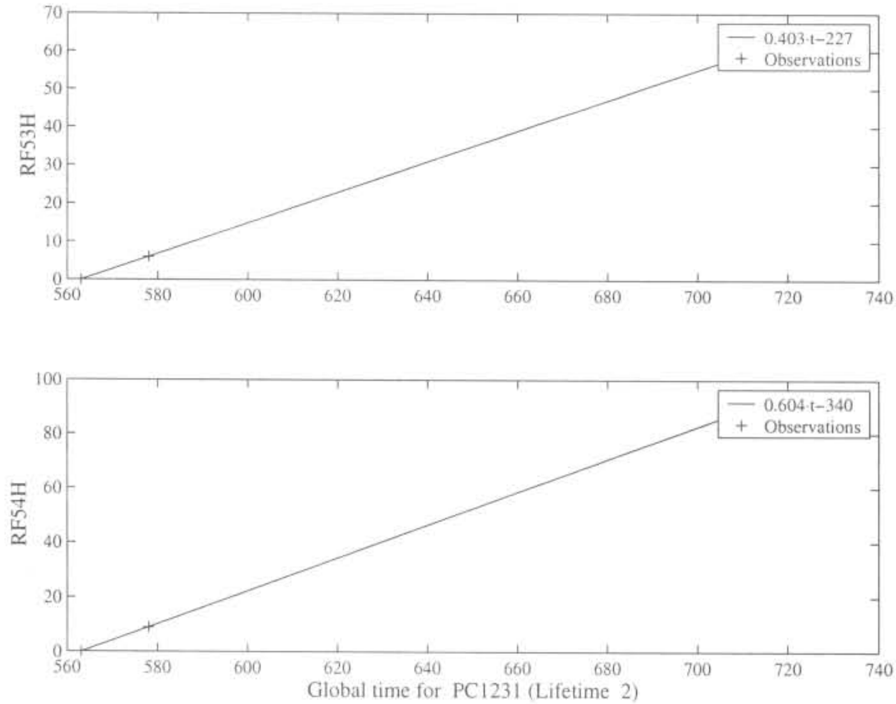

Figure E.12: Approximation of RF53H and RF54H measured on PC1231 during Lifetime 2

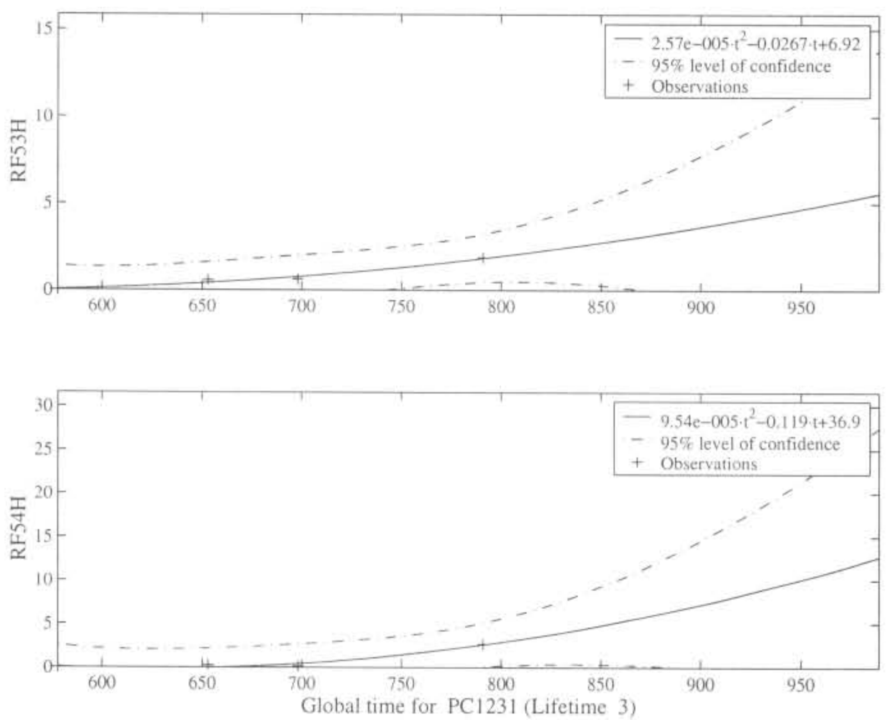

Figure E.13: Approximation of RF53H and RF54H measured on PC1231 during Lifetime 3

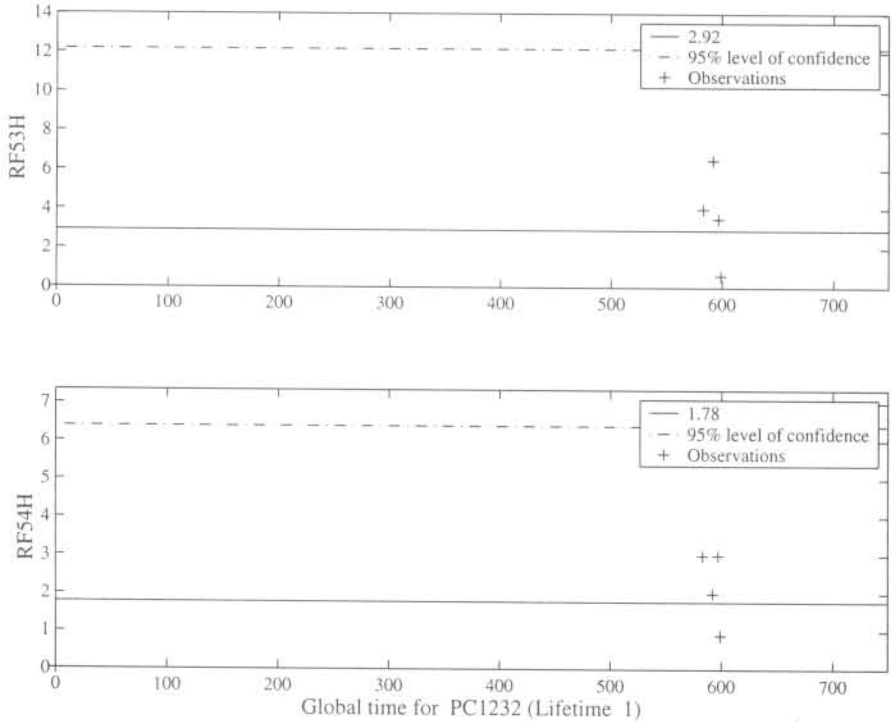

Figure E.14: Approximation of RF53H and RF54H measured on PC1232 during Lifetime 1

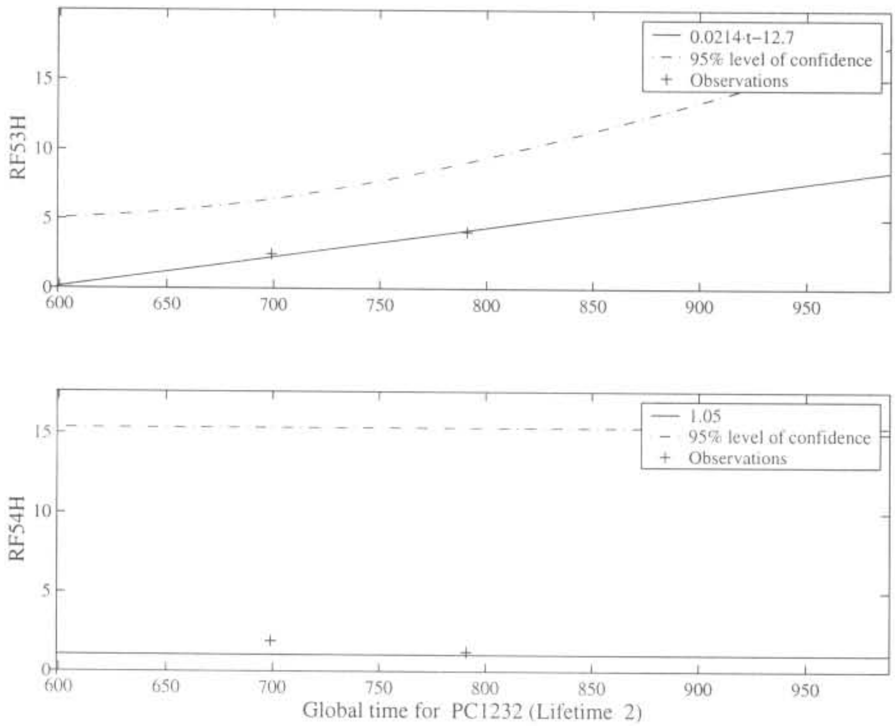

Figure E.15: Approximation of RF53H and RF54H measured on PC1232 during Lifetime 2

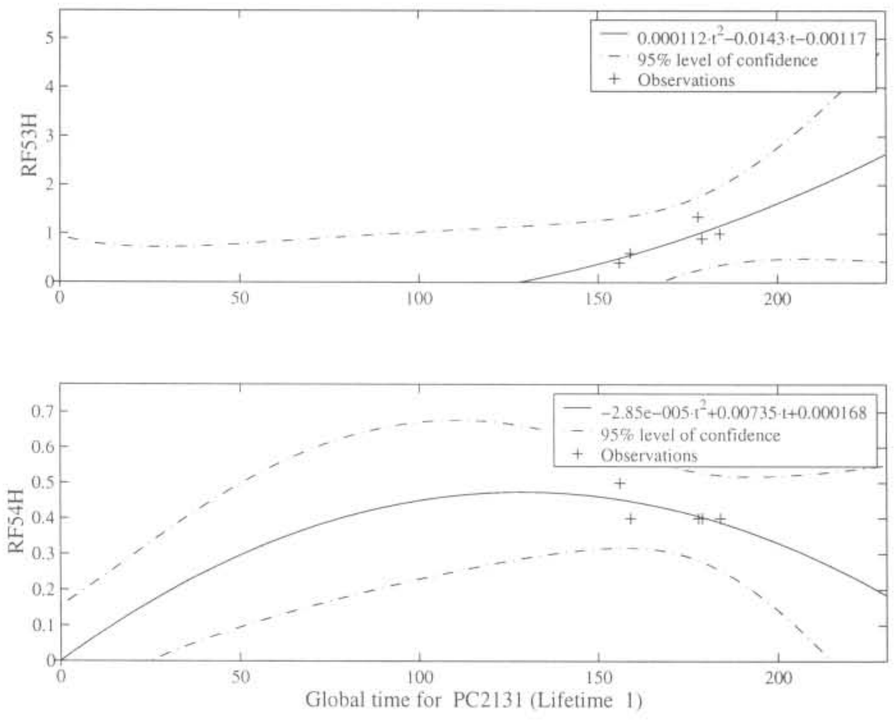

Figure E.16: Approximation of RF53H and RF54H measured on PC2131 during Lifetime 1

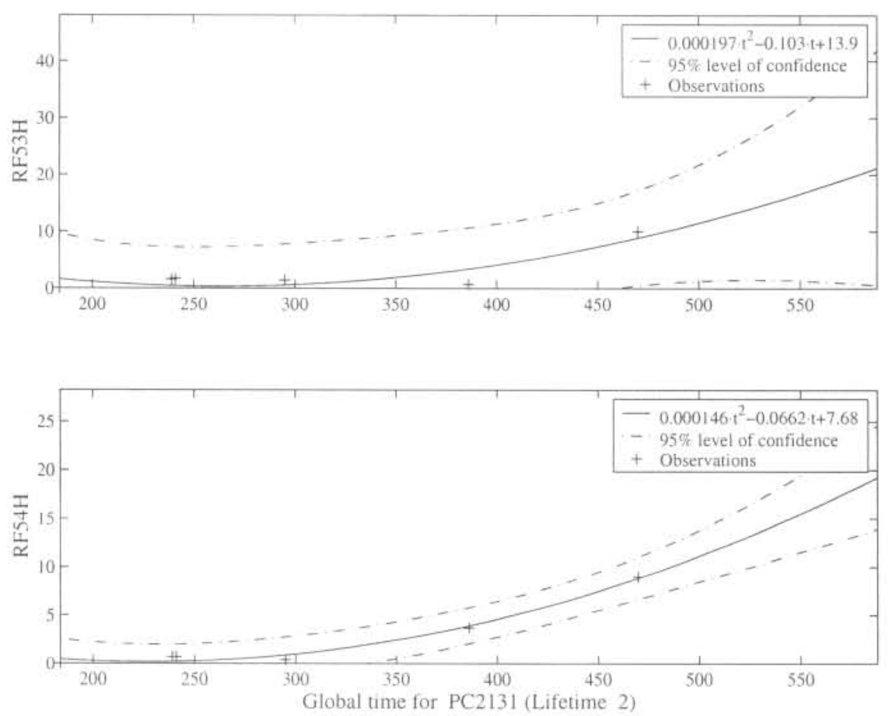

Figure E.17: Approximation of RF53H and RF54H measured on PC2131 during Lifetime 2

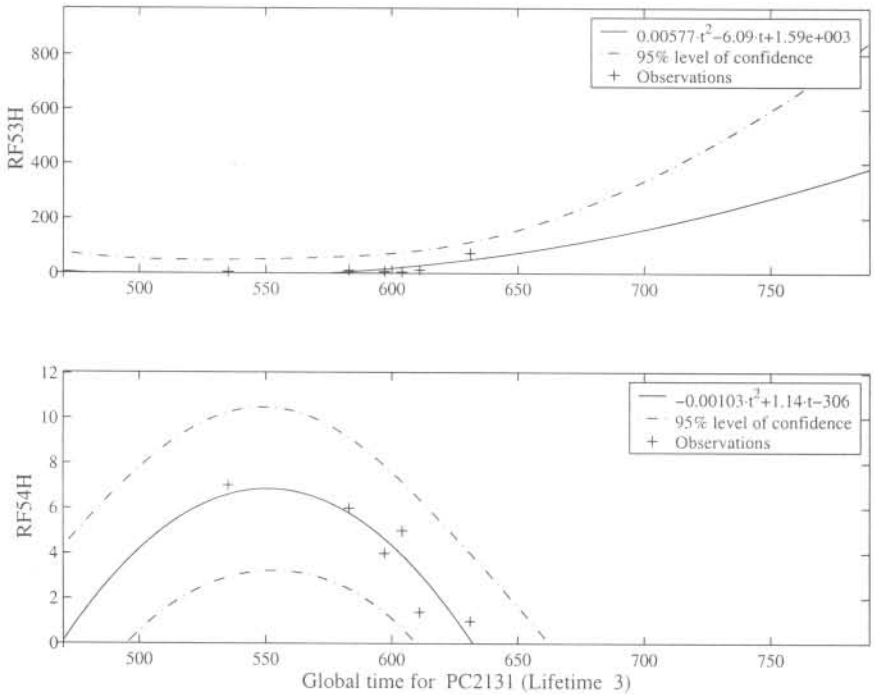

Figure E.18: Approximation of RF53H and RF54H measured on PC2131 during Lifetime 3

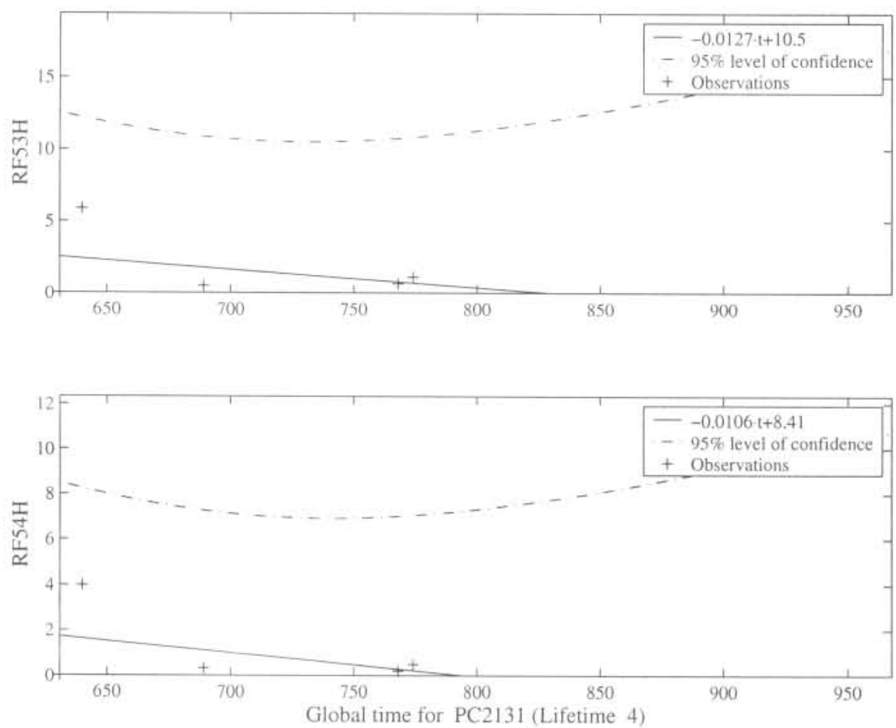

Figure E.19: Approximation of RF53H and RF54H measured on PC2131 during Lifetime 4

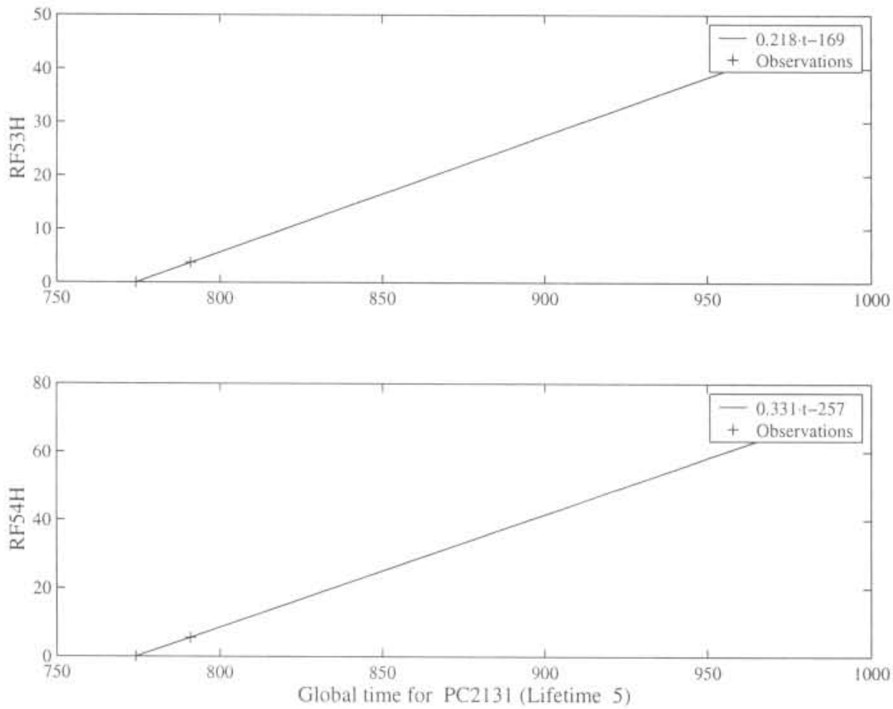

Figure E.20: Approximation of RF53H and RF54H measured on PC2131 during Lifetime 5

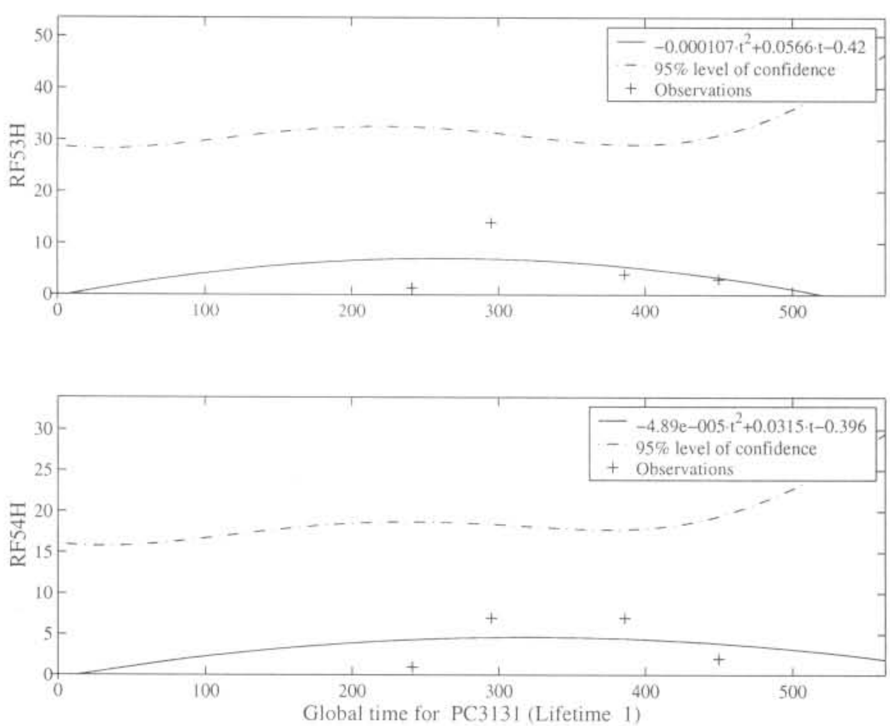

Figure E.21: Approximation of RF53H and RF54H measured on PC3131 during Lifetime 1

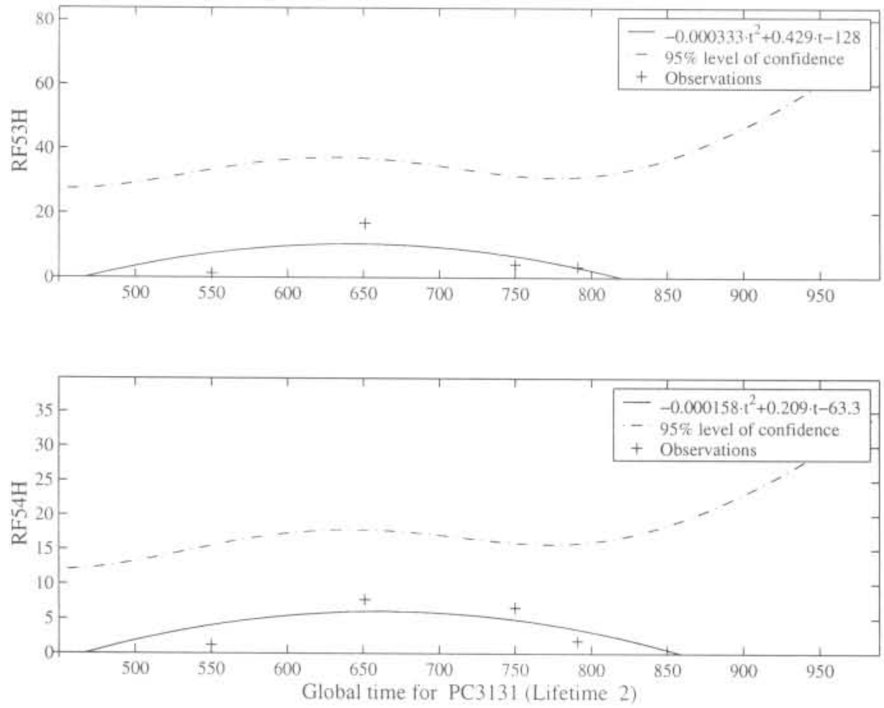

Figure E.22: Approximation of RF53H and RF54H measured on PC3131 during Lifetime 2

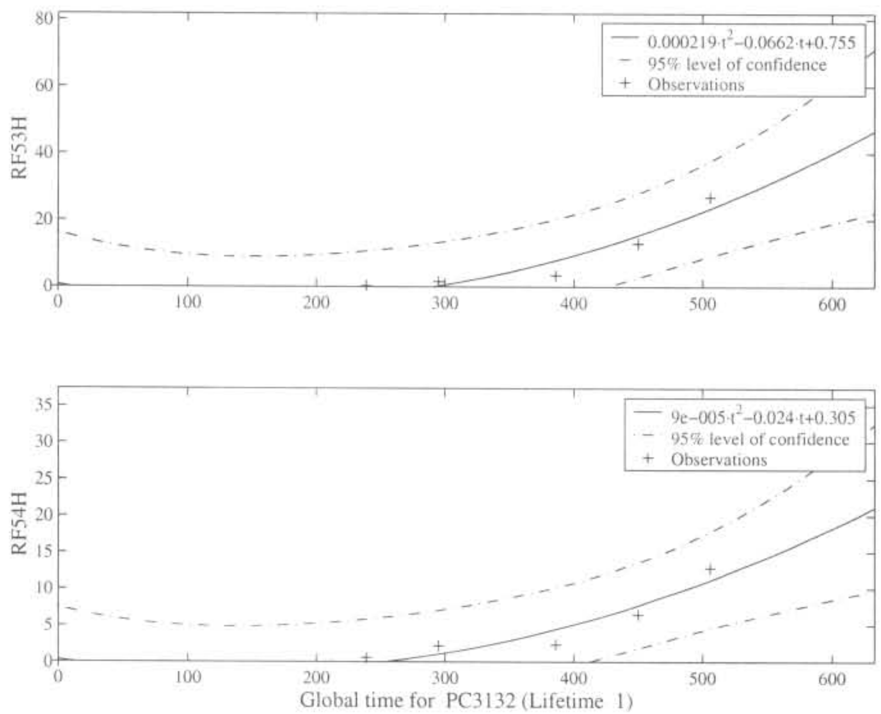

Figure E.23: Approximation of RF53H and RF54H measured on PC3132 during Lifetime 1

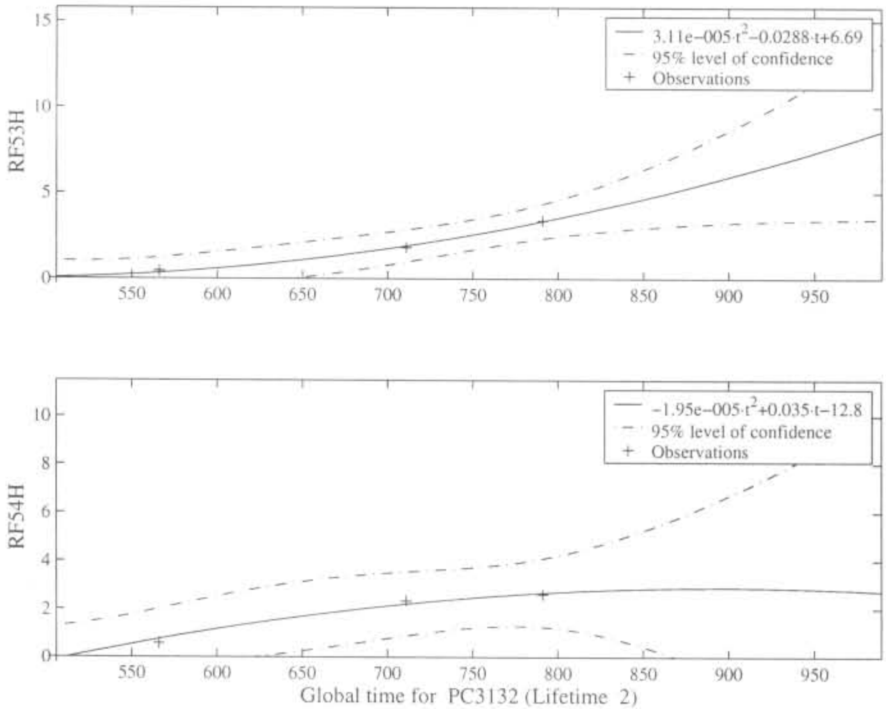

Figure E.24: Approximation of RF53H and RF54H measured on PC3132 during Lifetime 2

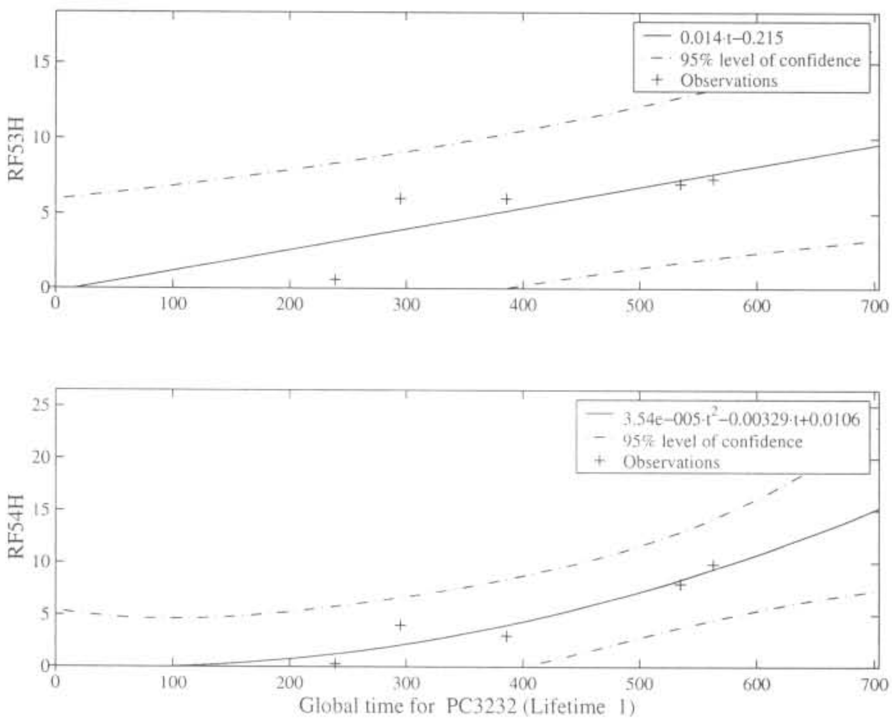

Figure E.25: Approximation of RF53H and RF54H measured on PC3232 during Lifetime 1

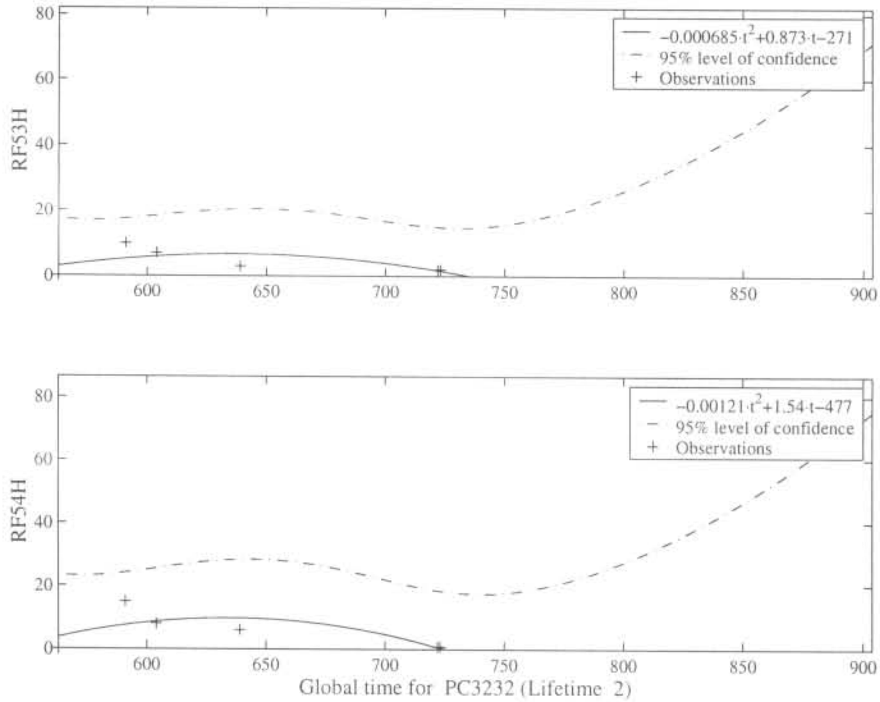

Figure E.26: Approximation of RF53H and RF54H measured on PC3232 during Lifetime 2

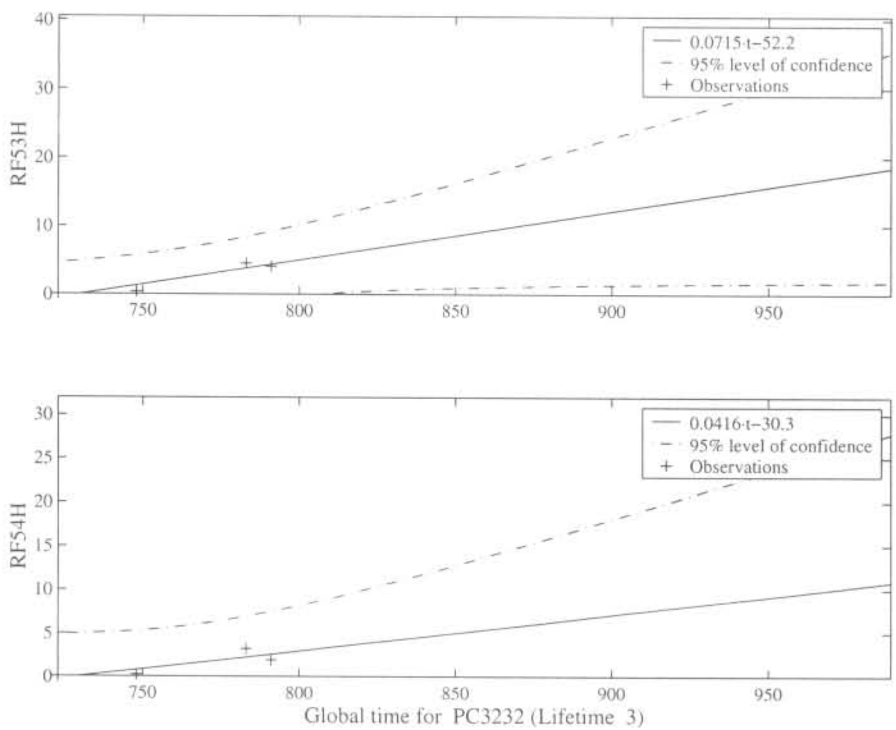

Figure E.27: Approximation of RF53H and RF54H measured on PC3232 during Lifetime 3