Quantum statistical mechanics, KMS states and Tomita-Takesaki theory

by

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Titel: Kwantum statistiese meganika, KMS-toestande en Tomita-Takesaki-teorie.
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'n Kort uiteensetting van kwantum statistiese meganika vir sekere eenvoudige fisiese stelsels word in Hoofstuk 2 gegee. In besonder word gekyk na die ewewigstoestande van sulke stelsels. In Hoofstuk 3 word aangetoon dat die ewewigstoestande van hierdie stelsels presies die sogenaamde KMS-toestande is. KMS-toestande word dan voorgestel as die ewewigstoestande van meer algemene stelsels. Hoofstuk 4 word gewy aan die Tomita-Takesaki-teorie. Dié teorie lyk aanvanklik heeltemal verwyder van KMS-toestande, maar in afdeling 4.4 word aangetoon dat daar in werklikheid 'n hegte verband tussen die twee is. Dié verband kan beskou word as die belangrikste resultaat in hierdie vehandeling want dit stel ons in staat om die fisiese betekenis van 'n groot deel van die Tomita-Takesaki-teorie te begryp, naamlik dat die tydevolusie van 'n fisiese stelsel in ewewig in terme van die Tomita-Takesaki-teorie uitgedruk kan word. Dat 'n abstrakte wiskundige teorie soos dié van Tomita-Takesaki 'n eenvoudige fisiese interpretasie het, is die motivering vir hierdie verhandeling.

Summary

Title: Quantum statistical mechanics, KMS states and Tomita-Takesaki theory.
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A short exposition of quantum statistical mechanics for certain simple physical systems is given in Chapter 2. In particular the equilibrium states of such systems are discussed. In Chapter 3 it is shown that the equilibrium states of these systems are exactly the so-called KMS states. KMS states are then suggested as the equilibrium states of more general systems. Chapter 4 is devoted to the Tomita-Takesaki theory. This theory initially seems completely separate from KMS states, but in section 4.4 it is shown that there is actually a strong connection between the two. This connection can be viewed as the central result of this dissertation since it enables us to understand the physical meaning of a large part of the Tomita-Takesaki theory, namely that the time-evolution of a physical system in equilibrium can be expressed in terms of the Tomita-Takesaki theory. That an abstract mathematical theory such as the Tomita-Takesaki theory has a simple physical interpretation, is the motivation for this dissertation.

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Contents

0.1 List of symbols

General symbols

$\emptyset$ is the empty set.
$A \subset B$ means: $x \in A \Rightarrow x \in B$.
$\mathbb{C}$ is the set of complex numbers.
$\mathbb{N} = \{0, 1, 2, ...\}$.
$\mathbb{R}$ is the set of real numbers.
$\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.
$[n] = \{1, ..., n\}$ for $n = 1, 2, 3, ...$
$\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ if $x \neq y$.

Symbols defined in the text

$\langle \cdot \rangle$, 2.2
$\langle \cdot, \cdot \rangle$, 1.1
$|\cdot|$, 1.1.4
$\langle \cdot, \cdot \rangle_{\mathbb{R}}$, $\|\cdot\|_{\mathbb{R}}$, 1.2
A_+, A a hermitian element of a C*-algebra, 1.8
G^*, G_+, G_-, G_0, G a region in $\mathbb{C}$, 1.6
$\overline{S}$, S a set in a metric space, 1.1
$X + Y$, X, Y subsets of a vector space, 4.1
$x \otimes y$, 1.9.1
$x_1 \otimes ... \otimes x_N$,1.10.2
$\mathfrak{H}_1 \otimes ... \otimes \mathfrak{H}_N$, 1.10.6
$\bigotimes_N \mathfrak{H}$, 1.10.6
$\boxtimes$, 1.10

$\mathfrak{A}_+$, 1.1
$\mathfrak{A}_1$, 1.4
$\mathfrak{A}_s$, 1.1
Aut, 3.1
β, 2.3.1
$B_\infty(K)$, 1.8

0.2 Introduction

In this dissertation we study the connection between quantum statistical mechanics and the Tomita-Takesaki theory in operator algebras.

A short discussion of quantum mechanics and quantum statistical mechanics for certain simple physical systems is given in Chapter 2. Specifically we look at equilibrium states in these systems. In Chapter 3 we show that the equilibrium states of these systems are exactly the so-called Kubo-Martin-Schwinger (or KMS) states. KMS states are then conjectured to represent equilibrium in more general physical systems as well.

In Chapter 4 the Tomita-Takesaki theory is developed. Initially this theory seems very abstract and far removed from physics, but in 4.4 it is shown to be intimately related to KMS states. This can be considered our most important result since it allows us to give a simple physical interpretation of

a large portion of the Tomita-Takesaki theory. That an abstract mathematical theory such as the Tomita-Takesaki theory has a straightforward physical interpretation, is the motivation for this dissertation.

A few historical remarks are in order (if for no other reason than to give credit where credit is due):

KMS states were first formulated in 1967 by Haag, Hugenholtz and Winnink [Ha], stimulated by earlier work done by R. Kubo (in 1957), and P.C. Martin and J. Schwinger (in 1959), in statistical mechanics.

Meanwhile, motivated by purely mathematical considerations, Minoru Tomita developed what would later come to be known as the Tomita-Takesaki theory (or modular theory). This took place from 1957 to 1967, but the results of his research (along with further developments) were only published in 1970 as a set of lecture notes by Takesaki [T].

The all important connection between the Tomita-Takesaki theory and KMS states (and hence also statistical mechanics) was first proved by Takesaki [T].

Chapter 1

Background and preliminary results

The basics of operator algebras (C*- and von Neumann algebras) and measure theory are assumed (see [M, Chapters 1 to 4], [Br, Sections 2.1 to 2.4] and [Ru, Chapters 1,2,6 and 8]). We discuss here topics of specific interest to us. This chapter should be viewed as nothing more than a tool-box. The main attraction starts with Chapter 2.

In this chapter most results for which easily accessible references are available are quoted without proof (our main references being [Br], [Con], [K], [M], [Ro], [Ru] and [S]); these results are all part of the standard theory of complex analysis, functional analysis, topological vector spaces, operator algebras and measure theory. All other results are proven in full.

1.1 Operators on Hilbert spaces

Unless otherwise stated, all vector spaces in this dissertation will be assumed to be complex. Inner products will be denoted by $\langle \cdot, \cdot \rangle$ which is linear in the second variable and conjugate linear in the first.

For normed spaces X, Y (over $\mathbb{R}$ or $\mathbb{C}$) $\mathfrak{L}(X, Y)$ will denote the set of bounded linear operators $X \to Y$, and we write $\mathfrak{L}(X) := \mathfrak{L}(X, X)$.

We denote the hermitian (i.e. self-adjoint) and positive elements of a C*-algebra $\mathfrak{A}$ by $\mathfrak{A}_s$ and $\mathfrak{A}_+$ respectively. If $\mathfrak{H}$ is a Hilbert space, $\mathfrak{L}(\mathfrak{H})$ is a C*-algebra (where A^* is defined by $\langle x, Ay \rangle = \langle A^* x, y \rangle$, $x, y \in \mathfrak{H}$, for all $A \in \mathfrak{L}(\mathfrak{H})$), and the hermitian and positive elements of $\mathfrak{L}(\mathfrak{H})$ are defined in

this context.

Since the case of real Hilbert spaces is less familiar, we will consider it in more detail.

For a real Hilbert space $\mathfrak{H}$ we define $A^* \in \mathfrak{L}(\mathfrak{H})$ for every $A \in \mathfrak{L}(\mathfrak{H})$ exactly as in the complex case above, and we call A *hermitian* (or *self-adjoint*) if $A^* = A$. See [K, 3.9-2] for a proof that this is a good definition. Following the complex case, we denote the hermitian elements of $\mathfrak{L}(\mathfrak{H})$ by $\mathfrak{L}(\mathfrak{H})_s$.

1.1.1 Definition Let $\mathfrak{H}$ be a real Hilbert space and let $A \in \mathfrak{L}(\mathfrak{H})_s$. We call A *positive* and write $A \geq 0$ if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathfrak{H}$. We denote the positive elements of $\mathfrak{L}(\mathfrak{H})$ by $\mathfrak{L}(\mathfrak{H})_+$. If $A, B \in \mathfrak{L}(\mathfrak{H})$ and $A - B \geq 0$, we write $A \geq B$ or $B \leq A$.■

(Note that 1.1.1 can be used in the complex case as well, but then the assumption $A \in \mathfrak{L}(\mathfrak{H})_s$ would be unnecessary; $A \in \mathfrak{L}(\mathfrak{H})$ would be enough because from the polarisation identity and $\langle x, Ax \rangle \in \mathbb{R}$ we can then in fact deduce $A \in \mathfrak{L}(\mathfrak{H})_s$.)

In the following definition and proposition we quote [K, 9.3-1 and 9.4-2]. [K, 9.4-2] is stated only for the complex case but the proof given by [K] holds for the real case as well.

1.1.2 Definition Let $\mathfrak{H}$ be a real or complex Hilbert space and let $A \in \mathfrak{L}(\mathfrak{H})_+$. The unique element of $\mathfrak{L}(\mathfrak{H})_+$, denoted by $A^{1/2}$, such that $(A^{1/2})^2 = A$, is called the *positive square root of* A.■

(Of course 1.1.2 can be extended to any C*-algebra. But the point of 1.1.2 is that it works for a real Hilbert space $\mathfrak{H}$, in which case $\mathfrak{L}(\mathfrak{H})$ is not a C*-algebra.)

1.1.3 Proposition *Let $\mathfrak{H}$ be a real or complex Hilbert space and consider any $A \in \mathfrak{L}(\mathfrak{H})_+$ and $B \in \mathfrak{L}(\mathfrak{H})$ such that $AB = BA$. Then $A^{1/2}B = BA^{1/2}$. If we also have $B \geq 0$, then $AB \geq 0$.*■

For $A \in \mathfrak{L}(\mathfrak{H})$, where $\mathfrak{H}$ is a real Hilbert space, we clearly have $A^*A \geq 0$ from 1.1.1, so we can give the next definition (inspired by the corresponding definition in C*-algebras):

1.1.4 Definition Let $\mathfrak{H}$ be a real Hilbert space and let $A \in \mathfrak{L}(\mathfrak{H})$. We define $|A| := (A^*A)^{1/2}$.■

We know projections in C*-algebras are positive (if p is such a projection then $p = p^2 = p^*p$). This is also true for projections on real Hilbert spaces:

1.1.5 Proposition *Let $\mathfrak{H}$ be a real Hilbert space and $\mathcal{K}$ a closed vector subspace of $\mathfrak{H}$. Let P be the orthogonal projection of $\mathfrak{H}$ on $\mathcal{K}$, then $P \geq 0$.*

Proof. Let $x_1, x_2 \in \mathfrak{H}$ and write $x_1 = y_1 + z_1$, $x_2 = y_2 + z_2$ where $y_1, y_2 \in \mathcal{K}$ and $z_1, z_2 \in \mathcal{K}^\perp$. Then $\langle x_1, Px_2\rangle = \langle y_1 + z_1, y_1\rangle = \langle y_1, y_2\rangle = \langle Px_1, x_2\rangle$ so $P \in \mathfrak{L}(\mathfrak{H})_s$, and setting $x_2 = x_1$, $\langle x_1, Px_1\rangle = \langle y_1, y_1\rangle \geq 0$, so $P \geq 0$.■

We end this section with two useful results:

1.1.6 Proposition *Let $\mathfrak{H}$ be a real or complex Hilbert space and let $A \in \mathfrak{L}(\mathfrak{H})$. Then $\ker(A^*) = (A\mathfrak{H})^\perp$.*

Proof. $x \in \ker(A^*) \iff \langle A^*x, y\rangle = 0$ for all $y \in \mathfrak{H} \iff \langle x, Ay\rangle = 0$ for all $y \in \mathfrak{H} \iff x \in (A\mathfrak{H})^\perp$.■

For a set S in a metric space we denote by $\overline{S}$ the closure of S in the metric space.

1.1.7 Lemma *Let $\mathfrak{H}$ be a real or complex Hilbert space and let $A \in \mathfrak{L}(\mathfrak{H})_s$ be injective. Then $\overline{A\mathfrak{H}} = \mathfrak{H}$.*

Proof. Since $\ker(A)$ is closed it follows from 1.1.6 that

$$\mathfrak{H} = \ker(A) \oplus (\ker(A))^\perp = \ker(A) \oplus \overline{A^*\mathfrak{H}} = \overline{A\mathfrak{H}}.\blacksquare$$

1.2 Real subspaces of a Hilbert space

In this section $\mathfrak{H}$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

1.2.1 Definition Set $\mathfrak{H}_\mathbb{R} = \mathfrak{H}$ as an additive group. Define $\langle x, y\rangle_\mathbb{R} := \operatorname{Re}\langle x, y\rangle$ and $\|x\|_\mathbb{R} := \langle x, x\rangle_\mathbb{R}^{1/2}$ for all $x, y \in \mathfrak{H}_\mathbb{R}$.■

We'll use $\mathfrak{H}_\mathbb{R}$, $\langle\cdot,\cdot\rangle_\mathbb{R}$ and $\|\cdot\|_\mathbb{R}$ as standard notation and its significance is the following:

1.2.2 Proposition *$\mathfrak{H}_\mathbb{R}$ is a real Hilbert space with the inner product given by $\langle\cdot,\cdot\rangle_\mathbb{R}$. Also, $\|\cdot\|_\mathbb{R} = \|\cdot\|$.*

Proof. That $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is a real inner product on $\mathfrak{H}_{\mathbb{R}}$ is easy to see. So $\mathfrak{H}_{\mathbb{R}}$ is a real inner product space with norm $\|\cdot\|_{\mathbb{R}}$. For all $x \in \mathfrak{H}_{\mathbb{R}}$ we have $\|x\|_{\mathbb{R}} = \langle x, x \rangle_{\mathbb{R}}^{1/2} = (\operatorname{Re} \langle x, x \rangle)^{1/2} = \langle x, x \rangle^{1/2} = \|x\|$. So $\mathfrak{H}_{\mathbb{R}}$ has the same norm as $\mathfrak{H}$, but $\mathfrak{H}$ is complete in this norm since it is a Hilbert space, therefore $\mathfrak{H}_{\mathbb{R}}$ is also complete in this norm, hence it is a Hilbert space.■

1.2.3 Proposition *$\langle x, y \rangle = \langle x, y \rangle_{\mathbb{R}} - i \langle x, iy \rangle_{\mathbb{R}}$ for all $x, y \in \mathfrak{H}$.*

Proof. $\langle x, y \rangle_{\mathbb{R}} - i \langle x, iy \rangle_{\mathbb{R}} = \operatorname{Re} \langle x, y \rangle - i \operatorname{Re} \langle x, iy \rangle = \operatorname{Re} \langle x, y \rangle - i \operatorname{Re} (i \langle x, y \rangle) = \operatorname{Re} \langle x, y \rangle + i \operatorname{Im} \langle x, y \rangle = \langle x, y \rangle$.■

1.2.4 Proposition *Consider any $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})$ such that $A \in \mathfrak{L}(\mathfrak{H})$ ($\mathfrak{H}_{\mathbb{R}}$ and $\mathfrak{H}$ are the same sets so $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})$ gives a function $\mathfrak{H} \to \mathfrak{H}$). Then we have the following:*
(1) *If $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_s$, then $A \in \mathfrak{L}(\mathfrak{H})_s$.*
(2) *If $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_+$, then $A \in \mathfrak{L}(\mathfrak{H})_+$.*

Proof. (1) Assume $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_s$ and let $x, y \in \mathfrak{H}$. Then

$$\begin{aligned} \langle x, Ay \rangle &= \langle x, Ay \rangle_{\mathbb{R}} - i \langle x, iAy \rangle_{\mathbb{R}} \quad \text{(according to 1.2.3)} \\ &= \langle x, Ay \rangle_{\mathbb{R}} - i \langle x, A(iy) \rangle_{\mathbb{R}} \quad \text{(since } A \in \mathfrak{L}(\mathfrak{H})\text{)} \\ &= \langle Ax, y \rangle_{\mathbb{R}} - i \langle Ax, iy \rangle_{\mathbb{R}} \quad \text{(since } A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_s\text{)} \\ &= \langle Ax, y \rangle \quad \text{(according to 1.2.3).} \end{aligned}$$

In other words, $A \in \mathfrak{L}(\mathfrak{H})_s$.

(2) Assume $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_+$, so by (1) and 1.1.1 $A \in \mathfrak{L}(\mathfrak{H})_s$. Let $x \in \mathfrak{H}$, then $\langle x, Ax \rangle \in \mathbb{R}$ since $A \in \mathfrak{L}(\mathfrak{H})_s$. Therefore $\langle x, Ax \rangle = \operatorname{Re} \langle x, Ax \rangle = \langle x, Ax \rangle_{\mathbb{R}} \geq 0$ since $A \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_+$.■

1.2.5 Definition A vector subspace $\mathcal{K}$ of $\mathfrak{H}_{\mathbb{R}}$ is called a *real subspace* of $\mathfrak{H}$. If $\mathcal{K}$ is closed in $\mathfrak{H}$ it is called a *closed real subspace* of $\mathfrak{H}$.■

Since $\|\cdot\|_{\mathbb{R}} = \|\cdot\|$ according to 1.2.2, $\mathcal{K}$ in 1.2.5 is closed in $\mathfrak{H}$ if and only if it is closed in $\mathfrak{H}_{\mathbb{R}}$.

1.2.6 Proposition *Let $\mathcal{K}$ be a closed real subspace of $\mathfrak{H}$, then $i\mathcal{K}$ is also a closed real subspace of $\mathfrak{H}$.*

Proof. $i\mathcal{K}$ is clearly a vector subspace of $\mathfrak{H}_{\mathbb{R}}$ since $\mathcal{K}$ is a vector subspace of $\mathfrak{H}_{\mathbb{R}}$. Let $x \in \overline{i\mathcal{K}}$, then there exists a sequence (x_n) in $\mathcal{K}$ such that $\|x_n - i(-x)\| = \|ix_n - x\| \to 0$. Hence $i(-x) \in \mathcal{K}$ since $\mathcal{K}$ is closed in $\mathfrak{H}$, and so $x = ii(-x) \in i\mathcal{K}$. Thus $i\mathcal{K}$ is closed in $\mathfrak{H}$.■

1.2.7 Definition Let $\mathcal{K}$ be a closed real subspace of $\mathfrak{H}$. $\mathcal{K}^{\perp}$ and $(i\mathcal{K})^{\perp}$ will denote the orthogonal complements of $\mathcal{K}$ and $i\mathcal{K}$ in $\mathfrak{H}_{\mathbb{R}}$.■

1.2.8 Proposition *In 1.2.7 we have* $i\mathcal{K}^{\perp} \subset (i\mathcal{K})^{\perp}$.

Proof. Let $x \in \mathcal{K}^{\perp}$, then for every $y \in \mathcal{K}$ we have $\langle ix, iy\rangle_{\mathbb{R}} = \operatorname{Re}\langle ix, iy\rangle = \operatorname{Re}\langle x, y\rangle = \langle x, y\rangle_{\mathbb{R}} = 0$, so $ix \in (i\mathcal{K})^{\perp}$.■

1.3 Topological vector spaces

The vector spaces considered in this section are real or complex.

1.3.1 Definition A vector space with a topology on it is called a *topological vector space* if vector addition and scalar multiplication are continuous in the topology.■

It is easy to prove that if $\mathcal{U}$ is a neighbourhood base at the origin of a topological vector space, then $\mathcal{U} + a$ is a neighbourhood base at the point a of the space ([Ro, I.3 Proposition 1]).

1.3.2 Definition A topological vector space is called a *convex space* if there is a neighbourhood base at the origin consisting of convex sets.■

1.3.3 Theorem ([Ro, I.4 Theorem 3]) *Let Q be a set of seminorms on a vector space X, then there is a coarsest topology τ on X making X a topological vector space on which every seminorm in Q is continuous. Under this topology X is a convex space, and a base of closed neighbourhoods at the origin of X is given by the sets*

$$\{x \in X : p_k(x) \leq \varepsilon \text{ for } k = 1, \ldots, n\},\ (\varepsilon > 0; p_1, \ldots, p_n \in Q).$$■

The topology given in 1.3.3 is said to be *determined* by Q.

1.3.4 Proposition ([Ro, I.4 Proposition 8]) *The convex space X given by 1.3.3 is a Hausdorff space if and only if for each $x \in X\backslash\{0\}$ there is a $p \in Q$ such that $p(x) > 0$.*■

1.3.5 Example Let $\mathfrak{H}$ be a complex Hilbert space, and $(x_n), (y_n)$ sequences in $\mathfrak{H}$ such that $\sum_n \|x_n\|^2 < \infty$ and $\sum_n \|y_n\|^2 < \infty$. Then $A \mapsto \sum_n |\langle x_n, Ay_n\rangle|$ defines a seminorm on $\mathfrak{L}(\mathfrak{H})$ (as can easily be confirmed). The topology on $\mathfrak{L}(\mathfrak{H})$ determined by the set of seminorms of this form (by means of 1.3.3) is called the *σ-weak topology*. The σ-weak topology on a C*-subalgbra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{H})$ is found by viewing $\mathfrak{A}$ as as a topological subspace of $\mathfrak{L}(\mathfrak{H})$ with this topology. We'll use this topology in 1.4.■

The following theorem follows from the Hahn-Banach theorem:

1.3.6 Theorem ([Ro, II.2 Corollary 1 of Proposition 5]) *Let Y be a convex subset of a convex space X, and let $x \in X\backslash\overline{Y}$ where $\overline{Y}$ is the closure of Y in X. Then there is a continuous linear functional f on X such that $f(x) \notin \overline{f(Y)}$.*■

1.3.7 Definition Let X be a topological vector space. The vector space consisting of all continuous linear functionals on X is called the *dual* of X, and denoted by X^*.■

1.3.8 Definition Let X and Y be vector spaces over the same scalar field and let b be a bilinear form on (X, Y) (in other words $b(\alpha x_1 + \beta x_2, y_1) = \alpha b(x_1, y_1) + \beta b(x_2, y_1)$ and $b(x_1, \alpha y_1 + \beta y_2) = \alpha b(x_1, y_1) + \beta b(x_1, y_2)$ for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and scalars α, β). (X, Y) is called a *dual pair* if the following two properties hold:
(1) For each $x \in X\backslash\{0\}$ there is a $y \in Y$ with $b(x, y) \neq 0$.
(2) For each $y \in Y\backslash\{0\}$ there is an $x \in X$ with $b(x, y) \neq 0$.■

When the bilinear form of a dual pair is not specified the natural bilinear form is assumed. This happens when the one vector space consists of linear functionals on the other vector space.

1.3.9 Definition Let (X, Y) be a dual pair and denote the corresponding bilinear form by b as in 1.3.8. Each $y \in Y$ defines a seminorm p_y on X by $p_y(x) := |b(x, y)|$. The topology on X determined by $\{p_y : y \in Y\}$ (by means of 1.3.3) is called the *weak topology on X determined by Y*, and it is denoted by $\sigma(X, Y)$. (This topology is clearly Hausdorff by 1.3.4 and 1.3.8.)■

1.3.10 Proposition ([Ro, II.3 Proposition 7]) *Let (X,Y) be a dual pair, then the dual of X under $\sigma(X,Y)$ is Y, i.e.*

$$X^* = \{b(\cdot, y) : y \in Y\}$$

where b is the bilinear form of the dual pair as in 1.3.8.■

1.3.11 Alaoglu's theorem ([Ro, III.7 Corollary 2]) *Let X be a normed space and set $N := \{x \in X^* : \|x\| \leq 1\}$. Then N is $\sigma(X^*, X)$-compact. (It is known that (X^*, X) is a dual pair, [K 4.3-3].)*■

1.4 C*-algebras and von Neumann algebras

Denote by $\sigma(A)$ the spectrum of an element A of a C*-algebra. Denote by $\mathrm{Inv}(\mathfrak{A})$ the invertible elements of a C*-algebra $\mathfrak{A}$.

1.4.1 Proposition *Let $\mathfrak{A}$ be a C*-algebra and let $A \in \mathfrak{A}_s$. Then $A \leq \|A\|$ (if $\mathfrak{A}$ is unital) and $A \leq |A|$.*

Proof. Let $\lambda \in \sigma(\|A\| - A)$, then $\lambda \in \mathbb{R}$ because $\|A\| - A \in \mathfrak{A}_s$. By definition $\lambda - (\|A\| - A) \notin \mathrm{Inv}(\mathfrak{A})$ so $(\|A\| - \lambda) - A \notin \mathrm{Inv}(\mathfrak{A})$ which by definition means $\|A\| - \lambda \in \sigma(A)$.

$$\therefore |\|A\| - \lambda| \leq \|A\|$$

$$\therefore 0 \leq \lambda \leq 2\|A\|$$

We conclude $\sigma(\|A\| - A) \subset \mathbb{R}^+$ and so $\|A\| - A \geq 0$.

Let $\varphi : C^*(A) \to C_0(K)$ be the Gelfand representation of the abelian C*-algebra $C^*(A)$ generated by A ($C_0(K)$ is the functions from K to $\mathbb{C}$ that vanish at infinity, K being some locally compact Hausdorff space, see [M, Theorem 2.1.10]). Then $\varphi(|A| - A) = |\varphi(A)| - \varphi(A) \geq 0$ since φ is a $*$-isomorphism, and so $|A| - A \geq 0$.■

Let $\mathfrak{H}$ a Hilbert space. For $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ we define

$$\mathfrak{M}' := \{A \in \mathfrak{L}(\mathfrak{H}) : AB = BA \text{ for every } B \in \mathfrak{M}\}.$$

It is clear that $\mathfrak{M} \subset \mathfrak{M}''$ and $\mathfrak{M}' = \mathfrak{M}'''$. It is also easily seen that if $\mathfrak{M}$ is self-adjoint (i.e. $\{A^* : A \in \mathfrak{M}\} \subset \mathfrak{M}$), then $\mathfrak{M}'$ is a C*-algebra. To avoid confusion we state that in this case

$$\mathfrak{M}'_s := (\mathfrak{M}')_s.$$

If $\mathfrak{M}$ is self-adjoint and $\mathfrak{M}'' = \mathfrak{M}$, we call $\mathfrak{M}$ a *von Neumann algebra on* $\mathfrak{H}$. Hence it is clear that a von Neumann algebra on $\mathfrak{H}$ is a C*-subalgebra of $\mathfrak{L}(\mathfrak{H})$.

For $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ and $S \subset \mathfrak{H}$, $\mathfrak{H}$ being a Hilbert space, we denote by $[\mathfrak{M}S]$ the closure in $\mathfrak{H}$ of the linear span of the set $\mathfrak{M}S$.

1.4.2 Definition Let $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ where $\mathfrak{H}$ is a Hilbert space, and let $S \subset \mathfrak{H}$.
S is called *cyclic for* $\mathfrak{M}$ if $[\mathfrak{M}S] = \mathfrak{H}$.
S is called *separating for* $\mathfrak{M}$ if for every $A \in \mathfrak{M}$ the following holds: If $AS = \{0\}$, then $A = 0$.
We call $\Omega \in \mathfrak{H}$ a *cyclic vector for* $\mathfrak{M}$ if $\{\Omega\}$ is cyclic for $\mathfrak{M}$.
We call $\Omega \in \mathfrak{H}$ a *separating vector for* $\mathfrak{M}$ if $\{\Omega\}$ is separating for $\mathfrak{M}$.■

1.4.3 Proposition ([Br, 2.5.3]) *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let $S \subset \mathfrak{H}$. Then the following two conditions are equivalent:*
(1) S is cyclic for $\mathfrak{M}$.
(2) S is separating for $\mathfrak{M}'$.

Proof. (1)$\Longrightarrow$(2): Assume S is cyclic for $\mathfrak{M}$. Consider any $A \in \mathfrak{M}'$ such that $AS = \{0\}$. Then for any $B \in \mathfrak{M}$ and $x \in S$ we have $ABx = BAx = 0$, so $A[\mathfrak{M}S] = \{0\}$. But $[\mathfrak{M}S] = \mathfrak{H}$ since S is cyclic for $\mathfrak{M}$, hence $A = 0$. So S is separating for $\mathfrak{M}'$.

(2)$\Longrightarrow$(1): Let P be the orthogonal projection of $\mathfrak{H}$ on $[\mathfrak{M}S]$. We first prove $P \in \mathfrak{M}'$. Let $A \in \mathfrak{M}$ and $x \in \mathfrak{H}$. Write $x = y + z$ where $y \in [\mathfrak{M}S]$ and $z \in [\mathfrak{M}S]^{\perp}$. $Ay \in [\mathfrak{M}S]$ since $\mathfrak{M}$ is an algebra. So $APy = Ay = PAy$. Let $v \in [\mathfrak{M}S]$, then $A^*v \in [\mathfrak{M}S]$ since $\mathfrak{M}$ is a $*$-algebra. Therefore $\langle v, Az\rangle = \langle A^*v, z\rangle = 0$ since $z \in [\mathfrak{M}S]^{\perp}$. This means $Az \in [\mathfrak{M}S]^{\perp}$ and it follows that $APz = A0 = 0 = PAz$. We conclude that $APx = PAx$, and so $AP = PA$. Thus $P \in \mathfrak{M}'$.

Assume now that S is separating for $\mathfrak{M}'$. $S \subset [\mathfrak{M}S]$ since $1 \in \mathfrak{M}$, so $(1-P)S = \{0\}$ by the definition of P. $1 - P \in \mathfrak{M}'$ since $1, P \in \mathfrak{M}$, therefore $1 - P = 0$ since S is separating for $\mathfrak{M}'$. From the definition of P we conclude $[\mathfrak{M}S] = \mathfrak{H}$, i.e. S is cyclic for $\mathfrak{M}$.■

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$.

The vector space of all σ-weakly continuous linear functionals on $\mathfrak{M}$ is called the *predual of* $\mathfrak{M}$ and denoted by $\mathfrak{M}_*$ (see 1.3.5). We assign the norm topology to $\mathfrak{M}_*$ (this is possible since the σ-weak topology is clearly

weaker than the norm topology on $\mathfrak{L}(\mathfrak{H})$ by 1.3.5, 1.3.3 and the Cauchy-Schwarz inequality, hence the elements of $\mathfrak{M}_*$ are in fact norm continuous, i.e. they are bounded linear functionals on $\mathfrak{M}$). Then $\mathfrak{M}_*$ is a Banach space, and $(\mathfrak{M}_*)^* = \mathfrak{M}$ as normed spaces (i.e. their norms are equal) where $A(\omega) := \omega(A)$ defines A as a linear functional on $\mathfrak{M}_*$ for every $A \in \mathfrak{M}$, ([Br, 2.4.18]).

The weak* topology on $\mathfrak{M}$ is defined to be the topology determined by means of 1.3.3 by the set of seminorms $\{p_\omega : \omega \in \mathfrak{M}_*\}$ on $\mathfrak{M}$ defined by $p_\omega(A) := |\omega(A)|$.

The weak topology on $\mathfrak{M}_*$ is defined to be the topology determined by means of 1.3.3 by the set of seminorms $\{p_A : A \in \mathfrak{M}\}$ on $\mathfrak{M}_*$ defined by $p_A(\omega) := |\omega(A)|$.

1.4.4 Proposition *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$. Then $\mathfrak{M}_s$ is weakly* closed in $\mathfrak{M}$.*

Proof. Let $A \in \overline{\mathfrak{M}_s}$ (the weak* closure of $\mathfrak{M}_s$ in $\mathfrak{M}$). Then there is a net (A_λ) in $\mathfrak{M}_s$ such that $A_\lambda \to A$ in the weak* topology. Given $x, y \in \mathfrak{H}$, the linear functional ω on $\mathfrak{M}$ defined by $\omega(B) = \langle x, By\rangle$ is in $\mathfrak{M}_*$ as can easily be seen from 1.3.5. So from the definition of the weak* topology we find

$$\langle x, A_\lambda y\rangle \to \langle x, Ay\rangle \text{ for all } x, y \in \mathfrak{H}.$$

So $\langle x, A_\lambda y\rangle = \langle A_\lambda x, y\rangle \to \langle Ax, y\rangle$ for all $x, y \in \mathfrak{H}$ since $A_\lambda \in \mathfrak{M}_s$. It follows that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathfrak{H}$, i.e. $A \in \mathfrak{M}_s$. We conclude $\mathfrak{M}_s$ is weak* closed in $\mathfrak{M}$.■

For a C*-algebra $\mathfrak{A}$, let $\mathfrak{A}_1 := \{A \in \mathfrak{A} : \|A\| \leq 1\}$.

1.4.5 Proposition *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space. Then $(\mathfrak{M}, \mathfrak{M}_*)$ is a dual pair (see 1.3.8) and the following holds:*
(1) The weak topology on $\mathfrak{M}$ is the $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-topology, hence $\mathfrak{M}_1$ is weakly* compact.*
(2) The weak topology on $\mathfrak{M}_*$ is the $\sigma(\mathfrak{M}_*, \mathfrak{M})$-topology, hence it is Hausdorff.

Proof. $\mathfrak{M} = (\mathfrak{M}_*)^*$ when $\mathfrak{M}_*$ has the norm topology, hence $(\mathfrak{M}, \mathfrak{M}_*)$ is a dual pair ([K, 4.3-3]). So clearly by 1.3.9 the weak* topology on $\mathfrak{M}$ is the $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-topology, and the weak topology on $\mathfrak{M}_*$ is the $\sigma(\mathfrak{M}_*, \mathfrak{M})$-topology. Hence these topologies are Hausdorff (see 1.3.9). Since $\mathfrak{M} = (\mathfrak{M}_*)^*$

as normed spaces when $\mathfrak{M}_*$ has the norm topology, it follows from 1.3.11 that N is $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-compact, i.e. weakly* compact.■

Let $\mathfrak{H}$ be a Hilbert space, then we can define a seminorm p_x on $\mathfrak{L}(\mathfrak{H})$ by $p_x(A) := \|Ax\|$ for every $x \in \mathfrak{H}$. The topology on $\mathfrak{L}(\mathfrak{H})$ determined by $\{p_x : x \in \mathfrak{H}\}$ (by means of 1.3.3) is called the *strong topology*. The strong topology on a C*-subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{H})$ is found by viewing $\mathfrak{A}$ as a topological subspace of $\mathfrak{L}(\mathfrak{H})$ with this topology.

It is known that any $*$-homomorphism between C*-algebras is norm decreasing ([M, Theorem 2.1.7]), hence the following proposition can be given:

1.4.6 Proposition ([Br, 2.4.1 and 2.4.23]) *Let $\mathfrak{M}$ and $\mathfrak{N}$ be von Neumann algebras on Hilbert spaces, and π a $*$-homomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. Then the restriction $\pi : \mathfrak{M}_1 \to \mathfrak{N}_1$ is strongly continuous.*■

We will largely be concerned with von Neumann algebras later on. We now show $\mathfrak{L}(\mathfrak{H})$ (where $\mathfrak{H}$ is a Hilbert space) is a von Neumann algebra. This example will be very important to us in Chapters 2 and 3.

1.4.7 Proposition *Let $\mathfrak{H}$ be a Hilbert space, then $\mathfrak{L}(\mathfrak{H})' = \mathbb{C}$.*

Proof. Let E be a total orthonormal set in $\mathfrak{H}$, i.e. E is an orthonormal set in $\mathfrak{H}$ such that $\overline{\text{span}(E)} = \mathfrak{H}$. ($E$ exists by [K, 4.1-8].)

Consider any $A \in \mathfrak{L}(\mathfrak{H})'$. Let $u \in E$ and let P be the orthogonal projection of $\mathfrak{H}$ onto $\mathbb{C}u$. Then for every $v \in E \backslash \{u\}$,

$$\langle v, Au\rangle = \langle v, APu\rangle = \langle v, PAu\rangle = \langle Pv, Au\rangle = \langle 0, Au\rangle = 0$$

so $Au = a(u)u$ for some $a(u) \in \mathbb{C}$ since E is a total orthonormal set. Now let $u, v \in E$ with $u \neq v$, then

$$a(u)u + a(v)v = Au + Av = A(u + v) = a(u + v)(u + v)$$

hence

$$a(u) = \langle u, a(u)u + a(v)v\rangle = \langle u, a(u + v)(u + v)\rangle = a(u + v).$$

In the same way $a(v) = a(u + v) = a(u)$. Hence, setting $c := a(u)$, we find $Ax = cx$ for all $x \in E$. It follows that $Ax = cx$ for all $x \in \overline{\text{span}(E)} = \mathfrak{H}$, i.e. $A = c$.■

1.4.8 Corollary *$\mathfrak{L}(\mathfrak{H})$ is a von Neumann algebra on the Hilbert space $\mathfrak{H}$.*

Proof. By 1.4.7, $\mathfrak{L}(\mathfrak{H})'' = (\mathbb{C}1)' = \mathfrak{L}(\mathfrak{H})$.■

For any $A \in \mathfrak{A}$, where $\mathfrak{A}$ is a C*-algebra, there exist unique $B, C \in \mathfrak{A}_s$ such that $A = B + iC$. Hence we can give

1.4.9 Definition Let $\mathfrak{A}$ be a C*-algebra. For every $A \in \mathfrak{A}$ we define $\operatorname{Re} A, \operatorname{Im} A \in \mathfrak{A}_s$ by $A = \operatorname{Re} A + i \operatorname{Im} A$.■

1.5 Polar decomposition

For a real or complex Hilbert space $\mathfrak{H}$, we call $U \in \mathfrak{L}(\mathfrak{H})$ a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \ker(U)^{\perp}$.

1.5.1 Polar decomposition *Let $V \in \mathfrak{L}(\mathfrak{H})$ where $\mathfrak{H}$ is a real or complex Hilbert space. Then there exists a unique partial isometry $U \in \mathfrak{L}(\mathfrak{H})$ such that*

$$V = U|V| \text{ and } \ker(U) = \ker(V).$$

U has the following properties:
(1) $U^*V = |V|$.
(2) *If $V^* = V$, then $V = |V|U$.*
(3) *If $V^* = V$ and $\ker(V) = \{0\}$, then $U^* = U$ and $U^2 = 1$.*
(4) *If $\mathfrak{H}$ is complex and $V^* = V$, then $U^* = U$.*
(5) *If $\mathfrak{M}$ is a von Neumann algebra on $\mathfrak{H}$ such that $V \in \mathfrak{M}$, then $U \in \mathfrak{M}$.*

Proof. The existence and uniqueness of U, as well as (1), is given by [M, Theorem 2.3.4] (the proof given in [M] works for the case of a real Hibert space as well, even though it's only stated for the complex case). (5) is given by [M, Theorem 4.1.10]. We now prove (2),(3) and (4).

$\ker(|V|) \subset \ker(V)$ since $V = U|V|$, and $\ker(V) \subset \ker(|V|)$ since $U^*V = |V|$. So

$$\ker(|V|) = \ker(V)$$

Now assume $V^* = V$.
$|V| = (V^2)^{1/2}$ and $VV^2 = V^2V$, so

$$V|V| = |V|V$$

by 1.1.3.

(2) By 1.1.6 we have $\overline{|V|\mathfrak{H}}^{\perp} = \ker(|V|) = \ker(V) = \ker(U)$. Let $x \in \mathfrak{H}$ and write $x = y + z$ where $y \in \overline{|V|\mathfrak{H}}$ and $z \in \overline{|V|\mathfrak{H}}^{\perp}$, so $Ux = Uy$ and $Vx = Vy$. Consider any sequence (y_n) in $\mathfrak{H}$ such that $|V|y_n \to y$. Then $Ux = \lim_{n\to\infty} U|V|y_n = \lim_{n\to\infty} Vy_n$ so

$$|V|Ux = \lim_{n\to\infty} |V|Vy_n = \lim_{n\to\infty} V|V|y_n = Vy = Vx.$$

We conclude $V = |V|U$.

(3) Assume $\ker(V) = \{0\}$. It follows that $\ker(|V|) = \{0\}$. We already know $|V|U^* = (U|V|)^* = V^* = V = |V|U$, so $U^* = U$ since $\ker(|V|) = \{0\}$. From this we have $U^2V = UU^*V = U|V| = V$ which means $U^2x = x$ for all $x \in V\mathfrak{H}$, but $V^* = V$ and V is injective ($\ker(V) = \{0\}$) so $\overline{V\mathfrak{H}} = \mathfrak{H}$ by 1.1.7. Since U is continuous it follows that $U^2 = 1$.

(4) U^* is a partial isometry since U is a partial isometry ([M, Theorem 2.3.3(1) and (4)], the proof of this requires the fact that if $\langle x, Ax\rangle = \langle x, Bx\rangle$ for all $x \in \mathfrak{H}$, where $A, B \in \mathfrak{L}(\mathfrak{H})$, then $A = B$, but this follows from the polarisation identity, hence the need for $\mathfrak{H}$ to be complex).

$V = V^* = (U|V|)^* = |V|U^*$, so $\ker(U^*) \subset \ker(V)$.

$U|V|\mathfrak{H} = V\mathfrak{H}$, so $U\left(\overline{|V|\mathfrak{H}}\right) \subset \overline{V\mathfrak{H}}$ since U is continuous, but $U\left(\overline{|V|\mathfrak{H}}^{\perp}\right) = U(\ker(U)) = \{0\}$ (as in the proof of (2)), therefore $U\mathfrak{H} = U\left(\overline{|V|\mathfrak{H}} \oplus \overline{|V|\mathfrak{H}}^{\perp}\right) \subset \overline{V\mathfrak{H}}$. It follows from 1.1.6 that $\ker(V) = \overline{V\mathfrak{H}}^{\perp} \subset (U\mathfrak{H})^{\perp} = \ker(U^*)$. Hence

$$\ker(U^*) = \ker(V)$$

$V = U^*|V|$ by (2), but U is the unique partial isometry such that $V = U|V|$ and $\ker(U) = \ker(V)$, so $U^* = U$.■

1.6 Complex analysis

1.6.1 Definition Let $G \subset \mathbb{C}$ be open. A function $f : G \to \mathbb{C}$ is called *analytic* if it is differentiable. f is called an *entire funtion* if it is analytic and $G = \mathbb{C}$.■

1.6.1 is motivated by the fact that if $f : G \to \mathbb{C}$ is differentable (with G as in 1.6.1), then it is infinitely differentiable ([Con, IV.8(Goursat's theorem), III.2.3 and IV.2.12]).

If X is a metric space and $S \subset X$, then we call $x \in X$ a *limit point* of S if there is a sequence (x_n) of distinct points in S such that $x_n \to x$.

A connected open subset of $\mathbb{C}$ is called a *region*.

1.6.2 Theorem ([Con, IV.3.7]) *Let G be a region and $f : G \to \mathbb{C}$ an analytic function such that $\{z \in G : f(z) = 0\}$ has a limit point in G. Then $f = 0$.*■

For a region G we define $G^* = \{\overline{z} : z \in G\}$, $G_+ = \{z \in G : \operatorname{Im} z > 0\}$, $G_- = \{z \in G : \operatorname{Im} z < 0\}$ and $G_0 = \{z \in G : \operatorname{Im} z = 0\}$.

1.6.3 Schwarz reflection principle ([Con, IX.1.1]) *Let G be a region such that $G^* = G$. Let $f : G_+ \cup G_0 \to \mathbb{C}$ be a continuous function, analytic on G_+, such that $f(G_0) \subset \mathbb{R}$. Then the function $g : G \to \mathbb{C}$ defined by*

$$g(z) = f(z) \quad \text{for } z \in G_+ \cup G_0$$

and

$$g(z) = \overline{f(\overline{z})} \quad \text{for } z \in G_-$$

is analytic.■

For $a, b \in \mathbb{R}$ we write

$$\mathbb{C}_a^b := \{z \in \mathbb{C} : a < \operatorname{Im} z < b\}$$

and

$$\overline{\mathbb{C}_a^b} := \{z \in \mathbb{C} : a \leq \operatorname{Im} z \leq b\}.$$

The strips $\mathbb{C}_a^b$ and $\overline{\mathbb{C}_a^b}$ play a key role in the formulation of KMS states in Chapter 3. We'll need the following two properties concerning these strips:

1.6.4 Corollary *Consider a bounded continuous function $f : \overline{\mathbb{C}_0^1} \to \mathbb{C}$, analytic on $\mathbb{C}_0^1$, and real-valued on $\overline{\mathbb{C}_0^1} \backslash \mathbb{C}_0^1$. Then f is constant.*

Proof. Say f is bounded by $M \in \mathbb{R}$, i.e. $|f(z)| < M$ for all $z \in \overline{\mathbb{C}_0^1}$.

Let $g : \overline{\mathbb{C}_0^1} \to \mathbb{C} : z \mapsto \overline{f\left(\overline{z-i}\right)}$, then g is continuous and bounded by M, it is analytic on $\mathbb{C}_0^1$, and $g(\mathbb{R}) \subset \mathbb{R}$. Define $g_1 : \overline{\mathbb{C}_{-1}^1} \to \mathbb{C}$ by $g_1(z) = g(z)$ for $z \in \overline{\mathbb{C}_0^1}$, and $g_1(z) = \overline{g(\overline{z})}$ for $z \in \overline{\mathbb{C}_{-1}^0}$. Clearly g_1 is continuous and bounded by M, and by 1.6.3 it is analytic on $\mathbb{C}_{-1}^1$.

Now let $f_1 : \overline{\mathbb{C}_0^2} \to \mathbb{C} : z \mapsto \overline{g_1\left(\overline{z-i}\right)}$, then f_1 is continuous and bounded by M, and it is analytic on $\mathbb{C}_0^2$. Also, for all $z \in \overline{\mathbb{C}_0^1}$, $f_1(z) = \overline{g_1\left(\overline{z-i}\right)} = \overline{g\left(\overline{z-i}\right)} = \overline{\overline{f\left(\overline{z-i-i}\right)}} = f(z)$.

In effect we have applied 1.6.3 to f along the line $\overline{\mathbb{C}_1^1}$, instead of along $\mathbb{R}$, to get f_1. This was posssible since $f\left(\overline{\mathbb{C}_1^1}\right) \subset \mathbb{R}$.

Note that $f_1\left(\overline{\mathbb{C}_2^2}\right) = f(\mathbb{R}) \subset \mathbb{R}$, so we can apply 1.6.3 on f_1 along the line $\overline{\mathbb{C}_2^2}$ to find a function $f_2 : \overline{\mathbb{C}_0^4} \to \mathbb{C}$ which is continuous and bounded by M, and analytic on $\mathbb{C}_0^4$, such that $f_2(z) = f_1(z)$ for $z \in \overline{\mathbb{C}_0^2}$.

By repeating this process (and setting $f_0 := f$) we find for every $n \in \mathbb{N}$ a function $f_n : \overline{\mathbb{C}_0^{2^n}} \to \mathbb{C}$ which is continuous and bounded by M, and analytic on $\mathbb{C}_0^{2^n}$, such that $f_{n+1}(z) = f_n(z)$ for $z \in \overline{\mathbb{C}_0^{2^n}}$.

Thus we can define $F : \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \to \mathbb{C}$ by $F(z) = f_n(z)$ for any n such that $z \in \overline{\mathbb{C}_0^{2^n}}$. For every $n \in \mathbb{N}$, F is continuous and bounded by M on $\overline{\mathbb{C}_0^{2^n}}$, and analytic on $\mathbb{C}_0^{2^n}$. It follows that F is continuous and bounded by M, and it is analytic on $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Clearly $G : \mathbb{C} \to \mathbb{C}$ defined by $G(z) = F(z)$ for $\operatorname{Im} z \geq 0$, and $G(z) = \overline{F(\overline{z})}$ for $\operatorname{Im} z < 0$, is bounded by M, and by 1.6.3 it is analytic. So G is constant by Liouville's theorem. But $f(z) = G(z)$ for $z \in \overline{\mathbb{C}_0^1}$, so f is constant.■

The following result is a consequence of the maximum modulus theorem:

1.6.5 Lemma ([Con, VI.3.9]) *Let $a < b \in \mathbb{R}$ and consider a continuous function $f : \overline{\mathbb{C}_a^b} \to \mathbb{C}$ which is analytic on $\mathbb{C}_a^b$. Then*

$$|f(z)| \leq \sup\left\{|f(w)| : w \in \overline{\mathbb{C}_a^b} \backslash \mathbb{C}_a^b\right\} \text{ for all } z \in \overline{\mathbb{C}_a^b}.\blacksquare$$

Let L^1 be the Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$$

(where dx here and in the sequel refers to Lebesgue measure).

For $f \in L^1$ we call the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixt} f(x) dx$$

the *Fourier transform of* f. $\hat{f}$ is well defined exactly because $f \in L^1$.

1.6.6 Fourier inversion theorem ([Ru, 9.11]) *Consider any* $f \in L^1$ *such that* $\hat{f} \in L^1$, *and define* $g : \mathbb{R} \to \mathbb{C}$ *by*

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixt} \hat{f}(t) dt.$$

Then $g = f$ *almost everywhere. In particular, for any* $f \in L^1$ *such that* $\hat{f} = 0$, *we have* $f = 0$ *almost everywhere.*■

1.6.6 will be needed in 4.3.7. We close this section with two results which we will need in 4.3.3 and 4.3.6:

1.6.7 Lemma *Let* W *be a closed convex subset of* $\mathbb{C}$ *and let* $z \in \mathbb{C} \backslash W$. *Then there is an* $r \in \mathbb{C}$ *with* $|r| = 1$ *such that* $\mathrm{Re}(rz) > \mathrm{Re}(rw)$ *for every* $w \in W$.

Proof. Since W is closed and convex there exists a $v \in W$ such that

$$|z - v| \leq |z - w| \text{ for every } w \in W, \tag{1}$$

([K, 3.3-1]). Set $r := \overline{z - v} / |z - v|$. We now show r has the required properties.

Clearly $|r| = 1$. By the definition of r we have

$$rz - rv = r(z - v) = |z - v|. \tag{2}$$

Let $w \in W$ and suppose

$$\mathrm{Re}(rw) > \mathrm{Re}(rv). \tag{3}$$

We assign the dot product of $\mathbb{R}^2$ to $\mathbb{C}$, i.e.

$$x \cdot y := (\mathrm{Re}\, x)(\mathrm{Re}\, y) + (\mathrm{Im}\, x)(\mathrm{Im}\, y) \tag{4}$$

for all $x, y \in \mathbb{C}$. Set

$$p = [(rz - rv) \cdot (rw - rv)] \frac{rw - rv}{|rw - rv|^2} + rv = arw + (1 - a)rv \tag{5}$$

where $a := [(rz - rv) \cdot (rw - rv)] / |rw - rv|^2$, i.e. $p - rv$ is the orthogonal projection of $rz - rv$ on $rw - rv$ (in the $\mathbb{R}^2$ sense), hence $rz - p$ and $p - rv$ are orthogonal in $\mathbb{R}^2$.

$rz - rv > 0$ because of (2), therefore $(rz - rv) \cdot (rw - rv) = (rz - rv)\,\mathrm{Re}(rw - rv) > 0$ by (3) and (4). Hence $a > 0$. Set $b := \min\{1, a\}$ and

$$u := brw + (1 - b)rv, \tag{6}$$

then $0 < b \leq a$, and $u \in rW$ since rW is convex, $0 < b \leq 1$ and $rw, rv \in rW$. Since u lies on the line from rv to rw, it follows that $rz - p$ and $p - u$ are orthogonal in $\mathbb{R}^2$. Now

$$\begin{aligned} |p - rv| &= |p - u + u - rv| = |a(rw - rv)| \quad \text{(by (5) and (6))} \\ &= a\,|rw - rv| \quad \text{(since } a > 0\text{)} \\ &= (a - b)\,|rw - rv| + b\,|rw - rv| \\ &= |(a - b)(rw - rv)| + |b(rw - rv)| \quad \text{(since } 0 < b \leq a\text{)} \\ &= |p - u| + |u - rv| \quad \text{(by (5) and (6)).} \end{aligned}$$

Hence, since $(rz - p) \perp (p - rv)$ in $\mathbb{R}^2$, we find

$$\begin{aligned} |rz - rv|^2 &= |rz - p|^2 + |p - u + u - rv|^2 \\ &= |rz - p|^2 + (|p - u| + |u - rv|)^2 \\ &> |rz - p|^2 + |p - u|^2 \quad \text{(since } u \neq rv \text{ by (6) and (3))} \\ &= |rz - u|^2 \quad \text{(since } (rz - p) \perp (p - u) \text{ in } \mathbb{R}^2\text{).} \end{aligned}$$

Therefore $|z - v| > \left|z - \frac{1}{r}u\right|$, while $\frac{1}{r}u \in W$ as pointed out just after (6). This contradicts (1), so we conclude $\mathrm{Re}(rw) \leq \mathrm{Re}(rv) < \mathrm{Re}(rz)$ by (2).■

For use in the proof of the next lemma (given as Lemma 4.6 in [R]), note that for $z = x + iy$, $x, y \in \mathbb{R}$, $\sin z = \sin x \cosh y + i \cos x \sinh y$. Therefore

$$|\sin z| \geq |\sin x| \quad \text{and} \quad |\sin z| \geq \min\{\cosh y, |\sinh y|\}.$$

1.6.8 Lemma *Let $\lambda = e^{i\varphi/2}$ where $-\pi < \varphi < \pi$. Let f be a complex-valued function defined, bounded and continuous on the strip $i\overline{\mathbb{C}^{1/2}_{-1/2}}$, and analytic inside this strip. Then*

$$f(0) = \int_{-\infty}^{+\infty} e^{-\varphi t}(e^{\pi t} + e^{-\pi t})^{-1} \left(\lambda f(it + 1/2) + \overline{\lambda} f(it - 1/2)\right) dt.$$

Proof. Let $S := i\overline{\mathbb{C}^{1/2}_{-1/2}}$ and $Z := \{z \in S : \sin \pi z \neq 0\}$. So we can define $g : Z \to \mathbb{C}$ by $g(z) = \pi e^{i\varphi z} f(z)/\sin \pi z$. It is easy to see that $Z = S\backslash\{0\}$. Since sine has a simple zero in 0, i.e. $\left[\frac{d}{dz} \sin z\right]_{z=0} \neq 0$, the residue of g in 0 is

$$\operatorname{res}(g, 0) = \left[\pi e^{i\varphi z} f(z)/\frac{d}{dz} \sin \pi z\right]_{z=0} = f(0).$$

Let a be any positive real number, and n any integer greater than 2. By the residue theorem we now have (integrating along the rectangle $-(\frac{1}{2} - \frac{1}{n}) - ia$, $(\frac{1}{2} - \frac{1}{n}) - ia$, $(\frac{1}{2} - \frac{1}{n}) + ia$, $-(\frac{1}{2} - \frac{1}{n}) + ia$)

$$\begin{aligned} 2\pi i f(0) &= \int_{-(\frac{1}{2}-\frac{1}{n})}^{\frac{1}{2}-\frac{1}{n}} g(t - ia)dt + \int_{-a}^{a} g\left(\frac{1}{2} - \frac{1}{n} + it\right) idt \\ &\quad - \int_{-(\frac{1}{2}-\frac{1}{n})}^{\frac{1}{2}-\frac{1}{n}} g(t + ia)dt - \int_{-a}^{a} g\left(-\left(\frac{1}{2} - \frac{1}{n}\right) + it\right) idt. \end{aligned}$$

$|\sin z| \geq |\sin(\operatorname{Re} z)|$, therefore $\left|\sin \pi(\frac{1}{2} - \frac{1}{n} + it)\right| \geq \frac{1}{2}$ since $n > 2$, hence $\left|g(\frac{1}{2} - \frac{1}{n} + it)\right| \leq 2\pi \left|f(\frac{1}{2} - \frac{1}{n} + it)\right|$ for real t. But f is continuous and bounded so from Lebesgue's dominated convergence theorem ([Ru, 1.34]) we have

$$\begin{aligned} \lim_{n\to\infty} \int_{-a}^{a} g\left(\frac{1}{2} - \frac{1}{n} + it\right) idt &= \int_{-a}^{a} \left(\lim_{n\to\infty} g\left(\frac{1}{2} - \frac{1}{n} + it\right)\right) idt \\ &= i \int_{-a}^{a} g\left(it + 1/2\right) dt. \end{aligned}$$

Similarly, $\lim_{n\to\infty} \int_{-a}^{a} g\left(-\left(\frac{1}{2} - \frac{1}{n}\right) + it\right) idt = i \int_{-a}^{a} g(it - 1/2)dt$, hence

$$\begin{aligned} 2\pi i f(0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t - ia)dt + i \int_{-a}^{a} g(it + 1/2)dt \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t + ia)dt - i \int_{-a}^{a} g(it - 1/2)dt. \end{aligned}$$

$|\sin z| \geq \min\{\cosh(\operatorname{Im} z), |\sinh(\operatorname{Im} z)|\}$ and f is bounded, so

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} g(t - ia)dt\right| \leq \sup\left\{|g(t - ia)| : -\frac{1}{2} \leq t \leq \frac{1}{2}\right\} \to 0 \text{ as } |a| \to \infty$$

by the definition of g. Likewise $\int_{-1/2}^{1/2} g(t+ia)dt \to 0$ as $|a| \to \infty$. Therefore

$$2\pi i f(0) = i \int_{-\infty}^{+\infty} (g(it+1/2) - g(it-1/2))\, dt.$$

So by the definition of g and λ

$$\begin{aligned} f(0) &= \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{e^{i\varphi(it+1/2)}f(it+1/2)}{\sin\pi(it+1/2)} - \frac{e^{i\varphi(it-1/2)}f(it-1/2)}{\sin\pi(it-1/2)}\right] dt \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\varphi t}(\cos i\pi t)^{-1}\left[e^{i\varphi/2}f(it+1/2) + e^{-i\varphi/2}f(it-1/2)\right] dt \\ &= \int_{-\infty}^{+\infty} e^{-\varphi t}(e^{\pi t}+e^{-\pi t})^{-1}\left(\lambda f(it+1/2) + \overline{\lambda} f(it-1/2)\right) dt. \blacksquare \end{aligned}$$

1.7 Differentiation in normed spaces

1.7.1 Definition Let X be a normed space and consider a function $f : S \to X$ where $S \subset \mathbb{C}$. Let z be an interior point of S. If the limit

$$f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h},\ (h \in \mathbb{C})$$

exists, we call $f'(z)$ the *derivative* of f at z, and say f is *differentiable at* z. We also write

$$\frac{d}{dz}f(z) = f'(z).$$

If S is open and f is differentiable at every point of S, we call f *differentiable.* $\blacksquare$

Clearly if f is differentiable at z in 1.7.1, then it is continuous at z.
The proof of the following is easy (it's the same as for real functions):

1.7.2 Product rule *Let $\mathfrak{A}$ be a normed algebra and consider functions $f, g : S \to \mathfrak{A}$, where $S \subset \mathbb{C}$, such that $f'(z)$ and $g'(z)$ exist for some interior point z of S. Then $(fg)'(z)$ exists and*

$$(fg)' = f'(z)g(z) + f(z)g'(z)$$

where $fg : S \to \mathfrak{A}$ is defined by $(fg)(z) = f(z)g(z)$. $\blacksquare$

1.7.3 Definition Consider a function $A : S \to \mathfrak{L}(X,Y)$ where $S \subset \mathbb{C}$, and X, Y are normed spaces. Let $x \in X$, and let z be and interior point of S. If $\frac{d}{dz}(A(z)x)$ exists, we write

$$\mathbf{D}(x,z)A := \frac{d}{dz}\left(A(z)x\right). \blacksquare$$

1.7.4 Proposition *Let X, Y be normed spaces. Consider the functions $A : S \to \mathfrak{L}(X,Y)$ and $f : S \to X$ where $S \subset \mathbb{C}$, and assume A is bounded (in the norm of $\mathfrak{L}(X,Y)$). Then we have:*
(1) If $A(\cdot)f(z)$ is continuous for every $z \in S$, and f is continuous, then

$$Af : S \to Y \ : z \mapsto A(z)f(z)$$

is continuous.
(2) If S is open, and $A(\cdot)x$ and f are differentiable for every $x \in X$, then Af is differentiable and

$$\frac{d}{dz}\left(A(z)f(z)\right) = \mathbf{D}\left(f(z),z\right)A + A(z)f'(z)$$

for every $z \in S$.

Proof. (1) For $w, z \in S$ we have

$$\|A(w)f(w) - A(z)f(z)\| \leq \|A(w)\|\,\|f(w) - f(z)\| + \|A(w)f(z) - A(z)f(z)\|$$

so clearly Af is contiuous since A is bounded.

(2) Let $z \in S$ be fixed but arbitrary. Define $g : S \to X$ by

$$g(w) = \frac{f(w) - f(z)}{w - z} \text{ if } w \neq z, \quad \text{and} \quad g(z) = f'(z).$$

g is contiuous by the definition of $f'(z)$ and the fact that f is continuous (since it is differentiable).

$A(\cdot)x$ is continuous (since it is differentiable) for every $x \in X$. So by (1)

$$\lim_{w \to z} A(w)g(w) = A(z)g(z) = A(z)f'(z).$$

It follows that

$$\begin{aligned}\lim_{w \to z} \frac{A(w)f(w) - A(z)f(z)}{w - z} &= \lim_{w \to z} A(w)g(w) + \lim_{w \to z} \frac{A(w)f(z) - A(z)f(z)}{w - z} \\ &= A(z)f'(z) + \mathbf{D}\left(f(z),z\right)A\end{aligned}$$

by the definition of $\mathbf{D}(f(z), z)A$, and this proves (2).■

1.7.4 will be a useful tool in our development of the Tomita-Takesaki theory in Chapter 4.

1.8 The Borel functional calculus

1.8.1 Definition Let K be a compact Hausdorff space and $\mathfrak{H}$ a Hilbert space. A *spectral measure relative to* $(K, \mathfrak{H})$ is a map E from the σ-algebra B of all Borel sets of K to the set of projections in $\mathfrak{L}(\mathfrak{H})$ such that
(1) $E(\emptyset) = 0$ and $E(K) = 1$;
(2) $E(S_1 \cap S_2) = E(S_1)E(S_2)$ for all $S_1, S_2 \in B$;
(3) for all $x, y \in \mathfrak{H}$, the function $E_{x,y} : B \to \mathbb{C}$ given by $E_{x,y}(S) := \langle x, E(S)y\rangle$ is a regular Borel complex measure on K.■

1.8.2 Proposition *Let E be as in 1.8.1, then $\|E_{x,y}\| \leq \|x\|\,\|y\|$.*

Proof. By definition $\|E_{x,y}\| = |E_{x,y}|(K)$, see [Ru, 6.5]. Consider any $S_1, ..., S_n \in B$, with B as in 1.8.1, such that $S_1, ..., S_n$ are mutually disjoint, and $K = S_1 \cup ... \cup S_n$. Then

$$\begin{aligned}
\sum_{j=1}^{n} |E_{x,y}(S_j)| &= \sum_{j=1}^{n} |\langle x, E(S_j)y\rangle| \\
&= \sum_{j=1}^{n} |\langle E(S_j)x, E(S_j)y\rangle| \qquad \text{(since } E(S_j) \text{ is a projection)} \\
&\leq \sum_{j=1}^{n} \|E(S_j)x\|\,\|E(S_j)y\| \\
&\leq \left(\sum_{j=1}^{n} \|E(S_j)x\|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|E(S_j)y\|^2\right)^{\frac{1}{2}} \\
&= \left\|\sum_{j=1}^{n} E(S_j)x\right\| \left\|\sum_{j=1}^{n} E(S_j)y\right\| \\
&\leq \|x\|\,\|y\|\,.
\end{aligned}$$

The second to last step above follows from the fact that $E(S_1)x, ..., E(S_n)x$ are orthogonal by 1.8.1(1) and (2) since $S_k \cap S_l = \emptyset$ for $k \neq l$, (and likewise for y), while the last step follows from the fact that $\sum_{j=1}^{n} E(S_j)$ is a projection since $E(S_k)E(S_l) = 0$ for $k \neq l$ by 1.8.1(1) and (2).

For a partition $(S_j)_{j=1}^{\infty}$ of K (where by definition of a partition $S_j \in B$) we therefore have

$$\sum_{j=1}^{n} |E_{x,y}(S_j)| \leq \sum_{j=1}^{n} |E_{x,y}(S_j)| + \left| E_{x,y}\left(K \setminus \bigcup_{j=1}^{n} S_j \right) \right| \leq \|x\| \, \|y\|$$

since $S_1, ..., S_n, K \setminus \bigcup_{j=1}^{n} S_j$ are mutually disjoint and $K = S_1 \cup ... \cup S_n \cup \left(K \setminus \bigcup_{j=1}^{n} S_j \right)$. Hence

$$\sum_{j=1}^{\infty} |E_{x,y}(S_j)| \leq \|x\| \, \|y\|$$

and we conclude by the definition of $|E_{x,y}|(K)$ (it is the supremum of $\sum_{j=1}^{\infty} |E_{x,y}(S_j)|$ over all partitions $(S_j)_{j=1}^{\infty}$ of K, [Ru, 6.1]) that $|E_{x,y}|(K) \leq \|x\| \, \|y\|$.■

Denote by $B_\infty(K)$ the C*-algebra of all bounded Borel measurable complex-valued functions on the compact Hausdorff space K.

1.8.3 Proposition ([M, Theorem 2.5.3]) *Let K, $\mathfrak{H}$ and E be as in 1.8.1, then for each $f \in B_\infty(K)$ there is a unique $A \in \mathfrak{L}(\mathfrak{H})$ such that*

$$\langle x, Ay \rangle = \int f dE_{x,y} \quad \text{for all } x, y \in \mathfrak{H}.\blacksquare$$

1.8.3 allows us to give the following definition:

1.8.4 Definition Let K, $\mathfrak{H}$ and E be as in 1.8.1. For every $f \in B_\infty(K)$ we define the *integral $\int f dE$ of f with respect to E* as the element of $\mathfrak{L}(\mathfrak{H})$ such that

$$\left\langle x, \left(\int f dE \right) y \right\rangle = \int f dE_{x,y} \quad \text{for all } x, y \in \mathfrak{H}.\blacksquare$$

For any set S, we define a real-valued function χ_S (called the *characteristic function of* S) by

$$\chi_S(x) = 1 \text{ if } x \in S, \quad \text{and} \quad \chi_S(x) = 0 \text{ if } x \notin S.$$

We call this function the *characteristic function* of S.

Note that in 1.8.4 we have

$$\int \chi_S dE = E(S)$$

for every Borel set $S \subset K$, by 1.8.1(3).

1.8.5 Proposition ([M, Theorem 2.5.4]) *With K, $\mathfrak{H}$ and E as in 1.8.1, the map*

$$B_\infty(K) \to \mathfrak{L}(\mathfrak{H}) : f \mapsto \int f dE$$

is a unital $*$-homomorphism.■

1.8.6 Spectral theorem ([M, Theorem 2.5.6]) *Let $\mathfrak{H}$ be a Hilbert space and $A \in \mathfrak{L}(\mathfrak{H})$ a normal operator. Then there is a unique spectral measure E relative to $(\sigma(A), \mathfrak{H})$ such that*

$$A = \int z dE$$

where $z : \sigma(A) \to \mathbb{C} : \lambda \mapsto \lambda$.■

1.8.7 Definition For A and E as in 1.8.6, E is called the *resolution of the identity for* A. If A is hermitian we define

$$E_\lambda := E\left(\sigma(A) \cap (-\infty, \lambda]\right)$$

for all real λ.■

Using 1.8.6 we can give the following definition:

1.8.8 Borel functional calculus For $\mathfrak{H}$, A and E as in 1.8.6 we define $f(A) \in \mathfrak{L}(\mathfrak{H})$ by

$$f(A) = \int f dE$$

for every $f \in B_\infty(\sigma(A))$.■

Note that for $A \geq 0$ in 1.8.8 we have $\sqrt{A} = A^{1/2}$, where $\sqrt{A}$ is given by 1.8.8 and $A^{1/2}$ by 1.1.2, as can easily be seen from 1.8.5.

1.8.9 Proposition ([M, p.72]) *Let $\mathfrak{H}$ and A be as in 1.8.6, and consider any $B \in \mathfrak{L}(\mathfrak{H})$ such that $AB = BA$ and $A^*B = BA^*$. Then*

$$f(A)B = Bf(A) \quad \text{for every } f \in B_\infty(\sigma(A)). \blacksquare$$

For a hermitian element A of a C*-algebra we write

$$A_+ = (|A| + A)/2.$$

Using 1.8.4, 1.8.6 and 1.8.7, and [K, 9.7-1, 9.8-3, 9.9-1 and 9.10-1] (a different formulation of the spectral theorem) one can show the following:

1.8.10 Proposition *Let $A \in \mathfrak{L}(\mathfrak{H})_s$ where $\mathfrak{H}$ is a Hilbert space, and let E be the resolution of the identity for A. Then E_λ is the orthogonal projection of $\mathfrak{H}$ on $\ker((A-\lambda)_+)$ for every $\lambda \in \mathbb{R}$.* $\blacksquare$

1.8.11 Corollary *If A in 1.8.10 is positive and injective, then*

$$\chi_{(0,+\infty)}(A) = 1.$$

Proof. By 1.8.10 $E_0 = 0$ since A is injective and positive. $\sigma(A) \subset \mathbb{R}$ since A is hermitian. So by 1.8.4 and 1.8.5 we have for all $x, y \in \mathfrak{H}$

$$\begin{aligned}
\langle x, y \rangle &= \left\langle x, \left(\int 1 dE \right) y \right\rangle = \int 1 dE_{x,y} \\
&= \int \chi_{(0,+\infty)} dE_{x,y} \quad (\text{since } E_{x,y}(\sigma(A) \cap (-\infty, 0]) = \langle x, E_0 y \rangle = 0) \\
&= \left\langle x, \left(\int \chi_{(0,+\infty)} dE \right) y \right\rangle.
\end{aligned}$$

We conclude that $1 = \int \chi_{(0,+\infty)} dE = \chi_{(0,+\infty)}(A)$, by 1.8.8. $\blacksquare$

We will now discuss at some length an example of the Borel functional calculus that will play a central role in our development of the Tomita-Takesaki theory in Chapter 4.

1.8.12 Example Define the function $f_z : \mathbb{R}^+ \to \mathbb{C}$ by

$$f_z(0) = 0, \quad \text{and} \quad f_z(\lambda) = \lambda^z \text{ if } \lambda > 0$$

for every $z \in \mathbb{C}$. For $\lambda > 0$, $f_z(\lambda) = e^{(\operatorname{Re} z)\ln\lambda} e^{i(\operatorname{Im} z)\ln\lambda}$. So f_z is continuous on $\mathbb{R}^+\backslash\{0\}$ and hence Borel measurable on $\mathbb{R}^+$. We also see that f_z is bounded on bounded subsets of $\mathbb{R}^+$ if $\operatorname{Re} z \geq 0$.

For $A \in \mathfrak{L}(\mathfrak{H})_+$ (with $\mathfrak{H}$ a Hilbert space), $\sigma(A) \subset \mathbb{R}^+$, so it follows from 1.8.8 that we can define

$$A^z := f_z(A) \in \mathfrak{L}(\mathfrak{H}) \quad \text{for } \operatorname{Re} z \geq 0$$

because $\sigma(A)$ is a bounded set since A is bounded. In particular $f_{1/2}(A)$ is the positive square root of A as noted after 1.8.8, hence the notation $A^{1/2}$ for both causes no confusion.■

1.8.13 Proposition *Let A be as in 1.8.12, and consider any $w, z \in \mathbb{C}$ with $\operatorname{Re} w \geq 0$ and $\operatorname{Re} z \geq 0$. Then:*
(1) $A^w A^z = A^{w+z}$.
(2) $(A^z)^* = A^{\bar{z}}$.
(3) $A^0 = 1$ *and* $A^1 = A$ *(where A^0 and A^1 are defined by 1.8.12).*

Proof. Let f be as in 1.8.12. Then by 1.8.8 and 1.8.5 we have

$$A^w A^z = f_w(A) f_z(A) = (f_w f_z)(A) = f_{w+z}(A) = A^{w+z}$$

and

$$(A^z)^* = (f_z(A))^* = \overline{f_z}(A) = f_{\bar{z}}(A) = A^{\bar{z}}.$$

From 1.8.11 we get $A^0 = f_0(A) = \chi_{(0,+\infty)}(A) = 1$. Clearly $f_1(\lambda) = \lambda$ for all λ, so $A^1 = f_1(A) = A$ by 1.8.8 and 1.8.6.■

It's straightforward to confirm that the Borel functional calculus for hermitian operators given in [S, 2.20] is the same as 1.8.8. Hence we can quote

1.8.14 Theorem ([S, 2.30]) *Let $A \in \mathfrak{L}(\mathfrak{H})_+$ (with $\mathfrak{H}$ a Hilbert space) and $x \in \mathfrak{H}$. Then*

$$z \mapsto A^z x \in \mathfrak{H}$$

is continuous on $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ and differentiable on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ (see 1.7.1).■

1.8.15 Proposition *Let $A \in \mathfrak{L}(\mathfrak{H})_+$ where $\mathfrak{H}$ is a Hilbert space, then*

$$\overline{\mathbb{C}^0_{-a}} \to \mathfrak{L}(\mathfrak{H}) : z \mapsto A^{iz}$$

is bounded in the norm of $\mathfrak{L}(\mathfrak{H})$ for every $a \in \mathbb{R}^+$.

Proof. Let f be as in 1.8.12, let E be the resolution of the identity for A, and set $s := \sup \sigma(A) + 1 \geq 1$. For $z \in \overline{\mathbb{C}^0_{-a}}$ and $\lambda \in (0, s]$, we have $|\lambda^{iz}| = |e^{iz \ln \lambda}| = e^{-(\operatorname{Im} z) \ln \lambda} = \lambda^{-\operatorname{Im} z} \leq s^{-\operatorname{Im} z}$. So $|f_{iz}(\lambda)| \leq s^{-\operatorname{Im} z} \leq s^a$ for $\lambda \in [0, s]$. It follows that $\|f_{iz}\|_\infty \leq s^a$, $\|f_{iz}\|_\infty$ being the sup-norm of f_{iz} on $[0, s]$. So for $x, y \in \mathfrak{H}$, we have

$$\begin{aligned}
|\langle x, A^{iz} y \rangle| &= \left| \int f_{iz} dE_{x,y} \right| \quad \text{(by 1.8.12, 1.8.8 and 1.8.4)} \\
&\leq \int |f_{iz}| \, d\, |E_{x,y}| \quad \text{(by [Ru, 1.33 and 6.18])} \\
&\leq \|f_{iz}\|_\infty \, |E_{x,y}| \, (\sigma(A)) \\
&\leq s^a \, \|E_{x,y}\| \\
&\leq s^a \, \|x\| \, \|y\| \quad \text{(by 1.8.2).}
\end{aligned}$$

For $x = A^{iz} y$, we obtain $\|A^{iz} y\|^2 \leq s^a \, \|A^{iz} y\| \, \|y\|$. This implies $\|A^{iz}\| \leq s^a$ by the definition of the norm of $\mathfrak{L}(\mathfrak{H})$. Since s does not depend on z, the result follows. ∎

1.8.16 Proposition *Let $A \in \mathfrak{L}(\mathfrak{H})_s$ where $\mathfrak{H}$ is a Hilbert space. Consider any $x \in \mathfrak{H}$ such that $Ax = x$, and let $f \in B_\infty(\sigma(A))$. Then*

$$f(A)x = f(1)x.$$

Proof. Clearly we may assume $x \neq 0$.

$(A - 1)x = 0$, therefore $A - 1 \notin \operatorname{Inv}(\mathfrak{L}(\mathfrak{H}))$ since $x \neq 0$. This means that $1 \in \sigma(A)$, so $f(1)$ is defined.

Using 1.8.8 we write $f(A) = \int f dE$.

We now prove the result in steps by considering special cases of f and building up to the most general case.

(a) Assume f is a polynomial, say $f(\lambda) = \sum_{n=1}^{N} a_n \lambda^n$ where $a_n \in \mathbb{C}$. Then

$$f(A)x = \left(\sum_{n=0}^{N} a_n A^n \right) x = \sum_{n=0}^{N} a_n A^n x = \sum_{n=0}^{N} a_n x = f(1)x$$

by 1.8.5 and 1.8.6.

(b) Assume f is continuous, then by the Stone-Weierstrass theorem there is a sequence of polynomials (p_n) such that $p_n \to f$ uniformly on $\sigma(A)$ (which is compact since A is bounded). Any $*$-homomorphism between two C*-algebras is norm decreasing ([M, Theorem 2.1.7]), so from 1.8.5 it follows that $\|(p_n - f)(A)\| \leq \|p_n - f\|_\infty \to 0$, hence $p_n(A)x \to f(A)x$. But $p_n(A)x = p_n(1)x$ by (a), and $p_n(1) \to f(1)$ since $p_n \to f$, so $f(A)x = f(1)x$.

(c) Assume $f = \chi_S$ for $S = \sigma(A) \cap I$ where I is any open interval in $\mathbb{R}$ (remember, $\sigma(A) \subset \mathbb{R}$ since A is hermitian). Clearly there is a sequence (g_n) of continuous real-valued functions on $\sigma(A)$ such that $g_n \to f$ pointwise and $0 \leq g_n \leq f$. So

$$\begin{aligned} \|g_n(A)x - f(A)x\|^2 &= \langle (g_n - f)(A)x, (g_n - f)(A)x \rangle \\ &= \langle x, (g_n - f)^2(A)x \rangle \quad \text{(by 1.8.5)} \\ &= \int (g_n - f)^2 dE_{x,x} \quad \text{(by 1.8.8 and 1.8.4)} \\ &\to 0 \end{aligned}$$

by Lebesgue's dominated convergence theorem and using the definition of integration with respect to a complex measure ([Ru, 1.34 and 6.18]). But $g_n(A)x = g_n(1)x$ by (b), and $g_n(1) \to f(1)$, so $f(A)x = f(1)x$.

(d) Let $\mathfrak{S}$ be the collection of all Borel sets S in $\sigma(A)$ such that $\chi_S(A)x = \chi_S(1)x$. We show that $\mathfrak{S}$ is a σ-algebra.

(d.i) By 1.8.5 (the $*$-homomorphism is unital) and the fact that $1 \in \sigma(A)$, we have $\chi_{\sigma(A)}(A)x = 1(A)x = 1x = \chi_{\sigma(A)}(1)x$. So $\sigma(A) \in \mathfrak{S}$.

(d.ii) Consider any $S \in \mathfrak{S}$. Let S^c be the compliment of S in $\sigma(A)$.

$$\begin{aligned} \chi_{S^c}(A)x &= (1 - \chi_S)(A)x = 1(A)x - \chi_S(A)x \quad \text{(by 1.8.8 and 1.8.5)} \\ &= 1x - \chi_S(1)x \quad \text{(by 1.8.5 and since } S \in \mathfrak{S}) \\ &= (1 - \chi_S)(1)x \\ &= \chi_{S^c}(1)x. \end{aligned}$$

Thus $S^c \in \mathfrak{S}$.

(diii) Consider any $S_1, S_2 \in \mathfrak{S}$.

$$\begin{aligned} \chi_{S_1 \cap S_2}(A)x &= (\chi_{S_1}\chi_{S_2})(A)x \quad \text{(by the definition of } \chi) \\ &= \chi_{S_1}(A)\chi_{S_2}(A)x \quad \text{(by 1.8.8 and 1.8.5)} \end{aligned}$$

$$\begin{aligned} &= \chi_{S_1}(1)\chi_{S_2}(1)x \quad (\text{since } S_1, S_2 \in \mathfrak{S}) \\ &= \chi_{S_1 \cap S_2}(1)x \end{aligned}$$

Thus $S_1 \cap S_2 \in \mathfrak{S}$.

(d.iv) Consider any $S_1, S_2 \in \mathfrak{S}$. $S_1 \cup S_2 = S_1 \cup (S_2 \backslash S_1)$ and $S_2 \backslash S_1 = S_2 \cap S_1^c \in \mathfrak{S}$ by (d.ii) and (d.iii). So

$$\begin{aligned} \chi_{S_1 \cup S_2}(A)x &= (\chi_{S_1} + \chi_{S_2 \backslash S_1})(A)x \quad (\text{since } S_1 \text{ and } S_2 \backslash S_1 \text{ are disjoint}) \\ &= \chi_{S_1}(A)x + \chi_{S_2 \backslash S_1}(A)x \quad (\text{by 1.8.8 and 1.8.5}) \\ &= \chi_{S_1}(1)x + \chi_{S_2 \backslash S_1}(1)x \quad (\text{since } S_1, S_2 \backslash S_1 \in \mathfrak{S}) \\ &= (\chi_{S_1} + \chi_{S_2 \backslash S_1})(1)x \\ &= \chi_{S_1 \cup S_2}(1)x. \end{aligned}$$

This means that $S_1 \cup S_2 \in \mathfrak{S}$. It follows that $S_1 \cup \ldots \cup S_n \in \mathfrak{S}$ for $S_1, \ldots, S_n \in \mathfrak{S}$.

(d.v) Let $S_n \in \mathfrak{S}$ for $n = 1, 2, 3, \ldots$ and set $S := \bigcup_{n=1}^{\infty} S_n$. Set $V_1 := S_1$ and $V_n := S_n \backslash \bigcup_{j=1}^{n-1} S_j = S_n \cap \left(\bigcup_{j=1}^{n-1} S_j \right)^c$ for $n > 1$, then $V_n \in \mathfrak{S}$ by (d.ii, iii and iv), $\bigcup_{n=1}^{\infty} V_n = S$, and $\chi_S = \sum_{n=1}^{\infty} \chi_{V_n}$ by the definition of χ since $V_1, V_2, V_3, \ldots$ are mutually disjoint. It follows for all $y \in \mathfrak{H}$ that

$$\begin{aligned} &\langle y, \chi_S(A)x \rangle \\ = &\int \chi_S dE_{y,x} = \int \left(\sum_{n=1}^{\infty} \chi_{V_n} \right) dE_{y,x} \\ = &\sum_{n=1}^{\infty} \int \chi_{V_n} dE_{y,x} \quad (\text{Lebesgue's dominated convergence theorem}) \\ = &\sum_{n=1}^{\infty} \langle y, \chi_{V_n}(A)x \rangle \\ = &\sum_{n=1}^{\infty} \langle y, \chi_{V_n}(1)x \rangle \quad (\text{since } V_n \in \mathfrak{S}) \\ = &\langle y, \chi_S(1)x \rangle \quad \left(\text{since } \chi_S = \sum_{n=1}^{\infty} \chi_{V_n}\right). \end{aligned}$$

So $\chi_S(A)x = \chi_S(1)x$, in other words $S \in \mathfrak{S}$.

(d.vi) According to (d.i, ii and v) $\mathfrak{S}$ is a σ-algebra. By (c) $\mathfrak{S}$ contains all sets $\sigma(A) \cap I$ with I and open interval in $\mathbb{R}$, but every open set in $\mathbb{R}$ is a countable union of open intervals ([B, 9.11]), hence $\mathfrak{S}$ contains all open sets of the topological space $\sigma(A) \subset \mathbb{R}$ since $\mathfrak{S}$ is a σ-algebra. This implies that $\mathfrak{S}$ contains all the Borel sets of $\sigma(A)$ since by definition the collection of Borel sets is the smallest σ-algebra containing all the open sets. Thus

$$\chi_S(A)x = \chi_S(1)x \quad \text{for all Borel sets } S \subset \sigma(A).$$

(e) Assume f is positive. Then there is a sequence (s_n) of simple functions on $\sigma(A)$ such that $s_n \to f$ pointwise and $0 \leq s_n \leq f$, ([Ru, 1.17]). Using exactly the same argument as in (c) we see that $s_n(A)x \to f(A)x$ (just replace g_n in (c) by s_n). But from (d) and 1.8.5 we deduce that $s_n(A)x = s_n(1)x$, since, by definition of a simple funtion, s_n is a linear combination of characteristic functions. Since $s_n(1) \to f(1)$ we conclude that $f(A)x = f(1)x$.

(f) Now we consider the general case for f. Then we can write

$$f = f_1 - f_2 + if_3 - if_4$$

where $f_1, f_2, f_3, f_4 \in B_\infty(\sigma(A))_+$, ([Ru, 1.9(b) and 1.14(b)]). By (e) and 1.8.5 we conclude $f(A)x = f(1)x$.■

1.9 A functional calculus for finite dimensions

We now present a functional calculus that is much simpler than the Borel functional calculus of 1.8. It will turn out to be very useful in Chapter 2 where we work mostly with finite dimensional Hilbert spaces.

Throughout this section $\mathfrak{H}$ is a finite dimensional Hilbert space.

1.9.1 Definition For $x, y \in \mathfrak{H}$ we define $x \otimes y \in \mathfrak{L}(\mathfrak{H})$ by

$$(x \otimes y)z = x \langle y, z \rangle .■$$

Clearly we have the following two propositions:

1.9.2 Proposition $(v \otimes w)(x \otimes y) = \langle w, x \rangle v \otimes y$.■

1.9.3 Proposition *Let $b_1, ..., b_N$ be an orthonormal basis for $\mathfrak{H}$, then $\{b_m \otimes b_n : m, n \in [N]\}$ is a basis for $\mathfrak{L}(\mathfrak{H})$. In particular,* $\dim(\mathfrak{L}(\mathfrak{H})) = N^2$.■

The following result is known from linear algebra:

1.9.4 Theorem *Let $A \in \mathfrak{L}(\mathfrak{H})_s$. Then there exists an orthonormal basis $b_1, ..., b_N$ of $\mathfrak{H}$ consisting of eigenvectors of A. Let λ_n be the eigenvalue of A corresponding to b_n. Then $\lambda_1, ..., \lambda_N$ are all the eigenvalues of A, and $\lambda_1, ..., \lambda_N \in \mathbb{R}$. Furthermore,*

$$A = \sum_{n=1}^{N} \lambda_n b_n \otimes b_n.■$$

1.9.4 allows us to give the following:

1.9.5 Definition(Functional calculus) Consider any $A \in \mathfrak{L}(\mathfrak{H})_s$ and let $b_1, ..., b_N$ and $\lambda_1, ..., \lambda_N$ be as in 1.9.4. Let $f : X \to \mathbb{C}$ be any function such that $\lambda_1, ..., \lambda_N \in X \subset \mathbb{R}$. Then we define

$$f(A) = \sum_{n=1}^{N} f(\lambda_n) b_n \otimes b_n.■$$

It is easily seen that $f(A)$ is independent of the choice of $b_1, ..., b_N$ in 1.9.5 by considering the eigenspace of each λ_n separately and using the fact that $\left(\sum_{n=1}^{N} f(\lambda_n) b_n \otimes b_n\right) b_m = f(\lambda_m) b_m$ by 1.9.1.

From 1.9.2 and 1.9.5 we immediately see

1.9.6 Proposition *Let $A \in \mathfrak{L}(\mathfrak{H})_s$, and let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be functions such that $f(A)$ and $g(A)$ are defined (by 1.9.5). Then*

$$f(A)g(A) = g(A)f(A).■$$

1.9.7 Proposition *For A and f as in 1.9.5 we have:*
(1) $(f(A))^* = \overline{f}(A)$.
(2) *If $1/f(x)$ exists in $\mathbb{C}$ for every $x \in X$, then $(f(A))^{-1} = (1/f)(A)$.*
(3) *If $f \geq 0$, then $f(A) \geq 0$.*

Proof. (1) Let $b_1, ..., b_N$ and $\lambda_1, ..., \lambda_N$ be as in 1.9.5.

$$\begin{aligned}\langle b_l, f(A)b_m\rangle &= \langle b_l, f(\lambda_m)b_m\rangle = f(\lambda_m)\langle b_l, b_m\rangle = f(\lambda_l)\langle b_l, b_m\rangle = \langle \overline{f}(\lambda_l)b_l, b_m\rangle \\ &= \langle \overline{f}(A)b_l, b_m\rangle\end{aligned}$$

because $b_1, ..., b_N$ are orthonormal. It follows that $\overline{f}(A) = (f(A))^*$.

(2) From 1.9.5 and 1.9.2 we have

$$\begin{aligned}f(A)(1/f)(A) &= \sum_{k=1}^{N}\sum_{n=1}^{N} f(\lambda_k)\,(1/f(\lambda_n))\,(b_k \otimes b_k)(b_n \otimes b_n) \\ &= \sum_{k=1}^{N} f(\lambda_k)\,(1/f(\lambda_k))\, b_k \otimes b_k \\ &= \sum_{k=1}^{N} b_k \otimes b_k \\ &= 1\end{aligned}$$

and so, using 1.9.6, $(f(A))^{-1} = (1/f)(A)$.

(3) $(f(A))^* = f(A)$ by (1) since $f \geq 0$, and from 1.9.5 it is clear that $f(A)$'s spectrum is $\{f(\lambda_1), ..., f(\lambda_N)\} \subset \mathbb{R}^+$. By definition this implies that $f(A) \geq 0$.■

The following proposition is clear:

1.9.8 Proposition *Let A, $b_1, ..., b_N$ and $\lambda_1, ..., \lambda_N$ be as in 1.9.4. Consider a function $f : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$. By 1.9.5 we have the function $f(A, \cdot) : \mathbb{C} \to \mathfrak{L}(\mathfrak{H})$. For each $z \in \mathbb{C}$ such that $\frac{d}{dz}f(\lambda_1, z), ..., \frac{d}{dz}f(\lambda_N, z)$ exist, we have*

$$\frac{d}{dz}f(A, z) = \sum_{n=1}^{N}\left(\frac{d}{dz}f(\lambda_n, z)\right) b_n \otimes b_n$$

(see 1.7.1).■

1.9.9 Example Using 1.9.5 and 1.9.8 it is seen that

$$\frac{d}{dz}e^{iAz} = iAe^{iAz} \text{ for all } A \in \mathfrak{L}(\mathfrak{H})_s \text{ and } z \in \mathbb{C}.$$■

1.10 Tensor products of Hilbert spaces

This section is adapted from [C, II.F.2.a] and [D, V.1.01 and 1.03].

1.10.1 Definition Let $X_1, ..., X_N, Y$ be vector spaces. A function

$$f : X_1 \times ... \times X_N \to Y$$

is called a *conjugate multilinear form* if

$$f(x_1, ..., \alpha x_k + \beta y_k, ...x_N) = \overline{\alpha} f(x_1, ..., x_N) + \overline{\beta} f(x_1, ..., y_k, ..., x_N)$$

for all $x_n \in X_n$, $y_k \in X_k$, $k \in [N]$ and $\alpha, \beta \in \mathbb{C}$. The vector space of all such forms is denoted by $\mathfrak{C}(X_1, ..., X_N, Y)$.■

1.10.2 Definition Let $\mathfrak{H}_1, ..., \mathfrak{H}_N$ be inner product spaces. For $x_k \in \mathfrak{H}_k$ we define

$$x_1 \otimes ... \otimes x_N \in \mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$$

by

$$x_1 \otimes ... \otimes x_N(y_1, ..., y_N) = \langle y_1, x_1 \rangle ... \langle y_N, x_N \rangle .■$$

1.10.2 uses the same notation as 1.9.1, but the former will be used only in 2.1 while the latter will only be used later on, so there won't be any confusion.

1.10.3 Proposition *Let $\mathfrak{H}_1, ..., \mathfrak{H}_N$ be finite dimensional Hilbert spaces and set $d(k) := \dim(\mathfrak{H}_k) > 0$. Let $b_{k,1}, ..., b_{k,d(k)}$ be an orthonormal basis for $\mathfrak{H}_k$. Then*

$$B := \{b_{1,l_1} \otimes ... \otimes b_{N,l_N} : l_k \in [d(k)]\}$$

is a basis for $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$. In particular,

$$\dim(\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})) = d(1)...d(N).$$

Proof. Let

$$\sum_{l_1=1}^{d(1)} ... \sum_{l_N=1}^{d(N)} \alpha_{l_1,...,l_N} b_{1,l_1} \otimes ... \otimes b_{N,l_N} = 0 \text{ where } \alpha_{l_1,...,l_N} \in \mathbb{C}.$$

For $l'_k \in [d(k)]$ it now follows

$$\begin{aligned} 0 &= \sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} \alpha_{l_1,\dots,l_N} b_{1,l_1} \otimes \dots \otimes b_{N,l_N}(b_{1,l'_1},\dots,b_{N,l'_N}) \\ &= \sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} \alpha_{l_1,\dots,l_N} \langle b_{1,l'_1}, b_{1,l_1}\rangle \dots \langle b_{N,l'_N}, b_{N,l_N}\rangle \quad \text{(by 1.10.2)} \\ &= \alpha_{l'_1,\dots,l'_N} \quad \text{(since } b_{k,1},\dots,b_{k,d(k)} \text{ are orthonormal).} \end{aligned}$$

This means B is linearly independent.

Now consider any $f \in \mathfrak{C}(\mathfrak{H}_1,\dots,\mathfrak{H}_N,\mathbb{C})$. Set

$$\alpha_{l_1,\dots,l_N} := f(b_{1,l_1},\dots,b_{N,l_N}) \quad \text{for } l_k = 1,\dots,d(k) \text{ and } k = 1,\dots,N. \tag{1}$$

For any $x_k \in \mathfrak{H}_k$ we then have

$$\begin{aligned} &\left(\sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} \alpha_{l_1,\dots,l_N} b_{1,l_1} \otimes \dots \otimes b_{N,l_N}\right)(x_1,\dots,x_N) \\ &= \sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} \alpha_{l_1,\dots,l_N} \langle x_1, b_{1,l_1}\rangle \dots \langle x_N, b_{N,l_N}\rangle \quad \text{(by 1.10.2)} \\ &= \sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} f\left(\langle b_{1,l_1}, x_1\rangle b_{1,l_1},\dots,\langle b_{N,l_N}, x_N\rangle b_{N,l_N}\right) \quad \text{(by (1) and 1.10.1)} \\ &= f\left(\sum_{l_1=1}^{d(1)} \langle b_{1,l_1}, x_1\rangle b_{1,l_1},\dots,\sum_{l_N=1}^{d(N)} \langle b_{N,l_N}, x_N\rangle b_{N,l_N}\right) \\ &= f(x_1,\dots,x_N) \quad \text{(since } b_{k,1},\dots,b_{k,d(k)} \text{ is an orthonormal basis for } \mathfrak{H}_k). \end{aligned}$$

Thus

$$f = \sum_{l_1=1}^{d(1)} \dots \sum_{l_N=1}^{d(N)} \alpha_{l_1,\dots,l_N} b_{1,l_1} \otimes \dots \otimes b_{N,l_N}.$$

This means that B is a basis for $\mathfrak{C}(\mathfrak{H}_1,\dots,\mathfrak{H}_N,\mathbb{C})$, and so

$$\dim(\mathfrak{C}(\mathfrak{H}_1,\dots,\mathfrak{H}_N,\mathbb{C})) = |B| = d(1)\dots d(N).\blacksquare$$

Because of 1.10.3 we can give the following definition:

1.10.4 Definition Consider the situation described in 1.10.3. We define an inner product on $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ by using the basis given in 1.10.3:

$$\langle b_{1,l_1} \otimes ... \otimes b_{N,l_N}, b_{1,m_1} \otimes ... \otimes b_{N,m_N} \rangle = \langle b_{1,l_1}, b_{1,m_1} \rangle ... \langle b_{N,l_N}, b_{N,m_N} \rangle .$$ ■

1.10.5 Proposition *Consider the situation in 1.10.3. The inner product defined by 1.10.4 is the unique inner product on* $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ *such that*

$$\langle x_1 \otimes ... \otimes x_N, y_1 \otimes ... \otimes y_N \rangle = \langle x_1, y_1 \rangle ... \langle x_N, y_N \rangle$$

for all $x_k, y_k \in \mathfrak{H}_k$. *The basis* B *for* $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ *given by 1.10.3 is orthonormal in this inner product.*

Proof. That the basis given in 1.10.3 is orthonormal is clear from 1.10.4.

For $x_k, y_k \in \mathfrak{H}_k$ we can write $x_k = \sum_{l_k=1}^{d(k)} \alpha_{l_k} b_{k,l_k}$ and $y_k = \sum_{l_k=1}^{d(k)} \beta_{l_k} b_{k,l_k}$ $(\alpha_{l_k}, \beta_{l_k} \in \mathbb{C})$. Hence by 1.10.4

$$\begin{aligned}
&\langle x_1 \otimes ... \otimes x_N, y_1 \otimes ... \otimes y_N \rangle \\
&= \left\langle \sum_{l_1=1}^{d(1)} ... \sum_{l_N=1}^{d(N)} \alpha_{l_1} ... \alpha_{l_N} b_{1,l_1} \otimes ... \otimes b_{N,l_N}, \sum_{l'_1=1}^{d(1)} ... \sum_{l'_N=1}^{d(N)} \beta_{l'_1} ... \beta_{l'_N} b_{1,l'_1} \otimes ... \otimes b_{N,l'_N} \right\rangle \\
&= \sum_{l_1=1}^{d(1)} ... \sum_{l_N=1}^{d(N)} \sum_{l'_1=1}^{d(1)} ... \sum_{l'_N=1}^{d(N)} \overline{\alpha_{l_1}} ... \overline{\alpha_{l_N}} \beta_{l'_1} ... \beta_{l'_N} \langle b_{1,l_1}, b_{1,l'_1} \rangle ... \langle b_{N,l_N}, b_{N,l'_N} \rangle \\
&= \left\langle \sum_{l_1=1}^{d(1)} \alpha_{l_1} b_{1,l_1}, \sum_{l'_1=1}^{d(1)} \beta_{l'_1} b_{1,l'_1} \right\rangle ... \left\langle \sum_{l_N=1}^{d(N)} \alpha_{l_N} b_{N,l_N}, \sum_{l'_N=1}^{d(N)} \beta_{l'_N} b_{N,l'_N} \right\rangle \\
&= \langle x_1, y_1 \rangle ... \langle x_N, y_N \rangle .
\end{aligned}$$

Let $\langle \cdot, \cdot \rangle'$ be any inner product on $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ such that

$$\langle x_1 \otimes ... \otimes x_N, y_1 \otimes ... \otimes y_N \rangle' = \langle x_1, y_1 \rangle ... \langle x_N, y_N \rangle$$

for all $x_k, y_k \in \mathfrak{H}_k$. Then it is clear from 1.10.4 that $\langle x, y \rangle' = \langle x, y \rangle$ for all $x, y \in B$, hence $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle$ on $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ which proves uniqueness. ■

1.10.5 enables us to give the following definition which is independent of any particular basis:

1.10.6 Definition Let $\mathfrak{H}_1, ..., \mathfrak{H}_N$ be finite dimensional Hilbert spaces and consider the unique inner product on $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ such that

$$\langle x_1 \otimes ... \otimes x_N, y_1 \otimes ... \otimes y_N \rangle = \langle x_1, y_1 \rangle ... \langle x_N, y_N \rangle$$

for all $x_k, y_k \in \mathfrak{H}_k$. $\mathfrak{C}(\mathfrak{H}_1, ..., \mathfrak{H}_N, \mathbb{C})$ with this inner product is called the *tensor product of* $\mathfrak{H}_1, ..., \mathfrak{H}_N$ and it is denoted by $\mathfrak{H}_1 \otimes ... \otimes \mathfrak{H}_N$. If $\mathfrak{H}_1 = ... = \mathfrak{H}_N = \mathfrak{H}$ we write

$$\mathfrak{H}_1 \otimes ... \otimes \mathfrak{H}_N = \bigotimes_N \mathfrak{H}. \blacksquare$$

The inner product space $\mathfrak{H}_1 \otimes ... \otimes \mathfrak{H}_N$ in 1.10.6 is a Hilbert space since it is a finite dimensional by 1.10.3.

Finally we note that

$$\boxtimes : \mathfrak{H}_1 \times ... \times \mathfrak{H}_N \to \mathfrak{H}_1 \otimes ... \otimes \mathfrak{H}_N : (x_1, ..., x_N) \mapsto x_1 \otimes ... \otimes x_N$$

is multilinear by 1.10.2, and that's why we define $x_1 \otimes ... \otimes x_N$ to be a conjugate multilinear form. If $x_1 \otimes ... \otimes x_N$ was multilinear, $\boxtimes$ would be conjugate multilinear. In 2.1 we will use tensor products to set up a simple quantum mechanical model, and there the linearity of $\boxtimes$ will be seen to fit naturally into the physical ideas.

Chapter 2

Quantum statistical mechanics

In this chapter we will give the postulates of quantum mechanics in the case of mathematically simple physical systems (meaning the Hilbert spaces involved are finite dimensional). A similar but more general development can be found in [Bo, Chapter II] where the postulates are discussed using the harmonic oscillator as an example. We will consider a model consisting of many particles in a magnetic field because this requires only finite dimensional Hilbert spaces while the harmonic oscillator requires infinite dimensional Hilbert spaces. Systems consisting of many particles are also the whole point of quantum *statistical* mechanics.

2.1 A simple quantum mechanical model

2.1.1 A spin 1/2 particle in a magnetic field

In nature there are particles, called spin 1/2 particles, that have the following property: If you put such a particle in a magnetic field and measure its spin, you can get only two values called *up* and *down* respectively (or 1/2 and −1/2). We view the particle as a system, and the magnetic field as an external condition applied to the system. If the particle's spin is measured as up, we say the system is in the *up state*. If the spin is measured as down, we say the system is in the *down state*.

In terms of classical physics we can describe it as follows: We view the spin 1/2 particle as a little magnet. Measuring the spin of the particle is analogous to measuring the direction in which the little magnet is pointing. The fact that there are only two possible values for the spin means that the little

magnet can only point in two directions, namely parallel and antiparallel to the magnetic field. This is a purely quantum mechanical effect and it can not be explained by classical physics. According to classical physics the magnet can point in *any* direction. A simple example of this is a magnetic compass used to find the direction of the earth's magnetic north pole. The needle of a compass is a magnet, and if you shake the compass the needle will start to oscillate around the direction of the earth's magnetic north, so the needle is pointing in different directions as time goes on. If you were to build a compass using a spin half particle as the needle, the needle would only be able to switch in a binary fashion between pointing north and pointing south when you shake the compass. There will not be any oscillations.

Examples of spin 1/2 particles are electrons, protons and neutrons ([Su, appendix II]), and silver atoms in their ground state ([C, p.392]).

The up and down states of a system consisting of a spin 1/2 particle in a magnetic field are called the spin eigenstates of the system. The quantum mechanical description of the physical states of this system (ignoring all aspects of the particle other than the spin) consists of any two dimensional Hilbert space (called the *state space* of the system) and two non-zero orthogonal vectors in the space. The one vector represents the up state, the other the down state. For simplicity we choose $\mathbb{C}^2 := \mathbb{C} \times \mathbb{C}$ with its usual inner product as the Hilbert space, $u = (1, 0)$ as the vector representing the up state, and $d = (0, 1)$ as the vector representing the down state. It is important to notice that u, d forms an orthonormal basis for $\mathbb{C}^2$.

Every non-zero element of $\mathbb{C}^2$ describes a physical state of the system (i.e. a state that can occur in nature), and every physical state of the system is completely described by a non-zero element of $\mathbb{C}^2$. Hence we call the non-zero elements of $\mathbb{C}^2$ *state vectors*. Two non-zero linearly dependent elements of $\mathbb{C}^2$ describe the same state of the system, therefore we can without loss of generality use vectors of norm one to describe the states of the system.

The physical meaning of all this is as follows: Let the state of the system be described by $v \in \mathbb{C}^2$. We may assume $\|v\| = 1$ as just mentioned. Write $v = \lambda u + \mu d$ where $\lambda, \mu \in \mathbb{C}$. If we now measure the spin of the particle, the probability of getting spin up is

$$|\langle u, v \rangle|^2 = |\lambda|^2 \tag{1}$$

and the probability of getting spin down is

$$|\langle d, v \rangle|^2 = |\mu|^2 . \tag{2}$$

Note that $|\lambda|^2 + |\mu|^2 = 1$ because $\|v\| = 1$, this is why we use normalized elements of $\mathbb{C}^2$. (It is important to understand that (1) and (2) cannot be derived, these probabilities are part of the postulates of quantum mechanics which we will discuss in more detail in succeeding sections.)

This illustrates the probabilistic nature of quantum mechanics. It also illustrates the so-called *superposition principle*, namely $v = \lambda u + \mu d$ is a superposition of u and d. Any linear combination of state vectors is called a *superposition*, and if this superposition is non-zero it is of course itself a state vector (i.e. it describes some physical state of the system).

Measuring the spin of the particle forces the system into one of the spin eigenstates, no matter what state the system was in before. If our measurement gives spin up, then the system is forced into the up state. Measuring the spin a second time immediately afterwards (before the physical state of the system has time to change) then yields spin up with probability 1. Similarly for the down state.

To complete our quantum mechanical description of the system we still have to consider its energy. We will assume the particle has a fixed position (hence it has no kinetic energy) and that it has no internal energy. We are only going to consider the energy of the system because of the particle's spin.

If the particle is in the up state, it has energy $E > 0$, say, and if it is in the down state, it has energy $-E$. E depends on the strength of the magnetic field. When we measure the system's energy, these values (namely E and $-E$) turn out to be the only ones we can get (similar to the two spin values $1/2$ and $-1/2$). If we get the value E, the system is forced into the up state, if we get $-E$, the system is forced into the down state (again similar to the measurement of the spin). We say that u and d are the energy eigenstates of the system, and E and $-E$ the corresponding energy eigenvalues. If the system is in the state $v \in \mathbb{C}^2$ where $\|v\| = 1$, then the probability of getting E is given by (1), and the probability of getting $-E$ is given by (2), exactly as for the spin eigenvalues.

Classically this can be understood as follows: In the up state the particle is "against" the magnetic field. This analogous to the north end of a compass needle pointing south. The earth's magnetic field will exert a turning force on the needle to get the north end of the needle to point north. Hence the needle has a high potential energy, corresponding to the spin energy E of the particle. In the down state the particle is "with" the magnetic field. This is analogous to the north end of the compass needle pointing north. In this case the earth's magnetic field exerts no turning force on the needle, hence

it has a low potential energy, corresponding to the spin energy $-E$ of the particle.

We can define a linear operator $H \in \mathfrak{L}(\mathbb{C}^2)$ by $Hu = Eu$ and $Hd = -Ed$. Then u and d are the eigenvectors of H with corresponding eigenvalues E and $-E$. Clearly H is hermitian because $E \in \mathbb{R}$. H is called the *Hamiltonian* of the system.

The pair $(\mathbb{C}^2, H)$ gives a complete quantum mechanical description of the system since it contains all the information on the possible physical states of the system (namely the non-zero elements of $\mathbb{C}^2$) and the energy of the system (namely the energy eigenvalues and the energy eigenstates which are just the eigenvalues and eigenvectors of H).

See [C, Chapter IV Sections A and B] for a more complete description of a spin 1/2 particle and its energy in a magnetic field.

2.1.2 Many spin 1/2 particles in a magnetic field

Consider N identical spin 1/2 particles in a magnetic field. Each particle has a fixed position in space. For the moment the spatial arrangement of the particles is not important. We also assume that the magnetic field is the same at each particle, and that the particles do not interact in any way (i.e. they do not exert forces on each other). Similar to 2.1.1, we view the particles as a system, and the magnetic field as an external condition applied to the system.

As in 2.1.1, we want to find a quantum mechanical description of this system, in other words a Hilbert space $\mathfrak{H}$ whose non-zero elements represent the physical states of the system (the state space), and an $H \in \mathfrak{L}(\mathfrak{H})_s$ whose eigenvalues are the energies the system can have (the Hamiltonian).

Measuring the energy of the system is the same as measuring the energy of each particle on its own and then adding all these energies together, because the particles do not interact. We already know from 2.1.1 that if we measure the energy of a single spin 1/2 particle we get $E > 0$ (corresponding to the up state) or $-E$ (corresponding to the down state). These energies are the same for each particle since the magnetic field is the same at each particle and the particles are identical. So, when we measure the system's energy we will find some of the particles in the up state and some in the down state, and we just have to add the corresponding E's and $-E$'s together.

Look at the special case where $N = 2$. The possible results of measuring the energy of the system are as follows:

(u, u) with energy $E + E = 2E$

(u,d) with energy $E-E=0$
(d,u) with energy $-E+E=0$
(d,d) with energy $-E-E=-2E$

where u represents the up state and d the down state for a single particle. For example (u,d) means the first particle is in the up state and the second in the down state. We thus get four energy eigenstates, namely $(u,u),(u,d),(d,u),(d,d)$ with corresponding energy eigenvalues $2E,0,0,-2E$.

In the same way we find that for N particles the energy eigenstates are represented by all strings of u's and d's of length N. These states are the eigenvectors of the Hamiltonian of the system and they form a basis for the state space of the system (exactly as for the case of a single spin 1/2 particle described in 2.1.1).

As in 2.1.1 we take the state space of a single particle as the Hilbert space $\mathbb{C}^2$ (with its usual inner product), but in order to simplify the expression for the Hamiltonian later on, we denote the up state by $s(1)=(1,0)$ and the down state by $s(-1)=(0,1)$.

The tensor products described in 1.10 now form a natural setting for our model. Look again at the case $N=2$. The strings $(u,u),(u,d),(d,u),(d,d)$ (representing the energy eigenstates which form a basis for the state space) can be represented by $s(1)\otimes s(1), s(1)\otimes s(-1), s(-1)\otimes s(1), s(-1)\otimes s(-1)$ respectively. The latter forms an orthonormal basis for $\mathbb{C}^2\otimes\mathbb{C}^2$ by 1.10.3 and 1.10.5. So $\mathbb{C}^2\otimes\mathbb{C}^2$ can be taken as the state space for two particles.

Similarly in the general case, the state space for N particles can be taken as the Hilbert space $\mathfrak{H}:=\bigotimes_N\mathbb{C}^2$, and the energy eigenstates are

$$B:=\{s(l_1)\otimes\ldots\otimes s(l_N):l_1,\ldots,l_N\in\{-1,1\}\}$$

which is an orthonormal basis for $\mathfrak{H}$ by 1.10.3 and 1.10.5. Defining the Hamiltonian $H\in\mathfrak{L}(\mathfrak{H})$ of the system is now easy:

$$Hs(l_1)\otimes\ldots\otimes s(l_N):=(l_1+\ldots+l_N)Es(l_1)\otimes\ldots\otimes s(l_N) \tag{1}$$

because when the system is in the state $s(l_1)\otimes\ldots\otimes s(l_N)$, the k'th particle is in the state $s(l_k)$ and so has energy l_kE. It is easy to see H is hermitian using the facts that $E\in\mathbb{R}$ and that B is orthonormal.

And so we have a complete quantum mechanical description of N identical spin 1/2 particles in a homogeneous magnetic field.

Note that we interpret the state space of the system as follows: If the k'th particle is in the state v_k, then the system is in the state $v_1\otimes\ldots\otimes v_N$.

If one of these particles, say the n'th, is in the superposition $v_n = \lambda x + \mu y$ where $x, y \in \mathbb{C}^2$ and $\lambda, \mu \in \mathbb{C}$, then by the multilinearity of $\boxtimes$ mentioned at the end of 1.10 we have

$$v_1 \otimes ... \otimes v_N = \lambda v_1 \otimes ... \otimes x \otimes ... \otimes v_N + \mu v_1 \otimes ... \otimes y \otimes ... \otimes v_N$$

where x and y are in the n'th position in both terms. So the superposition of the single particle carries over to the whole system, which seems very natural physically.

It is also interesting to note that not all the states of the system are of the form $v_1 \otimes ... \otimes v_N$. Consider the case $N = 2$ and the state $s(1) \otimes s(-1) + s(-1) \otimes s(1)$ of the system (this is indeed a state of the system because it is non-zero since B is a basis). Say

$$s(1) \otimes s(-1) + s(-1) \otimes s(1) = v \otimes w \tag{2}$$

for some $v, w \in \mathbb{C}^2$. Since $s(1), s(-1)$ is a basis for $\mathbb{C}^2$ we can write $v = \alpha s(1) + \lambda s(-1)$ and $w = \gamma s(1) + \mu s(-1)$ for some $\alpha, \lambda, \gamma, \mu \in \mathbb{C}$. Then by the multilinearity of $\boxtimes$ we have

$$v \otimes w = \alpha\gamma s(1) \otimes s(1) + \alpha\mu s(1) \otimes s(-1) + \lambda\gamma s(-1) \otimes s(1) + \lambda\mu s(-1) \otimes s(-1).$$

So by (2) and the fact that $s(1) \otimes s(1), s(1) \otimes s(-1), s(-1) \otimes s(1), s(-1) \otimes s(-1)$ is a basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ we see that $\alpha\gamma = 0, \alpha\mu = 1, \lambda\gamma = 1$ and $\lambda\mu = 0$. The second of these implies $\alpha \neq 0$, hence the first implies $\gamma = 0$ which contradicts the third. This means (2) is impossible, hence not all states of the system are of the form $v_1 \otimes v_2$. Physically this means that the state of the system can *not* be specified by simply giving the state of each particle separately, even though the particles do not interact in any way. This is another instance where quantum mechanics deviates from classical mechanics and our everyday experience. As illustrated using (2), this is essentially a consequence of the superposition principle. It therefore seems that it is inherent to the the structure of quantum mechanics that a system is described as a whole. This makes perfect sense if one considers that the quantum mechanical descriptions we gave for a single spin 1/2 particle and a collection of many spin 1/2 particles in a magnetic field are exactly the same, namely both consist of a state space (a Hilbert space) and a Hamiltonian (a hermitian operator on the state space). This will become even clearer when we present the postulates of quantum mechanics in the succeeding sections.

Finally we point out that the model presented in this section can be made more realistic by bringing in an interaction (i.e. some sort of force) between the particles. Now the spatial arrangement of the particles becomes important (for example, we expect the strength of the interaction between two particles to depend on the distance between them). If we assume the interactions are weak enough that the magnetic field at each particle is not altered significantly, then the state space is still $\bigotimes_N \mathbb{C}^2$ since each particle still has the two states up and down which are not affected by the other particles. A simple model of this type is where we have the N particles equally spaced in a straight line and we assume only adjacent particles interact. This interaction has to be built into the Hamiltonian of the system. For example we might assume that if two adjacent particles have the same spin, their interaction gives an energy ε, while if they have opposite spin their interaction gives an energy $-\varepsilon$. These energies then have to be added to the Hamiltonian, in other words we will have to modify (1). This is called a one dimensional *Ising* model. For more details on Ising models refer to [P, 12.5].

In the rest of this chapter we formalize and extend the ideas of this section.

2.2 Observables and states

In 2.1 we described a quantum mechanical model in terms of a finite dimensional Hilbert space (the state space) and a hermitian operator (the Hamiltonian) on this space. This inspires the next definition:

2.2.1 Definition A *finite dimensional quantum system* (abbreviated *fdqs*) is a pair $(\mathfrak{H}, H)$, where $\mathfrak{H}$ is a finite dimensional Hilbert space called the *state space* of the system, and $H \in \mathfrak{L}(\mathfrak{H})_s$ is called the *Hamiltonian* of the system.■

In this chapter we restrict our discussion of quantum mechanics to physical systems that can be described by fdqs's, because of their mathematical simplicity. In 4.5 we will outline a generalization of some of these ideas.

In 2.1 we saw that the eigenvalues of the Hamiltonian are the energies the system can have. So the operator H in 2.2.1 represents the energy of the system. The energy of a system is not necessarily the only measurable quantity associated with the system (witness the spin in 2.1.1). It turns out that the following postulate regarding measurable quantities (of which energy is an example) and the physical state of a system holds:

2.2.2 Postulate I Consider a fdqs $(\mathfrak{H}, H)$. The physical state of this system is completely determined (or described or represented) by a non-zero element x of $\mathfrak{H}$; conversely, every non-zero element of $\mathfrak{H}$ determines some physical state of the system. Any two non-zero linearly dependent elements of $\mathfrak{H}$ determine the same physical state of the system, hence we can completely describe any physical state of the system by a $x \in \mathfrak{H}$ with $\|x\| = 1$.

Every attribute of the system that can be physically measured to yield a real number as result, is called an *observable* of the system. Every observable of the system is represented by an $A \in \mathfrak{L}(\mathfrak{H})_s$ in the following sense: The eigenvalues of A (which are real since A is hermitian, see 1.9.4) are the values we can get when measuring the observable. If $a_1, ..., a_N$ is an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of A (see 1.9.4) with corresponding eigenvalues $\alpha_1, ..., \alpha_N$, and we write $x = c_1 a_1 + ... + c_N a_N$, $c_1, ..., c_N \in \mathbb{C}$, where $x \in \mathfrak{H}$ with $\|x\| = 1$ is the state of the system, then the probability of getting the value α_k when measuring the observable is

$$\|P_k x\|^2 = \sum_{n=1}^{N} |c_n|^2 \, \delta(\alpha_k, \alpha_n)$$

where P_k is the orthogonal projection of $\mathfrak{H}$ on the eigenspace of α_k. (This is a generalization of the probabilistic interpretation given in 2.1.1.) If this measurement yields the value α_k, then the physical state of the system immediately after the measurement is represented by $P_k x$.

Conversely, every $A \in \mathfrak{L}(\mathfrak{H})_s$ represents an observable of the system in the way described above. For this reason we also refer to the elements of $\mathfrak{L}(\mathfrak{H})_s$ as the observables of the system.■

So we can call the Hamiltonian the energy observable.

We mention that in more advanced quantum mechanics, $\mathfrak{L}(\mathfrak{H})$ in 2.2.2 is replaced by more general $*$-algebras. It turns out that these algebras are the important objects in a rigorous mathematical study of quantum mechanics. The Hilbert space acts merely as a "carrier" of the algebra. As we develop the theory further it will be clear that we work with the C*-algebra $\mathfrak{L}(\mathfrak{H})$ rather than $\mathfrak{H}$ itself. Even the physical state of the system can be described without refering to the Hilbert space as is done in 2.2.2. For this reason

2.2.2 should be seen as a preliminary postulate to help us set up the theory of quantum mechanics in such a way that it fits into the theory of operator algebras. The first step towards this is the concept of expectation values.

Consider a large number of identical fdqs's $(\mathfrak{H}, H)$, each in the same physical state namely $x \in \mathfrak{H}$ with $\|x\| = 1$. Let A be an observable of $(\mathfrak{H}, H)$, as in 2.2.2. We measure A for each system and take the average of the results. The *expectation value* $\langle A \rangle$ of A is intuitively seen as the limit of this average as the number of systems goes to infinity. Using 2.2.2 (and its notation) we make this more precise:

$$\langle A \rangle := \sum_{n=1}^{N} |c_n|^2 \alpha_n = \langle x, Ax \rangle .$$

We now define a linear functional ω on $\mathfrak{L}(\mathfrak{H})$ by $\omega(A) = \langle x, Ax \rangle$ (for every $A \in \mathfrak{L}(\mathfrak{H})$). $\omega(A)$ is then the expectation value for every observable A. Clearly $\omega(A^*A) = \langle Ax, Ax \rangle \geq 0$ and $\omega(1) = 1$. This leads us naturally to the next definition:

2.2.3 Definition(State) A linear functional ω on a unital C*-algebra $\mathfrak{A}$ is called a *state* on $\mathfrak{A}$, if $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and $\omega(1) = 1$.■

We note that an equivalent definition of a state would be to replace $\omega(1) = 1$ in 2.2.3 by $\|\omega\| = 1$ ([Br, 2.3.11]), and so the notion of a state can be extended to non-unital C*-algebras. The latter is of course the more usual definition of a state known from C*-algebra theory and we will use it in 3.2.4.

Instead of using elements of $\mathfrak{H}$ to represent physical states as in 2.2.2, it will prove more useful to represent physical states by states on $\mathfrak{L}(\mathfrak{H})$ (hence the name *state* in 2.2.3). This is in keeping with our move away from Hilbert spaces and towards operator algebras. The main advantage of using states on $\mathfrak{L}(\mathfrak{H})$ instead of elements of $\mathfrak{H}$, is that the former allows us to describe the physical state of the system even if we do not have complete knowledge of its physical state, while the latter can only be used to describe the physical state of the system if its physical state is known completely. States on $\mathfrak{L}(\mathfrak{H})$ therefore introduce a *statistical* nature to our theory above and beyond the probabilistic interpretation given in 2.2.2. This of course is essential in quantum statistical mechanics where we work with systems consisting of many particles and complete knowledge of the physical state of the system is impossible in practice. We describe this in more detail in the next section.

2.3 Equilibrium and the Gibbs state

From now on we will use the notion of temperature. We will not define it, but it has the usual intuitive physical meaning (hotter means higher temperature) and it is measured using a thermometer. For a rigorous definition of temperature see [Th, (2.3.16)].

It is important to note that we will not regard temperature as an observable of the physical system under consideration (see 2.2.2), but rather as an external condition applied to the system. This is similar to the magnetic field in the model described in 2.1. For example, the system might be put in a refrigerator to cool it down, or in an oven to heat it up.

As it turns out it is easier to work with the inverse temperature, which we now define:

2.3.1 Definition(Inverse temperature) We measure temperature T on the absolute temperature scale (SI unit: Kelvin, K). The *inverse temperature* of a temperature T is defined as

$$\beta := 1/k_B T$$

for all $T \in \mathbb{R} \cup \{-\infty, +\infty\} \backslash \{0\}$, where k_B is Boltzmann's constant. (In particular, $\beta = 0$ if $T = -\infty$ or $T = +\infty$.) The SI unit of β is J^{-1}, where J stands for joule.■

In 2.3.1 we allow negative temperatures, even though strictly speaking 0K is the theoretical lower bound for the temperature of any object. However, negative values of β fits naturally into our mathematical analysis later on. (We can also note that the concept of negative temperature is often useful in physics when considering entropy or non-equilibrium conditions in systems that have an upper bound for its energy (for example the model described in 2.1.2). In such cases negative temperatures have been verified experimentally (see [P, 3.9] and [To, 2.1.4(c) and 2.2.2]). The energy of a system at a negative temperature is *higher* than when it is at a positive temperature. To understand this one has to study entropy and the rigorous definition of temperature in terms of entropy. Further details can be found in the references already given.)

We are now going to look at the notion of equilibrium in some detail.

Consider any physical system at some fixed temperature T. We also assume that the system is not changing as time goes on, more precisely,

the expectation value of every observable of the system is constant in time (observable here having the same meaning as in 2.2.2, namely an attribute of the system yielding a real number when measured). We also want the system to be stable. We can describe this intuitively as follows: If the system is disturbed slightly in some way without the temperature changing, the system only changes slightly and then starts to return to its previous physical state as time passes. This disturbance would be described mathematically by a temporary change in the Hamiltonian of the system. In the model described in 2.1.2 the most obvious example of a disturbance would be to change the magnetic field (and hence the Hamiltonian) for a few moments and then restoring it. See [Br, Section 5.4.1] and [H, V.3.2] for further discussion on stability.

If a system has the three properties described above, we say it is in *equilibrium*. Obviously this is not a very clear definition of equilibrium because we did not give a mathematically rigorous definition of stability, but at least we now have a good idea what equilibrium means physically. The question is: If the only thing we know about a system is that it is in equilibrium at some given temperature, how can we describe the physical state of the system mathematically?

Let $(\mathfrak{H}, H)$ be a fdqs in equilibrium at inverse temperature $\beta > 0$, with $b_1, ..., b_N$ an orthonormal basis for $\mathfrak{H}$ consisting of eigenvectors of H, and $E_1, ..., E_N$ the corresponding eigenvalues (i.e. the energies of the physical states described by the vectors $b_1, ..., b_N$). It is important to note that this is the only information we have regarding the system.

If we could measure the energy of this system precisely, the probability of getting E_n would be

$$e^{-\beta E_n} / \sum_{k=1}^{N} e^{-\beta E_k}. \tag{1}$$

This is called the *Gibbs distribution* and it forms the basis of all our further analysis of equilibrium. Arguments in favor of this distribution as a discription of equilibrium can be found in [F, Section 40-1] and [P, 3.1 and 3.2]. We only mention that as the temperature goes down (i.e. β rises) the lower energies tend to become more probable in our measurements. This makes sense intuitively since we associate low temperatures with low energies. For example, say E_1 is the lowest energy ($E_1 < E_k$ for $k = 2, ..., N$) then the

probability of getting E_1 during a measurement is (by (1))

$$1/\left(1+\sum_{k=2}^{N} e^{-\beta(E_k-E_1)}\right)$$

which goes to 1 as the temperature goes to 0 (i.e. as β goes to infinity). This means that at absolute zero temperature, the system is in its lowest energy. On the other hand, as the temperature goes to infinity (i.e. β goes to 0) we see from (1) that all the energies become equally probable.

The expectation value for the energy (i.e the average energy after many measurements, see 2.2) is of course just the expectation value $\langle H\rangle$ of H since H is the energy observable. From (1) it is therefore clear that

$$\langle H\rangle = \left(\sum_{k=1}^{N} e^{-\beta E_k}E_k\right)/\sum_{k=1}^{N} e^{-\beta E_k} = \mathrm{Tr}(e^{-\beta H}H)/\mathrm{Tr}(e^{-\beta H})$$

(see 1.9.5). If we now define a linear functional ω on $\mathfrak{L}(\mathfrak{H})$ by

$$\omega(A) = \mathrm{Tr}(\rho A) \text{ where } \rho := e^{-\beta H}/\mathrm{Tr}(e^{-\beta H}), \tag{2}$$

then $\omega(H) = \langle H\rangle$.

$\rho \geq 0$ by 1.9.7(3), and clearly $\mathrm{Tr}(\rho) = 1$. (This holds even if $\beta \leq 0$.) So $\omega(1) = 1$, and for any $A \in \mathfrak{L}(\mathfrak{H})$ we have

$$\omega(A^*A) = \mathrm{Tr}(A\rho A^*) = \mathrm{Tr}(A\rho^{1/2}\rho^{1/2}A^*) = \mathrm{Tr}(A\rho^{1/2}(A\rho^{1/2})^*) \geq 0 \tag{3}$$

using the elementary properties of the trace. Therefore ω is a state on $\mathfrak{L}(\mathfrak{H})$ by 2.2.3.

Because $\omega(H) = \langle H\rangle$, we might guess $\omega(A)$ is the expectation value of any observable A. This turns out to be the case.

We now generalize and formalize these ideas.

2.3.2 Definition(Density operator) Let $\mathfrak{H}$ be a Hilbert space. An operator $\rho \in \mathfrak{L}(\mathfrak{H})_+$ with $\mathrm{Tr}(\rho) = 1$ is called a *density operator* (or *statistical operator*) on $\mathfrak{H}$.■

If ρ is a density operator on a Hilbert space $\mathfrak{H}$, and $\mathfrak{A}$ is any C*-subalgebra of $\mathfrak{L}(\mathfrak{H})$ that contains the unit of $\mathfrak{L}(\mathfrak{H})$,then we can define a linear functional ω on $\mathfrak{A}$ by $\omega(A) = \mathrm{Tr}(\rho A)$, (linearity of course follows directly from the definition of the trace). ω is then a state on $\mathfrak{A}$ exactly as in (3), by 2.2.3.

2.3.3 Definition(Normal state) Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let ρ be a density operator on $\mathfrak{H}$. Then the state ω on $\mathfrak{M}$ defined by $\omega(A) = \mathrm{Tr}(\rho A)$ is called a *normal state*.■

Keep in mind that $\mathfrak{L}(\mathfrak{H})$ is a von Neumann algebra on the Hilbert space $\mathfrak{H}$ by 1.4.8.

2.3.4 Definition(Gibbs states) Let $(\mathfrak{H}, H)$ be a fdqs and take any $\beta \in \mathbb{R}$. The normal state ω on $\mathfrak{L}(\mathfrak{H})$ defined by

$$\omega(A) = \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}(e^{-\beta H})$$

(as in (2)) is called the *β-Gibbs state of* $(\mathfrak{H}, H)$.■

Our discussion of the Gibbs distribution above suggests the following two postulates:

2.3.5 Postulate II Consider any fdqs $(\mathfrak{H}, H)$. The physical state of this system is described by a state ω on $\mathfrak{L}(\mathfrak{H})$ in the following sense: $\omega(A)$ is the expectation value of A for every observable $A \in \mathfrak{L}(\mathfrak{H})_s$. (The expectation value of an observable A is by definition the limit of the average value we get when measuring A in n copies of $(\mathfrak{H}, H)$, each in the state ω, as n goes to infinity.) Furthermore, ω is a normal state, i.e. it is determined by some density operator ρ in the following way: $\omega(A) = \mathrm{Tr}(\rho A)$ for every $A \in \mathfrak{L}(\mathfrak{H})$.■

2.3.6 Postulate III The physical state of a fdqs $(\mathfrak{H}, H)$ in equilibrium at inverse temperature $\beta > 0$ is described by the β-Gibbs state of $(\mathfrak{H}, H)$.■

It should be noted that postulate II cannot be proven mathematically (at least not without making other assumptions). We have made this postulate seem plausible by considering the special case of the energy observable (the Hamiltonian) and the Gibbs state, and by our discussion in 2.2 (based on 2.2.2). Its only real justification is that it can be verified experimentally. Attempts to prove postulate III leads one into ergodic theory (see [To, 2.1.1 and 2.4.1]), but we will take it as our starting point for further analysis (see Chapter 3).

2.3.7 Proposition *Let $(\mathfrak{H}, H)$ be a fdqs, $\beta \in \mathbb{R}$, and ω the β-Gibbs state of $(\mathfrak{H}, H)$. Then $\omega(A) > 0$ for every $A \in \mathfrak{L}(\mathfrak{H})_+ \backslash \{0\}$.*

Proof. Let $A \in \mathfrak{L}(\mathfrak{H})_+ \backslash \{0\}$. From 1.9.1 and 1.9.3 we see

$$A = \sum_{m=1}^{N} \sum_{n=1}^{N} \langle b_m, Ab_n \rangle \, b_m \otimes b_n$$

where $b_1, ..., b_N$ is an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of H (see 1.9.4). Let E_n be the eigenvalue of H corresponding to b_n. So

$$\begin{aligned}
& \mathrm{Tr}(e^{-\beta H} A) \\
= \; & \mathrm{Tr}\left(\sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{n=1}^{N} e^{-\beta E_l} \langle b_m, Ab_n \rangle \, (b_l \otimes b_l)(b_m \otimes b_n) \right) \quad \text{(by 1.9.5)} \\
= \; & \mathrm{Tr}\left(\sum_{l=1}^{N} \sum_{n=1}^{N} e^{-\beta E_l} \langle b_l, Ab_n \rangle \, b_l \otimes b_n \right) \quad \text{(by 1.9.2)} \\
= \; & \sum_{k=1}^{N} \left\langle b_k, \left(\sum_{l=1}^{N} \sum_{n=1}^{N} e^{-\beta E_l} \langle b_l, Ab_n \rangle \, b_l \otimes b_n \right) b_k \right\rangle \quad \text{(definition of Tr)} \\
= \; & \sum_{k=1}^{N} e^{-\beta E_k} \langle b_k, Ab_k \rangle \quad \text{(by 1.9.1).}
\end{aligned}$$

$\mathrm{Tr}(A) = \sum_{n=1}^{N} \langle b_n, Ab_n \rangle$, but $\mathrm{Tr}(A)$ is also the sum of the eigenvalues of A (multiplicities included) as can be seen from 1.9.4 and 1.9.1 and the definition of Tr. Since $A \geq 0$, the spectrum of A lies in $\mathbb{R}^+$, i.e. the eigenvalues of A are non-negative, but since $A \neq 0$ it follows from 1.9.4 that at least one of A's eigenvalues is non-zero. Hence $\mathrm{Tr}(A) > 0$, and since $\langle b_n, Ab_n \rangle \geq 0$ for all n (because $A \geq 0$) it follows that $\langle b_n, Ab_n \rangle > 0$ for at least one n. Since clearly $e^{-\beta E_k} > 0$ for all k, we find from the expression for $\mathrm{Tr}(e^{-\beta H} A)$ given above that $\omega(A) = \mathrm{Tr}(e^{-\beta H} A)/\mathrm{Tr}(e^{-\beta H}) > 0$.■

2.3.7, and the fact that $\mathfrak{L}(\mathfrak{H})$ is a von Neumann algebra by 1.4.8, inspires

2.3.8 Definition(Faithful state) Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space. A state ω on $\mathfrak{M}$ is called *faithful* if $\omega(A) > 0$ for all $A \in \mathfrak{M}_+ \backslash \{0\}$.■

Combining 2.3.4 and 2.3.7 we get

2.3.9 Proposition *Let $(\mathfrak{H}, H)$ be a fdqs, $\beta \in \mathbb{R}$, and ω the β-Gibbs state of $(\mathfrak{H}, H)$. Then ω is a faithful normal state on $\mathfrak{L}(\mathfrak{H})$.*■

If in 2.3.5 ρ is the orthogonal projection of $\mathfrak{H}$ on $\mathbb{C}x$ for some $x \in \mathfrak{H}$ with $\|x\| = 1$, then clearly $\omega(A) = \mathrm{Tr}(\rho A) = \langle x, Ax\rangle$ for every $A \in \mathfrak{L}(\mathfrak{H})$ (just compute $\mathrm{Tr}(\rho A)$ using an orthonormal basis of $\mathfrak{H}$ containing x). But if the system's state is given by x as in 2.2.2, then $\langle x, Ax\rangle = \langle A\rangle$ (see 2.2), so we conclude that a state given by a density operator (as in 2.3.5) is a generalization of the description of the physical state of $(\mathfrak{H}, H)$ by an element of $\mathfrak{H}$. As mentioned in 2.2.2 an element of $\mathfrak{H}$ gives a complete description of the physical state of the system. For this reason we call a state ω over $\mathfrak{L}(\mathfrak{H})$ *pure* if it is given by $\omega(A) = \mathrm{Tr}(\rho A)$ where ρ is the projection of $\mathfrak{H}$ on $\mathbb{C}x$ for some $x \in \mathfrak{H}\backslash\{0\}$. If ω is not pure it is called a *mixture*. The reason for this name is that various pure states are combined with different weights, for example in the Gibbs state the eigenvectors of H are combined by using weights of the form $e^{-\beta E}$ where $E \in \mathbb{R}$. Mixtures are used when we do not have complete knowledge of the physical state of the sytem, and pure states when we do.

From now on we will refer to the physical state of a fdqs $(\mathfrak{H}, H)$ simply as the state, meaning the state over $\mathfrak{L}(\mathfrak{H})$.

Only one part of our theory is still missing, namely how a fdqs evolves with time. We turn to this next.

2.4 The dynamical law

2.4.1 Postulate IV Consider an observable $A \in \mathfrak{L}(\mathfrak{H})_s$ of a fdqs $(\mathfrak{H}, H)$ which is in the state ω, at time 0. Then the expectation value of the observable at time $t \in \mathbb{R}$ is given by

$$\omega(e^{iHt} A e^{-iHt})$$

where we measure t in multiples of $\hbar$, i.e. if t' is the time in seconds then $t = t'/\hbar$. ($\hbar$ is Planck's constant devided by 2π, and the SI unit of t is J^{-1}, where J stands for joule.)■

We can interpret 2.4.1 in two ways. Firstly we can view the state ω as evolving with time. Secondly we can view the hermitian operator A, representing the observable, as evolving with time (note that $e^{iHt} A e^{-iHt}$ is

hermitian by 1.9.7(1), and therefore represents an observable by 2.2.2, for every $t \in \mathbb{R}$). In keeping with our approach of setting up quantum mechanics in terms of operator algebras (in this case $\mathfrak{L}(\mathfrak{H})$) we choose the second interpretation. More on this in 3.1.

2.4.1 is our dynamical law. We cannot really give any explanation for this law except to say that $t \mapsto e^{-iHt}$ satisfies the Schrödinger equation

$$\frac{d}{dt}A(t) = -iHA(t)$$

by 1.9.9 (here A is a function $\mathbb{R} \to \mathfrak{L}(\mathfrak{H})$). Refer to [Su, 3.1 and 3.4] for more about this.

In 2.3 we said that if a system is in equilibrium, the expectation value of each of its observables is constant in time. We can now test this for the β-Gibbs state using 2.4.1 in order to lend some more credibility to 2.3.6. Consider a fdqs $(\mathfrak{H}, H)$ in the β-Gibbs state as given by 2.3.4. For every $A \in \mathfrak{L}(\mathfrak{H})$ and $t \in \mathbb{R}$ we then have

$$\begin{aligned}
\omega(e^{iHt}Ae^{-iHt}) &= \mathrm{Tr}(e^{-\beta H}e^{iHt}Ae^{-iHt})/\mathrm{Tr}(e^{-\beta H}) \\
&= \mathrm{Tr}(e^{-iHt}e^{-\beta H}e^{iHt}A)/\mathrm{Tr}(e^{-\beta H}) \\
&= \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}(e^{-\beta H}) \quad \text{(by 1.9.6 and 1.9.7(2))} \\
&= \omega(A).
\end{aligned}$$

In particular, if A is an observable, this says (according to 2.4.1) that the expectation value of the observable does not change with time (see 2.3.5). This is of course not a complete characterization of equilibrium (any normal state whose density operator is a funtion only of H (by means of 1.9.4) ensures the constancy of the expectation values in time, the same proof as above applies) but it certainly is a minimum requirement.

Chapter 3

KMS states

We are now going to discuss the so-called Kubo-Martin-Schwinger (KMS for short) states. These states turn out to be exactly the equilibrium states of fdqs's, but because of the way they are expressed they are easy to extend to infinite dimensional systems. They also fit naturally into the Tomita-Takesaki theory as we will see in 4.4.

The key to arriving at the KMS states is the following simple observation concerning a fdqs $(\mathfrak{H}, H)$:

$$e^{iHt} = e^{-\beta H} \text{ if } t = i\beta \text{ where } \beta \in \mathbb{R}.$$

This connects the dynamical law 2.4.1 with the equilibrium states given by 2.3.4 and 2.3.6. It certainly is a mathematical trick rather than a big physical insight, but it will take us a long way.

3.1 C*- and W*-dynamical systems

The group of all $*$-automorphisms of a C*-algebra $\mathfrak{A}$ will be denoted by $\mathrm{Aut}(\mathfrak{A})$. (A $*$-automorphism of $\mathfrak{A}$ is a $*$-isomorphism from $\mathfrak{A}$ onto itself.)

The following two definitions are distilled from [Br, 2.7.1]:

3.1.1 Definition(One-parameter $*$-automorphism group) Let $\mathfrak{A}$ be a C*-algebra. A function of the form

$$\tau : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A}) : t \mapsto \tau_t$$

is called a *one-parameter $*$-automorphism group of* $\mathfrak{A}$ if τ_0 is the identity map on $\mathfrak{A}$ and $\tau_s\tau_t = \tau_{s+t}$.■

3.1.2 Definition(C*-dynamical system) Let $\mathfrak{A}$ be a C*-algebra and τ a one-parameter $*$-automorphism group of $\mathfrak{A}$. If $\mathbb{R} \to \mathfrak{A} : t \mapsto \tau_t(A)$ is continuous in the norm of $\mathfrak{A}$ for every $A \in \mathfrak{A}$, we call $(\mathfrak{A}, \tau)$ a *C*-dynamical system.*■

In Chapter 4 we will work with von Neumann algebras and there we will need a slighty different type of dynamical system (also see [Pe, 7.4.2]):

3.1.3 Definition(W*-dynamical system) Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and τ a one-parameter $*$-automorphism group of $\mathfrak{M}$. If $\mathbb{R} \to \mathfrak{H} : t \mapsto \tau_t(A)x$ is continuous for every $A \in \mathfrak{M}$ and $x \in \mathfrak{H}$, then we call $(\mathfrak{M}, \tau)$ a *W*-dynamical system.*■

Clearly we have

3.1.4 Proposition *If $(\mathfrak{A}, \tau)$ is a C*-dynamical system and $\mathfrak{A}$ is a von Neumann algebra on a Hilbert space, then $(\mathfrak{A}, \tau)$ is a W*-dynamical system.*■

We will only work with W*-dynamical systems later on (see 4.4), but in this chapter we also consider C*-dynamical systems since it doesn't require much extra effort.

The motivation for 3.1.2 and 3.1.3 is the fact that the dynamical law of a fdqs (2.4.1) fits into these settings as we will now see.

3.1.5 Definition(Time evolution) Let $(\mathfrak{H}, H)$ be a fdqs and define

$$\tau : \mathbb{R} \to \mathrm{Aut}(\mathfrak{L}(\mathfrak{H})) : t \mapsto \tau_t$$

by

$$\tau_t(A) = e^{iHt} A e^{-iHt}.$$

τ is called the *time evolution* of $(\mathfrak{H}, H)$ because of 2.4.1. Note that τ can be naturally extended from $\mathbb{R}$ to $\mathbb{C}$ by

$$\tau_z(A) = e^{iHz} A e^{-iHz} \quad \text{for } z \in \mathbb{C}.$$■

3.1.6 Proposition *The time evolution τ of a fdqs $(\mathfrak{H}, H)$ is well-defined (in fact $\tau_z(A) \in Aut(\mathfrak{L}(\mathfrak{H}))$ for every $z \in \mathbb{C}$), and $(\mathfrak{L}(\mathfrak{H}), \tau)$ is a C*- and W*-dynamical system.*

Proof. (a) Let $z \in \mathbb{C}$ and look at the function τ_z on $\mathfrak{L}(\mathfrak{H})$. τ_z is clearly linear, and by 1.9.7(1) and (2) we have $\tau_z(AB) = e^{iHz}Ae^{-iHz}e^{iHz}Be^{-iHz} = \tau_z(A)\tau_z(B)$ and $\tau_z(A^*) = ((e^{-iHz})^*A(e^{iHz})^*)^* = (e^{iHz}Ae^{-iHz})^* = (\tau_z(A))^*$. If $\tau_z(A) = 0$ then $A = e^{-iHz}\tau_z(A)e^{iHz} = 0$, and clearly $A = \tau_z(e^{-iHz}Ae^{iHz})$ for every $A \in \mathfrak{L}(\mathfrak{H})$, hence τ_z is a bijection. We conclude $\tau_z \in \mathrm{Aut}(\mathfrak{L}(\mathfrak{H}))$.

(b) From 1.9.5 and 1.9.1 we see that $\tau_0(A) = e^{iH0}Ae^{-iH0} = 1A1 = A$. Also, $\tau_s\tau_t(A) = e^{iHs}e^{iHt}Ae^{-iHt}e^{-iHs} = e^{iH(s+t)}Ae^{-iH(s+t)} = \tau_{s+t}(A)$ for every $A \in \mathfrak{L}(\mathfrak{H})$ and $s, t \in \mathbb{R}$ as is easily seen from 1.9.5 and 1.9.2. So τ is a one-parameter $*$-automorphism group of $\mathfrak{L}(\mathfrak{H})$.

(c) Let $A \in \mathfrak{L}(\mathfrak{H})$ and define $f : \mathbb{R} \to \mathfrak{L}(\mathfrak{H})$ by $f(t) = \tau_t(A)$. We want to prove that f is continuous in the norm of $\mathfrak{L}(\mathfrak{H})$, but for $r, s \in \mathbb{R}$

$$f(r) - f(s) = e^{iH(r-s)}\tau_s(A)e^{iH(s-r)} - \tau_s(A),$$

so we only have to prove $\left\|e^{iHt} - 1\right\| \to 0$ as $t \in \mathbb{R}$ goes to 0.

Let $b_1, \ldots, b_N$ be an orthonormal basis for $\mathfrak{H}$ consisting of eigenvectors of H with $E_1, \ldots, E_N$ the corresponding eigenvalues. By 1.9.5 $\left\|e^{iHt} - 1\right\| \leq \sum_{n=1}^{N} \left|e^{iE_nt} - 1\right| \|b_n \otimes b_n\| \to 0$ as $t \to 0$. We conclude that f is continuous and so $(\mathfrak{L}(\mathfrak{H}), \tau)$ is a C*-dynamical system according to 3.1.2. This proves the proposition by 3.1.3, 1.4.8 and 3.1.4.■

3.2 KMS states

The following definition is adapted from [Br, 5.3.1]:

3.2.1 Definition(KMS states with respect to time evolution) Let τ be the time evolution (as given by 3.1.5) of a fdqs $(\mathfrak{H}, H)$. Let $\beta \in \mathbb{R}$. A state ω on $\mathfrak{L}(\mathfrak{H})$ is called a (τ, β)-*KMS state* if

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \quad \text{for all } A, B \in \mathfrak{L}(\mathfrak{H}).\blacksquare$$

The importance of 3.2.1 is the following proposition which says that the KMS states with respect to time evolution are exactly the equilibrium states of a fdqs (see 2.3.6 and 2.3.4):

3.2.2 Proposition *Let τ be the time evolution of a fdqs $(\mathfrak{H}, H)$, ω a state on $\mathfrak{L}(\mathfrak{H})$, and $\beta \in \mathbb{R}$. Then ω is a (τ, β)-KMS state (in the sense of 3.2.1) if and only if it is the β-Gibbs state of $(\mathfrak{H}, H)$.*

Proof. (a) Assume ω is the β-Gibbs state of $(\mathfrak{H}, H)$. For any $A, B \in \mathfrak{L}(\mathfrak{H})$ we then have

$$\begin{aligned}\omega(A\tau_{i\beta}(B)) &= \omega(Ae^{-\beta H}Be^{\beta H}) = \mathrm{Tr}(e^{-\beta H}Ae^{-\beta H}Be^{\beta H})/\mathrm{Tr}(e^{-\beta H}) \\ &= \mathrm{Tr}(Ae^{-\beta H}B)/\mathrm{Tr}(e^{-\beta H}) = \mathrm{Tr}(e^{-\beta H}BA)/\mathrm{Tr}(e^{-\beta H}) \\ &= \omega(BA)\end{aligned}$$

so ω is a (τ, β)-KMS state.

(b) Now assume ω is a (τ, β)-KMS state. Let $b_1, ..., b_N$ be an orthonormal basis for $\mathfrak{H}$ consisting of eigenvectors of H with corresponding eigenvalues $E_1, ..., E_N$ (see 1.9.4). Let $l, m, n \in [N]$, then

$$\begin{aligned}\omega((b_m \otimes b_n)(b_l \otimes b_m)) &= \omega((b_l \otimes b_m)\tau_{i\beta}(b_m \otimes b_n)) \\ &= \omega((b_l \otimes b_m)e^{-\beta H}(b_m \otimes b_n)e^{\beta H}) \\ &= \omega((b_l \otimes b_m)e^{-\beta E_m}(b_m \otimes b_n)e^{\beta E_n})\end{aligned}$$

by 1.9.5 and 1.9.2, so $e^{-\beta E_n}\langle b_n, b_l\rangle \omega(b_m \otimes b_m) = e^{-\beta E_m}\omega(b_l \otimes b_n)$ by 1.9.2. Hence

$$e^{-\beta E_n}\langle b_n, b_l\rangle \omega\left(\sum_{m=1}^{N} b_m \otimes b_m\right) = \left(\sum_{m=1}^{N} e^{-\beta E_m}\right)\omega(b_l \otimes b_n).$$

But $\sum_{m=1}^{N} b_m \otimes b_m = 1$ as is easily seen from 1.9.1, and $\omega(1) = 1$ by 2.2.3, so

$$\omega(b_l \otimes b_n) = e^{-\beta E_n}\langle b_n, b_l\rangle / Tr(e^{-\beta H})$$

since $\mathrm{Tr}(e^{-\beta H}) = \sum_{m=1}^{N} e^{-\beta E_m}$ by 1.9.5 and the definition of Tr. Let $A \in \mathfrak{L}(\mathfrak{H})$, then by 1.9.3 we can write $A = \sum_{l=1}^{N}\sum_{n=1}^{N} a_{l,n}b_l \otimes b_n$ where $a_{l,n} \in \mathbb{C}$, so we find

$$\begin{aligned}\omega(A) &= \sum_{l=1}^{N}\sum_{n=1}^{N} a_{l,n}\omega(b_l \otimes b_n) = \sum_{l=1}^{N}\sum_{n=1}^{N} a_{l,n}e^{-\beta E_n}\langle b_n, b_l\rangle / \mathrm{Tr}(e^{-\beta H}) \\ &= \sum_{l=1}^{N} e^{-\beta E_l}a_{l,l}/\mathrm{Tr}(e^{-\beta H}) \quad \text{(since } b_1, ..., b_N \text{ are orthonormal)} \\ &= \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}(e^{-\beta H}) \quad \text{(by 1.9.5 and 1.9.2).}\end{aligned}$$

In other words ω is the β-Gibbs state of $(\mathfrak{H}, H)$.■

We are now going to give another characterization of KMS states which we will then use to define KMS states for any C*- or W*-dynamical system:

3.2.3 Proposition *Let τ be the time evolution of a $(\mathfrak{H}, H)$, ω a state on $\mathfrak{L}(\mathfrak{H})$, and $\beta \in \mathbb{R}$.*
Set

$$\mathfrak{D}_\beta := \mathbb{C}_0^\beta \quad \text{and} \quad \overline{\mathfrak{D}_\beta} := \overline{\mathbb{C}_0^\beta} \quad \text{if } \beta \geq 0,$$

and

$$\mathfrak{D}_\beta := \mathbb{C}_\beta^0 \quad \text{and} \quad \overline{\mathfrak{D}_\beta} := \overline{\mathbb{C}_\beta^0} \quad \text{if } \beta < 0.$$

Then the following two conditions are equivalent:
(1) ω is a (τ, β)-KMS state (as defined by 3.2.1).
(2) For every pair $A, B \in \mathfrak{L}(\mathfrak{H})$ their exists a bounded continuous function

$$F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$$

which is analytic on $\mathfrak{D}_\beta$ and has the following properties:

$$F_{A,B}(t) = \omega(A\tau_t(B))$$

and

$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A)$$

for every $t \in \mathbb{R}$.

Proof. (a) Assume ω is a (τ, β)-KMS state. So ω is the β-Gibbs state of $(\mathfrak{H}, H)$ by 3.2.2. We define

$$F_{A,B} : \mathbb{C} \to \mathbb{C} : z \mapsto \omega(A\tau_z(B))$$

for any $A, B \in \mathfrak{L}(\mathfrak{H})$. So, for $t \in \mathbb{R}$, $F_{A,B}(t) = \omega(A\tau_t(B))$ and

$$\begin{aligned} F_{A,B}(t + i\beta) &= \mathrm{Tr}(e^{-\beta H} A e^{iH(t+i\beta)} B e^{-iH(t+i\beta)})/\mathrm{Tr}(e^{-\beta H}) \\ &= \mathrm{Tr}(e^{-\beta H} A e^{-\beta H} e^{iHt} B e^{-iHt} e^{\beta H})/\mathrm{Tr}(e^{-\beta H}) \quad \text{(by 1.9.5)} \\ &= \mathrm{Tr}(e^{-\beta H} e^{iHt} B e^{-iHt} e^{\beta H} e^{-\beta H} A)/\mathrm{Tr}(e^{-\beta H}) \\ &= \omega(\tau_t(B)A) \quad \text{(by 1.9.7(2))}. \end{aligned}$$

ω is bounded (with norm 1) since it is a state, hence it is continuous. So from the definition of $F_{A,B}$ it follows by 3.1.5, 1.9.9 and 1.7.2 that $F_{A,B}$ is

differentiable (i.e. analytic), namely $\frac{d}{dz}F_{A,B}(z) = \omega\left(\frac{d}{dz}(A\tau_z(B))\right)$. Hence $F_{A,B}$ is continuous. Let $b_1, ..., b_N$ be any orthonormal basis for $\mathfrak{H}$ consisting of eigenvectors of H with corresponding eigenvalues $E_1, ..., E_N$. Let $z \in \overline{\mathfrak{D}_\beta}$ and write $z = x+iy$ where $x, y \in \mathbb{R}$. Since $\|\omega\| = 1$, $|F_{A,B}(z)| \leq \|A\tau_z(B)\| \leq \|A\| \left\|e^{iHz}\right\| \|B\| \left\|e^{-iHz}\right\|$, but by 1.9.5

$$\left\|e^{iHz}\right\| \leq \sum_{n=1}^{N} \left|e^{iE_n z}\right| \|b_n \otimes b_n\| = \sum_{n=1}^{N} e^{-E_n y} \leq \sum_{n=1}^{N} e^{|E_n \beta|}$$

since $|y| \leq |\beta|$ and $\|b_n \otimes b_n\| = 1$ by 1.9.1. A similar inequality holds for $\left\|e^{-iHz}\right\|$. It follows that $F_{A,B}$ is bounded on $\overline{\mathfrak{D}_\beta}$. We have thus proven that (1) implies (2).

(b) Our proof that (2) implies (1) is loosely based on the proof of [Br, 5.3.7(3)⇒(1)]. Assume (2) holds. Consider any $A, B \in \mathfrak{L}(\mathfrak{H})$ and let

$$G_{A,B} : \mathbb{C} \to \mathbb{C} : z \mapsto \omega(A\tau_z(B)).$$

We will now prove that $G_{A,B}(z) = F_{A,B}(z)$ for $z \in \overline{\mathfrak{D}_\beta}$. By 1.7.2, 1.9.9 and 3.1.5 we know that $\frac{d}{dz}(A\tau_z(B))$ exists for all $z \in \mathbb{C}$. Clearly $\frac{d}{dz}\omega(A\tau_z(B)) = \omega\left(\frac{d}{dz}(A\tau_z(B))\right)$ for all $z \in \mathbb{C}$since ω is continuous. So $G_{A,B}$ is analytic and therefore also continuous. Let

$$f := G_{A,B} - F_{A,B} \quad \text{on } \overline{\mathfrak{D}_\beta}.$$

f is continuous on $\overline{\mathfrak{D}_\beta}$ and analytic on $\mathfrak{D}_\beta$ since this is the case for $G_{A,B}$ and $F_{A,B}$. We want to show $f = 0$.

Clearly $f(t) = 0$ for $t \in \mathbb{R}$ by the definition of $G_{A,B}$ and the first of the two equalities given in (2). So we may assume $\beta \neq 0$ since $\overline{\mathfrak{D}_\beta} = \mathbb{R}$ for $\beta = 0$.

First assume $\beta > 0$. We will now use the terminology and notation given in 1.6. Set $D := \mathfrak{D}_{-\beta} \cup \mathbb{R} \cup \mathfrak{D}_\beta$, then $D^* = D$ is a region, $D_0 = \mathbb{R}$, $D_+ = \mathfrak{D}_\beta$ and $D_+ \cup D_0 \subset \overline{\mathfrak{D}_\beta}$. From 1.6.3 it now follows that there is an analytic function $g : D \to \mathbb{C}$ such that $g(z) = f(z)$ for $z \in D_+ \cup \mathbb{R}$. In particular $g(t) = 0$ for $t \in \mathbb{R}$, so $g = 0$ by 1.6.2. Therefore $f = 0$ since it is continuous and D_+ is dense in $\overline{\mathfrak{D}_\beta}$.

We now consider the case $\beta < 0$. Let $h : \overline{\mathfrak{D}_{-\beta}} \to \mathbb{C} : z \mapsto \overline{f(\overline{z})}$. It is easily seen that h is continuous on $\overline{\mathfrak{D}_{-\beta}}$ and analytic on $\mathfrak{D}_{-\beta}$ since f is continuous on $\overline{\mathfrak{D}_\beta}$ and analytic on $\mathfrak{D}_\beta$. Also $h(t) = \overline{f(t)} = 0$ for $t \in \mathbb{R}$. Furthermore, $-\beta > 0$, so we can replace f with h, and β with $-\beta$ in the argument above (for $\beta > 0$) to find $h = 0$. So $f = 0$.

We conclude that $F_{A,B}(z) = G_{A,B}(z) = \omega(A\tau_z(B))$ for $z \in \overline{\mathfrak{D}_\beta}$. In particular $\omega(A\tau_{i\beta}(B)) = F_{A,B}(i\beta) = \omega(\tau_0(B)A) = \omega(BA)$ by the second equality given in (2). So ω is a (τ,β)-KMS state by 3.2.1.

This proves that (2) implies (1).■

In our approach to KMS states thus far we have not referred to C*- or W*-dynamical systems. Since we worked with a concrete example of a one-parameter $*$-automorphism group (namely the time evolution) we could prove 3.2.2 and 3.2.3 by using the definition of the time evolution. However, in the general case C*- and W*-dynamical systems become important. The following definition is motivated by 3.1.6, 2.3.9, 3.2.2 and 3.2.3. (Also see [Br, 5.3.7].)

3.2.4 Definition(KMS states) Let $(\mathfrak{A},\tau)$ be a C*- or W*-dynamical system, ω a state on $\mathfrak{A}$, and $\beta \in \mathbb{R}$. In the W*-case we assume that ω is normal (see 2.3.3).

Set

$$\mathfrak{D}_\beta := \mathbb{C}_0^\beta \quad \text{and} \quad \overline{\mathfrak{D}_\beta} := \overline{\mathbb{C}_0^\beta} \quad \text{if } \beta \geq 0,$$

and

$$\mathfrak{D}_\beta := \mathbb{C}_\beta^0 \quad \text{and} \quad \overline{\mathfrak{D}_\beta} := \overline{\mathbb{C}_\beta^0} \quad \text{if } \beta < 0.$$

ω is called a (τ,β)-*KMS state* if for every pair $A, B \in \mathfrak{A}$ there exists a bounded continuous function

$$F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C}$$

which is analytic on $\mathfrak{D}_\beta$ and has the following properties:

$$F_{A,B}(t) = \omega(A\tau_t(B))$$

and

$$F_{A,B}(t + i\beta) = \omega(\tau_t(B)A)$$

for every $t \in \mathbb{R}$. A $(\tau,-1)$-KMS state is called a τ-*KMS state*.■

3.2.5 Proposition *Let $(\mathfrak{H},H)$ be a fdqs, $\beta \in \mathbb{R}$, ω the β-Gibbs state of $(\mathfrak{H},H)$, and τ the time evolution of $(\mathfrak{H},H)$, (see 2.3.4 and 3.1.5). Then ω is the only (τ,β)-KMS state on $\mathfrak{L}(\mathfrak{H})$ (in both the C*- and W*-dynamical settings) as defined by 3.2.4.*

Proof. This is clear from 3.1.6, 3.2.2, 3.2.3 and 2.3.9.■

The point of 3.2.4 is that we want to be able to describe equilibrium in physical systems that are more general than the fdqs's we have been working with. Instead of the observables of the system being given by $\mathfrak{L}(\mathfrak{H})_s$ where $\mathfrak{H}$ is a finite dimensional Hilbert space, it will be given by $\mathfrak{A}_s$ where $\mathfrak{A}$ is a C*-algebra. We would then represent a physical system by a C*- or W*-dynamical system $(\mathfrak{A}, \tau)$ where τ is the time evolution of the system. A physical state of the system is then described by a state ω on $\mathfrak{A}$ such that $\omega(A)$ is the expectation value of the observable A. Faithful KMS states as defined by 3.2.4 and 2.3.8 would then be a reasonable guess as a mathematical description of the equilibrium states of the physical system since by 2.3.6, 2.3.9 and 3.2.5 they describe the equilibrium states of fdqs's.

As it turns out the KMS states fit naturally into an important part of the theory of von Neumann algebras, namely the Tomita-Takesaki theory (also known as modular theory) to which we turn in Chapter 4. For this reason KMS states have "become an indisputable point of interaction between theoretical physics and pure mathematics" ([Co, p.42]).

We now point out the usefulness of the τ-KMS states defined at the end of 3.2.4:

3.2.6 Proposition *Let $(\mathfrak{A}, \tau)$ be a C*-dynamical system (respectively W*-dynamimal system), ω a state on $\mathfrak{A}$, and $\beta \in \mathbb{R}$. Let*

$$\alpha_t(A) := \tau_{-\beta t}(A)$$

for every $t \in \mathbb{R}$ and $A \in \mathfrak{A}$, then $(\mathfrak{A}, \alpha)$ is a C-dynamical system (respectively W*-dynamical system) and we have the following:*
(1) If ω is a (τ, β)-KMS state, then it is an α-KMS state.
(2) For $\beta \neq 0$, ω is a (τ, β)-KMS state if and only if it is an α-KMS state.

Proof. It is straightforward to show that $(\mathfrak{A}, \alpha)$ is a C*-dynamical system (respectively W*-dynamical system). Let $\mathfrak{D}_\beta$ and $\overline{\mathfrak{D}_\beta}$ be as in 3.2.4.

Assume ω is a (τ, β)-KMS state. Consider any $A, B \in \mathfrak{A}$ and let $F_{A,B}$ be as in 3.2.4. Since $\mathfrak{D}_\beta = -\beta\mathfrak{D}_{-1}$ and $\overline{\mathfrak{D}_\beta} = -\beta\overline{\mathfrak{D}_{-1}}$, we can define

$$G_{A,B} : \overline{\mathfrak{D}_{-1}} \to \mathbb{C} : z \mapsto F_{A,B}(-\beta z)$$

and then clearly $G_{A,B}$ is analytic on $\mathfrak{D}_{-1}$, and bounded and continuous on $\overline{\mathfrak{D}_{-1}}$. Also,

$$G_{A,B}(t) = \omega(A\tau_{-\beta t}(B)) = \omega(A\alpha_t(B))$$

and

$$G_{A,B}(t-i) = \omega(\tau_{-\beta t}(B)A) = \omega(\alpha_t(B)A)$$

for every $t \in \mathbb{R}$. So ω is an α-KMS state, proving (1).

Now assume that $\beta \neq 0$ and that ω is an α-KMS state. Consider any $A, B \in \mathfrak{A}$. By 3.2.4 there exists a bounded continuous function $G_{A,B} : \overline{\mathfrak{D}_{-1}} \to \mathbb{C}$, analytic on $\mathfrak{D}_{-1}$, such that

$$G_{A,B}(t) = \omega(A\alpha_t(B)) \quad \text{and} \quad G_{A,B}(t-i) = \omega(\alpha_t(B)A)$$

for every $t \in \mathbb{R}$. Setting

$$F_{A,B} : \overline{\mathfrak{D}_\beta} \to \mathbb{C} : z \mapsto G_{A,B}(-z/\beta)$$

we see that $F_{A,B}$ has all the properties required in 3.2.4, so ω is a (τ,β)-KMS state, proving (2).■

3.2.6(1) says that if we can prove a property of the α-KMS states, then the property can be extended to the (τ,β)-KMS states. So we can often restrict ourselves to the case $\beta = -1$ when we study general properties of KMS states, and this is indeed what we will do in Chapter 4.

We illustrate 3.2.6(1) in the proof of the next proposition which says that a (τ,β)-KMS state is *τ-invariant* if $\beta \neq 0$.

3.2.7 Proposition *Let $(\mathfrak{A},\tau)$ be a C*- or W*-dynamical system where $\mathfrak{A}$ is unital. Let ω be a (τ,β)-KMS state on $\mathfrak{A}$ where $\beta \neq 0$. For $A \in \mathfrak{A}$ we then have*

$$\omega(\tau_t(A)) = \omega(A) \quad \textit{for every } t \in \mathbb{R}.$$

Proof. $\omega(\tau_t(A)) = \omega(\tau_t(\operatorname{Re} A)) + i\omega(\tau_t(\operatorname{Im} A))$, see 1.4.9, so we may assume without loss of generality that A is hermitian. Since $\beta \neq 0$, we may assume $\beta = -1$ by 3.2.6(1). According to 3.2.4 there exists a bounded continuous complex-valued function $f := F_{1,A}$ on $\overline{\mathbb{C}^0_{-1}}$, analytic on $\mathbb{C}^0_{-1}$, such that

$$f(t) = \omega(\tau_t(A)) = f(t-i) \quad \text{for every } t \in \mathbb{R}.$$

Since $\overline{\omega(\tau_t(A))} = \omega(\tau_t(A)^*) = \omega(\tau_t(A^*)) = \omega(\tau_t(A))$, (see [Br, 2.3.11(a)]), it follows that f is real-valued on $\overline{\mathbb{C}^0_{-1}} \backslash \mathbb{C}^0_{-1}$. Hence f is constant by 1.6.4

applied to $z \mapsto f(z-i)$. (Note that for $\beta = 0$ this argument would break down even if we didn't use 3.2.6(1) because a bounded continuous function on $\mathbb{R}$ isn't necessarily constant.) Thus

$$\omega(\tau_t(A)) = f(t) = f(0) = \omega(\tau_0(A)) = \omega(A) \quad \text{for every } t \in \mathbb{R}.\blacksquare$$

As described in 2.3 and 2.4, one of the properties of a physical system in equilibrium is that it doesn't change with time (i.e. the expectation value of each of its observables is constant in time). So 3.2.7 lends support to our conjecture that a (τ, β)-KMS state is an equilibrium state of a general physical system (not just a fdqs) at inverse temperature β with the time evolution given by τ. This is because $\omega(A)$ is the expectation value of the observable A at time 0, and $\omega(\tau_t(A))$ is the expectation value of the same observable at time t, when the system is in the state ω.

Chapter 4

Tomita-Takesaki theory

This chapter (except for 4.5 and 4.6) is based on [R, sections 2, 3 and 4]. The two central results are 4.3.11 (the Tomita-Takesaki theorem), and 4.4.11 which gives the connection with KMS states. In 4.5 we discuss the physical significance of the Tomita-Takesaki theory, bringing together the physical ideas of Chapter 2 and the mathematical ideas of this chapter, using the KMS states of Chapter 3 as a bridge. The main theme of our approach to the Tomita-Takesaki theory is the use of real subspaces of Hilbert spaces which allows us to avoid unbounded operators entirely.

4.1 A preliminary definition and result

Let $\mathfrak{H}$ be a real Hilbert space, and $\mathcal{K}$, $\mathcal{L}$ two closed vector subspaces of $\mathfrak{H}$ such that

$$\mathcal{K} \cap \mathcal{L} = \{0\},$$

and

$$\mathcal{K} + \mathcal{L} \text{ is dense in } \mathfrak{H}.$$

(Notation: $X + Y := \{x + y : x \in X, y \in Y\}$ for any subsets X, Y of a real or complex vector space.)

4.1.1 Definition Let P and Q be the orthogonal projections of $\mathfrak{H}$ on $\mathcal{K}$ and $\mathcal{L}$ respectively. Set

$$R := P + Q \quad \text{and} \quad T := |P - Q| \quad \text{(see 1.1.4)}$$

and let J be the partial isometry given by the polar decomposition, 1.5.1, such that

$$JT = P - Q \quad \text{and} \quad \ker(J) = \ker(P - Q). \blacksquare$$

We have $P, Q, R, T \in \mathfrak{L}(\mathfrak{H})_+$ by 1.1.1, 1.1.2, 1.1.4 and 1.1.5.

4.1.2 Proposition *In 4.1.1 we have:*

(1) *R and $2 - R$ are injective and $0 \leq R \leq 2$.*
(2) *$T = R^{1/2}(2 - R)^{1/2}$ and T is injective.*
(3) *J is hermitian and isometric, and $J^2 = 1$.*
(4) *T commutes with P, Q, R and J.*
(5) *$JP = (1 - Q)J$, $JQ = (1 - P)J$ and $JR = (2 - R)J$.*
(6) *$J\mathcal{K} = \mathcal{L}^\perp$.*

Proof. (1) As already mentioned, $R \geq 0$. Also, $1 - P$ and $1 - Q$ are projections since P and Q are, so $2 - R = (1 - P) + (1 - Q) \geq 0$ by 1.1.5, i.e. $R \leq 2$.

Suppose $Rx = 0$, where $x \in \mathfrak{H}$. Then

$$\begin{aligned} \|Px\|^2 + \|Qx\|^2 &= \langle Px, Px\rangle + \langle Qx, Qx\rangle \\ &= \langle x, Px\rangle + \langle x, Qx\rangle \quad \text{(since } P \text{ and } Q \text{ are projections)} \\ &= \langle x, Rx\rangle \\ &= 0. \end{aligned}$$

So $Px = 0 = Qx$ which implies $x \in \mathcal{K}^\perp \cap \mathcal{L}^\perp$. But $\mathcal{K}^\perp \cap \mathcal{L}^\perp = (\mathcal{K} + \mathcal{L})^\perp = \{0\}$ since $\mathcal{K} + \mathcal{L}$ is dense in $\mathfrak{H}$, therefore R is injective.

$\overline{\mathcal{K}^\perp + \mathcal{L}^\perp} = \left(\mathcal{K}^\perp + \mathcal{L}^\perp\right)^{\perp\perp} = \left(\mathcal{K}^{\perp\perp} \cap \mathcal{L}^{\perp\perp}\right)^\perp = (\mathcal{K} \cap \mathcal{L})^\perp = \{0\}^\perp = \mathfrak{H}$, and $\mathcal{K}^\perp \cap \mathcal{L}^\perp = \{0\}$ as mentioned above. $1 - P$ and $1 - Q$ are the orthogonal projections of $\mathfrak{H}$ on $\mathcal{K}^\perp$ and $\mathcal{L}^\perp$ respectively, so by replacing $P, Q, R, \mathcal{K}, \mathcal{L}$ in the above argument by $1 - P, 1 - Q, 2 - R, \mathcal{K}^\perp, \mathcal{L}^\perp$ we see that $2 - R$ is injective.

(2) By definition

$$\begin{aligned} T^2 &= (P - Q)^*(P - Q) = P - PQ - QP + Q = (P + Q)(2 - P - Q) \\ &= R(2 - R) \\ &= (R^{1/2}(2 - R)^{1/2})^2 \quad \text{(by 1.1.3).} \end{aligned}$$

Moreover, $T^2 \geq 0$ and $R^{1/2}(2-R)^{1/2} \geq 0$ by 1.1.3, so $T = R^{1/2}(2-R)^{1/2}$ by 1.1.2 since $T \geq 0$.

Suppose $Tx = 0$, where $x \in \mathfrak{H}$, then $R(2-R)x = T^2x = 0$, so $x = 0$ by (1). Therefore T is injective.

(3) Suppose $(P-Q)x = 0$, where $x \in \mathfrak{H}$, then $T^2x = (P-Q)^*(P-Q)x = 0$, so $x = 0$ by (2). Therefore $\ker(P-Q) = \{0\}$. Hence $J^* = J$ and $J^2 = 1$ by 1.5.1(3). Since J is a partial isometry, we have by definition (see 1.5) that $\|Jx\| = \|x\|$ for every $x \in \ker(J)^\perp = \ker(P-Q)^\perp = \mathfrak{H}$. In other words, J is isometric.

(4) Note that $JT = TJ$ by 1.5.1(2). Also $T^2P = (P-Q)^2P = P-PQP = P(P-Q)^2 = PT^2$, hence $TP = PT$ by 1.1.3. Similarly $TQ = QT$, and so $TR = RT$ since $R = P+Q$.

(5) By (4) we have $TJP = (P-Q)P = (1-Q)(P-Q) = (1-Q)TJ = T(1-Q)J$, so $JP = (1-Q)J$ since T is injective by (2). Hence, by (3), $PJ = (JP)^* = ((1-Q)J)^* = J(1-Q)$, so $JQ = J - PJ = (1-P)J$. Therefore $JR = JP + JQ = (1-Q)J + (1-P)J = (2-R)J$.

(6) By (5), the definition of P and Q, and the fact that J is surjective (since $J^2 = 1$), $J\mathcal{K} = JP\mathfrak{H} = (1-Q)J\mathfrak{H} = (1-Q)\mathfrak{H} = \mathcal{L}^\perp$.■

4.2 The operators of the theory

In this section we obtain a unitary group Δ_t and a conjugate linear isometry J that will form the main ingredients of the Tomita-Takesaki theorem in the next section.

Let $\mathfrak{H}$ be a Hilbert space, and $\mathcal{K}$ a closed real subspace of $\mathfrak{H}$ (see 1.2.5) such that

$$\mathcal{K} \cap i\mathcal{K} = \{0\},$$

and

$$\mathcal{K} + i\mathcal{K} \text{ is dense in } \mathfrak{H}.$$

$\mathfrak{H}$ and $\mathfrak{H}_\mathbb{R}$ has the same norm by 1.2.2, so $\mathcal{K}+i\mathcal{K}$ is dense in $\mathfrak{H}_\mathbb{R}$. $i\mathcal{K}$ is a closed real subspace of $\mathfrak{H}$ by 1.2.6. Therefore, replacing $\mathfrak{H}$,$\mathcal{K}$,$\mathcal{L}$ in 4.1.1 by $\mathfrak{H}_\mathbb{R}$,$\mathcal{K}$,$i\mathcal{K}$, we can define

$$P, Q, R, T, J \in \mathfrak{L}(\mathfrak{H}_\mathbb{R})$$

and then 4.1.2 still holds. In particular P and Q are the orthogonal projections of $\mathfrak{H}_\mathbb{R}$ on $\mathcal{K}$ and $i\mathcal{K}$ respectively. $\mathcal{K}^\perp$ and $(i\mathcal{K})^\perp$ are defined by 1.2.7.

4.2.1 Proposition *We have:*

(1) $R, 2-R, T \in \mathfrak{L}(\mathfrak{H})_+$.

(2) J *is a conjugate linear isometry on* $\mathfrak{H}$.

(3) $\langle x, Jy\rangle = \langle y, Jx\rangle$ *for all* $x, y \in \mathfrak{H}$.

(4) $(JAJ)^* = JA^*J$ *for all* $A \in \mathfrak{L}(\mathfrak{H})$.

Proof. (1) Let $x \in \mathfrak{H}_{\mathbb{R}}$ and write $x = y + z$ where $y \in \mathcal{K}$ and $z \in \mathcal{K}^\perp$. Then $iz \in (i\mathcal{K})^\perp$ by 1.2.8. $ix = iy + iz$ and $iy \in i\mathcal{K}$, hence by the definition of P and Q we have $iPx = iy = Q(ix)$. This implies

$$iP = Qi \quad \text{and so} \quad Pi = -i(iP)i = -i(Qi)i = -iQ(-1) = iQ.$$

Therefore $R(ix) = P(ix) + Q(ix) = iQx + iPx = iRx$. Combining this with $R \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})$ we see that $R \in \mathfrak{L}(\mathfrak{H})$. So $R, 2-R \in \mathfrak{L}(\mathfrak{H})_+$ by 4.1.2(1) and 1.2.4(2). Therefore the positive square roots of R and $2-R$ exist in $\mathfrak{L}(\mathfrak{H})$.

$T = R^{1/2}(2-R)^{1/2}$ by 4.1.2(2), where $R^{1/2}$ and $(2-R)^{1/2}$ denote the positive square roots of R and $2-R$ on $\mathfrak{H}_{\mathbb{R}}$. But positive square roots are unique on $\mathfrak{H}_{\mathbb{R}}$, so the positive square roots of R and $2-R$ in $\mathfrak{L}(\mathfrak{H})$ must be equal to $R^{1/2}$ and $(2-R)^{1/2}$ (respectively) as functions since $\mathfrak{H}$ and $\mathfrak{H}_{\mathbb{R}}$ are the same set. It follows that $R^{1/2}, (2-R)^{1/2} \in \mathfrak{L}(\mathfrak{H})$. So $T = R^{1/2}(2-R)^{1/2} \in \mathfrak{L}(\mathfrak{H})$. Therefore $T \in \mathfrak{L}(\mathfrak{H})_+$ by 4.1.1 and 1.2.4(2).

(2) From 4.1.1 and 4.1.2(4) it follows that

$$\begin{aligned} TJ(ix) &= (P-Q)(ix) = P(ix) - Q(ix) = iQx - iPx = -i(P-Q)x \\ &= -iTJx \\ &= T(-iJx) \quad (\text{since } T \in \mathfrak{L}(\mathfrak{H})). \end{aligned}$$

Therefore $J(ix) = -iJx$ since T is injective by 4.1.2(2). But J is linear on $\mathfrak{H}_{\mathbb{R}}$, hence it is conjugate linear on $\mathfrak{H}$. By 4.1.2(3) and 1.2.2, J is an isometry on $\mathfrak{H}$.

(3) Let $x, y \in \mathfrak{H}$. Then

$$\begin{aligned} \langle x, Jy\rangle &= \langle x, Jy\rangle_{\mathbb{R}} - i\langle x, iJy\rangle_{\mathbb{R}} \quad (\text{by 1.2.3}) \\ &= \langle x, Jy\rangle_{\mathbb{R}} - i\langle x, J(-iy)\rangle_{\mathbb{R}} \quad (\text{by (2)}) \\ &= \langle Jx, y\rangle_{\mathbb{R}} - i\langle Jx, -iy\rangle_{\mathbb{R}} \quad (\text{since } J \in \mathfrak{L}(\mathfrak{H}_{\mathbb{R}})_s \text{ by 4.1.2(3)}) \\ &= \langle y, Jx\rangle_{\mathbb{R}} - i\langle -iy, Jx\rangle_{\mathbb{R}} \quad (\text{by 1.2.1}) \\ &= \langle y, Jx\rangle_{\mathbb{R}} - i\langle y, iJx\rangle_{\mathbb{R}} \quad (\text{by 1.2.1}) \\ &= \langle y, Jx\rangle \quad (\text{by 1.2.3}). \end{aligned}$$

(4) By (3) we have $\langle x, JAJy\rangle = \langle AJy, Jx\rangle = \langle Jy, A^*Jx\rangle = \langle JA^*Jx, y\rangle$.■

From now on we will view T, R and J as operators on $\mathfrak{H}$ (as given by 4.2.1) rather than on $\mathfrak{H}_{\mathbb{R}}$.

For future use we note that

$$JRJ = 2 - R \quad \text{and} \quad R = J(2 - R)J$$

by 4.1.2(3) and (5).

4.2.2 Proposition We have $\sigma(2 - R) = \sigma(R)$.

Proof. Let $\lambda \in \mathbb{R}$. Since $J^2 = 1$ by 4.1.2(3), we have that J is a bijection and $J(\lambda - R)J = J\lambda J - JRJ = \lambda - (2 - R)$. Therefore $\lambda - R$ is a bijection if and only if $\lambda - (2 - R)$ is a bijection. By the open mapping theorem it follows that $\lambda - R \in \mathrm{Inv}(\mathfrak{L}(\mathfrak{H}))$ if and only if $\lambda - (2 - R) \in \mathrm{Inv}(\mathfrak{L}(\mathfrak{H}))$. By definition this means that $\sigma(R) = \sigma(2 - R)$.■

Because of 4.2.1(1), R^z and $(2-R)^z$ are defined by 1.8.12 for every $z \in \mathbb{C}$ with $\mathrm{Re}\, z \geq 0$. In particular, R^{it} and $(2 - R)^{it}$ are defined for every $t \in \mathbb{R}$.

4.2.3 Proposition We have $JR^{it}J = (2 - R)^{-it}$ *for every* $t \in \mathbb{R}$.

Proof. Throughout this proof x and y are arbitrary elements of $\mathfrak{H}$.

By 1.8.6 there is a unique spectral measure such that $R = \int z dE$, where $z : \sigma(R) \to \mathbb{C} : \lambda \mapsto \lambda$. By 1.8.5 we have $\int 1 dE = 1$ and

$$2 - R = \int (2 - z) dE \tag{1}$$

(where by definition $(2 - z)(\lambda) = 2 - \lambda$ for $\lambda \in \sigma(R)$). By 4.2.2

$$(2 - z)(\sigma(R)) = \sigma(2 - R) = \sigma(R), \tag{2}$$

so $2 - z$ is a homeomorphism from $\sigma(R)$ to itself. Thus we can define a function F on the collection of Borel sets of $\sigma(R)$ by

$$F = JE \circ (2 - z)J,$$

i.e. $F(S) = JE((2 - z)(S))J$ for every Borel set $S \subset \sigma(R)$. We now show that F is a spectral measure relative to $(\sigma(R), \mathfrak{H})$; see 1.8.1.

For every projection $P \in \mathfrak{L}(\mathfrak{H})$ we have $(JPJ)^2 = JPJ^2PJ = JPJ$ and $(JPJ)^* = JPJ$ by 4.1.2(3) and 4.2.1(4). So JPJ is also a projection in $\mathfrak{L}(\mathfrak{H})$ since it is bounded and linear by 4.2.1(2). This means that the values of the function F lie in the set of projections in $\mathfrak{L}(\mathfrak{H})$. We have

$$F(\emptyset) = JE((2-z)(\emptyset))J = JE(\emptyset)J = 0 \quad \text{by 1.8.1(1)}$$

and

$$F(\sigma(R)) = JE(\sigma(R))J = 1 \quad \text{by 1.8.1(1) and 4.1.2(3).}$$

For Borel sets $S_1, S_2 \subset \sigma(R)$ we have

$$\begin{aligned} F(S_1 \cap S_2) &= JE((2-z)(S_1) \cap (2-z)(S_2))J \quad \text{(since } 2-z \text{ is injective)} \\ &= JE((2-z)(S_1))JJE((2-z)(S_2))J \quad \text{(by 1.8.1(2) and 4.1.2(3))} \\ &= F(S_1)F(S_2). \end{aligned}$$

For every Borel set $S \subset \sigma(R)$, we write $F_{x,y}(S) := \langle x, F(S)y\rangle$. Then

$$\begin{aligned} F_{x,y}(S) &= \langle x, JE((2-z)(S))Jy\rangle = \langle E((2-z)(S)Jy, Jx\rangle \quad \text{(by 4.2.1(3))} \\ &= \langle Jy, E((2-z)(S))Jx\rangle = E_{Jy,Jx}((2-z)(S)) \quad \text{(by 1.8.1)} \end{aligned}$$

so $F_{x,y}$ is a regular Borel complex measure on $\sigma(R)$ because of 1.8.1(3) and the fact that $2-z$ is a homeomorphism.

We conclude that F is a spectral measure relative to $(\sigma(R), \mathfrak{H})$. As shown above

$$F_{x,y} = E_{Jy,Jx} \circ (2-z). \tag{3}$$

We now see that

$$\begin{aligned} \langle x, Ry\rangle &= \langle x, J(2-R)Jy\rangle = \langle Jy, (2-R)Jx\rangle \quad \text{(by 4.2.1(3) and (1))} \\ &= \left\langle Jy, \left(\int (2-z)dE\right)Jx\right\rangle \quad \text{(by (1))} \\ &= \int (2-z)dE_{Jy,Jx} \quad \text{(by 1.8.4)} \\ &= \int (2-z)\circ(2-z)d(E_{Jy,Jx}\circ(2-z)) \quad \text{(because of (2))} \\ &= \int z dF_{x,y} \quad \text{(by (3) since } (2-z)\circ(2-z) = z) \\ &= \left\langle x, \left(\int z dF\right)y\right\rangle \quad \text{(by 1.8.4).} \end{aligned}$$

Therefore $R = \int z dF$ which implies that $F = E$ by the uniqueness of E given in 1.8.6.

Consider any $t \in \mathbb{R}$ and let f be as in 1.8.12. We now find

$$\begin{aligned}
\langle x, JR^{it}Jy\rangle &= \langle Jy, R^{-it}Jx\rangle \quad \text{(by 4.2.1(3) and 1.8.13(2))} \\
&= \langle Jy, f_{-it}(R)Jx\rangle \quad \text{(by 1.8.12)} \\
&= \int f_{-it} dE_{Jy,Jx} \quad \text{(by 1.8.8 and 1.8.4)} \\
&= \int f_{-it} \circ (2-z) d(E_{Jy,Jx} \circ (2-z)) \quad \text{(because of (2)} \\
&= \int f_{-it} \circ (2-z) dE_{x,y} \quad \text{(by (3) since } F = E).
\end{aligned}$$

Set $G := E \circ (2-z)$ on the collection of Borel sets of $\sigma(R)$ and let $G_{x,y}(S) := \langle x, G(S)y\rangle$ for all $x, y \in \mathfrak{H}$ and every Borel set $S \subset \sigma(R)$. Then G is a spectral measure relative to $(\sigma(R), \mathfrak{H})$ (the proof is like the one given for F, just remove the J's). Therefore by (1)

$$\begin{aligned}
\langle x, (2-R)y\rangle &= \int (2-z) dE_{x,y} = \int (2-z) \circ (2-z) d(E_{x,y} \circ (2-z)) \\
&= \int z dG_{x,y} = \left\langle x, \left(\int z dG\right) y \right\rangle \quad \text{(by 1.8.4)}.
\end{aligned}$$

So $2 - R = \int z dG$ which means G is the resolution of the identity for $2 - R$, see 1.8.7. It follows from 1.8.8 that $(2-R)^{-it} = f_{-it}(2-R) = \int f_{-it} dG$. Therefore

$$\begin{aligned}
\langle x, (2-R)^{-it} y\rangle &= \int f_{-it} dG_{x,y} = \int f_{-it} \circ (2-z) d(G_{x,y} \circ (2-z)) \\
&= \int f_{-it} \circ (2-z) dE_{x,y} \quad \text{(since } (2-z) \circ (2-z) = z) \\
&= \langle x, JR^{it}Jy\rangle \quad \text{(as was shown above)}.
\end{aligned}$$

We conclude that $JR^{it}J = (2-R)^{-it}$.■

4.2.4 Definition A function $\mathbb{R} \to \mathfrak{L}(\mathfrak{H}) : t \mapsto U_t$ is called a *one-parameter unitary group on* $\mathfrak{H}$ if the following properties are satisfied for all $s, t \in \mathbb{R}$:

(1) $U_s U_t = U_{s+t}$;

(2) $(U_t)^* = U_{-t}$;
(3) $U_0 = 1$.
(As a shorthand we will denote the one-parameter unitary group given above simply by U_t. Note that $t \mapsto U_t$ is bounded, in fact $\|U_t\|^2 = \|U_t^* U_t\| = \|1\| = 1$.)■

4.2.5 Definition Let $\Delta_t := (2-R)^{it} R^{-it}$ for every $t \in \mathbb{R}$.■

4.2.6 Proposition *We have that Δ_t is a one-parameter unitary group on $\mathfrak{H}$ with the following properties:*

$$J\Delta_t = \Delta_t J, \quad T\Delta_t = \Delta_t T \quad \text{and} \quad \Delta_t \mathcal{K} = \mathcal{K}$$

for every $t \in \mathbb{R}$.

Proof. $R^{is}(2-R)^{it} = (2-R)^{it}R^{is}$ for all $s, t \in \mathbb{R}$ by 1.8.9 and 1.8.12. So Δ_t is a one-parameter unitary group on $\mathfrak{H}$ by 4.2.5, 4.2.4 and 1.8.13.

By 4.1.2(3) and 4.2.3

$$\begin{aligned} J\Delta_t &= J(2-R)^{it}R^{-it} = J(2-R)^{it}J^2R^{-it} = R^{-it}JR^{-it} \\ &= R^{-it}JR^{-it}J^2 = R^{-it}(2-R)^{it}J = \Delta_t J. \end{aligned}$$

$T\Delta_t = \Delta_t T$ and $R\Delta_t = \Delta_t R$ by 4.2.5, 4.1.2(4), 1.8.12 and 1.8.9. So $\Delta_t(P-Q) = \Delta_t JT = JT\Delta_t = (P-Q)\Delta_t$ and $\Delta_t(P+Q) = (P+Q)\Delta_t$ by 4.1.1, which implies that $\Delta_t P = P\Delta_t$. It follows that $\Delta_t \mathcal{K} = \Delta_t P\mathfrak{H} = P\Delta_t\mathfrak{H} = P\mathfrak{H} = \mathcal{K}$ since P is the projection of $\mathfrak{H}_{\mathbb{R}}$ on $\mathcal{K}$ while $\mathfrak{H}$ and $\mathfrak{H}_{\mathbb{R}}$ are the same sets (see 1.2.1) and Δ_t is surjective since $\Delta_t\Delta_{-t} = \Delta_0 = 1$.■

4.2.7 Definition A one-parameter unitary group U_t on $\mathfrak{H}$ is called *strongly continuous* if the function $\mathbb{R} \to \mathfrak{H} : t \mapsto U_t x$ is continuous for every $x \in \mathfrak{H}$. (By 1.3.3 it is clear that U_t is strongly continuous if and only if it is continuous in the strong topology on $\mathfrak{L}(\mathfrak{H})$ defined in 1.4.)■

4.2.8 Proposition *The one-parameter unitary group Δ_t is strongly continuous.*

Proof. The function $t \mapsto (2-R)^{it}$ is bounded on $\mathbb{R}$ by 1.8.15. The functions $t \mapsto (2-R)^{it}x$ and $t \mapsto R^{-it}x$ are continuous on $\mathbb{R}$ for every $x \in \mathfrak{H}$ by 1.8.14.

It follows from 1.7.4(1) that the function $t \mapsto (2-R)^{it}R^{-it}x = \Delta_t x$ is continuous on $\mathbb{R}$ for every $x \in \mathfrak{H}$.■

4.3 The Tomita-Takesaki theorem

In this section we consider a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$, and we assume that there exists a $\Omega \in \mathfrak{H}$ such that Ω is cyclic and separating for $\mathfrak{M}$ (see 1.4.2).

Set $\mathcal{K} := \overline{\mathfrak{M}_s\Omega}$, the closure of $\mathfrak{M}_s\Omega$ in $\mathfrak{H}$.

4.3.1 Proposition *$\mathcal{K}$ is a closed real subspace of $\mathfrak{H}$, $\mathcal{K}\cap i\mathcal{K} = \{0\}$, $\mathcal{K}+i\mathcal{K}$ is dense in $\mathfrak{H}$, and $\mathfrak{M}'_s\Omega \subset (i\mathcal{K})^\perp$ (see 1.2.7).*

Proof. $\mathcal{K}$ is clearly a closed vector subspace of $\mathfrak{H}_\mathbb{R}$ (i.e. a closed real subspace of $\mathfrak{H}$) because of 1.2.2.

For any $A \in \mathfrak{M}$ we can write $A = B + iC$ where $B, C \in \mathfrak{M}_s$, so $A\Omega \in \mathfrak{M}_s\Omega + i\mathfrak{M}_s\Omega \subset \mathcal{K}+i\mathcal{K}$. However, $\mathfrak{M}\Omega$ is dense in $\mathfrak{H}$ since Ω is cyclic for $\mathfrak{M}$, therefore $\mathcal{K}+i\mathcal{K}$ is dense in $\mathfrak{H}$.

Consider any $A \in \mathfrak{M}_s$ and $B \in \mathfrak{M}'_s$, then $\langle A\Omega, B\Omega\rangle = \langle BA\Omega, \Omega\rangle = \langle AB\Omega, \Omega\rangle = \langle B\Omega, A\Omega\rangle$, so $\langle A\Omega, B\Omega\rangle \in \mathbb{R}$. Therefore $\langle A\Omega, iB\Omega\rangle_\mathbb{R} = \mathrm{Re}\,\langle A\Omega, iB\Omega\rangle = 0$. This implies that $i\mathfrak{M}'_s\Omega \subset \mathcal{K}^\perp$, so $\mathfrak{M}'_s\Omega \subset -i\mathcal{K}^\perp = i\mathcal{K}^\perp \subset (i\mathcal{K})^\perp$ by 1.2.8.

For any $A \in \mathfrak{M}'$ we can write $A = B + iC$ where $B, C \in \mathfrak{M}'_s$, so $A\Omega \in \mathfrak{M}'_s\Omega + i\mathfrak{M}'_s\Omega \subset i\mathcal{K}^\perp + ii\mathcal{K}^\perp = i\mathcal{K}^\perp + \mathcal{K}^\perp \subset (i\mathcal{K})^\perp + \mathcal{K}^\perp \subset (\mathcal{K}\cap i\mathcal{K})^\perp$ where $(\mathcal{K}\cap i\mathcal{K})^\perp$ is taken in $\mathfrak{H}_\mathbb{R}$. The vector Ω is separating for $\mathfrak{M} = \mathfrak{M}''$, so it is cyclic for $\mathfrak{M}'$ by 1.4.3, i.e. $\mathfrak{M}'\Omega$ is dense in $\mathfrak{H} = \mathfrak{H}_\mathbb{R}$ (see 1.2.1 and 1.2.2). Hence $(\mathcal{K}\cap i\mathcal{K})^\perp = \mathfrak{H}_\mathbb{R}$ and we conclude that $\mathcal{K}\cap i\mathcal{K} = \{0\}$.■

Using 4.3.1 we can now define P and Q on $\mathfrak{H}_\mathbb{R}$, and R, T, J and Δ_t on $\mathfrak{H}$ exactly as in 4.2 in terms of $\mathcal{K} = \overline{\mathfrak{M}_s\Omega}$.

4.3.2 Definition We call J the *modular conjugation associated to* $(\mathfrak{M}, \Omega)$. We call Δ_t the *unitary group associated to* $(\mathfrak{M}, \Omega)$.■

In what follows we give a series of intermediate results culminating in the Tomita-Takesaki theorem (4.3.11). These intermediate results are nothing more than a breakdown of the proof of 4.3.11 into smaller parts, except for $\Delta_t\Omega = \Omega$ in 4.3.9 which will be used in 4.4.

4.3.3 Lemma *Let $B \in \mathfrak{M}'_s$ and consider any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$. Then there is an $A \in \mathfrak{M}_s$ such that*

$$\langle B\Omega, C\Omega\rangle = \mathrm{Re}(\lambda\,\langle A\Omega, C\Omega\rangle)$$

for every $C \in \mathfrak{M}_s$.

Proof. Since every $U \in \mathfrak{M}$ can be written in a unique way as $U = V + iW$ where $V, W \in \mathfrak{M}_s$, we can define linear functionals ψ and ψ_A^λ on $\mathfrak{M}$ by

$$\psi(C) = \langle B\Omega, C\Omega \rangle \quad \text{and} \quad \psi_A^\lambda(C) = \mathrm{Re}(\lambda \langle A\Omega, C\Omega \rangle) \quad \text{for } C \in \mathfrak{M}_s$$

for every $A \in \mathfrak{M}$. Note that $\psi(C) \in \mathbb{R}$ for $C \in \mathfrak{M}_s$ since $B \in \mathfrak{M}_s'$.

We are now going to show that $\psi = \psi_A^\lambda$ for some $A \in \mathfrak{M}_s$, and this will complete our proof. We break the proof down into several parts.

(a) Clearly ψ is σ-weakly continuous by 1.3.5 and 1.3.3. For every $A \in \mathfrak{M}$ and $C \in \mathfrak{M}_s$

$$\psi_A^\lambda(C) = \frac{1}{2}\lambda \langle A\Omega, C\Omega \rangle + \frac{1}{2}\overline{\lambda} \langle C\Omega, A\Omega \rangle = \frac{1}{2}\lambda \langle A\Omega, C\Omega \rangle + \frac{1}{2}\overline{\lambda} \langle \Omega, CA\Omega \rangle$$

so $\psi_A^\lambda(U) = \frac{1}{2}\lambda \langle A\Omega, U\Omega \rangle + \frac{1}{2}\overline{\lambda} \langle \Omega, UA\Omega \rangle$ for every $U \in \mathfrak{M}$. It follows from 1.3.5 and 1.3.3 that ψ_A^λ is σ-weakly continuous.

In other words $\psi, \psi_A^\lambda \in \mathfrak{M}_*$ for every $A \in \mathfrak{M}$ (see 1.4).

(b) Suppose there is an $A' \in \mathfrak{M}_s$ such that

$$\psi(C) = \mathrm{Re}\left(\frac{\lambda}{\mathrm{Re}\,\lambda} \langle A'\Omega, C\Omega \rangle\right) \quad \text{for every } C \in \mathfrak{M}_s.$$

Then, setting $A := A'/\mathrm{Re}\,\lambda$, we find that $\psi(C) = \psi_A^\lambda(C)$ for every $C \in \mathfrak{M}_s$ as wanted. So we may assume that $\mathrm{Re}\,\lambda = 1$.

(c) Suppose there is an $A' \in \mathfrak{M}_s$ such that

$$\langle (B + \|B\|)\Omega, C\Omega \rangle = \mathrm{Re}(\lambda \langle A'\Omega, C\Omega \rangle) \quad \text{for every } C \in \mathfrak{M}_s.$$

It then follows for every $C \in \mathfrak{M}_s$ that

$$\langle B\Omega, C\Omega \rangle = \mathrm{Re}(\lambda \langle (A' - \|B\|)\Omega, C\Omega \rangle)$$

since $\mathrm{Re}\,\lambda = 1$ and $\langle \|B\|\,\Omega, C\Omega \rangle \in \mathbb{R}$ as $C^* = C$. Therefore $A := A' - \|B\|$ is the wanted operator. $B + \|B\| \geq 0$ by 1.4.1, so we may assume $B \geq 0$.

(d) If $B = 0$ the result follows by setting $A := 0$. So we may assume that $B \neq 0$. Suppose there is an $A' \in \mathfrak{M}_s$ such that

$$\left\langle \frac{B}{\|B\|}\Omega, C\Omega \right\rangle = \mathrm{Re}(\lambda \langle A'\Omega, C\Omega \rangle) \quad \text{for every } C \in \mathfrak{M}_s.$$

Then $A := \|B\|\,A'$ is the wanted operator. Since $B/\|B\| \leq \|B/\|B\|\| = 1$ by 1.4.1, we may assume $B \leq 1$.

(e) By (a) we may define $g : \mathfrak{M}_s \to \mathfrak{M}_* : A \mapsto \psi_A^\lambda$. We show that g is continuous with the weak* topology on $\mathfrak{M}$ and the weak topology on $\mathfrak{M}_*$ (see 1.4 for the definitions of these topologies). Let $A \in \mathfrak{M}_s$ and consider any basic closed weak neighbourhood of $g(A)$ as given by 1.3.3, say

$$N := \{\omega \in \mathfrak{M}_* : |(\omega - g(A))(M_k)| \leq \varepsilon \text{ for } k = 1, ..., n\}$$

where $\varepsilon > 0$ and $M_1, ..., M_n \in \mathfrak{M}$. We have to show there exists a weak* neighbourhood N' of A such that $g(N') \subset N$.

Set $\omega_k := \psi^{\overline{\lambda}}_{\operatorname{Re} M_k} + i\psi^{\overline{\lambda}}_{\operatorname{Im} M_k} \in \mathfrak{M}_*$ (by (a)) for $k = 1, ..., n$. (See 1.4.9 for the definition of Re and Im.) Let

$$N' := \{A' \in \mathfrak{M}_s : |\omega_k(A' - A)| \leq \varepsilon \text{ for } k = 1, ..., n\}$$

then N' is a weak* neighbourhood of A by 1.3.3. Consider any $A' \in N'$. Then, for $k = 1, ..., n$, we have

$$\begin{aligned}
& |(g(A') - g(A))(M_k)| \\
= \; & |\psi^\lambda_{A'}(\operatorname{Re} M_k) + i\psi^\lambda_{A'}(\operatorname{Im} M_k) - \psi^\lambda_A(\operatorname{Re} M_k) - i\psi^\lambda_A(\operatorname{Im} M_k)| \\
= \; & |\operatorname{Re}(\lambda \langle A'\Omega, (\operatorname{Re} M_k)\Omega\rangle) + i \operatorname{Re}(\lambda \langle A'\Omega, (\operatorname{Im} M_k)\Omega\rangle \\
& - \operatorname{Re}(\lambda \langle A\Omega, (\operatorname{Re} M_k)\Omega\rangle) - i \operatorname{Re}(\lambda \langle A\Omega, (\operatorname{Im} M_k)\Omega\rangle)| \\
= \; & |\operatorname{Re}(\lambda \langle (A' - A)\Omega, (\operatorname{Re} M_k)\Omega\rangle) + i \operatorname{Re}(\lambda \langle (A' - A)\Omega, (\operatorname{Im} M_k)\Omega\rangle)| \\
= \; & |\operatorname{Re}(\overline{\lambda} \langle (\operatorname{Re} M_k)\Omega, (A' - A)\Omega\rangle) + i \operatorname{Re}(\overline{\lambda} \langle (\operatorname{Im} M_k)\Omega, (A' - A)\Omega\rangle)| \\
= \; & |\omega_k(A' - A)| \quad \text{(by the definition of } \omega_k) \\
\leq \; & \varepsilon \quad \text{(by the definition of } N').
\end{aligned}$$

So $g(A') \in N$ and therefore $g(N') \subset g(N)$ as needed.

(f) From now on we write

$$\psi_A := \psi_A^\lambda.$$

Let $V := \{\psi_A : A \in \mathfrak{M}_s \text{ and } \|A\| \leq 1\} \subset \mathfrak{M}_*$. We show that V is weakly closed and convex.

Let $N := \{A \in \mathfrak{M} : \|A\| \leq 1\}$, then $V = g(\mathfrak{M}_s \cap N)$. The set N is weakly* compact according to 1.4.5(1). Also, $\mathfrak{M}_s$ is weakly* closed by 1.4.4. Therefore $\mathfrak{M}_s \cap N$ is weakly* compact. Because of the continuity of g given in (e), we conclude that $g(\mathfrak{M}_s \cap N)$ is weakly compact, i.e. V is weakly compact. Hence V is weakly closed since the weak topology on $\mathfrak{M}_*$ is Hausdorff by 1.4.5(2).

Consider any $t \in [0,1]$ and $A, A' \in \mathfrak{M}_s \cap N$. For every $C \in \mathfrak{M}_s$

$$\begin{aligned}(tg(A)+(1-t)g(A'))(C) &= t\,\mathrm{Re}(\lambda\langle A\Omega, C\Omega\rangle) + (1-t)\,\mathrm{Re}(\lambda\langle A'\Omega, C\Omega\rangle)\\ &= \mathrm{Re}(\lambda\langle (tA+(1-t)A'\Omega, C\Omega\rangle)\\ &= g(tA+(1-t)A')(C)\end{aligned}$$

which means that $tg(A)+(1-t)g(A') = g(tA+(1-t)A')$. Clearly $tA+(1-t)A' \in \mathfrak{M}_s \cap N$, therefore $tg(A)+(1-t)g(A') \in V$, so V is convex.

(g) If $\psi \in V$, the result follows. So suppose $\psi \notin V$. Then by 1.3.6 there exists a weakly continuous linear functional f on $\mathfrak{M}_*$ such that $f(\psi) \notin \overline{f(V)}$ since V is convex and weakly closed by (f). Also, $\overline{f(V)}$ is convex since f is linear and V is convex, so by 1.6.7 there exists an $r \in \mathbb{C}$ with $|r| = 1$ such that

$$\mathrm{Re}(rf(\psi)) > \mathrm{Re}(rw) \quad \text{for every } w \in \overline{f(V)}. \tag{1}$$

But by 1.4.5(2), 1.3.10 and 1.3.7 there exists an $M \in \mathfrak{M}$ such that

$$f(\omega) = \omega(M) \text{ for all } \omega \in \mathfrak{M}_*.$$

Therefore $\mathrm{Re}(r\psi(M)) > \mathrm{Re}(r\omega(M))$ for every $\omega \in V$ by (1). It follows that $\mathrm{Re}(r\psi(M)) > \mathrm{Re}(r\psi_A(M))$ for every $A \in \mathfrak{M}_s \cap N$ by the definitions of V and N given in (f). In other words

$$(\mathrm{Re}\,r)\psi(\mathrm{Re}\,M) - (\mathrm{Im}\,r)\psi(\mathrm{Im}\,M) > (\mathrm{Re}\,r)\psi_A(\mathrm{Re}\,M) - (\mathrm{Im}\,r)\psi_A(\mathrm{Im}\,M)$$

for every $A \in \mathfrak{M}_s \cap N$, since by definition $\psi(C), \psi_A(C) \in \mathbb{R}$ for $C \in \mathfrak{M}_s$. Writing $D := (\mathrm{Re}\,r)\,\mathrm{Re}\,M - (\mathrm{Im}\,r)\,\mathrm{Im}\,M \in \mathfrak{M}_s$, we get $\psi(D) > \psi_A(D)$ for every $A \in \mathfrak{M}_s \cap N$, i.e.

$$\langle B\Omega, D\Omega\rangle > \mathrm{Re}(\lambda\langle A\Omega, D\Omega\rangle) \quad \text{for every } A \in \mathfrak{M}_s \cap N \tag{2}$$

by the definition of ψ and ψ_A. By 1.5.1 we have a partial isometry $U \in \mathfrak{L}(\mathfrak{H})$ such that $D = U|D|$. By 1.5.1(4) $U^* = U$, so by 1.5.1(1) $UD = U^*D = |D| = |D|^* = (U^*D)^* = DU$. Hence $\langle U\Omega, D\Omega\rangle = \langle \Omega, UD\Omega\rangle = \langle \Omega, DU\Omega\rangle = \langle D\Omega, U\Omega\rangle$ which implies that $\langle U\Omega, D\Omega\rangle \in \mathbb{R}$. Since U is a partial isometry $\|U\| \leq 1$, while $U \in \mathfrak{M}$ by 1.5.1(5), thus $U \in \mathfrak{M}_s \cap N$. Note that $B^{1/2} \in \mathfrak{M}'$ since $B \geq 0$ by (c) and $B \in \mathfrak{M}'$ which is a C*-algebra. Therefore, setting $A = U$ in (2), we find

$$\mathrm{Re}(\lambda\langle U\Omega, D\Omega\rangle) \;<\; \langle B\Omega, D\Omega\rangle$$

$$
\begin{aligned}
&= \langle B^{1/2}\Omega, B^{1/2}D\Omega\rangle \\
&= \langle B^{1/2}\Omega, DB^{1/2}\Omega\rangle \quad (\text{since } B^{1/2} \in \mathfrak{M}') \\
&\leq \langle B^{1/2}\Omega, |D|B^{1/2}\Omega\rangle \quad (\text{since } D \leq |D| \text{ by } 1.4.1) \\
&= \langle |D|^{1/2}\Omega, B|D|^{1/2}\Omega\rangle \quad (\text{since } B^{1/2} \in \mathfrak{M}' \text{ and } D \in \mathfrak{M}) \\
&\leq \langle |D|^{1/2}\Omega, |D|^{1/2}\Omega\rangle \quad (\text{since } B \leq 1 \text{ by (d)}) \\
&= \langle \Omega, |D|\Omega\rangle \\
&= \langle \Omega, U^*D\Omega\rangle \quad (\text{by } 1.5.1(1)) \\
&= \langle U\Omega, D\Omega\rangle \\
&= \operatorname{Re}(\lambda \langle U\Omega, D\Omega\rangle)
\end{aligned}
$$

since $\operatorname{Re}\lambda = 1$ by (b) and $\langle U\Omega, D\Omega\rangle \in \mathbb{R}$. But this is absurd, hence our supposition $\psi \notin V$ must be wrong.■

4.3.4 Corollary *For every $B \in \mathfrak{M}'$ there is an $A \in \mathfrak{M}$ such that*

$$JTB\Omega = A\Omega \text{ and } JTB^*\Omega = A^*\Omega.$$

Proof. (a) First assume $B \in \mathfrak{M}'_s$. Apply 4.3.3 with $\lambda = 1$ to obtain an $A \in \mathfrak{M}_s$ such that $\langle B\Omega, C\Omega\rangle = \operatorname{Re}\langle A\Omega, C\Omega\rangle$ for all $C \in \mathfrak{M}_s$, i.e. $\langle B\Omega, C\Omega\rangle_{\mathbb{R}} = \langle A\Omega, C\Omega\rangle_{\mathbb{R}}$ for all $C \in \mathfrak{M}_s$. So $\langle B\Omega - A\Omega, C\Omega\rangle_{\mathbb{R}} = 0$ for all $C \in \mathfrak{M}_s$. Hence $B\Omega - A\Omega \in \overline{\mathfrak{M}_s\Omega}^{\perp} = \mathcal{K}^{\perp}$, but $A\Omega \in \mathcal{K}$ since $A \in \mathfrak{M}_s$, therefore

$$0 = P(B\Omega - A\Omega) = PB\Omega - A\Omega$$

since by definition P is the projection of $\mathfrak{H}_{\mathbb{R}}$ on $\mathcal{K}$. $B\Omega \in (i\mathcal{K})^{\perp}$ by 4.3.1, so $QB\Omega = 0$ since by definition Q is the projection of $\mathfrak{H}_{\mathbb{R}}$ on $i\mathcal{K}$. Thus

$$JTB\Omega = (P - Q)B\Omega = A\Omega \text{ according to } 4.1.1.$$

(b) Now consider any $B \in \mathfrak{M}'$. Then by (a) there exist $A, A' \in \mathfrak{M}_s$ such that $JT(\operatorname{Re}B)\Omega = A\Omega$ and $JT(\operatorname{Im}B)\Omega = A'\Omega$. By 4.2.1(2) we then have $JT(i\operatorname{Im}B)\Omega = -iA'\Omega$. Hence

$$JTB\Omega = JT(\operatorname{Re}B + i\operatorname{Im}B)\Omega = (A - iA')\Omega$$

and

$$JTB^*\Omega = JT(\operatorname{Re}B - i\operatorname{Im}B)\Omega = (A + iA')\Omega = (A - iA')^*\Omega.$$

So $A := A - iA' \in \mathfrak{M}$ is the operator we wanted.■

4.3.5 Lemma *Let $B \in \mathfrak{M}'$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Then there is an $A \in \mathfrak{M}$ such that*

$$TJBJT = \lambda(2-R)AR + \overline{\lambda}RA(2-R).$$

Proof. Suppose there are $A, A' \in \mathfrak{M}$ such that

$$TJ(\operatorname{Re}B)JT = \lambda(2-R)AR + \overline{\lambda}RA(2-R)$$

and

$$TJ(\operatorname{Im}B)JT = \lambda(2-R)A'R + \overline{\lambda}RA'(2-R).$$

Then

$$\begin{aligned} TJBJT &= TJ(\operatorname{Re}B)JT - iTJ(\operatorname{Im}B)JT \quad \text{(by 4.2.1(2))} \\ &= \lambda(2-R)(A-iA')R + \overline{\lambda}R(A-iA')(2-R) \end{aligned}$$

while $A - iA' \in \mathfrak{M}$. So we may assume that B is hermitian. Therefore, by 4.3.3, there is an $A \in \mathfrak{M}_s$ such that for all $C \in \mathfrak{M}_s$

$$\begin{aligned} \langle B\Omega, C\Omega\rangle &= \operatorname{Re}(\lambda\langle 2A\Omega, C\Omega\rangle) = 2\operatorname{Re}(\lambda\langle A\Omega, C\Omega\rangle) \\ &= \lambda\langle A\Omega, C\Omega\rangle + \overline{\lambda}\langle C\Omega, A\Omega\rangle. \end{aligned}$$

So for all $C \in \mathfrak{M}$ we have

$$\begin{aligned} \langle B\Omega, C\Omega\rangle &= \langle B\Omega, (\operatorname{Re}C)\Omega\rangle + i\langle B\Omega, (\operatorname{Im}C)\Omega\rangle \\ &= \lambda\langle A\Omega, (\operatorname{Re}C + i\operatorname{Im}C)\Omega\rangle + \overline{\lambda}\langle(\operatorname{Re}C - i\operatorname{Im}C)\Omega, A\Omega\rangle \\ &= \lambda\langle A\Omega, C\Omega\rangle + \overline{\lambda}\langle C^*\Omega, A\Omega\rangle. \end{aligned}$$

For all $C, D \in \mathfrak{M}$ it follows (substituting D^*C for C in the equality above) that

$$\langle BD\Omega, C\Omega\rangle = \langle B\Omega, D^*C\Omega\rangle = \lambda\langle DA\Omega, C\Omega\rangle + \overline{\lambda}\langle D\Omega, CA\Omega\rangle \qquad (1)$$

since $B \in \mathfrak{M}'$. Consider any $C', D' \in \mathfrak{M}'$, then by 4.3.4 there exist $C, D \in \mathfrak{M}$ such that

$$JTC'\Omega = C\Omega \text{ and } JTD'\Omega = D\Omega \qquad (2)$$

and

$$JTC'^*\Omega = C^*\Omega \text{ and } JTD'^*\Omega = D^*\Omega. \qquad (3)$$

Substituting (2) into (1) we get

$$\langle BJTD'\Omega, JTC'\Omega\rangle = \lambda\langle DA\Omega, JTC'\Omega\rangle + \overline{\lambda}\langle JTD'\Omega, CA\Omega\rangle$$

but $B^* = B$, so using 4.2.1(3) we have

$$\langle JBJTC'\Omega, TD'\Omega\rangle = \lambda\langle TC'\Omega, JDA\Omega\rangle + \overline{\lambda}\langle JCA\Omega, TD'\Omega\rangle .$$

Since $T^* = T$ by 4.2.1(1), it follows that

$$\langle TJBJTC'\Omega, D'\Omega\rangle = \lambda\langle C'\Omega, TJDA\Omega\rangle + \overline{\lambda}\langle TJCA\Omega, D'\Omega\rangle \qquad (4)$$

For every $U \in \mathfrak{M}_s$, we have $U\Omega \in \mathcal{K}$ (by definition) so $PU\Omega = U\Omega$ by definition of P, hence by 4.1.1 and 4.1.2(4)

$$TJU\Omega = (P-Q)U\Omega = (2-P-Q)U\Omega = (2-R)U\Omega. \qquad (5)$$

For every $U' \in \mathfrak{M}'_s$, we have $U'\Omega \in (i\mathcal{K})^{\perp}$ by 4.3.1 so $QU'\Omega = 0$ by definition of Q, hence by 4.1.1 and 4.1.2(4)

$$TJU'\Omega = (P-Q)U'\Omega = (P+Q)U'\Omega = RU'\Omega. \qquad (6)$$

Using (5) and 4.2.1(2) we find that for every $U \in \mathfrak{M}$

$$TJU\Omega = TJ(\mathrm{Re}\,U)\Omega - iTJ(\mathrm{Im}\,U)\Omega = (2-R)U^*\Omega. \qquad (7)$$

Similarly we find using (6) that

$$TJU'\Omega = RU'^*\Omega \quad \text{for every } U' \in \mathfrak{M}'. \qquad (8)$$

By (7) and (3) we have (setting $U = DA$ in (7))

$$\begin{aligned} TJDA\Omega &= (2-R)AD^*\Omega = (2-R)AJTD'^*\Omega \\ &= (2-R)ARD'\Omega \quad \text{(by (8) and 4.1.2(4))}. \end{aligned}$$

Likewise $TJCA\Omega = (2-R)ARC'\Omega$. Substituting these two expressions in (4) we get

$$\begin{aligned} \langle TJBJTC'\Omega, D'\Omega\rangle &= \lambda\langle C'\Omega, (2-R)ARD'\Omega\rangle + \overline{\lambda}\langle (2-R)ARC'\Omega, D'\Omega\rangle \\ &= \langle \overline{\lambda}RA(2-R)C'\Omega, D'\Omega\rangle + \langle \lambda(2-R)ARC'\Omega, D'\Omega\rangle \end{aligned}$$

since $R^* = R$ by 4.2.1(1). Since Ω is cyclic for $\mathfrak{M}'$ as pointed out in the proof of 4.3.1, it follows that

$$TJBJT = \overline{\lambda}RA(2-R) + \lambda(2-R)AR$$

by the definition of a cyclic vector, 1.4.2.■

4.3.6 Lemma *Let $\lambda = e^{i\varphi/2}$ where $-\pi < \varphi < \pi$, and let A and B be as in 4.3.5 (this is possible since $\operatorname{Re}\lambda = \cos\frac{\varphi}{2} > 0$). Then*

$$\langle x, Ay\rangle = \int_{-\infty}^{+\infty} e^{-\varphi t}(e^{\pi t} + e^{-\pi t})^{-1} \langle x, \Delta_t JBJ\Delta_{-t} y\rangle \, dt$$

for all $x, y \in \mathfrak{H}$.

Proof. Set $Z := \{z \in \mathbb{C} : |\operatorname{Re} z| \leq \frac{1}{2}\}$ and consider any $x, y \in \mathfrak{H}$. Let

$$f : Z \to \mathbb{C} : z \mapsto \left\langle x, R^{-z+1/2}(2-R)^{z+1/2} A R^{z+1/2}(2-R)^{-z+1/2} y\right\rangle$$

(this is well-defined by 1.8.12 and 4.2.1(1) since $\operatorname{Re}(-z+1/2), \operatorname{Re}(z+1/2) \in \mathbb{R}^+$ for $z \in Z$). By 1.8.15, 1.8.14 and 1.7.4 it is clear that f is bounded and continuous on Z, and differentiable (i.e. analytic, see 1.6.1) inside Z. For every $t \in \mathbb{R}$ we have by 4.2.5, 1.8.9 and 1.8.13(1) and (3) that

$$\begin{aligned}
\lambda f(it + \tfrac{1}{2}) + \overline{\lambda} f(it - \tfrac{1}{2}) &= \lambda \langle x, \Delta_t(2-R)AR\Delta_{-t} y\rangle + \overline{\lambda}\langle x, \Delta_t RA(2-R)\Delta_{-t} y\rangle \\
&= \langle x, \Delta_t TJBJT\Delta_{-t} y\rangle \quad \text{(by 4.3.5)} \\
&= \langle Tx, \Delta_t JBJ\Delta_{-t} Ty\rangle \quad \text{(by 4.2.6 and 4.2.1(1))}.
\end{aligned}$$

Moreover, $f(0) = \langle x, TATy\rangle = \langle Tx, ATy\rangle$ by 4.1.2(2), 1.8.12 and 4.2.1(1), so from 1.6.8 it follows that

$$\langle Tx, ATy\rangle = \int_{-\infty}^{+\infty} e^{-\varphi t}(e^{\pi t} + e^{-\pi t})^{-1} \langle Tx, \Delta_t JBJ\Delta_{-t} Ty\rangle \, dt. \tag{1}$$

By 4.2.1(1), 4.1.2(2) and 1.1.7 we have that $T\mathfrak{H}$ is dense in $\mathfrak{H}$, hence there exist sequences (x_n) and (y_n) in $T\mathfrak{H}$ such that

$$x_n \to x \quad \text{and} \quad y_n \to y. \tag{2}$$

Let $G : \mathbb{R} \to \mathbb{R} : t \mapsto e^{-\varphi t}(e^{\pi t} + e^{-\pi t})^{-1}$. Define $g, h, g_n, h_n : \mathbb{R} \to \mathbb{C}$ by

$$g(t) = \langle x, \Delta_t JBJ\Delta_{-t} y\rangle, \quad h(t) = G(t)g(t) \tag{3a}$$

$$g_n(t) = \langle x_n, \Delta_t JBJ\Delta_{-t} y_n\rangle, \quad h_n(t) = G(t)g_n(t) \tag{3b}$$

for $n = 1, 2, 3, \ldots$. So by (1)

$$\langle x_n, Ay_n\rangle = \int_{-\infty}^{+\infty} h_n(t)dt. \tag{4}$$

JBJ is a bounded linear operator by 4.2.1(2), and (x_n) and (y_n) are bounded by (2), hence since $t \mapsto \Delta_t$ is bounded (see 4.2.4 and 4.2.6) it follows that there exists an $M \in \mathbb{R}$ such that $|g_n(t)| < M$ for all n and t. Clearly

$$G(t) = 1/(e^{(\pi+\varphi)t} + e^{-(\pi-\varphi)t}). \tag{5}$$

From (2) and (3) it is clear that $g_n(t) \to g(t)$ for every t, hence by (3)

$$h_n(t) \to h(t) \quad \text{for every } t.$$

Also, $|h_n(t)| \leq MG(t)$ for all n and t, while

$$\begin{aligned} \int_{-\infty}^{+\infty} G(t)dt &\leq \int_{-\infty}^{0} e^{(\pi-\varphi)t}dt + \int_{0}^{+\infty} e^{-(\pi+\varphi)t}dt \quad \text{(using (5))} \\ &= 1/(\pi-\varphi) + 1/(\pi+\varphi) \\ &< \infty \quad (\text{since } -\pi < \varphi < \pi). \end{aligned}$$

So from Lebesgue's dominated convergence theorem ([Ru, 1.34]) it follows that

$$\int_{-\infty}^{+\infty} h(t)dt = \lim_{n\to\infty} \int_{-\infty}^{+\infty} h_n(t)dt.$$

By (2) and (4) we conclude that $\langle x, Ay\rangle = \int_{-\infty}^{+\infty} h(t)dt$.■

4.3.7 Lemma *Let $B \in \mathfrak{M}'$. For every $t \in \mathbb{R}$ we then have*

$$\Delta_t JBJ\Delta_{-t} \in \mathfrak{M}.$$

Proof. Set $\lambda = e^{i\varphi/2}$ where $-\pi < \varphi < \pi$ (so $\operatorname{Re}\lambda = \cos\frac{\varphi}{2} > 0$) and let A then be given by 4.3.5. Consider any $C \in \mathfrak{M}'$ and $x, y \in \mathfrak{H}$. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} g(t) &= \langle x, (C\Delta_t JBJ\Delta_{-t} - \Delta_t JBJ\Delta_{-t}C)y\rangle \\ &= \langle C^*x, \Delta_t JBJ\Delta_{-t}y\rangle - \langle x, \Delta_t JBJ\Delta_{-t}Cy\rangle\,. \end{aligned}$$

Then according to 4.3.6

$$\int_{-\infty}^{+\infty} e^{-\varphi t}(e^{\pi t} + e^{-\pi t})^{-1} g(t)dt = \langle C^*x, Ay\rangle - \langle x, ACy\rangle = 0 \tag{1}$$

since $A \in \mathfrak{M}$ and $C \in \mathfrak{M}'$. Let $Z := \{z \in \mathbb{C} : |\operatorname{Re} z| < \pi\}$ and

$$f : Z \to \mathbb{C} : z \mapsto \int_{-\infty}^{+\infty} e^{-zt}(e^{\pi t} + e^{-\pi t})^{-1} g(t)dt \tag{2}$$

The function f is well-defined. Indeed, since $t \mapsto \Delta_t$ is bounded (see 4.2.4), we have that g is bounded (say by $M \in \mathbb{R}$) and continuous by 4.2.1(2), 4.2.8 and 1.7.4(1)). Furthermore

$$\int_{-\infty}^{+\infty} \left|e^{-zt}(e^{\pi t} + e^{-\pi t})^{-1} g(t)\right| dt \leq M \int_{-\infty}^{+\infty} e^{-(\operatorname{Re} z)t}(e^{\pi t} + e^{-\pi t})^{-1} dt < \infty$$

in the same way as $\int_{-\infty}^{+\infty} G(t)dt < \infty$ in the proof of 4.3.6.

Let S be any triangular path in Z, then by applying Fubini's theorem ([Ru, 8.8]) to each edge of S separately we find from (2) using Cauchy's integral theorem that

$$\int_S f = \int_{-\infty}^{+\infty} \left(\int_S e^{-zt}(e^{\pi t} + e^{-\pi t})^{-1} g(t)dz \right) dt = 0$$

since $z \mapsto e^{-iz}$ is analytic. Hence f is analytic by Morera's theorem ([Con, IV 5.10]) since f is continuous as can easily be confirmed from the definition of f using Lebesgue's dominated convergence theorem. According to (1), $f(z) = 0$ if z is real, thus $f = 0$ by 1.6.2. In particular $f(is) = 0$ for real s, but then by (2) and 1.6.6

$$(e^{\pi t} + e^{-\pi t})^{-1} g(t) = 0 \quad \text{for all } t \in \mathbb{R} \backslash X$$

where X is some set in $\mathbb{R}$ with Lebesgue measure 0. Since g is continuous we conclude that $g = 0$. So, by the definition of g,

$$C\Delta_t JBJ\Delta_{-t} = \Delta_t JBJ\Delta_{-t} C \quad \text{for every } C \in \mathfrak{M}',$$

i.e. $\Delta_t JBJ\Delta_{-t} \in \mathfrak{M}'' = \mathfrak{M}$ for all $t \in \mathbb{R}$ since JBJ is a bounded linear operator by 4.2.1(2).■

4.3.8 Corollary *$J\mathfrak{M}'J \subset \mathfrak{M}$.*

Proof. $JBJ = \Delta_0 JBJ\Delta_0 \in \mathfrak{M}$ for $B \in \mathfrak{M}'$ by 4.2.6 and 4.3.7 .■

4.3.9 Proposition *$J\Omega = \Omega$ and $\Delta_t\Omega = \Omega$.*

Proof. $1 \in \mathfrak{M}_s$ and $1 \in \mathfrak{M}'_s$, so $\Omega \in \mathfrak{M}_s\Omega \subset \mathcal{K}$ and $\Omega \in \mathfrak{M}'_s\Omega \subset (i\mathcal{K})^\perp$ by 4.3.1. Therefore $P\Omega = \Omega$ and $Q\Omega = 0$ by the definitions of P and Q. So $T^2\Omega = (P-Q)^2\Omega = \Omega$ and $JT\Omega = (P-Q)\Omega = \Omega$ by 4.1.1, 1.1.4 and 1.1.5. Also, $T \geq 0$ and $T^2 \geq 0$ by 4.2.1 and 1.1.3. Therefore $T\Omega = (T^2)^{1/2}\Omega = (1)^{1/2}\Omega = \Omega$ by 1.8.16 and 1.8.12, and so $J\Omega = J^2T\Omega = T\Omega = \Omega$ by 4.1.2(3).

$R\Omega = (P+Q)\Omega = \Omega$, so $(2-R)\Omega = \Omega$. Therefore $\Delta_t\Omega = \Omega$ by 4.2.5, 1.8.12 and 1.8.16.■

4.3.10 Lemma $J\mathfrak{M}J \subset \mathfrak{M}'$.

Proof. We break the proof down into several parts.

(a) Consider any $A, B \in \mathfrak{M}_s$. Then $JA\Omega \in (i\mathcal{K})^\perp$ since $J\mathcal{K} = (i\mathcal{K})^\perp$ by 4.1.2(6). So $\operatorname{Re}\langle JA\Omega, iB\Omega\rangle = \langle JA\Omega, iB\Omega\rangle_{\mathbb{R}} = 0$ by the definition of $\mathcal{K}$ and 1.2.7. Hence $\operatorname{Im}\langle JA\Omega, B\Omega\rangle = -\operatorname{Re}(i\langle JA\Omega, B\Omega\rangle) = -\operatorname{Re}\langle JA\Omega, iB\Omega\rangle = 0$ which means that $\langle JA\Omega, B\Omega\rangle \in \mathbb{R}$. Therefore $\langle JA\Omega, B\Omega\rangle = \langle B\Omega, JA\Omega\rangle = \langle A\Omega, JB\Omega\rangle$ by 4.2.1(3). So

$$\langle BJA\Omega, \Omega\rangle = \langle JA\Omega, B\Omega\rangle = \langle A\Omega, JB\Omega\rangle = \langle \Omega, AJB\Omega\rangle .$$

(b) Consider any $A \in \mathfrak{M}_s$ and $B \in \mathfrak{M}$, then it follows from (a) that

$$\begin{aligned}\langle BJA\Omega, \Omega\rangle &= \langle (\operatorname{Re}B)JA\Omega, \Omega\rangle - i\langle (\operatorname{Im}B)JA\Omega, \Omega\rangle \\ &= \langle \Omega, AJ(\operatorname{Re}B)\Omega\rangle - i\langle \Omega, AJ(\operatorname{Im}B)\Omega\rangle \\ &= \langle \Omega, AJB\Omega\rangle \quad \text{(by 4.2.1(2))}.\end{aligned}$$

(c) Consider any $A, B \in \mathfrak{M}_s$ and $C \in \mathfrak{M}'$, then $BJCJ \in \mathfrak{M}$ by 4.3.8, so from (b) we get $\langle (BJCJ)JA\Omega, \Omega\rangle = \langle \Omega, AJ(BJCJ)\Omega\rangle$. Therefore

$$\langle BJCA\Omega, \Omega\rangle = \langle \Omega, AJBJC\Omega\rangle \tag{1}$$

by 4.3.9 and 4.1.2(3). Now,

$$\begin{aligned}\langle AJBJ\Omega, C\Omega\rangle &= \langle JBJ\Omega, AC\Omega\rangle \\ &= \langle \Omega, JBJAC\Omega\rangle \quad \text{(by 4.2.1(4))} \\ &= \langle BJAC\Omega, J\Omega\rangle \quad \text{(by 4.2.1(3))} \\ &= \langle BJAC\Omega, \Omega\rangle \quad \text{(by 4.3.9)} \\ &= \langle \Omega, AJBJC\Omega\rangle \quad \text{(by (1) since } AC = CA) \\ &= \langle A\Omega, JBJC\Omega\rangle \\ &= \langle JBJA\Omega, C\Omega\rangle \quad \text{(by 4.2.1(4))}.\end{aligned}$$

Since Ω is cyclic for $\mathfrak{M}'$ (as mentioned in the proof of 4.3.1) it follows that

$$AJBJ\Omega = JBJA\Omega.$$

(d) Now consider any $A, B, C \in \mathfrak{M}$. Since $A = \operatorname{Re} A + i \operatorname{Im} A$ and $B = \operatorname{Re} B + i \operatorname{Im} B$, we see from (c) and the conjugate linearity of J (4.2.1(2)) that

$$AJBJ\Omega = JBJA\Omega. \tag{2}$$

Replacing A in (2) by AC and then by C, we find that

$$JBJ(AC)\Omega = (AC)JBJ\Omega = AJBJC\Omega.$$

Since Ω is cyclic for $\mathfrak{M}$, and $JBJA$ and $AJBJ$ are continuous linear operators by 4.2.1(2), it follows that $JBJA = AJBJ$, i.e. $JBJ \in \mathfrak{M}'$.■

4.3.11 Tomita-Takesaki theorem *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let $\Omega \in \mathfrak{H}$ be cyclic and separating for $\mathfrak{M}$ (see 1.4.2). Let J and Δ_t be the modular conjugation and unitary group associated to $(\mathfrak{M}, \Omega)$ (see 4.3.2). Then:*
(1) $J\mathfrak{M}J = \mathfrak{M}'$.
(2) $\Delta_t \mathfrak{M} \Delta_{-t} = \mathfrak{M}$ *for every* $t \in \mathbb{R}$.

Proof. By 4.1.2(3) and 4.3.8 we have $\mathfrak{M}' = J^2\mathfrak{M}'J^2 \subset J\mathfrak{M}J$. So from 4.3.10 we get (1). Now 4.3.7 implies

$$\Delta_t \mathfrak{M} \Delta_{-t} = \Delta_t J\mathfrak{M}'J\Delta_{-t} \subset \mathfrak{M}$$

since $J^2 = 1$ by 4.1.2(3). Of course this still holds if we replace t by $-t$, so from 4.2.4 and 4.2.6 we find that

$$\mathfrak{M} = \Delta_0 \mathfrak{M} \Delta_0 = \Delta_t \Delta_{-t} \mathfrak{M} \Delta_t \Delta_{-t} \subset \Delta_t \mathfrak{M} \Delta_{-t}.$$■

Usually in Tomita-Takesaki theory an operator $\Delta = (2 - R)R^{-1}$, called the *modular operator*, is defined. (This operator may be unbounded which is why we avoided it.) Then one considers Δ^{it} (which requires a functional calculus for unbounded operators). Comparing this with 4.2.5 we see that we have replaced Δ^{it} by Δ_t in our approach to the theory.

We will need 4.3.11(2) in the proof of 4.4.11.

4.4 The KMS condition and the modular group

In this section we present the connection between Δ_t in the Tomita-Takesaki theorem (4.3.11) and KMS states (3.2.4).

But first we return to the situation described in 4.2. In 4.4.1 to 4.4.8 we study this situation further, 4.4.8 being the main result. So let $\mathfrak{H}$, $\mathcal{K}$, P, Q, R, T, J and Δ_t be as in 4.2.

4.4.1 Definition(KMS function) A bounded continuous function $f : \overline{\mathbb{C}^0_{-1}} \to \mathbb{C}$ which is analytic on $\mathbb{C}^0_{-1}$ is called a *KMS function.*■

4.4.2 Definition(KMS condition) Let $\mathcal{L}$ be a real subspace of $\mathfrak{H}$ (see 1.2.5). A one-parameter unitary group U_t on $\mathfrak{H}$ is said to satisfy the *KMS condition with respect to* $\mathcal{L}$ if for any given $x, y \in \mathcal{L}$ there is a KMS function f such that

$$f(t) = \langle x, U_t y \rangle \quad \text{and} \quad f(t - i) = \langle U_t y, x \rangle$$

for all $t \in \mathbb{R}$.■

4.4.3 Proposition The function f *in 4.4.2 is unique.*

Proof. Let g also be a KMS function satisfying 4.4.2 for the given x, y. Then clearly $f - g$ is a KMS function and $f(t) - g(t) = 0$ for all real t. So by 1.6.3 there is an analytic function $F : \mathbb{C}^1_{-1} \to \mathbb{C}$ such that $F(z) = f(z) - g(z)$ for $z \in \overline{\mathbb{C}^0_{-1}} \cap \mathbb{C}^1_{-1}$. In particular, $F(t) = 0$ for real t, hence $F = 0$ by 1.6.2 which means $f - g = 0$ since $f - g$ is continuous.■

We now give a useful alternative formulation of the KMS condition:

4.4.4 Proposition *Let $\mathcal{L}$ be a real subspace of $\mathfrak{H}$, and U_t a one-parameter unitary group on $\mathfrak{H}$. U_t satisfies the KMS condition with respect to $\mathcal{L}$ if and only if given any $x, y \in \mathcal{L}$ there is a bounded continuous function $f : \overline{\mathbb{C}^0_{-1/2}} \to \mathbb{C}$, analytic on $\mathbb{C}^0_{-1/2}$, such that*

$$f(t) = \langle x, U_t y \rangle \quad \textit{and} \quad f(t - i/2) \in \mathbb{R}$$

for all $t \in \mathbb{R}$.

Proof. First we prove that these conditions imply the KMS condition. Similar to the proof of 1.6.4 we apply 1.6.3 along the line $\overline{\mathbb{C}^{-1/2}_{-1/2}}$ to find that the function $g : \overline{\mathbb{C}^0_{-1}} \to \mathbb{C}$ defined by

$$g(z) = f(z) \quad \text{and} \quad g(z - i/2) = \overline{f(\overline{z} - i/2)} \quad \text{for } z \in \overline{\mathbb{C}^0_{-1/2}}$$

is analytic on $\mathbb{C}^0_{-1}$. The function g is bounded and continuous since f is. (What makes all this work is the fact that it is given that $f\left(\overline{\mathbb{C}^{-1/2}_{-1/2}}\right) \subset \mathbb{R}$.) So g is a KMS function that extends f, and

$$g(t - i) = g(t - i/2 - i/2) = \overline{f\left(\overline{t - i/2} - i/2\right)} = \overline{f(t)} = \langle U_t y, x \rangle$$

for all real t. Hence U_t satisfies the KMS condition with respect to $\mathcal{L}$.

Conversely, assume U_t satisfies the KMS condition with respect to $\mathcal{L}$. Consider any $x, y \in \mathcal{L}$ and let f be as in 4.4.2. Clearly the function

$$g : \overline{\mathbb{C}^0_{-1}} \to \mathbb{C} : z \mapsto \overline{f(\overline{z} - i)}$$

is a KMS function since f is. Also

$$g(t) = \overline{f(t - i)} = \langle x, U_t y \rangle \quad \text{and} \quad g(t - i) = \overline{f(t)} = \langle U_t y, x \rangle$$

for all real t by the definitions of g and f. So by 4.4.3 $g = f$, hence $f(t - i/2) = g(t - i/2) = \overline{f\left(\overline{t - i/2} - i\right)} = \overline{f(t - i/2)}$, i.e. $f(t - i/2) \in \mathbb{R}$, for all real t. ■

Δ_t is a one-parameter unitary group on $\mathfrak{H}$ by 4.2.6. Hence we can state

4.4.5 Proposition *Δ_t satisfies the KMS condition with respect to $\mathcal{K}$.*

Proof. Let $x, y \in \mathcal{K}$. By 4.1.1, 4.1.2(4) and the definition of P (as given in 4.2) we have

$$\begin{aligned} 2y &= 2Py = (P + Q)y + (P - Q)y = (R + JT)y \\ &= (R + TJ)y = R^{1/2}(R^{1/2} + (2 - R)^{1/2}J)y \end{aligned} \tag{1}$$

by 4.1.2(2) and the fact that $R^{1/2}$ (respectively $(2 - R)^{1/2}$) is the same in $\mathfrak{L}(\mathfrak{H})$ and $\mathfrak{L}(\mathfrak{H}_{\mathbb{R}})$, as mentioned in the proof of 4.2.1(1).

Set

$$v := (R^{1/2} + (2-R)^{1/2}J)y/2$$

then $y = R^{1/2}v$ and

$$\Delta_t y = \Delta_t R^{1/2} v = (2-R)^{it} R^{-it+1/2} v \tag{2}$$

by 4.2.5, 1.8.12 and 1.8.13(1). Let

$$f : \overline{\mathbb{C}^0_{-1/2}} \to \mathbb{C} : z \mapsto \left\langle x, (2-R)^{iz} R^{-iz+1/2} v \right\rangle .$$

By 4.2.1(1), 1.8.12, 1.8.14, 1.8.15 and 1.7.4 f is well-defined, bounded and continuous on $\overline{\mathbb{C}^0_{-1/2}}$ and analytic on $\mathbb{C}^0_{-1/2}$. Clearly

$$f(t) = \langle x, \Delta_t y \rangle \text{ for all } t \in \mathbb{R} \tag{3}$$

by (2). Consider any $t \in \mathbb{R}$. By 1.8.9 and 4.2.4(2)

$$f(t - i/2) = \left\langle x, (2-R)^{it+1/2} R^{-it} v \right\rangle = \left\langle \Delta_{-t} x, (2-R)^{1/2} v \right\rangle .$$

But by 4.1.2(2), 1.1.3 and the definition of v

$$\begin{aligned} 2(2-R)^{1/2} v &= (T + (2-R)J)y = (T + JR)y \quad \text{(by 4.1.2(5))} \\ &= J(JT + R)y = 2Jy \quad \text{(by (1) and 4.1.2(3)).} \end{aligned}$$

So $f(t - i/2) = \langle \Delta_{-t} x, Jy \rangle$. However, $\Delta_{-t} x \in \mathcal{K}$ by 4.2.6, and $Jy = JPy = (1-Q)Jy \in (i\mathcal{K})^{\perp}$ by 4.1.2(5) and the definition of P and Q (as given in 4.2), hence

$$\operatorname{Im} \langle \Delta_{-t} x, Jy \rangle = \operatorname{Im}(i \langle i\Delta_{-t} x, Jy \rangle) = \operatorname{Re} \langle i\Delta_{-t} x, Jy \rangle = \langle i\Delta_{-t} x, Jy \rangle_{\mathbb{R}} = 0.$$

Therefore $f(t - i/2) \in \mathbb{R}$ for all real t, so the result from (3) and 4.4.4.■

4.4.6 Definition Let U_t be a one-parameter unitary group on $\mathfrak{H}$. $x \in \mathfrak{H}$ is called a *weak entire vector for* U_t if there is a function $h : \mathbb{C} \to \mathfrak{H}$ such that
(1) $h(t) = U_t x$ for all $t \in \mathbb{R}$;
(2) $\langle y, h(\cdot) \rangle$ is an entire function for every $y \in \mathfrak{H}$;
(3) h is bounded on every bounded subset of $i\mathbb{R}$.
We denote the set of all weak entire vectors for U_t by $W(U_t)$.■

4.4.7 Lemma *Let $\mathcal{L}$ be a closed real subspace of $\mathfrak{H}$, and U_t a strongly continuous one-parameter unitary group on $\mathfrak{H}$ (see 4.2.7) such that $U_t\mathcal{L} \subset \mathcal{L}$. Then $\mathcal{L} \cap W(U_t)$ is dense in $\mathcal{L}$.*

Proof. Consider any $x \in \mathcal{L}$. For $n = 1, 2, 3, \ldots$ let

$$x_n := \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} U_t x dt. \tag{1}$$

This is a well-defined Riemann integral since $t \mapsto U_t x$ is bounded and continuous on $\mathbb{R}$ (boundedness follows from $\|U_t x\| \leq \|U_t\| \, \|x\|$, see 4.2.4, while the continuity is clear from the fact that U_t is strongly continuous).

$U_t x \in \mathcal{L}$ since $x \in \mathcal{L}$, so $e^{-nt^2} U_t x \in \mathcal{L}$ (since $e^{-nt^2} \in \mathbb{R}$) for $t \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$. Since $\mathcal{L}$ is closed in $\mathfrak{H}$ it is now clear from (1) that

$$x_n \in \mathcal{L} \quad \text{for all } n. \tag{2}$$

It is easy to confirm that $(n/\pi)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} dt = 1$ by squaring it, writing the result as a double integral and changing to polar coordinates). So it follows from (1) that

$$x_n \to U_0 x = 1x = x \tag{3}$$

by 4.2.4(3). (We will show (3) in detail at the end of this proof.)

For $n = 1, 2, 3, \ldots$ let

$$h_n : \mathbb{C} \to \mathfrak{H} : z \mapsto \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-n(t-z)^2} U_t x dt \tag{4}$$

(a well-defined Riemann integral since $e^{-n(t-z)^2} = e^{n(\operatorname{Im} z)^2} e^{-n(t-\operatorname{Re} z)^2} e^{2in(t-\operatorname{Re} z)\operatorname{Im} z}$, and the factor $e^{-n(t-\operatorname{Re} z)^2}$ causes the integral to converge similar to the argument given after (1)). Let $y \in \mathfrak{H}$. By definition of a Riemann integral there is a sequence of sums converging to the integral in (4), hence by the continuity of inner products

$$\langle y, h_n(z) \rangle = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-n(t-z)^2} \langle y, U_t x \rangle \, dt.$$

By an argument like the one given in the proof of 4.3.7 (using Fubini's theorem, Cauchy's integral theorem, Lebesgue's dominated convergence theorem

and Morera's theorem) we see that $\langle y, h_n(\cdot)\rangle$ is an entire function for all n. Take any $s \in \mathbb{R}$. From (4) we see

$$\|h_n(is)\| \leq \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} \left\| e^{-n(t-is)^2} U_t x \right\| dt = \left(\frac{n}{\pi}\right)^{1/2} e^{ns^2} \int_{-\infty}^{+\infty} e^{-nt^2} \|U_t x\| \, dt,$$

so h_n is bounded on every bounded subset of $i\mathbb{R}$. Changing the integration variable from t to $s+t$ we also see from (4) that

$$\begin{aligned} h_n(s) &= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-n(t-s)^2} U_t x dt = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} U_{s+t} x dt \\ &= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} U_s U_t x dt \quad \text{(by 4.2.4(1))} \\ &= U_s \left(\left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} U_t x dt \right) \quad \text{(the operator } U_s \text{ is continuous)} \\ &= U_s x_n \quad \text{(by (1))}. \end{aligned}$$

We have thus shown x_n is a weak entire vector for U_t, i.e. $x_n \in W(U_t)$. Combining this with (2) and (3) completes the proof.

We now show (3). Keep in mind that $\int_{-\infty}^{+\infty} e^{-nt^2} dt = (\pi/n)^{1/2}$ as pointed out above. Consider any $\varepsilon > 0$. Since U_t is strongly continuous, there is a $\delta > 0$ such that $\|U_t x - U_0 x\| < \varepsilon$ if $|t| < \delta$. So

$$\left\| \int_{-\delta}^{\delta} e^{-nt^2} (U_t x - U_0 x) dt \right\| \leq \varepsilon \int_{-\delta}^{\delta} e^{-nt^2} dt \leq \varepsilon \left(\frac{\pi}{n}\right)^{1/2}.$$

$e^{-nt^2} = 1/e^{nt^2} = 1/(1 + nt^2 + \frac{1}{2}(nt^2)^2 + \ldots) \leq 1/nt^2$ for $t \neq 0$. So

$$\int_{\delta}^{+\infty} e^{-nt^2} dt \leq \int_{\delta}^{+\infty} \frac{1}{nt^2} dt = \frac{1}{n\delta}.$$

Thus

$$\begin{aligned} &\|x_n - x\| \\ &= \left\| \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} U_t x dt - \left(\left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-nt^2} dt \right) U_0 x \right\| \\ &= \left(\frac{n}{\pi}\right)^{1/2} \left\| \int_{-\infty}^{+\infty} e^{-nt^2} (U_t x - U_0 x) dt \right\| \end{aligned}$$

$$\leq \left(\frac{n}{\pi}\right)^{1/2} \|x\| \left(\int_{-\infty}^{-\delta} e^{-nt^2} \|U_t - U_0\| \, dt + \left\| \int_{\delta}^{+\infty} e^{-nt^2} \|U_t - U_0\| \, dt \right\| \right) + \varepsilon$$

$$\leq 4 \left(\frac{n}{\pi}\right)^{1/2} \|x\| \int_{\delta}^{+\infty} e^{-nt^2} dt + \varepsilon \quad \text{(since } \|U_t\| = 1, \text{ see 4.2.4)}$$

$$\leq \frac{4\|x\|}{(\pi n)^{1/2}\delta} + \varepsilon.$$

Hence $x_n \to x$.■

The following result gives a characterization of Δ_t in terms of the KMS condition, and as such it is the first important step in obtaining the connection between Δ_t and KMS states.

4.4.8 Theorem *Δ_t is the unique strongly continuous one-parameter unitary group on $\mathfrak{H}$ satisfying the KMS condition with respect to $\mathcal{K}$ such that $\Delta_t \mathcal{K} \subset \mathcal{K}$.*

Proof. We know by 4.2.6, 4.2.8 and 4.4.5 that Δ_t has the stated properties.

Conversely, let U_t be a strongly continuous one-parameter group on $\mathfrak{H}$ satisfying the KMS condition with respect to $\mathcal{K}$ such that $U_t \mathcal{K} \subset \mathcal{K}$. Consider any $x \in \mathcal{K} \cap W(U_t)$ and let h be as in 4.4.6. We now break the proof into several parts. s and t will be arbitrary real numbers throughout the proof.

(a) Consider any $y \in \mathfrak{H}$. Then

$$z \mapsto \langle y, h(t+iz) \rangle \quad \text{and} \quad z \mapsto \langle y, U_t h(iz) \rangle = \langle U_t^* y, h(iz) \rangle$$

are entire functions (by the definition of h) which are equal for imaginary z (since $h(t+s) = U_{t+s}x = U_t U_s x = U_t h(s)$ by the definitions of h and U_t) and hence equal on $\mathbb{C}$ by 1.6.2. In particular $\langle y, h(t+is) \rangle = \langle y, U_t h(is) \rangle$ for all $y \in \mathfrak{H}$. So $h(t+is) = U_t h(is)$. Thus

$$\|h(t+is)\| \leq \|U_t\| \, \|h(is)\| = \|h(is)\| \quad \text{since } \|U_t\| = 1, \text{ see 4.2.4.}$$

Therefore h is bounded on sets of the form $\mathbb{C}_a^b$, $a, b \in \mathbb{R}$, because of 4.4.6(3).

(b) Consider any $y \in \mathcal{K}$. In this section of the proof we show that

$$\langle \Delta_t J y, U_t x \rangle = \langle Jy, x \rangle .$$

Let v be as in the proof of 4.4.5, so

$$y = R^{1/2} v \text{ and } Jy = (2-R)^{1/2} v. \tag{1}$$

Let

$$g : \overline{\mathbb{C}^0_{-1/2}} \to \mathbb{C} : z \mapsto \left\langle J(2-R)^{iz} R^{-iz+1/2} v, h(z) \right\rangle .$$

The function $f : \overline{\mathbb{C}^0_{-1/2}} \to \mathfrak{H} : z \mapsto (2-R)^{iz} R^{-iz+1/2} v$ is well-defined, bounded and continuous on $\overline{\mathbb{C}^0_{-1/2}}$, and differentiable on $\mathbb{C}^0_{-1/2}$, by 1.8.12, 1.8.14, 1.8.15 and 1.7.4. So g is well-defined and bounded since J is isometric by 4.2.1(2). For $u \in \mathfrak{H}$ and $z \in \mathbb{C}$

$$|\langle Ju, h(z)\rangle| \leq \|Ju\| \, \|h(z)\| = \|u\| \, \|h(z)\| .$$

Hence $\langle J\cdot, h(z)\rangle \in \mathfrak{L}(\mathfrak{H}, \mathbb{C})$ since J is conjugate linear, and

$$\|\langle J\cdot, h(z)\rangle\| \leq \|h(z)\| .$$

So $A : \overline{\mathbb{C}^0_{-1/2}} \to \mathfrak{L}(\mathfrak{H}, \mathbb{C}) : z \mapsto \langle J\cdot, h(z)\rangle$ is bounded because of (a). Also, for every $u \in \mathfrak{H}$, the function $A(\cdot)u = \langle Ju, h(\cdot)\rangle$ is entire (and therefore continuous) by 4.4.6(2). Putting all this together we conclude from 1.7.4 that $y = Af$ is continuous on $\overline{\mathbb{C}^0_{-1/2}}$ and analytic on $\mathbb{C}^0_{-1/2}$.

By (1), 4.2.5, 1.8.13(1), 1.8.12, 4.4.6(1) and 4.2.6

$$g(t) = \left\langle J\Delta_t R^{1/2} v, U_t x \right\rangle = \langle J\Delta_t y, U_t x\rangle = \langle \Delta_t J y, U_t x\rangle . \tag{2}$$

$\Delta_t y \in \mathcal{K}$ since $y \in \mathcal{K}$, so $J\Delta_t y \in (i\mathcal{K})^\perp$ by 4.1.2(6). $U_t x \in \mathcal{K}$ since $x \in \mathcal{K}$, therefore $\operatorname{Im}\langle J\Delta_t y, U_t x\rangle = -\operatorname{Re}\langle J\Delta_t y, iU_t x\rangle = -\langle J\Delta_t y, iU_t x\rangle_{\mathbb{R}} = 0$. Thus

$$g(t) \in \mathbb{R}. \tag{3}$$

Similar to (2), we have (using (1), 1.8.9 and 4.1.2(3)) that

$$g(t - i/2) = \left\langle J\Delta_t (2-R)^{1/2} v, h(t-i/2)\right\rangle = \langle \Delta_t y, h(t-i/2)\rangle . \tag{4}$$

Let s be fixed. Since U_t satisfies the KMS condition with respect to $\mathcal{K}$, and $\Delta_s y \in \mathcal{K}$, we know from 4.4.4 that there is a bounded continuous function $f : \overline{\mathbb{C}^0_{-1/2}} \to \mathbb{C}$, analytic on $\mathbb{C}^0_{-1/2}$, such that

$$f(t) = \langle \Delta_s y, U_t x\rangle \quad \text{and} \quad f(t - i/2) \in \mathbb{R}. \tag{5}$$

Let $F : \overline{\mathbb{C}^0_{-1/2}} \to \mathbb{C} : z \mapsto \langle \Delta_s y, h(z)\rangle - f(z)$. Then F is continuous on $\overline{\mathbb{C}^0_{-1/2}}$ and analytic on $\mathbb{C}^0_{-1/2}$, and $F(t) = 0$, by the definition of h and f. So by

1.6.3 there is an analytic function $G : \mathbb{C}^{1/2}_{-1/2} \to \mathbb{C}$ which agrees with F on $\mathbb{C}^0_{-1/2} \cup \mathbb{R}$. Hence $G(t) = 0$ which implies $G = 0$ by 1.6.2. It follows that $F = 0$ since it is continuous, i.e.

$$\langle \Delta_s y, h(z) \rangle = f(z) \quad \text{for all } z \in \overline{\mathbb{C}^0_{-1/2}}.$$

So $\langle \Delta_s y, h(t - i/2) \rangle \in \mathbb{R}$ by (5). In particular,

$$g(t - i/2) = \langle \Delta_t y, h(t - i/2) \rangle \in \mathbb{R} \quad \text{by (4).}$$

This combined with (3) implies g is constant according to 1.6.4. So $\langle \Delta_t Jy, U_t x \rangle = g(t) = g(0) = \langle Jy, x \rangle$ by (2) and 4.2.4(3).

(c) Consider any $y \in \mathcal{K}$, then

$$\langle U_{-t} \Delta_t Jy, x \rangle = \langle Jy, x \rangle \quad \text{by (b) and 4.2.4(2).} \tag{6}$$

$\mathcal{K}$ is total in $\mathfrak{H}$ (i.e. the span of $\mathcal{K}$ is dense in $\mathfrak{H}$, [K, 3.6-1]) since $\mathcal{K} + i\mathcal{K}$ is dense in $\mathfrak{H}$ by definition. Hence $\mathcal{K} \cap W(U_t)$ is total in $\mathfrak{H}$ by 4.4.7. So by (6) and the definition of x we have

$$U_{-t} \Delta_t Jy = Jy. \tag{7}$$

J is surjective (since $J^2 = 1$ by 4.1.2(3)) and continuous (since it is a bounded linear operator on $\mathfrak{H}_{\mathbb{R}}$ by 4.1.1), therefore

$$\mathfrak{H} = J\left(\overline{\mathcal{K} + i\mathcal{K}}\right) \subset \overline{J(\mathcal{K} + i\mathcal{K})}.$$

But $J\mathcal{K} + iJ\mathcal{K} = J(\mathcal{K} - i\mathcal{K}) = J(\mathcal{K} + i\mathcal{K})$ since J is conjugate linear, so we deduce $J\mathcal{K}$ is total in $\mathfrak{H}$. Thus $U_{-t}\Delta_t = 1$ by (7), i.e. $U_t = \Delta_t$ by 4.2.4.■

4.4.8 completes our study of the situation presented in 4.2. We now give three results concerning C*-algebras and von Neumann algebras in general. The third of these (4.4.11) is our ultimate mathematical goal, the first two (4.4.9 and 4.4.10) are just some more tools we need in order to prove 4.4.11 (the other important tools being 4.3.11 and 4.4.8).

Before we proceed we quickly recall the GNS construction ([Br, 2.3.16]): Given a state ω on a C*-algebra $\mathfrak{A}$ there exists a cyclic representation $(\mathfrak{G}, \pi, \Omega)$ of $\mathfrak{A}$ (i.e. a Hilbert space $\mathfrak{G}$, a $*$-homomorphism $\pi : \mathfrak{A} \to \mathfrak{L}(\mathfrak{G})$, and a vector $\Omega \in \mathfrak{G}$ which is cyclic for the C*-algebra $\pi(\mathfrak{A})$, see 1.4.2) such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathfrak{A}$. This representation is unique up to unitary equivalence and is called the *cyclic representation of* $\mathfrak{A}$ *associated to* ω.

The following result and its proof is adapted from [Br, 2.5.6]:

4.4.9 Proposition *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and ω a faithful normal state on $\mathfrak{M}$ (see 2.3.3 and 2.3.8). Let $(\mathfrak{G}, \pi, \Omega)$ be the cyclic representation of $\mathfrak{M}$ associated to ω, and set $\mathfrak{N} := \pi(\mathfrak{M})$. Then $\mathfrak{N}$ is a von Neumann algebra on $\mathfrak{G}$, $\pi : \mathfrak{M} \to \mathfrak{N}$ is a $*$-isomorphism, and Ω is a cyclic and separating vector for $\mathfrak{N}$ such that*

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$$

for all $A \in \mathfrak{M}$.

Proof. We know $\mathfrak{N} := \pi(\mathfrak{M})$ is a von Neumann algebra on $\mathfrak{G}$ (since ω is normal, [Br, 2.4.24]) with cyclic vector Ω such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathfrak{M}$.

Consider any $A \in \mathfrak{M}$ such that $\pi(A)\Omega = 0$. Then

$$\omega(A^*A) = \langle \Omega, \pi(A^*A)\Omega \rangle = \langle \pi(A)\Omega, \pi(A)\Omega \rangle = 0$$

since by definition π is a $*$-homomorphism. Hence $A^*A = 0$ since ω is faithful. So $\|A\|^2 = \|A^*A\| = 0$, which means $A = 0$. Therefore $\pi(A) = 0$, since π is linear, so by 1.4.2 we conclude Ω is separating for $\mathfrak{N}$.

As a special case of the foregoing argument we see that $\pi(A) = 0$ implies $A = 0$, hence π is injective since it is linear. This means $\pi : \mathfrak{M} \to \mathfrak{N}$ is a $*$-isomorphism.■

4.4.10 Proposition *Let $\mathfrak{A}$ be a C^*-algebra, ω a state on $\mathfrak{A}$, and τ a one-parameter $*$-automorphism group of $\mathfrak{A}$ (see 3.1.1) such that*

$$\omega(\tau_t(A)) = \omega(A) \text{ for all } t \in \mathbb{R} \text{ and } A \in \mathfrak{A}.$$

Let $(\mathfrak{G}, \pi, \Omega)$ be the cyclic representation of $\mathfrak{A}$ associated to ω. Then their exists a unique one-parameter unitary group U_t on $\mathfrak{G}$ such that

$$U_t\Omega = \Omega \text{ and } \pi(\tau_t(A)) = U_t\pi(A)U_{-t}$$

for all $t \in \mathbb{R}$ and $A \in \mathfrak{A}$. Furthermore, if $(\mathfrak{A}, \tau)$ is a W^*-dynamical system (see 3.1.3) and $\pi(\mathfrak{A})$ is a von Neumann algebra on $\mathfrak{G}$, then U_t is strongly continuous.

Proof. For every $t \in \mathbb{R}$ there is a unique unitary operator $U_t \in \mathfrak{L}(\mathfrak{G})$ such that

$$U_t\Omega = \Omega \quad \text{and} \quad \pi(\tau_t(A)) = U_t\pi(A)U_t^{-1} \text{ for all } A \in \mathfrak{A} \tag{1}$$

(this follows from the uniqueness of the cyclic representation associated to ω, see [Br, 2.3.17]). We now show that U_t is a one-parameter unitary group.

$U_sU_t\pi(A)U_t^{-1}U_s^{-1} = \pi(\tau_s(\tau_t(A))) = \pi(\tau_{s+t}(A)) = U_{s+t}\pi(A)U_{s+t}^{-1}$ while we know U_sU_t is unitary (namely $(U_sU_t)^*(U_sU_t) = U_t^*U_s^*U_sU_t = 1$, and similarly $U_sU_t(U_sU_t)^* = 1$, since U_s and U_t are unitary) and clearly $U_sU_t\Omega = \Omega$ by (1). Hence, using the uniqueness of U_t in (1), we see $U_sU_t = U_{s+t}$.

$U_0\pi(A)U_0^{-1} = \pi(\tau_0(A)) = \pi(A)$, hence $U_0 = 1$ by the uniqueness in (1).

We now see $U_tU_{-t} = U_{t-t} = 1$ so $U_t^* = U_t^*U_tU_{-t} = U_{-t}$.

Assuming $(\mathfrak{A}, \tau)$ is a W*-dynamical system and $\pi(\mathfrak{A})$ a von Neumann algebra, it remains only to show that U_t is strongly continuous. Consider any $A \in \mathfrak{A}\backslash\{0\}$. Then

$$\begin{aligned}\lim_{s\to t} U_s\pi(A)\Omega &= \lim_{s\to t} U_s\pi(A)U_{-s}\Omega = \lim_{s\to t}\pi(\tau_s(A))\Omega \quad \text{(by (1))}\\ &= \|A\|\lim_{s\to t}\pi(\tau_s(A/\|A\|)\Omega\\ &= \|A\|\,\pi(\tau_t(A/\|A\|))\Omega \quad \text{(by 1.4.6)}\end{aligned}$$

since $\|\tau_s(A/\|A\|)\| = \|A/\|A\|\| = 1$ (τ_s is a $*$-isomorphism from $\mathfrak{A}$ to itself, see [M, Theorem 2.1.7]) and since $s \mapsto \tau_s(A/\|A\|)$ is strongly continuous on $\mathbb{R}$ by 3.1.3, 1.3.3 and the definition of the strong topology in 1.4. Thus

$$\lim_{s\to t} U_s\pi(A)\Omega = \pi(\tau_t(A))\Omega = U_t\pi(A)U_{-t}\Omega = U_t\pi(A)\Omega. \tag{2}$$

By definition Ω is cyclic for $\pi(\mathfrak{A})$, i.e. $\pi(\mathfrak{A})\Omega$ is dense in $\mathfrak{H}$. Take any sequence (t_n) in $\mathbb{R}$ converging to $t \in \mathbb{R}$, then by (2)

$$U_{t_n}x \to U_tx \quad \text{for all } x \in \pi(\mathfrak{A})\Omega.$$

Furthermore, the sequence (U_{t_n}) is clearly bounded (every member of this sequence is unitary and so has norm 1), hence

$$U_{t_n}x \to U_tx \quad \text{for all } x \in \mathfrak{H}, \text{ ([K, 4.9-6]).}$$

It follows that $t \mapsto U_tx$ is continuous on $\mathbb{R}$ for every $x \in \mathfrak{H}$, i.e. U_t is strongly continuous (see 4.2.7).■

We are now in a position to formulate and prove our main result:

4.4.11 Theorem *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and ω a faithful normal state on $\mathfrak{M}$ (see 2.3.3 and 2.3.8). Then there*

is a unique W-dynamical system $(\mathfrak{M}, \alpha)$ such that ω is an α-KMS state (see 3.1.3 and 3.2.4). α is given by*

$$\alpha_t(A) = \pi^{-1}(\Delta_t) A \pi^{-1}(\Delta_{-t}) \tag{1}$$

for all $t \in \mathbb{R}$ and $A \in \mathfrak{M}$, where $(\mathfrak{G}, \pi, \Omega)$ and $\mathfrak{N}$ are as in 4.4.9, and Δ_t is the unitary group associated to $(\mathfrak{N}, \Omega)$ (see 4.3.2).

Proof. Define α by means of (1). By 4.3.11(2) we then have $\pi(\alpha_t(A)) = \Delta_t \pi(A) \Delta_{-t} \in \mathfrak{N}$, hence $\alpha_t(A) \in \mathfrak{M}$, for all $t \in \mathbb{R}$ and $A \in \mathfrak{M}$. Since π is a $*$-isomorphism and Δ_t a one-parameter unitary group, it is therefore clear from 3.1.1 and 4.2.4 that α is a one-parameter $*$-automorphism group of $\mathfrak{M}$.

Δ_t is strongly continuous by 4.2.8, and $\|\Delta_t\| = 1$ since Δ_t is unitary. Hence $t \mapsto \pi^{-1}(\Delta_t)$ is strongly continuous on $\mathbb{R}$ by 1.4.6, i.e. $t \mapsto \pi^{-1}(\Delta_t)x$ is continuous on $\mathbb{R}$ for every $x \in \mathfrak{H}$ (see 4.2.7). Also, $\|\pi^{-1}(\Delta_t)\| = \|\Delta_t\| = 1$ since π is a $*$-isomorphism ([M, Theorem 2.1.7]), so from (1) and 1.7.4(1) it follows that $t \mapsto \alpha_t(A)x$ is continuous on $\mathbb{R}$ for every $A \in \mathfrak{M}$ and $x \in \mathfrak{H}$. Therefore $(\mathfrak{M}, \alpha)$ is a W*-dynamical system by 3.1.3.

Given any $A, B \in \mathfrak{N}_s$, we know by 4.4.5 and 4.4.2 that there is a KMS function f such that for all real t

$$f(t) = \langle A\Omega, \Delta_t B\Omega \rangle = \langle \Omega, A\Delta_t B \Delta_{-t}\Omega \rangle \quad \text{by 4.3.9}$$

and

$$f(t-i) = \langle A\Delta_t B\Delta_{-t}\Omega, \Omega \rangle = \langle \Omega, \Delta_t B \Delta_{-t} A\Omega \rangle \quad \text{by 4.2.4(2).}$$

Given any $A, B \in \mathfrak{N}$, it now follows using 1.4.9 that there is a KMS function f such that

$$f(t) = \langle \Omega, A\Delta_t B\Delta_{-t}\Omega \rangle \quad \text{and} \quad f(t-i) = \langle \Omega, \Delta_t B\Delta_{-t} A\Omega \rangle \quad \text{for } t \in \mathbb{R}$$

since a linear combination of KMS functions is again a KMS function (this is clear from 4.4.1). Therefore, given any $A, B \in \mathfrak{M}$, there is a KMS function f such that for all real t

$$\begin{aligned} f(t) &= \langle \Omega, \pi(A)\Delta_t \pi(B)\Delta_{-t}\Omega \rangle = \langle \Omega, \pi(A\pi^{-1}(\Delta_t) B \pi^{-1}(\Delta_{-t}))\Omega \rangle \\ &= \omega(A\alpha_t(B)) \quad \text{(by (1) and 4.4.9),} \end{aligned}$$

and $f(t-i) = \omega(\alpha_t(B)A)$ by a similar argument. So ω is an α-KMS state by 3.2.4.

Now we have to show the uniqueness of the W*-dynamical system. So, let $(\mathfrak{M}, \alpha)$ be any W*-dynamical system making ω an α-KMS state. We have to prove that (1) now holds.

According to 3.2.7, 4.4.10 and 4.4.9 there is a strongly continuous one-parameter unitary group U_t on $\mathfrak{G}$ such that

$$U_t\Omega = \Omega \quad \text{and} \quad \pi(\alpha_t(A)) = U_t\pi(A)U_{-t}. \tag{2}$$

Hence we only have to prove $U_t = \Delta_t$. We will use 4.4.8 to do this.

Set $\mathcal{K} := \overline{\mathfrak{N}_s\Omega}$ (the closure of $\mathfrak{N}_s\Omega$ in $\mathfrak{G}$). Consider any $A \in \mathfrak{N}_s$, then $(U_tAU_{-t})^* = U_{-t}^*A^*U_t^* = U_tAU_{-t}$, so $U_tAU_{-t} = \pi(\alpha_t(\pi^{-1}(A))) \in \mathfrak{N}_s$. Therefore $U_tA\Omega = U_tAU_{-t}\Omega \in \mathfrak{N}_s\Omega$ according to (2), which implies

$$U_t\mathcal{K} \subset \mathcal{K}. \tag{3}$$

Consider any $A, B \in \mathfrak{N}_s$. Since ω is an α-KMS state it follows from 3.2.4 that there is a KMS function f such that for all real t

$$\begin{aligned} f(t) &= \omega(\pi^{-1}(A)\alpha_t(\pi^{-1}(B))) = \omega(\pi^{-1}(A\pi(\alpha_t(\pi^{-1}(B))))) \\ &= \omega(\pi^{-1}(AU_tBU_{-t})) = \langle\Omega, AU_tBU_{-t}\Omega\rangle \quad \text{(by (2) and 4.4.9)} \\ &= \langle A\Omega, U_tB\Omega\rangle \end{aligned}$$

and

$$\begin{aligned} f(t-i) &= \omega(\alpha_t(\pi^{-1}(B))\pi^{-1}(A)) = \langle\Omega, U_tBU_{-t}A\Omega\rangle = \langle U_tBU_{-t}\Omega, A\Omega\rangle \\ &= \langle U_tB\Omega, A\Omega\rangle\,. \end{aligned}$$

Now take any $x \in \mathcal{K}$, then there is a sequence (A_n) in $\mathfrak{N}_s$ such that $A_n\Omega \to x$, by the definition of $\mathcal{K}$. Hence by the equalities given above, there is a sequence (f_n) of KMS functions such that

$$f_n(t) = \langle A_n\Omega, U_tB\Omega\rangle \quad \text{and} \quad f_n(t-i) = \langle U_tB\Omega, A_n\Omega\rangle \tag{4}$$

for all real t. Hence by 1.6.5 and the fact that $\|U_t\| = 1$ (since U_t is unitary),

$$|f_m(z) - f_n(z)| \leq \|(A_m - A_n)\Omega\|\,\|B\Omega\| \quad \text{for all } m, n \text{ and } z.$$

This means (f_n) is a Cauchy sequence in the complete normed space $C_b\left(\overline{\mathbb{C}_{-1}^0}\right)$ (the space of all bounded continuous complex-valued functions on $\overline{\mathbb{C}_{-1}^0}$, the

norm being the sup-norm $\|\cdot\|_\infty$). Hence (f_n) converges uniformly to an $f \in C_b\left(\overline{\mathbb{C}^0_{-1}}\right)$. Since every f_n is analytic on $\mathbb{C}^0_{-1}$, we know that f must also be analytic on $\mathbb{C}^0_{-1}$, ([Con, VII 2.1 and 1.10(b)]). In other words f is a KMS function, and from (4) and the definition of A_n it follows that for all real t

$$f(t) = \lim_{n\to\infty} f_n(t) = \langle x, U_t B\Omega\rangle \quad \text{and} \quad f(t-i) = \langle U_t B\Omega, x\rangle\,. \tag{5}$$

By repeating the foregoing argument for $y \in \mathcal{K}$ and a sequence (B_n) in $\mathfrak{N}_s$ such that $B_n\Omega \to y$, it follows from (5) that there is a KMS function f such that

$$f(t) = \langle x, U_t y\rangle \quad \text{and} \quad f(t-i) = \langle U_t y, x\rangle \quad \text{for all } t \in \mathbb{R}.$$

This means U_t satisfies the KMS condition with respect to $\mathcal{K}$ (4.4.2), hence we conclude from (3) and 4.4.8 that $U_t = \Delta_t$.■

4.4.11 gives the connection between KMS states and the theory developed in 4.1, 4.2 and 4.3. In this sense 4.4.11 can be considered the most important result in this dissertation since it binds together almost everything we have done so far, except for 4.3.11(1). 4.4.11 is by no means a trivial result. This whole section was devoted to its proof, and in its proof we also used the Tomita-Takesaki theorem (specifically 4.3.11(2)) which in turn took us the whole of 4.3 to prove. In 4.5 we will discuss the physical implication of 4.4.11.

We end this section with one last definition:

4.4.12 Definition(Modular group) Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space, and ω a faithful normal state on $\mathfrak{M}$. The one-parameter $*$-automorphism group α of $\mathfrak{M}$ given by 4.4.11 is called the *modular group associated to* ω.■

4.5 The physical interpretation

In Chapter 2 we considered physical systems for which the observables were given by $\mathfrak{L}(\mathfrak{H})_s$, with $\mathfrak{H}$ a finite dimensional Hilbert space. $\mathfrak{L}(\mathfrak{H})$ is a von Neumann algebra by 1.4.8, and according to 3.1.6 $(\mathfrak{L}(\mathfrak{H}), \tau)$ is a W*-dynamical system where τ is the time evolution of the system as given by 3.1.5. In 2.3.6 and 2.3.9 we saw that the equilibrium states of such a system are faithful

normal states on $\mathfrak{L}(\mathfrak{H})$, and according to 3.2.5 the equilibrium state at inverse temperature β is exactly the (τ, β)-KMS state on $\mathfrak{L}(\mathfrak{H})$.

We now generalize these ideas. Consider a physical system for which the observables are given by $\mathfrak{M}_s$, where $\mathfrak{M}$ is a von Neumann algebra on some Hilbert space. Assume $(\mathfrak{M}, \tau)$ is a W*-dynamical system, τ being the time evolution of the system. A physical state of the system is represented by a normal state ω on $\mathfrak{M}$ such that $\omega(A)$ is the expectation value of any observable $A \in \mathfrak{M}_s$ (this is a generalization of 2.3.5). If we consider an observable A at time 0, its expectation value at time t is then $\omega(\tau_t(A))$. We further assume that the equilibrium states of this system are precisely the faithful (τ, β)-KMS states on $\mathfrak{M}$ (these states are normal by definition, see 3.2.4). In other words any (τ, β)-KMS state on $\mathfrak{M}$ is an equilibrium state of the system at inverse temperature β. (The question arises whether a (τ, β)-KMS state exists on $\mathfrak{M}$ for any given $\beta \in \mathbb{R}$, and if it is unique if it does exist. We know by 3.2.5 that the answer is yes in both instances for the fdqs's of Chapter 2. The situation is not nearly this simple in the general case; the answer can be no in both instances. See for example [Br, Section 5.3.2 and Example 5.3.2]. We will not pursue this question any further.)

Now assume the system described above is in equilibrium at inverse temperature $\beta \in \mathbb{R}$, and that its state is ω (i.e. ω is a faithful (and normal) (τ, β)-KMS state on $\mathfrak{M}$).

Let

$$\alpha_t(A) := \tau_{-\beta t}(A) \quad \text{for all } t \in \mathbb{R} \text{ and } A \in \mathfrak{M}, \tag{1}$$

then $(\mathfrak{M}, \alpha)$ is a W*-dynamical system and ω is an α-KMS state by 3.2.6(1). So by 4.4.11 and 4.4.12 we know that α is the modular group associated to ω, and

$$\alpha_t(A) = \pi^{-1}(\Delta_t) A \pi^{-1}(\Delta_{-t}) \quad \text{for all } t \in \mathbb{R} \text{ and } A \in \mathfrak{M} \tag{2}$$

where $(\mathfrak{G}, \pi, \Omega)$ and Δ_t are as in 4.4.11. Hence the modular group and Δ_t has a direct physical meaning:

4.5.1 Conclusion *For a physical system represented by the W*-dynamical system $(\mathfrak{M}, \tau)$ which is in an equilibrium state ω on $\mathfrak{M}$ at inverse temperature $\beta \neq 0$, the time evolution τ is determined by means of (1) where α is the modular group associated to ω. α in turn is determined by (2) where Δ_t is unitary group associated to $(\pi(\mathfrak{M}), \Omega)$, $(\mathfrak{G}, \pi, \Omega)$ being the cyclic representation of $\mathfrak{M}$ associated to ω.*■

This conclusion can not be over-emphasized. It represents the sum total of almost everything we have done in this dissertation. The Tomita-Takesaki theorem, 4.3.11, seems to be a rather abstract theorem, and it certainly has a long and intricate proof. Yet Δ_t, which appears in 4.3.11(2), has a straightforward physical meaning as spelled out by 4.5.1. This surprising connection between theoretical physics and pure mathematics is the main motivation for this dissertation. It is important to note that although we have not really applied the Tomita-Takesaki theorem (except in the proof of 4.4.11), it does indeed have applications in mathematics; more about this in 4.6.

The physical meaning of J (the modular conjugation associated to $(\pi(\mathfrak{M}), \Omega)$) in 4.3.11 is not as clear. However, it is known that if two observables commute then the measurement of one does not in any way affect the result of a measurement of the other (this is not the case for non-commuting observables), see for example [C, III.C.6] or [Su, 2.4]. Since $\pi : \mathfrak{M} \to \mathfrak{N} := \pi(\mathfrak{M})$ is a $*$-isomorphism according to 4.4.9, we can consider $\mathfrak{M}$ and $\mathfrak{N}$ to be the same $*$-algebra. Let us assume for the moment that we can in fact view $\mathfrak{N}$ as our algebra of observables (this is not so far-fetched since $\mathfrak{N}$ is a von Neumann algebra according to 4.4.9, the question is only if $\mathfrak{N}$ can be considered to be the same von Neumann algebra as $\mathfrak{M}$), i.e. the elements of $\mathfrak{N}_s$ are now our observables. 4.3.11(1) says that $J\mathfrak{N}J = \mathfrak{N}'$, and it would now seem that this has no direct physical implications, because if we view any $A \in \mathfrak{N}'_s$ as an observable, and measure it in our system, then it will not have any effect on the measurements of any of the system's observables (namely $\mathfrak{N}_s$), since A commutes with every element of $\mathfrak{N}_s$. We could say that $J\mathfrak{N}J = \mathfrak{N}'$ effectively throws us out of our physical system.

4.6 Further remarks

In this dissertation we concentrated on showing the connection between the Tomita-Takesaki theory and equilibrium states in quantum statistical mechanics. One might be led to believe that this is the only significance of the Tomita-Takesaki theory. This is not the case. It turns out that the Tomita-Takesaki theory is a very important tool in the classification of the so-called type III von Neumann algebras (see for example [Pe, 8.15]).

It is a remarkable fact that the type III von Neumann algebras are of considerable importance in algebraic quantum field theory (the operator alge-

braic approach to relativistic quantum physics). See for example [H, III.2.1, V.2.4 and V.6]. This is an indication that the relevance of the Tomita-Takesaki theory to physics is far greater than just the equilibrium states of quantum statistical mechanics that we discussed.

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